Riemannian geometry[1]

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[1] Based on the lecture notes [Ho1] whose main sources were [Ca] and [Le1].

Contents

1 Differentiable manifolds, a brief review

1.1 Definitions and examples

Definition 1.2. A topological space M is called a **topological n-manifold**, $n \in \mathbb{N}$, if

1. M is Hausdorff,
2. M has a countable base (i.e. M is N_2),
3. M is locally homeomorphic to $\mathbb{R}^n$.

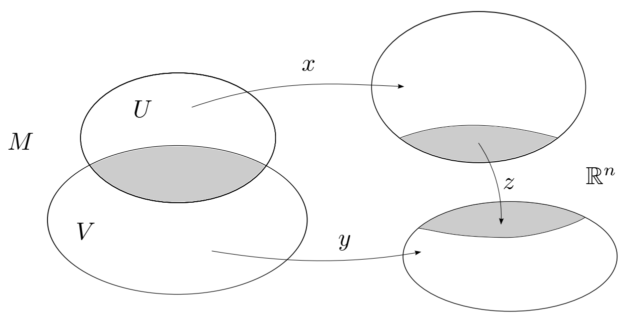
Let M be a topological n-manifold. A **chart** of M is a pair (U, x), where

1. $U \subset M$ is open,
2. $x \colon U \to xU \subset \mathbb{R}^n$ is a homeomorphism, $xU \subset \mathbb{R}^n$ open.

We say that charts (U, x) and (V, y) are C^∞**-compatible** if $U \cap V = \emptyset$ or

$$z = y \circ x^{-1} | x(U \cap V) \colon x(U \cap V) \to y(U \cap V)$$

is a C^∞-diffeomorphism.



A C^∞**-atlas**, $\mathcal{A}$, of M is a set of C^∞-compatible charts such that

$$M = \bigcup_{(U,x)\in\mathcal{A}} U.$$

A C^∞-atlas $\mathcal{A}$ is **maximal** if $\mathcal{A} = \mathcal{B}$ for all C^∞-atlases $\mathcal{B} \supset \mathcal{A}$. That is, $(U, x) \in \mathcal{A}$ if it is C^∞-compatible with every chart in $\mathcal{A}$.

Lemma 1.3. *Let M be a topological manifold. Then*

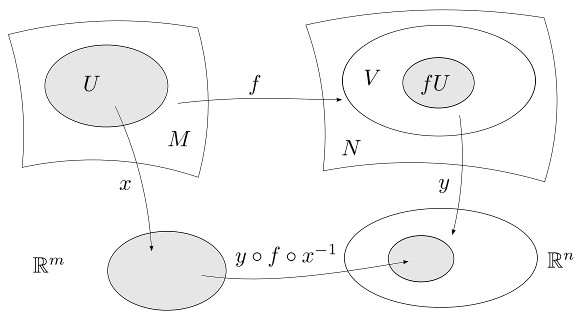
1. *every C^∞-atlas, $\mathcal{A}$, of M belongs to a unique maximal C^∞-atlas (denoted by $\bar{\mathcal{A}}$).*
2. *C^∞-atlases $\mathcal{A}$ and $\mathcal{B}$ belong to the same maximal C^∞-atlas if and only if $\mathcal{A} \cup \mathcal{B}$ is a C^∞-atlas.*

Proof. Exercise □

Definition 1.4. A **differentiable n-manifold** (or a smooth n-manifold) is a pair $(M, \mathcal{A})$, where M is a topological n-manifold and $\mathcal{A}$ is a maximal C^∞-atlas of M, also called a **differentiable structure** of M.

We abbreviate M or M^n and say that M is a C^∞-manifold, a differentiable manifold, or a smooth manifold.

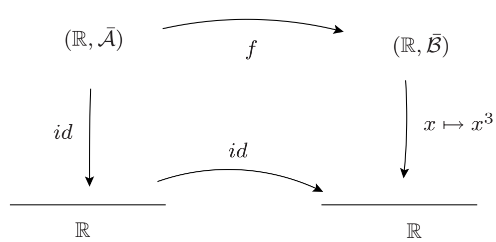
Definition 1.5. Let $(M^m, \mathcal{A})$ and $(N^n, \mathcal{B})$ be C^∞-manifolds. We say that a mapping $f\colon M \to N$ is C^∞ (or smooth) if each **local representation** of f (with respect to $\mathcal{A}$ and $\mathcal{B}$) is C^∞. More precisely, if the composition $y \circ f \circ x^{-1}$ is a smooth mapping $x(U \cap f^{-1}V) \to yV$ for every charts $(U, x) \in \mathcal{A}$ and $(V, y) \in \mathcal{B}$. We say that $f\colon M \to N$ is a C^∞-diffeomorphism if f is C^∞ and it has an inverse f^{-1} that is C^∞, too.



Remark 1.6. Equivalently, $f\colon M \to N$ is C^∞ if, for every $p \in M$, there exist charts (U, x) in M and (V, y) in N such that $p \in U$, $fU \subset V$, and $y \circ f \circ x^{-1}$ is $C^\infty(xU)$.

Examples 1.7. 1. $M = \mathbb{R}^n$, $\mathcal{A} = \{id\}$, $\bar{\mathcal{A}} =$ canonical structure.

2. $M = \mathbb{R}$, $\mathcal{A} = \{id\}$, $\mathcal{B} = \{x \overset{h}{\mapsto} x^3\}$. Now $\bar{\mathcal{A}} \neq \bar{\mathcal{B}}$ since $id \circ h^{-1}$ is not C^∞ at the origin. However, $(\mathbb{R}, \bar{\mathcal{A}})$ and $(\mathbb{R}, \bar{\mathcal{B}})$ are diffeomorphic by the mapping $f\colon (\mathbb{R}, \bar{\mathcal{A}}) \to (\mathbb{R}, \bar{\mathcal{B}})$, $f(y) = y^{1/3}$. Note: f is diffeomorphic with respect to structures $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ since id is the local representation of f.



3. If M is a differentiable manifold and $U \subset M$ is open, then U is a differentiable manifold in a natural way.

4. *Finite dimensional vector spaces.* Let V be an n-dimensional (real) vector space. Every norm on V determines a topology on V. This topology is independent of the choice of the norm since any two norms on V are equivalent (V finite dimensional). Let $E_1, \ldots, E_n$ be a basis of V and $E\colon \mathbb{R}^n \to V$ the isomorphism

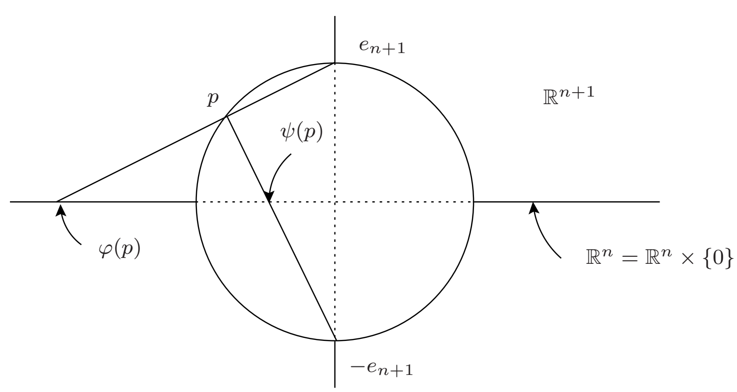
$$E(x) = \sum_{i=1}^{n} x^1 E_i, \quad x = (x^1, \ldots, x^n).$$

Then E is a homeomorphism (V equipped with the norm topology) and the (global) chart (V, E^{-1}) determines a smooth structure on V. Furthermore, these smooth structures are independent of the choice of the basis $E^1, \ldots, E_n$.

5. *Matrices.* Let $M(n \times m, \mathbb{R})$ be the set of all (real) $n \times m$-matrices. It is a nm-dimensional vector space and thus it is asmooth nm-manifold. A matrix $A = (a_{ij}) \in M(n \times m, \mathbb{R}),\ i = 1, \ldots, n,\ j = 1, \ldots, m$
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$
can be identified in a natural way with the point
$$(a_{11}, a_{12}, \ldots, a_{1m}, a_{21}, \ldots, a_{2m}, \ldots, a_{n1}, \ldots, a_{nm}) \in \mathbb{R}^{nm}$$
giving a global chart. If $n = m$, we abbreviate $M(n, \mathbb{R})$.

6. $GL(n, \mathbb{R})$ = *general linear group*
$$\begin{aligned} &= \{L \colon \mathbb{R}^n \to \mathbb{R}^n \text{ linear isomorphism}\} \\ &= \{A = (a_{ij}) \colon \text{ invertible (non-singular) } n \times n\text{-matrix}\} \\ &= \{A = (a_{ij}) \colon \det A \neq 0\}. \end{aligned}$$
[Note: an $n \times n$-matrix A is invertible (or non-singular) if it has an inverse matrix A^{-1}.]
By the identification above, we may interprete $GL(n, \mathbb{R}) \subset M(n, \mathbb{R}) = \mathbb{R}^{n^2}$. Equip $M(n, \mathbb{R})$ with the relative topology (induced by the inclusion $GL(n, \mathbb{R}) \subset M(n, \mathbb{R}) = \mathbb{R}^{n^2}$). Now the mapping $\det \colon M(n, \mathbb{R}) \to \mathbb{R}$ is continuous (a polynom of a_{ij} of degree n), and therefore $G(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ is open (as a preimage of an open set $\mathbb{R} \setminus \{0\}$ under a continuous mapping).

7. *Sphere* $\mathbb{S}^n = \{p \in \mathbb{R}^{n+1} \colon |p| = 1\}$. Let $e_1, \ldots, e_{n+1}$ be the standard basis of $\mathbb{R}^{n+1}$, let
$$\begin{aligned} \varphi &\colon \mathbb{S}^n \setminus \{e_{n+1}\} \to \mathbb{R}^n \\ \psi &\colon \mathbb{S}^n \setminus \{-e_{n+1}\} \to \mathbb{R}^n \end{aligned}$$
be the stereografic projections, and $\mathcal{A} = \{\varphi, \psi\}$. Details are left as an exercise.



8. *Projective space* $\mathbb{R}P^n$. The real n-dimensional projective space $\mathbb{R}P^n$ is the set of all 1-dimensional linear subspaces of $\mathbb{R}^{n+1}$, i.e. the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin. It can also be obtained by identifying points $x \in \mathbb{S}^n$ and $-x \in \mathbb{S}^n$. More precisely, define an equivalence relation
$$x \sim y \iff x = \pm y,\ x, y \in \mathbb{S}^n.$$

Then $\mathbb{R}P^n = \mathbb{S}^n/{\sim} = \{[x] \colon x \in \mathbb{S}^n\}$. Equip $\mathbb{R}P^n$ with so called quotient topology to obtain $\mathbb{R}P^n$ as a topological n-manifold. Details are left as an exercise.

9. *Product manifolds.* Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be differentiable manifolds and let $p_1 \colon M \times N \to M$ and $p_2 \colon M \times N \to N$ be the projections. Then
$$\mathcal{C} = \big\{ \big(U \times V, (x \circ p_1, y \circ p_2)\big) \colon (U, x) \in \mathcal{A}, (V, y) \in \mathcal{B} \}$$
is a C^∞-atlas on $M \times N$. Example
   (a) Cylinder $\mathbb{R}^1 \times \mathbb{S}^1$
   (b) Torus $\mathbb{S}^1 \times \mathbb{S}^1 = T^2$.

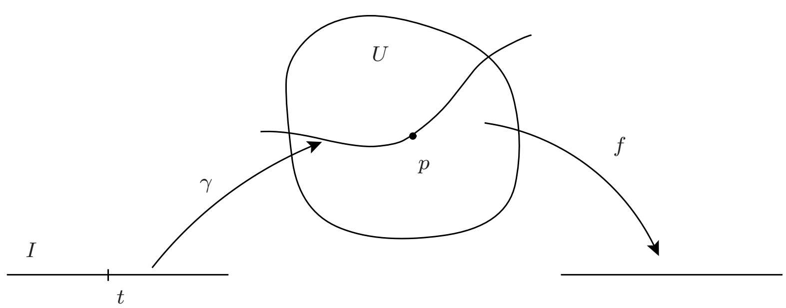
10. *Lie groups.* A Lie group is a group G which is also a differentiable manifold such that the group operations are C^∞, i.e.
$$(g, h) \mapsto gh^{-1}$$
is a C^∞-mapping $G \times G \to G$. For example, $GL(n, \mathbb{R})$ is a Lie group with composition as the group operation.

Remark 1.8. 1. Replacing C^∞ by, for example, C^k, C^ω (= real analytic), or complex analytic (in which case, $n = 2m$) we may equip M with other structures.

2. There are topological n-manifolds that do not admit differentiable structures. (Kervaire, $n = 10$, in the 60's; Freedman, Donaldson, $n = 4$, in the 80's). The Euclidean space $\mathbb{R}^n$ equipped with an arbitrary atlas is diffeomorphic to the canonical structure whenever $n \neq 4$ ("Exotic" structures of $\mathbb{R}^4$ were found not until in the 80's).

1.9 Tangent space

Let M be a differentiable manifold, $p \in M$, and $\gamma \colon I \to M$ a C^∞-path such that $\gamma(t) = p$ for some $t \in I$, where $I \subset \mathbb{R}$ is an open interval.



Write
$$C^\infty(p) = \{f \colon U \to \mathbb{R} \mid f \in C^\infty(U),\ U \text{ some neighborhood of } p\}.$$

Note: Here U may depend on f, therefore we write $C^\infty(p)$ instead of $C^\infty(U)$.

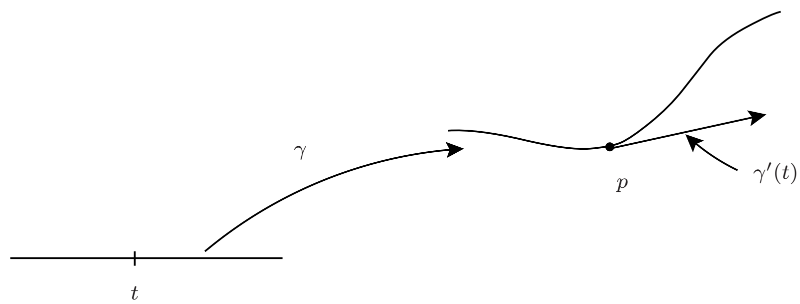
Now the path γ defines a mapping $\dot\gamma_t \colon C^\infty(p) \to \mathbb{R}$,
$$\dot\gamma_t f = (f \circ \gamma)'(t).$$

Note: The real-valued function $f \circ \gamma$ is defined on some neighborhood of $t \in I$ and $(f \circ \gamma)'(t)$ is its usual derivative at t.

Interpretation: We may interprete $\dot\gamma_t f$ as "a derivative of f in the direction of γ at the point p".

Example 1.10. $M = \mathbb{R}^n$
If $\gamma = (\gamma_1, \dots, \gamma_n)\colon I \to \mathbb{R}^n$ is a smooth path and $\gamma'(t) = \big(\gamma_1'(t), \dots, \gamma_n'(t)\big) \in \mathbb{R}^n$ is the derivative of γ at t, then

$$\dot\gamma_t f = (f \circ \gamma)'(t) = f'(p)\gamma'(t) = \gamma'(t) \cdot \nabla f(p).$$



In general: The mapping $\dot\gamma_t$ satisfies:
Suppose $f, g \in C^\infty(p)$ and $a, b \in \mathbb{R}$. Then

a) $\dot\gamma_t(af + bg) = a\dot\gamma_t f + b\dot\gamma_t g$,

b) $\dot\gamma_t(fg) = g(p)\dot\gamma_t f + f(p)\dot\gamma_t g$.

We say that $\dot\gamma_t$ is a **derivation**.

Motivated by the discussion above we define:

Definition 1.11. A **tangent vector** of M at $p \in M$ is a mapping $v\colon C^\infty(p) \to \mathbb{R}$ that satisfies:

(1) $v(af + bg) = av(f) + bv(g), \quad f, g \in C^\infty(p),\ a, b \in \mathbb{R}$;

(2) $v(fg) = g(p)v(f) + f(p)v(g)$ (cf. the "Leibniz rule").

The **tangent space** at p is the ($\mathbb{R}-$)linear vector space of tangent vector at p, denoted by T_pM or M_p.

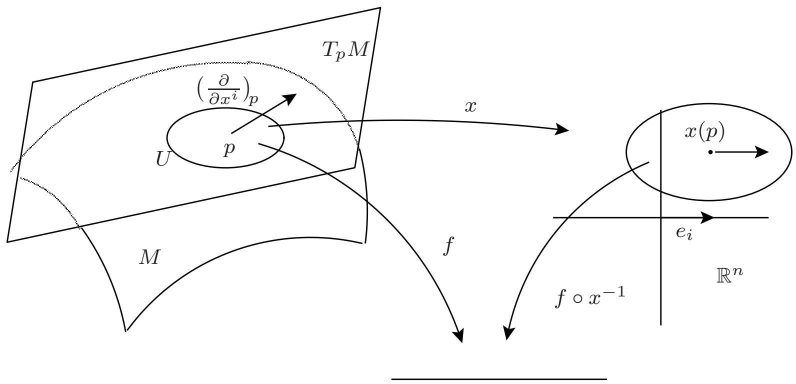
Remarks 1.12.
1. If $v, w \in T_pM$ and $c, d \in \mathbb{R}$, then $cv + dw$ is (of course) the mapping $(av + bw)\colon C^\infty(p) \to \mathbb{R}$,
$$(cv + dw)(f) = cv(f) + dw(f).$$
It is easy to see that $cv + dw$ is a tangent vector at p.
2. We abbreviate $vf = v(f)$.
3. Claim: If $v \in T_pM$ and $c \in C^\infty(p)$ is a constant function, then $cv = 0$. (Exerc.)
4. Let U be a neighborhood of p interpreted as a differentiable manifold itself. Since we use functions in $C^\infty(p)$ in the definition of T_pM, the spaces T_pM and T_pU can be identified in a natural way.

Let $(U, x),\ x = (x^1, x^2, \dots, x^n)$, be a chart at p. We define a tangent vector (so-called **coordinate vector**) $\left(\frac{\partial}{\partial x^i}\right)_p$ at p by setting

$$\left(\frac{\partial}{\partial x^i}\right)_p f = D_i\big(f \circ x^{-1}\big)\big(x(p)\big), \quad f \in C^\infty(p).$$

Here D_i is the partial derivative with respect to i^{th} variable. We also denote

$$(\partial_i)_p = D_{x_i}(p) = \left(\frac{\partial}{\partial x^i}\right)_p.$$



Remarks 1.13. 1. It is easy to see that $(\partial_i)_p$ is a tangent vector at p.

2. If (U, x), $x = (x^1, \ldots, x^n)$, is a chart at p, then $(\partial_i)_p x^j = \delta_{ij}$.

Next theorem shows (among others) that T_pM is n-dimensional.

Lemma 1.14. *If $f \in C^k(B)$, $k \geq 1$, is a real-valued function in a ball $B = B^n(0, r) \subset \mathbb{R}^n$, then there exist functions $g_i \in C^{k-1}(B), i = 1, \ldots, n$, such that $g_i(0) = D_i f(0)$ and*

$$f(y) - f(0) = \sum_{i=1}^{n} y_i g_i(y)$$

for all $y = (y_1, \ldots, y_n) \in B$.

Proof. For $y \in B$ we have

$$\begin{aligned} f(y) - f(0) &= f(y) - f(y_1, \ldots, y_{n-1}, 0) \\ &\quad + f(y_1, \ldots, y_{n-1}, 0) - f(y_1, \ldots, y_{n-2}, 0, 0) \\ &\quad + f(y_1, \ldots, y_{n-2}, 0, 0) - f(y_1, \ldots, y_{n-3}, 0, 0 \\ &\quad\ \vdots \\ &\quad + f(y_1, 0, \ldots, 0) - f(0) \\ &= \sum_{i=1}^{n} \int_0^1 f(y_1, \ldots, y_{i-1}, ty_i, 0, \ldots, 0) \\ &= \sum_{i=1}^{n} \int_0^1 \frac{d}{dt}\big(f(y_1, \ldots, y_{i-1}, ty_i, 0, \ldots, 0)\big) dt \\ &= \sum_{i=1}^{n} \int_0^1 D_i f(y_1, \ldots, y_{i-1}, ty_i, 0, \ldots, 0) y_i dt. \end{aligned}$$

Define

$$g_i(y) = \int_0^1 D_i f(y_1, \ldots, y_{i-1}, ty_i, 0, \ldots, 0) dt.$$

Then $g_i \in C^{k-1}(B)$ (since $f \in C^k(B)$) and $g_i(0) = D_i f(0)$. $\square$

Theorem 1.15. *If (U, x), $x = (x^1, \ldots, x^n)$, is a chart at p and $v \in T_pM$, then*

$$v = \sum_{i=1}^{n} v x^i (\partial_i)_p.$$

Furthermore, the vectors $(\partial_i)_p$, $i = 1, \ldots, n$, form a basis of T_pM and hence $\dim T_pM = n$.

Proof. For $u \in U$ we write $x(u) = y = (y^1, \dots, y^n) \in \mathbb{R}^n$, so $x^i(u) = y^i$. We may assume that $x(p) = 0 \in \mathbb{R}^n$. Let $f \in C^\infty(p)$. Since $f \circ x^{-1}$ on C^∞, there exist (by Lemma 1.14) a ball $B = B^n(0,r) \subset xU$ and functions $g_i \in C^\infty(B)$ such that

$$\big(f \circ x^{-1}\big)(y) = \big(f \circ x^{-1}\big)(0) + \sum_{i=1}^n y_i g_i(y) \quad \forall y \in B$$

and $g_i(0) = D_i\big(f \circ x^{-1}\big)(0) = (\partial_i)_p f$. Thus

$$f(u) = f(p) + \sum_{i=1}^n x^i(u) h_i(u),$$

where $h_i = g_i \circ x$ and

$$h_i(p) = g_i(0) = (\partial_i)_p f.$$

Hence

$$\begin{aligned} vf &= \underbrace{v\big(f(p)\big)}_{=0} + \sum_{i=1}^n \underbrace{x^i(p)}_{=0} v h_i + \sum_{i=1}^n (vx^i) h_i(p) \\ &= \sum_{i=1}^n vx^i (\partial_i)_p f. \end{aligned}$$

This holds for every $f \in C^\infty(p)$, and therefore

$$v = \sum_{i=1}^n vx^i (\partial_i)_p.$$

Hence the vectors $(\partial_i)_p,\ i = 1, \dots, n$, span T_pM. To prove the linear independence of these vectors, suppose that

$$w = \sum_{i=1}^n b_i (\partial_i)_p = 0.$$

Then

$$0 = wx^j = \sum_{i=1}^n b_i \underbrace{(\partial_i)_p x^j}_{=\delta_{ij}} = b_j$$

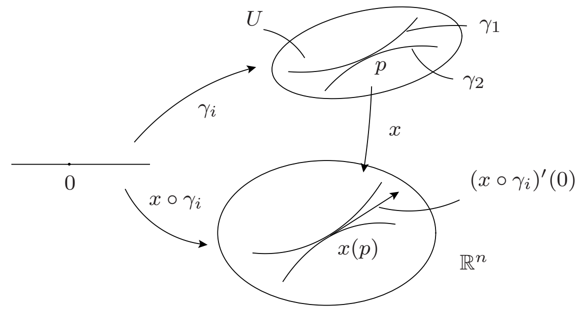
for all $j = 1, \dots, n$,, and so vectors $(\partial_i)_p,\ i = 1, \dots, n$, are linearly independent. □

Remark 1.16. Our definition for tangent vectors is useful only for C^∞-manifolds. Reason: If M is a C^k-manifold, then the functions h_i in the proof of Theorem 1.15 are not necessarily C^k-smooth (only C^{k-1}-smothness is granted).

Another definition that works also for C^k-manifolds, $k \geq 1$, is the following: Let M be a C^k-manifold and $p \in M$. Let $\gamma_i \colon I_i \to M$ be C^1-paths, $0 \in I_i \subset \mathbb{R}$ open intervals, and $\gamma_i(0) = p,\ i = 1, 2$. Define an equivalence relation $\gamma_1 \sim \gamma_2 \iff$ for every chart (U, x) at p we have

$$\big(x \circ \gamma_1\big)'(0) = \big(x \circ \gamma_2\big)'(0)$$

Def.: Equivalence classes = tangent vectors at p. In the case of a C^∞-manifold this definition coincides with the earlier one ($[\gamma] = \dot\gamma_0$).



1.17 Tangent map

Definition 1.18. Let M^m and N^n be differentiable manifolds and let $f\colon M \to N$ be a C^∞ map. The **tangent map** of f at p is a linear map $f_*\colon T_pM \to T_{f(p)}N$ defined by

$$(f_* v)g = v(g \circ f), \quad \forall g \in C^\infty\big(f(p)\big),\ v \in T_pM.$$

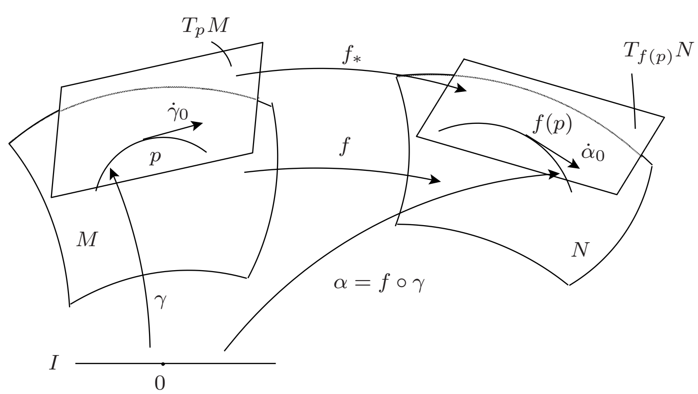
We also write f_{*p} or T_pf.

Remarks 1.19. 1. It is easily seen that f_*v is a tangent vector at $f(p)$ for all $v \in T_pM$ and that f_* is linear.

2. If $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, then $f_{*p} = f'(p)$ (see the canonical identification $T_p\mathbb{R}^n = \mathbb{R}^n$ below).

3. "Chain rule": Let $M,\ N$, and L be differentiable manifolds and let $f\colon M \to N$ and $g\colon N \to L$ be C^∞-maps. Then
$$(g \circ f)_{*p} = g_{*f(p)} \circ f_{*p}$$
for all $p \in M$. (Exerc.)

4. An interpretation of a tangent map using paths:
Let $v \in T_pM$ and let $\gamma\colon I \to M$ be a C^∞-path such that $\gamma(0) = p$ and $\dot\gamma_0 = v$. Let $f\colon M \to N$ be a C^∞-map and $\alpha = f \circ \gamma\colon I \to N$. Then $f_*v = \dot\alpha_0$. (Exerc.)



Let $x = (x^1, \dots, x^m)$ be a chart at $p \in M^m$ and $y = (y^1, \dots, y^n)$ a chart at $f(p) \in N^n$. What is the matrix of $f_*\colon T_pM \to T_{f(p)}N$ with respect to bases $\big(\frac{\partial}{\partial x^i}\big)_p, i = 1, \dots, m$, and $\big(\frac{\partial}{\partial y^j}\big)_{f(p)}, j = 1, \dots, n$,? By Theorem 1.15,

$$f_*\left(\frac{\partial}{\partial x^j}\right)_p = \sum_{i=1}^n f_*\left(\frac{\partial}{\partial x^j}\right)_p y^i \left(\frac{\partial}{\partial y^i}\right)_{f(p)}, \quad 1 \le j \le m.$$

Thus we obtain an $n \times m$ matrix (a_{ij}),

$$a_{ij} = f_* \left(\frac{\partial}{\partial x^j} \right)_p y^i = \frac{\partial}{\partial x^j} (y^i \circ f).$$

This is called the Jacobian matrix of f at p (with respect to given bases). As a matrix it is the same as the matrix of the linear map $g'\big(x(p)\big),\ g = y \circ f \circ x^{-1}$, with respect to standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$.

Recall that $f\colon M^m \to N^n$ is a diffeomorphism if f and its inverse f^{-1} are C^∞. A mapping $f\colon M \to N$ is a **local diffeomorphism** at $p \in M$ if there are neighborhoods U of p and V of $f(p)$ such that $f\colon U \to V$ is a diffeomorphism.
Note: Then necessarily $m = n$. (Exerc.)

Theorem 1.20. *Let $f\colon M \to N$ be C^∞ and $p \in M$. Then f is a local diffeomorphism at p if and only if $f_*\colon T_pM \to T_{f(p)}N$ is an isomorphism.*

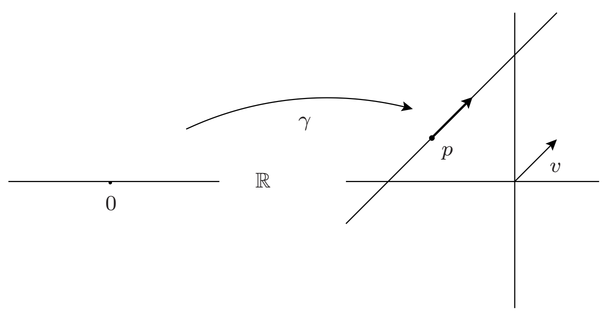
Proof. Apply the inverse function theorem (of $\mathbb{R}^n$). Details are omitted, □

Tangent space of an n-dimensional vector space. Let V be an n-dimensional (real) vector space. Recall that any (linear) isomorphism $x\colon V \to \mathbb{R}^n$ induces the same C^∞-structure on V. Thus we may identify V and T_pV in a natural way for any $p \in V$: If $p \in V$, then there exists a canonical isomorphism $i\colon V \to T_pV$. Indeed, let $v \in V$ and $\gamma\colon \mathbb{R} \to V$ the path

$$\gamma(t) = p + tv.$$

We set

$$i(v) = \dot\gamma_0.$$

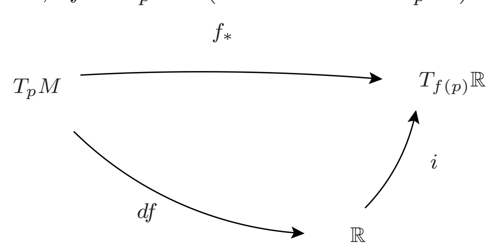


Example: $V = \mathbb{R}^n, \quad T_p\mathbb{R}^n \cong \mathbb{R}^n$ canonically.

If $f\colon M \to \mathbb{R}$ is C^∞ and $p \in M$, we define the **differential** of f, $df\colon T_pM \to \mathbb{R}$, by setting

$$df\, v = vf, \quad v \in T_pM.$$

(Also denoted by df_p.)
By the isomorphism $i\colon \mathbb{R} \to T_{f(p)}\mathbb{R}$ as above, we obtain $df = i^{-1} \circ f_*$. Usually we identify $df = f_*$.
Note: Since $df\colon T_pM \to \mathbb{R}$ is linear, $df \in T_pM^*$ (= the dual of T_pM).



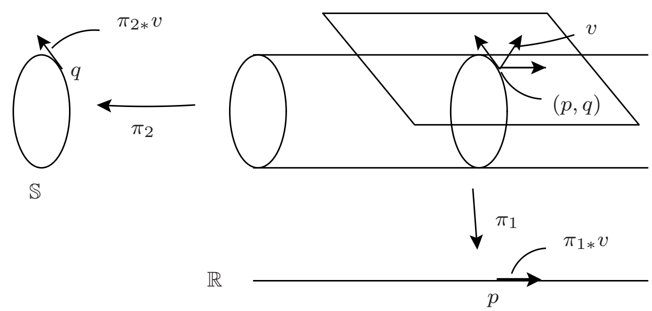
Tangent space of a product manifold. Let M and N be differentiablemanifolds and let

$$\pi_1\colon M\times N\to M,$$
$$\pi_2\colon M\times N\to N$$

be the projections. Using these projections we may identify $T_{(p,q)}(M\times N)$ and $T_pM\oplus T_qN$ in a natural way: Define a canonical isomorphism

$$\tau\colon T_{(p,q)}(M\times N)\to T_pM\oplus T_qN,$$
$$\tau v=\underbrace{\pi_{1*}v}_{\in T_pM}+\underbrace{\pi_{2*}v}_{\in T_pN},\quad v\in T_{(p,q)}(M\times N).$$

Example: $M=\mathbb{R},\ N=\mathbb{S}^1$



Let $f\colon M\times N\to L$ be a C^∞-mapping, where L is a differentiable manifold. For every $(p,q)\in M\times N$ we define mappings

$$f_p\colon N\to L,\quad f^q\colon M\to L,$$
$$f_p(q)=f^q(p)=f(p,q).$$

Thus, for $v\in T_pM$ and $w\in T_qN$, we have

$$f_*(v+w)=(f^q)_*v+(f_p)_*w.\quad\text{(Exerc.)}$$

1.21 Tangent bundle

Let M be a differentiable manifold. We define the **tangent bundle** TM of M as a disjoint union of all tangent spaces of M, i.e.

$$TM=\bigsqcup_{p\in M}T_pM.$$

Points in TM are thus pairs (p,v), where $p\in M$ and $v\in T_pM$. We usually abbreviate $v=(p,v)$, because the condition $v\in T_pM$ determines $p\in M$ uniquely.

Let $\pi\colon TM\to M$ be the projection

$$\pi(v)=p,\quad\text{if } v\in T_pM.$$

The tangent bundle TM has a canonical structure of a differentiable manifold.

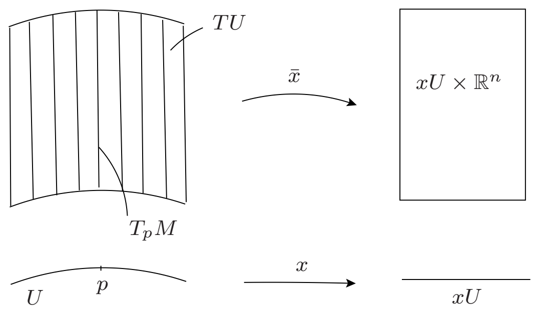
Theorem 1.22. *Let M be a differentiable n-manifold. The tangent bundle TM of M can be equipped with a natural topology and a C^∞-structure of a smooth $2n$-manifold such that the projection $\pi\colon TM\to M$ is smooth.*

Proof. (Idea): Let (U, x), $x = (x^1, \dots, x^n)$, be a chart on M. Define a one-to-one mapping

$$\bar{x} \colon TU \to xU \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$$

as follows. [Here $TU = \bigsqcup_{p \in U} T_pU = \bigsqcup_{p \in U} T_pM$.] If $p \in U$ and $v \in T_p$, we set

$$\bar{x}(v) = \big(\underbrace{x^1(p), \dots, x^n(p)}_{\in \mathbb{R}^n}, \underbrace{vx^1, \dots, vx^n}_{\in \mathbb{R}^n}\big)$$



First we transport the topology of $\mathbb{R}^n \times \mathbb{R}^n$ into TM by using maps $\bar{x}$ and then we verify that pairs $(TU, \bar{x})$ form an atlas of TM. We obtain a C^∞-structure for TM. [Details are left as an exercise.] □

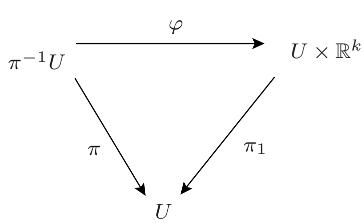
In the sequel the tangent bundle of M means TM equipped with this C^∞-structure. It is an example of a vector bundle over M.

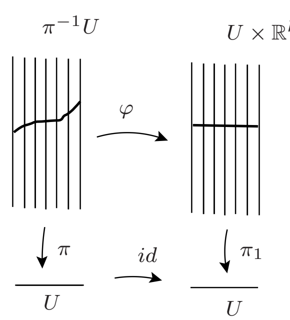
Let $\pi \colon TM \to M$ be the projection ($\pi(v) = p$ for $v \in T_pM$). Then $\pi^{-1}(p) = T_pM$ is a **fibre** over p. If $A \subset M$, then a map $s \colon A \to TM$, with $\pi \circ s = id$, is a **section** of TM in A (or a **vector field**).

Smooth vector bundles. Let M be a differentiable manifold. A smooth **vector bundle of rank** k **over** over M is a pair (E, π), where E is a smooth manifold and $\pi \colon E \to M$ is a smooth surjective mapping (projection) such that:

(a) for every $p \in M$, the set $E_p = \pi^{-1}(p) \subset E$ is a k-dimensional real vector space (= a fiber of E over p);

(b) for every $p \in M$ there exist a neighborhood $U \ni p$ and a diffeomorphism $\varphi \colon \pi^{-1}U \to U \times \mathbb{R}^k$ (= local trivialization of E over U) such that the following diagram commutes





[above $\pi_1 \colon U \times \mathbb{R}^k \to U$ is the projection] and that $\varphi | E_q \colon E_q \to \{q\} \times \mathbb{R}^k$ is a linear isomorphism for every $q \in U$.

The manifold E is called the *total space* and M is called the *base* of the bundle. If there exists a local trivialization of E over the whole manifold M, $\varphi\colon \pi^{-1}M \to M \times \mathbb{R}^k$, then E is a *trivial bundle.*

A **section** of E is any map $\sigma\colon M \to E$ such that $\pi \circ \sigma = id\colon M \to M$. A **smooth section** is a section that is smooth as a map $\sigma\colon M \to E$ (note that M and E are smooth manifolds). **Zero section** is a map $\zeta\colon M \to E$ such that

$$\zeta(p) = 0 \in E_p \quad \forall p \in M.$$

A **local frame** of E over an open set $U \subset M$ is a k-tuple $(\sigma_1, \dots, \sigma_k)$, where each σ_i is a smooth section of E (over U) ssuch that $\big(\sigma_1(p), \sigma_2(p), \dots, \sigma_k(p)\big)$ is a basis of E_p for all $p \in U$. If $U = M$, $(\sigma_1, \dots, \sigma_k)$ is called a **global frame**.

1.23 Submanifolds

Definition 1.24. Let M and N be differentiable manifolds and $f\colon M \to N$ a C^∞-map. We say that :

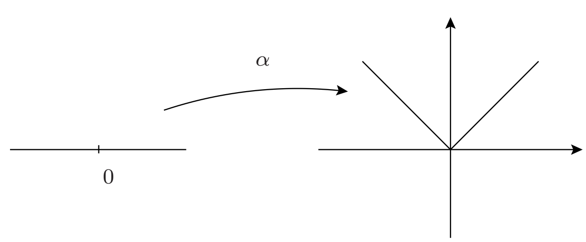
1. f is a **submersion** if $f_{*p}\colon T_pM \to T_{f(p)}N$ is surjective $\forall p \in M$.
2. f is an **immersion** if $f_{*p}\colon T_pM \to T_{f(p)}N$ is injective $\forall p \in M$.
3. f is an **embedding** if f is an immersion and $f\colon M \to fM$ is homeomorphici (note relative topology in fM).

If $M \subset N$ and the inclusion $i\colon M \hookrightarrow N,\ i(p) = p$, is an embedding, we say that M is a **submanifold** of N.

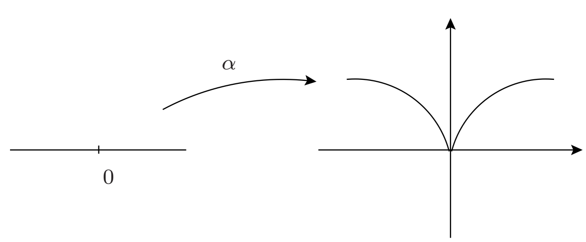
Remark 1.25. If $f\colon M^m \to N^n$ is an immersion, then $m \le n$ and $n - m$ is the **codimension** of f.

Examples 1.26. (a) If $M_1, \dots, M_k$ are smooth manifolds, then all projections $\pi_i\colon M_1 \times \cdots \times M_k \to M_i$ are submersions.

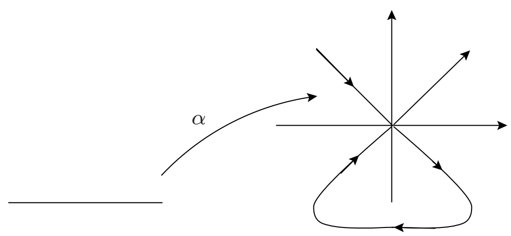
(b) $(M = \mathbb{R},\ N = \mathbb{R}^2)$ $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ \alpha(t) = (t, |t|)$ is not differentiable at $t = 0$.



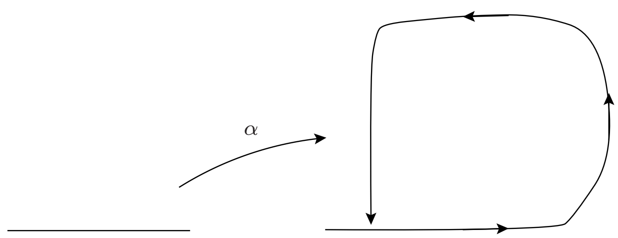
(c) $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ \alpha(t) = (t^3, t^2)$ is C^∞ but not an immersion since $\alpha'(0) = 0$.



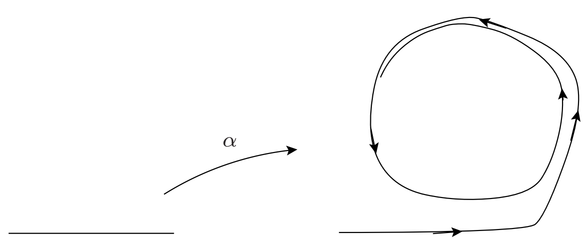
(d) $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ \alpha(t) = (t^3 - 4t, t^2 - 4)$ is C^∞ and an immersion but not an embedding ($\alpha(\pm 2) = (0,0)$).



(e) The map α (in the picture below) has an inverse but it is not an embedding since the inverse in not continuous (in the relative topology of the image).



(f) The following α is an embedding.



Remark 1.27. The notion of a submanifold has different meanings in the literature. For instance, Bishop-Crittenden [BC] allows the case (e) in the definition of a submanifold.

Theorem 1.28. *Let $f\colon M^m \to N^n$ be an immersion. Then each point $p \in M^m$ has a neighborhood U such that $f|U\colon U \to N^n$ is an embedding.*

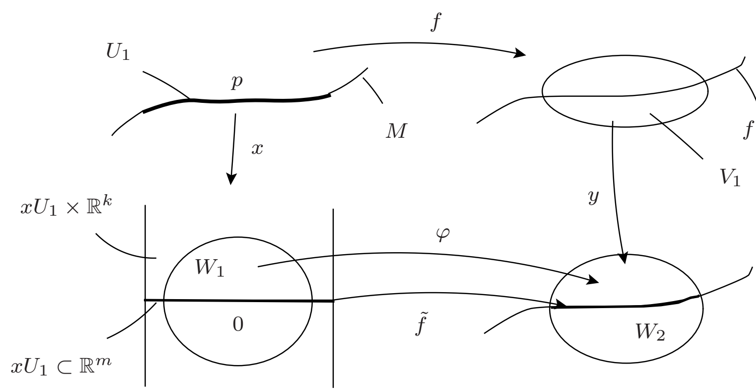
Proof. Fix $p \in M$. We have to find a neighborhood $U \ni p$ such that $f|U\colon U \to fU$ is a homeomorphism when fU is equipped with the relative topology. Let (U_1, x) and (V_1, y) be charts at points p and $f(p)$, respectivel, such that $fU_1 \subset V_1,\ x(p) = 0\ (\in \mathbb{R}^m)$, and $y\big(f(p)\big) = 0\ (\in \mathbb{R}^n)$. Write $\tilde f = y \circ f \circ x^{-1},\ \tilde f = (\tilde f_1, \ldots, \tilde f_n)$. Since f is an immersion, $\tilde f'(0)\colon \mathbb{R}^m \to \mathbb{R}^n$ is injective. We may assume that $\tilde f'(0)\mathbb{R}^m = \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^k,\ k = n - m$ (otherwise, apply a rotation in $\mathbb{R}^n$). Then $\det \tilde f'(0) \neq 0$, when $\tilde f'(0)$ is interpreted as a linear map $\mathbb{R}^m \to \mathbb{R}^m$. Define a mapping

$$\begin{aligned}
&\varphi\colon xU_1 \times \mathbb{R}^k \to \mathbb{R}^n,\\
&\varphi(\tilde x, t) = \big(\tilde f_1(\tilde x), \tilde f_2(\tilde x), \ldots, \tilde f_m(\tilde x), \tilde f_{m+1}(\tilde x) + t_1, \ldots, \tilde f_{m+k}(\tilde x) + t_k)\big),\\
&\tilde x \in xU_1, \quad t = (t_1, \ldots, t_k) \in \mathbb{R}^k.
\end{aligned}$$

The matrix of $\varphi'(0,0)\colon \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$ is

$$\begin{pmatrix} \frac{\partial \tilde{f}_i(0)}{\partial \tilde{x}_j} & 0 \\ * & I_k \end{pmatrix},$$

and therefore $\det \varphi'(0,0) = \det \tilde{f}'(0) \neq 0$. By the inverse mapping theorem, there are neighborhoods $0 \in W_1 \subset xU_1 \times \mathbb{R}^k$ and $0 \in W_2 \subset \mathbb{R}^n$ such that $\varphi|W_1\colon W_1 \to W_2$ is a diffeomorphism. Write $\tilde{U} = W_1 \cap xU_1$ and $U = x^{-1}\tilde{U}$ $(\subset U_1)$. Since $\varphi|xU_1 \times \{0\} = \tilde{f}$, we have $\varphi|\tilde{U} = \tilde{f}$. In particular, $f|U\colon U \to fU$ is a homeomorphism, when fU is equipped with the relative topology. □



Example 1.29. Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^∞-function such that $\nabla f(p) = \big(D_1 f(p), \dots, D_{n+1} f(p)\big) \neq 0$ for every $p \in M = \{x \in \mathbb{R}^{n+1}\colon f(x) = 0\} \neq \emptyset$. Then M is an n dimensional submanifold of $\mathbb{R}^{n+1}$.

Proof of the claim above. (Idea): Let $p \in M$ be arbitrary. Applying a transformation and a rotation if necessary we may assume that $p = 0$ and

$$\nabla f(0) = \Big(0, \dots, 0, \frac{\partial f}{\partial x_{n+1}}(0)\Big).$$

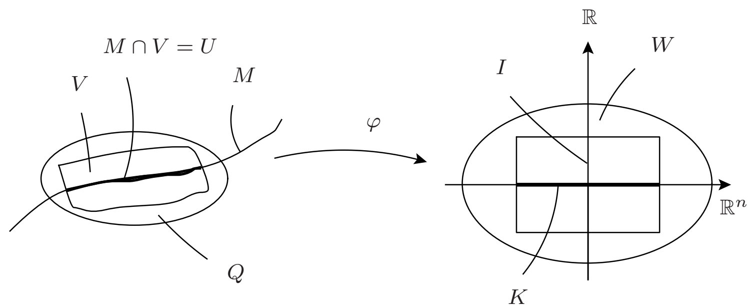
Then $\frac{\partial f}{\partial x_{n+1}}(0) \neq 0$. Define a mapping $\varphi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

$$\varphi(x) = \big(x_1, \dots, x_n, f(x)\big), \quad x = (x_1, \dots, x_n, x_{n+1}).$$

Then

$$\det \varphi'(0) = \begin{vmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{\partial f}{\partial x_{n+1}}(0) \end{vmatrix} = \frac{\partial f}{\partial x_{n+1}}(0) \neq 0.$$

By the inverse mapping theorem, there exist neighborhoods $Q \ni p$ and $W \ni \varphi(0) = (0,0) \in \mathbb{R}^n \times \mathbb{R}$ such that $\varphi\colon Q \to W$ is a diffeomorphism.



Choose an open set $K \subset \mathbb{R}^n$, $0 \in K$, and an open interval $I \subset \mathbb{R}$, $0 \in I$, such that $K \times I \subset W$. Let $V = \varphi^{-1}(K \times I) \cap Q$ and $U = V \cap M$. Then $\varphi\colon V \to K \times I$ is a diffeomorphism. Let $y = \varphi|U$. Repeat the above for all $p \in M$ and conclude that pairs (U, y) form a C^∞-atlas of M. Since the inclusion $i\colon M \hookrightarrow \mathbb{R}^{n+1}$ satisfies

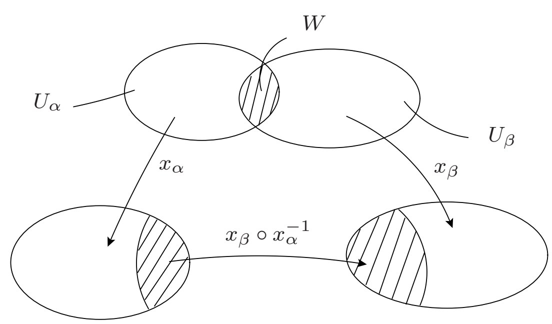
$$i|U = y^{-1} \circ \varphi|U,$$

i is an embedding. □

1.30 Orientation

Definition 1.31. A smooth manifold M is **orientable** if it admits a smooth atlas $\{(U_\alpha, x_\alpha)\}$ such that for every α and β, with $U_\alpha \cap U_\beta = W \neq \emptyset$, the Jacobian determinant of $x_\beta \circ x_\alpha^{-1}$ is positive at each point $q \in x_\alpha W$, i.e.

$$\det\big(x_\beta \circ x_\alpha^{-1}\big)'(q) > 0, \quad \forall q \in x_\alpha W. \tag{1.32}$$



In the opposite case M is **nonorientable.** If M is orientable, then an atlas satisfying (1.32) is called an **orientation** of M. Furthermore, M (equipped with such atlas) is said to be **oriented**. We say that two atlases satisfying (1.32) determine **the same orientation** if their union satisfies (1.32), too.

Remarks 1.33. 1. Warning: The notion of a smooth structure has different meanings in the literature (e.g. do Carmo [Ca]). What goes wrong if we define orientability by saying: "M is orientable if it admits a C^∞-structure such that (1.32) holds?" (Exerc.)

2. An is orientable and connected smooth manifold has exactly two distinct orientations. (Exerc.)

3. If M and N are smooth manifolds and $f\colon M \to N$ is a diffeomorphism, then

$$M \text{ is orientable} \iff N \text{ is orientable}.$$

4. Let M and N be connected oriented smooth manifolds and $f\colon M \to N$ a diffeomorphism. Then f induces an orientation on N. If the induced orientation of N is the same as the initial one, we say that f is **sense-preserving** (or f preserves the orientation). Otherwise, f is called **sense-reversing** (or f reverses the orientation).

Examples 1.34. 1. Suppose that there exists an atlas $\{(U,x),(V,y)\}$ of M such that $U \cap V$ is connected. Then M is orientable.
Proof. The mapping $y \circ x^{-1}\colon x(U\cap V) \to y(U \cap V)$ is diffeopmorphic, so

$$\det\big(y \circ x^{-1}\big)'(q) \neq 0 \quad \forall q \in x(U \cap V).$$

Since $q \mapsto \det\big(y\circ x^{-1}\big)'(q)$ is continuous and $x(U \cap V)$ is connected, the determinat can not change its sign. If the sign is positive, we are done. If the sign is negative, replace the chart $(V,y),\ y = (y_1,\dots,y_n)$, by a chart $(V,\tilde y),\ \tilde y = (-y_1, y_2, \dots, y_n)$. Then the atlas $\{(U,x),(V,\tilde y)\}$ satisfies (1.32). □

2. In particular, the sphere S^n is orientable.

1.35 Vector fields

Let M be a differentiable manifold and $A \subset M$. Recall that a mapping $V\colon A \to TM$ such that $X(p) \in T_pM$ for all $p \in M$ is called a **vector field** in A. We usually write $X_p = X(p)$. If $A \subset M$ is open and $X\colon A \to TM$ is a C^∞-vector field, we write $X \in \mathcal{T}(A)$. Clearly $\mathcal{T}(A)$ is a real vector space, where addition and multiplication by a scalar are defined pointwise: If $X, Y \in \mathcal{T}(A)$ and $a, b \in \mathbb{R}$, then $aX + bY,\ p \mapsto aX_p + bY_p$, is a smooth vector field. Furthermore, a vector field $V \in \mathcal{T}(A)$ can be multiplied by a smooth (real-valued) function $f \in C^\infty(A)$ producing a smooth vector field $fV,\ p \mapsto f(p)V_p$.

Let M be a differentable n-manifold and $A \subset M$ open. We say that vector fields $V^1, \dots, V^n$ in A form a **local frame** (or a **frame** in A) if the vectors $V^1_p, \dots, V^n_p$ form a basis of T_pM for every $p \in A$. In the case $A = M$ we say that vector fields $V^1, \dots, V^n$ form a **globalin frame**. Furthermore, M is called **parallelizable** if it admits a smooth global frame. This is equivalent to TM being a trivial bundle.[1]

Definition 1.36. (**Einstein summation convention**) If in a term the same index appears twice, both as upper and a lower index, that term is assumed to be summed over all possible values of that index (usually from 1 to the dimension).

Let $(U,x),\ x = (x^1,\dots,x^n)$, be a chart and $(\partial_i)_p = \big(\frac{\partial}{\partial x^i}\big)_p,\ i = 1,\dots,n$, the corresponding coordinate vectors at $p \in U$. Then the mappings

$$\partial_i\colon U \to TM,\ p \mapsto (\partial_i)_p = \left(\frac{\partial}{\partial x^i}\right)_p,$$

are vector fields in U, so-called **coordinate vector fields**. Since the vector fields ∂_i form a frame, so-called **coordinate frame**, in U, we can write any vector field V in U as

$$V_p = v^i(p)(\partial_i)_p, \quad p \in U,$$

where $v^i\colon U \to \mathbb{R}$. Functions v^i are called the **component functions** of V with respect to (U,x).

[1] Every Lie group is parallelizable; $\mathbb{S}^1, \mathbb{S}^3$, and $\mathbb{S}^7$ are the only parallelizable spheres; $\mathbb{R}P^1,\ \mathbb{R}P^3$, and $\mathbb{R}P^7$ are the only parallelizable projective spaces; a product $\mathbb{S}^n \times \mathbb{S}^m$ is parallelizable if at least one of the numbers $n > 0$ or $m > 0$ is odd.

Lemma 1.37. *Let V be a vector field on M. Then the following are equivalent:*

(a) $V \in \mathcal{T}(M)$*;*

(b) the component functions of V with respect to any chart are smooth;

(c) If $U \subset M$ is open and $f\colon U \to \mathbb{R}$ is smooth, then the function $Vf\colon U \to \mathbb{R}$, $(Vf)(p) = V_p f$, is smooth.

Proof. Exercise. □

Remark 1.38. In particular, coordinate vector fields are smooth by (b).

Suppose that $A \subset M$ is open and $V, W \in \mathcal{T}(A)$. If $f \in C^\infty(p)$, where $p \in A$, then $Vf \in C^\infty(p)$ and thus $W_p(Vf) \in \mathbb{R}$ (= "the derivative of Vf in the direction of W_p"). The function $A \to \mathbb{R}$, $p \mapsto W_p(Vf)$, is denoted by WVf. Thus $(WVf)(p) = W_p(Vf)$. We also denote $(WV)_p f = W_p(Vf)$.

Remark 1.39. $(WV)_p$ is <u>not</u> a derivation, so $(WV)_p \notin T_p(M)$, in general. <u>Reason</u>: Leibniz's rule (2) does not hold (choose $f = g$).

Definition 1.40. Suppose that $A \subset M$ is open and $V, W \in \mathcal{T}(A)$. We define the **Lie bracket** of V and W by setting

$$[V, W]_p f = V_p(Wf) - W_p(Vf), \quad p \in A,\ f \in C^\infty(p).$$

Theorem 1.41. *Let $A \subset M$ be open and $V, W \in \mathcal{T}(A)$. Then*

(a) $[V, W]_p \in T_p M$*;*

(b) $[V, W] \in \mathcal{T}(A)$ *and it satisfies*

$$[V, W]f = V(Wf) - W(Vf),\ f \in C^\infty(A); \tag{1.42}$$

(c) if v^i and w^i are the component functions of vector fields V and W, respectively, with respect to a chart $x = (x^1, \ldots, x^n)$, then

$$[V, W] = \left(v^i \partial_i w^j - w^i \partial_i v^j\right) \partial_j. \tag{1.43}$$

<u>Note</u>: The formula (1.43) can be written as

$$[V, W] = \left(V w^j - W v^j\right) \partial_j.$$

Proof. (a) We have to prove that $[V, W]_p$ satisfies conditions (1) and (2) in the definition of a tangent vector.
Condition (1) is clear.
Condition (2): Let $f, g \in C^\infty(p)$. Then

$$\begin{aligned}
[V, W]_p(fg) &= V_p\big(W(fg)\big) - W_p\big(V(fg)\big) \\
&= V_p\big(fWg + gWf\big) - W_p\big(fVg + gVf\big) \\
&= f(p)V_p(Wg) + (W_p g)(V_p f) + g(p)V_p(Wf) + (W_p f)(V_p g) \\
&\quad - f(p)W_p(Vg) - (V_p g)(W_p f) - g(p)W_p(Vf) - (V_p f)(W_p g) \\
&= f(p)\big(V_p(Wg) - W_p(Vg)\big) + g(p)\big(V_p(Wf) - W_p(Vf)\big) \\
&= f(p)[V, W]_p g + g(p)[V, W]_p f.
\end{aligned}$$

(b) Formula (1.42) follows immediately from the definition of a Lie bracket. Let $f \in C^\infty(A)$. Now functions Wf, Vf, $V(Wf)$, and $W(Vf)$ are smooth by Lemma 1.37 (c) since $V, W \in \mathcal{T}(A)$. Hence also $[V,W]f = V(Wf) - W(Vf)$ is a smooth function and therefore $[V,W] \in \mathcal{T}(A)$.

(c) If $V = v^i \partial_i$, $W = w^j \partial_j$, and f is smooth, we obtain by a direct computation that

$$\begin{aligned}[V,W]f &= V(Wf) - W(Vf) = v^i \partial_i (w^j \partial_j f) - w^j \partial_j (v^i \partial_i f) \\ &= v^i (\partial_i w^j)(\partial_j f) + v^i w^j \partial_i (\partial_j f) - w^j (\partial_j v^i)(\partial_i f) - w^j v^i \partial_j (\partial_i f) \\ &= v^i (\partial_i w^j)(\partial_j f) - w^j (\partial_j v^i)(\partial_i f).\end{aligned}$$

In the last step we used the fact that $\partial_j(\partial_i f) = \partial_i(\partial_j f)$ for a smooth function f. Changing the roles of indices i and j in the last sum we obtain (1.43).

□

Lemma 1.44. *The Lie bracket satisfies:*

(a) Bilinearity:

$$\begin{aligned}[a_1 X_1 + a_2 X_2, Y] &= a_1 [X_1, Y] + a_2 [X_2, Y] \quad \textit{ja} \\ [X, a_1 Y_1 + a_2 Y_2] &= a_1 [X, Y_1] + a_2 [X, Y_2]\end{aligned}$$

for $a_1, a_2 \in \mathbb{R}$;

(b) Antisymmetry: $[X,Y] = -[Y,X]$.

(c) Jacobi identity:

$$\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0.$$

(d)

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

Proof. (a) Follows directly from the definition.

(b) Follows directly from the definition.

(c)

$$\begin{aligned}\big[X,[Y,Z]\big]f &= X\big([Y,Z]f\big) - [Y,Z](Xf) \\ &= X\big(Y(Zf) - Z(Yf)\big) - Y\big(Z(Xf)\big) + Z\big(Y(Xf)\big) \\ &= X\big(Y(Zf)\big) - X\big(Z(Yf)\big) - Y\big(Z(Xf)\big) + Z\big(Y(Xf)\big)\end{aligned}$$

$$\big[Y,[Z,X]\big]f = Y\big(Z(Xf)\big) - Y\big(X(Zf)\big) - Z\big(X(Yf)\big) + X\big(Z(Yf)\big)$$

$$\big[Z,[X,Y]\big]f = Z\big(X(Yf)\big) - Z\big(Y(Xf)\big) - X\big(Y(Zf)\big) + Y\big(X(Zf)\big).$$

Adding up both sides yields

$$\big[X,[Y,Z]\big]f + \big[Y,[Z,X]\big]f + \big[Z,[X,Y]\big]f = 0.$$

(d)

$$
\begin{aligned}
[fX, gY]h &= fX(gYh) - gY(fXh) \\
&= fgX(Yh) + f(Xg)(Yh) - gfY(Xh) - g(Yf)(Xh) \\
&= fg[X,Y]h + f(Xg)Yh - g(Yf)Xh.
\end{aligned}
$$

□

Lemma 1.45. *Let (U, x), $x = (x^1, \ldots, x^n)$, be a chart and ∂_i, $i = 1, \ldots, n$, the corresponding coordinate vector fields. Then*

$$[\partial_i, \partial_j] = 0 \quad \forall i, j.$$

Proof. Let $p \in U$ and $f \in C^\infty(p)$. Then

$$
\begin{aligned}
(\partial_i)_p(\partial_j f) &= (\partial_i)_p \left[\big(D_j\big(f \circ x^{-1}\big)\big) \circ x \right] \\
&= D_i \left[\big(D_j(f \circ x^{-1}) \circ x\big) \circ x^{-1} \right] \big(x(p)\big) = D_i D_j \big(f \circ x^{-1}\big)\big(x(p)\big).
\end{aligned}
$$

Since $D_i D_j g = D_j D_i g$ for a smooth function g, we obtain the claim. □

2 Riemannian metrics

2.1 Tensors and tensor fields

Let $V_1, \ldots, V_k$, and W be (real) vector spaces. Recall that a mapping $F \colon V_1 \times \cdots \times V_k \to W$ is **multi linear** (more precisely, **k-linear**) if it is linear in each variable, i.e.

$$F(v_1, \ldots, av_i + bv_i', \ldots, v_k) = aF(v_1, \ldots, v_i, \ldots, v_k) + bF(v_1, \ldots, v_i', \ldots, v_k)$$

for all $i = 1, \ldots, k$ and $a, b \in \mathbb{R}$.

Let V be a finite dimensional (real) vector space. A linear map $\omega \colon V \to \mathbb{R}$ is called a **covector** on V and the vector space of all covectors (on V) is called the **dual** of V and denoted by V^*.

We will adopt the following notation

$$\langle \omega, v \rangle = \langle v, \omega \rangle = \omega(v) \in \mathbb{R}, \quad \omega \in V^*, \ v \in V.$$

Lemma 2.2. *Let V be an n-dimensional vector space and let $(v_1, \ldots, v_n)$ be its basis. Then covectors $\omega^1, \ldots, \omega^n$, with*

$$\omega^j(v_i) = \delta_i^j,$$

form a basis of V^*. In particular, $\dim V^* = \dim V$.

Proof. (Exerc.) □

[Note: Above δ_i^j is the Kronecker delta, i.e. $\delta_i^j = 1$, if $i = j$, and $\delta_i^j = 0$, whenever $i \neq j$.]

Definition 2.3. 1. A **k-covariant tensor** on V is a k-linear map

$$V^k \to \mathbb{R}, \quad V^k = \underbrace{V \times \cdots \times V}_{k \text{ copies}}.$$

2. An **l-contravariant tensor** on V is an l-linear map

$$V^{*l} \to \mathbb{R}, \quad V^{*l} = \underbrace{V^* \times \cdots \times V^*}_{l \text{ copies}}.$$

3. A **k-covariant, l-contravariant tensor** on V (or a **(k,l)-tensor**) is a $(k+l)$-linear map

$$V^k \times V^{*l} \to \mathbb{R}.$$

Denote

$$\begin{aligned} T^k(V) = &\text{ the space of all } k\text{-covariant tensors on } V, \\ T_l(V) = &\text{ the space of all } l\text{-contravariant tensors on } V, \\ T^k_l(V) = &\text{ the space of all } k\text{-covariant, } l\text{-contravariant tensors on } V \text{ (i.e. } (k,l)\text{-tensors).} \end{aligned}$$

Remarks 2.4. 1. $T^k(V)$, $T_l(V)$, and $T^k_l(V)$ are vector spaces in a natural way.

2. We make a convention that both 0-covariant and 0-contravariant tensors are real numbers, i.e. $T^0(V) = T_0(V) = \mathbb{R}$.

Examples 2.5. 1. Any linear map $\omega \colon V \to \mathbb{R}$ is a 1-covariant tensor. Thus $T^1(V) = V^*$. Similarly, $T_1(V) = V^{**} = V$.

2. If V is an inner product space, then any inner product on V is a 2-covariant tensor (a bilinear real-valued mapping, i.e. a **bilinear form**).

3. The determinantti $\det \colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is an n-covariant tensor on $\mathbb{R}^n$.
Interpretation: For $v_1, \dots, v_n \in \mathbb{R}^n$, $v_i = (v_i^1, \dots, v_i^n)$,

$$\det(v_1, \dots, v_n) = \det \begin{pmatrix} v_1^1 & \cdots & v_n^1 \\ \vdots & \ddots & \vdots \\ v_1^n & \cdots & v_n^n \end{pmatrix}.$$

Definition 2.6. The **tensor product** of tensors $F \in T^k_l(V)$ and $G \in T^p_q(V)$ is the tensor $F \otimes G \in T^{k+p}_{l+q}(V)$,

$$F \otimes G(v_1, \dots, v_{k+p}, \omega^1, \dots, \omega^{l+q}) = F(v_1, \dots, v_k, \omega^1, \dots, \omega^l) G(v_{k+1}, \dots, v_{k+p}, \omega^{l+1}, \dots, \omega^{l+q}).$$

Lemma 2.7. *If $(v_1, \dots, v_n)$ is a basis of V and $(\omega^1, \dots, \omega^n)$ the corresponding dual basis of V^* (i.e. $\omega^i(v_j) = \delta^i_j$), then the tensors*

$$\omega^{i_1} \otimes \cdots \omega^{i_k} \otimes v_{j_1} \otimes \cdots \otimes v_{j_l}, \quad 1 \le j_p, i_q \le n,$$

form a basis of $T^k_l(V)$. Consequently, $\dim T^k_l(V) = n^{k+l}$.

Proof. (Exerc.) □

Remark 2.8. Since $T_1(V) = V^{**} = V$ (that is, every vector $v \in V$ is a 1-contravariant tensor) and $T^1(V) = V^*$ (every covector is a 1-covariant tensor), we have

$$\omega^{i_1} \otimes \cdots \omega^{i_k} \otimes v_{j_1} \otimes \cdots \otimes v_{j_l} \in T^k_l(V),$$

i.e. it is a (k,l)-tensor.

2.9 Cotangent bundle

We defined earlier that the differential of a function $f \in C^\infty(p)$ at p is a linear map $df_p \colon T_pM \to \mathbb{R}$,

$$df_p v = vf, \quad v \in T_pM.$$

Hence $df_p \in T_pM^*$ (= the dual of T_pM). We call T_pM^* the **cotangent space** of M at p. If (U, x), $x = (x^1, \ldots, x^n)$, is a chart at p and $\big((\partial_1)_p, \ldots, (\partial_n)_p\big)$ is the basis of T_pM consisting of coordinate vectors, then differentials dx^i_p, $i = 1, \ldots, n$, of functions x^i at p form the dual basis of T_pM^*. Hence the differential (at p) of a function $f \in C^\infty(p)$ can be written as

$$df_p = (\partial_i)_p f \, dx^i_p. \qquad \text{(Exerc.) [Note: Einsteinin summation]}$$

We define the **cotangent bundle** of M as a disjoint union of all cotangent spaces of M

$$TM^* = \bigsqcup_{p \in M} T_pM^*$$

equipped with the natural C^∞-structure (defined similarly to that of TM). Furthermore, let $\pi \colon TM^* \to M$, $T_pM^* \ni \omega \mapsto p \in M$ be the canonical projection. We call sections of TM^*, i.e. mappings $\omega \colon M \to TM^*$, with $\pi \circ \omega = id$, **covector fields** on M or **(differential) 1-forms**. We denote by $\mathcal{T}^1(M)$ (or $\mathcal{T}^1_0(M)$, $\mathcal{T}^*(M)$, $\mathcal{T}^{0,1}(M)$) the set of all smooth covector fields on M. The **differential** of a function $f \in C^\infty(M)$ is the (smooth) covector field

$$df \colon M \to TM^*, \quad df(p) = df_p \colon T_pM \to \mathbb{R}.$$

If (U, x), $x = (x^1, \ldots, x^n)$, is a chart and ω is a covector field on U, there are functions $\omega_i \colon U \to \mathbb{R}$, $i = 1, \ldots, n$, such that

$$\omega = \omega_i dx^i.$$

Functions ω_i are called the **component functions** of ω with respect to the chart (U, x). As in the case of vector fields we have:

Lemma 2.10. *Let ω be a covector field on M. Then the following are equivalent:*

(a) $\omega \in \mathcal{T}^1(M)$;

(b) the component functions of ω (with resect to any chart) are smooth functions;

(c) if $U \subset M$ is open and $V \in \mathcal{T}(U)$ is a smooth vector field in U, then the function $p \mapsto \omega_p(V_p)$ is smooth.

Proof. Exercise [cf. Lemma 1.37] □

2.11 Tensor bundles

Let M be a smooth manifold.

Definition 2.12. We define tensor bundles on M as disjoint unions:

1. **k-covariant tensor bundle**

$$T^kM = \bigsqcup_{p \in M} T^k(T_pM),$$

2. **l-contravariant tensor bundle**

$$T_l M = \bigsqcup_{p \in M} T_l(T_pM), \quad \text{and}$$

3. **(k,l)-tensor bundle**

$$T^k_l M = \bigsqcup_{p \in M} T^k_l(T_pM)$$

equipped with natural C^∞-structures.

We identify:

$$\begin{aligned} T^0M &= T_0M = M \times \mathbb{R}, \\ T^1M &= TM^*, \\ T_1M &= TM, \\ T^k_0M &= T^kM, \\ T^0_lM &= T_lM. \end{aligned}$$

Since all tensor bundles are smooth manifolds, we may consider their smooth sections. We say that a section $s\colon M \to T^k_lM$ is a (k,l)**-tensor field** (recall that $\pi \circ s = id_M$, and so $s(p) \in T^k_l(T_pM)$). A smooth (k,l)-tensor field is a smooth section $M \to T^k_lM$. Similarly, we define (smooth) k**-covariant tensor fields** and l**-contravariant tensor fields**. Since 0-covariant and 0-contravariant tensors are real numbers, (smooth) 0-covariant tensor fields and (smooth) 0-contravariant tensor fields are (smooth) real-valued functions.

Denote

$$\begin{aligned} \mathcal{T}^k(M) &= \{\text{smooth sections on } T^kM\} \\ &= \{\text{smooth } k\text{-covariant tensor fields}\} \\ \mathcal{T}_l(M) &= \{\text{smooth sections on } T_lM\} \\ &= \{\text{smooth } l\text{-contravariant tensor fields}\} \\ \mathcal{T}^k_l(M) &= \{\text{smooth sections on } T^k_lM\} \\ &= \{\text{smooth } (k,l)\text{-tensor fields}\}. \end{aligned}$$

If (U,x), $x = (x^1, \ldots, x^n)$, is a chart and σ is a tensor field in U, we may write

$$\begin{aligned} \sigma &= \sigma_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}, \quad \text{if } \sigma \text{ is a } k\text{-covariant tensor field,} \\ \sigma &= \sigma^{j_1\cdots j_l} \partial_{j_1} \otimes \cdots \otimes \partial_{j_l}, \quad \text{if } \sigma \text{ is an } l\text{-contravariant tensor field, or} \\ \sigma &= \sigma^{j_1\cdots j_l}_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \partial_{j_1} \otimes \cdots \otimes \partial_{j_l}, \quad \text{if } \sigma \text{ is a } (k,l)\text{-tensor field.} \end{aligned}$$

Functions $\sigma_{i_1\cdots i_k}$, $\sigma^{j_1\cdots j_l}$ and $\sigma^{j_1\cdots j_l}_{i_1\cdots i_k}$ are called the **component functions** of σ with respect to the cahrt (U,x). Again we have:

Lemma 2.13. *Let σ be a (k,l)-tensor field on M. Then the following are equivalent:*

(a) $\sigma \in \mathcal{T}^k_l(M)$;

(b) the component functions of σ (with respect to any chart) are smooth;

(c) *if $U \subset M$ is open and $X_1, \ldots, X_k \in \mathcal{T}(U)$ are smooth vector fields in U and $\omega^1, \ldots, \omega^l \in \mathcal{T}^1(M)$ are smooth covector fields in U, then the function*

$$p \mapsto \sigma\big(X_1, \ldots, X_k, \omega^1, \ldots, \omega^l\big)_p \in \mathbb{R}$$

is smooth.

Proof. Exercise [cf. Lemma 1.37 and Lemma 2.10.] □

2.14 Riemannian metric tensor

Definition 2.15. Let M be a C^∞-manifold. A **Riemannian metric (tensor)** on M is a 2-covariant tensor field $g \in \mathcal{T}^2(M)$ that is symmetric (i.e. $g(X,Y) = g(Y,X)$) and positive definite (i.e. $g(X_p, X_p) > 0$ if $X_p \neq 0$). A C^∞-manifold M with a given Riemannian metric g is called a **Riemannian manifold** (M, g).

A Riemannian metric thus defines an inner product on each T_pM, written as $\langle v, w\rangle = \langle v, w\rangle_p = g(v,w)$ for $v, w \in T_pM$. The inner product varies smoothly in p in the sense that for every $X, Y \in \mathcal{T}(M)$, the function $M \to \mathbb{R}$, $p \mapsto g(X_p, Y_p)$, is C^∞.

The **length** (or **norm**) of a vector $v \in T_pM$ is

$$|v| = \langle v, v\rangle^{1/2}.$$

The **angle** between non-zero vectors $v, w \in T_pM$ is the unique $\vartheta \in [0, \pi]$ such that

$$\cos\vartheta = \frac{\langle v, w\rangle}{|v||w|}.$$

Vectors $e_1, \ldots, e_k \in T_pM$ are **orthonormal** if they are of length 1 and pairwise orthogonal, in other words, $\langle e_i, e_j\rangle = \delta_{ij}$.

Recall that vector fields $E_1, \ldots, E_n \in \mathcal{T}(U)$ in an open set $U \subset M$ form a local frame if $(E_1)_p, \ldots, (E_n)_p$ form a basis of T_pM for each $p \in U$. Associated to a local frame is the **coframe** $\varphi^1, \ldots, \varphi^n \in \mathcal{T}^1(U)$ (=differentiable 1-forms on U) such that $\varphi^i(E_j) = \delta_{ij}$.

Now, if $E_1, \ldots, E_n$ is any (smooth) local frame, and $\varphi^1, \ldots, \varphi^n$ its coframe, the Riemannian metric g can be written locally as

$$g = g_{ij}\varphi^i \otimes \varphi^j. \tag{2.16}$$

The coefficient matrix, defined by $g_{ij} = \langle E_i, E_j\rangle$, is symmetric in i and j, and the function

$$p \mapsto g_{ij}(p) := \langle E_i, E_j\rangle_p$$

is C^∞ for all i, j.

Example 2.17. If (U, x), $x = (x^1, \ldots, x^n)$ is a chart, then $\partial_1, \ldots, \partial_n$, where $\partial_i = \frac{\partial}{\partial x_i}$, form a coordinate frame and differentials $dx^1, \ldots, dx^n$ its coframe. The Riemannian metric can then be written (in U) as

$$g = g_{ij} dx^i \otimes dx^j = g_{ij} dx^i dx^j.$$

(If ω and η are 1-forms, we write $\omega\eta = \frac{1}{2}(\omega \otimes \eta + \eta \otimes \omega)$ (= symmetric product).)

Remark 2.18. If $p \in M$, then there exists a local orthonormal frame in the neighborhood of p, i.e. a local frame $E_1, \ldots, E_n$ that forms an orthonormal basis of T_qM for all q in this neighborhood. **Warning**: In general, it is **not** possible to find a chart (U, x) at p so that the coordinate frame $\partial_1, \ldots, \partial_n$ would be an orthonormal frame. In fact, this is possible only if the metric g is locally isometric to the Euclidean metric.

Definition 2.19. Let (M, g) and (N, h) be Riemannian manifolds. A diffeomorphism $f : M \to N$ is called an **isometry** if $f^*h = g$, i.e.

$$f^*h(v, w) = h(f_*v, f_*w) = g(v, w)$$

for all $v, w \in T_pM$ and $p \in M$. A C^∞-map $f : M \to N$ is a **local isometry** if, for each $p \in M$, there are neighborhoods U of p and V of $f(p)$ such that $f|U : U \to V$ is an isometry.

Examples 2.20. (1) If $M = \mathbb{R}^n$, then the **Euclidean metric** is the usual inner product on each tangent space $T_p\mathbb{R}^n \cong \mathbb{R}^n$. The standard coordinate frame is $\partial_1, \ldots, \partial_n$, where

$$\partial_i = e_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0),$$

$\langle \partial_i, \partial_j \rangle = \delta_{ij}$, and the metric can be written as

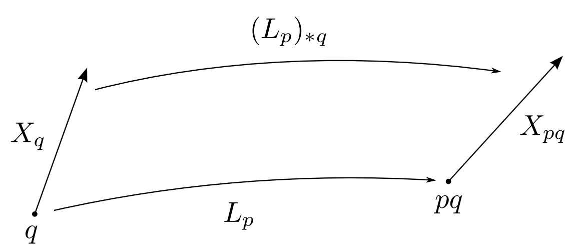
$$g = \sum_i dx^i dx^i = \delta_{ij} dx^i dx^j.$$

(2) Let $f : M^n \to N^{n+k}$ be an immersion, that is, f is C^∞ and $f_{*p} : T_pM \to T_{f(p)}N$ is injective for all $p \in M$. If N has a Riemannian metric g, then f^*g defines a Riemannian metric on M:

$$f^*g(v, w) = g(f_*v, f_*w)$$

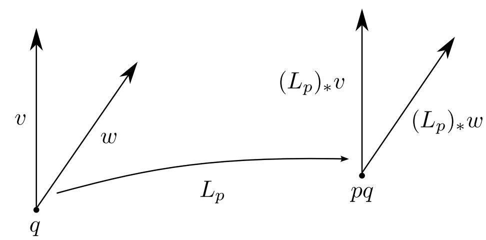
for all $v, w \in T_pM$ and $p \in M$. Since f_{*p} is injective, f^*g is positive definite. The metric f^*g is called the **induced metric**.

(3) Recall that a Lie group G is a group which is also a C^∞-manifold such that $G \times G \to G$, $(p, q) \mapsto pq^{-1}$, is C^∞. For fixed $p \in G$, the map $L_p : G \to G$, $L_p(q) = pq$, is called a **left translation**. A vector field X is called **left-invariant** if $X = (L_p)_*X$ for every $p \in G$, i.e. $X_{pq} = (L_p)_{*q}X_q$ for all $p, q \in G$.



If X is left-invariant, then $X \in \mathcal{T}(G)$ (is a smooth vector field) and it is completely determined by its value at a single point of G (e.g. by X_e). If X and Y are left-invariant, then so is $[X, Y]$. The set of left-invariant vector fields on G forms a vector space. This vector space together with the bracket $[\cdot, \cdot]$ is called a **Lie algebra** $\mathfrak{g}$. Thus $\mathfrak{g} \cong T_eG$.

A Riemannian metric $\langle \cdot, \cdot \rangle$ on G is called **left-invariant** if $(L_p)^*\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$ for all $p \in G$, i.e. if $\langle (L_p)_{*q}v, (L_p)_{*q}w \rangle_{pq} = \langle v, w \rangle_q$ for all $v, w \in T_qG$ and all $p, q \in G$.



To construct a left-invariant Riemannian metric on G, it is enough to give an arbitrary inner product $\langle\cdot,\cdot\rangle_e$ on T_eG. Similarly, we can define right-invariant Riemannian metrics for right translations $R_p : G \to G$, $R_p(q) = qp$.

(4) If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, the product $M_1 \times M_2$ has a natural Riemannian metric $g = g_1 \oplus g_2$, the **product metric**, defined by

$$g(X_1 + X_2, Y_1 + Y_2) := g_1(X_1, Y_1) + g_2(X_2, Y_2),$$

where $X_i, Y_i \in \mathcal{T}(M_i)$ and $T_{(p,q)}(M_1 \times M_2) = T_pM_1 \oplus T_qM_2$ for all $(p, q) \in M_1 \times M_2$.

If $(x^1, \ldots, x^n)$ is a chart on M_1 and $(x^{n+1}, \ldots, x^{n+m})$ is a chart on M_2, then $(x_1, \ldots, x^{n+m})$ is a chart on $M_1 \times M_2$. In these coordinates the product metric can be written as $g = g_{ij}dx^idx^j$, where (g_{ij}) is the block matrix

$$\begin{bmatrix} (g_1)_{11} & \cdots & (g_1)_{1n} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ (g_1)_{n1} & \cdots & (g_1)_{nn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (g_2)_{11} & \cdots & (g_2)_{1m} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & (g_2)_{m1} & \cdots & (g_2)_{mm} \end{bmatrix}.$$

As an example one can consider the **flat torus**:

$$\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$$

together with the product metric, where each $\mathbb{S}^1 \subset \mathbb{R}^2$ has the induced metric from $\mathbb{R}^2$.

Definition 2.21. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ a C^∞-path, where $I \subset \mathbb{R}$ an open interval. The **length** of $\gamma|[a, b]$, where $[a, b] \subset I$, is defined by

$$\ell(\gamma|[a, b]) := \int_a^b |\dot{\gamma}_t|\, dt = \int_a^b g(\dot{\gamma}_t, \dot{\gamma}_t)^{1/2}\, dt.$$

The length of a piecewise C^∞-path is the sum of the lengths of the pieces.

Let M be connected and $p, q \in M$. Define

$$d(p, q) = \inf_\gamma \ell(\gamma),$$

where inf is taken over all piecewise C^∞-paths from p to q. Then $d\colon M \times M \to \mathbb{R}$ is a metric whose topology is the same as the original topology of M (this will be proven later).

2.22 Integration on Riemannian manifolds

We start with a discussion on a partition of unity.

Definition 2.23. Let M be a C^∞-manifold. A **(C^∞-)partition of unity** on M is a collection $\{\varphi_i \colon i \in I\}$ of C^∞-functions on M such that

(a) the collection of supports $\{\operatorname{supp} \varphi_i \colon i \in I\}$ is locally finite,

(b) $\varphi_i(p) \geq 0$ for all $p \in M$ and $i \in I$, and

(c) for all $p \in M$

$$\sum_{i\in I} \varphi_i(p) = 1.$$

A partition of unity $\{\varphi_i \colon i \in I\}$ is **subordinate to** a cover $\{U_\alpha \colon \alpha \in A\}$ ($M = \cup_\alpha U_\alpha$) if, for each $i \in I$ there is $\alpha \in A$ such that $\operatorname{supp} \varphi_i \subset U_\alpha$.

Remarks 2.24. 1. Above I and A are arbitrary (not necessary countable) index sets.

2. The **support** of a function $f \colon M \to \mathbb{R}$ is the set

$$\operatorname{supp} f = \overline{\{p \in M \colon f(p) \neq 0\}}.$$

3. A collection $\{A_i \colon i \in I\}$ of of sets A_i is **locally finite** if each $p \in M$ has a neighborhood $U \ni p$ such that $U \cap A_i \neq \emptyset$ for only finitely many i.

4. The sum in (c) makes sense since only finitely many terms $\varphi_i(p)$ are nonzero for every $p \in M$.

Theorem 2.25. *Let M be a C^∞-manifold and $\{U_\alpha \colon \alpha \in A\}$ an open cover of M. Then there exists a countable C^∞-partition of unity $\{\varphi_i \colon i \in \mathbb{N}\}$ subordinate to $\{U_\alpha \colon \alpha \in A\}$, with* $\operatorname{supp} \varphi_i$ *compact for each i.*

Proof. See, for instance, [Le2], Theorem 2.25. □

As a simple application we obtain the existence of a Riemannian metric.

Theorem 2.26. *Every C^∞-manifold M admits a Riemannian metric.*

Proof. Let (U, x), $x = (x^1, \ldots, x^n)$, be a chart and $\partial_1, \ldots, \partial_n$ a (local) coordinate frame. We define a Riemannian metric $\tilde{g}$ on U as the pull-back of the Euclidean metric under x, in other words,

$$\tilde{g}(\partial_i, \partial_j) = \delta_{ij} \quad (\tilde{g} = \delta_{ij} dx^i dx^j). \tag{2.27}$$

Let $\{U_\alpha \colon \alpha \in A\}$ be an open cover of M by charts (U_α, x_α) and let φ_k, $k = 1, 2, \ldots,$ be a C^∞-partition of unity subordinate to $\{U_\alpha \colon \alpha \in A\}$. For each $k \in \mathbb{N}$ choose $\alpha \in A$ such that $\operatorname{supp} \varphi_k \subset U_\alpha$ and let $\tilde{g}_k$ be a Riemannian metric on U_α given by (2.27). Then

$$g = \sum_k \varphi_k \tilde{g}_k$$

is a Riemannian metric on M. Thus

$$g(v, w) = \sum_k \varphi_k(p) \tilde{g}_k(v, w)$$

for all $p \in M$ and $v, w \in T_pM$. □

Integration. Recall the change of variables formula for the (Lebesgue) integral (see e.g. [Jo]): Suppose that Ω_1 and Ω_2 are open subsets of $\mathbb{R}^n$ and that $\varphi\colon \Omega_1 \to \Omega_2$ is a diffeomorphism. Let $f\colon \Omega_2 \to \dot{\mathbb{R}}$ be (Lebesgue-)measurable. Then $f \circ \varphi$ is measurable and

$$\int_{\Omega_2} f\,dm = \int_{\Omega_1} (f \circ \varphi)|J_\varphi|\,dm. \tag{2.28}$$

The formula is valid in the following sense: If $f \geq 0$, then (2.28) always holds. In general, $f \in L^1(\Omega_2)$ if and only if $(f \circ \varphi)|J_\varphi| \in L^1(\Omega_1)$, and then (2.28) holds.

Suppose that (M, g) is a Riemannian n-manifold. Let (U, x), $x = (x^1, \ldots, x^n)$, and (U, y), $y = (y^1, \ldots, y^n)$, be charts. The Riemannian metric $g = \langle\, , \rangle$ can be written in U as

$$g = g^x_{ij} dx^i dx^j, \quad g^x_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle,$$

or

$$g = g^y_{ij} dy^i dy^j, \quad g^y_{ij} = \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle.$$

Denote $\varphi = y \circ x^{-1}\colon xU \to yU$. We want to define (first) $\int_U d\mu$, where $d\mu$ is a "volume element", by using a chart in such a way that the definition would be independent of the chosen chart. Write $G^x(p) = \big(g^x_{ij}(p)\big)$ and $G^y(p) = \big(g^y_{ij}(p)\big)$ for $p \in U$, and let $A(q)$ be the matrix of $\varphi'(q)$ with respectt the standard basis of $\mathbb{R}^n$. Since g is positive definite and symmetric, we have

$$\det G^x(p) > 0$$

for all $p \in$. We claim that

$$\sqrt{\det G^x(p)} = \sqrt{\det G^y(p)}\big|J_\varphi\big(x(p)\big)\big| \tag{2.29}$$

for all $p \in U$. If this is true, then

$$\begin{aligned}
\int_{xU} \big(\sqrt{\det G^x}\big) \circ x^{-1} &= \int_{xU} \sqrt{\det G^x\big(x^{-1}(q)\big)}\,dq \\
&\overset{(2.29)}{=} \int_{xU} \sqrt{\det G^y\big(y^{-1}(\varphi(q))\big)}\,|J_\varphi(q)|\,dq \\
&\overset{(2.28)}{=} \int_{yU} \sqrt{\det G^y\big(y^{-1}(m)\big)}\,dm \\
&= \int_{yU} \big(\sqrt{\det G^y}\big) \circ y^{-1}.
\end{aligned}$$

so, the definition

$$\int_U d\mu := \int_{xU} \big(\sqrt{\det G^x}\big) \circ x^{-1} \tag{2.30}$$

is independent of the chosen map x. Similarly,

$$\int_U f\,d\mu := \int_{xU} \big(f\sqrt{\det G^x}\big) \circ x^{-1}$$

is independent of x for all Borel functions $f\colon U \to \mathbb{R}$. Next pick an atlas $\mathcal{A} = \{(U_\alpha, x_\alpha)\colon \alpha \in I\}$ and a (countable) C^∞-partition of unity $\{\varphi_i\}$ subordinate to $\mathcal{A}$. For each i, let $\alpha_i \in I$ be such that $\operatorname{supp}\varphi_i \subset U_{\alpha_i}$. Then we define, for any Borel set $A \subset M$,

$$\mu(A) := \int_A d\mu = \sum_i \int_{U_{\alpha_i}\cap A} \varphi_i \, d\mu = \sum_i \int_{U_{\alpha_i}} \varphi_i \chi_A \, d\mu.$$

This is independent of the chosen atlas and partition of unity. After this we can develop a theory of measure ($= \mu$) and integration on M.

Proof of (2.29). Let

$$A = \left(A^i_j\right) = \left(D_j\varphi^i\right)$$

be the Jacobian matrix of $\varphi = y \circ x^{-1}$ with respect to the standard basis of $\mathbb{R}^n$. Then it is the matrix of id_* with respect to coordinate frames $\{\partial/\partial x^i\}$ and $\{\partial/\partial y^j\}$. Hence

$$\frac{\partial}{\partial x^i} = \sum_{j=1}^n D_i\varphi^j \frac{\partial}{\partial y^j}.$$

So,

$$\begin{aligned}
g^x_{ij} &= \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \\
&= \left\langle \sum_{k=1}^n D_i\varphi^k \frac{\partial}{\partial y^k}, \sum_{\ell=1}^n D_j\varphi^\ell \frac{\partial}{\partial y^\ell} \right\rangle \\
&= \sum_{k,\ell} D_i\varphi^k D_j\varphi^\ell \left\langle \frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^\ell} \right\rangle \\
&= \sum_{k,\ell} D_i\varphi^k D_j\varphi^\ell g^y_{k\ell}.
\end{aligned}$$

That is,

$$G^x = A^T G^y A,$$

and so

$$\det G^x = \det A^T \cdot \det G^y \cdot \det A.$$

Since

$$\det A = \det A^T = J_\varphi,$$

we obtain (2.29).

3 Connections

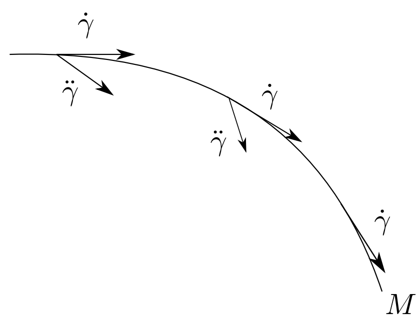
3.1 Motivation

We want to study geodesics which are Riemannian generalizations of straight lines. One possibility is to define geodesics as curves that minimize length between nearby points. However, this property is technically difficult to work with as a definition. Another approach:
In $\mathbb{R}^n$ straight lines are curves $\alpha : \mathbb{R} \to \mathbb{R}^n$,

$$\alpha(t) = p + tv, \qquad p, v \in \mathbb{R}^n.$$

(We do not consider e.g. $\gamma(t) = p + t^3 v$ as a straight line, although $\gamma(\mathbb{R}) = \alpha(\mathbb{R})$.)
The **velocity** vector of α is $\dot{\alpha}_t = \alpha'(t) = v$, and the **acceleration** of α is $\ddot{\alpha}_t = \alpha''(t) = 0$; so straight lines are curves α with $\ddot{\alpha} \equiv 0$.

Let $M^m \subset \mathbb{R}^n$ be a submanifold, $m < n$, with induced Riemannian metric.
Take a C^∞-path $\gamma : I \to M$, $\gamma = (\gamma^1, \dots, \gamma^n)$. Then $\dot{\gamma}_t = (\dot{\gamma}^1_t, \dots, \dot{\gamma}^n_t) \in \mathbb{R}^n$ but also $\dot{\gamma}_t \in T_{\gamma(t)}M$ and it has a coordinate-independent meaning. On the other hand, $\ddot{\gamma}_t = (\ddot{\gamma}^1_t, \dots, \ddot{\gamma}^n_t) \in \mathbb{R}^n$ but $\ddot{\gamma}_t \notin T_{\gamma(t)}M$, in general.



To measure the "straightness" of γ we project $\ddot{\gamma}_t$ orthogonally to $T_{\gamma(t)}M$ and obtain $\ddot{\gamma}^T_t$, the "tangential acceleration". Hence, we could define geodesics as curves γ, with $\ddot{\gamma}^T \equiv 0$.
Problem: For an abstract Riemannian manifold, there is no canonical ambient Euclidean space, where to differentiate. So the method does not work as such.
We face the following problem:
To differentiate (intrinsicly, i.e. within M) $\dot{\gamma}_t$ with respect to t we need to write the difference quotient of $\dot{\gamma}_t$ for $t \neq t_0$ but these vectors live in different vector spaces, so $\dot{\gamma}_t - \dot{\gamma}_{t_0}$ does **not** make sense.
To do so, we need a way to "connect" nearby tangent spaces. This will be the role of a **connection**.

3.2 Affine connections

First a general definition.

Definition 3.3. Let (E, π) be a C^∞ vector bundle over M, and let $\mathcal{E}(M)$ denote the space of $\mathcal{C}^\infty$-sections of E. A **connection** in E is a map

$$\nabla : \mathcal{T}(M) \times \mathcal{E}(M) \longrightarrow \mathcal{E}(M),$$

denoted by $(X, Y) \mapsto \nabla_X Y$, satisfying

(C1) $\nabla_X Y$ is linear over $C^\infty(M)$ in X:

$$\nabla_{fX_1 + gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y, \qquad f, g \in C^\infty(M);$$

(C2) $\nabla_X Y$ is linear over $\mathbb{R}$ in Y:

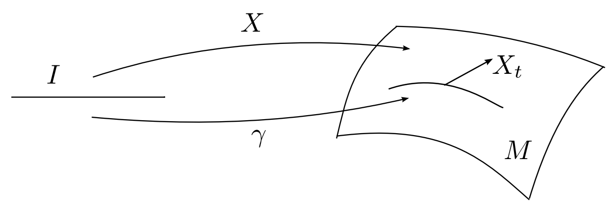
$$\nabla_X (aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2, \qquad a, b \in \mathbb{R};$$

(C3) ∇ satisfies the following product rule:

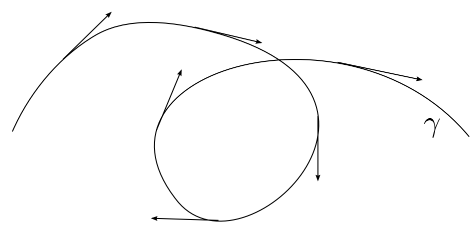
$$\nabla_X (fY) = f\nabla_X Y + (Xf)Y, \qquad f \in C^\infty(M).$$

We say that $\nabla_X Y$ is the **covariant derivative** of Y in the direction of X.

In the case $E = TM$ the connection ∇ is called an **affine connection**. Thus $\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \longrightarrow \mathcal{T}(M)$. From now on ∇ will be an affine connection on M. Let $\gamma : I \to M$ be a C^∞-path. We say that a C^∞-map $X : I \to TM$ is a C^∞**-vector field along** γ if $X_t = X_{\gamma(t)} \in T_{\gamma(t)}M$ for every $t \in I$.



Denote by $\mathcal{T}(\gamma)$ the space of all C^∞-vector fields along γ. Observe that $X \in \mathcal{T}(\gamma)$ cannot necessarily be extended to $\widetilde{X} \in \mathcal{T}(U)$, where U is an open set such that $\gamma : I \to U$. For example:



Lemma 3.4. $(\nabla_X Y)_p$ *depends only on* X_p *and the values of* Y *along a* C^∞*-path* γ*, with* $\dot\gamma_0 = X_p$ *(and, of course, on* ∇*).*

Remark 3.5. This innocent looking result will be very important since it makes it possible to define a notion of covariant derivative of a vector field along a smooth path, and therefore a parallel transport along a smooth path; see Theorem 3.7 and Definition 3.14 below.

Proof. Let (U, x) be a chart at p, and let $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, 2, \ldots, n$, be the corresponding coordinate frame. Let

$$X = a^i \partial_i, \quad Y = b^j \partial_j.$$

Using the axioms of connection, we gain

$$\begin{aligned}(\nabla_X Y)_p &= (\nabla_X b^j \partial_j)_p = b^j(p)(\nabla_X \partial_j)_p + (X_p b^j)(\partial_j)_p = b^j(p)(\nabla_{a^i\partial_i} \partial_j)_p + (X_p b^j)(\partial_j)_p \\ &= b^j(p) a^i(p)(\nabla_{\partial_i} \partial_j)_p + (X_p b^j)(\partial_j)_p,\end{aligned}$$

where terms $b^j(p)a^i(p)$ depend only on Y_p and X_p and terms $X_p b^j$ depend only on the values of Y along γ with $\dot\gamma_0 = X_p$. □

Let $\{E_i\}$ be a local frame on an open set $U \subset M$. Writing

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k,$$

we get functions $\Gamma^k_{ij} \in C^\infty(U)$ called the **Christoffel symbols** of ∇ with respect to $\{E_i\}$. As in the proof Lemma 3.4, we get

$$\nabla_X Y = a^i b^j \Gamma^k_{ij} E_k + X b^j E_j = (a^i b^j \Gamma^k_{ij} + X b^k) E_k. \tag{3.6}$$

Theorem 3.7. *Let ∇ be an affine connection on M, and let $\gamma : I \to M$ be a C^∞-path. Then there exists a unique map $D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ satisfying:*

(a) linearity over $\mathbb{R}$:

$$D_t(aV + bW) = aD_tV + bD_tW, \qquad a, b \in \mathbb{R};$$

(b) product rule:

$$D_t(fV) = \dot{f}V + fD_tV, \qquad f \in C^\infty(I);$$

(c) if V is induced by $Y \in \mathcal{T}(M)$ (V is "extendible"), i.e. $V_t = Y_{\gamma(t)}$, then

$$D_tV = \nabla_{\dot{\gamma}}Y.$$

The vector field D_tV is called the **covariant derivative of V along γ**.

Proof. Note that the last line in (c) makes sense by Lemma 3.4. We follow a typical scheme in the proof: first we prove the uniqueness and obtain a formula that can be used to define the object we are looking for.
Uniqueness Suppose that D_t exists with the properties (a), (b) and (c). Let $V \in \mathcal{T}(\gamma)$, $t_0 \in I$, and let $x = (x^1, \ldots, x^n)$ be a chart at $p = \gamma(t_0)$. Then for all t sufficiently close to t_0, say $|t - t_0| < \varepsilon$, we have

$$\dot{\gamma}_t = (x^i \circ \gamma)'(t)(\partial_i)_{\gamma(t)} = \dot{\gamma}^i(t)(\partial_i)_{\gamma(t)}$$

and

$$V_t = v^j(t)(\partial_j)_{\gamma(t)},$$

where $\dot{\gamma}^i = (x^i \circ \gamma)'$ and $v^j \in C^\infty(t_0 - \varepsilon, t_0 + \varepsilon)$. Using (a) and (b), we have

$$D_tV = D_t(v^j\partial_j) = \dot{v}^j\partial_j + v^jD_t\partial_j.$$

Because ∂_j is extendible, we have

$$D_t\partial_j \overset{(c)}{=} \nabla_{\dot{\gamma}}\partial_j = \nabla_{\dot{\gamma}^i\partial_i}\partial_j \overset{(C1)}{=} \dot{\gamma}^i\nabla_{\partial_i}\partial_j = \dot{\gamma}^i\Gamma^k_{ij}\partial_k.$$

Therefore,

$$D_tV = \dot{v}^j\partial_j + v^j\dot{\gamma}^i\Gamma^k_{ij}\partial_k = (\dot{v}^k + v^j\dot{\gamma}^i\Gamma^k_{ij})\partial_k. \tag{3.8}$$

By (3.8), if $D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ exists and satisfies (a), (b) and (c), then it is unique.
Existence If $\gamma(I)$ is contained in a single chart, we can define D_t by (3.8). In the general case, cover $\gamma(I)$ by charts and define D_tV by (3.8). The uniqueness implies that the definitions agree whenever two charts overlap. □

When do the affine connections exist?

Example 3.9. The Euclidean connection in $\mathbb{R}^n$ is defined as follows. Let $X, V \in \mathcal{T}(\mathbb{R}^n)$, $V = (v^1, \ldots, v^n) = v^i\partial_i$, where $v^i \in C^\infty(\mathbb{R}^n)$ and $\partial_1, \ldots, \partial_n$ is the standard basis of $\mathbb{R}^n$. Then we define

$$\overline{\nabla}_XV = (Xv^j)\partial_j,$$

i.e. $\overline{\nabla}_XV$ is a vector field whose components are the derivatives of V in the direction X. Note that the Christoffel symbols of $\overline{\nabla}$ (w.r.t. the standard basis of $\mathbb{R}^n$) vanish.

Lemma 3.10. *Suppose M can be covered by a single chart. Then there is a one-to-one correspondence between affine connections on M and the choises of n^3 functions $\Gamma_{ij}^k \in C^\infty(M)$, by the rule*

$$\nabla_X Y = (a^i b^j \Gamma_{ij}^k + X b^k)\partial_k, \tag{3.11}$$

where $X = a^i \partial_i$, $Y = b^i \partial_i$, and $\partial_1, \dots, \partial_n$ is the coordinate frame associated to the chart.

Proof. For every affine connection there are functions $\Gamma_{ij}^k \in C^\infty(M)$, namely the Christoffel symbols, such that (3.11) holds.
Conversely, given functions Γ_{ij}^k, $i, j, k = 1, 2, \dots, n$, then (3.11) defined an affine connection. (Exercise) □

Theorem 3.12. *Every C^∞-manifold M admits an affine connection*

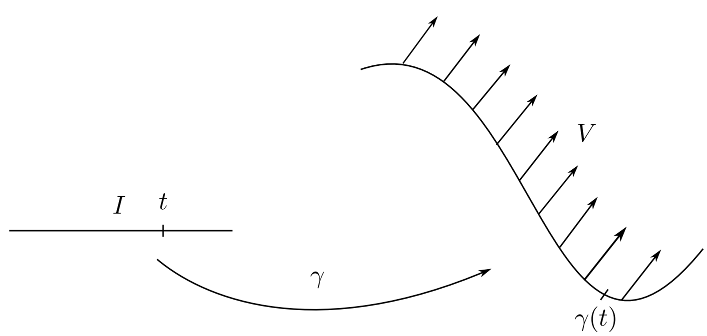
Proof. Cover M with charts $\{U_\alpha\}$. Then by Lemma 3.10 each U_α has a connection ∇^α. Choose a partition of unity $\{\varphi_\alpha\}$ subordinate to $\{U_\alpha\}$. Define

$$\nabla_X Y := \sum_\alpha \varphi_\alpha \nabla_X^\alpha Y.$$

Check that this defines a connection. □

Remark 3.13. If ∇^1 and ∇^2 are connections, then neither $\frac{1}{2}\nabla^1$ nor $\nabla^1 + \nabla^2$ satisfies the product rule (C3).

Definition 3.14. Let ∇ be an affine connection on M, $\gamma : I \to M$ a C^∞-path, and $D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ given by Theorem 3.7. We say that $V \in \mathcal{T}(\gamma)$ is **parallel** along γ if $D_t V = 0$.



Exercise 3.15. Let $\gamma : I \to \mathbb{R}^n$ be a C^∞-path. Show that a vector field $V \in \mathcal{T}(\gamma)$ is parallel (with respect to the Euclidean connection) if and only if its components are constants.

Theorem 3.16. *Let ∇ be an affine connection on M, $\gamma : I \to M$ a C^∞-path, $t_0 \in I$, and $v_0 \in T_{\gamma(t_0)}M$. Then there exists a unique parallel $V \in \mathcal{T}(\gamma)$ such that $V_{t_0} = v_0$. The vector field V is called the* ***parallel transport*** *of v_0 along γ.*

Before we prove this theorem, we state the following lemma about the existence and uniqueness for linear ODEs.

Lemma 3.17. *Let $I \subset \mathbb{R}$ be an interval and let $a_j^k : I \to \mathbb{R}$, $1 \le j, k \le n$, be C^∞-functions. Then the linear initial-value problem*

$$\begin{cases} \dot{v}^k(t) & = a_j^k(t) v^j(t); \\ v^k(t_0) & = b^k, \end{cases}$$

has a unique solution on all of I for any $t_0 \in I$ and $(b^1, \dots, b^n) \in \mathbb{R}^n$.

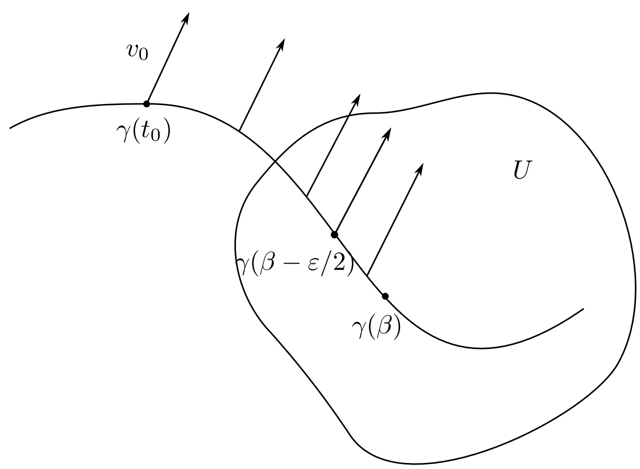
Proof of Theorem 3.16. Suppose first that $\gamma(I) \subset U$, where (U, x) is a chart. Then $V = v^j \partial_j \in \mathcal{T}(\gamma)$ is parallel along γ if and only if $D_t V \overset{(3.8)}{=} (\dot{v}^k + v^j \dot{\gamma}^i \Gamma_{ij}^k)\partial_k = 0$, that is, if and only if

$$\dot{v}^k(t) = -v^j \dot{\gamma}^i(t) \Gamma_{ij}^k(\gamma(t)), \qquad 1 \leq k \leq n.$$

This is a linear system of ODEs for $(v^1(t), \dots, v^n(t))$. Lemma 3.17 implies that there exists a unique solution on all of I for any initial condition $V_{t_0} = v_0$.
General case: ($\gamma(I)$ is not necessarily covered by a single chart)
Write

$$\beta := \sup\{b > t_0 : \text{there exists a unique parallel transport of } v_0 \text{ along } [t_0, b]\}.$$

Clearly, $\beta > t_0$, since for small enough $\varepsilon > 0$ the set $\gamma(t_0 - \varepsilon, t_0 + \varepsilon)$ is contained in a single chart, and the first part of the proof applies. Hence, a unique parallel transport V of v_0 exists on $[t_0, \beta)$. If $\beta \in I$, choose a chart U at $\gamma(\beta)$ such that $\gamma(\beta - \varepsilon, \beta + \varepsilon) \subset U$ for some $\varepsilon > 0$. The first part of the proof implies that there exists a unique parallel transport $\widetilde{V}$ of $V_{\beta - \varepsilon/2}$ along $(\beta - \varepsilon, \beta + \varepsilon)$. By uniqueness $V = \widetilde{V}$ on $(\beta - \varepsilon, \beta)$, and hence $\widetilde{V}$ is an extension of V past β, which is a contradiction. So $\beta \notin I$. Similarly, we can analyze the "lower end" of I. □



The parallel transport along $\gamma : I \to M$ defines for $t_0, t \in I$ a linear isomorphism $P_{t_0,t} : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$ by

$$P_{t_0,t} v_0 = V_t,$$

where $V \in \mathcal{T}(\gamma)$ is the parallel transport of $v_0 \in T_{\gamma(t_0)}M$ along γ.

Definition 3.18. Let ∇ be an affine connection on M. A C^∞-path $\gamma : I \to M$ is a **geodesic** if

$$D_t \dot{\gamma} = 0.$$

By Theorem 3.7(c), this can also be written as

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0,$$

provided that $\dot{\gamma}$ is extendible.

Theorem 3.19. *Let M be a C^∞-manifold with an affine connection ∇. Then for each $p \in M$, $v \in T_pM$, and $t_0 \in \mathbb{R}$, there exist an open interval $I \ni t_0$ and a geodesic $\gamma : I \to M$ satisfying $\gamma(t_0) = p$ and $\dot{\gamma}(t_0) = v$. Any two such geodesics agree on their common interval.*

Proof. Let (U, x), $x = (x^1, \dots, x^n)$, be a chart at p and $\{\partial_i\}$ the corresponding coordinate frame. If $\gamma : J \to U$ is a C^∞-path, with $\gamma(t_0) = 0$ and $\dot\gamma(t_0) = v$, then

$$\dot\gamma = (x^i \circ \gamma)'\partial_i = \dot\gamma^i \partial_i$$

and

$$D_t\dot\gamma \overset{(3.8)}{=} (\ddot\gamma^k + \dot\gamma^j\dot\gamma^i\Gamma^k_{ij})\partial_k.$$

Hence, $\gamma : I \to U$, $t_0 \in I \subset J$, is a geodesic, with $\gamma(t_0) = p$ and $\dot\gamma(t_0) = v$, which is equivalent to

$$\begin{cases} \ddot\gamma^k + \dot\gamma^j\dot\gamma^i\Gamma^k_{ij} = 0, \qquad k = 1, 2, \dots, n; \\ \gamma(t_0) = p; \\ \dot\gamma(t_0) = v. \end{cases}$$

The theory of ODEs implies that there exists a unique local solution to this. □

It follows from the uniqueness that, for each $p \in M$ and $v \in T_pM$, there exists a unique **maximal** geodesic $\gamma : I \to M$, with $\gamma(0) = p$ and $\dot\gamma_0 = v$, denoted by γ^v. By "maximal" we mean that I is the largest possible interval of definition. We will return to this later.

Remark 3.20. Above and also in the proof of Theorem 3.7 we have abused the notation by writing Γ^k_{ij} instead of $\Gamma^k_{ij} \circ \gamma$. We will continue to do so also in the sequel.

3.21 Riemannian connection

Let M be a C^∞-manifold and ∇ an affine connection on M. Define a map $T : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Then $T \in \mathcal{T}^2_1(M)$ (Exercise). It is called the **torsion tensor** of ∇. We say that ∇ is **symmetric** if $T \equiv 0$.

Remark 3.22. ∇ is symmetric if and only if the Christoffel symbols with respect to any coordinate frame are symmetric, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$ (Exercise).

Definition 3.23. Let M be a Riemannian manifold with the Riemannian metric $g = \langle \cdot, \cdot \rangle$. An affine connection ∇ is **compatible** with g if

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for every $X, Y, Z \in \mathcal{T}(M)$.

Lemma 3.24. *The following are equivalent*

(a) ∇ is compatible with g;

(b) If $\gamma : I \to M$ is a C^∞-path and $V, W \in \mathcal{T}(\gamma)$, then

$$\langle V, W\rangle' := \frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle;$$

(c) If $V, W \in \mathcal{T}(\gamma)$ are parallel, then $\langle V, W\rangle$ is constant.

Proof. $\boxed{(a) \Longrightarrow (b)}$ Let $\gamma : I \to M$ be a C^∞-curve, $p = \gamma(t)$, and $x = (x^1, \dots, x^n)$ a chart at p. Let $\partial_1, \dots, \partial_n$ be the coordinate frame associated to x. It is enough to show that (a) implies

$$\langle \partial_i, \partial_j \rangle'(t) = \langle D_t \partial_i, \partial_j \rangle(t) + \langle \partial_i, D_t \partial_j \rangle(t) \tag{3.25}$$

for every $t \in I$. By the definition of compatibility, (a) implies

$$\partial_k \langle \partial_i, \partial_j \rangle = \langle \nabla_{\partial_k} \partial_i, \partial_j \rangle + \langle \partial_i, \nabla_{\partial_k} \partial_j \rangle = \langle \Gamma^l_{ki} \partial_l, \partial_j \rangle + \langle \partial_i, \Gamma^l_{kj} \partial_l \rangle = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il}.$$

For the left-hand side of (3.25), we then have

$$\langle \partial_i, \partial_j \rangle'(t) = (g_{ij} \circ \gamma)'(t) = \dot\gamma_t(g_{ij}) = \dot\gamma^k_t \partial_k(g_{ij}) = \dot\gamma^k_t \Gamma^l_{ki} g_{lj} + \dot\gamma^k_t \Gamma^l_{kj} g_{il}.$$

For the right-hand side of (3.25), the identity (3.8) gives us $D_t \partial_i = \dot\gamma^k \Gamma^l_{ki} \partial_l$ and $D_t \partial_j = \dot\gamma^k \Gamma^l_{kj} \partial_l$. Therefore,

$$\langle D_t \partial_i, \partial_j \rangle(t) + \langle \partial_i, D_t \partial_j \rangle(t) = \langle \dot\gamma^k_t \Gamma^l_{ki} \partial_l, \partial_j \rangle(t) + \langle \partial_i, \dot\gamma^k_t \Gamma^l_{kj} \partial_l \rangle(t) = \dot\gamma^k_t \Gamma^l_{ki} g_{lj} + \dot\gamma^k_t \Gamma^l_{kj} g_{il},$$

which is equal to the left-hand side.

$\boxed{(b) \Longrightarrow (a)}$ Let $X, Y, Z \in \mathcal{T}(M)$, $p \in M$. Let γ be an integral curve of X starting at p. Then Y and Z induce vector fields $\widetilde{Y}, \widetilde{Z} \in \mathcal{T}(\gamma)$ by $\widetilde{Y}_t = Y_{\gamma(t)}$ and $\widetilde{Z}_t = Z_{\gamma(t)}$. Now

$$\begin{aligned} X_p \langle Y, Z \rangle = \dot\gamma_0 \langle \widetilde{Y}, \widetilde{Z} \rangle = \frac{d}{dt} \langle \widetilde{Y}, \widetilde{Z} \rangle(0) &\overset{(b)}{=} \langle D_t \widetilde{Y}, \widetilde{Z} \rangle(0) + \langle \widetilde{Y}, D_t \widetilde{Z} \rangle(0) \\ &\overset{3.7(c)}{=} \langle \nabla_{\dot\gamma} Y, Z \rangle_p + \langle Y, \nabla_{\dot\gamma} Z \rangle_p = \langle \nabla_X Y, Z \rangle_p + \langle Y, \nabla_X Z \rangle_p. \end{aligned}$$

$\boxed{(b) \Longrightarrow (c)}$ Since $V, W \in \mathcal{T}(\gamma)$ are parallel, we have by definition $D_t V = 0 = D_t W$. Using (b) this implies $\langle V, W \rangle' \equiv 0$, that is, $\langle V, W \rangle$ is a constant.

$\boxed{(c) \Longrightarrow (b)}$ Choose an orthonormal basis $\{E_1(t_0), \dots, E_n(t_0)\}$ of $T_{\gamma(t_0)} M$, where $t_0 \in I$. Let E_i be the parallel transport of $E_i(t_0)$ along γ, see Theorem 3.16. Now (c) implies that $\{E_1(t), \dots, E_n(t)\}$ is orthonormal for every $t \in I$. If $V, W \in \mathcal{T}(\gamma)$, we can therefore write

$$V = v^i E_i \quad \text{and} \quad W = w^i E_i.$$

Then $D_t V = v^i D_t E_i + \dot v^i E_i = \dot v^i E_i$ and $D_t W = \dot w^i E_i$. This gives

$$\langle D_t V, W \rangle + \langle V, D_t W \rangle = \langle \dot v^i E_i, w^j E_j \rangle + \langle v^i E_i, \dot w^j E_j \rangle = \dot v^i w^j \delta_{ij} + v^i \dot w^j \delta_{ij} = \frac{d}{dt}(v^i w^j \delta_{ij}) = \langle V, W \rangle'.$$

□

Definition 3.26. Let M be a Riemannian manifold with the Riemannian metric $g = \langle \cdot, \cdot \rangle$. An affine connection ∇ is called a **Riemannian** (or **Levi-Civita**) **connection** on M if

$$\nabla \text{ is symmetric: } \nabla_X Y - \nabla_Y X = [X, Y]; \tag{3.27}$$

and

$$\nabla \text{ is compatible with } g\text{: } X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle. \tag{3.28}$$

Theorem 3.29. *Given a Riemannian manifold M, there exists a unique Riemannian connection on M.*

Proof. *Uniqueness* Suppose such ∇ exists. Then

$$X\langle Y,Z\rangle \overset{(3.28)}{=} \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \overset{(3.28)}{=} \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X,Z]\rangle.$$

Similarly,

$$Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle + \langle Z, [Y,X]\rangle;$$

and

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z,Y]\rangle.$$

Hence,

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = 2\langle \nabla_X Y, Z\rangle + \langle Y,[X,Z]\rangle + \langle Z,[Y,X]\rangle - \langle X,[Z,Y]\rangle.$$

This gives

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}\Big(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle - \langle Z,[Y,X]\rangle + \langle X,[Z,Y]\rangle\Big). \tag{3.30}$$

Suppose ∇^1 and ∇^2 are Riemannian connections. Since the right-hand side of (3.30) is independent of the connection, we have

$$\langle \nabla^1_X Y - \nabla^2_X Y, Z\rangle = 0$$

for every $X,Y,Z \in \mathcal{T}(M)$. However, this is true only if $\nabla^1_X Y = \nabla^2_X Y$ for every $X,Y \in \mathcal{T}(M)$, that is, $\nabla^1 = \nabla^2$.

Existence We use (3.30) or, more precisely, its coordinate version to define ∇ and then show that ∇ is a Riemannian connection. It suffices to show that such ∇ exists in each coordinate chart since the uniqueness guarantees that connections agree if the charts overlap.

Let (U,x), $x = (x^1,\dots,x^n)$, be a chart. Using (3.30) and $[\partial_i,\partial_j]=0$, we have

$$\langle \nabla_{\partial_i}\partial_j, \partial_k\rangle = \frac{1}{2}\Big(\partial_i\langle \partial_j,\partial_k\rangle + \partial_j\langle \partial_k,\partial_i\rangle - \partial_k\langle \partial_i,\partial_j\rangle\Big).$$

This is the same as

$$\Gamma^l_{ij} g_{lk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

Let (g^{ij}) be the inverse matrix of (g_{ij}), i.e. $g_{lk}g^{km} = \delta_{lm}$. Multiplying both sides of the above equality by g^{km} and summing over $k = 1,2,\dots,n$, we get

$$\Gamma^m_{ij} = \frac{1}{2} g^{km}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}). \tag{3.31}$$

This formula defines ∇ in U. Furthermore, from (3.31) we get $\Gamma^m_{ij} = \Gamma^m_{ji}$, i.e. ∇ is symmetric. To show that ∇ (defined by (3.30) or its coordinate version (3.31)) is compatible with g is left as an exercise. □

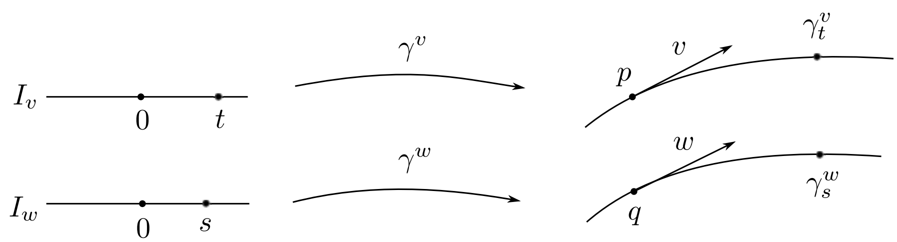
4 Geodesics

4.1 Geodesic flow

Let M be a Riemannian manifold with the Riemannian metric $g = \langle\cdot,\cdot\rangle$ and the Riemannian connection ∇. Recall that a C^∞-path $\gamma : I \to M$ is a **geodesic** if

$$D_t\dot\gamma \equiv 0.$$

If we want to emphasize that γ is a geodesic with respect to a Riemannian connection, we call γ a **Riemannian geodesic**. Recall that for every $p \in M$ and $v \in T_pM$, there exists a unique maximal geodesic $\gamma^v : I_v \to M$, with $\gamma^v_0 = p$ and $\dot\gamma^v_0 = v$. Next we "show" that γ^v_t depends C^∞-smoothly on p, v and t.



For that purpose we recall following facts on the tangent bundle. Let (U, x), $x = (x^1, \dots, x^n)$, be a chart and $v \in TU$. Then $v \in T_pM$ for some $p \in U$ and v can be uniquely written as $v = v^i(p)(\partial_i)_p$, with $\big(v^1(p), \dots, v^n(p)\big) \in \mathbb{R}^n$. Thus $TU = U \times \mathbb{R}^n$ and we local coordinates for $v \in TU$:

$$\overline{x}(v) = \big(x^1(p), \dots, x^n(p), v^1(p), \dots, v^n(p)\big) \in \mathbb{R}^{2n}.$$

Since $(TU, \overline{x})$, $\overline{x} = (x^1, \dots, x^n, v^1, \dots, v^n)$, is a chart on TM, we get a basis $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial v^i}$, $i = 1, 2, \dots, n$ for $T_{(p,v)}(TM) = T_pM \oplus \mathbb{R}^n$.

Let $G \in \mathcal{T}(U)$ be the following vector field on TU:

$$G_v = \sum_{k=1}^n v^k \frac{\partial}{\partial x^k} - \sum_{i,j,k=1}^n v^i v^j \Gamma^k_{ij}(p) \frac{\partial}{\partial v^k}. \tag{4.2}$$

We want to find out the integral curves $\overline{\gamma} : I \to TU$ of G. We can "lift" a C^∞-path $\gamma : I \to U$ to a C^∞-path $\overline{\gamma} : I \to TU$ by setting

$$\overline{\gamma}(t) = \dot\gamma_t.$$

Using local coordinates $\overline{x} = (x, v)$ we get a C^∞-path $\overline{x} \circ \overline{\gamma} : I \to \mathbb{R}^{2n}$,

$$\overline{x} \circ \overline{\gamma}(t) = \big(\gamma^1(t), \dots, \gamma^n(t), v^1(t), \dots, v^n(t)\big),$$

where $\gamma^i = x^i \circ \gamma$ and $v^i = \dot\gamma^i = (x^i \circ \gamma)'$. Now $\overline{\gamma}$ is an integral curve of G if and only if $\dot{\overline{\gamma}}_t = G_{\overline{\gamma}(t)}$ for all $t \in I$, that is, if and only if

$$\dot{\overline{\gamma}} = \sum_{k=1}^n \Big(\dot\gamma^k \frac{\partial}{\partial x^k} + \dot v^k \frac{\partial}{\partial v^k}\Big) = G_{\overline{\gamma}}.$$

Taking into account (4.2) we finally see that $\overline{\gamma}$ is an integral curve of G if and only if

$$\begin{cases} \dot\gamma^k = v^k, & 1 \le k \le n; \\ \dot v^k = -v^i v^j \Gamma^k_{ij}. \end{cases} \tag{4.3}$$

This is a first-order system equivalent to the second-order geodesic equation in the proof of Theorem 3.19 under substitution $v^k = \dot\gamma^k$.

Conclusion: Integral curves of G project to geodesics in projection $\pi : TM \to M$. Conversely, any geodesic $\gamma : I \to U$ lifts to an integral curve $\overline{\gamma}$ of G.

Since the geodesic equations are independent of the choice of local coordinates, we conclude that (4.2) defines a **global** vector field G, so called **geodesic field**, on TM. More precisely:

Lemma 4.4. *There exists a unique vector field G on TM whose integral curves project to geodesics under $\pi : TM \to M$.*

Proof. $\boxed{\textit{Uniqueness}}$ If G exists, then its integral curves project to geodesics and therefore satisfy (4.3) locally. Hence, G is unique if it exists.
$\boxed{\textit{Existence}}$ Define G locally by (4.2). Then uniqueness implies that various definitions of G in overlapping charts agree. □

The theory of flows implies that there exists an open neighborhood $\mathcal{D}(G) \subset \mathbb{R}\times TM$ of $\{0\}\times TM$ and a C^∞-map $\alpha : \mathcal{D}(G) \to TM$, called the **geodesic flow**, such that each curve

$$t \mapsto \alpha(t, v)$$

is the integral curve of G starting at $v \in TM$ and defined on an open interval $I_v \ni 0$. Since α is C^∞, also $\pi \circ \alpha : \mathcal{D}(G) \to M$ is C^∞. Now

$$t \mapsto (\pi \circ \alpha)(t, v)$$

is the geodesic γ^v, with $\gamma^v_0 = p$ and $\dot\gamma^v_0 = v$. We have shown that $\gamma^v_t = (\pi \circ \alpha)(t, v)$ depends C^∞-smoothly on t, p and $v \in T_pM$.

4.5 Appendix

Let N^n and M^m be C^∞-manifolds and $f : N \to M$ a C^∞-map. A C^∞-map $V : N \to TM$ is said to be a **vector field along** f if $V_p := V(p) \in T_pM$ for all $p \in N$, i.e. $\pi \circ V = f$.

Theorem 4.6. *If $f : N \to M$ is an embedding and V is a C^∞ vector field along f, there exists $\tilde V \in \mathcal{T}(M)$ such that $V_p = \tilde V_{f(p)}$ for all $p \in N$, i.e. V is "extendible".*

Proof. The proof is based on the following: For each $q \in fN \subset M$ there exists a neighborhood U of q in M and a chart $x : U \to \mathbb{R}^m$ such that

$$x^{n+1} \equiv \cdots \equiv x^m = 0 \tag{4.7}$$

in $U \cap fN$. These are so called **slice coordinates** (cf. Theorem 1.28).
How to construct the extension of V?
<u>Sketch</u>: Cover fN by charts $\{U_\alpha\}$ with the property (4.7). In $f^{-1}\big(fN \cap U_\alpha\big)$ we have

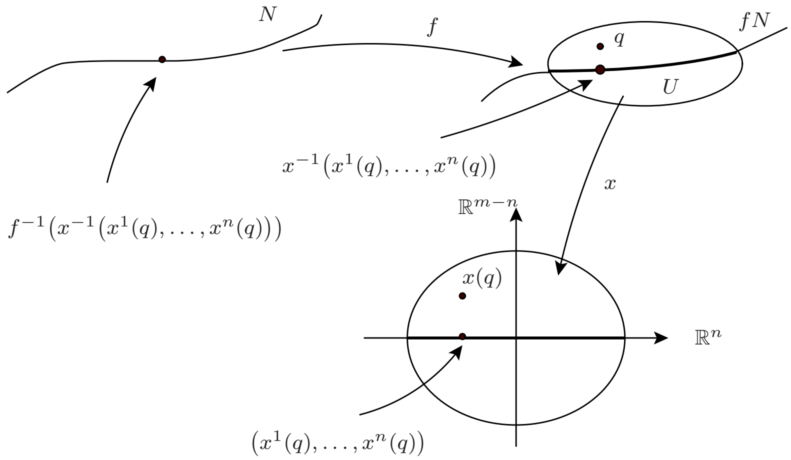
$$V_p = \sum_{i=1}^m v_i^\alpha(p)(\partial_i)_{f(p)}.$$

Define in U_α a vector field $\tilde V^\alpha$ by setting

$$\tilde V^\alpha_q = \sum_{i=1}^m w_i^\alpha(q)(\partial_i)_q,$$

where

$$w_i^\alpha(q) = v_i^\alpha\left(f^{-1}\left(x^{-1}\left(x^1(q), \dots, x^n(q)\right)\right)\right).$$



Then take all charts $\{U_\beta\}$ such that $U_\beta \cap fN = \emptyset$ for all β and

$$M = \bigcup_{\alpha,\beta} \big(U_\alpha \cup U_\beta\big).$$

Define $\tilde{V}^\beta \in \mathcal{T}(U_\beta)$ by $\tilde{V}^\beta \equiv 0$. Rename $U_\alpha,\ \tilde{V}^\alpha,\ U_\beta$, and $\tilde{V}^\beta$ as U_i and $\tilde{V}^i,\ i \in I$. Finally, take a C^∞ partition of unity $\{\varphi_i\}$ subordinate to $\{U_i\}$ and define

$$\tilde{V} = \sum_{i\in I} \varphi_i \tilde{V}^i.$$

□

The assumption "f embedding" is crucial: For example, $\gamma\colon \mathbb{R} \to \mathbb{R}^2,\ \gamma(t) = (t^3, 0)$ is a C^∞-path but not embedding. Now $\dot\gamma \in \mathcal{T}(\gamma),\ \dot\gamma_t = 3t^2(\partial_1)_{\gamma(t)}$, but $\dot\gamma \notin \mathcal{T}(\mathbb{R})$ since $\dot\gamma$ considered as a vector field in $\mathbb{R}$ is given by $\dot\gamma_u = 3u^{2/3}\partial_1$ which is not differentiable at $u = 0$.

4.8 Exponential map

Lemma 4.9. *All Riemannian geodesics have constant* ***speed****, i.e. for every Riemannian geodesic γ there is a constant c such that*

$$|\dot\gamma_t| = \langle \dot\gamma_t, \dot\gamma_t\rangle^{1/2} = c$$

for every $t \in I$.

Proof. Lemma 3.24 implies that $\langle \dot\gamma, \dot\gamma\rangle' = 2\langle D_t\dot\gamma, \dot\gamma\rangle = 0$, since by definition $D_t\dot\gamma = 0$. □

Lemma 4.9 implies that the length of $\gamma|[t_0, t]$ is

$$\ell\big(\gamma|[t_0, t]\big) = \int_{t_0}^{t} |\dot\gamma_t|\, dt = c(t - t_0). \tag{4.10}$$

If $c = 1$, we say that γ is a **normalized geodesic** (or of **unit speed**, or **parametrized by arc length**).
Let I_v be the maximal interval where γ^v is defined, and let $[0, \ell_v)$ be the nonnegative part of I_v.

Lemma 4.11. *For every $\alpha > 0$ and $0 \le t < \ell_{\alpha v}$*

$$\gamma_t^{\alpha v} = \gamma_{\alpha t}^v.$$

In particular, $\ell_{\alpha v} = \frac{1}{\alpha}\ell_v$.

Proof. The claim holds if $|\dot\gamma^v| \equiv 0$, so we may assume that $\dot\gamma^v_t \neq 0$. Let $I_v = (a,b)$ and $\widetilde{I}_{\alpha v} = \frac{1}{\alpha} I_v = (a/\alpha, b/\alpha)$. Define $\gamma : \widetilde{I}_{\alpha v} \to M$ by

$$\gamma(t) = \gamma^v(\alpha t).$$

Then $\dot\gamma_t = \alpha \dot\gamma^v_{\alpha t}$, and so

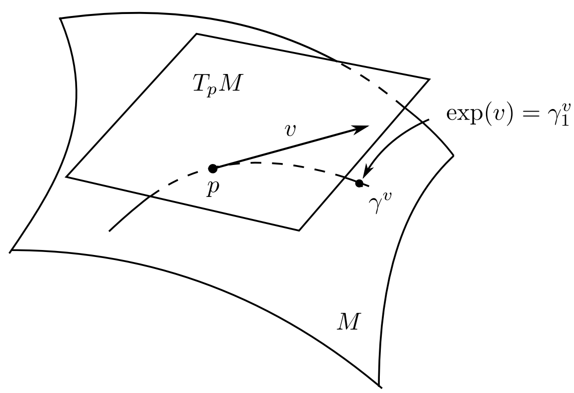
$$D_t \dot\gamma_t \overset{(*)}{=} \nabla_{\dot\gamma_t} \dot\gamma_t = \nabla_{\alpha \dot\gamma^v_{\alpha t}} \left(\alpha \dot\gamma^v_{\alpha t}\right) = \alpha^2 \nabla_{\dot\gamma^v_{\alpha t}} \left(\dot\gamma^v_{\alpha t}\right) = 0.$$

Hence, γ is a geodesic, with $\gamma_0 = \gamma^v_0$ and $\dot\gamma_0 = \alpha \dot\gamma^v_0 = \alpha v$. Furthermore, $\widetilde{I}_{\alpha v}$ is the maximal interval since I_v is. Uniqueness implies that $\gamma = \gamma^{\alpha v}$. The equality $(*)$ holds since the vector field $t \mapsto \dot\gamma_t$ (along γ) is locally extendible to a vector field on M (also denoted by $\dot\gamma$). This is seen as follows: Since $\dot\gamma^v_t \neq 0$, $\gamma : \widetilde{I}_{\alpha v} \to M$ is an immersion and therefore locally an embedding by Theorem 1.28. Then $t \mapsto \dot\gamma_t$ is locally extendible by Theorem 4.6. □

Let $\mathcal{E} \subset TM$ be the set of vectors v such that $\ell_v > 1$, i.e. $\gamma^v(t)$ is defined for all $t \in [0,1]$. The **exponential map** $\exp : \mathcal{E} \to M$ is defined by

$$\exp(v) := \gamma^v(1). \tag{4.12}$$

For $p \in M$, the exponential map at p is the map $\exp_p = \exp|\mathcal{E}_p$, where $\mathcal{E}_p = \mathcal{E} \cap T_pM$.



Theorem 4.13. *We have the following properties*

(a) $\mathcal{E} \subset TM$ *is open and contains the (image of the) zero section* $M \times \{0\} = \bigsqcup_{p \in M} 0_p$, *where* 0_p *is the zero element of* T_pM*;*

(b) each $\mathcal{E}_p$ *is star-shaped with respect to* 0 $(= 0_p)$*;*

(c) for each $v \in TM$, *the geodesic* γ^v *is given by*

$$\gamma^v(t) = \exp(tv)$$

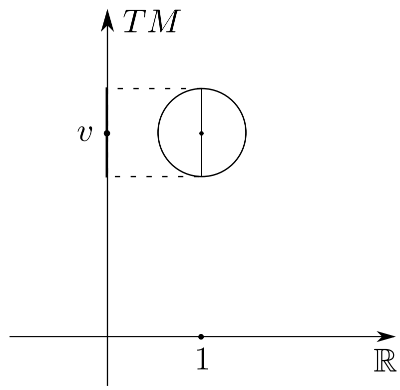
for all t *such that either side is defined;*

(d) the exponential map $\exp : \mathcal{E} \to M$ *is* C^∞.

Proof. The claim (c) follows from Lemma 4.11:

$$\exp(tv) = \gamma^{tv}_1 \overset{4.11}{=} \gamma^v_t.$$

(b): If $v \in \mathcal{E}_p$, then γ_t^v is defined for all $t \in [0,1]$. However, $\exp(tv) = \gamma_1^{tv} = \gamma_t^v$, so γ_1^{tv} is defined for all $t \in [0,1]$. This means that $\mathcal{E}_p$ is star-shaped with respect to 0.
(d): We have $\exp(v) = (\pi \circ \alpha)(1, v)$, where α is the geodesic flow. Hence, exp is C^∞.
(a): Suppose $v \in \mathcal{E}$. Then γ^v is defined at least on $[0,1]$. Therefore, also the integral curve $\overline{\gamma}^v$ of G starting at $v \in TM$ is defined on $[0,1]$. In particular, $\overline{\gamma}^v(1)$ is defined, hence $(1, v) \in \mathcal{D}(G)$. Because $\mathcal{D}(G)$ is an open subset of $\mathbb{R} \times TM$, there exists an open neighborhood of $(1, v)$ in $\mathbb{R} \times TM$ on which the flow α is defined.



In particular, there exists an open neighborhood of v in TM where $\gamma_1^w = \exp(w)$ is defined. This implies that $\mathcal{E}$ is open. If $0_p \in T_pM$ is the zero element, then γ^{0_p} is the constant path $\gamma^{0_p} = p$ for every $t \in \mathbb{R}$. In particular, $\gamma_t^{0_p}$ is defined for every $t \in [0,1]$. So, $\mathcal{E}$ contains the zero-section. □

Remark 4.14. If $v \in T_pM$, $v \neq 0$, then $\exp(v) = \gamma_1^v = \gamma_{|v|}^{v/|v|}$. Because $v/|v|$ is a unit vector, $\exp(v)$ is obtained by traveling from p of length $|v|$ along the unit speed geodesic passing through p with velocity $v/|v|$.

Theorem 4.15. *For any $p \in M$, there exist a neighborhood $\mathcal{V}$ of the origin in T_pM and a neighborhood U of p in M such that*

$$\exp_p : \mathcal{V} \to U$$

is a diffeomorphism.

Proof. The map $\exp_p$ is clearly C^∞ since exp is. We show that $(\exp_p)_{*0} : T_0(T_pM) \cong T_pM \to T_pM$ is invertible, in fact, the identity map. Let $v \in T_pM$. To compute $(\exp_p)_{*0}v$, choose a curve $\tau : I \to T_pM$ with $\tau(0) = 0 \in T_pM$ and $\dot{\tau}(0) = v$ and compute $\big((\exp_p) \circ \tau\big)'(0)$. An obvious choice is $\tau(t) = tv$. Then

$$(\exp_p)_{*0}v = \frac{d}{dt}\big((\exp_p) \circ \tau\big)(t)|_{t=0} = \frac{d}{dt}\exp_p(tv)|_{t=0} = \frac{d}{dt}\gamma_t^v|_{t=0} = \dot{\gamma}_0^v = v.$$

Hence, $(\exp_p)_{*0} : T_pM \to T_pM$ is identity, in particular, it is invertible. The inverse function theorem implies that $\exp_p$ is a local diffeomorphism on a neighborhood of $0 \in T_pM$. □

Remark 4.16. The name "exponential map" comes from following observation:
Let G be a Lie group. The **left-invariant connection** ∇^L is defined by the requirement

$$\nabla_X^L Y = 0$$

for every $X \in \mathcal{T}(G)$ and $Y \in \mathfrak{g}$, where $\mathfrak{g}$ is the set of all left-invariant vector fields ($\cong T_eG$). Geodesics with respect to ∇^L is the set of all integral curves of left-invariant vector fields.

Suppose that $G = \mathrm{GL}(n, \mathbb{R})$. Then one can show that $T_eG \cong \mathrm{gl}(n, \mathbb{R})$, the set of all linear maps $\mathbb{R}^n \to \mathbb{R}^n$ or $n \times n$ matrices. For $A \in \mathrm{gl}(n, \mathbb{R}) \cong T_eG$, we have

$$\exp_e A = e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

The natural identification for $T_eG \cong \mathrm{gl}(n, \mathbb{R})$ is given as follows. Let x_{ij}, $i, j = 1, 2, \ldots, n$, be the coordinate functions on $\mathrm{GL}(n, \mathbb{R})$, i.e. $x_{ij}(g)$ is the ijth entry of $g \in \mathrm{GL}(n, \mathbb{R})$. Define, for each $V \in \mathfrak{g}$, a matrix $(V_{ij}) \in \mathrm{gl}(n, \mathbb{R})$ by setting

$$V_{ij} = V_e(x_{ij}),$$

which gives the identification.

4.17 Normal neighborhoods

Let $\mathcal{V}$ and U be as in Theorem 4.15, i.e. so that $\exp_p : \mathcal{V} \to U$ is a diffeomorphism. Then U is called a **normal neighborhood** of p.
If $\varepsilon > 0$ is so small that $B(0, \varepsilon) := \{v \in T_pM : |v| < \varepsilon\} \subset \mathcal{V}$, then the image $\exp_p(B(0, \varepsilon))$ is called a **normal** (or **geodesic**) **ball**. Furthermore, if $\overline{B}(0, \varepsilon) \subset \mathcal{V}$, then $\exp_p(\overline{B}(0, \varepsilon))$ is called **closed normal** (or geodesic) **ball**, and $\exp_p(\partial B(0, \varepsilon))$ is called **normal** (or geodesic) **sphere** in M.
Any orthonormal basis $\{e_i\}$ of T_pM defines an isomorphism $E : \mathbb{R}^n \to T_pM$,

$$E(x^1, \ldots, x^n) := x^i e_i.$$

If U is a normal neighborhood of p, we get a coordinate chart $\varphi : U \to \mathbb{R}^n$ by defining

$$\varphi := E^{-1} \circ \exp_p^{-1}.$$

Then

$$\varphi : \exp_p(x^i e_i) \mapsto (x^1, \ldots, x^n), \quad \text{if} \quad x^i e_i \in \mathcal{V}. \tag{4.18}$$

We call the pair (U, φ) a **normal chart** and $(x^1, \ldots, x^n) \in \mathbb{R}^n$ are called (Riemannian) **normal coordinates** of the point $x = \exp_p(x^i e_i)$. We define the **radial distance function** $r : U \to \mathbb{R}$ by

$$r(x) := \left(\sum_{i=1}^{n} (x^i)^2 \right)^{1/2},$$

and **the unit radial vector field** $\frac{\partial}{\partial r} \in \mathcal{T}(U \setminus \{p\})$ by

$$\left(\frac{\partial}{\partial r} \right)_x := \frac{x^i}{r(x)} (\partial_i)_x.$$

Note that $r(x) = |\exp_p^{-1} x|$ since $\{e_i\}$ is orthonormal.

Lemma 4.19. *Let (U, φ) be a normal chart at p.*

(a) If $v = v^i e_i \in T_pM$, then the normal coordinates of $\gamma^v(t)$ are $(tv^1, \ldots, tv^n)$ whenever $tv \in \mathcal{V}$.

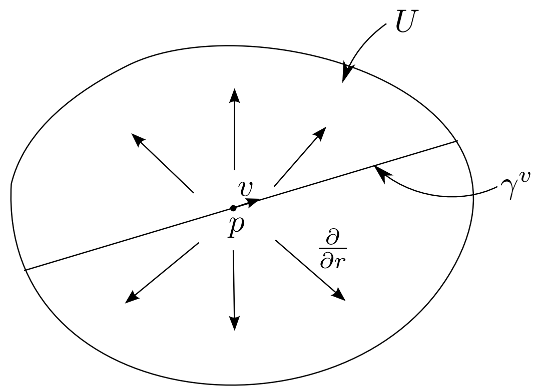
(b) The normal coordinates of p are $(0, \ldots, 0)$.

(c) *The components of the Riemannian metric* **at** p *are* $g_{ij} = \delta_{ij}$.

(d) *Any set* $\{x \in U : r(x) < \varepsilon\}$ *is a normal ball* $\exp_p(B(0, \varepsilon))$.

(e) *If* $q \in U \setminus \{p\}$, *then* $\left(\frac{\partial}{\partial r}\right)_q$ *is the velocity vector* ($\dot\gamma$) *of the unit speed geodesic from* p *to* q *(unique by (a)), and therefore* $\left|\frac{\partial}{\partial r}\right| = 1$.

(f) $\partial_k g_{ij}(p) = 0$ *and* $\Gamma^k_{ij}(p) = 0$.



Proofs are straightforward consequences of (4.18).
Geodesics γ^v starting at p and staying in U are called **radial geodesics** (because of (a)).
Warning: Geodesics that do not pass through p do not have, in general, a "simple" form in normal coordinates.

Definition 4.20. An open set $W \subset M$ is called **uniformly** (or **totally**) **normal** if there exists $\delta > 0$ such that for any $q \in W$ the map $\exp_q$ is diffeomorphism on $B(0, \delta) \subset T_qM$ and $W \subset \exp_q(B(0, \delta))$.

Lemma 4.21. *Given* $p \in M$ *and any neighborhood* U *of* p, *there exists a uniformly normal* W, *with* $p \in W$.

Proof. Let $\mathcal{E}$ be as in the definition of the exponential map ($\mathcal{E} \subset TM$ is open and contains the zero section). Denote the points of $\mathcal{E}$ by (q, v), $v \in T_qM \cap \mathcal{E} = \mathcal{E}_q$. Define a map $F : \mathcal{E} \to M \times M$ by

$$F(q, v) = (q, \exp_q v).$$

Clearly, F is C^∞. (Projections $\pi_i : M \times M \to M$, $\pi_i(q_1, q_2) = q_i$, $i = 1, 2$, are C^∞ and $\pi_1 \circ F = \pi_1|\mathcal{E}$, $\pi_2 \circ F = \exp$). We want to compute the Jacobian matrix of F at $(p, 0)$. Now

$$T_{(p,0)}\mathcal{E} = T_{(p,0)}(TM) = T_pM \oplus T_0(T_pM)$$

and

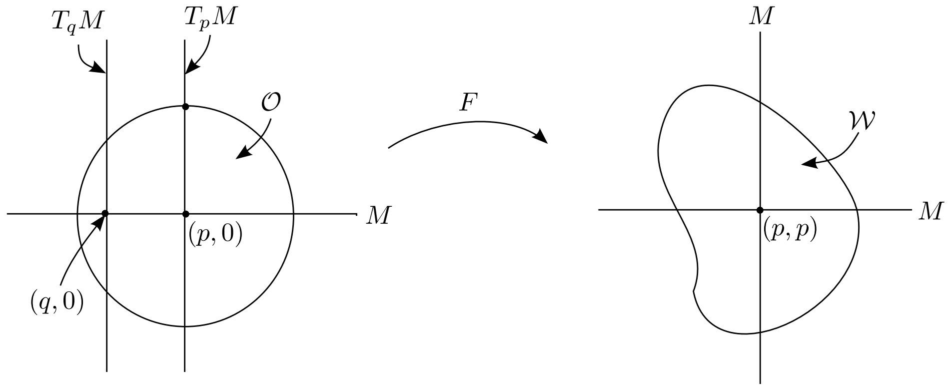
$$T_{F(p,0)}(M \times M) = T_{(p,p)}(M \times M) = T_pM \oplus T_pM.$$

Then the matrix of $F_* : T_{(p,0)}\mathcal{E} \to T_{F(p,p)}(M \times M)$ is

$$\begin{bmatrix} \mathrm{id} & 0 \\ * & (\exp_p)_* \end{bmatrix},$$

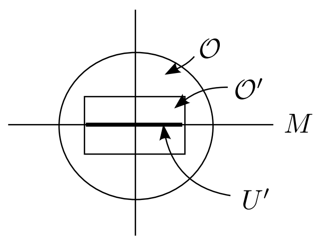
where in the upper left block we have id since the map $(q, v) \mapsto q$ is the identity w.r.t. q; in the upper right block we have 0 since $(q, v) \mapsto q$ is independent of v; the lower left block $*$ is irrelevant;

and in the lower right block we have $(\exp_p)_*$ since the map $(q, v) \mapsto \exp_q v$ is the exponential map $\exp_q$ w.r.t. v.
Hence, $F_{*(p,0)}$ is invertible. The inverse mapping theorem implies that there exist a neighborhood $\mathcal{O}$ of $(p, 0)$ in TM and $\mathcal{W}$ of (p, p) such that $F : \mathcal{O} \to \mathcal{W}$ is a diffeomorphism.



It is possible to choose another neighborhood $\mathcal{O}' \subset \mathcal{O}$ of $(p, 0)$ of the form

$$\mathcal{O}' = \{(q, v) : q \in U' \text{ and } |v| < \delta\}, \quad U' \ni p.$$



The topology of TM is generated by product open sets in local trivializations. Hence, there exists $\varepsilon > 0$ so that the set

$$\mathcal{X} = \{(q, v) : r(q) < 2\varepsilon \text{ and } |v|_{\overline{g}} < 2\varepsilon\}$$

is an open subset of $\mathcal{O}$, where $|\cdot|_{\overline{g}}$ is the Euclidean norm in the normal coordinates. The set

$$\mathcal{K} = \{(q, v) : r(q) \leq \varepsilon \text{ and } |v|_{\overline{g}} = \varepsilon\}$$

is compact, and the Riemannian norm $|\cdot|_g$ is continuous and nonvanishing on $\mathcal{K}$, so it is bounded from above and below by positive constants. Both norms $|\cdot|_{\overline{g}}$ and $|\cdot|_g$ are homogeneous ($|\lambda v| = \lambda|v|$, $\lambda > 0$), so $c_1|v|_{\overline{g}} \leq |v|_g \leq c_2|v|_{\overline{g}}$ whenever $v \in T_qM$, with $r(q) \leq \varepsilon$. Denoting $\delta := c_1\varepsilon$, we may then choose the set

$$\mathcal{O}' := \{(q, v) : r(q) < \varepsilon \text{ and } |v| < \delta\} \subset \mathcal{X}.$$

Now choose a neighborhood $W \subset U$ of p such that also $W \subset U'$ (=the set in the definition of $\mathcal{O}'$) and that $W \times W \subset F(\mathcal{O}')$. Next we show that W and δ satisfy the claim of the Lemma. Take $q \in W$. Because F is a diffeomorphism on $\mathcal{O}'$, we know that $\exp_q$ is a diffeomorphism on $B(0, \delta) \subset T_qM$.
Is $W \subset \exp_q(B(0, \delta))$? Take a point $y \in W$. Since $(q, y) \in W \times W \subset F(\mathcal{O}')$, there exists $v \in B(0, \delta) \subset T_qM$ such that $(q, y) = F(q, v)$, so $y = \exp_q v$. Hence, $W \subset \exp_q(B(0, \delta))$. □

4.22 Riemannian manifolds as metric spaces

Recall that the length of a C^∞-path $\gamma : [a,b] \to M$ is

$$\ell(\gamma) = \ell_g(\gamma) = \int_a^b |\dot\gamma_t|\, dt,$$

where g is the Riemannian metric on M. It is independent of parametrization: if $\varphi : [c,d] \to [a,b]$ is C^∞ with C^∞ inverse, then

$$\widetilde{\gamma} = \gamma \circ \varphi : [c,d] \to M$$

is called a reparametrization of γ (a forward reparametrization if $\varphi(c) = a$ and a backward reparametrization if $\gamma(c) = b$). Then (Exercise)

$$\ell(\gamma) = \ell(\widetilde{\gamma}).$$

A **regular** curve is a C^∞-path $\gamma : I \to M$ such that $\dot\gamma_t \neq 0$ for every $t \in I$. A path $\gamma : [a,b] \to M$ is **piecewise regular** if there exists $a_0 = a < a_1 < \cdots < a_k = b$ such that $\gamma|[a_{i-1}, a_i]$ is regular. The length of γ is then

$$\ell(\gamma) = \sum_{i=1}^k \ell(\gamma|[a_{i-1}, a_i]) = \int_a^b |\dot\gamma_t|\, dt,$$

which is well-defined since $\dot\gamma_t$ exists and is continuous outside the discrete set of points $t = a_i$. We say that γ is **admissible** if it is piecewise regular or $\gamma : \{a\} \to M$, $\gamma(a) = p \in M$.

Remark 4.23. The idea of reparametrization extends to admissible curves. The **arc length function** of an admissible curve $\gamma : [a,b] \to M$ is the function $s : [a,b] \to \mathbb{R}$,

$$s(t) = \ell(\gamma|[a,b]) = \int_a^t |\dot\gamma_u|\, du.$$

Furthermore, the derivative $s'(t)$ exists whenever $\dot\gamma_t$ exists and $s'(t) = |\dot\gamma_t|$.

Every admissible curve has a unit speed reparametrization: if $\gamma : [a,b] \to M$ is admissible and $\ell = \ell(\gamma)$, there exists a forward reparametrization $\widetilde{\gamma} : [0,\ell] \to M$ of γ such that $\widetilde{\gamma}$ is of unit speed (piecewise).

Now suppose that M is connected (hence path-connected). For $p, q \in M$, we define

$$d(p,q) := \inf_\gamma \ell(\gamma),$$

where inf is taken over all admissible paths γ from p to q ($\gamma : [a,b] \to M$, $\gamma(a) = p$, $\gamma(b) = q$).

Theorem 4.24. *Let M be a connected Riemannian manifold, and let d be as above. Then (M,d) is a metric space whose induced topology is the same as the given manifold topology.*

Proof. (i) $d(p,q)$ is finite for every $p, q \in M$ (exercise).

(ii) Clearly, $d(p,q) = d(q,p) \geq 0$ since $\ell(\gamma)$ is independent of parametrization (exercise).

(iii) $d(p,p) = 0$ since we can take the constant path $\gamma \equiv p$.

(iv) $d(p,q) \leq d(p,z) + d(z,q)$ (exercise)

So it remains to show:

(v) $p \neq q$ implies $d(p,q) > 0$.

(vi) metric space topology = manifold topology.

(v): Let $p \in M$ and let $(x^1, \ldots, x^n)$ be normal coordinates at p. As in the proof of Lemma 4.21, we can find a closed normal ball $\overline{B} = \exp_p(\overline{B}(0,\delta))$ and positive constants c_1 and c_2 such that

$$c_1|v|_{\overline{g}} \leq |v| \leq c_2|v|_{\overline{g}}$$

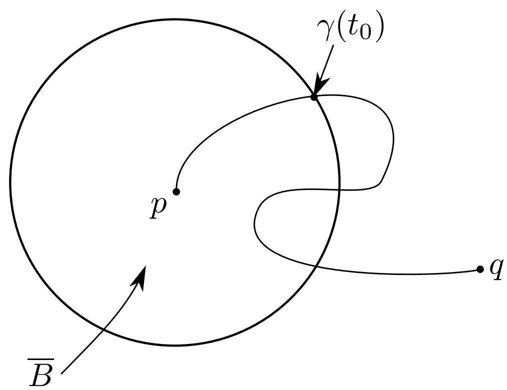
for every $v \in T_qM$ and $q \in \overline{B}$. This implies that for every piecewise regular $\gamma : I \to \overline{B}$ we have

$$c_1\ell_{\overline{g}}(\gamma) \leq \ell_g(\gamma) \leq c_2\ell_{\overline{g}}(\gamma). \tag{4.25}$$

Here $\ell_{\overline{g}}(\gamma)$ is the length w.r.t. the Euclidean metric $\overline{g}$ and $\ell_g(\gamma)$ is the length w.r.t. the Riemannian metric g. Now, if $p \neq q$, take $\delta > 0$ so small that $q \notin \overline{B}$. Then each admissible path γ from $p = \gamma(a)$ to q has to pass through $\partial\overline{B} = \exp_p(\partial B(0,\delta))$. Let t_0 be the smallest of those $t \geq a$ with $\gamma(t) \in \partial\overline{B}$. Then

$$\ell_g(\gamma) \geq \ell_g(\gamma|(a,t_0)) \geq c_1\ell_{\overline{g}}(\gamma|(a,t_0)) \geq c_1 d_{\overline{g}}(p,\gamma(t_0)) = c_1\delta > 0,$$

where $d_{\overline{g}}$ is the Euclidean distance.



Thus (v) is proven and (M,d) is indeed a metric space.

(vi): We need to show that for every $p \in M$ and for every neighborhood U of p in the manifold topology there exists a metric open ball $B(p,\varepsilon) = \{q \in M : d(p,q) < \varepsilon\} \subset U$, and conversely for every $p \in M$ and $\varepsilon > 0$ there exists a neighborhood U of p in the manifold topology such that $U \subset B(p,\varepsilon)$. This can be done for example by using (4.25). Details are left as an exercise. □

4.26 Minimizing properties of geodesics

Definition 4.27. An admissible curve γ is called **minimizing** if $\ell(\gamma) \leq \ell(\widetilde{\gamma})$ for any admissible $\widetilde{\gamma}$ with the same endpoints.

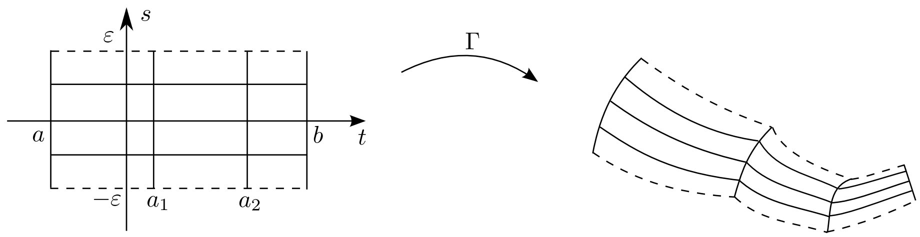
Remark 4.28. A curve γ is minimizing if and only if $\ell(\gamma) = d(p,q)$, where p and q are the end points of γ.

We shall show that minimizing curves, with unit speed parametrization, are geodesics.

Definition 4.29. An **admissible family of curves** is a continuous map $\Gamma : (-\varepsilon,\varepsilon) \times [a,b] \to M$ such that

1° Γ is C^∞ on each rectangle $(-\varepsilon, \varepsilon) \times [a_{i-1}, a_i]$ for some $a_0 = a < a_1 < \cdots < a_k = b$; and

2° for each $s \in (-\varepsilon, \varepsilon)$ the map $\Gamma_s : [a, b] \to M$, $\Gamma_s(t) = \Gamma(s, t)$, is an admissible curve.

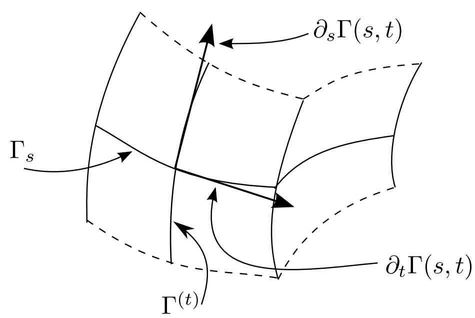


A **vector field along** Γ is a continuous map $V : (-\varepsilon, \varepsilon) \times [a, b] \to TM$ such that $V(s,t) \in T_{\Gamma(s,t)}M$ for every (s,t) and $V|(-\varepsilon, \varepsilon) \times [\widetilde{a}_{i-1}, \widetilde{a}_i]$ is C^∞ for some (possibly finer) subdivision $\widetilde{a}_0 = a < \widetilde{a}_1 < \cdots < \widetilde{a}_\ell = b$. Curves $\Gamma_s : [a, b] \to M$, $\Gamma_s(t) = \Gamma(s, t)$, are called the **main curves**. They are piecewise regular.
Curves $\Gamma^{(t)} : (-\varepsilon, \varepsilon) \to M$, $\Gamma^{(t)}(s) = \Gamma(s, t)$, are called the **transverse curves**. They are always C^∞. We define

$$\partial_t \Gamma(s,t) := \frac{d}{dt} \Gamma_s(t), \quad t \neq a_i;$$

and

$$\partial_s \Gamma(s,t) := \frac{d}{dt} \Gamma^{(t)}(s), \quad \text{for every } (s,t).$$



Then $\partial_s \Gamma$ is a vector field along Γ, but $\partial_t \Gamma$ can not necessarily be extended to a vector field along Γ. If V is a vector field along Γ, we write $D_t V$ as the covariant derivative of V along main curves and $D_s V$ as the covariant derivative of V along the transverse curves.

Lemma 4.30 (Symmetry Lemma)**.** *Let $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a family of admissible curves on a Riemannian manifold M. Then*

$$D_s \partial_t \Gamma = D_t \partial_s \Gamma.$$

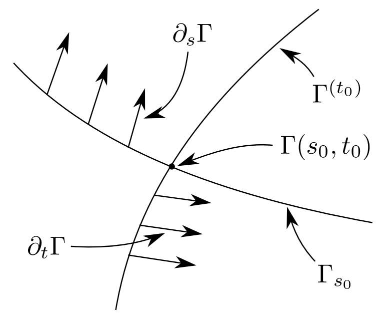
Remark 4.31. This is the point where the symmetry condition on ∇ is needed.

Proof of Lemma 4.30. Let x be a chart at $\Gamma(s_0, t_0)$. Writing

$$(x \circ \Gamma)(s,t) = (x^1(s,t), \ldots, x^n(s,t)),$$

we get

$$\partial_t \Gamma = \frac{\partial x^i}{\partial t} \partial_i \quad \text{and} \quad \partial_s \Gamma = \frac{\partial x^i}{\partial s} \partial_i.$$



Recall the equation (3.8) in Chapter III: $\dot\gamma = \dot\gamma^i\partial_i$ and $V = v^j\partial_j$ implies that

$$D_tV = (\dot v^k + v^j\dot\gamma^j\Gamma^k_{ij})\partial_k.$$

Now when calculating $D_s\partial_t\Gamma$ we can use $\dot\gamma = \partial_s\Gamma$ and $V = \partial_t\Gamma$; and similarly, when calculating $D_t\partial_s\Gamma$, we can use $\dot\gamma = \partial_t\Gamma$ and $V = \partial_s\Gamma$. Hence,

$$D_s\partial_t\Gamma = \left(\frac{\partial^2x^k}{\partial s\partial t} + \frac{\partial x^j}{\partial t}\frac{\partial x^i}{\partial s}\Gamma^k_{ij}\right)\partial_k$$

and

$$\begin{aligned} D_t\partial_s\Gamma &= \left(\frac{\partial^2x^k}{\partial t\partial s} + \frac{\partial x^j}{\partial s}\frac{\partial x^i}{\partial t}\Gamma^k_{ij}\right)\partial_k \overset{i\leftrightarrow j}{=} \left(\frac{\partial^2x^k}{\partial t\partial s} + \frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\Gamma^k_{ji}\right)\partial_k \\ &\overset{(*)}{=} \left(\frac{\partial^2x^k}{\partial t\partial s} + \frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\Gamma^k_{ij}\right)\partial_k = D_s\partial_t\Gamma. \end{aligned}$$

We have $(*)$ because $\Gamma^k_{ij} = \Gamma^k_{ji}$ due to the symmetricity of Γ. □

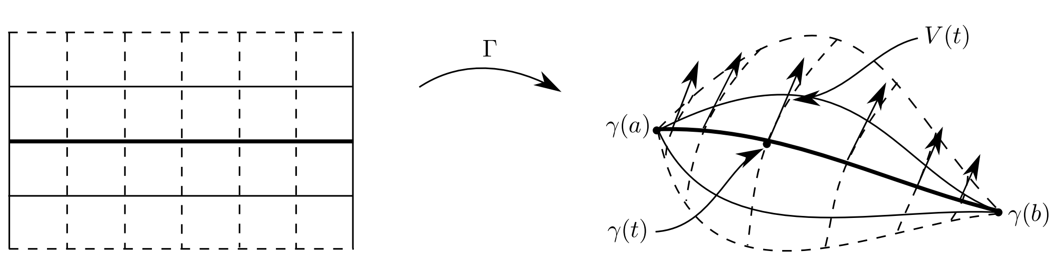
Remark 4.32. *Shorter proof of Lemma 4.30.* Let ∂_t and ∂_s be the standard coordinate vector fields in $\mathbb{R}^2$. Then

$$\partial_t\Gamma = \Gamma_*\partial_t \quad\text{and}\quad \partial_s\Gamma = \Gamma_*\partial_s.$$

Since $[\partial_t, \partial_s] = 0$, we have

$$\begin{aligned} D_sD_t\Gamma - D_tD_s\Gamma &= \nabla_{\Gamma_*\partial_s}\Gamma_*\partial_t - \nabla_{\Gamma_*\partial_t}\Gamma_*\partial_s \\ &= [\Gamma_*\partial_s, \Gamma_*\partial_t] \\ &= \Gamma_*[\partial_t, \partial_s] = 0. \end{aligned}$$

Definition 4.33. Let $\gamma : [a,b] \to M$ be an admissible curve. A **variation** of γ is an admissible family $\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \to M$ such that $\Gamma_0 = \gamma$. It is called a **proper variation** (or **fixed-endpoint variation**) if $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for every s. The **variation field** of Γ is the vector field $V(t) = \partial_s\Gamma(0,t)$. A vector field W along γ is **proper** if $W(a) = 0$ and $W(b) = 0$. (If Γ is proper variation of γ, the variation field of Γ is proper.



Lemma 4.34. *Let $\gamma : [a, b] \to M$ be admissible and V a continuous piecewise smooth vector field along γ. Then there exists Γ, a variation of γ, such that V is the variation field of Γ. If V is proper, then Γ can be taken to be proper as well.*

Proof. Define $\Gamma(s,t) := \exp(sV(t))$. (Exercise) □

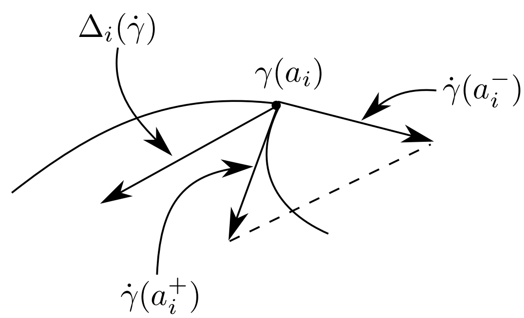
Theorem 4.35 (First variation formula)**.** *Let $\gamma : [a, b] \to M$ be a unit speed admissible curve, Γ a proper variation of γ, and V the variation field of Γ. Then*

$$\frac{d}{ds}\ell(\Gamma_s)|_{s=0} = -\int_a^b \langle V, D_t\dot\gamma\rangle\,dt - \sum_{i=1}^{k-1}\langle V(a_i), \Delta_i\dot\gamma\rangle, \tag{4.36}$$

where $\Delta_i\dot\gamma := \dot\gamma(a_i^+) - \dot\gamma(a_i^-)$ and a_i's are the subdivision points of $[a, b]$ associated to γ;

$$\dot\gamma(a_i^+) := \lim_{t\downarrow a_i}\dot\gamma(t) \quad \textit{and} \quad \dot\gamma(a_i^-) := \lim_{t\uparrow a_i}\dot\gamma(t).$$

Note: The unit speed assumption is not restrictive: each admissible curve has a unit speed reparametrization and the length is independent of parametrization.



Proof of Theorem 4.35. Write $T(s,t) = \partial_t\Gamma(s,t)$ and $S(s,t) = \partial_s\Gamma(s,t)$. Then

$$\frac{d}{ds}\ell(\Gamma_s|[a_{i-1},a_i]) = \frac{d}{ds}\int_{a_{i-1}}^{a_i}\langle\dot\Gamma_s(t),\dot\Gamma_s(t)\rangle^{1/2}\,dt = \frac{d}{ds}\int_{a_{i-1}}^{a_i}\langle T(s,t),T(s,t)\rangle^{1/2}\,dt = \int_{a_{i-1}}^{a_i}\frac{\partial}{\partial s}\langle T,T\rangle^{1/2}\,dt$$
$$= \int_{a_{i-1}}^{a_i}\frac{1}{2}\langle T,T\rangle^{-1/2}\frac{\partial}{\partial s}\langle T,T\rangle\,dt = \int_{a_{i-1}}^{a_i}\frac{1}{2}\langle T,T\rangle^{-1/2}2\langle D_sT,T\rangle\,dt \overset{4.30}{=} \int_{a_{i-1}}^{a_i}\frac{1}{|T|}\langle D_tS,T\rangle\,dt.$$

At $s = 0$, we have $T(0,t) = \partial_t\Gamma(0,t) = \dot\gamma_t$, $|T(0,t)| = |\dot\gamma_t| = 1$, and $S(0,t) = \partial_s\Gamma(0,t) = V(t)$. Hence,

$$\begin{aligned}\frac{d}{ds}\ell(\Gamma_s|[a_{i-1},a_i])|_{s=0} &= \int_{a_{i-1}}^{a_i}\langle D_tV,\dot\gamma\rangle\,dt = \int_{a_{i-1}}^{a_i}\left(\frac{d}{dt}\langle V,\dot\gamma\rangle - \langle V, D_t\dot\gamma\rangle\right)\,dt\\ &= \langle V(a_i),\dot\gamma(a_i^-)\rangle - \langle V(a_{i-1}),\dot\gamma(a_{i-1}^+)\rangle - \int_{a_{i-1}}^{a_i}\langle V,D_t\dot\gamma\rangle\,dt.\end{aligned}$$

Using $V(a_0) = V(a) = 0$ and $V(a_k) = V(b) = 0$, and summing over all $i = 1,\dots,k$, we get the claim. □

Theorem 4.37. *Every minimizing curve with unit speed is a geodesic.*

Proof. Let $\gamma : [a,b] \to M$ be minimizing, with $|\dot{\gamma}_t| \equiv 1$, and let $a_0 = a < a_1 < \cdots < a_k = b$ be the subdivision such that $\gamma|[a_{i-1}, a_i]$ is C^∞. If Γ is a proper variation of γ, then the minimizing property of γ implies that

$$\frac{d}{ds}\ell(\Gamma_s)|_{s=0} = 0. \tag{4.38}$$

Using Lemma 4.34, we know that every proper vector field V along γ is the variation field of some proper variation Γ of γ. Now using (4.36) and (4.38), we get

$$\int_a^b \langle V, D_t\dot{\gamma}\rangle\, dt + \sum_{i=1}^{k-1} \langle V(a_i), \Delta_i\dot{\gamma}\rangle = 0 \tag{4.39}$$

for every proper vector field V along γ.
$\boxed{1^\circ}$ Take an interval $[a_{i-1}, a_i]$ and choose a function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi > 0$ on (a_{i-1}, a_i) and $\varphi = 0$ elsewhere. Then (4.39) with $V = \varphi D_t\dot{\gamma}$ implies

$$\int_{a_{i-1}}^{a_i} \varphi |D_t\dot{\gamma}|^2\, dt = 0.$$

Hence, $D_t\dot{\gamma} \equiv 0$ on each (a_{i-1}, a_i), that is, γ is a "broken" geodesic.
$\boxed{2^\circ}$ For each $i = 1, \ldots, k-1$ one can construct, using local coordinates at $\gamma(a_i)$, a vector field V along γ such that $V(a_i) = \Delta_i\dot{\gamma}$ and $V(t) \equiv 0$ for every $t \notin (a_i - \varepsilon, a_i + \varepsilon)$, where $\varepsilon > 0$ is so small that $a_j \notin (a_i - \varepsilon, a_i + \varepsilon)$ if $j \neq i$. Using again (4.39) and 1°, we know that $|\Delta_i\dot{\gamma}|^2 = 0$, that is, $\Delta_i\dot{\gamma} = 0$. Hence,

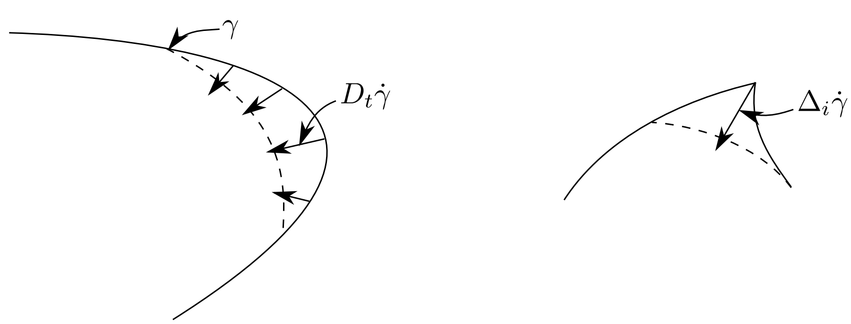
$$\dot{\gamma}(a_i^-) = \dot{\gamma}(a_i^+) \quad \text{for every } i = 1, \ldots, k-1.$$

The existence and uniqueness of geodesics imply that there exists a geodesic $\widetilde{\gamma} : I \to M$, $a_i \in I$, such that $\widetilde{\gamma}(a_i) = \gamma(a_i)$, $\dot{\widetilde{\gamma}}(a_i) = \dot{\gamma}(a_i^-) = \dot{\gamma}(a_i^+)$, and $\widetilde{\gamma} = \gamma$ on both $(a_{i-1}, a_i) \cap I$ and $(a_i, a_{i+1}) \cap I$. Hence, γ is a geodesic. □

Geometric interpretation: If $D_t\dot{\gamma} \neq 0$, then (4.36) with $V = \varphi D_t\dot{\gamma}$, where φ is as in 1°, gives

$$\frac{d}{ds}\ell(\Gamma_s)|_{s=0} = -\int_a^b \varphi |D_t\dot{\gamma}|^2\, dt < 0.$$

Thus deforming γ in the direction of its "acceleration vector" $D_t\dot{\gamma}$ **decreases** length.



Similarly, if $\Delta_i\dot{\gamma} \neq 0$, then the length of the broken geodesic γ decreases by deforming it in the direction of V, with $V(a_i) = \Delta_i\dot{\gamma}$.

Definition 4.40. We say that an admissible curve $\gamma : [a,b] \to M$ is a **critical point of the length functional** ℓ if

$$\frac{d}{ds}\ell(\Gamma_s)|_{s=0} = 0$$

for every proper variation Γ of γ.

Proof of Theorem 4.37 actually gives the following:

Corollary 4.41. *A unit speed admissible curve γ is a critical point of the length functional if and only if γ is a geodesic.*

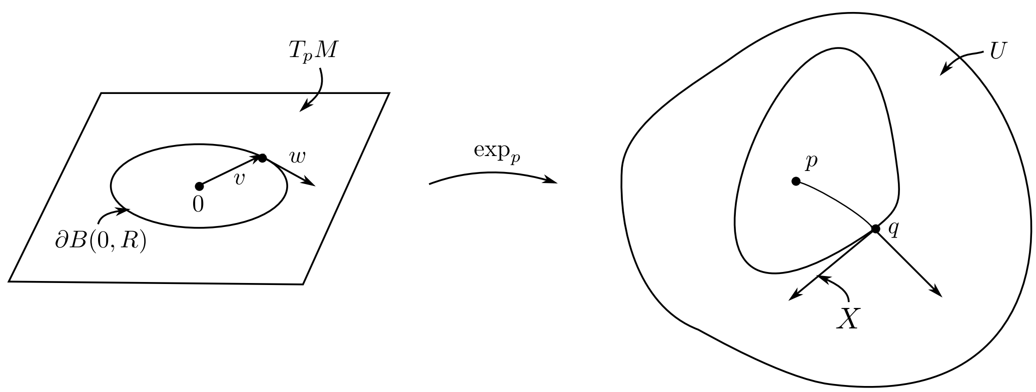
Proof. If γ is a critical point, then the proof of Theorem 4.37 implies that γ is a geodesic. Conversely, if γ is a geodesic, then the right-hand side of (4.36) has only a term

$$-\int_a^b \langle V, D_t\dot{\gamma}\rangle\, dt,$$

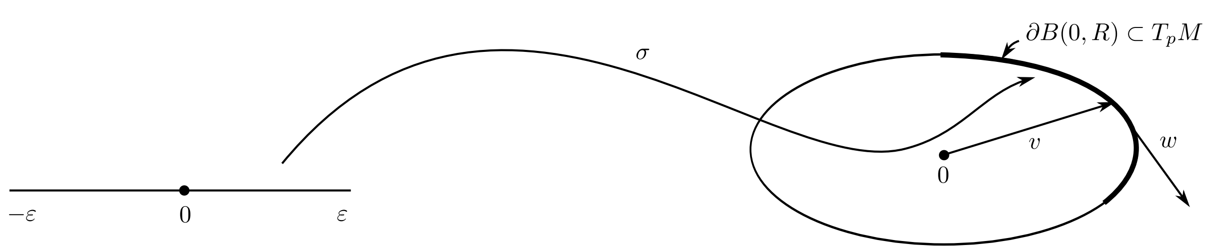
which vanishes since $D_t\dot{\gamma} \equiv 0$ by the definition of geodesic. Hence, γ is a critical point. □

Next we study the converse of Theorem 4.37 and prove that geodesics are **locally minimizing**.

Lemma 4.42 (Gauss lemma)**.** *Let U be a normal ball at $p \in M$. Then the unit radial vector field $\frac{\partial}{\partial r}$ is orthogonal to the normal spheres in U.*



Proof of the Gauss lemma. Let $q \in U \setminus \{q\}$. Since $\exp_p : B(0,r_0) \to U$ is a diffeomorphism for some $r_0 > 0$, there is $v \in T_pM$ such that $\exp_p v = q$. Let $X \in T_qM$ be tangent to the normal sphere through q, that is, $X \in T_q(\exp_p(\partial B(0,R)))$, $R = |v| > 0$. Let $w \in T_v(T_pM) = T_pM$ such that $(\exp_p)_* w = X$. Then $w \in T_v(\partial B(0,R))$. By Lemma 4.19, the radial geodesic from p to q is $\gamma(t) = \exp_p(tv)$ and $\dot{\gamma}_t = |v|\left(\frac{\partial}{\partial r}\right)_{\gamma(t)} = R\left(\frac{\partial}{\partial r}\right)_{\gamma(t)}$. Hence, $\dot{\gamma}_1 = R\left(\frac{\partial}{\partial r}\right)_q$.
We want to show that $X \perp \left(\frac{\partial}{\partial r}\right)_q$ or $\langle X, \dot{\gamma}_1\rangle = 0$. Let $\sigma : (-\varepsilon,\varepsilon) \to T_pM$, $\sigma(s) \in \partial B(0,R)$, be a C^∞-path such that $\sigma(0) = v$ and $\dot{\sigma}(0) = w$.



Let Γ be a variation of γ given by

$$\Gamma(s,t) = \exp_p(t\sigma(s)).$$

For each $s \in (-\varepsilon, \varepsilon)$, Γ_s is a geodesic with speed $|\sigma(s)| = R$. Write $S = \partial_s \Gamma$ and $T = \partial_t \Gamma$. Then

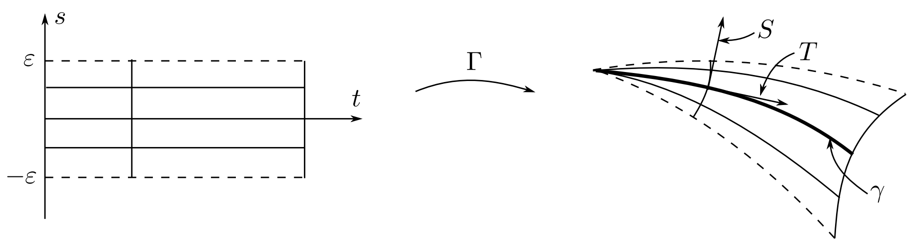
$$S(0,0) = \frac{d}{ds}\Gamma(s,0)|_{s=0} = \frac{d}{ds}\exp_p(0)|_{s=0} = 0;$$

$$T(0,0) = \frac{d}{dt}\Gamma(0,t)|_{t=0} = \frac{d}{dt}\exp_p(tv)|_{t=0} = v;$$

$$S(0,1) = \frac{d}{ds}\Gamma(s,1)|_{s=0} = \frac{d}{ds}\exp_p(\sigma(s))|_{s=0} = (\exp_p)_*(\dot\sigma(0)) = (\exp_p)_* w = X;$$

and

$$T(0,1) = \frac{d}{dt}\Gamma(0,t)|_{t=1} = \frac{d}{dt}\exp_p(tv)|_{t=1} = \dot\gamma(1).$$



Now $\langle S, T\rangle = 0$ at $(s,t) = (0,0)$ and $\langle S, T\rangle = \langle X, \dot\gamma(1)\rangle$ at $(s,t) = (0,1)$. Therefore, to prove that $\langle X, \dot\gamma(1)\rangle = 0$, it is enough to show that $\langle S, T\rangle$ is independent of t. Using the Symmetry lemma 4.30 and the fact that $\dot\Gamma_s$ is a geodesic with $\dot\Gamma_s = T$ we obtain

$$\frac{\partial}{\partial t}\langle S, T\rangle = \langle D_t S, T\rangle + \langle S, D_t T\rangle \overset{D_t T = 0}{=} \langle D_t S, T\rangle \overset{4.30}{=} \langle D_s T, T\rangle = \frac{1}{2}\frac{\partial}{\partial s}\langle T, T\rangle = \frac{1}{2}\frac{\partial}{\partial s}|T|^2 = 0,$$

since $|T| = |\dot\Gamma_s| \equiv R$ for every (s,t). ☐

Definition 4.43. Let $U \subset M$ be open and $f \in C^\infty(U)$. The **gradient** of f, denoted by ∇f or $\operatorname{grad} f$, is a C^∞-vector field on U, defined by

$$\langle \nabla f, X\rangle = df(X) = Xf$$

for every $X \in \mathcal{T}(U)$.

Corollary 4.44 (of the Gauss lemma)**.** *Let U be a normal ball centered at $p \in M$ and let $\frac{\partial}{\partial r} \in \mathcal{T}(U \setminus \{p\})$ be the unit radial vector field. Then $\nabla r = \frac{\partial}{\partial r}$ on $U \setminus \{p\}$.*

Recall that here $r : U \to \mathbb{R}$ is the radial distance function, defined in normal coordinates by

$$r(x) = \left(\sum_{i=1}^n (x^i)^2\right)^{1/2} = |\exp_p^{-1}(x)|$$

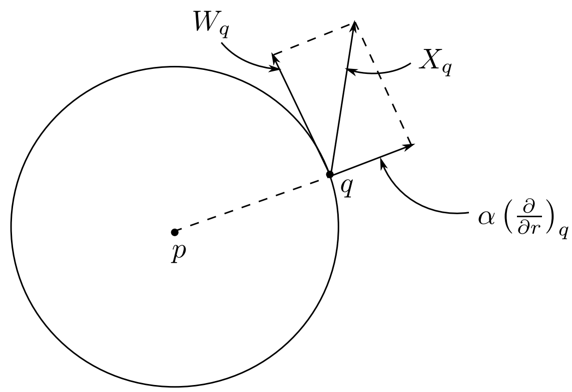
and

$$\left(\frac{\partial}{\partial r}\right)_x = \frac{x^i}{r(x)}(\partial_i)_x; \quad x = \exp_p(x^i e_i).$$

Proof of Corollary 4.44. Take $q \in U\setminus\{p\}$ and $X_q \in T_qM$. We need to show that $dr(X_q) = \langle \frac{\partial}{\partial r}, X_q\rangle$. Let $\exp_p(\partial B(0,R))$, $R = r(q)$, be the normal sphere through q. We decompose X_q as

$$X_q = W_q + \alpha \left(\frac{\partial}{\partial r}\right)_q, \quad \alpha \in \mathbb{R},$$

where W_q is tangent to the sphere $\exp_p(\partial B(0,R))$, i.e. $W_q \in T_q(\exp_p(\partial B(0,R)))$.



This can be done since $\left(\frac{\partial}{\partial r}\right)_q \notin T_q(\exp_p(\partial B(0,R)))$ by the Gauss lemma. Now $dr(W_q) = W_q r = 0$ since $W_q \in T_q(\exp_p(\partial B(0,R)))$ and $r \equiv R$ on $\exp_p(\partial B(0,R))$. A direct computation (in normal coordinates) gives

$$dr\left(\frac{\partial}{\partial r}\right) = \left(\frac{\partial}{\partial r}\right) r = 1,$$

see Remark 4.45 below. By Gauss lemma

$$\left\langle \frac{\partial}{\partial r}, W_q \right\rangle = 0.$$

Hence

$$dr(X_q) = dr(W_q) + \alpha dr \left(\frac{\partial}{\partial r}\right)_q = \alpha;$$

and

$$\left\langle \frac{\partial}{\partial r}, X_q \right\rangle = \left\langle \frac{\partial}{\partial r}, W_q \right\rangle + \alpha \left|\frac{\partial}{\partial r}\right|^2 = 0 + \alpha \cdot 1 = \alpha.$$

Therefore, $\left\langle \frac{\partial}{\partial r}, X_q \right\rangle = dr(X_q)$. □

Remark 4.45. Let $U = \exp_p(B(0,r_0))$ be a normal ball centered at p. We prove that

$$\left(\frac{\partial}{\partial r}\right) r = 1$$

in $U \setminus \{p\}$. Let $\gamma(t) = \exp_p(tv)$, $v = v^i e_i$, be a radial unit speed geodesic starting at p. Then

$$\left(\frac{\partial}{\partial r}\right)_{\gamma(t)} r = \dot{\gamma}_t r = (r \circ \gamma)'(t)$$

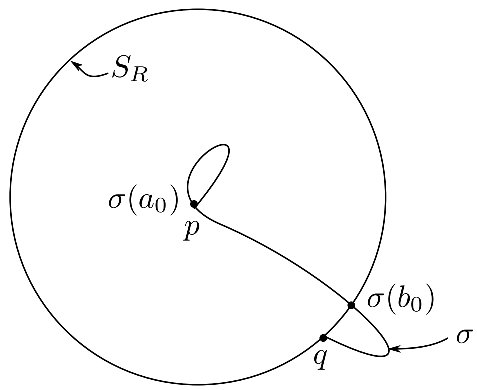
for all $t \in]0, r_0[$. Since the normal coordinates of $\gamma(t)$ are $(tv^1, \dots, tv^n)$, we have

$$(r\circ\gamma)(t) = r\big(\gamma(t)\big) = \sqrt{(tv^1)^2 + \cdots + (tv^n)^2} = t\sqrt{(v^1)^2 + \cdots + (v^n)^2} = t,$$

and therefore $(r \circ \gamma)'(t) = 1$.

Theorem 4.46. *Let U be a normal ball at $p \in M$. If $q \in U \setminus \{p\}$, then the radial geodesic from p to q is the unique minimizing curve from p to q in M up to reparametrization.*

Proof. Take $\varepsilon > 0$ such that $q \in \exp_p(B(0,\varepsilon)) \subset U$. Let $\gamma : [0,R] \to M$ be the unique radial geodesic from p to q, with unit speed and $R = r(q) = |\exp_p^{-1}(q)|$. Then $\gamma(t) = \exp_p(tv)$ for some unit vector $v \in T_pM$. Since γ has unit speed, $\ell(\gamma) = R$. Thus we need to show that $\ell(\sigma) > R$ whenever $\sigma : [0,b] \to M$ is an admissible unit speed curve from p to q, with $\sigma([0,b]) \neq \gamma([0,R])$. Let $a_0 \in [0,b]$ be the largest t such that $\sigma(t) = p$ and let $b_0 \in [a_0, b]$ be the smallest t such that $\sigma(t) \in S_R = \exp_p(\partial B(0,R))$.



For $t \in (a_0, b_0]$, we can decompose $\dot\sigma(t)$ as

$$\dot\sigma(t) = \alpha(t)\frac{\partial}{\partial r} + W(t),$$

where $W(t)$ is tangent to the normal sphere centered at p through $\sigma(t)$. The Gauss lemma implies that $\langle W(t), \left(\frac{\partial}{\partial r}\right)_{\sigma(t)}\rangle = 0$, so

$$|\dot\sigma(t)|^2 = \langle \dot\sigma(t), \dot\sigma(t)\rangle = \alpha(t)^2 + |W(t)|^2 \geq \alpha(t)^2.$$

Using Corollary 4.44 we know that

$$\alpha(t) = \left\langle \frac{\partial}{\partial r}, \dot\sigma(t)\right\rangle = dr(\dot\sigma(t)).$$

Hence,

$$\begin{aligned}\ell(\sigma) \geq \ell(\sigma|[a_0,b_0]) &= \lim_{\delta\to 0}\int_{a_0+\delta}^{b_0} |\dot\sigma(t)|\,dt \geq \lim_{\delta\to 0}\int_{a_0+\delta}^{b_0} \alpha(t)\,dt = \lim_{\delta\to 0}\int_{a_0+\delta}^{b_0} dr(\dot\sigma(t))\,dt \\ &= \lim_{\delta\to 0}\int_{a_0+\delta}^{b_0} \frac{d}{dt} r(\sigma(t))\,dt = r(\sigma(b_0)) - r(\sigma(a_0)) = R = \ell(\gamma).\end{aligned}$$

If $\ell(\sigma) = \ell(\gamma)$, then both inequalities above are equalities. Since σ is of unit speed, we must have $a_0 = 0$ and $b_0 = b = R$; and $W(t) \equiv 0$ and $\alpha(t) > 0$. So, $\dot\sigma(t) = \alpha(t)\frac{\partial}{\partial r}$ and since σ is of unit speed $\alpha(t) \equiv 1$. Thus both σ and γ are integral curves of $\frac{\partial}{\partial r}$, with $\sigma(R) = \gamma(R) = q$. Hence, $\sigma = \gamma$. □

Corollary 4.47. *Let U be a normal ball at p. Then $r(x) = d(x,p)$ for every $x \in U$.*

Proof. Exercise. □

Denote

$$B(p,r) := \{q \in M : d(p,q) < r\};$$

$$\overline{B}(p,r) := \{q \in M : d(p,q) \le r\};$$

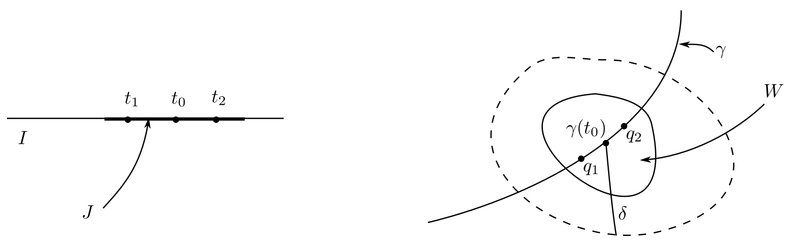
and

$$S(p,r) := \{q \in M : d(p,q) = r\}.$$

We say that an admissible curve $\gamma : I \to M$ is **locally minimizing** if each $t_0 \in I$ has a neighborhood $J \subset I$ such that $\gamma|J$ is minimizing between any pair of its points. Clearly, a minimizing curve is locally minimizing.

Theorem 4.48. *Every geodesic is locally minimizing.*

Proof. Let $\gamma : I \to M$ be a geodesic such that $I \subset \mathbb{R}$ is open. Let $t_0 \in I$ and let $W \subset M$ be a uniformly normal neighborhood of $\gamma(t_0)$, that is, there exists $\delta > 0$ such that for every $q \in W$ the map $\exp_q$ is a diffeomorphism in $B(0,\delta) \subset T_qM$ and $W \subset \exp_q(B(0,\delta)) = B(q,\delta)$.



Let $J \subset I$ be an open interval containing t_0 such that $\gamma(J) \subset W$. If $t_1, t_2 \in J$, then $q_2 = \gamma(t_2)$ belongs to a normal ball centered at $q_1 = \gamma(t_1)$ by the definition of uniformly normal neighborhood. Theorem 4.46 implies that the radial geodesic from q_1 to q_2 is the unique minimizing curve from q_1 to q_2. However, $\gamma|[t_1, t_2]$ is a geodesic from q_1 to q_2 and $\gamma([t_1, t_2])$ is contained in the same normal ball around q_1, so $\gamma|[t_1, t_2]$ is this minimizing radial geodesic. □

Remark 4.49. We need a uniformly normal neighborhood above to be able to place the center of the normal ball to any point $\gamma(t)$, with t in a neighborhood of t_0.

Another proof of 4.37 (without using the first variation formula). Let $\gamma : [a,b] \to M$ be a minimizing curve and let $t_0 \in (a,b)$. As above, there exists an interval $J = (t_0 - \varepsilon, t_0 + \varepsilon) \subset [a,b]$ and a uniformly normal neighborhood W such that $\gamma(J) \subset W$. As above, we conclude that for every $t_1, t_2 \in J$, the unique minimizing curve from $\gamma(t_1)$ to $\gamma(t_2)$ is the radial geodesic. Since the restriction of γ is such a minimizing curve, it coincides with the radial geodesic thus solving the geodesic equation in a neighborhoof of t_0. Since t_0 is arbitrary, γ is indeed a geodesic. □

4.50 Completeness

Definition 4.51. A Riemannian manifold M is said to be geodesically complete if every maximal geodesic is defined for all $t \in \mathbb{R}$.

Example 4.52. If $U \subsetneq \mathbb{R}^n$ is an open subset with the Euclidean metric, then U is not complete.

Theorem 4.53 (Hopf-Rinow)**.** *Let M be a connected Riemannian manifold. Then the following are equivalent:*

(a) *there exists $p \in M$ such that $\exp_p$ is defined on the whole of T_pM;*

(b) *for every* $p \in M$ *the map* $\exp_p$ *is define on the whole of* T_pM*;*

(c) M *is complete as a metric space;*

(d) M *is geodesically complete.*

Moreover, any of the above conditions implies that

(e) *if* $p, q \in M$*, then there exists a geodesic from* p *to* q *with* $\ell(\gamma) = d(p, q)$*, that is,* M *is a* ***geodesic metric space****.*

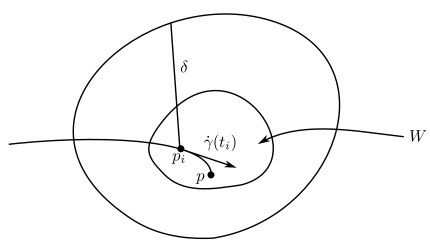
Proof. $\boxed{(c) \implies (d)}$ Suppose M is metrically complete but not geodesically complete. Then there exists a unit speed geodesic $\gamma : [0, b) \to M$ that extends **no** interval $[0, b + \varepsilon)$ for $\varepsilon > 0$. Let $t_i \uparrow b$ and write $p_i = \gamma(t_i)$. Since γ is of unit speed, we have

$$\ell(\gamma|[t_i, t_j]) = |t_j - t_i|,$$

which gives

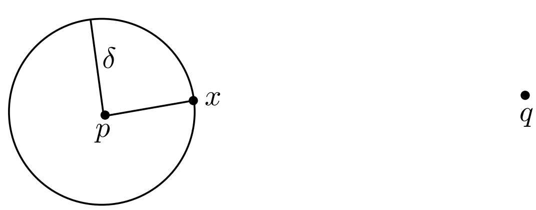
$$d(p_i, p_j) \le |t_j - t_i|.$$

Hence (p_i) is a Cauchy sequence in M. Because M is metrically complete, there exists $p \in M$ such that $d(p_i, p) \to 0$. Let W be a uniformly normal neighborhood of p and $\delta > 0$ such that for every $q \in W$, the map $\exp_q$ is diffeomorphism in $B(0, \delta) \subset T_qM$ and $W \subset B(q, \delta) = \exp_q(B(0, \delta))$. If $i \in \mathbb{N}$ is large enough, then $p_i \in W$ and $t_i > b - \delta/4$.



Because $\exp_{p_i}$ is diffeomorphism in $B(0, \delta) \subset T_{p_i}M$, we know that every geodesic σ starting at p_i (i.e. $\sigma(0) = p_i$) is defined at least on $[0, \delta)$. In particular, the geodesic σ, with $\dot\sigma(0) = \dot\gamma(t_i)$, is defined on $[0, \delta/2]$. The uniquesness of the geodesic implies that σ is a reparametrization of γ. Hence $\widetilde\gamma$, $\widetilde\gamma(t) = \sigma(t - t_i)$, is an extension of γ which is defined on $[t_i, t_i + \delta/2]$, with $t_i + \delta/2 > b + \delta/4$; a contradiction. Hence, M is geodesically complete.

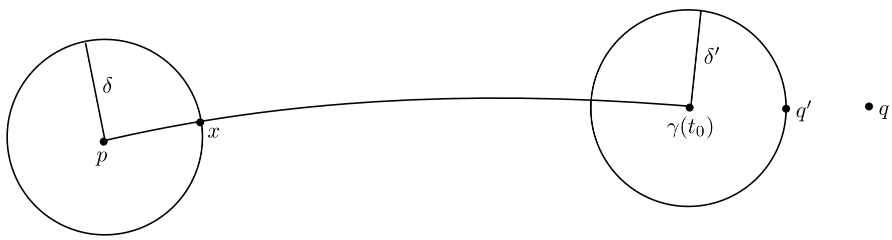
$\boxed{(a) \implies (c)}$ First of all, we will show that that every $q \in M$ can be joined to p by a geodesic of length $d(p, q)$, i.e. claim (e) when p is as in (a). Let $\overline{B}(p, \delta)$ be a closed ball at p. If $q \in \overline{B}(p, \delta)$, then there exists a minimizing geodesic from p to q by Theorem 4.46. Suppose $q \notin \overline{B}(p, \delta)$. Since $S(p, \delta) = \exp_p(\partial B(0, \delta))$ is compact and the distance function is continuous, there exists $x \in S(p, \delta)$ such that $d(x, q) = \min\{d(y, q) : y \in S(p, \delta)\}$.



Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic such that $\gamma|[0,\delta]$ is the unique radial geodesic from p to x. Hence, $\gamma(t) = \exp_p(tv)$, where $v = \exp_p^{-1}(x)/\delta$. (Note that the assumption (a) says that $\exp_p(tv)$, hence γ, is defined for all $t \in \mathbb{R}$.) We are going to show that $\gamma(r) = q$, where $r = d(p,q)$. Let $f : [0,r] \to \mathbb{R}$ be the continuous function $f(t) = t + d(\gamma(t), q)$ and let

$$T := \{t \in [0,r] : f(t) = r\} \ \big(= f^{-1}(r)\big)$$

Then $0 \in T$ and T is closed. Let $t_0 := \sup T$. Then $t_0 \in T$ since T is closed. If $t_0 = r$, we have $r + d(\gamma(r), q) = r$, and so $\gamma(r) = q$. Thus, we may assume that $t_0 < r$. Next we show that $t_0 + \delta' \in T$ if $\delta' > 0$ is so small that $t_0 + \delta' \leq r$. Let $\overline{B}(\gamma(t_0), \delta')$ be a closed normal ball and choose $q' \in S(\gamma(t_0), \delta')$ such that $d(q', q) = \min\{d(y,q) : y \in S(\gamma(t_0), \delta')\}$.



It suffices to show that $q' = \gamma(t_0 + \delta)$, because then

$$d(\gamma(t_0), q) \overset{(*)}{=} \delta' + \min\{d(y,q) : y \in S(\gamma(t_0), \delta')\} = \delta' + d(q', q) = \delta' + d(\gamma(t_0 + \delta), q),$$

($(*)$ is an exercise) and since $t_0 \in T$ implies $d(\gamma(t_0), q) = r - t_0$; we have

$$d(\gamma(t_0 + \delta'), q) = d(\gamma(t_0), q) - \delta' = r - t_0 - \delta' = r - (t_0 + \delta').$$

Hence, $t_0 + \delta' \in T$; a contradiction with the definition of t_0. To prove that $\gamma(t_0 + \delta') = q'$, observe that

$$d(p, q') \geq d(p,q) - d(q', q) = r - \big(d(\gamma(t_0), q) - \delta'\big) \overset{t_0 \in T}{=} r - (r - t_0 - \delta') = t_0 + \delta'.$$

On the other hand, the broken geodesic from p to q' that goes from p to $\gamma(t_0)$ by γ and then from $\gamma(t_0)$ to q' by a radial geodesic in $B(\gamma(t_0), \delta')$ has length $t_0 + \delta$. Hence, $d(p, q') \leq t_0 + \delta'$, and so this broken geodesic in minimizing, hence a geodesic. The uniqueness of geodesics implies that it coincides with $\gamma|[0, t_0 + \delta']$, so $\gamma(t_0 + \delta') = q'$. This completes the proof of the claim that every $q \in M$ can be joined to p by a geodesic of length $d(p,q)$.

Let then (q_i) be a Cauchy sequence in M. Let $\gamma_i : [0, t_i] \to M$, $\gamma_i(t) = \exp_p(tv_i)$, be a unit speed minimizing geodesic from p to q_i. Then

$$|t_i - t_j| = |d(p, q_i) - d(p, q_j)| =\leq d(q_i, q_j).$$

Hence, (t_i) is a Cauchy sequence in $\mathbb{R}$, in particular $t_i \leq R < \infty$ for every $i \in \mathbb{N}$. Since $|v_i| = 1$, the sequence $(t_i v_i)$ of T_pM is bounded. Therefore, a subsequence $(t_{i_k} v_{i_k})$ converges to $v \in T_pM$. The continuity of the exponential map $\exp_p$ implies that $q_{i_k} = \exp_p(t_{i_k} v_{i_k}) \to \exp_p v$. Because (q_i) is Cauchy, $q_i \to \exp_p v$, so (q_i) converges. This gives (c).

$\boxed{(b) \implies (a)}$ Trivial.

$\boxed{(d) \implies (b)}$ Obvious.

$\boxed{(b) \implies (e)}$ That was, in fact, proven in $\boxed{(a) \implies (c)}$.

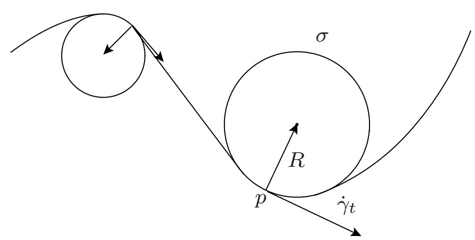
□

Remark 4.54. The condition (e) does not imply completeness (e.g. open ball in $\mathbb{R}^n$); all compact Riemannian manifolds are complete.

5 Curvature

5.1 What is curvature?

A Consider a C^∞-path $\gamma\colon I \to \mathbb{R}^2$ in the plane. Assume that $|\dot\gamma| \equiv 1$. Formally, the curvature of γ is defined by $\kappa(t) = |\ddot\gamma_t|$, the norm of the accelleration vector. Geometrically, the curvature has an interpretation:
Given a point $p = \gamma(t)$, there are many circles σ that are tangent to γ at p, i.e. $\sigma(t) = p$ and $\dot\sigma_t = \dot\gamma_t$ but exactly one such that also $\ddot\sigma_t = \ddot\gamma_t$. Call this the **osculating circle**. If $\ddot\gamma_t = 0$, take σ to be the straight line tangent to γ at p. Note that $\ddot\gamma_t \perp \dot\gamma_t$, since $|\dot\gamma| \equiv 1$ (γ has no accelleration in its own direction).



Then $\kappa(t) = 1/R$, where R is the radius of the osculating circle ($R = \infty$ and $\kappa(t) = 0$ if $\ddot\gamma_t = 0$). Choose a unit normal vector at some point of γ and let N be the corresponding (continuous) unit normal vector field along γ. Then the **signed curvature** κ_N is

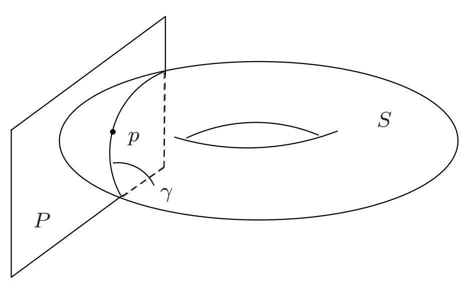
$$\kappa_N(t) = \begin{cases} \kappa(t), & \text{if } \ddot\gamma_t \uparrow\uparrow N_t \\ -\kappa(t), & \text{if } \ddot\gamma_t \uparrow\downarrow N_t. \end{cases}$$

B Suppose S is a (2-dimensional) smooth surface in $\mathbb{R}^3$. The curvature of S at $p \in S$ is described by two numbers, called the principal curvatures, as follows:

(i) Choose a plane P through $p \in S$ containing N, a unit normal vector to S at p; near p $S \cap P$ is a smooth plane curve γ ($\subset P$) passing through p.

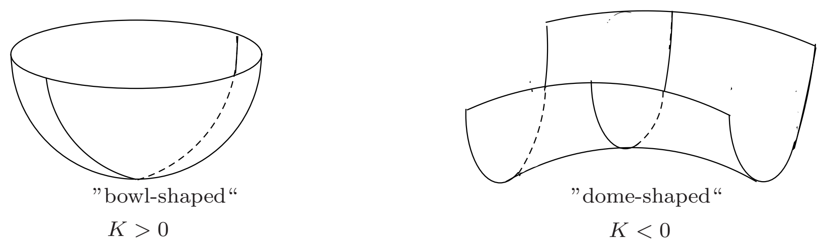
(ii) Compute κ_N of γ at p with respect to the chosen unit normal N.

(iii) Repeat this for all such planes P.



The **principal curvatures**, κ_1 and κ_2, of S at p are the minimum and the maximum signed curvatures obtained in (iii). Principal curvatures are not isometrically invariant; they are <u>not intrinsic</u> properties of S. For instance, a strip $S_1 = \{(x,y) \in \mathbb{R}^2\colon x \in \mathbb{R}, 0 < y < \pi\}$ and a half-cylinder $S_2 = \{(x,y,z) \in \mathbb{R}^3\colon x \in \mathbb{R}, y^2 + z^2 = 1, z > 0\}$ are isometric (by the map $(x,y) \mapsto (x, \cos y, \sin y)$), but the principal curvatures of S_1 are $\kappa_1 = \kappa_2 = 0$ whereas the principal curvatures of S_2 are $\kappa_1 = 0$ and $\kappa_2 = 1$.

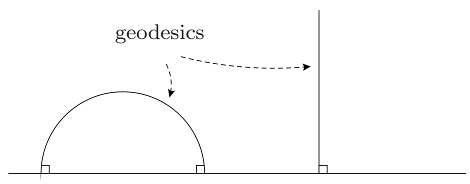
Gauss's Theorema Egregium ("remarkable theorem"), 1827: The product $K = \kappa_1\kappa_2$ is intrinsic, i.e. can be expressed in terms of the metric of S. The product K is called the **Gaussian curvature**.



Model surfaces.

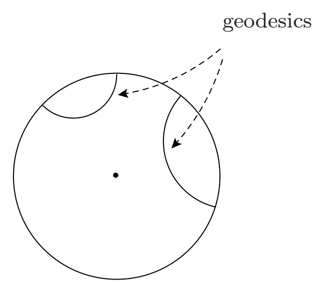
1. The plane $\mathbb{R}^2$, $\quad K \equiv 0$.
2. The sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \colon |x| = 1\}$ with induced metric, $K \equiv 1$.
3. The hyperbolic plane $\mathbb{H}^2$, $\quad K \equiv -1$.
   - Upper half-plane model: $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \colon y > 0\}$ with the Riemannian metric

$$g_H = y^{-2} g_E, \quad g_E = \text{ the Euclidean metric.}$$



   - Poincaré-disk model: $\mathbb{H}^2 = \{x \in \mathbb{R}^2 \colon |x| < 1\}$ with the Riemannian metric

$$g_H = \frac{4 g_E}{\left(1 - |x|^2\right)^2}.$$



Theorem 5.2 (Uniformization theorem)**.** *Every connected 2-manifold is diffeomorphic to a quotient space of either $\mathbb{R}^2$, $\mathbb{S}^2$, or $\mathbb{H}^2$ by a discrete group of isometries acting properly discontinuously without fixed points. Therefore, every connected 2-manifold has a complete Riemannian metric with constant Gaussian curvature.*

Theorem 5.3 (Gauss-Bonnet theorem)**.** *If S is a compact oriented 2-manifold with a Riemannian metric, then*

$$\int_S K = 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic of S.

The Euler characteristic of S is a topological invariant of S defined as

$$\chi(S) = \#\text{ vertices - }\#\text{ edges }+\#\text{ faces in any triangulation of } S.$$

$$X(S) = \begin{cases} 2, & \text{if } S = \text{sphere},\\ 0, & \text{if } S = \text{torus},\\ 2-2g, & \text{if } S = \text{ an oriented surface of genus } g. \end{cases}$$

For Gauss's Theorema Egregium and the Gauss-Bonnet theorem see e.g. [Le1].

C Curvature in higher dimensions.

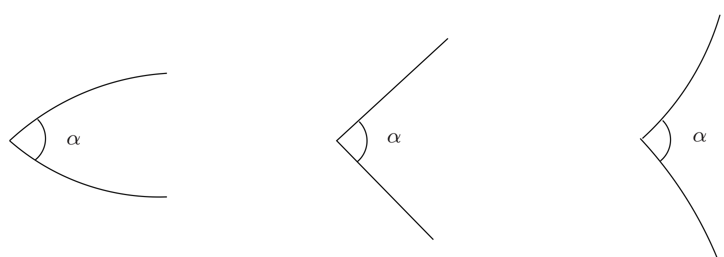
A recipe for computing "some curvatures" at $p \in M$:

1. Take a 2-dimensional subspace $P \subset T_pM$;
2. Take a ball $B(0,r) \subset T_pM$ such that $\exp_p$ is a diffeomorphism in a neighborhood of $\bar{B}(0,r)$. Then $\exp_p\big(P \cap B(0,r)\big)$ is a 2-dimensional submanifold of M. Call it S_P.
3. Compute the Gaussian curvature of S_P at p. Denote it by $K(P)$.

Thus "curvature" of M at p can be interpreted as a map

$$K\colon \{\text{2-planes in } T_pM\} \to \mathbb{R}.$$

A geometric description of curvature: Consider two geodesics intersecting at p in angle α. We will show later that the curvature has the following effect to the behavior of geodesics:



"curvature" > 0 (e.g. $\mathbb{S}^n$) "curvature" $= 0$ ($\mathbb{R}^n$) "curvature" < 0 (e.g. $\mathbb{H}^n$)

Model spaces with "constant curvature" will be: $\mathbb{R}^n$, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ with the induced metric, and the **hyperbolic space** $\mathbb{H}^n$.

- Upper half-space model for $\mathbb{H}^n$:

$$\{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n > 0\}, \quad g_H = x_n^{-2} g_E, \quad \text{where } g_E \text{ is the Euclidean metric.}$$

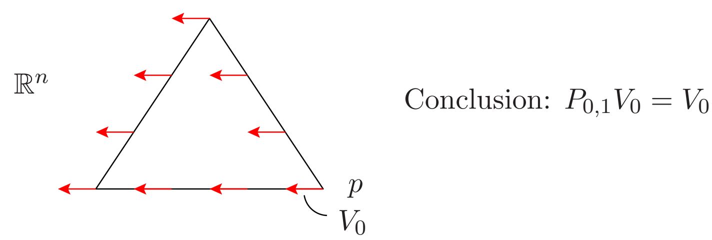
- Poincaré model for $\mathbb{H}^n$:

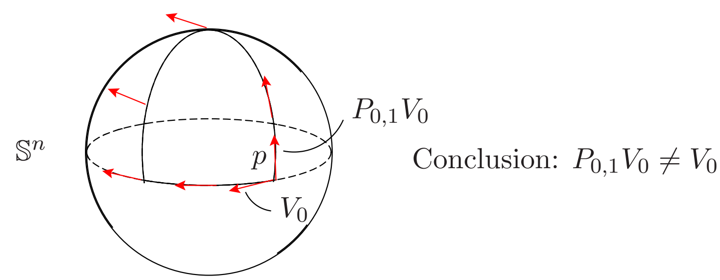
$$\{x \in \mathbb{R}^n : |x| < 1\}, \quad g_H = \frac{4g_E}{(1-|x|^2)^2}.$$

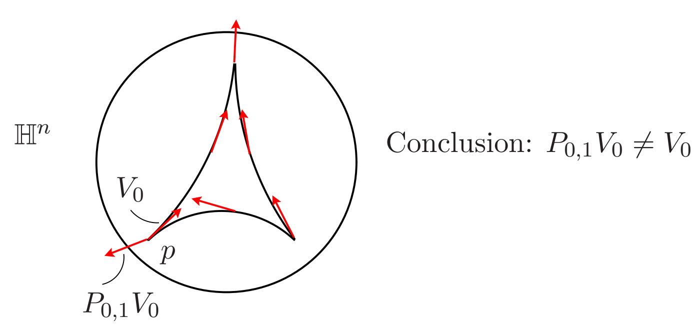
- Geodesics (in the models above) are as in the 2-dimensional case.

Remark 5.4. We say that a Riemannian metric $\tilde{g}$ is obtained from another Riemannian metric g by a **conformal change** of the metric if $\tilde{g} = fg$, where f is a positive C^∞-function. (Conformal = "angles are preserved".)

Consider next the parallel translation $P_{0,1}$ around a (piecewise smooth geodesic) triangle $\gamma\colon [0,1] \to M,\ p = \gamma(0) = \gamma(1)$, when $M = \mathbb{R}^n,\ \mathbb{S}^n$, or $\mathbb{H}^n$.







The phenomenon above is related to the question whether M is locally isometric to $\mathbb{R}^n$ at p. Indeed, a Riemannian manifold M is locally isometric to $\mathbb{R}^n$ at p if and only if $P_{0,1} = \mathrm{id}$ for every sufficient small loops γ, with $\gamma(0) = \gamma(1) = p$.
So, the curvature is a local invariant that in some sense measures how far away the affine conection (locally) is from the Euclidean connection.

5.5 Curvature tensor and Riemannian curvature

Let M be a C^∞-manifold with an affine connection ∇. The **curvature tensor field** of ∇ is the map $R\colon \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Warning: In some books the definition differs from above by sign. (e.g. in Do Carmo [Ca]).

Lemma 5.6. *R is 3-linear over $C^\infty(M)$: $\forall f, g \in C^\infty(M)$*

(i) $R(fX_1 + gX_2, Y)Z = fR(X_1, Y) + gR(X_2, Y)Z$;

(ii) $R(X, fY_1 + gY_2)Z = fR(X, Y_1)Z + gR(X, Y_2)Z$;

(iii) $R(X, Y)(fZ + gW) = fR(X, Y)Z + gR(X, Y)W$.

Proof. (Exercise). □

Thus $R \in \mathcal{T}_1^3(M)$. As a tensor field the value of $R(X, Y)Z$ at p depends only on X_p, Y_p, and Z_p (and, of course, on R itself).

Remark 5.7. (i) We immediately see that

$$R(X, Y)Z = -R(Y, X)Z. \tag{5.8}$$

(ii) If $M = \mathbb{R}^n$ with the standard connection, then $R(X, Y)Z = 0 \; \forall \; X, Y, Z \in \mathcal{T}(\mathbb{R}^n)$.

Let (U, x), $x = (x^1, \ldots, x^n)$, be a chart at p, with $\partial_1, \ldots, \partial_n$ the coordinate frame. Then $R \quad (\in \mathcal{T}_1^3(M))$ can be written in coordinates (x^i) as

$$R = R^\ell_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \partial_\ell,$$

where the functions R^ℓ_{ijk} are defined by

$$R(\partial_i, \partial_j)\partial_k = R^\ell_{ijk}\partial_\ell.$$

So, if

$$V = v^i \partial_i, \quad W = w^j \partial^j \text{ and } Z = z^k \partial_k,$$

then by linearity (over $C^\infty(U)$)

$$R(V, W)Z = R^\ell_{ijk} v^i w^j z^k \partial_\ell,$$

where we also see that $(R(V, W)Z)_p$ depends only on V_p, W_p, Z_p, and $R^l_{ijk}(p)$.

Since $[\partial_i, \partial_j] = 0$, we have

$$R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k = \cdots = \left(\Gamma^\ell_{jk}\Gamma^m_{i\ell} - \Gamma^\ell_{ik}\Gamma^m_{j\ell} + \partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik}\right)\partial_m.$$

Geometric interpretation for $R(X, Y)Z$: For small $t > 0$, define a piecewise regular curve $\gamma : [0, 4t] \to M$ as follows:

$$\begin{aligned} \gamma|[0, t] &= \text{the integral curve of } \partial_i \text{ starting at } p \in M; \\ \gamma|[t, 2t] &= \text{the integral curve of } \partial_j \text{ starting at } \gamma(t); \\ \gamma|[2t, 3t] &= \text{the integral curve of } -\partial_i \text{ starting at } \gamma(2t); \\ \gamma|[3t, 4t] &= \text{the integral curve of } -\partial_j \text{ starting at } \gamma(3t). \end{aligned}$$

Here ∂_i and ∂_j are coordinate vector fields corresponding to a chart (U, x) at p. Since ∂_i and ∂_j are coordinate vector fields, $\gamma(0) = \gamma(4t) = p$.

Let $P_{0,4t} : T_pM \to T_pM$ be the parallel translation along γ. Then for $v \in T_pM$, we have:

$$R(\partial_i, \partial_j)v = \lim_{t \searrow 0} \frac{(I - P_{0,4t})v}{t^2}, \tag{5.9}$$

where $I : T_pM \to T_pM$ is the identity map.

The proof of (5.9) is left as an exercise.

Assume that M is a Riemannian manifold, ∇ the Riemannian connection, and $\langle\, ,\, \rangle$ the Riemannian metric. Using the Riemannian metric we can change $R \in \mathcal{T}_1^3(M)$ to $R \in \mathcal{T}^4(M)$ by defining

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W\rangle \tag{5.10}$$

for $X, Y, Z, W \in \mathcal{T}(M)$. It is called the **Riemannian curvature tensor**. In coordinates it is written as

$$R = R_{ijk\ell} dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell,$$

where

$$R_{ijk\ell} = g_{\ell m} R^m_{ijk}.$$

Proposition 5.11. *Let M be a Riemannian manifold. Then*

(1) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ *(Bianchi identity);*

(2) $\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$*;*

(3) $\langle R(X, Y)Z, W\rangle = -\langle R(X, Y)W, Z\rangle$*.*

Proof. (Exercise) □

Remark 5.12. The value of $R(X, Y, Z, W)$ at p depends only on X_p, Y_p, Z_p and W_p (and, of course, on R).

5.13 Sectional curvature

For $u, v \in T_pM$, write

$$\begin{aligned} |u \wedge v| &= \sqrt{|u|^2|v|^2 - \langle u, v\rangle^2} \\ &= \text{ the area of the parallelogram spanned by } u \text{ and } v. \end{aligned}$$

If $|u \wedge v| \neq 0$, we define

$$K(u, v) = \frac{\langle R(u, v)v, u\rangle}{|u \wedge v|^2}. \tag{5.14}$$

Lemma 5.15. *Let $P \subset T_pM$ be a 2-dimensional subspace and let $u, v \in P$ be linearly independent. The $K(u, v)$ does not depend on the choice of u and v.*

Proof. Exercise. □

Definition 5.16. Given $p \in M$ and a 2-dimensional subspace $P \subset T_pM$, the number $K(P) = K(u, v)$, where $\{u, v\}$ is any basis of P, is called the **sectional curvature** of P at p.

Remark 5.17. This is the same as the Gaussian curvature of S_P described earlier in $\boxed{\mathbf{C}}$; see e.g. Lee [Le1, Chapter 8].

Lemma 5.18. *$\langle R(u,v)v,u\rangle$ determines the curvature completely, i.e. K and the metric defines R.*

Proof. We need to show that $(x,y,z,w)\mapsto\langle R(x,y)z,w\rangle$ is the only 4-linear form that satisfies conditions (5.8) and 5.11(1)-(3), and whose restriction to points (x,y,y,x) is equal to $\langle R(x,y)y,x\rangle$. Suppose that f and f' are two such maps (i.e. 4-linear maps $(x,y,z,w)\mapsto f(x,y,z,w)$ satisfying (5.8) and 5.11(1)-(3), and whose restrictions to points (x,y,y,x) are equal to $\langle R(x,y)y,x\rangle$). Then the 4-linear form $g=f-f'$ also satisfies (5.8) and 5.11(1)-(3). Since

$$g(u,v,v,u)=f(u,v,v,u)-f'(u,v,v,u)=\langle R(u,v)v,u\rangle-\langle R(u,v)v,u\rangle=0$$

for all u,v, we have $g(x+z,y,y,x+z)=0$, and by 4-linearity

$$\underbrace{g(x,y,y,x)}_{=0}+g(x,y,y,z)+g(z,y,y,x)+\underbrace{g(z,y,y,z)}_{=0}=0.$$

Thus

$$g(x,y,y,z)+g(z,y,y,x)=0$$

Using (5.8) and 5.11(2)-(3) we obtain

$$\begin{aligned}0&=g(x,y,y,z)+g(z,y,y,x)\\&\overset{(2)}{=}g(x,y,y,z)+g(y,x,z,y)\\&\overset{(3)}{=}g(x,y,y,z)+g(y,x,y,z)\\&\overset{(5.8)}{=}g(x,y,y,z)+g(x,y,y,z).\end{aligned}$$

Thus

$$g(x,y,y,z)=0.$$

Here replace y by $y+w$ to obtain first

$$g(x,y+w,y+w,z)=0$$

and then by 4-linearity

$$\underbrace{g(x,y,y,z)}_{=0}+g(x,y,w,z)+g(x,w,y,z)+\underbrace{g(x,w,w,z)}_{=0}=0.$$

Hence

$$g(x,w,y,z)=-g(x,y,w,z)$$

which by (2) and (3) (of 5.11) is the same as

$$g(y,z,x,w)=g(x,y,z,w).$$

We conclude that g does not change in cyclic permutations of the first 3 variables. By 5.11(1), the sum over such permutations vanishes, and therefore $g=0$. □

By using Lemma 5.18 one can characterize curvature tensors with constant sectional curvature.

Proposition 5.19. *Let M be a Riemannian manifold and $p \in M$. Then $K(P) = K$ for all 2-planes $P \subset T_pM$ if and only if*

$$R(x,y)z = K\left(\langle y,z\rangle x - \langle x,z\rangle y\right)$$

for all $x, y, z \in T_pM$.

Proof. $\boxed{\Rightarrow}$ Define multilinear maps $\tilde{R} : (T_pM)^3 \to T_pM$,

$$\tilde{R}(x,y)z = K\left(\langle y,z\rangle x - \langle x,z\rangle y\right),$$

and $\tilde{R} : (T_pM)^4 \to \mathbb{R}$,

$$\tilde{R}(x,y,z,w) = K\left(\langle y,z\rangle\langle x,w\rangle - \langle x,z\rangle\langle y,w\rangle\right).$$

Now $\tilde{R}$ satisfies (5.8) and 5.11(1)-(3). If $K(P) \equiv K$, we have

$$R(x,y,y,x) = K\left(|x|^2|x|^2 - \langle x,y\rangle^2\right) = \tilde{R}(x,y,y,x).$$

Lemma 5.18 then implies that $R = \tilde{R}$.

$\boxed{\Leftarrow}$ Obvious. □

5.20 Ricci curvature and scalar curvature

Definition 5.21. The **Ricci curvature** is a tensor field $\mathrm{Ric} \in \mathcal{T}^2(M)$ defined by

$$\mathrm{Ric}(x,y) = \text{ the trace of the linear map } z \mapsto R(z,x)y.$$

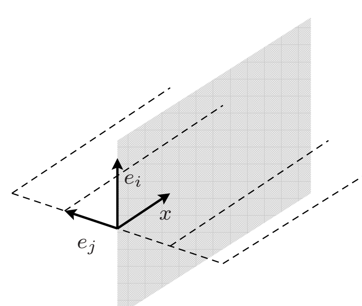
If $e_1, \ldots, e_n$ is an orthonormal basis of T_pM, then

$$\begin{aligned}\mathrm{Ric}(x,y) &= \sum_{i=1}^n \langle R(e_i,x)y, e_i\rangle \\ &= \sum_{i=1}^n \langle R(x,e_i)e_i, y\rangle.\end{aligned}$$

We set $\mathrm{Ric}(x) = \mathrm{Ric}(x,x)$. If $|x| = 1$, $\mathrm{Ric}(x)$ is called the **Ricci curvature in the direction** x. Hence if $|x| = 1$ and $e_1, \ldots, e_{n-1} \in T_pM$ such that $x, e_1, \ldots, e_{n-1}$ is an orthonormal basis of T_pM, we get

$$\begin{aligned}\mathrm{Ric}(x) &= \underbrace{\langle R(x,x)x, x\rangle}_{=0} + \sum_{i=1}^{n-1}\langle R(x,e_i)e_i, x\rangle \\ &\overset{(*)}{=} \sum_{i=1}^{n-1} K(P_i),\end{aligned}$$

where $P_i \subset T_pM$ is the plane spanned by x and e_i. Note that $(*)$ holds since $|x \wedge e_i| = 1$ for all $i = 1, \ldots, n-1$.



Remark 5.22. Lower bounds for the Ricci curvature $\mathrm{Ric}(x)$ give upper bounds for the volume growth. The Ricci curvature will be important in relations between curvature and topology.

The **scalar curvature** is a function S defined as the trace of Ric. Thus

$$S(p) = \sum_{i=1}^{n} \mathrm{Ric}(e_i),$$

where $e_1, \ldots, e_n$ is an orthonormal basis of T_pM.

6 Jacobi fields

Jacobi fields provide tools to study the effect of curvature on the behavior of nearby geodesics. They can also be used to characterize points where $\exp_p$ fails to be a local diffeomorphism.

In this chapter we assume that M is a Riemannian manifold.

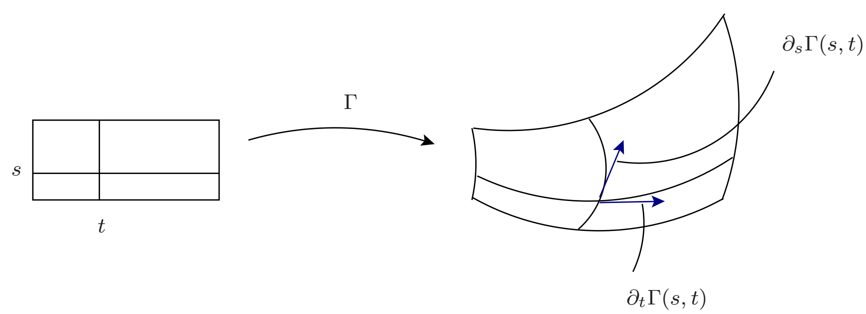
6.1 Jacobi equation

Lemma 6.2. *If Γ is a C^∞ admissible family of curves and if V is a C^∞ vector field along Γ, then*

$$D_sD_tV - D_tD_sV = R(S,T)V.$$

Recall that $\Gamma\colon]-\varepsilon,\varepsilon[\times[a,b] \to M$ and

$$\begin{aligned} T(s,t) &= \partial_t\Gamma(s,t), \\ S(s,t) &= \partial_s\Gamma(s,t), \\ D_tV &= \text{ the covariant derivative of } V \text{ along main curves } \Gamma_s, \\ D_sV &= \text{ the covariant derivative of } V \text{ along transverse curves } \Gamma^{(t)}. \end{aligned}$$



Proof. This is a local question, so we may compute in local coordinates. Let x be a chart at $\Gamma(s_0,t_0)$. Writing

$$V(s,t) = V^i(s,t)\partial_i,$$

we get

$$\begin{aligned} D_tV &= \frac{\partial V^i}{\partial t}\partial_i + V^iD_t\partial_i, \\ D_sD_tV &= \frac{\partial^2 V^i}{\partial s\partial t}\partial_i + \frac{\partial V^i}{\partial t}D_s\partial_i + \frac{\partial V^i}{\partial s}D_t\partial_i + V^iD_sD_t\partial_i \\ D_tD_sV &\overset{s\leftrightarrow t}{=} \cdots \qquad \cdots \qquad \cdots \qquad + V^iD_tD_s\partial_i. \end{aligned}$$

Thus
$$D_s D_t V - D_t D_s V = V^i (D_s D_t \partial_i - D_t D_s \partial_i).$$
Writing $(x \circ \Gamma)(s,t) = \big(x^1(s,t), \dots, x^n(s,t)\big)$, we have
$$T = \frac{\partial x^j}{\partial t} \partial_j\,, \quad S = \frac{\partial x^k}{\partial s} \partial_k.$$
Since ∂_i is extendible, we have
$$D_t \partial_i = \nabla_T \partial_i = \frac{\partial x^j}{\partial t} \nabla_{\partial_j} \partial_i.$$
Furthermore, since $\nabla_{\partial_j} \partial_i$ is extendible, we obtain
$$\begin{aligned} D_s D_t \partial_i &= \frac{\partial^2 x^j}{\partial s \partial t} \nabla_{\partial_j} \partial_i + \frac{\partial x^j}{\partial t} \nabla_S \big(\nabla_{\partial_j} \partial_i\big) \\ &= \frac{\partial^2 x^j}{\partial s \partial t} \nabla_{\partial_j} \partial_i + \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial s} \nabla_{\partial_k} \nabla_{\partial_j} \partial_i. \end{aligned}$$
Similarly (interchanging $s \leftrightarrow t$ and $j \leftrightarrow k$),
$$D_t D_s \partial_i = \frac{\partial^2 x^j}{\partial t \partial s} \nabla_{\partial_j} \partial_i + \frac{\partial x^k}{\partial s} \frac{\partial x^j}{\partial t} \nabla_{\partial_j} \nabla_{\partial_k} \partial_i.$$
Hence
$$\begin{aligned} D_s D_t \partial_i - D_t D_s \partial_i &= \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial s} \big(\nabla_{\partial_k} \nabla_{\partial_j} \partial_i - \nabla_{\partial_j} \nabla_{\partial_k} \partial_i\big) \\ &\overset{[\partial_k, \partial_j]=0}{=} \frac{\partial x^j}{\partial t} \frac{\partial x^k}{\partial s} R(\partial_k, \partial_j) \partial_i \\ &= R(S,T) \partial_i. \end{aligned}$$
So,
$$D_s D_t V - D_t D_s V = V^i R(S,T) \partial_i = R(S,T) V.$$
□

Remark 6.3. *Shorter proof of Lemma 6.2.*(Cf. Remark 4.32.) Since
$$[S,T] = [\Gamma_* \partial_s, \Gamma_* \partial_t] = \Gamma_* \underbrace{[\partial_s, \partial_t]}_{=0} = 0.$$
we obtain
$$\begin{aligned} R(S,T)V &= \nabla_S \nabla_T V - \nabla_T \nabla_S V - \nabla_{\underbrace{[S,T]}_{=0}} V \\ &= \nabla_{\Gamma_* \partial_s} \nabla_{\Gamma_* \partial_t} V - \nabla_{\Gamma_* \partial_t} \nabla_{\Gamma_* \partial_s} V \\ &= D_s D_t V - D_t D_s V. \end{aligned}$$

Let Γ be as above. We say that Γ is a **variation** of γ **through geodesics** if all main curves Γ_s are geodesics and $\Gamma_0 = \gamma$. Recall that the variation field of Γ is the vector field $V(t) = \partial_s \Gamma(0,t) = S(0,t)$.

Theorem 6.4. *Let γ be a geodesic and Γ a variation of γ through geodesics. If V is the variation field of Γ, then it satisfies the* ***Jacobi equation***

$$D_t^2 V + R(V, \dot{\gamma})\dot{\gamma} = 0. \tag{6.5}$$

Proof. Let $S(s,t) = \partial_s \Gamma(s,t)$ and $T(s,t) = \partial_t \Gamma(s,t)$ be as earlier. Since all main curves Γ_s are geodesics, we have

$$D_t T = D_t \dot{\Gamma} = 0.$$

By Lemma 6.2 and the Symmetry Lemma 4.30, we obtain

$$\begin{aligned} 0 = D_s D_t T &= D_t D_s T + R(S,T)T \\ &= D_t D_t S + R(S,T)T. \end{aligned}$$

At $s = 0$, $S(0,t) = V(t)$ and $T(0,t) = \dot{\gamma}_t$, so we get (6.5). □

Definition 6.6. Any vector field V along a geodesic γ that satisfies (6.5) is called a **Jacobi field**.

Let $\gamma\colon I \to M$ be a geodesic, $E_i \in \mathcal{T}(\gamma)$, $i = 1, \dots, n$, a parallel orthonormal frame along γ, and $E_n = \dot{\gamma}$. Let $V \in \mathcal{T}(\gamma)$,

$$V = v^i E_i.$$

Since E_i is parallel, $D_t V = \dot{v}^i E_i$ and

$$D_t^2 V = \ddot{v}^i E_i. \tag{6.7}$$

Writing $R(E_j, E_k)E_\ell = R^i_{jk\ell} E_i$, we get

$$R(V, \dot{\gamma})\dot{\gamma} = R(v^j E_j, E_n)E_n = v^j R^i_{jnn} E_i. \tag{6.8}$$

By definition, V is a Jacobi field if and only if it satisfies (6.5). Plugging-in (6.7) and (6.8) into (6.5), we conclude that

$$\begin{aligned} V \text{ is a Jacobi field } &\Leftrightarrow \ \ddot{v}^i E_i + v^j R^i_{jnn} E_i = 0 \\ &\Leftrightarrow \ \ddot{v}^i + v^j R^i_{jnn} = 0, \ \forall i = 1, \dots, n. \end{aligned}$$

This is a linear system of 2^{nd}-order ODEs. Theory of ODEs then imply the following:

Proposition 6.9. *Let $\gamma\colon I \to M$ be a geodesic, $t_0 \in I$, and $p = \gamma(t_0)$. Given any vectors $v, w \in T_pM$ there exists a unique Jacobi field V satisfying the initial conditions*

$$V_{t_0} = v \quad \text{and} \quad (D_t V)_{t_0} = w.$$

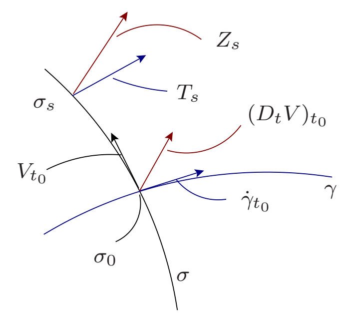
Corollary 6.10. *Given a geodesic γ, the set of all Jacobi fields along γ is a 2n-dimensional linear subspace of $\mathcal{T}(\gamma)$.*

Proof. Follows easily from 6.9 (Exercise) □

Lemma 6.11. *If $\gamma\colon I \to M$ is a geodesic and V is a Jacobi field along γ, then on every $[a,b] \subset I$, V is the variation field of some variation of $\gamma|[a,b]$ through geodesics.*

Proof. Let $\gamma\colon I \to M$ be a geodesic and V a Jacobi field along γ. Fix $[a,b] \subset I$ and $t_0 \in [a,b]$. Let σ be a C^∞-path such that $\dot{\sigma}_0 = V_{t_0}$. Let T and Z be parallel vector fields along σ such that

$$T_0 = \dot{\gamma}_{t_0} \quad \text{and} \quad Z_0 = (D_t V)_{t_0}.$$



For a sufficiently small $\varepsilon > 0$ define $\Gamma\colon]-\varepsilon, \varepsilon[\times[a,b] \to M$ by

$$\Gamma(s,t) = \exp_{\sigma(s)}\big[(t-t_0)(T_s + sZ_s)\big].$$

Then Γ is a variation of γ through geodesics. By Theorem 6.4,

$$t \mapsto \partial_s\Gamma(0,t)$$

is a Jacobi field along γ. We claim that $V_t = \partial_s\Gamma(0,t)$. To prove the claim, we observe that

$$\partial_s\Gamma(0,t_0) = \frac{d}{ds}\Gamma(s,t_0)_{|s=0} = \frac{d}{ds}\sigma(s)_{|s=0} = \dot\sigma_0 = V_{t_0}$$

and

$$\partial_t\Gamma(s,t_0) = \frac{d}{dt}\Gamma(s,t)_{|t=t_0} = T_s + sZ_0.$$

The Symmetry Lemma 4.30 and the assumption that T and Z are parallel along σ imply that

$$\begin{aligned} D_t\partial_s\Gamma(s,t_0) &\overset{4.30}{=} D_s\partial_t\Gamma(s,t_0) = D_s(T_s + sZ_s) \\ &= \underbrace{D_sT_s}_{=0} + s\underbrace{D_sZ_s}_{=0} + \underbrace{\frac{d}{ds}(s)}_{=1} Z_s \\ &= Z_s. \end{aligned}$$

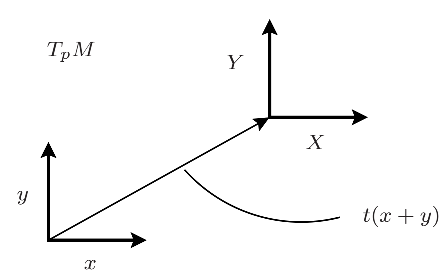
Hence at $s = 0$

$$D_t\partial_s\Gamma(0,t_0) = Z_0 = (D_tV)_{t_0}$$

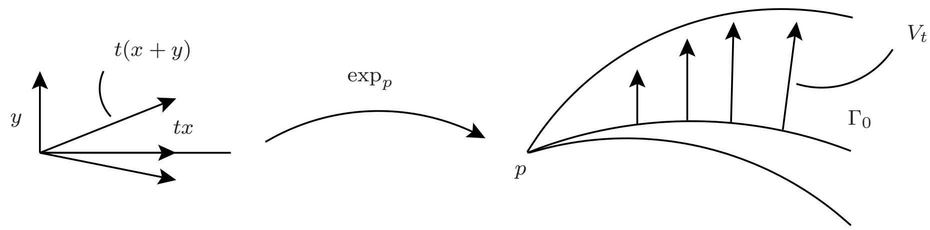
Since V and $\partial_s\Gamma(0,\cdot)$ have the same initial values, we get $V_t = \partial_s\Gamma(0,t)$ by Proposition 6.9. □

6.12 Effect of curvature on geodesics

Let $x, y \in T_pM$ be orthonormal and X, Y their parallel fields in T_pM.



Define $\Gamma(s,t) = \exp_p[t(x+sy)]$.



Then Γ is a variation of Γ_0 through geodesics and

$$V_t = \partial_s \Gamma(0,t) = \frac{d}{ds}\left(\exp_p\big[t(x+sy)\big]\right)_{|s=0} = (\exp_p)_*(tY) \tag{6.13}$$

is a Jacobi field. More precisely, $(\exp_p)_*(tY) = (\exp_p)_{*tx} \colon T_{tx}(T_pM) \to T_{\Gamma(0,t)}M$.

We want to study the Taylor expansion of $|V_t|^2$ at $t=0$. In what follows we denote the covariant derivative D_t by prime ('). Write $T_t = \partial_t\Gamma(0,t) = \Gamma_0(t)$. From (6.13), we get

$$V_0' = Y_0 = y \quad \text{and} \quad \langle V,V\rangle_0 = 0,$$

and consequently

$$\begin{aligned}
\langle V,V\rangle_0' &= 2\langle V',V\rangle_0 = 0\\
\langle V,V\rangle_0'' &= 2\underbrace{\langle V'',V\rangle_0}_{=0} + 2\langle V',V'\rangle_0 = 2\underbrace{|y|}_{=1}{}^{2} = 2\\
\langle V,V\rangle_0''' &= 2\langle V'',V\rangle_0' + 2\langle V',V'\rangle_0'\\
&= 2\underbrace{\langle V''',V\rangle_0}_{=0} + 2\langle V'',V'\rangle_0 + 4\langle V'',V'\rangle_0\\
&= 6\langle V'',V'\rangle_0.
\end{aligned}$$

Since V is a Jacobi field, we have $V'' = -R(V,T)T$, and therefore

$$V_0'' = -\big(R(V,T)T\big)_0 = 0,$$

and so

$$\langle V,V\rangle_0''' = 0.$$

Hence

$$V_0''' = -\big(R(V,T)T\big)_0' \overset{(*)}{=} -\big(R(V',T)T\big)_0 = -R(y,x)x.$$

Using this we compute

$$\begin{aligned}
\langle V,V\rangle_0'''' &= \big(2\langle V''',V\rangle + 6\langle V'',V'\rangle\big)_0'\\
&= 2\underbrace{\langle V'''',V\rangle_0}_{=0} + 2\langle V''',V'\rangle_0 + 6\langle V''',V'\rangle_0 + 6\underbrace{\langle V'',V''\rangle_0}_{=0}\\
&= 8\langle V''',V'\rangle_0\\
&= -8\langle R(y,x)x,y\rangle.
\end{aligned}$$

Putting these together, we obtain

$$|V_t|^2 = 2\frac{t^2}{2!} - \frac{8}{4!}\langle R(y,x)x,y\rangle t^4 + O(t^5).$$

Vectors $x, y \in T_pM$ are orthonormal, hence $\langle R(y,x)x, y\rangle = K(y,x)$, and therefore

$$|V_t|^2 = t^2 - \frac{1}{3}K(y,x)t^4 + O(t^5). \tag{6.14}$$

Let us prove the equality $(*)$:

$$(-R(V,T)T)'_0 = (-R(V',T)T)_0.$$

For every $W \in \mathcal{T}(\Gamma_0)$, we have at $t = 0$:

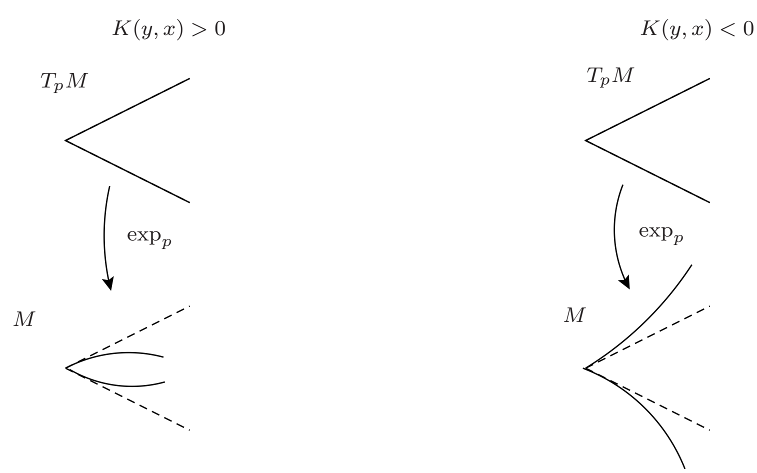
$$\langle R(V,T)T, W\rangle'_0 = \langle (R(V,T)T)', W\rangle_0 + \underbrace{\langle R(V,T)T, W'\rangle_0}_{=0 \text{ since } V_0=0}$$

Hence using Proposition 5.11(2)-(3) we obtain

$$\begin{aligned}
\langle (R(V,T)T)', W\rangle_0 &= \langle R(V,T)T, W\rangle'_0 \\
&\overset{(2),(3)}{=} -\langle R(T,W)T, V\rangle'_0 \\
&= -\underbrace{\langle (R(T,W)T)', V\rangle_0}_{=0} - \langle R(T,W)T, V'\rangle_0 \\
&\overset{(3)}{=} \langle R(T,W)V', T\rangle_0 \\
&\overset{(2)}{=} \langle R(V',T)T, W\rangle_0.
\end{aligned}$$

Since this holds for every $W \in \mathcal{T}(\Gamma_0)$, the equality $(*)$ follows.

Geometrical interpretation:

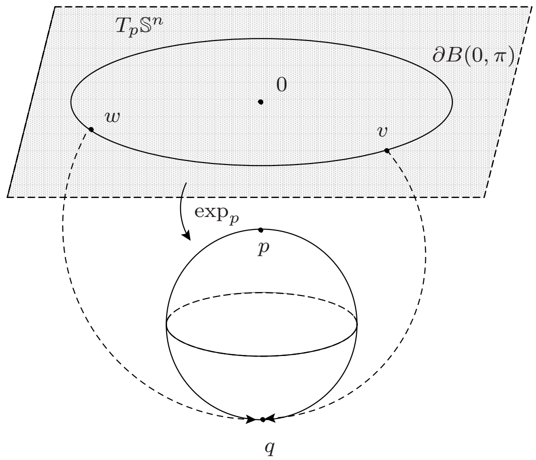


6.15 Conjugate points

In this section we study the relationship between singularities of the exponential map and Jacobi fields.

If M is complete, then $\exp_p$ is defined on all of T_pM and it is a local diffeomorphism near 0. However, it may fail to be a local diffeomophism at points far away.

Example 6.16. *The sphere* $\mathbb{S}^n$. For any $p \in \mathbb{S}^n$, all points on $\partial B(0,\pi) \subset T_p\mathbb{S}^n$ are mapped to the antipodal point $q \in \mathbb{S}^n$ (of p) by the exponential map $\exp_p$. Hence q is the critical value of $\exp_p$.



Definition 6.17. A point q is a **conjugate point** of $p \in M$ if q is a critical value of $\exp_p$. That is,

$$\exp_{p*v} \colon T_v(T_pM) \to T_qM$$

is singular for some $v \in T_pM$. (Note that then $q = \exp_p v$.) Moreover, q is **conjugate to** p **along a geodesic** γ if γ is a reparametrization of γ^v, where v is as above.

Suppose that $v \in T_pM$ and $\exp_{p*v} w = 0$ for some $0 \neq w \in T_v(T_pM) = T_pM$. Let

$$\Gamma(s,t) = \exp_p t(v + sw)$$

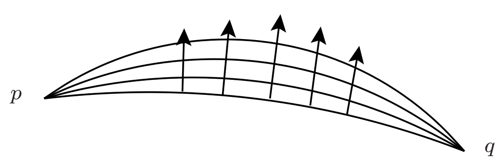
be the variation of $t \mapsto \exp_p(tv)$ through geodesics. The corresponding variation field

$$V_t = \partial_s\Gamma(0,t) = \exp_{p*tv} tW,$$

where W is the parallel field of w in T_pM, is a Jacobi field that vanishes at $t = 0$ and $t = 1$; $V_0 = \exp_{p*0} 0 = 0, V_1 = \exp_{p*v} w = 0$. Since $\exp_p$ is a local diffeomorphism at $0 \in T_pM$, V is non-trivial.

Theorem 6.18. *Let* $\gamma \colon [0,1] \to M$ *be a geodesic. Then* $q = \gamma_1$ *is conjugate to* $p = \gamma_0$ *along* γ *if and only if there exists a non-trivial Jacobi field* V *along* γ *such that* $V_0 = 0$ *and* $V_1 = 0$.

Proof. $\boxed{\Rightarrow}$ Proved above.



$\boxed{\Leftarrow}$ Suppose that V is a non-trivial Jacobi field along γ, with $V_0 = 0$ and $V_1 = 0$. Let

$$\Gamma(s,t) = \exp_p t(\dot\gamma_0 + sV_0').$$

Its variation field is V (see the proof of Lemma 6.11). Hence

$$\exp_{p*\gamma_0} V_0' = \partial_s\Gamma(0,1) = V_1 = 0.$$

Since $V_0 = 0$ and V is non-trivial, we must have $V_0' \neq 0$ (otherwise, $V_t \equiv 0$ by Proposition 6.9). It follows that $\exp_{p*\gamma_0}$ is singular, and therefore $q = \exp_p(\dot\gamma_0)$ is conjugate to p along γ. ∎

Remark 6.19. If $\gamma\colon [a,b] \to M$ is a geodesic, so does $\sigma\colon [a,b] \to M$, $\sigma(t) = \gamma(a+b-t)$. Furthermore, if V is a Jacobi field along γ, then

$$t \mapsto V_{a+b-t}$$

is a Jacobi field along σ. In conclusion,

$$q \text{ is conjugate to } p \Leftrightarrow p \text{ is conjugate to } q.$$

Theorem 6.20. *If V is a Jacobi field along a geodesic $\gamma\colon [a,b] \to M$, $V_a = 0$, and $V_b = 0$, then*

$$\langle V, \dot\gamma\rangle = \langle V', \dot\gamma\rangle = 0.$$

Proof. Since γ is a geodesic, $D_t\gamma = 0$, and so

$$\langle V', \dot\gamma\rangle' = \langle V'', \dot\gamma\rangle = \underbrace{-R(V,\dot\gamma)\dot\gamma,\dot\gamma\rangle}_{(=\langle R(V,\dot\gamma)\dot\gamma,\dot\gamma\rangle \text{ hence } =0)} = 0.$$

Thus $\langle V', \dot\gamma\rangle = c =$ constant. On the other hand,

$$\langle V, \dot\gamma\rangle' = \langle V', \dot\gamma\rangle = c,$$

and therefore

$$\langle V_t, \dot\gamma_t\rangle = ct + d,$$

where d is a constant. Since $\langle V_a, \dot\gamma_a\rangle = 0$ and $\langle V_b, \dot\gamma_b\rangle = 0$, we have $c = d = 0$, and consequently

$$\langle V, \dot\gamma\rangle = \langle V', \dot\gamma\rangle = 0.$$

□

Remark 6.21. We get from the proof above that every Jacobi field V satisfies

$$\langle V_t, \dot\gamma_t\rangle = \langle V_a, \dot\gamma_a\rangle + \langle V_a', \dot\gamma_a\rangle(t-a).$$

Theorem 6.22. *Let $\gamma\colon [a,b] \to M$ be a geodesic. If γ_a is not conjugate to γ_b and $v_1 \in T_{\gamma_a}M$, $v_2 \in T_{\gamma_b}M$, then there exists a unique Jacobi field V along γ such that $V_a = v_1$ and $V_b = v_2$.*

Proof. Let V and W be Jacobi fields such that $V_a = W_a = v_1$ and $V_b = W_b = v_2$. Then $Y = V - W$ is a Jacobi field along γ, with $Y_a = 0$ and $Y_b = 0$. Theorem 6.18 implies that $Y = 0$, hence V is unique (if exists). The proof of the existence of V is left as an exercise. □

Suppose that M has constant sectional curvature K. Let $\gamma\colon [0,b] \to M$ be a geodesic and E_i, $i = 1,\ldots,n$, be a parallel frame along γ. Let V be a Jacobi field along γ. Then by Proposition 5.19

$$\begin{aligned}\langle V'', E_i\rangle &= -\langle R(v,\dot\gamma)\dot\gamma, E_i\rangle \\ &= -K\big(\langle\dot\gamma,\dot\gamma\rangle\langle V, E_i\rangle - \langle V,\dot\gamma\rangle\langle\dot\gamma, E_i\rangle\big).\end{aligned}$$

If γ is of unit speed and $\langle V, \dot\gamma\rangle \equiv 0$, then

$$\langle V'', E_i\rangle = -K\langle V, E_i\rangle.$$

Solutions:

$$\begin{aligned}
K>0: &\quad V_t = \big(a^i \sin(\sqrt{K}t) + b^i \cos(\sqrt{K}t)\big)E_i(t);\\
K=0: &\quad V_t = \big(a^i t + b^i\big)E_i(t);\\
K<0: &\quad V_t = \big(a^i \sinh(\sqrt{|K|}t + b^i \cosh(\sqrt{|K|}t)\big)E_i(t),
\end{aligned}$$

where a^i and b^i are constants.
Conclusion:
If $K \le 0$, there are <u>no</u> conjugate points of $\gamma(0)$.
If $K > 0$, we get conjugate points of $\gamma(0)$ for $t = \ell\pi/\sqrt{K},\ \ell = 1, 2, \ldots$.

6.23 Second variation formula

Theorem 6.24 (The second variation formula)**.** *Let $\gamma\colon [a,b] \to M$ be a unit speed geodesic, Γ a proper variation of γ, and V its variation field. Then*

$$\frac{d^2}{ds^2}\ell(\Gamma_s)_{|s=0} = \int_a^b \Big(|D_t V^\perp|^2 - \langle R(V^\perp, \dot\gamma)\dot\gamma, V^\perp\rangle\Big)\, dt, \tag{6.25}$$

where $V^\perp$ is the normal component of V, i.e. $V = V^T + V^\perp,\ V^T = \langle V, \dot\gamma\rangle\dot\gamma$.

Proof. Write $T = \partial_t\Gamma,\ S = \partial_s\Gamma$. Assume Γ is smooth in $]-\varepsilon,\varepsilon[\times[a_{i-1},a_i]$. Then

$$\frac{d}{ds}\ell\left(\Gamma_s|[a_{i-1},a_i]\right) = \int_{a_{i-1}}^{a_i} \frac{1}{|T|}\langle D_t S, T\rangle dt.$$

The Symmetry lemma 4.30 and Lemma 6.2 imply that

$$\begin{aligned}
\frac{d^2}{ds^2}\ell\left(\Gamma_s|[a_{i-1},a_i]\right) &= \int_{a_{i-1}}^{a_i} \frac{\partial}{\partial s}\left(\frac{\langle D_t S, T\rangle}{|T|}\right) dt\\
&= \int_{a_{i-1}}^{a_i} \left(\frac{\langle D_s D_t S, T\rangle + \langle D_t S, D_s T\rangle}{|T|} - \frac{1}{2}\frac{\langle D_t S, T\rangle 2\langle D_s T, T\rangle}{|T|^3}\right) dt\\
&= \int_{a_{i-1}}^{a_i} \left(\frac{\langle D_t D_s S + R(S,T)S, T\rangle + |D_t S|^2}{|T|} - \frac{\langle D_t S, T\rangle^2}{|T|^3}\right) dt.
\end{aligned}$$

At $s = 0,\ |T| \equiv 1$, hence

$$\frac{d^2}{ds^2}\ell(\Gamma_s|[a_{a-1},a_i])_{|s=0} = \int_{a_{i-1}}^{a_i} \left(\langle D_t D_s S, T\rangle + \langle R(S,T)S,T\rangle + |D_t S|^2 - \langle D_t S, T\rangle^2\right) dt_{|s=0}.$$

Since $T(0,t) = \dot\gamma_t$, we have $D_t T = D_t\dot\gamma = 0$ at $s = 0$, and therefore

$$\begin{aligned}
\int_{a_{i-1}}^{a_i} \langle D_t D_s S, T\rangle dt_{|s=0} &= \int_{a_{i-1}}^{a_i} \frac{\partial}{\partial t}\langle D_s S, T\rangle dt_{|s=0}\\
&= \langle D_s S(0,a_i), \dot\gamma_{a_i}\rangle - \langle D_s S(0,a_{i-1}), \dot\gamma_{a_{i-1}}\rangle.
\end{aligned}$$

Since Γ is proper, $S(s,t) = 0$ for all s at the endpoints $t = a_0 = a$ and $t = a_k = b$. Hence $D_s S(s,a_0) = 0$ and $D_s S(s,a_k) = 0$. Furthermore, $D_s S$ is continuous at every (s,t), in particular, when $t = a_i$, and therefore

$$\sum_{i=1}^{k}\int_{a_{i-1}}^{a_i} \langle D_t D_s S, T\rangle dt_{|s=0} = \sum_{i=1}^{k}\left(\langle D_s S(0,a_i), \dot\gamma_{a_i}\rangle - \langle D_s S(0,a_{i-1}), \dot\gamma_{a_{i-1}}\rangle\right) = 0.$$

We obtain

$$\begin{aligned}\frac{d^2}{ds^2}\ell(\Gamma_s)_{|s=0} &= \int_a^b \big(|D_tS|^2 - \langle D_tS, T\rangle^2 - \langle R(S,T)T, S\rangle\big)\, dt_{|s=0} \\ &= \int_a^b \big(|D_tV|^2 - \langle D_tV, \dot\gamma\rangle^2 - \langle R(V,\dot\gamma)\dot\gamma, V\rangle\big)\, dt\end{aligned}$$

where the last equality holds since $S(0,t) = V_t$.
Write $V = V^T + V^\perp$, where $V^T = \langle V, \dot\gamma\rangle\dot\gamma$. Then

$$\begin{aligned}D_tV^T &= D_t(\langle V,\dot\gamma\rangle\dot\gamma) = \langle V,\dot\gamma\rangle \underbrace{D_t\dot\gamma}_{=0} + \frac{d}{dt}(\langle V,\dot\gamma\rangle)\dot\gamma \\ &= \langle D_tV,\dot\gamma\rangle\dot\gamma + \langle V, \underbrace{D_t\dot\gamma}_{=0}\rangle\dot\gamma \\ &= (D_tV)^T; \\ D_tV^\perp &= (D_tV)^\perp.\end{aligned}$$

Hence

$$|D_tV|^2 = |(D_tV)^T|^2 + |(D_tV)^\perp|^2 = \langle D_tV, \dot\gamma\rangle^2 + |D_tV^\perp|^2,$$

and so

$$|D_tV|^2 - \langle D_tV, \dot\gamma\rangle^2 = |D_tV^\perp|^2.$$

Also,

$$\begin{aligned}\langle R(V,\dot\gamma)\dot\gamma, V\rangle &= \langle \underbrace{R(\langle V,\dot\gamma\rangle\dot\gamma,\dot\gamma)\dot\gamma}_{=0}, V\rangle + \langle R(V^\perp,\dot\gamma)\dot\gamma, V\rangle \\ &= \underbrace{\langle R(V^\perp,\dot\gamma)\dot\gamma, \langle V,\dot\gamma\rangle\dot\gamma\rangle}_{=0} + \langle R(V^\perp,\dot\gamma)\dot\gamma, V^\perp\rangle \\ &= \langle R(V^\perp,\dot\gamma)\dot\gamma, V^\perp\rangle.\end{aligned}$$

□

We define a symmetric bilinear form, called the **index form**, on the space of continuous, piecewise C^∞ vector fields along γ by

$$I(V,W) = \int_a^b (\langle D_tV, D_tW\rangle - \langle R(V,\dot\gamma)\dot\gamma, W\rangle)\, dt.$$

Corollary 6.26. *If $\gamma\colon [a,b] \to M$ is a unit speed geodesic and if Γ is a proper variation of γ whose variation field V is normal, then*

$$\frac{d^2}{ds^2}\ell(\Gamma_s)_{|s=0} = I(V,V). \tag{6.27}$$

In particular, if γ is minimizing, then $I(V,V) \geq 0$ for any proper, normal vector field V along γ.

Proof. Since Γ is proper, also V is proper. Furthermore, since V is proper and normal, we obtain (6.27) from the second variation formula (6.25). To prove the second claim, suppose on the contrary that there exists a proper normal vector field V along γ such that $I(V,V) < 0$. Now V is the variation field of some proper variation Γ of γ. But then (6.27) implies that

$$\frac{d^2}{ds^2}\ell(\Gamma_s)_{|s=0} < 0,$$

and therefore γ can not be minimizing. □

Next we express $I(V,W)$ in another form involving the Jacobi equation.

Suppose that V and W are continuous, piecewise smooth vector fields along γ. Let $a = a_0 < a_1 < \cdots < a_k = b$ be such that V and W are C^∞ on each $[a_{i-1}, a_i]$. Then

$$\langle D_t V, W\rangle' = \langle D_t^2 V, W\rangle + \langle D_t V, D_t W\rangle.$$

Hence

$$\int_{a_{i-1}}^{a_i} \langle D_t V, D_t W\rangle dt = -\int_{a_{i-1}}^{a_i} \langle D_t^2 V, W\rangle dt + \Big/_{a_{i-1}}^{a_i} \langle D_t V, W\rangle.$$

By taking the sum over $i = 1, \ldots, k$ and obseving that W is continuous at points $t = a_i$ we get

$$I(V,W) = -\int_a^b \langle D_t^2 V + R(V,\dot\gamma)\dot\gamma, W\rangle dt - \sum_{i=0}^{k} \langle \Delta_i D_t V, W(a_i)\rangle, \tag{6.28}$$

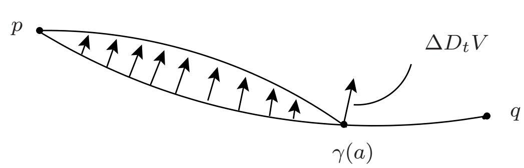
where

$$\begin{aligned}
\Delta_i D_t V &= \lim_{t\searrow a_i} D_t V(t) - \lim_{t\nearrow a_i} D_t V(t), \quad i = 1, \ldots, k-1;\\
\Delta_0 D_t V &= \lim_{t\searrow a} D_t V(t);\\
\Delta_k D_t V &= -\lim_{t\nearrow b} D_t V(t).
\end{aligned}$$

The next theorem says that no geodesic is minimizing past its first conjugate point.

Theorem 6.29. *Let $\gamma\colon [0,b] \to M$ be a unit speed geodesic from $p = \gamma(0)$ to $q = \gamma(b)$ such that $\gamma(a)$ is conjugate to p for some $a \in]0,b[$. Then there exists a proper normal vector field X along γ such that $I(X,X) < 0$. In particular, $\gamma|[a,c]$ is not minimizing for any $c \in]a,b[$.*

Proof. By Theorem 6.18 and Theorem 6.20, there exists a nontrivial normal Jacobi field J along $\gamma|[0,a]$ such that $J_0 = 0, J_a = 0$ since $\gamma(a)$ is conjugate to p.



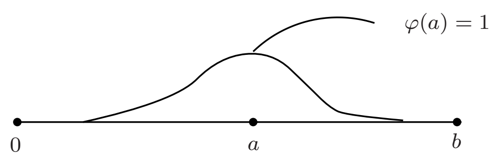
Define a vector field V along γ by

$$V_t = \begin{cases} J_t, & t \in [0,a] \\ 0, & t \in [a,b]. \end{cases}$$

Then V is proper, normal, and piecewise smooth. Let W be a smooth, proper, and normal vector field along γ such that

$$W_a = \Delta D_t V = \lim_{t \searrow a} \underbrace{D_t V(t)}_{=0} - \lim_{t \nearrow a} D_t V(t) = -D_t J(a) \neq 0.$$

Note that $D_t J(a) \neq 0$ otherwise $J \equiv 0$. Also $D_t J(a) \perp \dot{\gamma}_a$ by Theorem 6.20. Such W is easy to construct: take the parallel translation of $-D_t J(a)$ and then multiply by a smooth "bumb function" φ.



Define

$$X^\varepsilon = V + \varepsilon W, \quad \varepsilon > 0.$$

Then X^ε is a proper, normal, piecewise smooth vector field along γ, and

$$I(X^\varepsilon, X^\varepsilon) = I(V,V) + 2\varepsilon I(V,W) + \varepsilon^2 I(W,W).$$

Since V is a Jacobi field along $[0,a]$ and $[a,b]$, we get by (6.28) that

$$I(V,V) = -\langle \Delta D_t V, V_a \rangle = 0$$

and

$$I(V,W) = -\langle \Delta D_t V, W_a \rangle = -|W_a|^2 \neq 0.$$

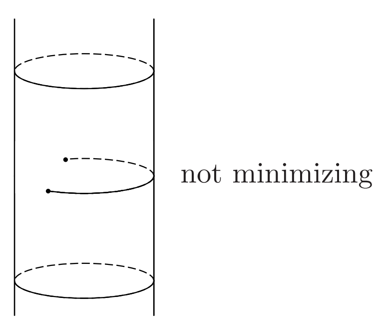
Hence

$$I(X^\varepsilon, X^\varepsilon) = -2\varepsilon \underbrace{|W_a|^2}_{\neq 0} + \varepsilon^2 I(W,W) < 0$$

if ε is small enough. □

Remark 6.30. A geodesic without conjugate points need not be minimizing.

Example 6.31. There are no conjugate points along any geodesic on a cylinder $S^1 \times \mathbb{R}$. However, no geodesic that wraps more than half way around the cylinder is minimizing.



7 Curvature and topology

7.1 Index lemma

Lemma 7.2 (Index Lemma). *Let $\gamma : [0, b] \to M$ be a unit speed geodesic from $p = \gamma(0)$ to $q = \gamma(b)$ without conjugate points to p along γ. Let W be a piecewise smooth vector field along γ with $W_0 = 0$ and let $V \in \mathcal{T}(\gamma)$ be the unique Jacobi field with $V_0 = W_0$ and $V_b = W_b$. Then*

$$I(V, V) \leq I(W, W)$$

and equality occurs if and only if $W = V$.

Proof. Let $v_1, \dots, v_n$ be a basis in T_qM and $V_1, \dots, V_n \in \mathcal{T}(\gamma)$ be Jacobi fields such that $V_i(0) = 0$ and $V_i(b) = v_i$. Then by Theorem 6.22 the fields V_i are unique. Because the Jabobi equation is linear, the set $\{V_i(t)\}$ is linearly independent for every $t \in (0, b]$. Because $W_0 = 0$, we know that $W = f^i V_i$, where f^i is piecewise smooth along γ. On the other hand, the equality $V_b = W_b = f^i(b)V_i(b)$ combined with the fact that V_i is a Jacobi field implies that $V = f^i(b)V_i$. Hence, due to the fact that V is a Jacobi field and (6.28), we have

$$I(V, V) = \langle V'(b), V(b)\rangle = f^i(b)f^j(b)\langle V_i'(b), V_j(b)\rangle. \tag{7.3}$$

Furthermore,

$$\begin{aligned}\big(\langle V_i', V_j\rangle - \langle V_i, V_j'\rangle\big)' &= \langle V_i'', V_j\rangle + \langle V_i', V_j'\rangle - \langle V_i', V_j'\rangle - \langle V_i, V_j''\rangle \\ &= \langle R(V_j, \dot\gamma)\dot\gamma, V_i\rangle - \langle R(V_i, \dot\gamma)\dot\gamma, V_j\rangle = 0.\end{aligned}$$

Hence,

$$\langle V_i', V_j\rangle - \langle V_i, V_j'\rangle = C, \tag{7.4}$$

where C is a constant. The constant $C = 0$ because $\langle V_i', V_j\rangle_0 - \langle V_i, V_j'\rangle_0 = 0$. On the other hand,

$$W' = \dot f^i V_i + f^i V_i' =: A + B,$$

so

$$I(W, W) = \int_0^b \big(\langle A, A\rangle + \langle A, B\rangle + \langle B, A\rangle + \langle B, B\rangle - \langle R(W, \dot\gamma)\dot\gamma, W\rangle\big)\, dt.$$

Integrating by parts, using the fact that V_i is a Jacobi field and (7.3), we have

$$\begin{aligned}\int_0^b \langle B, B\rangle\, dt &= \int_0^b f^i f^j \langle V_i', V_j'\rangle\, dt = \int_0^b f^i f^j \left(\langle V_i', V_j\rangle' - \langle V_i'', V_j\rangle\right)\, dt \\ &= f^i(b)f^j(b)\langle V_i'(b), V_j(b)\rangle - \int_0^b \Big(\dot f^i f^j\langle V_i', V_j\rangle + f^i \dot f^j \langle V_i', V_j\rangle - f^i f^j\langle R(V_i, \dot\gamma)\dot\gamma, V_j\rangle\Big)\, dt \\ &\overset{(7.3)}{=} I(V, V) - \int_0^b \big(\langle A, B\rangle + \langle B, A\rangle - \langle R(W, \dot\gamma)\dot\gamma, W\rangle\big)\, dt.\end{aligned}$$

Hence,

$$I(W, W) = \underbrace{\int_0^b \langle A, A\rangle\, dt}_{\geq 0} + I(V, V) \geq I(V, V).$$

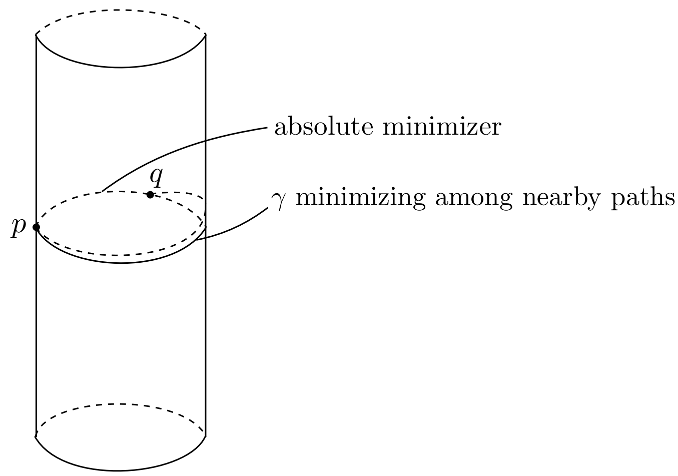
as required. From this we see that the equality occurs if and only if $A \equiv 0$, or equivalently if $\dot f^i \equiv 0$ for every i. However, this is possible if and only if $W = V$. □

Let γ be as in the assumptions of Lemma 7.2 and let Γ be a proper variation of γ whose variation field W is normal, that is, $\langle W, \dot\gamma\rangle \equiv 0$ and is non-trivial. Then Corollary 6.26 and the Index Lemma 7.2 implies

$$\tfrac{d^2}{ds^2}\ell(\Gamma_s)|_{s=0} = I(W,W) > I(V,V) = 0,$$

where V is the unique Jacobi field along γ with $V_0 = W_0 = 0$ and $V_b = W_b = 0$. Hence, $V \equiv 0$. Note that $\langle W, \dot\gamma\rangle \equiv 0$ is not a restriction: any proper variation Γ can be reparametrized such that $W \perp \dot\gamma$.

Conclusion: γ is minimizing among "nearby paths".
Warning: γ may **not** be minimizing among **all** paths joining $\gamma(0)$ and $\gamma(b)$. For example, consider the cylinder:



7.5 Bonnet's theorem and Myers' theorem

We write $K_M \geq H$ if the sectional curvature $K(P) \geq H$ for all 2-planes $P \subset T_pM$ and $p \in M$.

Theorem 7.6. *Let M be a complete connected Riemannian n-manifold. Suppose that there exists $H > 0$ such that*

(1) (Bonnet, 1855): $K_M \geq H$; or

(2) (Myers, 1941): $\mathrm{Ric}(x) \geq (n-1)H$ *for every $x \in TM$, $|x| = 1$.*

Then there are conjugate points on every geodesic γ of length at least $\pi/\sqrt{H}$. In particular,

$$\mathrm{diam}(M) \leq \tfrac{\pi}{\sqrt{H}}.$$

Proof. If suffices to prove (2): Let $\gamma : [0,b] \to M$ be a unit speed geodesic with $b \geq \pi/\sqrt{H}$. Let $E_1, \ldots, E_n$ be an orthonormal parallel frame along γ such that $E_n = \dot\gamma$. We define

$$W_i(t) = \sin(\tfrac{\pi i}{b})E_i(t),$$

for $i = 1, 2, \ldots, n-1$. Then $W_i \in \mathcal{T}(\gamma)$, $W_i(0) = 0$ and $W_i(b) = 0$. Then (6.28) gives

$$\begin{aligned} I(W_i, W_i) &= -\int_0^b \langle D_t^2 W_i + R(W_i, \dot\gamma)\dot\gamma, W_i\rangle\, dt = \int_0^b \sin^2(\tfrac{\pi t}{b})\langle \tfrac{\pi^2}{b^2}E_i - R(E_i,\dot\gamma)\dot\gamma, E_i\rangle\, dt \\ &= \int_0^b \sin^2(\tfrac{\pi t}{b})\left(\tfrac{\pi^2}{b^2} - \langle R(E_i,\dot\gamma)\dot\gamma, E_i\rangle\right)\, dt. \end{aligned}$$

Hence,

$$\sum_{i=1}^{n-1} I(W_i, W_i) = \int_0^b \sin^2(\tfrac{\pi t}{b}) \left((n-1)\tfrac{\pi^2}{b^2} - \mathrm{Ric}(\dot\gamma)\right) dt.$$

On the other hand, $\mathrm{Ric}(\dot\gamma) \geq (n-1)H$ and $\frac{\pi^2}{b^2} \leq H$, so

$$\sum_{i=1}^{n-1} I(W_i, W_i) \leq 0.$$

Therefore, there exists $j = 1, 2, \ldots, n-1$ such that $I(W_j, W_j) \leq 0$. Suppose that there are no conjugate points on γ. Let V be the unique Jacobi field along γ such that $V_0 = W_j(0) = 0$ and $V_b = W_j(b) = 0$; hence $V \equiv 0$. Index lemma and the fact that $W_j \neq V$ implies that

$$I(W_j, W_j) > I(V, V) = 0,$$

which is a contradiction. Hence, there are conjugate points on γ. Suppose $\mathrm{diam}(M) > \frac{\pi}{\sqrt{H}}$. Then there exists $p, q \in M$ and a minimizing geodesic γ from $p = \gamma(0)$ to $q = \gamma(b)$ of length $b > \pi/\sqrt{H}$. We just proved that p is conjugate to $\gamma(t)$ for some $0 < t \leq \pi/\sqrt{H}$. By Theorem 6.29 we see that $\gamma|[0, b]$ is not minimizing, which is a contradiction. □

Corollary 7.7. *Let M be as in Theorem 7.6. Then M is compact and the fundamental group $\pi_1 M$ is finite.*

Proof. Let $\widetilde{M}$ be the universal covering space of M. Because $\pi : \widetilde{M} \to M$ is a local diffeomorphism, we see that $\widetilde{g} = \pi^* g$ is a Riemannian metric on $\widetilde{M}$ such that π is a local isometry. Because $\widetilde{M}$ is complete and satisfies the same conditions (1) or (2) as M does, we see that

$$\mathrm{diam}(\widetilde{M}) \leq \pi/\sqrt{H}.$$

Hence, $\widetilde{M}$ is bounded. However, $\widetilde{M}$ is also complete so it must be compact. Similarly, M is compact. Furthermore, for every $p \in M$ the set $\pi^{-1}(p)$ is finite since it is compact and discrete. Hence $\pi_1 M$ is finite because there is a one-to-one correspondence between $\pi^{-1}(p)$ and $\pi_1 M$. □

7.8 Cartan-Hadamard theorem

Lemma 7.9. *Let M be a complete connected Riemannian manifold with $K(P) \leq 0$ for every 2-planes $P \subset T_pM$ and $p \in M$. Then for all $p \in M$ the exponential map $\exp_p : T_pM \to M$ is a local diffeomorphism.*

Proof. Let $\gamma : [0, \infty) \to M$, $\gamma(0) = p$, be a geodesic and V a non-trivial Jacobi field along γ with $V_0 = 0$. Show that $V_t \neq 0$ for every $t > 0$ and conclude that for every $t > 0$ the point $\gamma(t)$ is not a conjugate to p. Details are left as an exercise. □

Remark 7.10. Theorem 6.29 can be used here.

Lemma 7.11. *Let $\widetilde{M}$ and M be connected Riemannian manifolds such that $\widetilde{M}$ is complete and there is a local isometry $\pi : \widetilde{M} \to M$. Then M is complete and π is a covering map.*

Remark 7.12. To show that π is a covering map, we need to show that every $p \in M$ has a neighborhood U such that $\pi^{-1}U$ is a disjoint union of sets U_α and $\pi|U_\alpha : U_\alpha \to U$ is a diffeomorphism for every α.

We will prove Lemma 7.11 later.

Theorem 7.13 (Cartan-Hadamard theorem)**.** *Let M be a complete connected Riemannian manifold with $K_M \leq 0$. Then for every $p \in M$ the exponential map $\exp_p : T_pM \to M$ is a covering map. Hence, the universal covering space $\widetilde{M}$ of M is diffeomorphic to $\mathbb{R}^n$.*

Proof of Theorem 7.13. Lemma 7.9 implies that $\exp_p$ is a local diffeomorphism. Hence, there exists a Riemannian metric on T_pM such that $\exp_p : T_pM \to M$ is a local isometry. The space T_pM with this metric is complete since geodesics of T_pM passing through origin 0 are straight lines. Now Lemma 7.11 implies that $\exp_p$ is a covering map. Furthermore, since the fundamental group $\pi_1(T_pM) = 0$, we know that $\widetilde{M}$ is diffeomorphic to T_pM, that is, to $\mathbb{R}^n$. □

Proof of Lemma 7.11. Recall the path-lifting property of covering maps: any path γ in M lifts to a path $\widetilde{\gamma}$ in $\widetilde{M}$ such that $\pi \circ \widetilde{\gamma} = \gamma$. We prove first that π has the **path-lifting property for geodesics**: Let $p \in M$, $\widetilde{p} \in \pi^{-1}(p)$, $\gamma : I \to M$ a geodesic such that $\gamma(0) = p$. Let $\widetilde{v} = \pi_*^{-1}\dot{\gamma}_0 \in T_{\widetilde{p}}(\widetilde{M})$; recall that $\pi_* : T_{\widetilde{p}}(\widetilde{M}) \to T_pM$ is an isomorphism. Let $\widetilde{\gamma} : \mathbb{R} \to \widetilde{M}$ be the geodesic with $\dot{\widetilde{\gamma}}_0 = \widetilde{v}$; recall that $\widetilde{M}$ is complete. Because π is a local isometry, we see that geodesics are mapped to geodesics. Hence $\pi \circ \widetilde{\gamma} = \gamma$ on I. Therefore, γ extends to all of $\mathbb{R}$, which implies the completeness of M.

$\boxed{\pi \text{ is surjective}}$ Choose $\widetilde{p} \in \widetilde{M}$ and write $p = \pi(\widetilde{p})$. Let $q \in M$ be arbitrary and write $r = d(p, q)$. Because M is connected and complete we know that there exists a minimizing geodesic $\gamma : [0, r] \to M$ with $\gamma(0) = p$ and $\gamma(r) = q$. Let $\widetilde{\gamma}$ be the lift of γ with $\widetilde{\gamma}(0) = \widetilde{p}$. Then $\pi(\widetilde{\gamma}(r)) = \gamma(r) = q$. Hence π is surjective.

$\boxed{\pi \text{ is a covering map}}$ Fix $p \in M$ and let $\pi^{-1}(p) = \{\widetilde{p}_\alpha\}$. Choose $r > 0$ such that $\overline{U} = \overline{B}(p, r)$ is contained in a normal neighborhood of p. Let $\widetilde{U}_\alpha = B(\widetilde{p}_\alpha, r) \subset \widetilde{M}$. We will show that

(1) the sets $\widetilde{U}_\alpha$ are disjoint;

(2) $\pi^{-1}U = \bigcup_\alpha \widetilde{U}_\alpha$; and

(3) $\pi|\widetilde{U}_\alpha : \widetilde{U}_\alpha \to U$ is a diffeomorphism for every α,

which finishes the proof.

(1): Take any $\widetilde{p}_\alpha, \widetilde{p}_\beta \in \pi^{-1}(p)$, $\widetilde{p}_\alpha \neq \widetilde{p}_\beta$. Because $\widetilde{M}$ is complete, there exists a minimizing geodesic $\widetilde{\gamma}$ from $\widetilde{p}_\alpha$ to $\widetilde{p}_\beta$. Because $\gamma = \pi \circ \widetilde{\gamma}$ is a geodesic from p to p, such γ must leave U and re-enter it since all geodesics in U passing through p are radial geodesics. Hence γ (and therefore $\widetilde{\gamma}$) has length at least $2r$. Therefore, $d(\widetilde{p}_\alpha, \widetilde{p}_\beta) \geq 2r$ so $\widetilde{U}_\alpha \cap \widetilde{U}_\beta = \emptyset$ due to triangle inequality.

(2): Because π is a local isometry, we know that $\pi(\widetilde{U}_\alpha) \subset U$ for every α. Hence, $\bigcup_\alpha \widetilde{U}_\alpha \subset \pi^{-1}U$. Thus we need to show that $\pi^{-1}U \subset \bigcup_\alpha \widetilde{U}_\alpha$. Let $\widetilde{q} \in \pi^{-1}U$. Then $q := \pi(\widetilde{q}) \in U$, so there exists a minimizing geodesic γ in U from $q = \gamma(0)$ to $p = \gamma(\varepsilon)$, with $\varepsilon := d(p, q) < r$. If $\widetilde{\gamma}$ is the lift of γ starting at $\widetilde{q} = \widetilde{\gamma}(0)$, then $\pi(\widetilde{\gamma}(\varepsilon)) = \gamma(\varepsilon) = p$. Therefore, $\widetilde{\gamma}(\varepsilon) = \widetilde{p}_\alpha$ for some α and $d(\widetilde{p}_\alpha, \widetilde{q}) \leq \ell(\widetilde{\gamma}) = \varepsilon < r$. So $\widetilde{q} \in \widetilde{U}_\alpha$, and $\pi^{-1}U \subset \bigcup_\alpha \widetilde{U}_\alpha$.

(3): For each α the map $\pi|\widetilde{U}_\alpha : \widetilde{U}_\alpha \to U$ is a local diffeomorphism. Moreover, it is bijective since its inverse is the map sending each radial geodesic starting at p to its lift starting at $\widetilde{p}_\alpha$.

□

Remark 7.14. A complete, simply-connected Riemannian manifold with nonpositive sectional curvature is called a **Cartan-Hadamard manifold**.

Corollary 7.15. *A Cartan-Hadamard n-manifold is diffeomorphic to $\mathbb{R}^n$.*

8 Comparison geometry

8.1 Rauch comparison theorem

Theorem 8.2 (Rauch)**.** *Let M^n and $\widetilde{M}^{n+k}$, $k \geq 0$, be Riemannian manifolds and let $\gamma : [0,b] \to M$, $\widetilde{\gamma} : [0,b] \to \widetilde{M}$ be unit speed geodesics such that $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}$. Suppose that for every $t \in [0,b]$, $v \in T_{\gamma(t)}M$ and $\widetilde{v} \in T_{\widetilde{\gamma}(t)}\widetilde{M}$ we have*

$$K(\dot{\gamma}_t, v) \leq K(\dot{\widetilde{\gamma}}_t, \widetilde{v}).$$

Let J and $\widetilde{J}$ be non-trivial Jacobi fields along γ and $\widetilde{\gamma}$, respectively, such that

$$J_0 = \lambda\dot{\gamma}_0, \quad \widetilde{J}_0 = \lambda\dot{\widetilde{\gamma}}_0, \quad \langle J_0', \dot{\gamma}_0\rangle = \langle \widetilde{J}_0', \dot{\widetilde{\gamma}}_0\rangle, \quad \text{and} \quad |J_0'| = |\widetilde{J}_0'|,$$

where $\lambda \in \mathbb{R}$ is a constant. Then

$$|J_t| \geq |\widetilde{J}_t|$$

for every $t \in [0,b]$.

Proof. First a special case

$$\langle J_t, \dot{\gamma}_t\rangle \equiv 0 \equiv \langle \widetilde{J}_t, \dot{\widetilde{\gamma}}_t\rangle. \tag{8.3}$$

Since

$$\langle J_t, \dot{\gamma}_t\rangle = \langle J_0, \dot{\gamma}_0\rangle + t\langle J_0, \dot{\gamma}_0\rangle = \langle \widetilde{J}_0, \dot{\widetilde{\gamma}}_0\rangle + t\langle \widetilde{J}_0, \dot{\widetilde{\gamma}}_0\rangle = \langle \widetilde{J}_t, \dot{\widetilde{\gamma}}_t\rangle,$$

we get from (8.3) that $J_0 = 0$, $\widetilde{J}_0 = 0$, $\langle J_0', \dot{\gamma}_0\rangle = 0$, and $\langle \widetilde{J}_0', \dot{\widetilde{\gamma}}_0\rangle = 0$. Because J and $\widetilde{J}$ are non-trivial, we have $J_0' \neq 0$ and $\widetilde{J}_0' \neq 0$. Since $\widetilde{\gamma}(0)$ has no conjugate points along $\widetilde{\gamma}$, we have $\widetilde{J}_t \neq 0$ for every $t > 0$. On the other hand, $J_0 = 0$ and $\widetilde{J}_0 = 0$ so by the l'Hôpital's rule there exists a limit

$$\lim_{t \downarrow 0} \frac{|J_t|^2}{|\widetilde{J}_t|^2} = \frac{|J_0'|^2}{|\widetilde{J}_0'|^2} = 1.$$

Hence, in order to prove $|J_t| \geq |\widetilde{J}_t|$ it is enough to show that

$$\frac{d}{dt}\left(\frac{|J|^2}{|\widetilde{J}|^2}\right) \geq 0$$

for every $t > 0$, that is,

$$\langle J_t', J_t\rangle |\widetilde{J}_t|^2 - \langle \widetilde{J}_t', \widetilde{J}_t\rangle |J_t|^2 \geq 0. \tag{8.4}$$

We define

$$\varphi(\widetilde{J})_t = \frac{\langle \widetilde{J}_t', \widetilde{J}_t\rangle}{\langle \widetilde{J}_t, \widetilde{J}_t\rangle}$$

for $t \in]0,b]$ and

$$\varphi(J)_t = \frac{\langle J_t', J_t\rangle}{\langle J_t, J_t\rangle}$$

for those $t > 0$ for which $J_t \neq 0$. Fix $t_1 \in [0,b]$. If $J_{t_1} = 0$, then (8.4) holds trivially. Therefore, we may assume that $J_{t_1} \neq 0$ and then define vector fields W^{t_1} (along γ) and $\widetilde{W}^{t_1}$ (along $\widetilde{\gamma}$) by

$$W_t^{t_1} = \frac{J_t}{|J_{t_1}|} \quad \text{and} \quad \widetilde{W}_t^{t_1} = \frac{\widetilde{J}_t}{|\widetilde{J}_{t_1}|}.$$

Then $\varphi(J)_t = \varphi(W^{t_1})_t$ whenever defined and $\varphi(\widetilde{J})_t = \varphi(\widetilde{W}^{t_1})_t$ for $t \in]0, b]$. Now

$$\varphi(J)_{t_1} = \varphi(W^{t_1})_{t_1} = \langle W^{t_1\prime}, W^{t_1}\rangle_{t_1}$$

$$\begin{aligned}
\varphi(J)_{t_1} = \varphi(W^{t_1})_{t_1} = \langle W^{t_1\prime}, W^{t_1}\rangle_{t_1} &= \langle W^{t_1\prime}, W^{t_1}\rangle_{t_1} - \underbrace{\langle W^{t_1\prime}, W^{t_1}\rangle_0}_{=0 \text{ since } J_0 = 0} = \int_0^{t_1} \langle W^{t_1\prime}, W^{t_1}\rangle_t' \, dt \\
&= \int_0^{t_1} \langle W^{t_1\prime}, W^{t_1\prime}\rangle_t + \langle W^{t_1\prime\prime}, W^{t_1}\rangle_t \, dt = \int_0^{t_1} \langle W^{t_1\prime}, W^{t_1\prime}\rangle_t - \langle R(W^{t_1}, \dot\gamma)\dot\gamma, W^{t_1}\rangle_t \, dt \\
&= \int_0^{t_1} \langle W^{t_1\prime}, W^{t_1\prime}\rangle_t - K\left(\dot\gamma_t, \tfrac{W_t^{t_1}}{|W_t^{t_1}|}\right) |W_t^{t_1}|^2 \, dt.
\end{aligned}$$

Let $P_t : T_{\gamma(0)}M \to T_{\gamma(t)}M$ and $\widetilde{P}_t : T_{\widetilde{\gamma}(0)}\widetilde{M} \to T_{\widetilde{\gamma}(t)}\widetilde{M}$ be parallel transports along γ and $\widetilde{\gamma}$, respectively. Let $I : T_{\gamma(0)}M \to T_{\widetilde{\gamma}(0)}\widetilde{M}$ be an injective linear map that preserves the inner product. Denote $I_t = \widetilde{P}_t \circ I \circ P_t^{-1} : T_{\gamma(t)}M \to T_{\widetilde{\gamma}(t)}\widetilde{M}$. Suppose that I is chosen such that

$$I(\dot\gamma_0) = \dot{\widetilde{\gamma}}_0 \quad \text{and} \quad I_{t_1}(W_{t_1}^{t_1}) = \widetilde{W}_{t_1}^{t_1}.$$

Define $\widehat{W}^{t_1}$ to be a vector field along $\widetilde{\gamma}$ by

$$\widehat{W}_t^{t_1} := I_t W_t^{t_1}.$$

Note that now $\widehat{W}_{t_1}^{t_1} = \widetilde{W}_{t_1}^{t_1}$. Let $E_1, \ldots, E_n \in \mathcal{T}(\gamma)$ and $\widetilde{E}_1, \ldots, \widetilde{E}_n \in \mathcal{T}(\widetilde{\gamma})$ be parallel along γ and $\widetilde{\gamma}$ such that $E_n = \dot\gamma$, $\widetilde{E}_n = \dot{\widetilde{\gamma}}$, $I(E_i(0)) = \widetilde{E}_i(0)$, and that $\{E_1(t), \ldots, E_n(t)\}$ spans $T_{\gamma(t)}M$ for every t. Write $W^{t_1} = \sum_i f_i E_i$. Then

$$\widehat{W}^{t_1} = \sum_i f_i \widetilde{E}_i.$$

Hence,

(a) $|W_t^{t_1}| = |\widehat{W}_t^{t_1}|$ for every t.

Since $W^{t_1\prime} = \sum_i f_i' E_i$ and $\widehat{W}^{t_1\prime} = \sum_i f_i' \widetilde{E}_i$, we have

(b) $\langle W^{t_1\prime}, W^{t_1\prime}\rangle = \sum_{i,j} f_i' f_j' \langle E_i, E_j\rangle = \sum_{i,j} f_i' f_j' \langle \widetilde{E}_i, \widetilde{E}_j\rangle = \langle \widehat{W}^{t_1\prime}, \widehat{W}^{t_1\prime}\rangle$.

Now (a) and (b) together with the curvature assumption and the Index Lemma 7.2 imply

$$\begin{aligned}
\varphi(J)_{t_1} &= \int_0^{t_1} \langle W^{t_1\prime}, W^{t_1\prime}\rangle_t - K\left(\dot\gamma_t, \tfrac{W_t^{t_1}}{|W_t^{t_1}|}\right) |W_t^{t_1}|^2 \, dt \\
&\geq \int_0^{t_1} \langle \widehat{W}^{t_1\prime}, \widehat{W}^{t_1\prime}\rangle_t - K\left(\dot{\widetilde{\gamma}}_t, \tfrac{\widehat{W}_t^{t_1}}{|\widehat{W}_t^{t_1}|}\right) |\widehat{W}_t^{t_1}|^2 \, dt \\
&= \int_0^{t_1} \langle \widehat{W}^{t_1\prime}, \widehat{W}^{t_1\prime}\rangle_t - \langle \widetilde{R}(\widehat{W}_t^{t_1}, \dot{\widetilde{\gamma}}_t)\dot{\widetilde{\gamma}}_t, \widehat{W}_t^{t_1}\rangle \, dt \\
&\overset{7.2}{\geq} \int_0^{t_1} \langle \widetilde{W}^{t_1\prime}, \widetilde{W}^{t_1\prime}\rangle_t - \langle \widetilde{R}(\widetilde{W}_t^{t_1}, \dot{\widetilde{\gamma}}_t)\dot{\widetilde{\gamma}}_t, \widetilde{W}_t^{t_1}\rangle \, dt \\
&= \varphi(\widetilde{J})_{t_1},
\end{aligned}$$

that is, (8.4) holds at t_1. Because $t_1 \in [0,b]$ is arbitrary, we have $|J_t| \geq |\widetilde{J}_t|$ in the special case.

In general case,

$$J = J^{\perp} + \langle J, \dot\gamma\rangle\dot\gamma \quad \text{and} \quad \widetilde{J} = \widetilde{J}^{\perp} + \langle \widetilde{J}, \dot{\widetilde{\gamma}}\rangle\dot{\widetilde{\gamma}}.$$

The first part then gives $|J^{\perp}| \geq |\widetilde{J}^{\perp}|$. On the other hand,

$$\langle J, \dot\gamma\rangle_t = \underbrace{\langle J, \dot\gamma\rangle_0}_{=\lambda} + t\langle J', \dot\gamma\rangle_0 = \underbrace{\langle \widetilde{J}, \dot{\widetilde{\gamma}}\rangle_0}_{=\lambda} + t\langle \widetilde{J}', \dot{\widetilde{\gamma}}\rangle_0 = \langle \widetilde{J}, \dot{\widetilde{\gamma}}\rangle_t.$$

Hence, $|J| \geq |\widetilde{J}|$. □

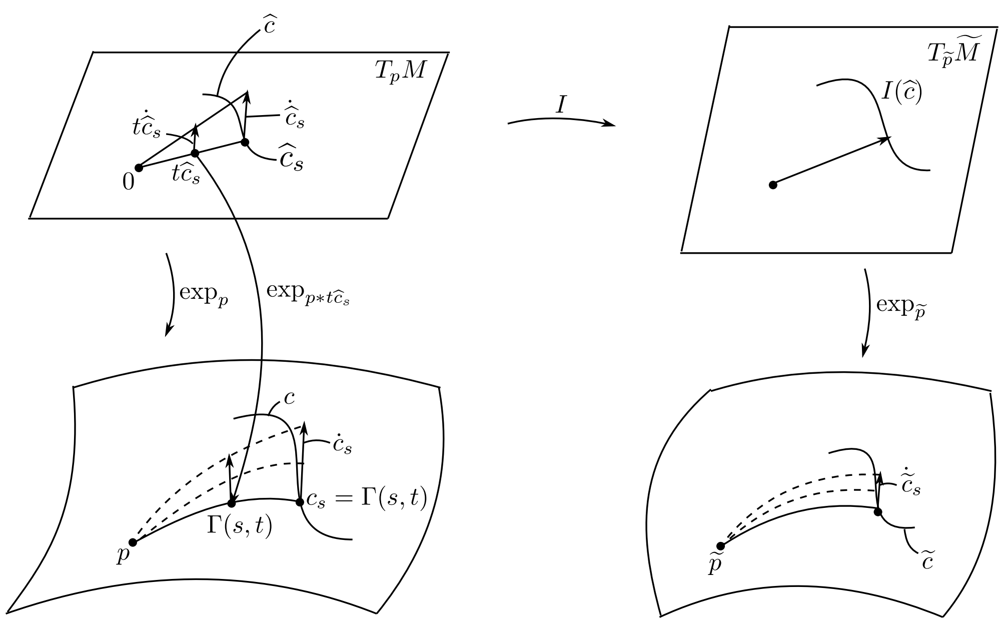
Corollary 8.5. *Let M and $\widetilde{M}$ be Riemannian manifolds with $\dim \widetilde{M} \geq \dim M$, and let $p \in M$ and $\widetilde{p} \in \widetilde{M}$. Assume $K_{\widetilde{M}} \geq K_M$ and let $I : T_pM \to T_{\widetilde{p}}\widetilde{M}$ be a linear injection preserving the inner product. Let $r > 0$ be so small that $\exp_p|B(0,r)$ is an embedding and $\exp_{\widetilde{p}}|B(0,r)$ is non-singular. Then for any piecewise C^∞-path $c : [0,1] \to \exp_p B(0,r)$ we have*

$$\ell(c) \geq \ell(\underbrace{\exp_{\widetilde{p}} \circ I \circ \exp_p^{-1} \circ c}_{=:\widetilde{c}}) = \ell(\widetilde{c}).$$

Above the assumption $\exp_{\widetilde{p}}|B(0,r)$ being non-singular means that $\exp_{\widetilde{p}} B(0,r)$ contains no conjugate points to $\tilde{p}$.

Proof. Denote $\widehat{c} : [0,1] \to B(0,r) \subset T_pM$ by

$$\widehat{c} := \exp_p^{-1} \circ c.$$



Consider the variation $\Gamma(s,t) = \exp_p(t\widehat{c}_s)$. For each fixed s, the variation field

$$V_t^s := \partial_s\Gamma(s,t)$$

is a Jacobi field along geodesic Γ_s, $t \mapsto \exp_p(t\widehat{c}_s)$. Then

$$V_t^s = \tfrac{d}{ds}\Gamma(s,t) = t\exp_{p*t\widehat{c}_s}(\dot{\widehat{c}}_s);$$

$$V_0^s = 0;$$
$$V_1^s = \frac{d}{ds}\underbrace{(\exp_p \widehat{c}_s)}_{=c_s} = \dot{c}_s; \quad \text{and}$$
$$(D_t V^s)_0 = D_t(t \exp_{p*t\widehat{c}_s}(\dot{\widehat{c}}_s))|_{t=0} = \dot{\widehat{c}}_s.$$

Consider next the variation
$$\widetilde{\Gamma}(s,t) = \exp_{\widetilde{p}}(I(t\widehat{c}_s)) = \exp_{\widetilde{p}}(tI(\widehat{c}_s)).$$
Again, for each fixed s, the variation field
$$\widetilde{V}_s^t = \partial_s \widetilde{\Gamma}(s,t)$$
is a Jacobi field along $\widetilde{\Gamma}_s$, $t \mapsto \exp_{\widetilde{p}}(tI(\widehat{c}_s))$, with
$$\widetilde{V}_0^s = 0, \quad \widetilde{V}_1^s = \dot{\widetilde{c}}_s, \quad \text{and} \quad (D_t\widetilde{V}^s)_0 = I(\dot{\widehat{c}}_s).$$
Since I preserves the inner product,
$$|(D_t V^s)_0| = |\dot{\widehat{c}}_s| = |I(\dot{\widehat{c}}_s| = |(D_t \widetilde{V}^s)_0|$$
and
$$\langle \dot{\Gamma}_s, D_t V^s \rangle_0 = \langle \dot{\Gamma}_s(0), \underbrace{(D_t V^s)_0}_{=\dot{\widehat{c}}_s} \rangle = \langle I(\dot{\Gamma}_s(0)), I(\dot{\widehat{c}}_s)\rangle$$
$$= \langle I(\widehat{c}_s), (D_t\widetilde{V}^s)_0\rangle = \langle \dot{\widetilde{\Gamma}}_s(0), (D_t\widetilde{V}^s)_0\rangle = \langle \dot{\widetilde{\Gamma}}_s, D_t\widetilde{V}^s\rangle_0.$$
Furthermore,
$$V_0^s = 0 \quad \text{and} \quad \widetilde{V}_0^s = 0.$$
The Rauch comparison theorem now implies
$$|\dot{c}_s| = |V_1^s| \geq |\widetilde{V}_1^s| = |\dot{\widetilde{c}}_s|.$$
Since s is arbitrary, we have the claim. □

Corollary 8.6. *Suppose that the sectional curvatures of M satisfy*
$$0 < \kappa \leq K_M \leq \delta$$
for some constants κ and δ. Let γ be a geodesic in M. Then the distance d between two consecutive conjugate points along γ satisfies
$$\frac{\pi}{\sqrt{\delta}} \leq d \leq \frac{\pi}{\sqrt{\kappa}}.$$

Proof. Let $\gamma : [0,\ell] \to M$ be a unit speed geodesic with $\gamma(0) = p$. Let J be a Jacobi field along γ, with $J_0 = 0$ and $\langle J, \dot{\gamma}\rangle \equiv 0$. Let $S^n(\delta)$ be the sphere with constant sectional curvature δ. Fix $\widetilde{p} \in S^n(\delta)$ and a unit speed geodesic $\widetilde{\gamma} : [0,\ell] \to S^n(\delta)$ with $\widetilde{\gamma}(0) = \widetilde{p}$. Let $\widetilde{J}$ be a Jacobi field along $\widetilde{\gamma}$ with $\widetilde{J}_0 = 0$, $\langle \widetilde{J}, \dot{\widetilde{\gamma}}\rangle \equiv 0$ and $|\widetilde{J}_0'| = |J_0'|$. Since $\widetilde{\gamma}$ has no conjugate pairs in $(0, \frac{\pi}{\sqrt{\delta}})$, we have
$$|J_t| \geq |\widetilde{J}_t| > 0$$
for any $t \in (0, \frac{\pi}{\sqrt{\delta}})$ by the Rauch comparison theorem. Therefore, the distance d from p to its first conjugate point along γ satisfies
$$d \geq \frac{\pi}{\sqrt{\delta}}.$$
If $d > \frac{\pi}{\sqrt{\kappa}}$, we get by applying the Rauch comparison theorem to M and $S^n(\kappa)$ that the distance between any pairs of conjugate points in $S^n(\kappa)$ is strictly greater than $\frac{\pi}{\sqrt{\kappa}}$, which is a contradiction. □

8.7 Hessian and Laplace comparison

Recall that the gradient, Hessian, and Laplacian of $f \in C^\infty(M)$ are defined by

$$\langle \nabla f, X\rangle := X(f), \quad \operatorname{Hess} f(X,Y) := \langle \nabla_X(\nabla f), Y\rangle, \quad \text{and} \quad \Delta f := \operatorname{div}(\nabla f) = \operatorname{tr}(v \mapsto \nabla_v(\nabla f)),$$

and that $\nabla f \in \mathcal{T}(M)$, $\operatorname{Hess} f \in \mathcal{T}^2(M)$, and $\Delta f \in C^\infty(M)$. Furthermore, $\operatorname{Hess} f$ is symmetric and

$$\operatorname{Hess} f(X,Y) = X(Yf) - (\nabla_X Y)f.$$

If $(V, \langle\cdot,\cdot\rangle)$ is an n-dimensional inner product space and $B : V \times V \to \mathbb{R}$ is bilinear, then the **trace** of B, the **determinant** of B, and the **norm** of B with respect to $\langle\cdot,\cdot\rangle$ are defined by

$$\operatorname{tr} B = \operatorname{tr} L, \quad \det B = \det L, \quad \text{and} \quad |B| = \sqrt{\operatorname{tr}(LL^*)},$$

where $L : V \to V$ is linear such that

$$B(x,y) = \langle Lx, y\rangle$$

for every $x, y \in V$.

Hence,

$$\Delta f = \operatorname{tr} \operatorname{Hess} f$$

with respect to Riemannian metric $\langle\cdot,\cdot\rangle$.

Definition 8.8. The **injectivity radius** at $p \in M$ is defined as

$$\operatorname{inj}(p) := \sup\{r \in \mathbb{R} : \exp_p |B(0,r) \text{ is diffeomorphism}\},$$

which is always positive since $\exp_p$ is a local diffeomorphism at $0 \in T_pM$.

Example 8.9. If M is a Cartan-Hadamard manifold, then $\operatorname{inj}(p) = +\infty$ for each $p \in M$.

Theorem 8.10 (Hessian comparison theorem)**.** *Let M^n and $\widetilde{M}^{n+k}$, $k \geq 0$, be Riemannian manifolds and let $\gamma : [0,b] \to M$ and $\widetilde{\gamma} : [0,b] \to \widetilde{M}$ be unit speed geodesics such that*

$$b < \min\{\operatorname{inj}(\gamma(0)), \operatorname{inj}(\widetilde{\gamma}(0))\}.$$

Suppose that

$$K(\dot{\gamma}_t, v) \leq K(\dot{\widetilde{\gamma}}_t, \widetilde{v})$$

for every $t \in [0,b]$, $v \in T_{\gamma(t)}M$, and $\widetilde{v} \in T_{\widetilde{\gamma}(t)}\widetilde{M}$. If $h : [0,\infty) \to \mathbb{R}$ is smooth and increasing, $r_M := d(\cdot, \gamma(0))$, and $r_{\widetilde{M}} := d(\cdot, \widetilde{\gamma}(0))$, then

$$\operatorname{Hess}(h \circ r_M)(X,X) \geq \operatorname{Hess}(h \circ r_{\widetilde{M}})(\widetilde{X}, \widetilde{X})$$

for all $t \in (0,b]$, $X \in T_{\gamma(t)}M$, and $\widetilde{X} \in T_{\widetilde{\gamma}(t)}\widetilde{M}$ such that $|X| = |\widetilde{X}|$ and $\langle \dot{\gamma}_t, X\rangle = \langle \dot{\widetilde{\gamma}}_t, \widetilde{X}\rangle$.

Proof. First of all, $h \circ r_M$ is smooth in $B(\gamma(0), b) \setminus \{\gamma(0)\}$, and $h \circ r_{\widetilde{M}}$ is smooth in $B(\widetilde{\gamma}(0), b) \setminus \{\widetilde{\gamma}(0)\}$, respectively. We may assume that $|X| = 1 = |\widetilde{X}|$ and that $t = b$.

1° Case $h(t) = t$. For every $v \in T_{\gamma(b)}M$,

$$\begin{aligned}\operatorname{Hess} r_M(v, \dot\gamma_b) &= v(\underbrace{\dot\gamma_b r_M}_{\equiv 1}) - (\nabla_v \dot\gamma) r_M = -(\nabla_v \dot\gamma) r_M \\ &= -\langle \nabla r_M, \nabla_v \dot\gamma \rangle_{\gamma(b)} = -\langle \dot\gamma_b, \nabla_v \dot\gamma \rangle_{\gamma(b)} = -\tfrac{1}{2} v \underbrace{\langle \dot\gamma_t, \dot\gamma_t \rangle}_{\equiv 1} = 0.\end{aligned}$$

Similarly,

$$\operatorname{Hess} r_{\widetilde M}(\widetilde v, \dot{\widetilde\gamma}_b) = 0$$

for every $\widetilde v \in T_{\widetilde\gamma(b)}\widetilde M$. Write $X = X^\top + X^\perp$ and $\widetilde X = \widetilde X^\top + \widetilde X^\perp$, where

$$X^\top := \langle X, \dot\gamma_b \rangle \dot\gamma_b \quad \text{and} \quad \widetilde X^\top := \langle \widetilde X, \dot{\widetilde\gamma}_b \rangle \dot{\widetilde\gamma}_b.$$

Then

$$\operatorname{Hess} r_M(X, X) = \operatorname{Hess} r_M(X^\perp, X^\perp) \quad \text{and} \quad \operatorname{Hess} r_{\widetilde M}(\widetilde X, \widetilde X) = \operatorname{Hess} r_{\widetilde M}(\widetilde X^\perp, \widetilde X^\perp).$$

Hence, we may assume $X = X^\perp$ and $\widetilde X = \widetilde X^\perp$. Let $\sigma := \gamma^X$, which is a geodesic such that $\dot\sigma_0 = X$ and $\widetilde\sigma := \gamma^{\widetilde X}$, which is a geodesic such that $\dot{\widetilde\sigma}_0 = \widetilde X$, respectively. Let $\Gamma : [-\varepsilon, \varepsilon] \times [0, b] \to M$ be the variation of γ through geodesics such that $\Gamma_s : t \mapsto \Gamma(s, t)$ is the geodesic from $\gamma(0)$ to $\sigma(s)$. Similarly, we define $\widetilde\Gamma : [-\varepsilon, \varepsilon] \times [0, b] \to \widetilde M$.

Then the variation field J of Γ and the variation field $\widetilde J$ of $\widetilde\Gamma$ are Jacobi fields. This implies that the mappings $s \mapsto \langle J_s, \dot\gamma_s \rangle$ and $s \mapsto \langle \widetilde J_s, \dot{\widetilde\gamma}_s \rangle$ are affine. Furthermore, because $J_0 = 0$, $J_b = X \perp \dot\gamma_b$ and $\widetilde J_0 = 0$ and $\widetilde J_b = \widetilde X \perp \dot{\widetilde\gamma}_b$, respectively, we have

$$J_s \perp \dot\gamma_s \quad \text{and} \quad \widetilde J_s \perp \dot{\widetilde\gamma}_s$$

for every $s \in [0, b]$. By an exercise

$$\operatorname{Hess} r_M(X, X) = (r_M \circ \sigma)''(0) = \tfrac{d^2}{ds^2} \ell(\Gamma_s)|_{s=0} = \int_0^b |D_t J|^2 - \langle R(J, \dot\gamma)\dot\gamma, J \rangle \, dt.$$

Similarly,

$$\operatorname{Hess} r_{\widetilde M}(\widetilde X, \widetilde X) = \int_0^b |D_t \widetilde J|^2 - \langle R(\widetilde J, \dot{\widetilde\gamma})\dot{\widetilde\gamma}, \widetilde J \rangle \, dt.$$

Fix orthonormal bases $\{e_i\}_{i=1}^n$ of $T_{\gamma(b)}M$ and $\{\widetilde e_i\}_{i=1}^{n+k}$ of $T_{\widetilde\gamma(b)}\widetilde M$ such that $e_1 = X$ and $\widetilde e_1 = \widetilde X$. Let E_i be the parallel transport of e_i along γ and $\widetilde E_i$ be the parallel transport of $\widetilde e_i$ along $\widetilde\gamma$. Then $\{E_i(t)\}_{i=1}^n$ is an orthonormal basis of $T_{\gamma(t)}M$ for every $t \in [0, b]$ and $\{\widetilde E_i(t)\}_{i=1}^{n+k}$ is an orthonormal basis of $T_{\widetilde\gamma(t)}\widetilde M$ for every $t \in [0, b]$. Define functions h_i, $1 \le i \le n$, by

$$h_i(t) := \langle J_t, E_i(t) \rangle_{\gamma(t)}.$$

Then

$$J_t = \sum_{i=1}^n h_i(t) E_i(t).$$

Define

$$\widetilde W := \sum_{i=1}^n h_i \widetilde E_i.$$

Since $J_0 = 0$, we have

$$\widetilde{W}_0 = \sum_{i=1}^{n} \underbrace{h_i(0)}_{=0} \widetilde{E}_i(0) = 0 = \widetilde{J}_0.$$

Furthermore, since $J_b = X = e_1 = E_1(b)$, we have $h_1(b) = 1$ and $h_i(b) = 0$ for $i \neq 1$, which gives

$$\widetilde{W}_b = \widetilde{E}_1(b) = \widetilde{e}_1 = \widetilde{X} = \widetilde{J}_b.$$

Since $b < \min\{\operatorname{inj}(\gamma(0)), \operatorname{inj}(\widetilde{\gamma}(0))\}$, there are no conjugate points of $\gamma(0)$ (resp. $\widetilde{\gamma}(0)$) along $\gamma|[0,b)$ (resp. $\widetilde{\gamma}|[0,b)$). The Index Lemma gives

(8.11)

$$\operatorname{Hess} r_{\widetilde{M}}(\widetilde{X}, \widetilde{X}) = \int_0^b |D_t \widetilde{J}|^2 - \langle R(\widetilde{J}, \dot{\widetilde{\gamma}})\dot{\widetilde{\gamma}}, \widetilde{J}\rangle \, dt \le I(\widetilde{W}, \widetilde{W}) = \int_0^b |D_t \widetilde{W}|^2 - \langle R(\widetilde{W}, \dot{\widetilde{\gamma}})\dot{\widetilde{\gamma}}), \widetilde{W}\rangle \, dt.$$

Furthermore, on $[0,b]$ we have

$$|D_t \widetilde{W}|^2 = \sum_{i=1}^{n} |h_i'|^2 = |D_t J|^2, \quad |\widetilde{W}| = |J|, \quad \widetilde{W} \perp \dot{\widetilde{\gamma}}, \quad \text{and} \quad J \perp \dot{\gamma}.$$

Hence, the assumption $K(v, \dot{\gamma}_t) \le K(\widetilde{v}, \dot{\widetilde{\gamma}}_t)$ implies

$$-\langle R(\widetilde{W}, \dot{\widetilde{\gamma}})\dot{\widetilde{\gamma}}, \widetilde{W}\rangle \le -\langle R(J, \dot{\gamma})\dot{\gamma}, J\rangle$$

on $[0,b]$. Thus we get from (8.11)

$$\operatorname{Hess} r_{\widetilde{M}}(\widetilde{X}, \widetilde{X}) \le \operatorname{Hess} r_M(X, X).$$

2° The general case, that is, h is smooth and increasing. As an exercise we have

$$\operatorname{Hess}(h \circ f) = (h'' \circ f) df \otimes df + (h' \circ f) \operatorname{Hess} f,$$

if $f : M \to \mathbb{R}$ and $h : R \to R$ are smooth mappings. Hence,

$$\begin{aligned}
\operatorname{Hess}(h \circ r_M)(X, X) &= (h'' \circ r_M)(b) dr_M \otimes dr_M(X, X) + (h' \circ r_M)(b) \operatorname{Hess} r_M(X, X) \\
&= h''(b)(\underbrace{dr_M(X)}_{= dr_{\widetilde{M}}(\widetilde{X})})^2 + \underbrace{(h' \circ r_M)(b)}_{= (h' \circ r_{\widetilde{M}})(b) \ge 0} \operatorname{Hess} r_M(X, X) \\
&\ge (h'' \circ r_{\widetilde{M}}) dr_{\widetilde{M}} \otimes dr_{\widetilde{M}}(\widetilde{X}, \widetilde{X}) + (h' \circ r_{\widetilde{M}})(b) \operatorname{Hess} r_{\widetilde{M}}(\widetilde{X}, \widetilde{X}) \\
&= \operatorname{Hess}(h \circ r_{\widetilde{M}})(\widetilde{X}, \widetilde{X}).
\end{aligned}$$

□

Corollary 8.12. *Let M^n and $\widetilde{M}^n$ be Riemannian n-manifolds, $\gamma : [0,b] \to M$ and $\widetilde{\gamma} : [0,b] \to \widetilde{M}$ be unit speed geodesics such that*

$$b < \min\{\operatorname{inj}(\gamma(0)), \operatorname{inj}(\widetilde{\gamma}(0))\}.$$

Suppose that for every $t \in [0,b]$, $v \in T_{\gamma(t)}M$ and $\widetilde{v} \in T_{\widetilde{\gamma(t)}}\widetilde{M}$, we have

$$K(\dot{\gamma}_t, v) \le K(\dot{\widetilde{\gamma}}_t, \widetilde{v}).$$

If $h : [0,\infty) \to \mathbb{R}$ is smooth and increasing, we have

$$\Delta(h \circ r_M)(\gamma(t)) \geq \Delta(h \circ r_{\widetilde{M}})(\widetilde{\gamma}(t))$$

for every $t \in [0,b]$, where $r_M := d(\cdot,\gamma(0))$ and $r_{\widetilde{M}} := d(\cdot,\widetilde{\gamma}(0))$.

Proof. Fix $t \in [0,b]$ and orthonormal bases $\{X_i\}_{i=1}^n$ of $T_{\gamma(t)}M$ and $\{\widetilde{X}_i\}_{i=1}^n$ of $T_{\widetilde{\gamma}(t)}\widetilde{M}$ such that $X_1 = \dot{\gamma}_t$ and $\widetilde{X}_1 = \dot{\widetilde{\gamma}}_t$. The Hessian comparison implies

$$\operatorname{Hess}(h \circ r_M)(X_i, X_i) \geq \operatorname{Hess}(h \circ r_{\widetilde{M}})(\widetilde{X}_i, \widetilde{X}_i)$$

for every $1 \leq i \leq n$. Thus

$$\Delta(h \circ r_M)(\gamma(t)) = \sum_{i=1}^n \operatorname{Hess}(h \circ r_M)(X_i, X_i) \geq \sum_{i=1}^n \operatorname{Hess}(h \circ r_{\widetilde{M}})(\widetilde{X}_i, \widetilde{X}_i) = \Delta(h \circ r_{\widetilde{M}})(\widetilde{\gamma}(t)).$$

□

8.13 Bochner-Weitzenböck-Lichnerowitz formula

Theorem 8.14. *Let M be a Riemannian manifold. Then for every $f \in C^\infty(M)$*

$$\tfrac{1}{2}\Delta(|\nabla f|^2) = |\operatorname{Hess} f|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + \operatorname{Ric}(\nabla f, \nabla f).$$

Proof. Fix $p \in M$ and let $E_1,\dots,E_n$ be a local geodesic frame at p, that is, $E_1,\dots,E_n \in \mathcal{T}(U)$, $U \ni p$ open, $\langle E_i, E_j\rangle = \delta_{ij}$ in U, and $(\nabla_{E_i}E_j)_p = 0$. Then

$$\nabla h = \sum_{i=1}^n E_i(h)E_i$$

for every $h \in C^\infty(U)$. Now **at** p we have

$$\tfrac{1}{2}\Delta(|\nabla f|^2) = \tfrac{1}{2}\operatorname{div}(\nabla(|\nabla f|^2)) = \tfrac{1}{2}\operatorname{tr}(T_pM \ni v \mapsto \nabla_v(\nabla(|\nabla f|^2))) = \frac{1}{2}\sum_{i=1}^n \langle \nabla_{E_i}(\nabla(|\nabla f|^2)), E_i\rangle.$$

Moreover,

$$\begin{aligned}\nabla_{E_i}(\nabla(|\nabla f|^2)) = \nabla_{E_i}\left(\sum_{j=1}^n E_j(|\nabla f|^2)E_j\right) &= \sum_{j=1}^n E_j(|\nabla f|^2)\underbrace{\nabla_{E_i}E_j}_{=0 \text{ at } p} + \sum_{j=1}^n E_i(E_j(|\nabla f|^2))E_j \\ &= \sum_{j=1}^n E_i(E_j(|\nabla f|^2))E_j.\end{aligned}$$

Hence,

$$\begin{aligned}
\tfrac{1}{2}\Delta(|\nabla f|^2) &= \frac{1}{2}\sum_{i=1}^{n}\left\langle \sum_{j=1}^{n} E_i(E_j(|\nabla f|^2))E_j, E_i\right\rangle = \frac{1}{2}\sum_{i=1}^{n} E_i(E_i(|\nabla f|^2)) \\
&= \frac{1}{2}\sum_{i=1}^{n} E_i(E_i\langle \nabla f, \nabla f\rangle) = \sum_{i=1}^{n} E_i\langle \nabla_{E_i}\nabla f, \nabla f\rangle = \sum_{i=1}^{n} E_i(\operatorname{Hess} f(E_i, \nabla f)) \\
&= \sum_{i=1}^{n} E_i(\operatorname{Hess} f(\nabla f, E_i)) = \sum_{i=1}^{n} E_i\langle \nabla_{\nabla f}\nabla f, E_i\rangle \\
&= \sum_{i=1}^{n}\Big(\langle \nabla_{E_i}\nabla_{\nabla f}\nabla f, E_i\rangle + \langle \nabla_{\nabla f}\nabla f, \underbrace{\nabla_{E_i}E_i}_{=0 \text{ at } p}\rangle\Big) = \sum_{i=1}^{n}\langle \nabla_{E_i}\nabla_{\nabla f}\nabla f, E_i\rangle \\
&= \underbrace{\sum_{i=1}^{n}\langle R(E_i, \nabla f)\nabla f, E_i\rangle}_{=:A} + \underbrace{\sum_{i=1}^{n}\langle \nabla_{\nabla f}\nabla_{E_i}\nabla f, E_i\rangle}_{=:B} + \underbrace{\sum_{i=1}^{n}\langle \nabla_{[E_i, \nabla f]}\nabla f, E_i\rangle}_{=:C}.
\end{aligned}$$

First of all,

$$A = \operatorname{Ric}(\nabla f, \nabla f).$$

Secondly,

$$\begin{aligned}
B &= \sum_{i=1}^{n}\Big((\nabla f)\langle \nabla_{E_i}\nabla f, E_i\rangle - \langle \nabla_{E_i}\nabla f, \underbrace{\nabla_{\nabla f}E_i}_{\overset{(*)}{=}0.}\rangle\Big) = (\nabla f)\underbrace{\sum_{i=1}^{n}\langle \nabla_{E_i}\nabla f, E_i\rangle}_{=\operatorname{tr}(v\mapsto \nabla_v\nabla f)=\Delta f} \\
&= (\nabla f)(\Delta f) = \langle \nabla f, \nabla(\Delta f)\rangle,
\end{aligned}$$

where $(*)$ is because

$$\nabla_{\nabla f}E_i = \nabla_{\sum_j E_j(f)E_j}E_i = \sum_j E_j(f)\underbrace{\nabla_{E_j}E_i}_{=0} = 0.$$

Lastly,

$$\begin{aligned}
C &= \sum_{i=1}^{n}\operatorname{Hess} f([E_i, \nabla f], E_i) = \sum_{i=1}^{n}\operatorname{Hess} f(\nabla_{E_i}\nabla f - \underbrace{\nabla_{\nabla f}E_i}_{=0}, E_i) \\
&= \sum_{i=1}^{n}\operatorname{Hess} f(\nabla_{E_i}\nabla f, E_i) = \sum_{i=1}^{n}\operatorname{Hess} f(E_i, \nabla_{E_i}\nabla f) \\
&= \sum_{i=1}^{n}\langle \nabla_{E_i}\nabla f, \nabla_{E_i}\nabla f\rangle \overset{(**)}{=} |\operatorname{Hess} f|^2.
\end{aligned}$$

Here $(**)$ holds since

$$\sum_{i=1}^{n}\langle \nabla_{E_i}\nabla f, \nabla_{E_i}\nabla f\rangle = \sum_{i=1}^{n}\langle LE_i, LE_i\rangle = \sum_{i=1}^{n}\langle LL^*E_i, E_i\rangle = \operatorname{tr}(LL^*),$$

where $L : T_pM \to T_pM$, $Lv = \nabla_v\nabla f$, is linear such that

$$\operatorname{Hess} f(v, w) = \langle Lv, w\rangle$$

for every $v, w \in T_pM$. □

Definition 8.15. Let $p \in M$.

(a) Let $v \in T_pM$, $|v| = 1$. The **distance to the cut point of p along γ^v** is

$$d(v) := \sup\{t > 0 : tv \in \mathcal{E}_p \text{ and } d(p, \gamma^v(t)) = t\}.$$

(b) The **cut locus of p in T_pM** is

$$C_p := \{d(v)v : v \in T_pM, \ |v| = 1, \text{ and } d(v) < \infty\}.$$

(c) The **cut locus of p in M** is

$$C(p) := \exp_p(C_p \cap \mathcal{E}_p).$$

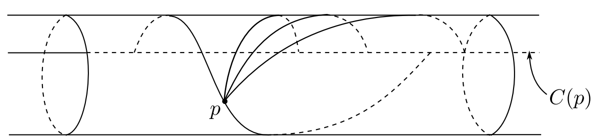
We write also

$$D_p := \{tv : v \in T_pM, \ |v| = 1, \text{ and } 0 \le t < d(v)\}$$

and

$$D(p) := \exp_p D_p.$$

Example 8.16. The cylinder $\mathbb{R} \times \mathbb{S}^1$: $C(p)$ is the line "opposite to p".



8.17 Riemannian volume form

This section is based on the pro gradu thesis of Aleksi Vähäkangas.

Let M and N be smooth oriented Riemannian n-manifolds and $f : M \to N$ smooth. The **Jacobian determinant** of f at $p \in M$ is

$$J_f(p) := \det D(y \circ f \circ x^{-1})(x(p)),$$

where x and y are orientation-preserving charts at p and $f(p)$, respectively, such that $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ and $\{\frac{\partial}{\partial y^i}\}_{i=1}^n$ form orthonormal bases of T_pM and $T_{f(p)}N$.

The Jacobian determinant $J_f(p)$ is well-defined, i.e. it does not depend on charts x and y (Exercise).

Let then (U, x) and (U, y) be charts on M. For the Jacobians of x and y, we have

$$J_y = (J_{y \circ x^{-1}} \circ x) J_x.$$

Hence,

$$\begin{aligned} dy^1 \wedge \cdots \wedge dy^n(\tfrac{\partial}{\partial x^1}, \ldots, \tfrac{\partial}{\partial x^n}) &= \det(dy^j(\tfrac{\partial}{\partial x^i})) = \det(D_i(y^j \circ x^{-1}) \circ x) \\ &= \det(D_i(y \circ x^{-1})^j \circ x) = J_{y \circ x^{-1}} \circ x = J_y / J_x. \end{aligned}$$

So,

$$\frac{1}{J_y} dy^1 \wedge \cdots \wedge d^n = \frac{1}{J_x} dx^1 \wedge \cdots \wedge dx^n.$$

Definition 8.18. The **Riemannian volume form** of M is the smooth n-form ω_M such that

$$\omega_M|U = \frac{1}{J_x} dx^1 \wedge \cdots \wedge dx^n$$

for every chart (U, x).

Lemma 8.19. *If M and N are oriented Riemannian n-manifolds and $f : M \to N$ is a diffeomorphism, then*

$$f^*\omega_N = J_f \omega_M. \tag{8.20}$$

Proof. Exercise. □

Let $p \in M$ and φ an orientation preserving chart at p such that $\{\partial_i\}_{i=1}^n$, $\partial_i = \frac{\partial}{\partial \varphi^i}$, is an orthonormal basis of T_pM. Then, by definition,

$$J_\varphi(p) = \det D(\mathrm{id} \circ \varphi \circ \varphi^{-1})(\varphi(p)) = 1.$$

If $v \in T_pM$, then

$$\langle v, \partial_i \rangle = \langle v(\varphi^j)\partial_j, \partial_i \rangle = v(\varphi^i).$$

Let $v_1, \ldots, v_n \in T_pM$. Then

$$\omega_M(v_1, \ldots, v_n) = \frac{1}{J_x(p)} d\varphi^1 \wedge \cdots \wedge d\varphi^n(v_1, \ldots, v_n) = \det(v_i(\varphi^j)) = \det(\langle v_i, \partial_j \rangle).$$

Because $\{\partial_i\}_{i=1}^n$ is orthonormal

$$\langle v_i, v_j \rangle = \left\langle v_i, \sum_{k=1}^n \langle v_j, \partial_k \rangle \partial_k \right\rangle = \sum_{k=1}^n \langle v_i, \partial_k \rangle \langle v_j, \partial_k \rangle.$$

Hence,

$$B = AA^T,$$

where $B = (\langle v_i, v_j \rangle)_{ij}$ and $A = (\langle v_i, \partial_j \rangle)_{ij}$. Therefore,

$$\det(\langle v_i, v_j \rangle) = (\det A)^2 = (\omega_M(v_1, \ldots, v_n))^2.$$

Let then (U, x) be an orientation preserving chart. Apply the formula above to $v_i = (\frac{\partial}{\partial x^i})_p$ to gain

$$\det \langle \tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j} \rangle_p = (\omega_M(\tfrac{\partial}{\partial x^1}, \ldots, \tfrac{\partial}{\partial x^n}))^2 = \left(\frac{1}{J_x(p)} dx^1 \wedge \cdots \wedge dx^n (\tfrac{\partial}{\partial x^1}, \ldots, \tfrac{\partial}{\partial x^n}) \right)^2.$$

Hence,

$$J_x(p) = \frac{1}{\sqrt{\det g_{ij}(p)}},$$

where $g_{ij}(p) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_p$. Thus the Riemannian volume form can be written in local coordinates as

$$\sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n, \tag{8.21}$$

where $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$.

Lemma 8.22. *Let M be an oriented Riemannian manifold, ω_M Riemannian volume form, and $V \in \mathcal{T}(M)$. Then the divergence of V, $\operatorname{div} V = \operatorname{tr}(X \mapsto \nabla_X V)$, satisfies*

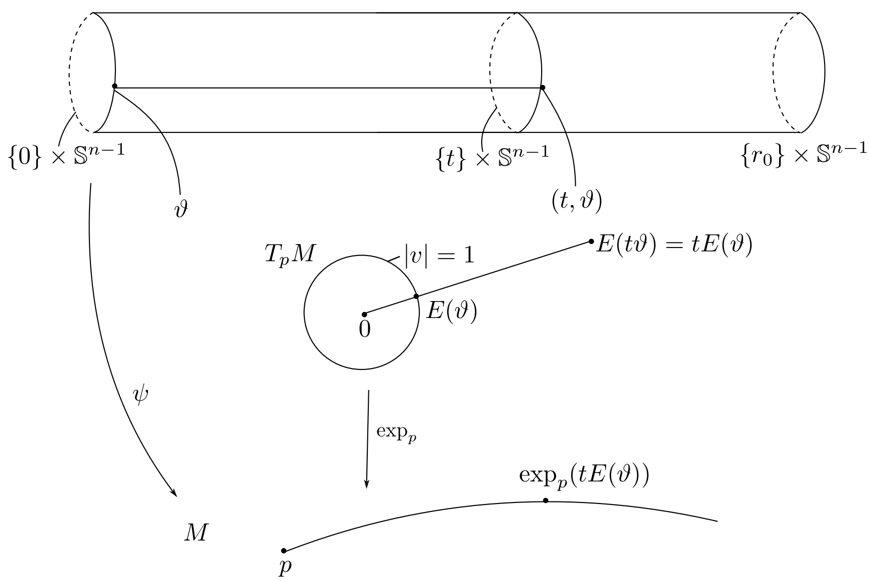
$$L_V \omega_M = (\operatorname{div} V)\omega_M.$$

Proof. Exercise. □

Let $p \in M$ (M oriented Riemannian n-manifold) and $r_0 = \operatorname{inj}(p)$. Let $C = (0, r_0) \times \mathbb{S}^{n-1}$ and $\psi : C \to B(p, r_0) \setminus \{p\}$,

$$\psi(t, \vartheta) := \exp_p(E(t\vartheta)),$$

where $E : \mathbb{R}^n \to T_pM$ is an isometric isomorphism. Then ψ is a diffeomorphism and (t, ϑ) are **geodesic polar coordinates** of $\psi(t, \vartheta) \in B(p, r_0) \setminus \{p\}$.



The Riemannian volume form of C can be written as

$$\omega_C = dt \wedge \omega_{\mathbb{S}^{n-1}} = \omega_R \wedge \omega_{\mathbb{S}^{n-1}},$$

where $t : (t, \vartheta) \mapsto t$. The form $\omega_{\mathbb{S}^{n-1}}$ can be interpreted as $\omega_{\mathbb{S}^{n-1}} \in \mathcal{A}^{n-1}(C)$ (= smooth differentiable $(n-1)$-forms on C) that is independent of t-variable of $(t, \vartheta) \in C$. More precisely, write $v \in T_{(t,\vartheta)}C = T_t\mathbb{R} \oplus T_\vartheta \mathbb{S}^{n-1}$ as $v = (v_t, v_\vartheta)$. Then

$$\underbrace{\omega_{\mathbb{S}^{n-1}}(v^1, \ldots, v^{n-1})}_{\in \mathcal{A}^{n-1}(C)} = \underbrace{\omega_{\mathbb{S}^{n-1}}(v_\vartheta^1, \ldots, v_\vartheta^{n-1})}_{\in \mathcal{A}^{n-1}(\mathbb{S}^{n-1})}.$$

We define the distance function $r : B(p, r_0) \to [0, r_0)$ by $r(x) = d(p, x)$. Then $r \in C^\infty(B(p, r_0) \setminus \{p\})$. Furthermore, let ∂_r be the radial vector field on $B(p, r_0) \setminus \{p\}$,

$$(\partial_r)_x = \dot{\gamma}_{r(x)},$$

where γ is the unique unit speed geodesic from p to x. Thus

$$\gamma(t) = \exp_p\Big(t \cdot \tfrac{\exp_p^{-1}(x)}{r(x)}\Big).$$

In fact, $\partial_r = \psi_* \frac{\partial}{\partial t} = \nabla r$. Define a smooth function $A : B(p, r_0) \setminus \{p\} \to \mathbb{R}$ by

$$A(x) := J_\psi(\psi^{-1}(x)).$$

Theorem 8.23. *In* $B(p, r_0) \setminus \{p\}$ *we have*

$$\frac{\partial_r A}{A} = \Delta r. \tag{8.24}$$

Remark 8.25. Since $\exp_p$ preserves radial distances, i.e. $d(\exp_p(tv), \exp_p(sv)) = |t-s||v|$, the value $A(x)$ describes the "size of the area element" of the geodesic sphere $S(p,x)$, $t = d(p,x)$, at x.

Proof of the Theorem 8.23. Since $J_{\psi^{-1}}\omega_M = (\psi^{-1})^*\omega_C$, we have

$$\omega_M = \frac{1}{J_{\psi^{-1}}}(\psi^{-1})^*\omega_C = (J_\psi \circ \psi^{-1})(\psi^{-1})^*\omega_C = A(\psi^{-1})^*\omega_C.$$

Hence, in $B(p, r_0) \setminus \{p\}$ we have

$$(\Delta r)\omega_M = (\operatorname{div} \partial_r)\omega_M = L_{\partial_r}\omega_M = L_{\partial_r}(A(\psi^{-1})^*\omega_C) = (\partial_r A)(\psi^{-1})^*\omega_C + AL_{\partial_r}(\psi^{-1})^*\omega_C.$$

Here

$$L_{\partial_r}(\psi^{-1})^*\omega_C = L_{\psi_*\frac{\partial}{\partial t}}(\psi^{-1})^*\omega_C \overset{(*)}{=} (\psi^{-1})^* L_{\frac{\partial}{\partial t}}\omega_C = 0,$$

since $\omega_C = dt \wedge \omega_{\mathbb{S}^{n-1}}$ is invariant in translation in t (= the flow of $\frac{\partial}{\partial t}$). Hence,

$$\frac{\partial r A}{A}\omega_M = \frac{\partial r A}{A} A(\psi^{-1})^*\omega_C = (\Delta r)\omega_M,$$

which implies (8.24). □

Another proof of $(*)$. We have

$$L_{\partial_r}(\psi^{-1})^*\omega_C = L_{\partial_r}(d(t\circ\psi^{-1}) \wedge (\psi^{-1})^*\omega_{\mathbb{S}^{n-1}}) = L_{\partial_r}(dr) \wedge (\psi^{-1})^*\omega_{\mathbb{S}^{n-1}} + dr \wedge L_{\partial_r}(\psi^{-1})^*\omega_{\mathbb{S}^{n-1}}.$$

Here the first term is zero because

$$L_{\partial_r}(dr) = d(\underbrace{\partial_r(r)}_{\equiv 1}) = 0.$$

Moreover, the second term is also zero because

$$\begin{aligned} L_{\partial_r}(\psi^{-1})^*\omega_{\mathbb{S}^{n-1}} &= i_{\partial_r} d((\psi^{-1})^*\omega_{\mathbb{S}^{n-1}}) + d i_{\partial_r}(\psi^{-1})^*\omega_{\mathbb{S}^{n-1}} \\ &= i_{\partial_r}(\psi^{-1})^* d\omega_{\mathbb{S}^{n-1}} + d(\psi^{-1})^* \underbrace{i_{\frac{\partial}{\partial t}}\omega_{\mathbb{S}^{n-1}}}_{=0} \\ &= (\psi^{-1})^* i_{\frac{\partial}{\partial t}} d\omega_{\mathbb{S}^{n-1}} = 0, \end{aligned}$$

since $d\omega_{\mathbb{S}^{n-1}} = 0$ giving the claim. Note that $d\omega_{\mathbb{S}^{n-1}} = 0$ holds since

$$\omega_{\mathbb{S}^{n-1}} = \omega d\vartheta^1 \wedge \cdots \wedge d\vartheta^{n-1},$$

where ω is independent of t, so

$$d\omega_{\mathbb{S}^{n-1}} = \underbrace{\frac{\partial\omega}{\partial t}}_{=0} dt \wedge d\vartheta^1 \wedge \cdots \wedge d\vartheta^{n-1} + \sum_{i=1}^{n-1} \frac{\partial\omega}{\partial\vartheta^i} \underbrace{d\vartheta^i \wedge d\vartheta^1 \wedge \cdots \wedge d\vartheta^{n-1}}_{=0} = 0.$$

□

Remark 8.26. Let M be complete. Then (8.24) can be generalized for all points $x \notin C(p)$, $x \neq p$: Take the unique minimizing unit speed geodesic γ from p to x. Then the geodesic polar coordinates of x are (t_x, ϑ_x), where $t_x := d(x,p)$ and $\vartheta_x := E^{-1}\dot\gamma_0$. The value $\psi(t,\vartheta) = \exp_p(tE(\vartheta))$ is defined for all $t > 0$ and $\vartheta \in \mathbb{S}^{n-1}$, and is a local diffeomorphism at (t_x, ϑ_x). Hence, we may also define

$$A(x) = J_\psi(t_x, \vartheta_x).$$

8.27 Ricci curvature comparisons

Let M be complete, $p \in M$, and $x \notin C(p) \cup \{p\}$. We denote $A(x)$ also by

$$A(x) = A(t, \vartheta),$$

where (t, ϑ) are geodesic polar coordinates of x.

Theorem 8.28. *Let M^n be complete, $p \in M$, and $\mathrm{Ric}(v,v) \geq (n-1)H$ for every $v \in TM$ with $|v| = 1$. Then in $M \setminus (C(p) \cup \{p\})$ we have*

$$\frac{A(t,\vartheta)}{A^H(t,\vartheta)} \text{ is decreasing in } t \text{ along radial geodesic } (= \vartheta \text{ is fixed}); \tag{8.29}$$

and

$$\Delta r \leq \Delta^H r = \begin{cases} (n-1)\sqrt{H}\cot(\sqrt{H}r), & H > 0; \\ (n-1)/r, & H = 0; \\ (n-1)\sqrt{-H}\coth(\sqrt{-H}r), & H < 0. \end{cases} \tag{8.30}$$

Here A^H and Δ^H refer to the corresponding notions in simply connected M_H with constant sectional curvature H.[2] *If $H > 0$, then $r \leq \pi/\sqrt{H}$ by Theorem 7.6.*

Proof. We apply the Bochner-Weitzenböck-Lichnerowitz formula with $f(x) = r(x)$ in $M \setminus (C(p) \cup \{p\})$, where r is smooth and $|\nabla r| = 1$. We have

$$|\operatorname{Hess} r|^2 + \tfrac{\partial}{\partial r}(\Delta r) + \mathrm{Ric}(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}) = 0.$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\operatorname{Hess} r$, i.e. eigenvalues of (self-adjoint) linear map

$$v \mapsto \nabla_v \nabla r.$$

Since $\nabla r(x) = (\frac{\partial}{\partial r})_x = \dot\gamma_{r(x)}$, where γ is the unique unit speed geodesic from p to x, we have

$$\nabla_{\nabla r} \nabla r = 0.$$

It follows that one of the eigenvalues, say λ_1, is zero. The Cauchy-Schwarz inequality gives

$$\frac{(\Delta r)^2}{n-1} = \frac{(\operatorname{tr}\operatorname{Hess} r)^2}{n-1} = \frac{(\lambda_2 + \cdots + \lambda_n)^2}{n-1} \leq \lambda_2^2 + \cdots + \lambda_n^2 = |\operatorname{Hess} r|^2.$$

Since $\mathrm{Ric}(\nabla r, \nabla r) \geq (n-1)H$, we get the **Riccati inequality**:

$$\frac{(\Delta r)^2}{n-1} + \frac{\partial}{\partial r}(\Delta r) + (n-1)H \leq 0. \tag{8.31}$$

[2] $A_H(\cdot, \vartheta)$ is independent of ϑ

Denote

$$S_H(t) := \begin{cases} \frac{1}{\sqrt{H}}\sin(\sqrt{H}t), & H > 0; \\ t, & H = 0; \\ \frac{1}{\sqrt{-H}}\sinh(\sqrt{-H}t), & H < 0, \end{cases}$$

$$Ct_H(t) := \frac{S_H'(t)}{S_H(t)},$$

and

$$\psi_H := (n-1)Ct_H.$$

Then the right-hand side of (8.30) equals to $\psi_H(r(x))$ and ψ_H satisfies the **Riccati equation**

$$\psi_H' + \frac{\psi_H^2}{n-1} + (n-1)H = 0.$$

Now $\frac{\psi_H^2}{n-1}+(n-1)H > 0$ on $(0, \pi/\sqrt{H})$ for $H > 0$ and on $(0, +\infty)$ for $H \leq 0$. Let $x \in M\backslash(C(p)\cup\{p\})$, γ be the unit speed geodesic from p to x, and $v := \dot{\gamma}_0$. Write $\varphi(t) = \Delta r(\gamma(t))$. Then φ satisfies

$$\varphi' + \frac{\varphi^2}{n-1} + (n-1)H \leq 0.$$

On the other hand,

$$\Delta r = \frac{n-1}{r} + \mathcal{O}(r), \quad \text{as} r \to 0, \tag{8.32}$$

i.e. $\varphi(t) = \frac{n-1}{t} + \mathcal{O}(t)$ (Exercise). Hence, there exists $r_0 \leq d(v)$ such that

$$\frac{\varphi^2(t)}{n-1} + (n-1)H > 0 \tag{8.33}$$

for every $t \in (0, r_0)$. Now (8.31) implies

$$\frac{-\varphi'}{\frac{\varphi^2}{n-1} + (n-1)H} \geq 1$$

on $(0, r_0)$. Hence,

$$\int_0^t \frac{-\varphi'}{\frac{\varphi^2}{n-1} + (n-1)H}\, ds \geq t$$

for every $t \in (0, r_0]$, which gives

$$\operatorname{arc} Ct_H\left(\frac{\varphi(t)}{n-1}\right) \geq t$$

for every $t \in (0, r_0]$. Here $\operatorname{arc} Ct_H$ is the inverse function of Ct_H. Now

$$\varphi(t) \leq (n-1)Ct_H(t) = \psi_H(t)$$

for every $0 < t \leq r_0$. Denote

$$t_0 := \sup\{0 < t \leq d(v) : \varphi \leq \psi_H \text{ on } (0, t)\}.$$

If $t_0 = d(v)$, we are done with (8.30). If $t_0 < d(v)$, then $\varphi(t_0) = \psi_H(t_0)$, and so

$$\frac{\varphi^2(t_0)}{n-1} + (n-1)H = \frac{\psi_H^2(t_0)}{n-1} + (n-1)H > 0.$$

But then (8.33) holds on $(0, t_0+\varepsilon)$ for some $\varepsilon > 0$, and hence $\varphi(t) \le \psi_H(t)$ for every $t \in (0, t_0+\varepsilon)$. This is a contradiction with the definition of t_0. Thus, $\varphi(t) \le \psi_H(t)$ for every $t \in (0, d(v))$. On M_H, the inequality (8.31) holds as an equality. Since $\Delta^H r$ satisfies (8.32), we have $\Delta^H r(x) = \psi_H(r(x))$. We have proved (8.30). By (8.33) and (8.30)

$$\frac{\frac{\partial}{\partial t} A(t,\vartheta)}{A(t,\vartheta)} \le \frac{\frac{\partial}{\partial t} A^H(t,\vartheta)}{A^H(t,\vartheta)}.$$

Hence,

$$\frac{\partial}{\partial t}(\log A(t,\vartheta) - \log A^H(t,\vartheta)) \le 0,$$

so

$$t \mapsto \log \frac{A(t,\vartheta)}{A^H(t,\vartheta)}$$

is decreasing when ϑ is fixed. This implies (8.29). □

Lemma 8.34. *Let $f, g : [a,b) \to [0,\infty)$, $g > 0$, be integrable on $[a,r]$ for every $a \le r < b$. Suppose that f/g is decreasing. Then*

$$r \mapsto \int_a^r f \Big/ \int_a^r g$$

is decreasing.

Proof. Let $a \le r < R < b$. Then

$$\Big(\int_a^r f\Big)\Big(\int_a^R g\Big) = \Big(\int_a^r f\Big)\Big(\int_a^r g\Big) + \Big(\int_a^r f\Big)\Big(\int_r^R g\Big)$$

and

$$\Big(\int_a^R f\Big)\Big(\int_a^r g\Big) = \Big(\int_a^r f\Big)\Big(\int_a^r g\Big) + \Big(\int_r^R f\Big)\Big(\int_a^r g\Big).$$

We want to show

$$\Big(\int_a^r f\Big)\Big(\int_a^R g\Big) \ge \Big(\int_a^R f\Big)\Big(\int_a^r g\Big)$$

or equivalently

$$\Big(\int_a^r f\Big)\Big(\int_r^R g\Big) \ge \Big(\int_r^R f\Big)\Big(\int_a^r g\Big)$$

Let $h = f/g$. Then h is decreasing and $f = gh$. Hence,

$$\begin{aligned}\Big(\int_a^r f\Big)\Big(\int_r^R g\Big) &= \Big(\int_a^r gh\Big)\Big(\int_r^R g\Big) \ge h(r)\Big(\int_a^r g\Big)\Big(\int_r^R g\Big)\\ &\ge \Big(\int_a^r g\Big)\Big(\int_r^R hg\Big) = \Big(\int_a^r g\Big)\Big(\int_r^R f\Big).\end{aligned}$$

□

We denote

$$\mathrm{Vol}(B(p,r)) = \int_{B(p,r)} \omega_M = \int_M \chi_{B(p,r)} \omega_M,$$

that is, the volume (measure) of $B(p,r) \subset M$.

Remark 8.35. The volume $\mathrm{Vol}(C(p)) = 0$ for every $p \in M$ since for every $x \in C(p)$ there exists $v \in T_pM$ with $|v| = 1$, and $t_0 \in \mathbb{R}$ such that $x = \gamma^v(t_v)$. Each $\{t_v\}$ is of zero one-dimensional measure, hence $\mathrm{Vol}(C(p)) = 0$ by Fubini's theorem.

Theorem 8.36. *Let M be complete, $p \in M$, and $\mathrm{Ric}(v,v) \geq (n-1)H$ for every $v \in T_qM$, $|v| = 1$, $q \in M$. Then for every $0 < r \leq R$ ($R \leq \pi/\sqrt{H}$ if $H > 0$)*

$$\frac{\mathrm{Vol}(B(p,R))}{\mathrm{Vol}(B(p,r))} \leq \frac{\mathrm{Vol}(B_H(R))}{\mathrm{Vol}(B_H(r))}$$

Here $\mathrm{Vol}(B_H(t))$ is the volume of any ball of radius t in M_H (= independent of the centre).

Proof. We set $A(t,\vartheta) = 0$ for every $t \geq d(E(\vartheta))$. Then

$$\mathrm{Vol}(B(p,r)) = \int_{\mathbb{S}^{n-1}} \int_0^r A(t,\vartheta)\, dt\, d\vartheta.$$

By (8.29) and Lemma 8.34

$$\frac{\int_0^r A(t,\vartheta)\, dt}{\int_0^r A^H(t,\vartheta)\, dt} \geq \frac{\int_0^R A(t,\vartheta)\, dt}{\int_0^R A^H(t,\vartheta)\, dt}.$$

Hence,

$$\int_0^r A(t,\vartheta)\, dt \geq \frac{\int_0^r A^H(t,\vartheta)\, dt}{\int_0^R A^H(t,\vartheta)\, dt} \cdot \int_0^R A(t,\vartheta)\, dt = \frac{\mathrm{Vol}(B_H(r))}{\mathrm{Vol}(B_H(R))} \cdot \int_0^R A(t,\vartheta)\, dt.$$

Integrating this over the sphere $\mathbb{S}^{n-1}$ we have the claim. □

Corollary 8.37. *Let M be as in Theorem 8.36. Then for every $p \in M$ and $r > 0$*

$$\mathrm{Vol}(B(p,r)) \leq \mathrm{Vol}(B_H(r)).$$

Proof. Let $t \in (0,r)$. Then

$$\mathrm{Vol}(B(p,r)) \leq \left(\frac{\mathrm{Vol}(B(p,t))}{\mathrm{Vol}(B_H(t))} \right) \mathrm{Vol}(B_H(r)).$$

On the other hand,

$$\frac{\mathrm{Vol}(B(p,t))}{\mathrm{Vol}(B_H(t))} \to 1 \quad \text{as} \quad t \to 0. \quad \text{(Exercise)}$$

This gives the claim. □

9 The sphere theorem

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