Higher Transcendental Functions

HIGHER TRANSCENDENTAL FUNCTIONS

Volume I

Based, in part, on notes left by

Harry Bateman

Late Professor of Mathematics, Theoretical Physics, and Aeronautics at the California Institute of Technology

and compiled by the

Staff of the Bateman Manuscript Project

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PREFACE

The late Professor Harry Bateman of the California Institute of Technology was one of those rare scientists who, responding to the interplay between mathematical analysis and physical understanding, made outstanding contributions to American applied mathematics. His contributions to aero- and fluid mechanics, to electro-magnetic theory, to thermodynamics, to geophysics, and to a host of other fields in which his adroit mathematical skills were applied, resulted in significant advances in these fields. During his last years he had embarked upon a project whose successful completion, he believed, would prove of great value to scientists in all fields. He planned an extensive compilation of "special functions" -- solutions of a wide class of mathematically and physically relevant functional equations. He intended to investigate and to tabulate properties of such functions, inter-relations between such functions, their representations in various forms, their macro- and microscopic behavior, and to construct tables of important definite integrals involving such functions.

It is true that much of this material was already in existence. However, anyone who has been faced with the task of handling and discussing and understanding in detail the solution to an applied problem which is described by a differential equation is painfully familiar with the disproportionately large amount of scattered research on special functions one must wade through in the hope of extracting the desired information. Professor Bateman was eminently qualified to embark on such a compilation, for he was unusually familiar -- and systematically so -- with existing mathematical literature on the subject; he was exceptionally adept in mathematical analysis; and he was ever conscious of the needs of the scientist who must so often use these functions. When his death cut short his work, the California Institute of Technology, in recognition of one of its great scientists, and the Office of Naval Research, in recognition of the extremely useful service such a compilation could render to both basic and applied science, pooled their efforts to continue the task initiated by Professor Bateman.

In 1948 arrangements were completed between the California Institute of Technology and the Office of Naval Research to employ at the California Institute of Technology four mathematical analysts of international reputation to complete Professor Bateman's work: Professors Arthur Erdélyi of the University of Edinburgh; Wilhelm Magnus of the University of Göttingen; Fritz Oberhettinger of the University of Mainz; and Francesco Tricomi of the University of Torino. It was not long after this team began work that it became apparent that not only would Professor Bateman's original project find its completion in their unusually competent hands, but that the activities of such a group would lead to significant mathematical investigations and advances in the general field of mathematical analysis, as well as in the more particular field of special functions. The present compendia bear undeniable witness to the success of the undertaking.

The Office of Naval Research is proud of its collaboration with the California Institute of Technology, not only for erecting this lasting memorial to Professor Bateman, but also for producing what it considers a significant contribution to general science. These compendia, which have taken their roots in Professor Bateman's "shoe boxes" (his repository for card files) have been nurtured into mathematical maturity under the deft minds and penetrating work of the members of the international team of Erdélyi, Magnus, Oberhettinger, and Tricomi. In addition, we are pleased to have been able to render support to several young American mathematicians who have not only contributed to these compendia but were able to avail themselves of the opportunity to work and study under the direction of distinguished scientists in a field that is sorely in need of young recruits. We feel that special thanks should be extended to both Dean E. C. Watson of the California Institute of Technology and to Professor Erdélyi; to the former, for his extremely helpful and untiring interest in seeing to the establishment and completion of this task; to the latter, for assuming, in addition to scientific participation, both the scientific administrative duties of the project and the general editorial responsibilites for the publication of these compendia.

MINA REES,
Director Mathematical Sciences Division
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FOREWORD

The late Professor Harry Bateman was one of the greatest authorities in that part of mathematics now usually described as classical analysis. His knowledge of the literature was encyclopedic and probably unsurpassed and his ability to utilize this knowledge for specific problems was extraordinary. Research workers in difficulties would often write to him and receive, by return mail, detailed answers to their questions together with a list of references which in many cases amounted to a complete bibliography.

It was natural for Bateman to want to make accessible in a systematic form the tremendous amount of material which he had collected in the course of the years. His book on *Partial Differential Equations* (1932) was an attempt to carry out this task in a restricted field. Although the book was received with enthusiasm, and, after twenty years, is still one of the most important books on its subject, Bateman was not satisfied with this method of providing information. For a number of years he made plan after plan to organize and prepare for publication his material, a task made extremely difficult by the very breadth of the field which he intended to cover.

At the time of Bateman's death (1946) his notes amounted to a veritable mountain of paper. His card-catalogue alone filled several dozen cardboard boxes (the famous "shoe-boxes"). His family, his friends, and his colleagues at the California Institute of Technology very naturally wished to have some of this material prepared for posthumous publication, thereby erecting a monument to one of the most distinguished and most versatile members of the faculty of the Institute. Professor A. D. Michal, for many years a friend and colleague of the deceased, undertook the sifting of Bateman's notes. He spent several months in this herculean task, sorted out those notes which might be considered for publication and made recommendations for proceeding further with the matter. Dr. A. Erdélyi, then of the University of Edinburgh in Scotland, was invited to prepare a detailed report and proposals, and spent the academic year 1947-48 in Pasadena for this purpose.

It turned out that Bateman's notes ranged over a wider field than even his friends had suspected and also that no single section of this wide field was in a state sufficiently advanced for immediate publication. Indeed the field was so wide that it appeared imperative to narrow it down if anything useful was to be accomplished. Notes for books on functional equations, integrals in potential theory, binomial coefficients and factorials, and many other matters had to be laid aside entirely. Of the remaining material the most important part was a projected trilogy on the higher transcendental functions, on definite integrals (especially those containing higher functions), and on numerical tables of functions occurring in applied mathematics. Since the appearance of the *Index of Mathematical Tables* by Fletcher, Miller, and Rosenhead, adequate information has been available on numerical tables, and so it was decided to concentrate on the first two parts, and these came to be called the handbook and the integral tables.

The Office of Naval Research recognized the great importance of such a work by giving generous financial support to it. Thus originated what at the California Institute came to be called the Bateman Manuscript Project. The Institute was fortunate indeed, not only in being able to persuade Professor Erdélyi to remain as its Director and as Editor of the forthcoming publications, but also in securing the services of Professor Wilhelm Magnus of the University of Göttingen (now of New York University), of Dr. Fritz Oberhettinger of the University of Mainz (now Professor at the American University, Washington, D.C.) and of Professor Francesco G. Tricomi of the University of Turin. These distinguished and internationally known scholars were assisted by a staff of younger mathematicians. The technical preparation of the vari-typescript suitable for reproduction by a photo-offset process was in the capable hands of Miss Rosemarie Stampfel.

The present volume is the first of three projected volumes on the higher transcendental functions. These three volumes will be followed by two volumes of integral tables.

The California Institute of Technology wishes to express its thanks both to the family of the late Professor Bateman for the gift of his notes and of his library, and to the Office of Naval Research and especially to Dr. Mina Rees, the Director of its Mathematical Sciences Division, for the generous support they have given to this work and for the understanding they have constantly shown for the difficulties encountered. The Institute also wishes to record its appreciation and thanks to the following persons and organizations: to Professor Michal for his preliminary survey of Bateman's notes; to the University of Edinburgh for

granting leave of absence to Dr. Erdélyi; to the Rockefeller Foundation for defraying travelling expenses for Dr. and Mrs. Erdélyi on their visit in 1947-48; to the University of Turin for granting leave of absence to Professor Tricomi; to Professors T.M. Apostol of the California Institute, R. C. Archibald of Brown University, E. D. Rainville of the University of Michigan, Mr. S. O. Rice of the Bell Telephone Laboratories, and Professor C. A. Truesdell of Indiana University for information or consultations in connection with the work; and to the McGraw-Hill Company for technical advice and publication. Last but not least, acknowledgments should be expressed to Dr. Erdélyi and the staff of the Bateman Manuscript Project for the faithful and highly competent performance of a difficult task.

E. C. WATSON

Dean of the Faculty
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INTRODUCTION

The work of which this book is the first volume might be described as an up-to-date version of *Part II. The Transcendental Functions* of Whittaker and Watson's celebrated "Modern Analysis". Bateman (who was a pupil of E. T. Whittaker) planned his "Guide to the Functions" on a gigantic scale. In addition to a detailed account of the properties of the most important functions, the work was to include the historic origin and definition of, the basic formulas relating to, and a bibliography for *all* special functions ever invented or investigated. These functions were to be catalogued and classified under twelve different headings according to their definition by power series, generating functions, infinite products, repeated differentiations, indefinite integrals, definite integrals, differential equations, difference equations, functional equations, trigonometric series, series of orthogonal functions, or integral equations. Tables of definite integrals representing each function and numerical tables of a few new functions were to form part of the "Guide". An extensive table of definite integrals and a Guide to numerical tables of special functions were planned as companion works.

The great importance of such a work hardly needs emphasis. Bateman's unparalleled knowledge of the mathematical literature, past and present, and his equally exceptional diligence, would have made the book an authoritative account of its vast subject, and in many respects a definitive account; a Greater Oxford Dictionary of special functions.

A realistic appraisal of our abilities and of the time at our disposal led to a drastic revision of Bateman's plans. Only Bateman himself had the erudition to give a reliable and accurate history of special functions, and the manpower available to us was insufficient for the inclusion of all functions. Thus we restricted ourselves to an account (probably far less detailed than that planned by Bateman) of the principal properties of those special functions which we considered the most important ones. The loss thus caused to mathematical scholarship is great, regrettable, and final but we venture to hope that it will be counterbalanced in some measure by the considerable reduction in size of the book, and by the gain in the clarity of its organization. We can only hope that although

NOTATION, REFERENCES

The notation presents peculiar difficulties. There are special functions, for instance Bessel functions of the first kind, for which there is a generally accepted standard notation. There are others, like confluent hypergeometric functions, for which there are several essentially different and independent notations. The most awkward problems present themselves with those functions for which more or less the same symbol is used with several different meanings. Hermite polynomials are usually denoted by $H_n(x)$ or $He_n(x)$, but this symbol sometimes refers to the polynomials derived by repeated differentiation of $\exp(-x^2)$, and sometimes to those derived from $\exp(-\frac{1}{2}x^2)$. Moreover, some authors include, others exclude a factor, $n!$. We attempted to use the same notations throughout our book. The most significant deviation from this principle is in the case of the confluent hypergeometric series for which the symbol ${}_1F_1$ is used mostly, except in Chapter VI (and some of the later chapters) where the symbol Φ is used for the same series (and Ψ for a second solution of the confluent hypergeometric equation).

Wherever possible we followed standard notations. In the case of Bessel functions we adopted the notations used by G. N. Watson in his monumental work, in the case of orthogonal polynomials we used Szegö's notation (with the exception of using C_n^ν for ultraspherical polynomials). With Legendre functions, we followed Jahnke-Emde, Magnus-Oberhettinger and some other authors in making a distinction between the definition of Legendre functions appropriate for the interval $(-1, 1)$ and the definition appropriate for the complex plane outside of this interval. In cases of doubt we usually decided upon that notation for which more convenient or more extensive numerical tables were available. We adhered to definitions used in numerical tables even in cases in which we thought that a different definition would be preferable from the mathematical point of view. All notations are explained where they occur for the first time. There is at the end of this volume an *Index of notations* which will help the reader to find the meaning of any notation used in the book, and a *Subject index* which gives the notation for any function which occurs in the text.

Many of our chapters may be read independently of the others, yet there are many cross references. Equations within the same section are referred to simply by number, equations in other sections are indicated by the section number followed by the number of the equation. Thus (3) means equation (3) in the section in which the reference occurs, 2.1(3) means equation (3) in section 2.1. References to the literature have the name of the author followed by the year of publication. They invariably refer to the list of references at the end of the chapter.

The size and complexity of our compilation make it vain to hope that errors of judgment, or mistakes have been avoided. The undersigned will be glad to receive corrections or suggestions for the improvement of the work should a second edition become desirable.

In conclusion I should like to express the thanks of the entire project staff to the California Institute of Technology, and especially to Dean E. C. Watson, for initiating this work and for the great understanding they have shown for the numerous problems we encountered. I should also like to thank my colleagues without whose assistance the present work could not have been carried out.

A. ERDELYI

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CHAPTER I

THE GAMMA FUNCTION

1.1. Definition of the gamma function

The function $\Gamma(z)$ can be defined by one of the following expressions:

(1) $$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt = \int_0^1 (\log 1/t)^{z-1}\, dt \qquad \operatorname{Re} z > 0,$$

(2) $$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)} = \lim_{n\to\infty} \frac{n^z}{z(1+z)(1+\tfrac{1}{2}z)\cdots(1+z/n)}$$

$$= z^{-1} \prod_{n=1}^{\infty} [(1+1/n)^z (1+z/n)^{-1}],$$

(3) $$1/\Gamma(z) = z e^{\gamma z} \prod_{n=1}^{\infty} [(1+z/n)\, e^{-z/n}],$$

where

(4) $$\gamma = \lim_{m\to\infty} \left(\sum_{n=1}^{m} 1/n - \log m \right) = 0.5772156649\cdots$$

denotes Euler's or Mascheroni's constant. The definition (1) was used by Euler, (2) (in a slightly different notation) by Gauss, and (3) by Weierstrass.

Replacing t by st in (1) (s real and positive) we get

(5) $$\Gamma(z) = s^z \int_0^\infty e^{-st} t^{z-1}\, dt \qquad \operatorname{Re} z > 0.$$

It can be shown [cf. 1.5(34)] that this formula holds for complex values of s and for a path of integration along the straight line from the origin to $\infty e^{i\delta}$. Thus we have

(6) $$\Gamma(z) = s^z \int_0^{\infty e^{i\delta}} e^{-st} t^{z-1}\, dt$$

$$-(\tfrac{1}{2}\pi + \delta) < \arg s < \tfrac{1}{2}\pi - \delta, \qquad \operatorname{Re} z > 0.$$

This equation holds for $\arg s + \delta = \pm \tfrac{1}{2}\pi$ provided $0 < \operatorname{Re} z < 1$.

From (2) and (3) it is seen that the gamma function is an analytic function of z whose only finite singularities are $z = 0, -1, -2, \ldots$. From (1) it follows that

$$(7)\quad \Gamma(z) = \int_0^1 e^{-t} t^{z-1}\, dt + \int_1^\infty e^{-t} t^{z-1}\, dt = P(z) + Q(z),$$

$Q(z)$ being an integral function. Expanding e^{-t} in a power series and integrating term by term:

$$(8)\quad P(z) = \sum_{n=0}^{\infty} (-1)^n [n!\,(z+n)]^{-1}.$$

Hence it follows that $(-1)^n/n!$ is the residue of $\Gamma(z)$ at the simple pole $z = -n$, $(n = 0, 1, 2, \ldots)$ [cf. 1.17(11)].

It will be shown that the expressions (1), (2), (3) represent the same function.

For a positive integer n and $\operatorname{Re} z > 0$ repeated integration by parts yields

$$\int_0^n (1 - t/n)^n t^{z-1}\, dt = \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)},$$

so that by Tannery's theorem

$$\lim_{n\to\infty} \int_0^n (1 - t/n)^n t^{z-1}\, dt = \int_0^\infty e^{-t} t^{z-1}\, dt = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}.$$

Thus (1) is equivalent to (2). Equation (3) can be deduced from (2) as follows. By (2) we have

$$1/\Gamma(z) = \lim_{n\to\infty} z(1+z)(1+\tfrac{1}{2}z)\cdots(1+z/n)\, e^{-z\log n}$$

or

$$1/\Gamma(z) = \lim_{n\to\infty} [z(1+z)\, e^{-z}(1+\tfrac{1}{2}z)\, e^{-\frac{1}{2}z}\cdots(1+z/n)\, e^{-z/n}$$
$$\times\, e^{z(1+\frac{1}{2}+\cdots+1/n-\log n)}],$$

and finally

$$1/\Gamma(z) = z e^{\gamma z} \prod_{n=1}^{\infty} [(1+z/n)\, e^{-z/n}].$$

If the real part of z is negative, and $n + 1 > \operatorname{Re}(-z) > n$, $(n = 0, 1, 2, \ldots)$, $\Gamma(z)$ can be represented by an integral due to Cauchy and Saalschütz (Whittaker-Watson, 1927, p. 243):

$$(9)\quad \Gamma(z) = \int_0^\infty [e^{-t} - \sum_{m=0}^{n} (-t)^m/m!]\, t^{z-1}\, dt \qquad -(n+1) < \operatorname{Re} z < -n.$$

1.2. Functional equations satisfied by $\Gamma(z)$.

Integrating 1.1(1) by parts,

$$\Gamma(z) = (1/z) \int_0^\infty e^{-t}\, t^z\, dt = (1/z)\, \Gamma(1+z),$$

or

(1) $\Gamma(1+z) = z\,\Gamma(z),$

and hence if n is a positive integer,

(2) $\Gamma(z+n) = z(z+1)(z+2)\cdots(z+n-1)\,\Gamma(z),$

whence follows

(3) $\Gamma(z)/\Gamma(z-n) = (z-1)(z-2)\cdots(z-n)$
$= (-1)^n\,\Gamma(-z+n+1)/\Gamma(-z+1),$

(4) $\Gamma(-z+n)/\Gamma(-z) = (-1)^n z(z-1)\cdots(z-n+1)$
$= (-1)^n\,\Gamma(z+1)/\Gamma(z-n+1).$

Since

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1,$$

we have

$$\Gamma(n+1) = 1\cdot 2\cdot 3\cdots n = n!.$$

From the expression 1.1(3),

$$\Gamma(z)\,\Gamma(-z) = -z^{-2} \prod_{n=1}^{\infty} (1 - z^2/n^2)^{-1}$$

and since

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} (1 - z^2/n^2)$$

(Bromwich, 1947, p. 294), we have

(5) $\Gamma(z)\,\Gamma(-z) = -\pi z^{-1} \csc(\pi z)$

so that

(6) $\Gamma(z)\,\Gamma(1-z) = \pi \csc(\pi z),$

or

(7) $\Gamma(\tfrac{1}{2}+z)\,\Gamma(\tfrac{1}{2}-z) = \pi \sec(\pi z).$

From (5) and (2)

$$(8)\quad \frac{\Gamma(n+z)\,\Gamma(n-z)}{[(n-1)!]^2} = \frac{\pi z}{\sin(\pi z)} \prod_{m=1}^{n-1} (1 - z^2/m^2) \qquad n = 1, 2, 3, \ldots .$$

From (7), (2), and (3)

$$(9)\quad \frac{\Gamma(n+\tfrac{1}{2}+z)\,\Gamma(n+\tfrac{1}{2}-z)}{[\Gamma(n+\tfrac{1}{2})]^2} = \frac{1}{\cos(\pi z)} \prod_{m=1}^{n} \left[1 - \frac{4z^2}{(2m-1)^2}\right] \qquad n = 1, 2, 3, \ldots .$$

From (6) and (1) with $z = \tfrac{1}{2}$, it follows that

$$(10)\quad \Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-v^2}\,dv = \sqrt{\pi}.$$

We shall next prove the *multiplication formula* of *Gauss* and *Legendre*:

$$(11)\quad \prod_{r=0}^{m-1} \Gamma(z + r/m) = (2\pi)^{\frac{1}{2}(m-1)}\, m^{\frac{1}{2} - mz}\, \Gamma(mz) \qquad m = 2, 3, 4, \ldots .$$

From 1.1(2),

$$(12)\quad H(z) = \prod_{r=0}^{m-1} \Gamma(z + r/m) = \lim_{n\to\infty} n^{mz + \frac{1}{2}(m-1)} (n!)^m N^{-1},$$

where

$$N = mz(mz+1)\cdots(mz+mn)(mz+mn+1)\cdots(mz+mn+m-1)\, m^{-m(n+1)}.$$

Since

$$\Gamma(mz) = \lim_{n\to\infty} (mn)^{mz} (mn)!\,[mz(mz+1)\cdots(mz+mn)]^{-1},$$

we have

$$(13)\quad m^{-mz}\,\Gamma(mz)/H(z) = \lim_{n\to\infty} n^{\frac{1}{2}(m+1)} (mn-1)!\,(n!)^{-m}\, m^{-mn} = 1/K, \text{ say.}$$

It is evident that K is independent of z and can be evaluated by putting, for instance, $z = 1/m$ in (13). Thus

$$\Gamma(1)\,K/m = H(1/m) = \Gamma(1/m)\,\Gamma(2/m)\cdots\Gamma[(m-1)/m]\,\Gamma(1)$$

or

$$K/m = \Gamma(1 - 1/m)\,\Gamma(1 - 2/m)\cdots\Gamma[1 - (m-1)/m].$$

Multiplying the last two equations and using (6),

$$m^2\,\pi^{m-1} = K^2 \prod_{r=1}^{m-1} \sin(\pi r/m)$$

so that

(14) $K^2 = m(2\pi)^{m-1}$.

Since K, as defined by (13), is certainly positive, (12), (13), and (14) prove (11).

The case $m = 2$ of (11) is Legendre's duplication formula

(15) $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\,\Gamma(z)\,\Gamma(z+1/2)$.

1.3. Expressions for some infinite products in terms of the gamma function

From 1.1(2)

$$\frac{2\,\Gamma(1/2)}{z\,\Gamma(1/2 z)\,\Gamma(1/2 - 1/2 z)} = \frac{(1-z)\prod_{n=1}^{\infty}\{[1+1/n]^{1/2}[1+\frac{1}{2}1/n]^{-1}\}}{\prod_{n=1}^{\infty}[1+1/n]^{1/2}[1+\frac{1}{2}1/n]^{-1}[1+\frac{1}{2}z/n]^{-1}[1-z/(2n+1)^{-1}]}$$

and since $\Gamma(1/2) = \sqrt{\pi}$,

(1) $$\frac{2\sqrt{\pi}}{z\,\Gamma(1/2 z)\,\Gamma(1/2 - 1/2 z)} = (1-z)\prod_{n=1}^{\infty}[1+\tfrac{1}{2}z/n][1-z/(2n+1)] = (1-z)(1+z/2)(1-z/3)\cdots.$$

From 1.1(3)

(2) $$\Gamma(u)/\Gamma(u+v) = (1+v/u)\,e^{\gamma v}\prod_{n=1}^{\infty}[1+v/(u+n)]e^{-v/n}$$

$$= e^{\gamma v}\prod_{n=0}^{\infty}[1+v/(u+n)]\,e^{-v/(n+1)},$$

and hence

(3) $$\Gamma(x+iy)/\Gamma(x) = e^{-i\gamma y}\,x(x+iy)^{-1}\prod_{n=1}^{\infty}\frac{e^{iy/n}}{1+iy/(n+x)}.$$

From (2),

(4) $$\frac{\Gamma(z_1)\,\Gamma(z_2)}{\Gamma(z_1+z_3)\,\Gamma(z_2-z_3)} = \prod_{n=0}^{\infty}[1+z_3/(z_1+n)][1-z_3/(z_2+n)].$$

Mellin's formula

$$(5)\quad e^{y\psi(x)}\Gamma(x)/\Gamma(x+y) = \prod_{n=0}^{\infty} [1+y/(n+x)]\, e^{-y/(n+x)}$$

can easily be proved as follows. From 1.7(3)

$$e^{y\psi(x)} = e^{-\gamma y - y/x} \prod_{n=1}^{\infty} e^{xy/[n(n+x)]},$$

and by means of (2) the required result (5) follows.

We now take $z = 2^{-1}v,\ 2^{-2}v,\ \ldots,\ 2^{-n}v$, respectively in Legendre's duplication formula 1.2(15) and multiply the n relations so obtained. After canceling some factors,

$$\Gamma(v) = 2^{2v(1-2^{-n})-n}\Gamma(2^{-n}v) \prod_{m=1}^{n} [\pi^{-1/2}\,\Gamma(1/2 + 2^{-m}v)],$$

or what is equivalent to this by 1.2(1),

$$\Gamma(1+v) = 2^{2v(1-2^{-n})}\Gamma(1+2^{-n}v) \prod_{m=1}^{n} [\pi^{-1/2}\,\Gamma(1/2 + 2^{-m}v)].$$

On making $n \to \infty$, *Knar's formula*

$$(6)\quad \Gamma(1+v) = 2^{2v} \prod_{m=1}^{\infty} [\pi^{-1/2}\,\Gamma(1/2 + 2^{-m}v)]$$

is obtained.

The relation

$$(7)\quad \prod_{n=1}^{\infty} [1-(z/n)^m] = -z^{-m}\Big[\prod_{r=1}^{m} \Gamma(-ze^{2\pi ir/m})\Big]^{-1} \qquad m = 2, 3, 4, \ldots$$

is a generalization of the well-known formula 1.2(4) and can easily be verified by introducing the expression 1.1(3) for each gamma function on the right-hand side and by taking into account the relations

$$\sum_{r=1}^{m} e^{2\pi ir/m} = 0 \text{ and } \prod_{r=1}^{m} e^{2\pi ir/m} = (-1)^{m-1}.$$

Finally we consider the expression

$$P = \prod_{n=1}^{\infty} \frac{(n-a_1)(n-a_2)\cdots(n-a_k)}{(n-b_1)(n-b_2)\cdots(n-b_k)} = \prod_{n=1}^{\infty} \frac{(1-a_1/n)\cdots(1-a_k/n)}{(1-b_1/n)\cdots(1-b_k/n)}$$

$b_1, b_2, b_3, \ldots, b_k$ not positive integers.

A necessary condition for the absolute convergence of this infinite product is $a_1 + a_2 + \cdots + a_k - b_1 - b_2 - \cdots - b_k = 0$. This condition is also sufficient, and if it is satisfied,

$$(8)\quad P = \prod_{n=1}^{\infty} \frac{(1-a_1/n)\, e^{a_1/n} \cdots (1-a_k/n)\, e^{a_k/n}}{(1-b_1/n)\, e^{b_1/n} \cdots (1-b_k/n)\, e^{b_k/n}}$$

$$= \prod_{m=1}^{k} \frac{\Gamma(1-b_m)}{\Gamma(1-a_m)}$$

by virtue of 1.1(3) and 1.2(1).

1.4. Some infinite sums connected with the gamma function

Dougall's formula

$$(1)\quad S = \sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)\,\Gamma(d+n)} = \pi^2 \csc(\pi a)\csc(\pi b)$$

$$\times \frac{\Gamma(c+d-a-b-1)}{\Gamma(c-a)\,\Gamma(d-a)\,\Gamma(c-b)\,\Gamma(d-b)}$$

$$\operatorname{Re}(a+b-c-d) < -1, \qquad a,\ b,\ \text{not integers}$$

can be proved as follows.

The series S is obviously the sum of the residues of the function

$$f(z) = \pi \operatorname{ctn}(\pi z)\,\Gamma(a+z)\,\Gamma(b+z)/[\Gamma(c+z)\,\Gamma(d+z)]$$

at the poles $z = 0, \pm 1, \pm 2, \ldots$ of $\operatorname{ctn}(\pi z)$. For large $|z|$, 1.18(4) and 1.2(5) show that

$$\Gamma(a+z)\,\Gamma(b+z)/[\Gamma(c+z)\,\Gamma(d+z)]$$

is represented asymptotically by $z^{a+b-c-d}$ if $-\pi < \arg z < \pi$, and by

$$(-z)^{a+b-c-d}\,\frac{\sin[\pi(z+c)]\,\sin[\pi(z+d)]}{\sin[\pi(z+a)]\,\sin[\pi(z+b)]}$$

if $-\pi < \arg(-z) < \pi$.

We can describe in the z-plane a circle of radius r as large as we please avoiding all zeros of $\sin[\pi(z+a)]$ or $\sin[\pi(z+b)]$ or $\sin(\pi z)$. On this circle

$$\sin(\pi z + \pi c)\,\sin(\pi z + \pi d)\,\csc(\pi z + \pi a)\,\csc(\pi z + \pi b)\,\operatorname{ctn}(\pi z)$$

is bounded, the bound being independent of r. Also the integral

$$\int |z^{a+b-c-d}|\, dz \qquad\qquad \operatorname{Re}(a+b-c-d) < -1$$

taken along the circle will tend to zero as r increases. In this case the sum of all the residues of $f(z)$ is zero, and thus $S = -$ sum of the residues at the poles of $\Gamma(z+a)$ and $\Gamma(z+b)$. The residue of $f(z)$ at the pole $z=-(a+m)$ $(m=0,1,2,)$ is

$$-\pi(-1)^m\,(m!)^{-1}\operatorname{ctn}(\pi a)\cdot\frac{\Gamma(b-a-m)}{\Gamma(c-a-m)\,\Gamma(d-a-m)},$$

and the sum of the residues at the poles of $\Gamma(z+a)$ is

$$-\pi\operatorname{ctn}(\pi a)\cdot\frac{\Gamma(b-a)}{\Gamma(c-a)\,\Gamma(d-a)}\;{}_2F_1(a-c+1,\,a-d+1;\,a-b+1;\,1)$$

$$=\frac{\pi^2\operatorname{ctn}(\pi a)}{\sin[\pi(a-b)]}\cdot\frac{\Gamma(c+d-a-b-1)}{\Gamma(c-a)\,\Gamma(d-a)\,\Gamma(c-b)\,\Gamma(d-b)}$$

by Gauss' formula 2.1(14).

For the sum of the residues at the poles of $\Gamma(b+z)$ we have only to interchange a and b, and the sum of these two expressions is (1).

The formula

$$(2)\quad \sum_{n=0}^{\infty}(-1)^n\binom{y-1}{n}(x+n)^{-1}=\Gamma(x)\,\Gamma(y)/\Gamma(x+y)=\mathrm{B}(x,y)$$

can easily be verified by expanding the integrand of 1.5(1) in a power series and integrating term by term.

Furthermore, we have the following formulas (formulas to be proved in Chapter 2.4):

$$(3)\quad \sum_{n=0}^{\infty}\{(a)_n\,(b)_n/[(1-b+a)_n\,n!]\}\,[(\tfrac{1}{2}a+n-z)^{-1}+(\tfrac{1}{2}a+n+z)^{-1}]$$

$$=\frac{\Gamma(\tfrac{1}{2}a-z)\,\Gamma(\tfrac{1}{2}a+z)}{\Gamma(1-b+\tfrac{1}{2}a-z)\,\Gamma(1-b+\tfrac{1}{2}a+z)}\cdot\frac{\Gamma(1-b+a)\,\Gamma(1-b)}{\Gamma(a)}$$

$\operatorname{Re} b<1,$

$$(4)\quad \frac{\Gamma(\tfrac{1}{2}a+z)\,\Gamma(\tfrac{1}{2}a-z)}{\Gamma(c-\tfrac{1}{2}a+z)\,\Gamma(1-b+\tfrac{1}{2}a-z)}$$

$$=\frac{\Gamma(a)}{\Gamma(c)\,\Gamma(1-b)}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,n!}\cdot(\tfrac{1}{2}a+n-z)^{-1}$$

$$+\frac{\Gamma(a)}{\Gamma(1+a-b)\,\Gamma(c-a)}\sum_{n=0}^{\infty}\frac{(a)_n\,(1-c+a)_n}{(1-b+a)_n\,n!}\cdot(\tfrac{1}{2}a+n+z)^{-1}$$

$\operatorname{Re}(a+b-c)<1.$

1.5. The beta function

The beta function is defined by the integral

(1) $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad \operatorname{Re} x > 0, \quad \operatorname{Re} y > 0.$

Substituting $t = v/(1+v)$, the relation

(2) $B(x, y) = \int_0^\infty v^{x-1}(1+v)^{-x-y}\,dv \qquad \operatorname{Re} x > 0, \quad \operatorname{Re} y > 0$

is obtained, and from this

(3) $B(x, y) = \int_0^1 (v^{x-1} + v^{y-1})(1+v)^{-x-y}\,dv \qquad \operatorname{Re} x > 0, \quad \operatorname{Re} y > 0.$

can be deduced. It follows that

(4) $B(x, y) = B(y, x)$.

If

$$\int_0^\infty e^{-(1+v)t}\, t^{x+y-1}\,dt = \frac{\Gamma(x+y)}{(1+v)^{x+y}}$$

[cf. 1.1(4)] is multiplied by v^{x-1}, integrated with respect to v between 0 and ∞, and if the order of integration is inverted, we have

$$\int_0^\infty dt \int_0^\infty e^{-t(v+1)}\, t^{x+y-1}\, v^{x-1}\,dv = \Gamma(x+y)\int_0^\infty v^{x-1}(1+v)^{-x-y}\,dv$$

or

(5) $B(x, y) = \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$,

the expression for the beta function in terms of the gamma function.

The following functional equations for the beta function can be deduced easily from (4) and (5). (See Section 1.2):

(6) $B(x, y+1) = (y/x)\, B(x+1, y) = [y/(x+y)]\, B(x, y)$,

(7) $B(x, y)\, B(x+y, z) = B(y, z)\, B(y+z, x) = B(z, x)\, B(x+z, y)$,

(8) $B(x, y)\, B(x+y, z)\, B(x+y+z, u) = \dfrac{\Gamma(x)\,\Gamma(y)\,\Gamma(z)\,\Gamma(u)}{\Gamma(x+y+z+u)}$,

(9) $\dfrac{1}{B(n, m)} = m\binom{n+m-1}{n-1} = n\binom{n+m-1}{m-1} \qquad n, m$, positive integers.

1.5.1. Definite integrals expressible in terms of the beta function

By means of suitable substitutions, a number of definite integrals, such as the following, are reducible to the beta function:

(10) $\mathrm{B}(x, y) = 2^{1-x-y} \int_0^1 [(1+t)^{x-1}(1-t)^{y-1} + (1+t)^{y-1}(1-t)^{x-1}]\, dt$

$\mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0,$

(11) $\int_0^1 t^{x-1}(1-t)^{y-1}(1+bt)^{-x-y}\, dt = (1+b)^{-x}\, \mathrm{B}(x, y)$

$b > -1, \quad \mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0,$

(12) $\int_0^\infty t^{x-1}(1+bt)^{-x-y}\, dt = b^{-x}\, \mathrm{B}(x, y)$

$b > 0, \quad \mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0,$

(13) $\int_b^a (t-b)^{x-1}(a-t)^{y-1}\, dt = (a-b)^{x+y-1}\, \mathrm{B}(x, y)$

$\mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0, \quad b < a,$

(14) $$\int_b^a \frac{(t-b)^{x-1}(a-t)^{y-1}}{(t-c)^{x+y}}\, dt = \frac{(a-b)^{x+y-1}}{(a-c)^x (b-c)^y}\, \mathrm{B}(x, y)$$

$\mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0, \quad c < b < a,$

(15) $$\int_b^a \frac{(t-b)^{x-1}(a-t)^{y-1}}{(c-t)^{x+y}}\, dt = \frac{(a-b)^{x+y-1}}{(c-a)^x (c-b)^y}\, \mathrm{B}(x, y)$$

$\mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0, \quad b < a < c,$

(16) $\int_0^\infty (1+bt^z)^{-y}\, t^x\, dt = z^{-1}\, b^{-(x+1)/z}\, \mathrm{B}[(x+1)/z,\, y-(x+1)/z]$

$z > 0, \quad b > 0, \quad 0 < \mathrm{Re}\, [(x+1)/z] < \mathrm{Re}\, y,$

(17) $\int_0^1 t^{x-1}(1-t^z)^{y-1}\, dt = z^{-1}\, \mathrm{B}(xz^{-1}, y)$

$z > 0, \quad \mathrm{Re}\, y > 0, \quad \mathrm{Re}\, x > 0,$

(18) $\int_{-1}^1 (1+t)^{2x-1}(1-t)^{2y-1}(1+t^2)^{-x-y}\, dt = 2^{x+y-2}\, \mathrm{B}(x, y)$

$\mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0.$

Substituting trigonometric and hyperbolic functions, we obtain a number of integrals involving trigonometric and hyperbolic functions:

(19) $\int_0^{\frac{1}{2}\pi} (\sin t)^{2x-1}(\cos t)^{2y-1}\, dt = \frac{1}{2}\, \mathrm{B}(x, y)$

$\mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0,$

(20) $$\int_0^{\frac{1}{2}\pi} \frac{(\sin t)^{2x-1}(\cos t)^{2y-1}}{(1+b\sin^2 t)^{x+y}}\, dt = \tfrac{1}{2}(1+b)^{-x}\, \mathrm{B}(x, y)$$

$\mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0, \quad b > -1,$

(21) $$\int_0^{\frac{1}{2}\pi} \frac{(\sin t)^{2x-1} (\cos t)^{2y-1}}{(\cos^2 t + b \sin^2 t)^{x+y}} dt = \tfrac{1}{2} b^{-x} \mathrm{B}(x, y)$$

$\mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0, \quad b > 0,$

(22) $$\int_0^\infty \cosh t\, (\sinh t)^{2x-1} (1 + b \sinh^2 t)^{-x-y}\, dt = \tfrac{1}{2} b^{-x} \mathrm{B}(x, y)$$

$\mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0, \quad b > 0,$

(23) $$\int_0^\infty (\sinh t)^\alpha (\cosh t)^{-\beta} dt = \tfrac{1}{2} \mathrm{B}(\tfrac{1}{2}\alpha + \tfrac{1}{2}, \tfrac{1}{2}\beta - \tfrac{1}{2}\alpha)$$

$\mathrm{Re}\, \alpha > -1, \quad \mathrm{Re}(\alpha - \beta) < 0,$

(24) $$\int_0^\infty e^{-xt} (1 - e^{-tz})^{y-1} dt = z^{-1} \mathrm{B}(x/z, y)$$

$\mathrm{Re}\, x/z > 0, \quad \mathrm{Re}\, z > 0, \quad \mathrm{Re}\, y > 0,$

(25) $$\int_0^\infty e^{-\alpha t} [\sinh (\beta t)]^\gamma dt = \beta^{-1} 2^{-1-\gamma} \mathrm{B}(\tfrac{1}{2} \tfrac{\alpha}{\beta} - \tfrac{1}{2} \gamma, 1 + \gamma)$$

$\mathrm{Re}\, \gamma > -1, \quad \mathrm{Re}\, \beta > 0, \quad \mathrm{Re}(\alpha/\beta) > \mathrm{Re}\, \gamma,$

(26) $$\int_0^\infty \frac{\cosh (2\alpha t)}{[\cosh (pt)]^{2\beta}} dt = 4^{\beta-1} p^{-1} \mathrm{B}(\beta + \alpha/p, \beta - \alpha/p)$$

$\mathrm{Re}(\beta \pm \alpha/p) > 0, \quad p > 0,$

(27) $$\int_0^\infty \cos (2zt) \operatorname{sech}(\pi t)\, dt = \tfrac{1}{2} \operatorname{sech} z \qquad |\mathrm{Im}\, z| < \tfrac{1}{2}\pi,$$

(28) $$\int_0^\infty \cosh (2zt) \operatorname{sech}(\pi t)\, dt = \tfrac{1}{2} \sec z \qquad |\mathrm{Re}\, z| < \tfrac{1}{2}\pi.$$

Formula (27) is known as Ramanujan's formula.

Formulas (12), (13), (17), (19) originate from (1); (11) from (2); (10) and (26) from (3); (14), (15), (20), and (21) from (11); (16) and (22) from (12); (18) from (16); (24) from (17); (23) from (22); (25) from (24); (27) and (28) from (26); all are obtained by easily recognizable substitutions or specializations of parameters. Evidently the range of validity of the formulas (11), (20), and (12), (16), (21), (22) with respect to b can be extended to any values of b in the complex b-plane supposed cut along the real axis from -1 to $-\infty$ and from 0 to $-\infty$ respectively.

By complex integration it is possible to express some further trigonometric integrals in terms of the gamma function. Consider

$$\int_C (z^{-1} - z)^\alpha z^{\beta-1} dz$$

where C is a contour consisting of the upper semi-circle $|z| = 1$ and its diameter. The contour is indented at $z = 0, \pm 1$, and the radius of each indentation is ϵ. On letting ϵ approach zero, one obtains (cf. Nielsen, 1906, p. 158) the following result:

$$(29)\quad \int_0^{\pi} (\sin t)^{\alpha} e^{i\beta t}\, dt = \frac{\pi}{2^{\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1+\frac{\alpha+\beta}{2})\,\Gamma(1+\frac{\alpha-\beta}{2})}\, e^{i\frac{1}{2}\pi\beta}$$

$\mathrm{Re}\,\alpha > -1.$

If C is a contour consisting of the semi-circle $|z| = 1$ in the right-half plane and the straight line joining the points $z = \pm i$, with indentations at $z = 0, \pm i$, and if the radii of indentation are made to approach 0, the evaluation of

$$\int_c (z^{-1} + z)^{\alpha} z^{\beta-1}\, dz$$

gives

$$(30)\quad \int_0^{\frac{1}{2}\pi} (\cos t)^{\alpha} \cos(\beta t)\, dt = \frac{\pi}{2^{\alpha+1}} \frac{\Gamma(1+\alpha)}{\Gamma(1+\frac{\alpha+\beta}{2})\,\Gamma(1+\frac{\alpha-\beta}{2})}$$

$\mathrm{Re}\,\alpha > -1.$

For other similar integrals see 2.4(6) to 2.4(10).

Next consider

$$\int_c z^{\alpha-1} e^{-cz}\, dz \qquad c > 0,$$

where the contour C consists of the real axis from $+\epsilon$ to $+R$, the arc of the circle $z = Re^{i\phi}$ from $\phi = 0$ to $\phi = \beta(-\frac{1}{2}\pi \le \beta \le \frac{1}{2}\pi)$, the straight line from $z = Re^{i\beta}$ to $\epsilon e^{i\beta}$, and the arc of the circle $z = \epsilon e^{i\phi}$ from $\phi = \beta$ to $\phi = 0$. Since the value of the contour integral is zero, on making $\epsilon \to 0$ and $R \to \infty$ it follows that

$$(31)\quad \int_0^{\infty} t^{\alpha-1} e^{-ct\cos\beta - ict\sin\beta}\, dt = \Gamma(\alpha)\, c^{-\alpha} e^{-i\alpha\beta}$$

$-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi, \quad \mathrm{Re}\,\alpha > 0, \text{ or } \beta = \pm\frac{1}{2}\pi, \quad 0 < \mathrm{Re}\,\alpha < 1.$

With $p = c\cos\beta$, $q = c\sin\beta$

$$(32)\quad \int_0^{\infty} t^{\alpha-1} e^{-pt-iqt}\, dt = \Gamma(\alpha)\,(p^2+q^2)^{-\frac{1}{2}\alpha} e^{-i\alpha\tan^{-1}(p/q)}$$

$p > 0, \quad \mathrm{Re}\,\alpha > 0, \text{ or } p = 0, \quad 0 < \mathrm{Re}\,\alpha < 1.$

With $p + iq = s$, $\tan^{-1}(q/p) = \arg s$

$$(33)\quad \int_0^{\infty} t^{\alpha-1} e^{-st}\, dt = \Gamma(\alpha)\, s^{-\alpha}$$

$\mathrm{Re}\,\alpha > 0, \quad \mathrm{Re}\,s > 0 \text{ or } \mathrm{Re}\,s = 0, \quad 0 < \mathrm{Re}\,\alpha < 1,$

and hence more generally

(34) $\int_0^{\infty e^{i\delta}} t^{\alpha-1} e^{-st} dt = \Gamma(\alpha) s^{-\alpha}$

$\text{Re } \alpha > 0, \quad -(\tfrac{1}{2}\pi + \delta) < \arg s < \tfrac{1}{2}\pi - \delta.$

From (32) one obtains

(35) $\int_0^{\infty} t^{\alpha-1} e^{-ct\cos\beta} \cos(ct \sin\beta)\, dt = \Gamma(\alpha) c^{-\alpha} \cos(\alpha\beta)$

$c > 0, \quad \text{Re } \alpha > 0, \quad -\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi,$

(36) $\int_0^{\infty} t^{\alpha-1} e^{-ct\cos\beta} \sin(ct \sin\beta)\, dt = \Gamma(\alpha) c^{-\alpha} \sin(\alpha\beta)$

$c > 0, \quad \text{Re } \alpha > -1, \quad -\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi.$

If β approaches $\tfrac{1}{2}\pi$ and c is greater than zero, then

(37) $\int_0^{\infty} t^{\alpha-1} \cos(ct)\, dt = c^{-\alpha}\Gamma(\alpha) \cos(\tfrac{1}{2}\pi\alpha) \qquad 0 < \text{Re } \alpha < 1,$

(38) $\int_0^{\infty} t^{\alpha-1} \sin(ct)\, dt = c^{-\alpha}\Gamma(\alpha) \sin(\tfrac{1}{2}\pi\alpha) \qquad -1 < \text{Re } \alpha < 1.$

Furthermore, one obtains

(39) $\int_0^{\infty} \cos(at^p)\, dt = (pa^{1/p})^{-1}\, \Gamma(1/p) \cos[\pi(2p)^{-1}] \qquad a > 0, \quad p > 1,$

(40) $\int_0^{\infty} \sin(at^p)\, dt = (pa^{1/p})^{-1}\, \Gamma(1/p) \sin[\pi(2p)^{-1}]$

1.6. The gamma and beta functions expressed as contour integrals

We use the notation $\int_{\zeta}^{(0+)} f(t)\, dt$ for an integral taken along a contour C which starts at a point ζ, encircles the origin once counter-clockwise and returns to its starting point, it being understood that all singularities of the integrand except $t = 0$ are outside C.

Consider $\int_{-\infty}^{(0+)} e^t t^{-z}\, dt$, the initial and final values of $\arg t$ being $-\pi$ and $+\pi$ respectively. Taking C to consist of the lower edge of the cut from $-\infty$ to $-\rho$, the circle $t = \rho e^{i\varphi}$ $(-\pi \le \varphi \le \pi)$, and the upper edge of the cut from $-\rho$ to $-\infty$, we find that

$$\int_{-\infty}^{(0+)} e^t t^{-z}\, dt = 2i \sin(\pi z) \int_{\rho}^{\infty} e^{-v} v^{-z}\, dv + I,$$

where I denotes the integral along the circle $|t| = \rho$. Since I tends to zero with ρ, provided that $\text{Re } z < 1$, we have, in view of 1.1(1),

(1) $\int_{-\infty}^{(0+)} e^t t^{-z}\, dt = 2i \sin(\pi z)\, \Gamma(1 - z)$

or, by means of 1.2(6), Hankel's representation

(2) $1/\Gamma(z) = 1/(2\pi i) \int_{-\infty}^{(0+)} e^t t^{-z}\, dt \qquad |\arg t| \le \pi.$

Since both sides of this equation represent entire functions of z, the equation is valid for all values of z.

If we replace z by $1 - z$ in (1) we obtain

$$(3)\quad 2i\,\sin(\pi z)\,\Gamma(z) = \int_{-\infty}^{(0+)} e^{t}\, t^{z-1}\, dt \qquad |\arg t| \le \pi.$$

Equation (3) may be written as

$$(4)\quad 2i\,\sin(\pi z)\,\Gamma(z) = -\int_{\infty}^{(0+)} (-t)^{z-1}\, e^{-t} dt \qquad |\arg(-t)| \le \pi.$$

In the same manner, a more general expression can be obtained by the aid of 1.5(34) if we consider the contour integral

$$\int_{\infty \exp i\delta}^{(0+)} t^{s-1}\, e^{-t\zeta}\, dt.$$

The initial and final values of arg t are now taken to be δ and $2\pi + \delta$. This leads to

$$(5)\quad \Gamma(s) = \zeta^{s}\,(e^{2\pi i s} - 1)^{-1} \int_{\infty \exp i\delta}^{(0+)} t^{s-1}\, e^{-t\zeta}\, dt$$

$$-(\tfrac{1}{2}\pi + \delta) < \arg \zeta < \tfrac{1}{2}\pi - \delta, \quad \delta \le \arg t \le 2\pi + \delta, \quad s \ne 0, \pm 1, \pm 2, \ldots,$$

or, by replacing s by $1 - s$ and using 1.2(6), we have

$$(6)\quad 2\pi i\,(\zeta e^{-i\pi})^{s-1}/\Gamma(s) = \int_{\infty \exp i\delta}^{(0+)} t^{-s}\, e^{-t\zeta}\, dt$$

$$-(\tfrac{1}{2}\pi + \delta) < \arg \zeta < \tfrac{1}{2}\pi - \delta, \quad \delta \le \arg t \le 2\pi + \delta,$$

which is valid for all values of s.

Finally, consider

$$\int_c t^{x-1}\,(1-t)^{y-1}\, dt = \int_c f(t)\, dt$$

taken around a closed contour which starts from a point A on the real t-axis between 0 and 1, and consists of a loop around $t = 1$ in the positive sense, a loop around $t = 0$ in the positive sense, a loop around $t = 1$ in the negative sense, and a loop around $t = 0$ in the negative sense, so that $f(t)$ returns to A with its initial value, which is positive real and taken with argument zero. Take the loop around 1 to consist of the line from A to $1 - \rho$, the small circle $|t - 1| = \rho$ and the line from $1 - \rho$ to A, and similarly with the other loops. On making $\rho \to 0$,

$$\int^{(1+,0+,1-,0-)} t^{x-1}\,(1-t)^{y-1}\, dt = (1 - e^{2\pi i x})\,(1 - e^{2\pi i y})\,\mathrm{B}(x, y)$$

$$\operatorname{Re} x > 0, \quad \operatorname{Re} y > 0.$$

Hence

$$(7)\quad \mathrm{B}(x, y) = \frac{-e^{-i\pi(x+y)}}{4\sin(\pi x)\,\sin(\pi y)} \int^{(1+,0+,1-,0-)} t^{x-1}\,(1-t)^{y-1}\, dt,$$

and by the theory of analytic continuation this formula, derived in the first instance for $\operatorname{Re} x > 0$, $\operatorname{Re} y > 0$, x, y, not integers, holds for all values of x and y except integers. It is due to Pochhammer.

In a similar manner $B(x, y)$ can be represented as a single loop integral

(8) $$B(x, y) = \tfrac{1}{2} \operatorname{csch}(\pi i y) \int_0^{(1+)} t^{x-1} (t-1)^{y-1}\, dt$$
$$\operatorname{Re} x > 0, \quad |\arg(t-1)| \le \pi, \quad y \neq 0, \pm 1, \pm 2, \ldots,$$

(9) $$B(x, y) = -\tfrac{1}{2} \operatorname{csch}(\pi i x) \int_1^{(0+)} (-t)^{x-1} (1-t)^{y-1}\, dt$$
$$\operatorname{Re} y > 0, \quad |\arg(-t)| \le \pi, \quad x \neq 0, \pm 1, \pm 2, \ldots.$$

1.7. The ψ function

The function $\psi(z)$ is the logarithmic derivative of the gamma function:

(1) $$\psi(z) = \frac{d \log \Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log \Gamma(z) = \int_1^z \psi(z)\, dz.$$

From equations 1.1 (2) and 1.1 (3) we obtain the representations

(2) $$\psi(z) = \lim_{n \to \infty} \left[\log n - \frac{1}{z} - \frac{1}{z+1} - \frac{1}{z+2} - \cdots - \frac{1}{z+n} \right],$$

(3) $$\psi(z) = -\gamma - (1/z) + \sum_{n=1}^{\infty} z/[n(z+n)]$$
$$= -\gamma + (z-1) \sum_{n=0}^{\infty} 1/[(n+1)(z+n)].$$

The ψ function is meromorphic with simple poles at $z = 0, -1, -2, \ldots$. Clearly

(4) $$\psi(1) = -\gamma.$$

From equation 1.3 (2) with $u = z$, $v = 1$ we have

(5) $$\log \frac{\Gamma(1+z)}{\Gamma(z)} = \log z = -\gamma + \sum_{n=0}^{\infty} \{(n+1)^{-1} - \log[1 + 1/(n+z)]\}.$$

From equations (3) and (5)

(6) $$\psi(z) = \log z - \sum_{n=0}^{\infty} \{(n+z)^{-1} - \log[1 + 1/(n+z)]\}.$$

From equation (6) and 1.1 (1)

(7) $$\gamma = -\psi(1) = \sum_{n=1}^{\infty} [n^{-1} - \log(1 + n^{-1})] = -\int_0^{\infty} e^{-t} \log t\, dt.$$

1.7.1. Functional equations for $\psi(z)$

From equations 1.2(1), 1.2(2), 1.2(6) and 1.2(11) we have

(8) $\psi(z) = \psi(1+z) - 1/z,$

(9) $$\psi(1+n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \gamma,$$

(10) $$\psi(z+n) = \frac{1}{z} + \frac{1}{z+1} + \cdots + \frac{1}{z+n-1} + \psi(z) \qquad n = 1, 2, 3, \ldots,$$

(11) $\psi(z) - \psi(1-z) = -\pi \operatorname{ctn}(\pi z),$

$\psi(z) - \psi(-z) = -\pi \operatorname{ctn}(\pi z) - 1/z,$

$\psi(1+z) - \psi(1-z) = z^{-1} - \pi \operatorname{ctn}(\pi z),$

$\psi(\tfrac{1}{2}+z) - \psi(\tfrac{1}{2}-z) = \pi \tan(\pi z),$

(12) $$\psi(mz) = m^{-1} \sum_{r=0}^{m-1} \psi(z + r/m) + \log m.$$

1.7.2. Integral representations for $\psi(z)$

The formula

(13) $$\psi(z) = -\gamma + \int_0^1 (1-t^{z-1})(1-t)^{-1}\, dt \qquad \operatorname{Re} z > 0,$$

is easily verified by expanding $(1-t)^{-1}$ into a series, integrating term by term, and using (3).

The substitution $t = e^{-x}$ gives

(14) $$\psi(z) = -\gamma + \int_0^\infty (e^{-t} - e^{-tz})(1-e^{-t})^{-1}\, dt \qquad \operatorname{Re} z > 0.$$

Hence we have

$$\psi[\tfrac{1}{2} + \tfrac{1}{2}(a+\beta)/b] - \psi[\tfrac{1}{2} + \tfrac{1}{2}(a-\beta)/b]$$
$$= 2b \int_0^\infty e^{-at} \sinh(\beta t)\, [\sinh(bt)]^{-1}\, dt \qquad \operatorname{Re}(a + b \pm \beta) > 0.$$

From (11) we obtain a formula for $\psi(z)$ valid for Re $z < 1$,

(15) $$\psi(z) = -\gamma - \pi \operatorname{ctn}(\pi z) + \int_0^1 (1-t^{-z})(1-t)^{-1}\, dt \qquad \operatorname{Re} z < 1$$

or

(16) $$\psi(z) = -\gamma - \pi \operatorname{ctn}(\pi z) + \int_0^\infty (1-e^{tz})(e^t - 1)^{-1}\, dt \qquad \operatorname{Re} z < 1.$$

Gauss' integral formula,

(17) $$\psi(z) = \int_0^\infty [t^{-1} e^{-t} - (1-e^{-t})^{-1} e^{-tz}]\, dt \qquad \operatorname{Re} z > 0,$$

can be proved as follows. Integrating

$$x^{-1} = \int_0^\infty e^{-xt}\, dt$$

with respect to x from 1 to n we have

(18) $\log n = \int_0^\infty (e^{-t} - e^{-nt})\, t^{-1}\, dt.$

Introducing this and $1/(z+m) = \int_0^\infty e^{-(m+z)t} dt$ in (2) we have

$$\psi(z) = \lim_{n\to\infty} \{ \int_0^\infty [(e^{-t} - e^{-nt})\, t^{-1} - e^{-tz} - e^{-t(z+1)} - \cdots - e^{-t(z+n)}]\, dt \}$$

$$= \lim_{n\to\infty} \{ \int_0^\infty [t^{-1} e^{-t} - (1-e^{-t})^{-1} e^{-tz}]\, dt$$

$$- \int_0^\infty e^{-nt} [t^{-1} - (1-e^{-t})^{-1} e^{-t(z+1)}]\, dt \}.$$

The first integral is independent of n, and the second tends to zero as $n \to \infty$. This proves (17).

Taking $z = 1$ in (17) an integral formula for Euler's constant is obtained:

(19) $\gamma = \int_0^\infty [(1-e^{-t})^{-1} - t^{-1}]\, e^{-t}\, dt.$

With $t = \log(1+x)$ and $\delta = \log(1+\Delta)$ we have from (17)

$$\psi(z) = \lim_{\delta\to 0} \int_\delta^\infty [t^{-1} e^{-t} - (1-e^{-t})^{-1} e^{-zt}]\, dt$$

$$= \lim_{\delta\to 0} \int_\delta^{e^\delta - 1} t^{-1} e^{-t}\, dt + \lim_{\Delta\to 0} \int_\Delta^\infty [e^{-x} - (1+x)^{-z}]\, x^{-1}\, dx.$$

Since the first limit is zero, Dirichlet's formula,

(20) $\psi(z) = \int_0^\infty [e^{-t} - (1+t)^{-z}]\, t^{-1}\, dt$ $\qquad$ $\operatorname{Re} z > 0,$

follows. Also we have

(21) $\gamma = -\psi(1) = -\int_0^\infty [e^{-t} - (1+t)^{-1}]\, t^{-1}\, dt$

$$= -\int_0^\infty [\cos t - (1+t^2)^{-1}]\, t^{-1}\, dt.$$

The first integral follows from equation (20), and the second can be obtained by integrating $t^{-1} e^{-t} - t^{-1}(1+t)^{-1}$ around a quadrant of a circle indented at the origin, the origin being the centre of the circle.

From equations (20) and (21) we obtain

$$\psi(z) = -\gamma + \int_0^\infty [(1+t)^{-1} - (1+t)^{-z}] t^{-1}\, dt \qquad \operatorname{Re} z > 0.$$

Binet's expressions,

(22) $\psi(z) = \log z + \int_0^\infty [t^{-1} - (1 - e^{-t})^{-1}] e^{-tz} dt$ $\qquad$ Re $z > 0$,

(23) $\psi(z) = \log z - \tfrac{1}{2} z^{-1} - \int_0^\infty [(1 - e^{-t})^{-1} - t^{-1} - \tfrac{1}{2}] e^{-tz} dt$

Re $z > 0$,

(24) $\psi(z) = \log z + \int_0^\infty [(1 - e^{t})^{-1} + t^{-1} - 1] e^{-tz} dt$ $\qquad$ Re $z > 0$,

(25) $\psi(z) = \log z - \tfrac{1}{2} z^{-1} - \int_0^\infty [(e^{t} - 1)^{-1} - t^{-1} + \tfrac{1}{2}] e^{-tz} dt$ $\qquad$ Re $z > 0$,

can easily be obtained from (17) and (18).

The more general expression

(26) $\psi(z) = \log z - \tfrac{1}{2} z^{-1} - \int_0^{\infty e^{i\beta}} [(e^{t} - 1)^{-1} - t^{-1} + \tfrac{1}{2}] e^{-tz} dt$

$-\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi, \quad -(\tfrac{1}{2}\pi + \beta) < \arg z < (\tfrac{1}{2}\pi - \beta)$

can be deduced from (25) by integrating

$$[(e^{t} - 1)^{-1} - t^{-1} + \tfrac{1}{2}] e^{-tz}$$

around a sector indented at the origin, as in the derivation of 1.5(31).

From 1.9(9) we obtain

(27) $\psi(z) = \dfrac{d \log \Gamma(z)}{dz} = \log z - \tfrac{1}{2} z^{-1} - 2\int_0^\infty (t^2 + z^2)^{-1} (e^{2\pi t} - 1)^{-1} t\, dt$

Re $z > 0$,

which is likewise due to Binet. Hence we have

(28) $\gamma = -\psi(1) = \tfrac{1}{2} + 2\int_0^\infty (t^2 + 1)^{-1} (e^{2\pi t} - 1)^{-1} t\, dt.$

1.7.3. The theorem of Gauss

Taking $z = p/q$ in (13), $0 < p < q$, p and q integers, and putting $t = v^q$ we obtain

$$\psi(p/q) = -\gamma + \int_0^1 R(v)\, dv, \qquad R(v) = q(v^{p-1} - v^{q-1})(v^q - 1)^{-1}.$$

Since

$$v^q - 1 = (v - 1) \prod_{n=1}^{q-1} [v - \exp(2\pi i n/q)],$$

we can decompose $R(v)$ into partial fractions:

$$R(v) = \sum_{n=1}^{q-1} [\exp(2\pi i p n/q) - 1] [v - \exp(2\pi i n/q)]^{-1}.$$

Introducing $R(v)$ and integrating it can be shown (Böhmer 1939, p. 77) that

$$(29)\quad \psi(p/q) = -\gamma - \log q - \tfrac{1}{2}\pi \operatorname{ctn}(\pi p/q) + \sum_{n=1}^{\leq \frac{1}{2} q}{}' \cos(2\pi p n/q) \log[2 - 2\cos(2\pi n/q)].$$

The prime indicates that in case of an even q only one-half of the last term shall be taken in the sum. Thus for a positive proper fraction z the value of $\psi(z)$ can be expressed as a finite combination of elementary functions. By means of (10) this result may be extended to every rational value of z. This is Gauss' theorem.

1.7.4. Some infinite series connected with the ψ function

If we define

$$\Delta f(z) = f(z+1) - f(z), \qquad \Delta^n f(z) = \sum_{m=0}^{n} \binom{n}{m} f(z+n-m)(-1)^m,$$

it follows from 1.7(8) that

$$\Delta \psi(a+z) = 1/(a+z),$$

so that we have

$$\Delta^2 \psi(a+z) = \Delta[1/(a+z)] = -1/[(a+z)(a+z+1)]$$

and

$$\Delta^n \psi(a+z) = \Delta^{n-1}[1/(a+z)]$$
$$= (-1)^{n-1}(n-1)!/[(a+z)(a+z+1)\cdots(a+z+n-1)].$$

Hence the development of $\psi(a+z)$ in a factorial series is convergent for $\mathrm{Re}(a+z) > 0$, a not a negative integer, and is of the form (Nörlund 1924, p. 261)

$$(30)\quad \psi(a+z) = \psi(a) + \frac{z}{a} - \frac{1}{2}\,\frac{z(z-1)}{a(a+1)} + \frac{1}{3}\,\frac{z(z-1)(z-2)}{a(a+1)(a+2)} - \cdots$$

The functional equation 1.7(10) is useful for summing some series. We have for instance:

$$(31)\quad \sum_{m=0}^{n} (a+mb)^{-1} = b^{-1} \sum_{m=0}^{n} (m+a/b)^{-1} = b^{-1}[\psi(n+1+a/b) - \psi(a/b)],$$

$$(32)\quad \frac{1}{a+b} - \frac{1}{a+2b} + \cdots - \frac{1}{a+2nb} = \frac{1}{4b} \sum_{m=1}^{n} \left(\frac{a-b}{2b} + m\right)^{-1} \left(\frac{a}{2b} + m\right)^{-1}$$
$$= \tfrac{1}{2} b^{-1} \left[\psi\left(\frac{a}{2b} + n + \frac{1}{2}\right) - \psi\left(\frac{a}{2b} + \frac{1}{2}\right) - \psi\left(\frac{a}{2b} + n + 1\right) + \psi\left(\frac{a}{2b} + 1\right)\right],$$

and, if $n \to \infty$,

(33) $$\frac{1}{a+b} - \frac{1}{a+2b} + \frac{1}{a+3b} - \cdots = \tfrac{1}{2} b^{-1}[\psi(1+\tfrac{1}{2}ab^{-1}) - \psi(\tfrac{1}{2}+\tfrac{1}{2}ab^{-1})].$$

1.8. The function $G(z)$

The function $G(z)$ is defined by

(1) $$G(z) = \psi(\tfrac{1}{2} + \tfrac{1}{2}z) - \psi(\tfrac{1}{2}z).$$

From 1.7(13) and 1.7(14) we have

(2) $$G(z) = 2\int_0^1 t^{z-1}(1+t)^{-1}\,dt \qquad \operatorname{Re} z > 0,$$

(3) $$G(z) = 2\int_0^\infty e^{-zt}(1+e^{-t})^{-1}\,dt \qquad \operatorname{Re} z > 0.$$

A consideration of $\int_c e^{-zt}(1+e^{-t})^{-1}\,dt$ extended over the contour used in deriving 1.5(31) yields the more general representation

(4) $$G(z) = 2\int_0^{\infty e^{i\beta}} e^{-zt}(1+e^{-t})^{-1}\,dt$$

$$-\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi, \quad -(\tfrac{1}{2}\pi+\beta) < \arg z < \tfrac{1}{2}\pi - \beta,$$

or

(5) $$G(z) = z^{-1} + \int_0^{\infty e^{i\beta}} \tanh(\tfrac{1}{2}t)\, e^{-zt}\,dt$$

$$-\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi, \quad -(\tfrac{1}{2}\pi+\beta) < \arg z < \tfrac{1}{2}\pi - \beta.$$

If we expand $(1+t)^{-1}$ in (2), and integrate term by term, we obtain

(6) $$G(z) = 2\sum_{n=0}^{\infty}(-1)^n(z+n)^{-1} = 2z^{-1}\,{}_2F_1(1, z; 1+z; -1).$$

The functional equations

(7) $$G(1+z) = 2z^{-1} - G(z),$$

(8) $$G(1-z) = 2\pi\csc(\pi z) - G(z),$$

(9) $$G(mz) = -(2/m)\sum_{r=0}^{m-1}(-1)^r\,\psi(z+r/m) \qquad m \text{ even},$$

(10) $$G(mz) = (1/m)\sum_{r=0}^{m-1}(-1)^r\,G(z+r/m) \qquad m \text{ odd},$$

follow from (1) in conjunction with 1.7(1).

1.9. Expressions for the function $\log\Gamma(z)$

From 1.7(17) we obtain Malmstén's formula

(1) $\log \Gamma(z) = \int_1^z \psi(z)\,dz = \int_0^\infty \left[(z-1) - \frac{1-e^{-(z-1)t}}{1-e^{-t}} \right] \frac{e^{-t}}{t}\,dt$ $\qquad \operatorname{Re} z > 0$

and from 1.7(25)

(2) $\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + 1$

$+ \int_0^\infty [(e^t - 1)^{-1} - t^{-1} + \tfrac{1}{2}] (e^{-tz} - e^{-t})\, t^{-1}\, dt$ $\qquad \operatorname{Re} z > 0.$

Since (Whittaker-Watson, 1927, p. 249)

(3) $\int_0^\infty [\tfrac{1}{2} - t^{-1} + (e^t - 1)^{-1}]\, t^{-1} e^{-t} dt = 1 - \tfrac{1}{2} \log(2\pi),$

we have Binet's first expression of $\log \Gamma(z)$,

(4) $\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log(2\pi)$

$+ \int_0^\infty [(e^t - 1)^{-1} - t^{-1} + \tfrac{1}{2}]\, t^{-1} e^{-tz}\, dt$ $\qquad \operatorname{Re} z > 0,$

or, more generally [cf. 1.5(1) and also 1.7(25), 1.7(26)],

(5) $\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log(2\pi)$

$+ \int_0^{\infty e^{i\beta}} [(e^t - 1)^{-1} - t^{-1} + \tfrac{1}{2}]\, t^{-1} e^{-tz}\, dt$

$-\tfrac{1}{2}\pi < \beta < \tfrac{1}{2}\pi, \quad -(\tfrac{1}{2}\pi + \beta) < \arg z < \tfrac{1}{2}\pi - \beta.$

From 1.2(6) one obtains

$\log \Gamma(z) = \log \pi - \log(\sin \pi z) - \log \Gamma(1 - z)$

and hence

(6) $\log \Gamma(z) = \log \pi - \log(\sin \pi z)$

$- \int_0^\infty [(e^{zt} - 1)(1 - e^{-t})^{-1} - z]\, t^{-1} e^{-t} dt$ $\qquad \operatorname{Re} z < 1.$

Adding (1) and (6) we have

(7) $\log \Gamma(z) = \tfrac{1}{2} \log \pi - \tfrac{1}{2} \log(\sin \pi z)$

$+ \tfrac{1}{2} \int_0^\infty \{\sinh[(\tfrac{1}{2} - z)t] \operatorname{csch}(t/2) - (1 - 2z) e^{-t}\}\, t^{-1}\, dt$

$0 < \operatorname{Re} z < 1.$

Since

$|[\tfrac{1}{2} - t^{-1} + (e^t - 1)^{-1}]\, t^{-1}| \le K \quad \text{for} \quad 0 \le t < \infty,$

it is easily seen from Binet's first expression (4) that

(8) $|\log \Gamma(z) - (z - \tfrac{1}{2}) \log z + z - \tfrac{1}{2} \log 2\pi| < K/x$ $\qquad z = x + iy.$

Finally, we derive Binet's second expression for $\log\Gamma(z)$

(9) $$\log\Gamma(z) = (z - \tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + 2\int_0^\infty \frac{\tan^{-1}(t/z)}{e^{2\pi t} - 1}\,dt$$

$\operatorname{Re} z > 0.$

From 1.7(3) we have

(10) $$\psi'(z) = \frac{d^2 \log\Gamma(z)}{dz^2} = \sum_{n=0}^{\infty} 1/(z+n)^2.$$

Now we make use of a summation formula due to Plana (Lindelöf, 1906, p. 61),

(11) $$\sum_{n=0}^{\infty} f(n) = \tfrac{1}{2}f(0) + \int_0^\infty f(\tau)\,d\tau + i\int_0^\infty [f(it) - f(-it)](e^{2\pi t} - 1)^{-1}\,dt,$$

valid if

1) $f(\zeta)$ is regular for $\operatorname{Re}\zeta \geq 0$, $\qquad \zeta = \tau + it,$

2) $\lim_{t\to\infty} e^{-2\pi|t|} f(\tau + it) = 0$ uniformly for $0 \leq \tau < \infty$,

3) $\lim_{\tau\to\infty} \int_{-\infty}^{\infty} e^{-2\pi|t|}\,|f(\tau+it)|\,dt = 0.$

Taking $f(\zeta) = 1/(z+\zeta)^2$ ($\operatorname{Re} z > 0$) in (11) we find that

(12) $$\sum_{n=0}^{\infty} 1/(z+n)^2 = \psi'(z) = \tfrac{1}{2}z^{-2} + z^{-1} + \int_0^\infty 4tz\,(t^2+z^2)^{-2}(e^{2\pi t}-1)^{-1}\,dt.$$

Integrating twice from 1 to z we obtain

(13) $$\log\Gamma(z) = (z - \tfrac{1}{2})\log z + z(A-1) + B + 2\int_0^\infty (e^{2\pi t} - 1)^{-1}\tan^{-1}(t/z)\,dt,$$

A and B being integration constants. To determine these, we note that $0 \leq \tan^{-1} x \leq x$ for $x \geq 0$ so that

$$|\log\Gamma(z) - (z-\tfrac{1}{2})\log z - (A-1)z - B| < (2/z)\int_0^\infty (e^{2\pi t} - 1)^{-1}\,t\,dt$$

for z real and positive. The right-hand side vanishes as $z \to \infty$ through positive real values, and by comparison with (8) we at once have $A = 0$, $B = \tfrac{1}{2}\log(2\pi)$. This proves (9).

1.9.1. Kummer's series for $\log \Gamma(z)$

The function $\log\Gamma(x)$, $0 < x < 1$, can be expanded in a Fourier series. We shall use the known Fourier expansions (Bromwich, 1947, pp. 356, 393, and 370 respectively):

$$\log(\sin \pi x) = -\log 2 - \sum_{n=1}^{\infty} (1/n) \cos(2\pi n x),$$

$$\operatorname{csch}(\tfrac{1}{2}t) \sinh(\tfrac{1}{2} - x)t = 8\pi \sum_{n=1}^{\infty} [n \sin(2\pi n x)]/(t^2 + 4\pi^2 n^2),$$

$$\pi(1 - 2x) = 2 \sum_{n=1}^{\infty} (1/n) \sin(2\pi n x).$$

If these are substituted in (7) with $z = x$, we have to evaluate the integral

$$\int_0^\infty \left(\frac{2\pi n}{t^2 + 4\pi^2 n^2} - \frac{e^{-t}}{2\pi n}\right)\frac{dt}{t} = \frac{1}{2\pi n}\int_0^\infty \left(\frac{1}{1+t^2} - e^{-2\pi n t}\right)\frac{dt}{t}$$

$$= \frac{1}{2\pi n}\left[\int_0^\infty \left(\frac{1}{1+t^2} - \cos t\right)\frac{dt}{t} + \int_0^\infty \frac{e^{-t} - e^{-2\pi n t}}{t}\, dt + \int_0^\infty (\cos t - e^{-t})\frac{dt}{t}\right]$$

and by means of 1.7(21) and 1.7(18) this is $(2\pi n)^{-1}[\gamma + \log(2\pi n)]$ since we have for the third integral:

$$\lim_{\delta \to 0} \int_\delta^\infty (\cos t - e^{-t})\, t^{-1}\, dt = \lim_{\delta \to 0} [\operatorname{Ei}(-\delta) - \operatorname{Ci}(\delta)] = 0.$$

Thus we have

(14) $$\log \Gamma(x) = \tfrac{1}{2}\log(2\pi)$$

$$+ \sum_{n=1}^{\infty} [(2n)^{-1} \cos(2\pi n x) + (\gamma + \log 2\pi n)(\pi n)^{-1} \sin(2\pi n x)],$$

$$\log \Gamma(x) = (\tfrac{1}{2} - x)(\gamma + \log 2) + (1 - x)\log \pi - \tfrac{1}{2}\log(\sin \pi x)$$

$$+ \sum_{n=1}^{\infty} (\pi n)^{-1} \log n \sin(2\pi n x) \qquad 0 < x < 1,$$

which is Kummer's series.

A similar representation for $\psi(x)$ is due to Lerch (Nielsen, 1906, p. 204),

(15) $$\psi(x) \sin(\pi x) = -\tfrac{1}{2}\pi \cos(\pi x) - (\gamma + \log 2\pi) \sin \pi x$$

$$+ \sum_{n=1}^{\infty} \log\left(\frac{n}{n+1}\right) \sin(2n+1)\pi x \qquad 0 < x < 1.$$

From (14) we obtain the integral formulas:

$$(16)\quad \int_0^1 \log\Gamma(x)\sin(2\pi nx)\,dx = \frac{\gamma + \log(2\pi n)}{2\pi n} \qquad n = 1, 2, 3, \ldots,$$

$$(17)\quad \int_0^1 \log\Gamma(x)\cos(2\pi nx)\,dx = \frac{1}{4n} \qquad n = 1, 2, 3, \ldots,$$

$$(18)\quad \int_0^1 \log\Gamma(x)\,dx = \tfrac{1}{2}\log(2\pi).$$

Furthermore, we have

$$(19)\quad \int_x^{x+1} \log\Gamma(t)\,dt = x\log x - x + \tfrac{1}{2}\log(2\pi).$$

This formula can be proved in the following way.

From the multiplication formula 1.2(11) we have

$$m^{-1}\log[\Gamma(mx)(2\pi)^{m/2-1/2}m^{1/2-mx}] = \sum_{r=0}^{m-1} m^{-1}\log\Gamma(x + r/m).$$

If we now let $m \to \infty$, replace $\Gamma(mx)$ by its asymptotic expression 1.18(1), and observe that

$$\lim_{m\to\infty}\sum_{r=0}^{m-1} m^{-1}\log\Gamma(x + r/m) = \int_0^1 \log\Gamma(x + y)\,dy = \int_x^{x+1}\log\Gamma(t)\,dt,$$

we obtain (19).

Replacing x by $x + 1, x + 2, x + 3, \ldots, x + n - 1$, respectively, in (19) and adding the equations, we have more generally

$$(20)\quad \int_x^{x+n}\log\Gamma(x)\,dx = x\log x + (x + 1)\log(x + 1) + \cdots$$
$$+ (x + n - 1)\log(x + n - 1) - nx - \tfrac{1}{2}n(n - 1) + \tfrac{1}{2}n\log(2\pi) \qquad n = 1, 2, 3, \ldots.$$

1.10. The generalized zeta function

The generalized zeta function is defined for Re $s > 0$ by the equation

$$(1)\quad \zeta(s, v) = \sum_{n=0}^{\infty}(v + n)^{-s} \qquad v \neq 0, -1, -2, \ldots.$$

It satisfies the functional equation

$$(2)\quad \gamma(s, v) = \gamma(s, m + v) + \sum_{n=0}^{m-1}(n + v)^{-s} \qquad m = 1, 2, 3, \ldots.$$

Since for Re $s > 0$ and Re $v > 0$ we have from 1.1(5)

$$(v + n)^{-s}\,\Gamma(s) = \int_0^{\infty} e^{-(v+n)t}\,t^{s-1}\,dt,$$

it follows that

(3) $\Gamma(s)\ \zeta(s,v) = \int_0^\infty t^{s-1} e^{-vt}(1-e^{-t})^{-1}\,dt$

$= \int_0^1 x^{v-1}(1-x)^{-1}(\log 1/x)^{s-1}\,dx$ $\qquad$ Re $s > 1$, $\quad$ Re $v > 0$.

Considering $\int_c t^{s-1} e^{-vt}(1-e^{-t})^{-1}\,dt$ taken around the complete boundary of a sector of a circle, indented at the origin [cf. 1.5(1)], we have the more general representation

(4) $\Gamma(s)\ \zeta(s,v) = \int_0^{\infty e^{i\beta}} t^{s-1} e^{-vt}(1-e^{-t})^{-1}\,dt$

Re $s > 1$, $\quad -\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$, $\quad -(\frac{1}{2}\pi + \beta) < \arg v < \frac{1}{2}\pi - \beta$.

With the notation of section 1.6 equation (3) can be converted into a contour integral,

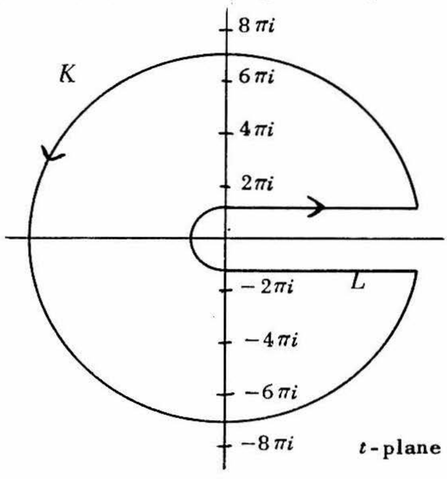
(5) $2\pi i\ \zeta(s,v) = -\Gamma(1-s)\int_\infty^{(0+)} (-t)^{s-1} e^{-vt}(1-e^{-t})^{-1}\,dt$

Re $v > 0$, $\quad |\arg(-t)| \le \pi$.

This integral gives a representation of $\zeta(s,v)$ valid over the whole s-plane with the exception of the points $s = 1, 2, 3, \ldots$. From it Hurwitz' series representation of $\zeta(s,v)$ can be obtained. Consider

$\int_c (-t)^{s-1} e^{-vt}(1-e^{-t})^{-1}\,dt$

taken around a closed contour C starting at the point $t = (2N+1)\pi$ and



consisting of a circle K and a loop L as indicated in the above figure.

The radius of the circle is $(2N+1)\pi$ (N an integer), and the loop L does not contain any of the points $t = \pm 2\pi i, \pm 4\pi i, \pm 6\pi i, \ldots$. In the region bounded by C the integrand of (5) is analytic and one valued except at the simple poles $\pm 2\pi i, \pm 4\pi i, \ldots, \pm 2N\pi i$. By the theorem of residues

$$\int_K \frac{(-t)^{s-1} e^{-vt}}{1-e^{-t}}\, dt + \int_L \frac{(-t)^{s-1} e^{-vt}}{1-e^{-t}}\, dt = 2\pi i \sum_{n=1}^{N} (R_n + R_n')$$

where R_n and R_n' are the residues of the integrand respectively at $2n\pi i$ and $-2n\pi i$,

$$R_n = (2n\pi)^{s-1} e^{-i\frac{1}{2}\pi(s-1)} e^{-2n\pi vi}, \quad R_n' = (2n\pi)^{s-1} e^{i\frac{1}{2}\pi(s-1)} e^{2n\pi vi}.$$

Letting $N \to \infty$ we find that the integral over K tends to zero provided $\operatorname{Re} s < 0$ and $0 < v \le 1$. By means of (5) we thus obtain Hurwitz' formula

$$(6) \quad \zeta(s, v) = 2(2\pi)^{s-1}\Gamma(1-s) \sum_{n=1}^{\infty} n^{s-1} \sin(2\pi n v + \tfrac{1}{2}\pi s)$$

$$\operatorname{Re} s < 0, \quad 0 < v \le 1.$$

Finally, we shall take $f(y) = (y+v)^{-s}$ in Plana's summation formula 1.9(11) to find

$$(7) \quad \zeta(s, v) = \frac{1}{2v^s} + \frac{v^{1-s}}{s-1} + 2\int_0^\infty \frac{\sin[s \tan^{-1}(t/v)]}{(v^2+t^2)^{\frac{1}{2}s}} \frac{dt}{e^{2\pi t}-1}$$

$$\operatorname{Re} v > 0,$$

which is Hermite's representation of $\zeta(s, v)$.

From (7) it can be seen that $\zeta(s, v)$ has only one singularity (a simple pole with residue 1) in the finite part of the s-plane. Furthermore we have [cf. 1.7(27)]

$$(8) \quad \zeta(0, v) = \tfrac{1}{2} - v,$$

$$(9) \quad \lim_{s \to 1}\left[\zeta(s, v) - \frac{1}{s-1}\right] = \frac{1}{2v} - \log v + 2\int_0^\infty \frac{t}{v^2+t^2} \frac{dt}{e^{2\pi t}-1}$$

$$= -\psi(v) \qquad \operatorname{Re} v > 0.$$

Differentiating (7) with respect to s, then putting $s = 0$, and using 1.9(9) we obtain

$$(10) \quad \left[\frac{d\,\zeta(s, v)}{ds}\right]_{s=0} = \log \Gamma(v) - \tfrac{1}{2}\log(2\pi).$$

In the special case when $s = -m\,(m = 0, 1, 2, \ldots)$, we have

$$(11)\quad \zeta(-m, v) = -\frac{B_{m+1}(v)}{m+1},$$

where $B_r(v)$ denotes the Bernoulli polynomial [cf. 1.13(3)]. To prove this, we note that if s is an integer, the integrand of (5) is a one-valued function of t, and we may apply Cauchy's theorem. If $s = -m\ (m = 0, 1, 2, \ldots)$, we have [cf. 1.13(2)]

$$(-t)^{-m-1}\frac{e^{-vt}}{1-e^{-t}} = (-1)^{-m-1}t^{-m-2}\frac{te^{-vt}}{1-e^{-t}}$$

$$= (-1)^{m-1}\sum_{n=0}^{\infty}(-1)^n B_n(v)\frac{t^{n-m-2}}{n!}.$$

Thus the residue of the integrand at $t = 0$ is $\dfrac{B_{m+1}(v)}{(m+1)!}$, and this proves (11).

1.11. The function $\Phi(z, s, v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n$

The function

$$(1)\quad \Phi(z, s, v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n \qquad |z| < 1, \quad v \neq 0, -1, -2, \ldots$$

satisfies the equation

$$(2)\quad y(z, s, v) = z^m\, y(z, s, m+v) + \sum_{n=0}^{m-1}(v+n)^{-s}z^n$$

$$m = 1, 2, 3, \ldots, \qquad v \neq 0, -1, -2, \ldots.$$

Since

$$(v+n)^{-s}z^n = [1/\Gamma(s)]\int_0^{\infty}e^{-vt}t^{s-1}(ze^{-t})^n\,dt$$

$$\operatorname{Re} v > 0, \qquad \operatorname{Re} s > 0,$$

From 1.1(5), we have the integral formula

$$(3)\quad \Phi(z, s, v) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}\,dt = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^t - z}\,dt$$

$\operatorname{Re} v > 0$ and either $|z| \le 1$, $z \neq 1$, $\operatorname{Re} s > 0$ or $z = 1$, $\operatorname{Re} s > 1$.

If a cut is made from 1 to ∞ along the positive real z-axis, Φ is an analytic function of z in the cut z-plane provided that $\operatorname{Re} s > 0$ and $\operatorname{Re} v > 0$.

Another representation by a definite integral can be obtained from the definition (1) and Plana's summation formula 1.9(11)

(4) $\Phi(z, s, v) = \tfrac{1}{2} v^{-s} + \int_0^\infty (v+t)^{-s} z^t \, dt$

$$- 2 \int_0^\infty \sin\{t \log z - s \tan^{-1}(t/v)\} (v^2 + t^2)^{-\frac{1}{2}s} (e^{2\pi t} - 1)^{-1} \, dt$$

$\operatorname{Re} v > 0.$

For $z = 1$ we have again Hermite's formula 1.10 (7).

Lipschitz's formula

$$2\Gamma(s) \sum_{n=1}^{\infty} e^{in\theta} (v+n)^{-s} = \int_0^\infty t^{s-1} e^{-vt} (e^{i\theta} - e^{-t}) (\cosh t - \cos\theta)^{-1} \, dt$$

$0 < \theta < 2\pi, \quad \operatorname{Re} s > 0, \quad \operatorname{Re} v > -1$

results from (3) by taking $z = e^{i\theta}$.

Φ can be represented as a contour integral

(5) $2\pi i \, \Phi(z, s, v) = -\Gamma(1-s) \int_\infty^{(0+)} (-t)^{s-1} e^{-vt} (1 - z e^{-t})^{-1} \, dt$

$\operatorname{Re} v > 0, \quad |\arg(-t)| \leq \pi$

assuming, as in the analogous work of 1.6, that the contour does not enclose any of the points $t = \log z \pm 2n\pi i (n = 0, 1, 2, \ldots)$, which are poles of the integrand of (5). Equation (5), for every fixed s which is not a positive integer, defines Φ as an analytic function of z regular in the cut plane, and for every fixed z in the cut plane, Φ as an analytic function of s regular, except possibly at the points $s = 1, 2, 3, \ldots$ (it being understood that $\operatorname{Re} v > 0$).

As in the preceding section our function can be represented by a series. In order to do so, consider

$$\int_C (-t)^{s-1} e^{-vt} (1 - z e^{-t})^{-1} \, dt$$

over the contour C consisting of a circle K of radius $(2N+1)\pi$ (N a positive integer) and a loop L round the origin. The center of the circle in this case is the point $t = \log z \, (z \neq 1)$, and all points $t = \log z \pm 2n\pi i$ $(n = 0, 1, 2, \ldots)$ are to be outside the loop. Letting $N \to \infty$, it is found that the integral over K tends to zero provided $\operatorname{Re} s < 0$ and $0 < v \leq 1$. Therefore

$$\Phi(z, s, v) = \Gamma(1-s) \sum_{n=-\infty}^{\infty} R_n,$$

where $R_n = z^{-1}(-t_n)^{s-1} e^{-(v-1)t_n}$ is the residue of the integrand at the pole $t = t_n = \log z + 2n\pi i$. Thus we have

(6) $\Phi(z, s, v) = z^{-v} \Gamma(1-s) \sum_{n=-\infty}^{\infty} (-\log z + 2n\pi i)^{s-1} e^{2n\pi iv}$

$0 < v \leq 1, \quad \operatorname{Re} s < 0, \quad |\arg(-\log z + 2n\pi i)| \leq \pi.$

Writing

$$\sum_{n=-\infty}^{\infty} (-\log z + 2n\pi i)^{s-1} e^{2n\pi iv} = \sum_{n=0}^{\infty} e^{-2n\pi iv} (-\log z - 2n\pi i)^{s-1}$$

$$+ \sum_{n=1}^{\infty} e^{2n\pi iv} (-\log z + 2n\pi i)^{s-1}$$

and comparing with (1) we obtain at once Lerch's transformation formula for the function $\Phi(z, s, v)$:

$$(7)\ \Phi(z, s, v) = iz^{-v}(2\pi)^{s-1}\Gamma(1-s)\{e^{-i\pi s/2}\Phi[e^{-2\pi iv}, 1-s, (\log z)/(2\pi i)]$$

$$- e^{i\pi(s/2+2v)}\Phi[e^{2\pi iv}, 1-s, 1-(\log z)/(2\pi i)]\}.$$

If in (6) we use the binomial expansions

$$(-\log z + 2n\pi i)^{s-1}$$
$$= -(2n\pi)^{s-1}\, ie^{i\pi s/2} \sum_{r=0}^{\infty} (-1)^r \binom{s-1}{r} [(\log z)/(2n\pi)]^r e^{-i\pi r/2},$$

$$(-\log z - 2n\pi i)^{s-1}$$
$$= (2n\pi)^{s-1}\, ie^{-i\pi s/2} \sum_{r=0}^{\infty} \binom{s-1}{r} [(\log z)/(2n\pi)]^r e^{-i\pi r/2},$$

we find

$$z^{v}\Phi(z, s, v)/\Gamma(1-s) = [\log(1/z)]^{s-1} + 2\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}(2\pi n)^{s-1}$$

$$\times\left\{(-1)^r\binom{s-1}{2r}\sin(\tfrac{1}{2}\pi s + 2n\pi v)\ [(\log z)/(2n\pi)]^{2r}\right.$$

$$\left.+ (-1)^r\binom{s-1}{2r+1}\cos(\tfrac{1}{2}\pi s + 2n\pi v)[(\log z)/(2\pi n)]^{2r+1}\right\}.$$

Summing with respect to n by means of Hurwitz' formula 1.10(6) we have

$$(8)\quad \Phi(z, s, v) = \frac{\Gamma(1-s)}{z^v}(\log 1/z)^{s-1} + z^{-v}\sum_{r=0}^{\infty}\zeta(s-r, v)\,\frac{(\log z)^r}{r!}$$

$$|\log z| < 2\pi, \quad s \neq 1, 2, 3, \ldots, \quad v \neq 0, -1, -2, \ldots.$$

If s is a positive integer $s = m$, we first put $s = m + \epsilon$ and we have from 1.17(11) and 1.10(9)

$$(\log 1/z)^{\epsilon} = 1 + \epsilon\log(\log 1/z) + O(\epsilon^2),$$

$$\Gamma(1-s) = \Gamma(1-m-\epsilon) = \frac{(-1)^m}{(m-1)!}\ [\epsilon^{-1} - \psi(m)] + O(\epsilon),$$

$$\zeta(1+\epsilon, v) = \epsilon^{-1} - \psi(v) + O(\epsilon).$$

Making $\epsilon \to 0$, we then obtain from (8)

$$(9)\quad \Phi(z, m, v) = z^{-v}\left\{\sum_{n=0}^{\infty}{}' \zeta(m-n, v)\frac{(\log z)^n}{n!} + \frac{(\log z)^{m-1}}{(m-1)!}[\psi(m) - \psi(v) - \log(\log 1/z)]\right\}$$

$m = 2, 3, 4, \ldots, \quad |\log z| < 2\pi, \quad v \neq 0, -1, -2, \ldots.$

The prime indicates that the term with $n = m - 1$ is to be omitted.

In the case where $s = 1$ we have simply

$$(10)\quad \Phi(z, 1, v) = \sum_{n=0}^{\infty}\frac{z^n}{n+v} = v^{-1}\,{}_2F_1(1, v; 1+v; z) \qquad |z| < 1.$$

From 1.8(6) we see that

$$G(v) = 2\,\Phi(-1, 1, v).$$

If s is a negative integer, $s = -m\,(m = 1, 2, 3, \ldots)$, we can use 1.10(11) in order to express Φ, as given by (8), in terms of Bernoulli's polynomials:

$$(11)\quad \Phi(z, -m, v) = \frac{m!}{z^v}(\log 1/z)^{-m-1} - \frac{1}{z^v}\sum_{r=0}^{\infty}\frac{B_{m+r+1}(v)(\log z)^r}{r!(m+r+1)}$$

$|\log z| < 2\pi.$

Finally from (8) and (10) we deduce

$$(12)\quad \lim_{z\to 1}(1-z)^{1-s}\Phi(z, s, v) = \Gamma(1-s) \qquad \text{Re } s < 1,$$

$$(13)\quad \lim_{z\to 1}\Phi(z, 1, v)/[-\log(1-z)] = 1.$$

The properties of the function

$$(14)\quad F(z, s) = \sum_{n=1}^{\infty}(z^n/n^s) = z\,\Phi(z, s, 1)$$

can easily be deduced from the equations (1) to (13). If $s = -m\,(m = 1, 2, 3, \ldots)$, we find from (11) and 1.13(7) that

$$(15)\quad F(z, -m) = m!\,(\log 1/z)^{-m-1} - \sum_{r=0}^{\infty}\frac{B_{m+r+1}}{(m+r+1)\,r!}(\log z)^r$$

$|\log z| < 2\pi,$

where B_{m+r+1} denotes the Bernoulli number.

From Lerch's transformation 1.11(7) we obtain Joncquière's relation

$$(16)\quad F(z, s) + e^{is\pi} F(1/z, s) = \frac{(2\pi)^s}{\Gamma(s)} e^{i\pi s/2} \zeta\left(1 - s, \frac{\log z}{2\pi i}\right).$$

Furthermore we have

$$(17)\quad F(z, -m) = (-1)^{m+1} F(1/z, -m) \qquad m = 1, 2, 3, \ldots,$$

$$(18)\quad F(z, m) + (-1)^m F(1/z, m) = -\frac{(2\pi i)}{m!} B_m\left(\frac{\log z}{2\pi i}\right) \qquad m = 2, 3, 4, \ldots.$$

These equations furnish the analytical continuation of the series (14) beyond its circle of convergence $|z| = 1$.

If $F_0(z)$ denotes the principal branch of $F(z)$ in the cut z-plane $[0 < \arg(z - 1) < 2\pi]$, the cut being imposed from 1 to ∞ along the real axis, the difference of the values of $F_0(z)$ between a point on the upper edge of the cut and a point on the lower edge of the cut is seen from (16) to be

$$(19)\quad F_0(x, s) - F_0(xe^{2i\pi}, s) = \frac{2\pi i}{\Gamma(s)} (\log x)^{s-1}.$$

Hence, if we cross the cut, from the upper half-plane to the lower half-plane, we obtain for the continuation $F_1(z)$ of $F_0(z)$

$$(20)\quad F_1(z) = F_0(z) + 2\pi i (\log z)^{s-1}/\Gamma(s).$$

The analogous formula for the inverse process of continuation is

$$(21)\quad F_2(z) = F_0(z) - 2\pi i (\log z)^{s-1}/\Gamma(s).$$

(For further discussions of the function $F(z, s)$ see Truesdell, 1945, p. 144.)

1.11.1. Euler's dilogarithm

Euler's dilogarithm is defined by

$$(22)\quad L_2(z) = \sum_{n=1}^{\infty} (z^n/n^2) = -\int_0^z z^{-1} \log(1 - z)\, dz = F(z, 2),$$

which is a special case of (14).

From (18) we get the equation

$$(23)\quad L_2(z) = -L_2(1/z) - \tfrac{1}{2}(\log z)^2 + \pi i \log z + \pi^2/3.$$

If we denote the principal branch of $L_2(z)$ by $L_2^*(z)$ $[0 < \arg(z - 1) < 2\pi]$, (19) and (20) show that for any branch

$$L_2(z) = L_2^*(z) + 2n\pi i \log z + 4m\pi^2 \qquad n, m = 0, \pm 1, \pm 2, \ldots.$$

(For a detailed discussion, see O. Hölder, 1928, p. 312. For other special cases of formula (14) see Ramanujan, 1927, p. 40, 336; Rogers, 1905; and Sandham, 1949.)

1.12. The zeta function of Riemann

Putting $v = 1$ in 1.10(1) we obtain Riemann's zeta-function

(1) $$\zeta(s) = \zeta(s, 1) = \Phi(1, s, 1) = \sum_{n=1}^{\infty} (1/n^s) \qquad \text{Re } s > 1.$$

Hence, we have

(2) $$\sum_{n=1}^{\infty} [(-1)^{n-1}/n^s] = (1 - 2^{1-s})\, \zeta(s) = \Phi(-1, s, 1) \qquad \text{Re } s > 0,$$

(3) $$\sum_{n=0}^{\infty} [1/(2n+1)^s] = (1 - 2^{-s})\, \zeta(s) = 2^{-s}\, \Phi(1, s, \tfrac{1}{2}) \qquad \text{Re } s > 1.$$

We therefore have the following integral expressions for $\zeta(s)$ [cf. 1.10(3) and 1.11(3)].

(4) $$\Gamma(s)\, \zeta(s) = \int_0^{\infty} t^{s-1} (e^t - 1)^{-1}\, dt = 2^{s-1} \int_0^{\infty} e^{-t} t^{s-1} \operatorname{csch} t\, dt \qquad \text{Re } s > 1,$$

(5) $$(1 - 2^{1-s})\, \Gamma(s)\, \zeta(s) = \int_0^{\infty} t^{s-1} (e^t + 1)^{-1}\, dt$$
$$= 2^{s-1} \int_0^{\infty} e^{-t} t^{s-1} \operatorname{sech} t\, dt \qquad \text{Re } s > 0,$$

(6) $$2\,\Gamma(s)\, (1 - 2^{-s})\, \zeta(s) = \int_0^{\infty} t^{s-1} \operatorname{csch} t\, dt \qquad \text{Re } s > 1.$$

From 1.11(1) and 1.11(3) we have

(7) $$\zeta(s) = \Phi_z(1, s+1, 0) = [2^{s-1}/\Gamma(s+1)] \int_0^{\infty} t^s (\operatorname{csch} t)^2\, dt \qquad \text{Re } s > 1,$$

(8) $$(1 - 2^{1-s})\, \zeta(s) = \Phi_z(-1, s+1, 0) = [2^{s-1}/\Gamma(s+1)] \int_0^{\infty} t^s (\operatorname{sech} t)^2\, dt \qquad \text{Re } s > -1.$$

The following representations of $\zeta(s)$ by means of contour integrals

(9) $$2\pi i\, \zeta(s) = -\Gamma(1-s) \int_{\infty}^{(0+)} (-t)^{s-1} (e^t - 1)^{-1}\, dt$$

(10) $$2\pi i (1 - 2^{1-s})\, \zeta(s) = -\Gamma(1-s) \int_{\infty}^{(0+)} (-t)^{s-1} (e^t + 1)^{-1}\, dt$$

(11) $$4\pi i (1 - 2^{-s})\, \zeta(s) = -\Gamma(1-s) \int_{\infty}^{(0+)} (-t)^{s-1} \operatorname{csch} t\, dt$$

with

$$s \neq 1, 2, 3, \ldots, \qquad |\arg(-t)| \leq \pi$$

follow from 1.10(5) and 1.11(5) by means of (1), (2), and (3). The contour in (9) and (11) contains none of the points $t = \pm 2n\pi i$ and in (10) none of the points $t = (2n-1)\pi i$.

From (1) and 1.10(7) we obtain

$$(12) \quad \zeta(s) = \tfrac{1}{2} + 1/(s-1) + 2\int_0^\infty (1+t^2)^{-s/2}(e^{2\pi t}-1)^{-1} \sin(s \tan^{-1} t)\, dt.$$

Furthermore (Lindelöf, 1905, p. 103) we have

$$(13) \quad \zeta(s) = \frac{2^{s-1}}{s-1} - 2^s \int_0^\infty (1+t^2)^{-s/2}(e^{2\pi t}+1)^{-1} \sin(s \tan^{-1} t)\, dt,$$

$$(14) \quad \zeta(s) = \frac{\pi\, 2^{s-2}}{s-1} \int_0^\infty (1+t^2)^{\frac{1}{2}(1-s)} \frac{\cos[(s-1)\tan^{-1} t]}{[\cosh(\frac{1}{2}\pi t)]^2}\, dt,$$

$$(15) \quad \zeta(s) = \frac{2^{s-1}}{1-2^{1-s}} \int_0^\infty (1+t^2)^{-s/2} \frac{\cos(s \tan^{-1} t)}{\cosh(\frac{1}{2}\pi t)}\, dt.$$

These formulas are due to Jensen. The integrals in (12) to (15) define an analytic function for all values of s.

Other integral representations are (Bruijn, 1937)

$$\zeta(s) = (s-1)^{-1} + \pi^{-1} \sin(\pi s) \int_0^\infty [\log(1+x) - \psi(1+x)]\, x^{-s}\, dx,$$

$$\zeta(1+s) = (\pi s)^{-1} \sin(\pi s) \int_0^\infty \psi'(1+x)\, x^{-s}\, dx$$

$$= \pi^{-1} \sin(\pi s) \int_0^\infty [\psi(1+x) + \gamma]\, x^{-1-s}\, dx,$$

$$\zeta(m+s) = (-1)^{m-1} \frac{\Gamma(s)\sin(\pi s)}{\pi\,\Gamma(m+s)} \int_0^\infty \psi^{(m)}(1+x)\, x^{-s}\, dx$$

$$m = 1, 2, 3, \ldots.$$

These formulas are valid for $0 < \operatorname{Re} s < 1$ and $\psi^{(m)}$ is defined in 1.16(1). Furthermore

$$\zeta(s) = (s-1)^{-1} + \frac{\sin(\pi s)}{\pi(s-1)} \int_0^\infty [\psi'(1+x) - (1+x)^{-1}]\, x^{1-s}\, dx$$

$$0 < \operatorname{Re} s < 2, \qquad s \neq 1.$$

Finally we prove Riemann's representation of $\zeta(s)$,

$$(16) \quad \pi^{-s/2}\Gamma(s/2)\, \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (t^{\frac{1}{2}(1-s)} + t^{s/2})\, t^{-1}\, \omega(t)\, dt$$

where

$$\omega(t) = \sum_{n=1}^{\infty} e^{-n^2\pi t} = \tfrac{1}{2}[\theta_3(0, it) - 1],$$

θ_3 being the elliptic theta function. The integral in (16) represents an analytic function of s for all values of s.

From 1.1(5) we have

$$\int_0^\infty e^{-n^2\pi t} t^{s/2-1}\, dt = \pi^{-s/2}\Gamma(s/2)\, n^{-s} \qquad \text{Re } s > 0.$$

Hence we obtain

$$\pi^{-s/2}\Gamma(s/2)\ \zeta(s) = \int_0^\infty \omega(t)\, t^{s/2-1}\, dt$$

$$= \int_0^1 \omega(t)\, t^{s/2-1}\, dt + \int_1^\infty \omega(t)\, t^{s/2-1}\, dt.$$

But by means of Jacobi's imaginary transformation of the theta functions (Whittaker and Watson, 1927, § 21.51) we have

$$\omega(t) = -\tfrac{1}{2} + \tfrac{1}{2} t^{-1/2} + t^{-1/2}\, \omega(1/t).$$

Introducing this expression into the integral, we obtain

$$\pi^{-s/2}\Gamma(s/2)\ \zeta(s)$$

$$= -\ (1/s) + 1/(s-1)\ + \int_0^1 \omega(1/t)\, t^{s/2-3/2}\, dt + \int_1^\infty \omega(t)\, t^{s/2-1}\, dt,$$

and substituting $1/t = t'$ in the first integral, we obtain (16). For further integral representations see Ramanujan, 1927, p. 72; Hardy, 1949, pp. 333, 337.

A power series expansion of $\zeta(s)$ is (Hardy, 1912, p. 215; Kluyver, 1927, p. 185)

$$(17)\quad \zeta(s) = (s-1)^{-1} + \gamma + \sum_{n=1}^{\infty} \gamma_n (s-1)^n$$

where

$$\gamma_n = \lim_{m\to\infty} \left[\ \sum_{l=1}^{m} l^{-1} (\log l)^n - (n+1)^{-1} (\log l)^{n+1}\right].$$

Putting $v = 1$ in 1.10(8) to 1.10(11) we obtain

$$(18)\quad \zeta(0) = -\tfrac{1}{2}, \qquad \zeta'(0) = -\tfrac{1}{2}\log(2\pi);$$

$$(19)\quad \lim_{s\to 1}\ [\ \zeta(s) - 1/(s-1)] = -\psi(1) = \gamma,$$

and [cf. 1.13(7)]

$$(20)\quad \zeta(-m) = -\frac{B_{m+1}}{m+1} \qquad m = 1, 2, 3, \ldots,$$

or

(21) $\zeta(-2m) = 0, \quad \zeta(2m) = (-1)^{m+1} (2\pi)^{2m} \dfrac{B_{2m}}{2(2m)!}$

$m = 1, 2, 3, \ldots,$

(22) $\zeta[-(2m-1)] = -\dfrac{B_{2m}}{2m}.$

Putting $v = 1$ in Hurwitz' equation, we obtain Riemann's functional equation for $\zeta(s)$

(23) $\zeta(s) = \dfrac{2\,\Gamma(1-s)}{(2\pi)^{1-s}} \sin(\pi s/2)\, \zeta(1-s)$

or in view of 1.2(6)

(24) $\zeta(1-s) = (2\pi)^{-s}\, 2\,\Gamma(s) \cos(\pi s/2)\, \zeta(s).$

Introducing a new function defined by

(25) $\xi(s) = \dfrac{s(s-1)}{2} \Gamma(s/2)\, \pi^{-s/2} \zeta(s)$

we have

(26) $\xi(1-s) = \xi(s).$

This function is known as Riemann's ξ function. For asymptotic representations of the zeta function see Hutchinson, 1925; Titchmarsh, 1935, 1936; for numerous other results, Titchmarsh, 1930.

If we consider the function

(27) $L(s) = \sum_{m=0}^{\infty} \dfrac{(-1)^m}{(2m+1)^s}$ $\qquad \operatorname{Re} s > 0,$

which is similar to the ζ function, we have by means of 1.11(1) and 1.11(3)

(28) $L(s) = 2^{-s}\, \Phi(-1, s, \tfrac{1}{2}) = \dfrac{1}{2\,\Gamma(s)} \displaystyle\int_0^\infty \dfrac{t^{s-1}}{\cosh t}\, dt$ $\qquad \operatorname{Re} s > 0.$

Putting $z = -1$, $v = \tfrac{1}{2}$ in Lerch's transformation 1.11(7), the following functional equation for $L(s)$ is found:

(29) $L(1-s) = \left(\dfrac{2}{\pi}\right)^s \Gamma(s) \sin(\pi s/2)\, L(s).$

(For further discussions see Lichtenbaum, 1931, p. 641.)

1.13. Bernoulli's numbers and polynomials

The Bernoulli numbers B_n are defined by the equation

(1) $z(e^z - 1)^{-1} = \sum_{n=0}^{\infty} B_n\, z^n/n!$ $\qquad z < 2\pi,$

and the Bernoulli polynomials $B_n(x)$ by means of

(2) $ze^{xz}(e^z-1)^{-1}=\sum_{n=0}^{\infty} B_n(x)\, z^n/n!$ $|z|<2\pi$.

Since the left-hand side of (2) is

$$\left\{\sum_{r=0}^{\infty} B_r\, z^r/r!\right\}\cdot\left\{\sum_{m=0}^{\infty}[(xz)^m/m!]\right\},$$

Cauchy's rule for multiplying power series gives

(3) $B_n(x)=x^n+\binom{n}{1}B_1\,x^{n-1}+\cdots+\binom{n}{n-1}B_{n-1}x+\binom{n}{n}B_n$

$$=\sum_{r=0}^{n}\binom{n}{r}B_r\,x^{n-r},$$

$$B_0(x)=1,\quad B_1(x)=x-\tfrac{1}{2},\quad B_2(x)=x^2-x+1/6,$$

$$B_3(x)=x^3-3/2x^2+\tfrac{1}{2}x,\quad B_4(x)=x^4-2x^3+x^2-1/30,\ \ldots.$$

Clearly we have

(4) $B_n(0)=B_n$.

Differentiating (2) with respect to x and comparing coefficients we obtain

(5) $B_n'(x)=n\,B_{n-1}(x)$.

From (2) it follows that

$$\sum_{n=0}^{\infty}[B_n(x+1)-B_n(x)]\frac{z^n}{n!}=ze^{xz}=\sum_{n=1}^{\infty}\frac{x^{n-1}z^n}{(n-1)!}.$$

Hence we have

$$B_0(x+1)=B_0(x),\quad B_1(x+1)-B_1(x)=1,$$

and in general

(6) $B_n(x+1)-B_n(x)=nx^{n-1}$, $n=2,3,4,\ldots,$

from which it follows that

(7) $B_n(1)=B_n(0)=B_n$. $n\geq 2$

Since we have

$$\sum_{n=0}^{\infty}B_n(x+1)\frac{z^n}{n!}=\frac{ze^{zx}e^z}{e^z-1}=\sum_{r=0}^{\infty}B_r(x)\frac{z^r}{r!}\sum_{m=0}^{\infty}\frac{z^m}{m!},$$

Cauchy's rule for multiplying power series gives a recurrence formula for the Bernoulli polynomial:

$$(8)\quad \sum_{r=0}^{n} \binom{n}{r} B_r(x) = B_n(x+1), \quad \text{or} \quad \sum_{r=0}^{n-1} \binom{n}{r} B_r(x) = nx^{n-1}$$

$$n = 2, 3, 4, \ldots .$$

From (5) and (6) we obtain

$$(9)\quad \int_x^y B_n(t)\,dt = \frac{B_{n+1}(y) - B_{n+1}(x)}{n+1}, \quad \int_x^{x+1} B_n(t)\,dt = x^n.$$

Hence it follows that

$$(10)\quad \sum_{r=0}^{m-1} r^n = \sum_{r=0}^{m-1} \int_r^{r+1} B_n(t)\,dt = \int_0^m B_n(t)\,dt = \frac{B_{n+1}(m) - B_{n+1}}{n+1}.$$

$$n = 2, 3, 4, \ldots .$$

From (6) we can obtain the multiplication theorem and the symmetry property of $B_n(x)$ (Fort pp. 32, 34)

$$(11)\quad B_n(mx) = m^{n-1} \sum_{r=0}^{m-1} B_n(x + r/m),$$

$$(12)\quad B_n(1-x) = (-1)^n B_n(x).$$

The Bernoulli polynomials are expressible in trigonometric series. For $B_1(x)$ we have from (3)

$$(13)\quad B_1(x) = x - ½ = -\sum_{r=1}^{\infty} (r\pi)^{-1} \sin(2\pi r x) \qquad 0 < x < 1.$$

The Fourier series of $B_k(x)$ for $k > 1$ can easily be obtained by the calculus of residues. Consider $\int_C f(z)\,dz$ with $f(z) = z^{-k} e^{xz}(e^z - 1)^{-1}$ (k an integer > 1), the contour C being a (large) circle with radius $(2N+1)\pi$ (N an integer), center at the origin. The poles of the integrand are $z_r = 2\pi i r$, ($r = 0, \pm 1, \pm 2, \ldots$). The residues of the function $f(z)$ for $r = \pm 1, \pm 2, \ldots$ are easily found to be $(2\pi i r)^{-k} e^{2\pi i r x}$, and from (2) the residue at $z = 0$ is seen to be $B_k(x)/k!$. The integral around the circle C tends to zero as $N \to \infty$ provided $0 \le x \le 1$, and by the theorem of residues we have

$$B_k(x)/k! = -\sum_{r=-\infty}^{\infty}{}' (2\pi i r)^{-k} e^{2\pi i r x}.$$

The prime indicates that the term corresponding to $r = 0$ must be omitted. This gives the expansions ($n = 1, 2, 3, \ldots ; 0 \le x \le 1$)

$$(14)\quad B_{2n}(x) = 2(-1)^{n+1}(2n)! \sum_{r=1}^{\infty} (2\pi r)^{-2n} \cos(2\pi r x),$$

(15) $B_{2n+1}(x) = 2(-1)^{n+1}(2n+1)! \sum_{r=1}^{\infty} (2\pi r)^{-2n-1} \sin(2\pi r x).$

Putting $x = 0$ we get the following expressions for the Bernoulli numbers (cf. also Schwatt, 1932, p. 143):

(16) $B_{2n} = 2(-1)^{n+1}(2n)! \sum_{r=1}^{\infty} (2\pi r)^{-2n}$ $\qquad n = 1, 2, 3, \ldots,$

(17) $B_{2n+1} = 0$ $\qquad n = 1, 2, 3, \ldots .$

Equations (4) and (8) give a recurrence formula for the Bernoulli numbers

(18) $\sum_{r=0}^{n-1} \binom{n}{r} B_r = 0$ $\qquad n = 2, 3, 4, \ldots .$

From (18) and (3) we have

(19) $B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42, \ldots,$

Numerical values of the B_{2n} up to B_{40} and recurrence relations can be found in Ramanujan, 1927, p. 1.

By using (14), (15), and 1.11(4), the following integral representations for the Bernoulli polynomials are obtained:

(20) $B_{2n}(x) = (-1)^{n+1}(2n) \int_0^{\infty} \frac{\cos(2\pi x) - e^{-2\pi t}}{\cosh(2\pi t) - \cos(2\pi x)} t^{2n-1}\, dt$

$0 < \operatorname{Re} x < 1, \qquad n = 1, 2, 3, \ldots,$

(21) $B_{2n+1}(x) = (-1)^{n+1}(2n+1) \int_0^{\infty} \frac{\sin(2\pi x)}{\cosh(2\pi t) - \cos(2\pi x)} t^{2n}\, dt$

$0 < \operatorname{Re} x < 1, \qquad n = 0, 1, 2, \ldots .$

In terms of Riemann's zeta-function 1.12(1) we have

(22) $B_{2n} = (-1)^{n+1}(2\pi)^{-2n}\, 2(2n)!\, \zeta(2n)$ $\qquad n = 0, 1, 2, \ldots,$

(23) $B_{2n} = -2n\, \zeta[-(2n-1)]$ $\qquad n = 1, 2, 3, \ldots,$

as is seen from (16) and 1.12(22).

From 1.12(4) to 1.12(8) we find integral representations for the B_{2n} $(n = 1, 2, 3, \ldots)$:

(24) $B_{2n} = (-1)^{n+1}\, 4n \int_0^{\infty} t^{2n-1} (e^{2\pi t} - 1)^{-1}\, dt$

$= (-1)^{n+1}\, 2n \int_0^{\infty} t^{2n-1} e^{-\pi t} \operatorname{csch}(\pi t)\, dt,$

(25) $B_{2n} = (-1)^{n+1}\, 4n(1-2^{1-2n})^{-1} \int_0^\infty t^{2n-1}(e^{2\pi t}+1)^{-1}\, dt$

$= (-1)^{n+1}\, 2n(1-2^{1-2n})^{-1} \int_0^\infty t^{2n-1} e^{-\pi t} \operatorname{sech}(\pi t)\, dt,$

(26) $B_{2n} = (-1)^{n+1}\, 2n(2^{2n}-1)^{-1} \int_0^\infty t^{2n-1} \operatorname{csch}(\pi t)\, dt,$

(27) $B_{2n} = (-1)^{n+1}\, \pi \int_0^\infty t^{2n} [\operatorname{csch}(\pi t)]^2\, dt,$

(28) $B_{2n} = (-1)^{n+1}\, \pi(1-2^{1-2n})^{-1} \int_0^\infty t^{2n} [\operatorname{sech}(\pi t)]^2 dt.$

(For other results cf. Nielsen, 1923, and Ramanujan, 1927, p.1.)

1.13.1. The Bernoulli polynomials of higher order

The Bernoulli numbers and polynomials of order m are defined respectively by

(29) $\alpha_1 \cdots \alpha_m\, z^m [(e^{\alpha_1 z}-1) \cdots (e^{\alpha_m z}-1)]^{-1}$

$= \sum_{n=0}^{\infty} B_n^{(m)}(\alpha_1 \cdots \alpha_m)\, z^n/n! \qquad |z| < 2\pi|\alpha_l|^{-1},$

(30) $\alpha_1 \cdots \alpha_m\, z^m [(e^{\alpha_1 z}-1) \cdots (e^{\alpha_m z}-1)]^{-1} e^{xz}$

$= \sum_{n=0}^{\infty} B_n^{(m)}(x|\alpha_1 \cdots \alpha_m)\, z^n/n! \qquad |z| < 2\pi|\alpha_l|^{-1}.$

Here m is a positive integer, $\alpha_1, \ldots, \alpha_m$ are arbitrary parameters, and

(31) $|\alpha_l| = \max[|\alpha_1|, \ldots, |\alpha_m|].$

For $m = 1$ and $\alpha_1 = 1$, (29) and (30) reduce to (1) and (2) respectively.

Clearly we have

(32) $B_n^{(m)}(0|\alpha_1 \cdots \alpha_m) = B_n^{(m)}(\alpha_1 \cdots \alpha_m),$

(33) $B_n^{(1)}(x|\alpha_1) = \alpha_1^n\, B_n(x/\alpha_1).$

From (29) and (30)

(34) $B_n^{(m)}(x|\alpha_1 \cdots \alpha_m) = \sum_{l=0}^{n} \binom{n}{l} x^l B_{n-l}^{(m)}(\alpha_1 \cdots \alpha_m).$

We denote

(35) $\xi = \tfrac{1}{2}(\alpha_1 + \cdots + \alpha_m)$

and

(36) $D_n^{(m)} = 2^n B_n^{(m)}(\xi|\alpha_1 \cdots \alpha_m).$

It can be shown that

(37) $D^{(m)}_{2n+1} = 0 \qquad n = 0, 1, 2, \ldots.$

We thus get from (30)

(38) $$(\alpha_1 \cdots \alpha_m) z^m [\sinh(\alpha_1 z) \cdots \sinh(\alpha_m z)]^{-1} = \sum_{n=0}^{\infty} D^{(m)}_{2n} z^{2n}/(2n)! \qquad |z| < \pi |\alpha_l|^{-1}.$$

The Bernoulli numbers and polynomials of order $-m$ ($m = 1, 2, 3, \ldots$) are defined respectively by

(39) $$(e^{\alpha_1 z} - 1) \cdots (e^{\alpha_m z} - 1)(\alpha_1 \cdots \alpha_m)^{-1} z^{-m} = \sum_{n=0}^{\infty} B^{(-m)}_n (\alpha_1 \cdots \alpha_m) z^n/n!,$$

(40) $$(e^{\alpha_1 z} - 1) \cdots (e^{\alpha_m z} - 1)(\alpha_1 \cdots \alpha_m)^{-1} z^{-m} e^{xz} = \sum_{n=0}^{\infty} B^{(-m)}_n (x|\alpha_1 \cdots \alpha_m) z^n/n!;$$

both expansions converge in the whole z-plane.

From (35) and (40) for $x = -\xi$ we have

(41) $$\sinh(\alpha_1 z) \cdots \sinh(\alpha_m z)(\alpha_1 \cdots \alpha_m)^{-1} z^{-m} = \sum_{n=0}^{\infty} D^{(-m)}_{2n} z^{2n}/(2n)!,$$

where

(42) $$D^{(-m)}_n = 2^n B^{(-m)}_n (-\xi|\alpha_1 \cdots \alpha_m).$$

Again we have

(43) $D^{(-m)}_{2n+1} = 0 \qquad n = 0, 1, 2, \ldots.$

For an exhaustive treatise of the Bernoulli numbers and polynomials of higher order see Nörlund, 1922 and 1924, Ch. VI.

The case $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 1$ is thoroughly discussed in Milne-Thomson, 1933, Ch. VI.

1.14. Euler numbers and polynomials

Euler numbers E_n and Euler polynomials $E_n(x)$ are defined by the equations:

(1) $$\operatorname{sech} z = 2e^z(e^{2z} + 1)^{-1} = \sum_{n=0}^{\infty} E_n z^n/n! \qquad |z| < \tfrac{1}{2}\pi,$$

(2) $$2e^{xz}(e^z + 1)^{-1} = \sum_{n=0}^{\infty} E_n(x) z^n/n! \qquad |z| < \pi.$$

Differentiating (2) with respect to x and equating coefficients of z^n we obtain

(3) $E_n'(x) = n\, E_{n-1}(x).$

If the left-hand side of (2) is written in the form

$$2e^{z/2}(e^z+1)^{-1}\, e^{2(x-\frac{1}{2})} = \sum_{r=0}^{\infty} E_r\, z^r (r!\,2^r)^{-1} \sum_{m=0}^{\infty} (x-\tfrac{1}{2})^m\, z^m/m!\,,$$

Cauchy's rule for multiplying power series gives

(4) $E_n(x) = \sum_{r=0}^{n} \binom{n}{r} 2^{-r} E_r\, (x-\tfrac{1}{2})^{n-r},$

hence taking $x = \tfrac{1}{2}$,

(5) $E_n = 2^n E_n(\tfrac{1}{2}).$

From (2) we have

$$\sum_{n=0}^{\infty} [E_n(x+1) + E_n(x)]\, z^n/n! = 2e^{xz} = 2\sum_{n=0}^{\infty} x^n\, z^n/n!\,,$$

and therefore

(6) $E_n(x+1) + E_n(x) = 2x^n.$

Writing

$$ze^{xz/2}(e^{z/2}+1)^{-1} = ze^{(x+1)z/2}(e^z-1)^{-1} - ze^{xz/2}(e^z-1)^{-1},$$

we obtain from (2) and 1.13(2)

(7) $E_{n-1}(x) = n^{-1}\, 2^n \{B_n[\tfrac{1}{2}(x+1)] - B_n(\tfrac{1}{2}x)\} = n^{-1}\, 2[B_n(x) - 2^n B_n(\tfrac{1}{2}x)].$

Hence from 1.13(11), 1.13(12) the following relations are obtained

(8) $E_n(mx) = m^n \sum_{r=0}^{m-1} (-1)^r E_n(x + r/m)$ $\qquad m$ odd,

(9) $E_n(mx) = -2m^n(n+1)^{-1} \sum_{r=0}^{m-1} (-1)^r B_{n+1}(x + r/m)$ $\qquad m$ even,

(10) $E_n(1-x) = (-1)^n E_n(x).$

From

$$2e^{(x+1)z}(e^z+1)^{-1} = \sum_{r=0}^{\infty} E_r(x)\, z^r/r!\, \sum_{m=0}^{\infty} z^m/m! = \sum_{n=0}^{\infty} E_n(x+1)\, z^n/n!$$

we obtain a recurrence formula

(11) $\sum_{r=0}^{n} \binom{n}{r} E_r(x) = E_n(x+1),$ $\quad$ or $\quad$ $\sum_{r=0}^{n} \binom{n}{r} E_r(x) + E_n(x) = 2x^n.$

In a manner similar as in 1.13 a representation of Euler's polynomials by means of Fourier series can be obtained. Here one considers the integral $2\int_c z^{-k-1}\, e^{xz}\,(e^z+1)^{-1}\,dz$ taken along a circle, center at the origin, radius $2N\pi$ (N an integer). From (2) the residue of the integrand at $z=0$ is easily seen to be $E_k(x)/k!$. The result is

$$(12)\quad E_{2n}(x) = (-1)^n\, 4(2n)! \sum_{r=0}^{\infty} [(2r+1)\pi]^{-2n-1} \sin[(2r+1)\pi x]$$

$$n = 1, 2, 3, \ldots, \quad 0 \le x \le 1,$$

$$(13)\quad E_{2n+1}(x) = (-1)^{n+1}\, 4(2n+1)! \sum_{r=0}^{\infty} [(2r+1)\pi]^{-2n-2} \cos[(2r+1)\pi x]$$

$$n = 0, 1, 2, \ldots, \quad 0 \le x \le 1.$$

From (5), (12), and (13) we have

$$(14)\quad E_{2n} = (-1)^n\, 2(2n)!\,(2/\pi)^{2n+1} \sum_{r=0}^{\infty} (-1)^r/(2r+1)^{2n+1} \qquad n = 0, 1, 2, \ldots,$$

$$(15)\quad E_{2n+1} = 0,$$

or, with the notation of 1.12(27)

$$(16)\quad E_{2n} = (-1)^n\, 2(2n)!\,(2/\pi)^{2n+1} L(2n+1) \qquad n = 0, 1, 2, \ldots.$$

The equation

$$(1/\cosh z)\cosh z = 1 = \sum_{n=0}^{\infty} E_{2n}\, z^{2n}/(2n)! \sum_{m=0}^{\infty} z^{2m}/(2m)!,$$

and the application of Cauchy's multiplication rule gives the recurrence formula for Euler's numbers:

$$(17)\qquad \sum_{r=0}^{n} \binom{2n}{2r} E_{2r} = 0 \qquad n > 0.$$

Using (14) we have

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385, \ldots.$$

An integral expression for E_{2n} can be obtained by replacing $L(2n+1)$ in (16) by the expression 1.12(28),

$$(18)\quad E_{2n} = (-1)^n\,(2/\pi)^{2n+1} \int_0^\infty t^{2n}\,\mathrm{sech}\, t\, dt$$

$$= (-1)^n\, 2^{2n+1} \int_0^\infty t^{2n}\,\mathrm{sech}(\pi t)\, dt \qquad n = 0, 1, 2, \ldots.$$

The Fourier expansions (12) and (13) can be replaced by integral expressions. The result is:

(19) $$E_{2n}(x) = (-1)^n\, 4 \int_0^\infty \frac{t^{2n} \sin(\pi x) \cosh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} dt$$

$n = 0, 1, 2, \ldots, \quad 0 < \operatorname{Re} x < 1,$

(20) $$E_{2n+1}(x) = (-1)^{n+1}\, 4 \int_0^\infty \frac{t^{2n+1} \cos(\pi x) \sinh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} dt$$

$n = 0, 1, 2, \ldots, \quad 0 < \operatorname{Re} x < 1.$

(For other results cf. Nielsen, 1923.)

1.14.1. The Euler polynomials of higher order

Euler's numbers and polynomials are defined respectively by

(21) $$2^m e^{z(\alpha_1 + \ldots + \alpha_m)} [(e^{2\alpha_1 z} + 1) \cdots (e^{2\alpha_m z} + 1)]^{-1}$$

$$= [\cosh(\alpha_1 z) \cdots \cosh(\alpha_m z)]^{-1} = \sum_{n=0}^{\infty} E_n^{(m)}(\alpha_1 \cdots \alpha_m)\, z^n/n!\,,$$

(22) $$2^m e^{xz} [(e^{\alpha_1 z} + 1) \cdots (e^{\alpha_m z} + 1)]^{-1} = \sum_{n=0}^{\infty} E_n^{(m)}(x|\alpha_1 \cdots \alpha_m)\, z^n/n!\,.$$

The series in formula (21) is convergent for $|z| < \frac{1}{2}\pi |\alpha_l|^{-1}$, and the series in (22) is convergent for $|z| < \pi |\alpha_l|^{-1}$ where $|\alpha_l|$ is defined in 1.13(31). Again in (21) and (22) m is a positive integer, and $\alpha_1, \ldots, \alpha_m$ are arbitrary parameters. The special case $m = 1$, $\alpha_1 = 1$ reduces to that discussed in 1.14.

Clearly from (21), (22), and 1.13(35) we have

(23) $$E_n^{(m)}(\alpha_1 \cdots \alpha_m) = 2^n E_n^{(m)}(\xi|\alpha_1 \cdots \alpha_m).$$

The Euler numbers and polynomials of order $-m$ ($m = 1, 2, 3, \ldots$) are defined respectively as follows:

(24) $$2^{-m} e^{-z(\alpha_1 + \ldots + \alpha_m)} [(e^{2\alpha_1 z} + 1) \cdots (e^{2\alpha_m z} + 1)]$$

$$= \cosh(\alpha_1 z) \cdots \cosh(\alpha_m z) = \sum_{n=0}^{\infty} E_n^{(-m)}(\alpha_1 \cdots \alpha_m)\, z^n/n!\,,$$

(25) $$2^{-m} e^{xz} (e^{\alpha_1 z} + 1) \cdots (e^{\alpha_m z} + 1) = \sum_{n=0}^{\infty} E_n^{(-m)}(x|\alpha_1 \cdots \alpha_m)\, z^n/n!\,;$$

both expansions are convergent in the whole z-plane. For more details see Nörlund, 1922 and 1924, Ch. VI. The case $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 1$ is thoroughly discussed in Milne-Thomson, 1933, Ch. VI.

1.15. Some integral formulas connected with the Bernoulli and Euler polynomials

Some integral relations can be deduced from the two preceding sections.

First, 1.13(1) can be written in the form

(1) $$(e^z - 1)^{-1} - z^{-1} + \tfrac{1}{2} = \sum_{n=1}^{\infty} B_{2n}\, z^{2n-1}/(2n)! \qquad |z| < 2\pi.$$

If the B_{2n} are replaced by 1.13(24) and 1.13(27) we find

(2) $$(e^z - 1)^{-1} = z^{-1} - \tfrac{1}{2} + 2\int_0^{\infty} (e^{2\pi t} - 1)^{-1} \sin(tz)\, dt \qquad |\operatorname{Im} z| < 2\pi,$$

(3) $$(e^{2z} - 1)^{-1} = (2z)^{-1} - \tfrac{1}{2} + \pi z^{-1} \int_0^{\infty} \sin^2(tz)\, \operatorname{csch}^2(\pi t)\, dt \qquad |\operatorname{Im} z| < \pi.$$

If in 1.13(2) the $B_r(x)$ are replaced by the expressions 1.13(20) and 1.13(21) and in 1.14(2) the $E_r(x)$ by the expressions 1.14(19) and 1.14(20), we find

(4) $$\frac{e^{xz}}{e^z - 1} = \frac{1}{z} + \int_0^{\infty} \frac{\cos(2\pi x) - e^{-2\pi t}}{\cosh(2\pi t) - \cos(2\pi x)} \sin(tz)\, dt$$
$$- \int_0^{\infty} \frac{\sin(2\pi x)}{\cosh(2\pi t) - \cos(2\pi x)} \cos(tz)\, dt$$
$$0 \le x < 1, \qquad |\operatorname{Im} z| < 2\pi,$$

(5) $$\frac{e^{xz}}{e^z + 1} = 2\int_0^{\infty} \frac{\sin(\pi x)\cosh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} \cos(tz)\, dt$$
$$- 2\int_0^{\infty} \frac{\cos(\pi x)\sinh(\pi t)}{\cosh(2\pi t) - \cos(2\pi x)} \sin(tz)\, dt$$
$$0 \le x < 1, \qquad |\operatorname{Im} z| < \pi.$$

1.16. Polygamma functions

We define

(1) $$\psi^{(n)}(z) = \frac{d^{n+1} \log\Gamma(z)}{dz^{n+1}} = \frac{d^n \psi(z)}{dz^n}, \qquad \psi^{(0)}(z) = \psi(z)$$
$$n = 1, 2, 3, \ldots,$$

(2) $$G^{(n)}(z) = \frac{d^n G(z)}{dz^n}, \qquad G^{(0)}(z) = G(z) \qquad n = 1, 2, 3, \ldots.$$

The following functional equations are consequences of the results of 1.7.1 and 1.8:

(3) $$\psi^{(n)}(z) - \psi^{(n)}(1 + z) = (-1)^{n+1}\, n!/z^{n+1}$$

(4) $$\psi^{(n)}(z) - (-1)^n \psi^{(n)}(1 - z) = -\pi \frac{d^n}{dz^n}[\operatorname{ctn}(\pi z)]$$

(5) $$\psi^{(n)}(mz) = m^{-n-1} \sum_{r=0}^{m-1} \psi^{(n)}(z + r/m) \qquad m = 1, 2, 3, \ldots,$$

(6) $2^n G^{(n)}(z) = \psi^{(n)}(\tfrac{1}{2}z + \tfrac{1}{2}) - \psi^{(n)}(\tfrac{1}{2}z)$,

(7) $G^{(n)}(1+z) + G^{(n)}(z) = 2(-1)^n n!/z^{n+1}$,

(8) $G^{(n)}(z) + (-1)^n G^{(n)}(1-z) = 2\pi \dfrac{d^n}{dz^n} [\csc(\pi z)]$.

We have also the expressions:

(9) $\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{r=0}^{\infty} (z+r)^{-n-1} = (-1)^{n+1} n!\ \zeta(n+1, z)$,

(10) $G^{(n)}(z) = 2(-1)^n n! \sum_{r=0}^{\infty} (-1)^r (z+r)^{-n-1} = 2(-1)^n n!\, \Phi(-1, n+1, z)$.

Hence, we may express $\psi^{(n)}(z)$ and $G^{(n)}(z)$ as definite integrals if we replace the functions ζ and Φ by their integral representations.

1.17. Some expansions for $\log\Gamma(1+z)$, $\psi(1+z)$, $G(1+z)$, and $\Gamma(z)$

The Taylor expansion of $\log\Gamma(1+z)$ is

(1) $$\log\Gamma(1+z) = \sum_{m=0}^{\infty} \left[\frac{d^m \log\Gamma(1+z)}{dz^m}\right]_{z=0} \frac{z^m}{m!}$$

$$= z\,\psi(1) + \sum_{m=2}^{\infty} z^m/m!\,[\psi^{(m-1)}(1+z)]_{z=0}$$

or

(2) $\log\Gamma(1+z) = -\gamma z + \sum_{m=2}^{\infty} (-1)^m\ \zeta(m)\, z^m/m \qquad |z| < 1,$

[cf. 1.16(9) and 1.12(1)].

Taking $z = 1$ we obtain the expression

(3) $\gamma = \sum_{m=2}^{\infty} (-1)^m\ \zeta(m)/m$

for Euler's constant.

If in

(4) $\psi(1+z) = -\gamma + \sum_{n=1}^{\infty} \left(\dfrac{1}{n} - \dfrac{1}{z+n}\right)$

[cf. 1.7(3)] we expand

$$\frac{1}{n} - \frac{1}{z+n} = \frac{z}{n^2} - \frac{z^2}{n^3} + \frac{z^3}{n^4} - \cdots \qquad |z| < 1,$$

we obtain

(5) $\psi(1+z) = -\gamma + \sum_{n=2}^{\infty} (-1)^n\ \zeta(n)\, z^{n-1} \qquad |z| < 1.$

Similarly, from 1.8(6) we have

(6) $$G(1+z) = 2\sum_{n=1}^{\infty} (-1)^{n-1}\,\sigma(n)\,z^{n-1}$$

$$= 2\sigma(1) + 2\sum_{n=2}^{\infty} (-1)^{n-1}(1-2^{1-n})\,\zeta(n)\,z^{n-1}$$

where $|z| < 1$, $\sigma(1) = \log 2$, and $\sigma(n) = \sum_{r=1}^{\infty} (-1)^{r-1}/r^n = (1-2^{1-n})\,\zeta(n)$ for $n > 1$.

If we form the expressions $\psi(1+z) + \psi(1-z)$ and $G(1+z) + G(1-z)$ by means of (5) and (6), and take into account 1.8(7), 1.8(8), 1.7(10), and 1.7(11) we obtain

(7) $$\psi(1+z) = (2z)^{-1} - \gamma - (\pi/2)\operatorname{ctn}(\pi z) - \sum_{n=1}^{\infty} \zeta(2n+1)\,z^{2n} \qquad |z| < 1,$$

(8) $$G(1+z) = z^{-1} - \pi\csc(\pi z) + 2\sigma(1) + 2\sum_{n=1}^{\infty} (1-2^{-2n})\,\zeta(2n+1)\,z^{2n} \qquad |z| < 1.$$

Using 1.7(1) we have from (7)

(9) $$\log\Gamma(1+z) = -\tfrac{1}{2}\log\left(\frac{\sin \pi z}{\pi z}\right) - \sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{2n+1}\,z^{2n+1} - \gamma z,$$

or, using the series

$$\tfrac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1},$$

we obtain

(10) $$\log\Gamma(1+z) = \tfrac{1}{2}\left\{\log\left[\frac{\pi z}{\sin(\pi z)}\right] - \log\left(\frac{1+z}{1-z}\right)\right\} + \sum_{n=1}^{\infty} \frac{1-\zeta(2n+1)}{2n+1}\,z^{2n+1} + (1-\gamma)\,z.$$

Formulas (9) and (10) are valid if $|z| < 1$.

Finally we give an expression of $\Gamma(z)$ and $\psi(z)$ near $z = -m$ ($m = 0, 1, 2, \ldots$). From 1.2(6) we have

$$\Gamma(z) = \pi(-1)^m/\{\Gamma(1-z)\sin[\pi(z+m)]\}.$$

Expanding $1/\Gamma(1-z)$ in a Taylor series near $z = -m$ and using 1.13(36) we obtain

(11) $$\Gamma(z) = [(-1)^m/m!]\{(z+m)^{-1} + \psi(m+1) + \tfrac{1}{2}(z+m)[(\pi^2/3) + \psi^2(m+1) - \psi'(m+1)] + O[(z+m)^2]\}.$$

Similarly from 1.7(11), 1.13(31), and 1.16(9) we have

(12) $\psi(z) = -(z+m)^{-1} + \psi(m+1) + \sum_{n=2}^{\infty} [(-1)^n \zeta(n) + \sum_{r=1}^{m} r^{-n}](z+m)^{n-1}$.

1.18. Asymptotic expansions

In 1.9(5) we replace the expression within the parentheses under the integral sign by the right-hand side of 1.15(1). Since the conditions of Watson's lemma are satisfied, we may integrate term by term and obtain the following asymptotic expansion (Stirling series)

(1) $$\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log(2\pi) + \sum_{n=1}^{m} B_{2n} / [(2n-1)(2n) z^{2n-1}] + O(z^{-2m-1})$$

$|\arg z| < \pi$.

This is equivalent to

(2) $$\Gamma(z) = e^{-z} e^{(z-\frac{1}{2})\log z} (2\pi)^{\frac{1}{2}} \left[1 + \frac{z^{-1}}{12} + \frac{z^{-2}}{288} - \frac{139\, z^{-3}}{51840} - O(z^{-4})\right]$$

$|\arg z| < \pi$.

[Formula (2) can be obtained directly from the loop integral 1.6(2) using the method of steepest descent. For this and for the remainder in (1) and (2) cf. Watson, 1920, p. 1.]

From (1) and (2) a number of asymptotic formulas can be derived, such as

(3) $\log \Gamma(z+\alpha) = (z+\alpha-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log(2\pi) + O(z^{-1})$,

(4) $\Gamma(z+\alpha)/\Gamma(z+\beta) = z^{\alpha-\beta}[1 + \tfrac{1}{2} z^{-1}(\alpha-\beta)(\alpha+\beta-1) + O(z^{-2})]$,

(5) $\lim_{|z|\to\infty} e^{-\alpha \log z}\, \Gamma(z+\alpha)/\Gamma(z) = 1$.

In connection with formula (3) see also (12), and in connection with (4) see (13). In (3), (4), and (5) α and β are fixed arbitrary complex numbers and $-\pi < \arg z < \pi$. We also have

(6) $\lim_{|y|\to\infty} |\Gamma(x+iy)|\, e^{\frac{1}{2}\pi|y|}\, |y|^{\frac{1}{2}-x} = (2\pi)^{\frac{1}{2}}$ $\qquad x, y$ real.

From (1) we obtain the asymptotic expansion for $\psi(z)$,

(7) $\psi(z) = \log z - (2z)^{-1} - \sum_{n=1}^{m} B_{2n} z^{-2n}/(2n) + O(z^{-2m-2})$.

The integrand of 1.10(4) can be written as [cf. 1.15(1)]

(8) $t^{s-1} e^{-vt}/(1-e^{-t}) = t^{s-1} e^{-vt}[t^{-1} + \tfrac{1}{2} + \sum_{n=1}^{\infty} B_{2n} t^{2n-1}/(2n)!]$.

Hence, from 1.10(3) we obtain the following asymptotic expansion of $\zeta(s, v)$ for large values of $|v|$ with $|\arg v| < \pi$:

(9) $$\zeta(s, v) = [1/\Gamma(s)]\{v^{1-s}\Gamma(s-1) + \tfrac{1}{2} v^{-s}\Gamma(s) + \sum_{n=1}^{m} B_{2n}\,\Gamma(s+2n-1)/[(2n)!\,v^{2n+s-1}] + O(v^{-2m-s-1})\}$$
$\operatorname{Re} s > 1.$

Putting $s = (n+1)$ we obtain an asymptotic expansion for $\psi^{(n)}(z)$ as given in 1.16(9).

Finally we derive an asymptotic expression of $\log \Gamma(z)$ due to Binet. In Binet's first expression 1.9(4) we write the integrand in the form

$$\tfrac{1}{2} e^{-tz} t^{-2} (e^{t} - 1)^{-1} [e^{t}(t-2) + t + 2]$$
$$= \tfrac{1}{2} \sum_{n=1}^{\infty} n t^{n} e^{-tz}/[(n+2)!\,(e^{t}-1)],$$

replacing e^{t} in the numerator on the left-hand side by its power series. Since, according to 1.10(3)

$$\int_0^{\infty} t^{n} e^{-tz} (e^{t}-1)^{-1}\,dt = \Gamma(n+1)\,\zeta(n+1, z+1),$$

we obtain

(10) $$\log \Gamma(z) = (z - \tfrac{1}{2}) \log z - z + \tfrac{1}{2}\log(2\pi) + \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)}\,\zeta(n+1, z+1).$$

This is Binet's formula.

A similar expression converging faster is Burnside's formula (Wilton, 1922, p. 90)

(11) $$\log \Gamma(z) = (z - \tfrac{1}{2}) \log (z - \tfrac{1}{2}) - z - \tfrac{1}{2} + \tfrac{1}{2}\log(2\pi) - \sum_{n=1}^{\infty} \zeta(2n, z)/[2^{2n}\, 2n\,(2n+1)]$$
$\operatorname{Re} z \geq -\tfrac{1}{2}.$

From the left-hand side of (3) and (4) complete asymptotic expansions can be given. These are

(12) $$\log \Gamma(z+a) = (z + a - \tfrac{1}{2}) \log z - z + \tfrac{1}{2}\log(2\pi) + \frac{B_2(a)\, z^{-1}}{1\cdot 2} - \cdots + \frac{(-1)^{n+1} B_{n+1}(a)\, z^{-n}}{n(n+1)} + O(z^{-n-1})$$
$|\arg z| < \pi, \quad n = 1, 2, 3, \ldots.$

(13) $\Gamma(z+\alpha_1)\,\Gamma(z+\alpha_2)/[\Gamma(z+\beta_1)\,\Gamma(z+\beta_2)]$

$$= z^{\alpha_1+\alpha_2-\beta_1-\beta_2}\left[1+\frac{c_1}{z+1}+\frac{c_2}{(z+1)(z+2)}+\cdots\right] \qquad |\arg z|<\pi.$$

These expansions are due to Barnes, 1899, p. 64, and Van Engen, 1938, respectively.

1.19. Mellin-Barnes integrals

Of all the integrals which contain gamma functions in their integrands the most important ones are the so-called Mellin-Barnes integrals. Such integrals were first introduced by S. Pincherle, in 1888; their theory has been developed by H. Mellin (1910, where there are references to earlier work), and they were used for a complete integration of the hypergeometric differential equation by E. W. Barnes (1908). See also section 2.1.3.

The integral

(1) $$f(z)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(a_1+A_1s)\cdots\Gamma(a_m+A_ms)}{\Gamma(c_1+C_1s)\cdots\Gamma(c_p+C_ps)}$$

$$\times\frac{\Gamma(b_1-B_1s)\cdots\Gamma(b_n-B_ns)}{\Gamma(d_1-D_1s)\cdots\Gamma(d_q-D_qs)}\,z^s\,ds$$

is a typical Mellin-Barnes integral. It will be assumed that γ is real, all the A_j, B_j, C_j, D_j are positive, and that the path of integration is a straight line parallel to the imaginary axis with indentations, if necessary, to avoid the poles of the integrand. The discussion given here is based on Dixon and Ferrar (1936).

The following notations will be used:

(2) $$\alpha=\sum_{j=1}^{m}A_j+\sum_{j=1}^{n}B_j-\sum_{j=1}^{p}C_j-\sum_{j=1}^{q}D_j$$

(3) $$\beta=\sum_{j=1}^{m}A_j-\sum_{j=1}^{n}B_j-\sum_{j=1}^{p}C_j+\sum_{j=1}^{q}D_j$$

(4) $$\lambda=\operatorname{Re}\left(\sum_{j=1}^{m}a_j-\tfrac12 m+\sum_{j=1}^{n}b_j-\tfrac12 n-\sum_{j=1}^{p}c_j+\tfrac12 p-\sum_{j=1}^{q}d_j+\tfrac12 q\right)$$

(5) $$\rho=\prod_{j=1}^{m}(A_j)^{A_j}\prod_{j=1}^{n}(B_j)^{-B_j}\prod_{j=1}^{p}(C_j)^{-C_j}\prod_{j=1}^{q}(D_j)^{D_j}.$$

The convergence of (1) can be investigated by means of the asymptotic representation of the gamma function 1.18(6). With

$$s=\sigma+it\quad(\sigma,\,t\text{ real}),\qquad z=Re^{i\Phi}\quad(R>0,\quad\Phi\text{ real})$$

the absolute value of the integrand is comparable with

$$(6) \quad e^{-\frac{1}{2}\alpha\pi|t|}\,|t|^{\beta\gamma+\lambda}R^{-\gamma}\,e^{\Phi t}\rho^{\gamma}$$

when $|t|$ is large. There are four types of convergent integrals (1).

First type: $\alpha > 0$. The integral converges absolutely for $|\Phi| < \alpha\pi/2$ and defines a function analytic in the sector $|\arg z| < \min(\pi, \alpha\pi/2)$. (The point $z = 0$ is tacitly excluded.)

Second type: $\alpha = 0$, $\beta \neq 0$. The integral (1) does not converge for complex z. For $z > 0$ it converges absolutely if γ is so chosen that

$$(7) \quad -\beta\gamma > 1 + \lambda;$$

and there exists an analytic function of z, defined over $|\arg z| < \pi$, whose values for positive z are given by (1).

Third type: $\alpha = \beta = 0$, $\lambda < -1$. Here (7) is satisfied for arbitrary γ. The integral converges absolutely for all positive z (but not for complex z) and represents a continuous function of z $(0 < z < \infty)$. There are now two analytic functions, one regular in any domain contained in $|\arg z| < \pi$, $|z| > \rho$ whose values for $z > \rho$ are represented by (1), and another regular in any domain contained in $|\arg z| < \pi$, $0 < |z| < \rho$ whose values for $0 < z < \rho$ are represented by (1). The two functions are in general distinct.

Fourth type: $\alpha = \beta = 0, -1 \leq \lambda < 0$. The integral converges (although not absolutely) for $0 < z < \rho$ and for $z > \rho$. There are two analytic functions of the same nature as in the preceding case. There is a discontinuity at $z = \rho$ and the integral does not exist there, though it may have a principal value. The nature of the discontinuity, and the principal value, are discussed in the paper by Dixon and Ferrar.

Multiple integrals of a similar structure occur occasionally.

An example for an integral of the Mellin-Barnes type is the following one (Whittaker-Watson, 1927, p. 289)

$$(8) \quad \int_{-i\infty}^{i\infty}\Gamma(\alpha+s)\,\Gamma(\beta+s)\,\Gamma(\gamma-s)\,\Gamma(\delta-s)\,ds$$

$$= 2\pi i\,\frac{\Gamma(\alpha+\gamma)\,\Gamma(\alpha+\delta)\,\Gamma(\beta+\gamma)\,\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}.$$

The path of integration is indented so that the poles of $\Gamma(\gamma - s)\ \Gamma(\delta - s)$ lie to the right and the poles of $\Gamma(\alpha + s)\ \Gamma(\beta + s)$ to the left of it, and it is supposed that α, β, γ, δ are such that no pole of the first set coincides with any pole of the second set. [For further examples cf. 2.1(15) and section 7.3.6 and Ramanujan, 1927, p. 216.]

1.20. Power series of some trigonometric functions

From 1.13(1) a number of trigonometric expansions can be deduced (cf. also similar expansions obtained by Schwatt, 1932) such as

(1) $z \coth z = 2z(e^{2z}-1)^{-1} + z = \sum_{n=0}^{\infty} 2^{2n} B_{2n} \, z^{2n}/(2n)!$

$= 2 \sum_{n=0}^{\infty} (-1)^{n+1} \zeta(2n) \, \pi^{-2n} z^{2n}$ $\qquad |z| < \pi,$

(2) $\tanh z = 2 \coth(2z) - \coth z = \sum_{n=1}^{\infty} 2^{2n}(2^{2n}-1) B_{2n} \, z^{2n-1}/(2n)!$

$= 2 \sum_{n=1}^{\infty} (-1)^{n+1} (2^{2n}-1) \zeta(2n) \, \pi^{-2n} z^{2n-1}$ $\qquad |z| < \pi/2,$

(3) $z \operatorname{ctn} z = \sum_{n=0}^{\infty} (-1)^n 2^{2n} B_{2n} \, z^{2n}/(2n)!$

$= -2 \sum_{n=0}^{\infty} \zeta(2n) \, \pi^{-2n} z^{2n}$ $\qquad |z| < \pi,$

(4) $\tan z = \sum_{n=1}^{\infty} (-1)^{n+1} 2^{2n} (2^{2n}-1) B_{2n} \, z^{2n-1}/(2n)!$

$= 2 \sum_{n=1}^{\infty} (2^{2n}-1) \zeta(2n) \, \pi^{-2n} z^{2n-1}$ $\qquad |z| < \pi/2,$

(5) $z/\sin z = z[\operatorname{ctn}(\tfrac{1}{2}z) - \operatorname{ctn} z] = 2 \sum_{n=0}^{\infty} (-1)^n (1-2^{2n-1}) B_{2n} \, z^{2n}/(2n)!$

$|z| < \pi,$

(6) $\log \cos z = -\int_0^z \tan z \, dz = \sum_{n=1}^{\infty} (-1)^n (2^{2n}-1) \, 2^{2n-1} B_{2n} \, z^{2n}/[n(2n)!]$

$|z| < \pi/2.$

We write

(7) $\tan z = \sum_{n=0}^{\infty} (-1)^{n+1} C_{2n+1} \, z^{2n+1}/(2n+1)!$ $\qquad |z| < \pi/2,$

(8) $z/\sin z = \sum_{n=0}^{\infty} (-1)^n D_{2n} \, z^{2n}/(2n)!$ $\qquad |z| < \pi.$

Then a comparison with (4) and (5) gives

(9) $C_{2n-1} = 2^{2n}(1-2^{2n}) B_{2n}/(2n),$

(10) $D_{2n} = 2(1-2^{2n-1}) B_{2n} \cdot$

Integralexpressions for C_{2n-1} and D_{2n} can be obtained from 1.13(24) to 1.13(28).

More general expansions than those listed before can be obtained from the results in sections 1.13.1 and 1.14.1 (Nörlund, 1922, p. 196). Two examples are

(11) $\cos(mt)\,(t/\sin t)^m = \sum_{n=0}^{\infty} (-1)^n (2t)^{2n} B_{2n}^{(m)}/(2n)!\,,$

(12) $\sin(mt)(t/\sin t)^m = \sum_{n=0}^{\infty} (-1)^{n+1} (2t)^{2n+1} B_{2n+1}^{(m)}/(2n+1)!\,.$

Both expansions converge for $|t| < \pi$. With the notation used in 1.13(1) we have

(13) $B_l^{(m)} = B_l^{(m)}(a_1 \cdots a_m) \qquad a_1 = \cdots = a_m = 1.$

1.21. Some other notations and symbols

Alternative notations for the gamma function and some related symbols are (cf. 1.2):

(1) (Factorial function) $\Pi(z) = z! = \Gamma(z+1)$;

(2) γ = Euler's constant 1.1(4);

(3) (Hankel's symbol)

$$(v,n) = 2^{-2n}\{(4v^2-1)(4v^2-3^2)\cdots[4v^2-(2n-1)^2]\}/n!$$
$$= \Gamma(\tfrac{1}{2}+v+n)/[n!\,\Gamma(\tfrac{1}{2}+v-n)] \qquad n = 1, 2, 3, \ldots;$$

(4) (Kramp's symbol)

$$c^{a/b} = c(c+b)(c+2b)\cdots[c+(a-1)b]$$
$$= b^{a-1}\Gamma(a+c/b)/\Gamma(c/b) \qquad a = 2, 3, 4, \ldots;$$

(5) (Pochhammer's symbol)

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a) \qquad n = 1, 2, 3, \ldots;$$

(6) (Binomial coefficient)

$$\binom{a}{m} = (-1)^m\,\Gamma(m-a)/[m!\,\Gamma(-a)] = \Gamma(1+a)/[m!\,\Gamma(1+a-m)].$$

The Bernoulli numbers B_n are often defined by the expansion

(7) $\tfrac{1}{2}z + z(e^z-1)^{-1} = \tfrac{1}{2}z\coth(\tfrac{1}{2}z) = 1 - \sum_{n=1}^{\infty} (-1)^n B_n\, z^{2n}/(2n)!\,.$

It follows from 1.13(1) and 1.13(16) for the B_n thus defined

(8) $B_n = 2(2n)!\,(2\pi)^{-2n} \sum_{r=1}^{\infty} r^{-2n}$

and hence

(9) $B_1 = 1/6, \quad B_2 = 1/30, \quad B_3 = 1/42, \quad B_4 = 1/30, \ldots.$

The Bernoulli polynomials are often denoted by $\Phi_n(x)$ and defined by

(10) $z(e^{xz} - 1)/(e^z - 1) = \sum_{n=1}^{\infty} \Phi_n(x) z^n/n!\,.$

With our notation 1.13 (2) we have

(11) $\Phi_n(x) = B_n(x) - B_n(0)$

and hence with 1.13 (3)

(12) $\Phi_1(x) = x, \quad \Phi_2(x) = x^2 - x, \quad \Phi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$

If the Euler numbers E_n are defined by

(13) $\operatorname{sech} z = \sum_{n=0}^{\infty} (-1)^n E_n z^{2n}/(2n)!\,,$

then it is obvious from 1.14 (1) and 1.14 (14) that

(14) $E_n = 2(2n)!\,(2/\pi)^{2n+1} \sum_{r=0}^{\infty} (-1)^r (2r+1)^{-2n-1}$

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CHAPTER II

THE HYPERGEOMETRIC FUNCTION

FIRST PART: THEORY

2.1. The hypergeometric series

2.1.1. The hypergeometric equation

If a homogeneous linear differential equation of the second order has at most three singularities we may assume that these are at $0, \infty, 1$. If all of these singularities are "regular" (cf. Poole, 1936), then the equation can be reduced to the form (cf. Poole, 1936)

$$(1) \quad z(1-z)\frac{d^2u}{dz^2} + [c-(a+b+1)z]\frac{du}{dz} - abu = 0$$

where a, b, c, are independent of z. This is the *hypergeometric equation.* We shall call a, b, c the parameters of the equation; they are arbitrary complex numbers.

We define

$$(a)_n = \Gamma(a+n)/\Gamma(a),$$

i.e.,

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \qquad n = 1, 2, 3, \ldots .$$

If $c \neq 0, -1, -2, \ldots$, then

$$(2) \quad u_1 = \sum_{n=0}^{\infty} (a)_n (b)_n z^n/[(c)_n n!] \equiv {}_2F_1(a, b; c; z) \equiv F(a, b; c; z)$$

is a solution of (1) which is regular at $z = 0$.

If $c = -n$, where $n = 0, 1, 2, \ldots$, then

$$(3) \quad u_1 = z^{n+1} \sum_{m=0}^{\infty} (a+n+1)_m (b+n+1)_m z^m/[(n+2)_m m!]$$

$$= z^{n+1} {}_2F_1(a+n+1, b+n+1; n+2; z)$$

is such a solution. The function ${}_2F_1(a,b;c;z)$ is called the hypergeometric series of variable z with parameters a, b, c. The subscripts in ${}_2F_1$ are usually omitted if there do not occur any other types of generalized hypergeometric series (cf. Chapters 4, 5) in the investigation.

We shall supplement the definition of the hypergeometric series in the case $c = -m$, $(m = 0, 1, 2, \ldots,)$, when (2) becomes meaningless.

If $a = -n$ or $b = -n$ where $n = 0, 1, 2, \ldots$, and if $c = -m$ where $m = n, n+1, n+2, \ldots$, then we define

$$(4)\qquad \begin{cases} F(-n,b;-m;z) = \sum_{r=0}^{n} (-n)_r\,(b)_r\, z^r/[(-m)_r\, r\,!] \\ F(a,-n;-m;z) = \sum_{r=0}^{n} (a)_r\,(-n)_r\, z^r/[(-m)_r\, m\,!] \end{cases}$$

Since (3) and (4) are solutions of (1), we see that the hypergeometric equation has a solution which is a polynomial of z whenever $-a$ or $-b$ is a non-negative integer. (If $a = -m$ or $b = -m$ and $c = -n$, where $n = 0, 1, 2, \ldots$, and $m = n+1, n+2, \ldots$, the series in (3) terminates.)

If a and b are different from $0, -1, -2, \ldots$, then the hypergeometric series (2) [or (3), in the case $c = -n$] converges absolutely for all values of $|z| < 1$. Since an application of 1.18(4) shows that

$$(5)\qquad \frac{(a)_n\,(b)_n}{(c)_n\, n\,!} = \frac{\Gamma(a+n)}{\Gamma(a)}\,\frac{\Gamma(b+n)}{\Gamma(b)}\,\frac{\Gamma(c)}{\Gamma(c+n)}\,\frac{\Gamma(1)}{\Gamma(n+1)}$$

$$= \frac{\Gamma(c)\,\Gamma(1)}{\Gamma(a)\,\Gamma(b)}\, n^{a+b-c-1}\,[1 + O(n^{-1})],$$

we see by Raabe's test (see e.g. Bromwich 1947, pp. 39, 241) that for $|z| = 1$ we have:

absolute convergence for $|z| = 1$ if $\mathrm{Re}(a+b-c) < 0$,
conditional convergence for $|z| = 1$, $z \neq 1$ if $0 \leq \mathrm{Re}(a+b-c) < 1$,
divergence if $|z| = 1$ and $1 \leq \mathrm{Re}(a+b-c)$.

2.1.2. Elementary relations

From the definition (2) we have

$$F(a,b;c;z) = F(b,a;c;z).$$

The six functions

$$F(a \pm 1, b; c; z), \quad F(a, b \pm 1; c; z), \quad F(a,b;c \pm 1; z)$$

are called *contiguous* to $F(a,b;c;z)$. Between $F(a,b;c;z)$ and any two functions contiguous to it there exists a linear relation with coefficients which are linear functions of z. There are 15 relations of this type which have been found by Gauss. For a complete list see 2.8(31) to 2.8(45). One of these relations is

(6) $$c\,F(a,b-1;c;z)+(a-b)\,z\,F(a,b;c+1;z) = c\,F(a-1,b;c;z).$$

To verify (6) we expand both sides in a power series. Then the coefficient of z^n on the left-hand side of (6) is

$$c\,\frac{(a)_n\,(b-1)_n}{(c)_n\,n!}+(a-b)\,\frac{(a)_{n-1}\,(b)_{n-1}}{(c+1)_{n-1}\,(n-1)!}$$

$$=\frac{(a)_{n-1}\,(b)_{n-1}}{(c+1)_{n-1}\,(n-1)!}\,[a-b+(b-1)\,(a+n-1)/n]$$

$$=\frac{c\,(a)_{n-1}\,(b)_{n-1}}{(c)_n\,n!}\,(a-1)(b+n-1)=c\,\frac{(a-1)_n\,(b)_n}{(c)_n\,n!},$$

which proves (6).

If l, m, n, are integers, then

$$F(a+l,\ b+m;\ c+n;\ z)$$

can be expressed by repeated applications of these relations as a linear combination of $F(a,b;c;z)$ and one of its contiguous functions with coefficients which are rational functions of a, b, c, z.

Of course we must assume that c and $c+n$ are different from $0, -1, -2, \ldots$. $F(a,b;c;z)$ and $F(a+l,\ b+m;\ c+n;\ z)$ are called *associated series*. It can be shown that any three associated series are connected by a linear homogeneous relation with polynomial coefficients provided that the values of the third parameter are different from $0, -1, -2, \ldots$ (cf. Poole, 1936, p. 91 ff).

We also have

(7) $$\frac{d^n}{dz^n}\,F(a,b;c;z)=(a)_n(b)_n\,[(c)_n]^{-1}\,F(a+n,\ b+n;c+n;z),$$

(8) $$(a)_n\,z^{a-1}\,F(a+n,\ b;c;z)=\frac{d^n}{dz^n}\,[z^{a+n-1}\,F(a,b;c;z)],$$

(9) $$(c)_n\,z^{c-1}\,(1-z)^{a+b-c}\,F(a,b;c;z) = \frac{d^n}{dz^n}\,[z^{n+c-1}\,(1-z)^{n+a+b-c}\,F(a+n,\ b+n;\ c+n;\ z)].$$

Relation (9) is due to Jacobi (1859). For a complete list of such relations see 2.8(20) to 2.8(27). To prove (8) and (9) we introduce the operators

$$\delta = z\,\frac{d}{dz}, \qquad D = \frac{d}{dz}.$$

We have

$$a\,F(a+1,b\,;\,c\,;\,z) = (\delta + a)\,F(a,b\,;\,c\,;\,z)$$

and since

$$(\delta + a)(\delta + a + 1)\cdots(\delta + a + n - 1)\,f(z) = z^{1-a}\,D^n[z^{a+n-1}\,f(z)]$$

for every analytic function $f(z)$ (cf. Poole, 1936, p. 93), this proves (8).

To obtain (9), we write (1) in the form

$$D\,[z(1-z)\,MD\,u] = abMu$$

where $M = z^{c-1}(1-z)^{a+b-c}$. According to (7), $D^{n-1}\,F(a,b\,;\,c\,;\,z)$ satisfies the hypergeometric equation with $a+n-1$, $b+n-1$, $c+n-1$ instead of a, b, c and from that we obtain the recurrence relation

$$D\,[z^n(1-z)^n\,MD^nF] = (a+n-1)(b+n-1)\,[z(1-z)]^{n-1}\,MD^{n-1}F$$

and therefore

$$D^n[z^n\,(1-z)^n\,MD^nF] = (a)_n\,(b)_n\,MF.$$

Using (7) again and assuming that F is not a polynomial of degree less than n, i. e., $(a)_n\,(b)_n \neq 0$, we finally obtain (9).

The general theory of Riemann's equation (cf. section 2.6.1, and Poole, 1936) indicates that in general there must exist 24 solutions of (1) which are of the type

$$z^{\rho}\,(1-z)^{\sigma}\,F(a',\,b';\,c';\,z')$$

where ρ, σ, a', b', c' are linear functions of a, b, c and where z and z' are connected by a homographic transformation. For a list of these 24 solutions (which are due to Kummer) see Goursat (1881), and 2.9(1) to 2.9(24). Any three of these solutions are connected by a linear relation with constant coefficients; for these see Goursat (1881) and 2.9(25) to 2.9(44). These relations can be used for the analytic continuation of the hypergeometric series, for a proof see 2.1.4.

2.1.3. The fundamental integral representations

If Re $c >$ Re $b > 0$, we have Euler's formula

(10) $$F(a,b\,;\,c\,;\,z) = \Gamma(c)\,[\Gamma(b)\,\Gamma(c-b)]^{-1}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt.$$

Here the right-hand side is a one-valued analytic function of z within the domain $|\arg\,(1-z)| < \pi$; therefore (10) gives also the analytic continuation of $F(a,b\,;\,c\,;\,z)$. To prove (10) for $|z| < 1$ we expand $(1-tz)^{-a}$ in a binomial series and integrate term by term; this leads to beta-integrals which can be evaluated by 1.5(1) to 1.5(5).

From the identity

$$(11)\quad \left\{z(1-z)\frac{\partial^2}{\partial z^2}+[c-(a+b+1)z]\frac{\partial}{\partial z}-ab\right\}$$
$$\times\,[t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}]=-a\frac{\partial}{\partial t}[t^{b}(1-t)^{c-b}(1-tz)^{-a-1}]$$

it follows that the right-hand side in (10) satisfies (1), and with $s=-t$ that

$$\int_0^\infty s^{b-1}(1+s)^{c-b-1}(1+sz)^{-a}\,ds$$

is a solution of (1) if Re $b>0$, Re$(a+1-c)>0$, and $|\arg z|<\pi$. With $s=\tau/(1-\tau)$ this becomes

$$\int_0^1 \tau^{b-1}(1-\tau)^{a-c}[1-\tau(1-z)]^{-a}\,dt,$$

and therefore

$$(12)\quad F(a,b;a+b+1-c;1-z)=\Gamma(a+b+1-c)\,[\Gamma(b)\,\Gamma(a+1-c)]^{-1}$$
$$\times\int_0^\infty s^{b-1}(1+s)^{c-b-1}(1+sz)^{-a}\,ds$$

is also a solution of the hypergeometric equation. Moreover, any integral

$$\int_C t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

is a solution of (1) if C is either closed on the Riemann surface of the integrand or terminates at zeros of $t^b(1-t)^{c-b}(1-tz)^{-a-1}$ Expanding $(1-tz)^{-a}$ in a binomial series and using the contour integrals 1.6(6) to 1.6(8) for the beta-function we find

$$F(a,b;c;z)=\frac{i\Gamma(c)\exp[i\pi(b-c)]}{\Gamma(b)\,\Gamma(c-b)\,2\sin\pi(c-b)}$$
$$\times\int_0^{(1+)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$
$$\text{Re } b>0,\quad |\arg(1-z)|<\pi,\quad c-b\neq 1,2,3,\ldots,$$

$$F(a,b;c;z)=\frac{-i\,\Gamma(c)\exp(-i\pi b)}{\Gamma(b)\,\Gamma(c-b)\,2\sin\pi b}$$
$$\times\int_1^{(0+)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$
$$\text{Re } c>\text{Re } b,\quad |\arg(-z)|<\pi,\quad b\neq 1,2,3,\ldots,$$

$$(13)\quad F(a,b;c;z)=\frac{-\Gamma(c)\exp(-i\pi c)}{\Gamma(b)\,\Gamma(c-b)\,4\sin\pi b\,\sin\pi(c-b)}$$
$$\times\int^{(1+,0+,1-,0-)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$
$$|\arg(-z)|<\pi,\quad b,\ 1-c,\ c-b\neq 1,2,3,\ldots.$$

In each case we assume that the path of integration starts at a point of the Riemann surface of $t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$ where t is real, $0 \le t \le 1$ and t^b, $(1-t)^{c-b}$ denote the principal values of these functions, and where $(1-tz)^{-a}$ is defined in such a way that $(1-tz)^{-a} \to 1$ if $z \to 0$.

If we put $z = 1$, the right-hand side of (10) becomes a beta-integral and we obtain form 1.5(1) and 1.5(5)

$$F(a,b;c;1) = \Gamma(c)\,\Gamma(c-a-b)\,[\Gamma(c-a)\,\Gamma(c-b)]^{-1} \tag{14}$$

$$\operatorname{Re} c > \operatorname{Re} b > 0, \qquad \operatorname{Re}(c-a-b) > 0.$$

We can show directly that (14) is valid only if $c \neq 0, -1, -2, \ldots$, and $\operatorname{Re}(c-a-b) > 0$. From the recurrence relation

$$\begin{aligned}(c-a)(c-b)\,z\,F(a,b;c+1;z) &= c\,[(2c-a-b-1)z-c+1]\,F(a,b;c;z)\\ &\quad + c(c-1)(1-z)\,F(a,b;c-1;z)\end{aligned}$$

and from the remarks after (5) of section 2.1.1 we find that, for $m = 1, 2, 3, \ldots$.

$$\begin{aligned}F(a,b;c;1) &= \frac{(c-a)(c-b)}{c(c-a-b)}\,F(a,b;c+1;1)\\ &= \frac{(c-a)_m\,(c-b)_m}{(c)_m\,(c-a-b)_m}\,F(a,b;c+m;1)\end{aligned}$$

provided that

$$\lim_{z\to 1}(1-z)\,F(a,b;c;z) = 0 \qquad \operatorname{Re}(c-a-b) > 0.$$

If this is true (as we will show presently), then we have for $m \to \infty$

$$\lim_{m\to\infty} F(a,b;c+m;1) = 1,$$

$$\lim_{m\to\infty}\frac{(c-a)_m\,(c-b)_m}{(c)_m\,(c-a-b)_m} = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \times \lim_{m\to\infty}\frac{\Gamma(c-a+m)\,\Gamma(c-b+m)}{\Gamma(c+m)\,\Gamma(c-a-b+m)},$$

and this together with 1.18(4) proves (14). Now

$$(1-z)\,F(a,b;c-1;z) = 1 + \sum_{n=1}^{\infty}(v_n - v_{n-1})\,z^n \to 0 \quad \text{for } z \to 1,$$

if

$$v_n = \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n-1)\,\Gamma(n+1)}\,\frac{\Gamma(c-1)}{\Gamma(a)\,\Gamma(b)} \to 0 \quad \text{for } n \to \infty.$$

This is true according to (5) if $\operatorname{Re}(c-a-b)>0$.

A second type of integral representations for the hypergeometric series is due to E. W. Barnes, (1908), who based the whole theory of the hypergeometric function on the representation

$$(15)\quad \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(c)}\,F(a,b;c;z) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(-s)}{\Gamma(c+s)}\,(-z)^s\,ds,$$

where $|\arg(-z|<\pi$ and where the path of integration is indented if necessary in such a manner as to separate the poles at $s=0,1,2,\ldots,$ from the poles at $s=-a-n$, $s=-b-n$ $(n=0,1,2,\ldots,)$ of the integrand. It is always possible to find such a path of integration provided that both a and b are different from $0,-1,-2,\ldots$. If we define

$$F(a,b;c;z)/\Gamma(c)$$

to be equal to

$$(16)\quad (a)_{n+1}\,(b)_{n+1}\,z^{n+1}\,F(a+n+1,\ b+n+1;\ n+2;\ z)/(n+1)!$$

when $c=-n$ $(n=0,1,2,\ldots,)$, then (15) remains valid also for these values of c.

To prove (15) we observe that in the case $|z|<1$ the integral on the right-hand side can be evaluated by the calculus of residues as the sum of the residues of the integrand at the poles $s=0,1,2,\ldots,$ (cf. 1.18 for the asymptotic formulas which describe the behavior of the integrand at infinity).

2.1.4. Analytic continuation of the hypergeometric series

The integrals in (10), (13), (15) define analytic functions of z which are one-valued in the domain $|\arg(-z)|<\pi$, that is, in the whole z plane with the exception of the points on the positive real axis, and may serve for effecting the analytic continuation of $F(a,b;c;z)$ to the domain $|\arg(-z)|<\pi$. We shall denote this analytic continuation of $F(a,b;c;z)$ again by $F(a,b;c;z)$ which then means a branch (the *principal branch*) of the analytic function generated by the hypergeometric series.

We exclude the polynomial case when a or b is equal to $0,-1,-2,\ldots,$ and define $F(a,b;c;z)/\Gamma(c)$ by (16) if $c=0,-1,-2,\ldots$. Evaluating the integral on the right-hand side of (15) by the calculus of residues as the sum of the residues of the integrand at the poles $s=-a-l$, $s=-b-k$, where $k,\ l=0,1,2,\ldots,$ we first assume that $a-b$ is not an integer so that these poles are simple poles and obtain

(17) $F(a, b; c; z)/\Gamma(c)$

$$= \Gamma(b-a)[\Gamma(b)\,\Gamma(c-a)]^{-1}(-z)^{-a} F(a, 1-c+a; 1-b+a; z^{-1})$$
$$+ \Gamma(a-b)[\Gamma(a)\,\Gamma(c-b)]^{-1}(-z)^{-b} F(b, 1-c+b; 1-a+b; z^{-1})$$

where $a-b$ is not an integer and where $|\arg(-z)| < \pi$.

If $b = a + m$ where $m = 0, 1, 2, \ldots$, then the integrand in (15) has simple poles at $s = -a - k$ ($k = 0, 1, \ldots, m-1$) (and no simple poles at all if $m = 0$); at $s = -a - m - l$ ($l = 0, 1, 2, \ldots,$) there are poles of the second order, and we have

(18) $$\frac{\Gamma(a+m)}{\Gamma(c)} F(a, a+m; c; z)$$

$$= \frac{(-z)^{-a-m}}{\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_{n+m}(1-c+a)_{n+m}}{n!\,(n+m)!} z^{-n} [\log(-z) + h_n]$$

$$+ (-z)^{-a} \sum_{n=0}^{m-1} \frac{(a)_n \Gamma(m-n)}{\Gamma(c-a-n)\, n!} z^{-n}$$

$$a \neq 0, -1, -2, \ldots, \qquad m = 0, 1, 2, \ldots, \qquad |\arg(-z)| < \pi,$$

and

$$h_n = \psi(1+m+n) + \psi(1+n) - \psi(a+m+n) - \psi(c-a-m-n)$$
$$= \psi(1+m+n) + \psi(1+n) - \psi(a+m+n) - \psi(1-c+a+m+n)$$
$$+ \pi \operatorname{ctn} \pi(c-a).$$

The summation $\sum_{n=0}^{m-1}$ in (18) means zero if $m = 0$. If $c = a + m + l$ where $l = 0, 1, 2, \ldots$, then (18) remains valid only after a passage to the limit the result of which is:

(19) $$\frac{\Gamma(a+m)}{\Gamma(a+m+l)} F(a, a+m; a+m+l; z)$$

$$= (-1)^{m+l}(-z)^{-a-m} \sum_{n=l}^{\infty} \frac{(a)_{n+m}(n-l)!}{(n+m)!\, n!} z^{-n}$$

$$+ \frac{(-z)^{-a-m}}{(l+m-1)!} \sum_{n=0}^{l-1} \frac{(a)_{n+m}(1-m-l)_{n+m}}{n!\,(n+m)!} z^{-n} [\log(-z) + h'_n]$$

$$+ (-z)^{-a} \sum_{n=0}^{m-1} \frac{(m-n-1)!\,(a)_n}{(m+l-n-1)!\, n!} z^{-n}$$

$$a + m \neq 0, -1, -2, \ldots, \qquad |\arg(-z) < \pi.$$

Here $\sum_{n=0}^{l-1}$, $\sum_{n=0}^{m-1}$ denote zero if $l = 0$ or $m = 0$, and

$$h'_n = \psi(1+m+n) + \psi(1+n) - \psi(a+m+n) - \psi(l-n).$$

If $c - a$ or $c - b$ is a negative integer, then $F(a,b\,;\,c\,;\,z)$ becomes an elementary function of z. In particular we have

(20) $F(a, a+m\,;\, a+m-l\,;\, z) = (1-z)^{-a-l} F(m-l, -l\,;\, a+m-l\,;\, z)$

where the hypergeometric series on the right-hand side is a polynomial if $l = 0, 1, 2, \ldots$. To prove (20) we observe that Euler's integral (10) or (12) transforms into an integral of the same type if we put

(21) $s = 1-t, \quad (1-t)/(1-tz), \quad t/(1-z+tz).$

From this we obtain

(22) $F(a,b\,;\,c\,;\,z) = (1-z)^{-a} F[a, c-b\,;\,c\,;\,z/(z-1)],$

$$= (1-z)^{-b} F[c-a, b\,;\,c\,;\,z/(z-1)],$$

and

(23) $F(a,b\,;\,c\,;\,z) = (1-z)^{c-a-b} F(c-a, c-b\,;\,c\,;\,z).$

Relation (22) holds if both $|z| < 1$ and $|z/(z-1)| < 1$. Since the right-hand side of (22) is defined for Re $z < ½$, this can be used to obtain an analytic continuation of $F(a,b\,;\,c\,;\,z)$ into the half plane Re $z < ½$. Of course (23) is valid only if $|z| < 1$, unless a, b, $c - a$ or $c - b$ is a non-positive integer.

From (17) to (23) and from combinations of these formulas (cf. sections 2.9 and 2.10) we can obtain the complete analytic continuation of $F(a,b\,;\,c\,;\,z)$ into the domain $|\arg(1-z)| < 1$. As a result, at any point $z = z_0$, $F(a,b\,;\,c\,;\,z)$ may be computed from a series which converges like a geometric series, unless $z_0 = \exp(\pm i\pi/3)$. In this case it may happen that all of the series in (17) to (23) converge only conditionally or like a series of the type $\Sigma\, z^n n^{-k}$ where k is a constant > 1.

2.1.5. Quadratic and cubic transformations

We may consider (17) to (23) as linear transformations of $F(a, b;\, c;\, z)$. If a, b, c are unrestricted then there exists no transformation of a higher order (i.e., a transformation in which the variables are connected by a non-linear relation).

If and only if the numbers

$$\pm(1-c), \qquad \pm(a-b), \qquad \pm(a+b-c)$$

have the property that one of them equals ½ or that two of them are equal, then there exists a so-called quadratic transformation. The fundamental formulas are those of Gauss and Kummer:

(24) $F[a,b;\, 2b;\, 4z/(1+z)^2] = (1+z)^{2a} F(a, a+½-b;\, b+½;\, z^2),$

(25) $F(a,b;\, 1+a-b;\, z)$

$$= (1-z)^{-a} F[½a, ½a+½-b\,;\, 1-a+b;\, -4z/(1-z)^2],$$

(26) $F(a, a+\frac{1}{2}; b; z) = 2^{2a}[1+(1-z)^{\frac{1}{2}}]^{-2a}$
$\times F\{2a, 2a-b+1; b; [1-(1-z)^{\frac{1}{2}}]/[1+(1-z)^{\frac{1}{2}}]$,

(27) $F[a, b; a+b+\frac{1}{2}; 4z(1-z)] = F(2a, 2b; a+b+\frac{1}{2}; z)$.

In (26), $(1-z)^{\frac{1}{2}}$ is defined in such a way that it becomes positive if z is real and $z < 1$. A consequence of (27) and 2.10 (1) is

(28) $$F(2a, 2b; a+b+\tfrac{1}{2}; \tfrac{1}{2}z+\tfrac{1}{2}) = \frac{\Gamma(a+b+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})\,\Gamma(b+\frac{1}{2})}\;F(a, b; \tfrac{1}{2}; z^2) - z\,\frac{\Gamma(a+b+\frac{1}{2})\,\Gamma(-\frac{1}{2})}{\Gamma(a)\;\Gamma(b)}\;F(a+\tfrac{1}{2}, b+\tfrac{1}{2}; 3/2; z^2).$$

The series on both sides of (24) to (28) converge in a certain neighborhood of $z=0$, and the formulas are valid in the largest connected domain which is such that it contains the point $z = 0$ and that the series in the formula under consideration are convergent within this domain. For instance, (27) holds if $|z| < (2^{\frac{1}{2}} - 1)/2$ but is not valid if z is real and $\frac{1}{2} < z < 1$, although both sides of (27) make sense in this case. Apart from these restrictions, (24) to (28) can also be used for the analytic continuation of one of the series involved, in particular if they are combined with the linear transformations.

For a complete list of quadratic transformations see E. Goursat, 1881, and 2.11(1) to 2.11(36).

The quadratic transformations are consequences of the general theory of Riemann's P-equation [cf. Poole, (1936) and 2.6(2)]. We could verify (24) to (28) by showing that both sides satisfy the hypergeometric equation. For instance, it is easy to show that $F[a, b; a+b+\frac{1}{2}; 4z(1-z)]$ satisfies (1) if we take $2a$, $2b$ and $a+b+\frac{1}{2}$ as the values of the parameters. Next we see that at $z = 0$ both sides of (27) have the same values and the same first derivative. Since we can deduce from 2.2(2) and 2.3(1) that (1) has only one solution which is one-valued and regular at $z = 0$ unless $c = 0, -1, -2, \ldots$, both sides of (27) must be equal, with the possible exception of the case where $a+b+\frac{1}{2} = 0, -1, -2, \ldots$.

By applying the linear transformations to (27) we can obtain the remaining formulas (24) to (28). There exist also direct proofs for these transformations. To prove, e.g., (25) we can proceed as follows (cf. Bailey 1935). We write (23) in the form

(29) $(1-z)^{a+b-c}\,F(a, b; c; z) = F(c-a, c-b; c; z)$,

expand both sides in a series of powers of z, and compare the coefficients of z^n. This gives

$$\sum_{r=0}^{n} \frac{(a)_r\,(b)_r}{(c)_r\, r!}\;\frac{(c-a-b)_{n-r}}{(n-r)\,!} = \frac{(c-a)_n\,(c-b)_n}{(c)_n\, n\,!}$$

and hence

$$\sum_{r=0}^{n} \frac{(a)_r\,(b)_r\,(-n)_r}{(c)_r\,(1+a+b-c-n)_r\, r\,!} = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}. \tag{30}$$

This is *Saalschütz's formula.* Now the right-hand side of (25) is

$$\sum_{r=0}^{\infty} \frac{(\tfrac12 a)_r\,(\tfrac12 a+\tfrac12-b)_r}{r\,!\,(1+a-b)_r}\,(-4z)^r\,(1-z)^{-a-2r},$$

and here the coefficient of z^n is

$$\sum_{r=0}^{n} \frac{(\tfrac12 a)_r\,(\tfrac12 a+\tfrac12-b)_r\,(-4)^r\,(a+2r)_{n-r}}{(1+a-b)_r\, r\,!\,(n-r)\,!}.$$

Because of the relations

$$\begin{aligned} &4^r\,(\tfrac12 a)_r\,(\tfrac12 a+\tfrac12)_r = (a)_{2r}, \qquad (-1)^r\,(-n)_r\,(n-r)\,! = n\,!\,, \\ &(a+2r)_{n-r} = (a+n)_r\,(a)_n/(a)_{2r}, \end{aligned} \tag{31}$$

and because of Saalschütz's formula (30), this is equal to

$$\frac{(a)_n}{n\,!}\sum_{r=0}^{n} \frac{(\tfrac12 a+\tfrac12-b)_r\,(a+n)_r\,(-n)_r}{(1+a-b)_r\,(\tfrac12 a+\tfrac12)_r\, r\,!} = \frac{(a)_n\,(b)_n}{n\,!\,(1+a-b)_n}.$$

This completes the proof of (25).

Applying (22) to the right-hand side of (25) we obtain

$$(1+z)^{-a}\,F[\tfrac12 a,\ \tfrac12 a+\tfrac12;\ 1+a-b;\ 4z/(1+z)^2] = F(a,\ b;\ 1+a-b;\ z),$$

and if we introduce $4z/(1+z)^2$ instead of z as a new variable, a relation equivalent to (26) is obtained.

To prove (24), we show first that

$$F(a,\ b;\ 2b;\ 4z/(1+z)^2] = (1+z)^{2a}\,(1+z^2)^{-a}\,F\{\tfrac12 a,\ \tfrac12 a+\tfrac12;\ b+\tfrac12;\ [2z/(1+z^2)]^2\}, \tag{32}$$

which is equivalent to

$$F(a,\ b;\ 2b;\ z) = (1-\tfrac12 z)^{-a}\,F\{\tfrac12 a,\ \tfrac12 a+\tfrac12;\ b+\tfrac12;\ [z/(2-z)]^2\}, \tag{33}$$

as may be seen by introducing $4z/(1+z)^2$ as a new variable instead of z. From (26) it follows that the right-hand side in (32) is equal to

$$(1+z)^{2a}\,F(a,\ a-b+\tfrac12;\ b+\tfrac12;\ z^2),$$

and therefore (24) follows from (26) and (32). To prove (32), we use (10) which gives:

(34) $F[a, b; 2b; 4z/(1+z)^2] = \Gamma(2b)[\Gamma(b)]^{-2}(1+z)^{2a}$

$$\times \int_0^1 [t(1-t)]^{b-1}[1+2z(1-2t)+z^2]^{-a}\,dt.$$

Since the integral on the right-hand side remains unchanged if we write $-z$ instead of z and $1-t$ instead of t, it is an even function of z, and by introducing $1-2t = \cos\theta$ we see that the right-hand side in (34) is equal to

$$2^{-2b+1}(1+z)^{2a}(1+z^2)^{-a}\Gamma(2b)[\Gamma(b)]^{-2}$$
$$\times \int_0^\pi (\sin\theta)^{2b-1}[1+2z\cos\theta/(1+z^2)]^{-a}\,d\theta.$$

Expanding the brackets [ ] in a series of powers of $\cos\theta$ and evaluating the resulting beta-integrals according to 1.5 (19), we obtain

$$2^{-2b+1}(1+z)^{2a}(1+z^2)^{-a}\Gamma(2b)[\Gamma(b)]^{-2}$$
$$\times \sum_{n=0}^{\infty} \frac{\Gamma(b)\,\Gamma(n+\tfrac12)\,(a)_{2n}}{\Gamma(b+n+\tfrac12)(2n)!}\left(\frac{2z}{1+z^2}\right)^{2n},$$

and because of

$$(a)_{2n} = 2^{2n}(\tfrac12 a)_n(\tfrac12 a+\tfrac12)_n, \qquad (2n)! = 2^{2n}\,n!\,(\tfrac12)_n,$$

we find that this is

$$\left[\frac{(1+z)^2}{1+z^2}\right]^a \frac{\Gamma(\tfrac12)\,\Gamma(2b)}{2^{2b-1}\,\Gamma(b)\,\Gamma(b+\tfrac12)}$$
$$\times \sum_{n=0}^{\infty} \frac{(\tfrac12 a)_n(\tfrac12 a+\tfrac12)_n}{(b+\tfrac12)_n\,n!}\left(\frac{2z}{1+z^2}\right)^{2n}$$

Applying the multiplication theorem of the gamma-function to the factor in front of the sum, we find that this is the right-hand side of (32).

If two of the parameters are unrestricted, there are only linear and quadratic transformations.

A cubic transformation of the hypergeometric equation exists if and only if either

$$1-c = \pm(a-b) = \pm(c-a-b)$$

or if two of the numbers

$$\pm(1-c), \quad \pm(a-b), \quad \pm(c-a-b)$$

are equal to one-third. For a proof of the main results, which are given in section 2.11, see E. Goursat (1881) and G. N. Watson (1909).

There exist transformations of the fourth and sixth degrees where one of the three parameters is unrestricted (cf. Goursat, 1881, and section

2.11). Transformations of other degrees can exist only if a, b, c are certain rational numbers; in these cases the solutions of the hypergeometric equation are algebraic functions (cf. section 2.7.2 and Goursat, 1938).

2.1.6. $F(a,b;c;z)$ as function of the parameters

In many cases it is convenient to prove a relation (e.g., a linear transformation) for the hypergeometric series under certain restrictions (e.g., inequalities) for the parameters; for instance, it is easier to deduce (22) from (10) with the restriction $\operatorname{Re} c > \operatorname{Re} b > 0$ than to use (13) for the proof of (22) without any restriction for the parameters. In this connection the method of analytic continuation (with respect to the parameters) is useful.

It is trivial that $F(a,b;c;z_0)/\Gamma(c)$ is an entire analytic function of a, b, c, if z_0 is fixed and $|z_0| < 1$ since the hypergeometric series then converges uniformly in every finite domain of the (complex) a, b, c space. From (22) it follows that the same is true for all z_0 for which $\operatorname{Re} z_0 < \frac{1}{2}$. Typical examples are:

$$F(2a, 1-2a; 2c; \tfrac{1}{2})/\Gamma(2c) = 2^{1-2c}\,\Gamma(\tfrac{1}{2})/[\Gamma(a+c)\Gamma(c-a+\tfrac{1}{2})],$$

$$F(1, 2a; 2a+1; -1)/\Gamma(2a+1) = 2[\psi(a+\tfrac{1}{2}) - \psi(a)]/\Gamma(2a)$$

for other formulas see 2.8(46) to 2.8(56). Most of the results of this type can be deduced from the formulas for the transformation of the hypergeometric series, from a direct evaluation of the integral representations, or from an expansion by partial fractions. There are a few other cases known where a more elaborate proof is necessary, e.g.,

$$F(2a, 2b; a+b+1; \tfrac{1}{2}) = \pi^{\frac{1}{2}}(b-a)^{-1}\,\Gamma(a+b+1)$$
$$\times\{[\Gamma(b)\,\Gamma(a+\tfrac{1}{2})]^{-1} - [\Gamma(a)\,\Gamma(b+\tfrac{1}{2})]^{-1}\}.$$

For this and for more general results see Mitra (1943).

2.2. The degenerate case of the hypergeometric equation

2.2.1. A particular solution

In general, the points $z = 0, \infty, 1$, are branch points of the solutions of the hypergeometric equation 2.1(1). If we start with an expansion in a series of powers of $z - z_0$ for a certain solution $u_1(z)$ of 2.1(1) and if we continue u_1 analytically along a closed curve L which encircles at least one of the branch points 0, 1 and returns to z_0, then we shall obtain a solution $u = \lambda_1 u_1 + \lambda_2 u_2$, where λ_1 and λ_2 are constants and where u_1 and u_2 are linearly independent solutions of 2.1(1). In general, λ_2 will be $\neq 0$, and this means that all solutions of 2.1(1) can be obtained from a single one and its analytic continuations. But it may happen that $\lambda_2 = 0$

for any L. If this happens we speak of a *degenerate case* and call the solution u_1 involved in this process a *degenerate solution.*

If the effect upon u_1 of a simple loop $L_0^{(+)}$ or $L_1^{(+)}$ (which goes around $z = 0$ or $z = 1$ in the positive sense) amounts to the multiplication of u_1 by a factor $e^{2\pi i\rho}$ or $e^{2\pi i\sigma}$ respectively then

$$z^{-\rho}(1-z)^{-\sigma} u_1(z) = u^*(z)$$

is a one-valued function of z which is regular for all finite z with the possible exception of $z = 0$ and $z = 1$ where u^* may have poles. According to the general theory of Fuchsian equations (cf. Poole 1936), u_1 and therefore u^* cannot have an essential singularity at $z = \infty$. Therefore u^* must be a rational function which can have poles only at $z = 0, \infty, 1$, and therefore we have that, in the degenerate case, u_1 is of the type

$$(1) \quad u_1(z) = z^{\lambda}(1-z)^{\mu} p_n(z)$$

where $p_n(z)$ denotes a polynomial of degree n, such that $p_n(0) \neq 0$ and $p_n(1) \neq 0$.

From the general theory of Riemann's P-equation it follows (cf. Winston 1895 and section 2.7.1) that 2.1(1) has a solution of type (1) if and only if at least one of the numbers

$$(2) \quad a, \; b, \; c-a, \; c-b$$

is an integer. This is equivalent to the condition that at least one of the eight numbers $\pm(c-1) \pm(a-b) \pm(a+b-c)$ is an odd integer.

If precisely one of the four numbers (2) is an integer and $c \neq 0, \pm 1, \pm 2, \ldots$, then one of the two functions

$$(3) \quad \begin{cases} u_1 = F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z) \\ u_5 = z^{1-c}(1-z)^{c-a-b} F(1-a, 1-b; 2-c; z) \\ \quad = z^{1-c} F(a+1-c, b+1-c; 2-c; z) \end{cases}$$

is of type (1), since one of the four series in (3) terminates.

2.2.2. The full solution in the degenerate case

We shall now give two linearly independent solutions of the hypergeometric equation in the degenerate case. This can be done by referring to the 24 series of Kummer [cf. 2.9(1) to 2.9(24)]. In order to obtain also the analytic continuation of these solutions throughout the domain $|\arg(-z)| < \pi$ we shall use the formula

$$(4) \quad F(n+1, n+m+1; n+m+l+2; z)$$
$$= \frac{(n+m+l+1)!\,(-1)^m}{l!\,n!\,(n+m)!\,(m+l)!} \frac{d^{n+m}}{dz^{n+m}} \left[(1-z)^{m+l} \frac{d^l}{dz^l} F(1,1;2;z)\right]$$
$$l, m, n = 0, 1, 2, \ldots$$

where $F(1,1;\ 2;\ z) = -z^{-1}\log(1-z)$. It is also useful to observe that

(5) $$(1-z)^{-b}\int z^{-a}(1-z)^{b-1}\,dz$$

is a solution of 2.1(1) if $c = a$, and that this becomes

(6) $$\sum_{r=0}^{m}(-1)^r\binom{l+m}{r}\frac{z^{m+1-r}(1-z)^{-(m+l+1)}}{m+1-r}$$
$$+(-1)^{m+1}\binom{l+m}{m+1}(1-z)^{-(m+l+1)}\log z$$
$$+\sum_{r=m+2}^{m+l}(-1)^r\binom{l+m}{r}\frac{z^{m+1-r}(1-z)^{-(m+l+1)}}{m+1-r}$$
$$b = l+m+1,\qquad c = a = m,\qquad l,\ m = 0,\ 1,\ 2,\ \ldots.$$

Here the last sum denotes zero if $l < 2$.

To prove (4) we only have to apply 2.1(7) to 2.1(9); and (6) is an elementary formula.

The choice of the two linearly independent solutions from Kummer's series will depend on the number of quantities in (2) which are integers.

In the survey of the various cases we shall use the following notations:

$l,\ m,\ n$	denote non-negative integers;
n.i.	indicates that a quantity is not an integer;
deg.	indicates that the solution is of type (1);
rat.	indicates that the solution is a rational function;
log 2.1(19)	indicates that the analytic continuation of the solution can be effected by 2.1(19) and leads to logarithms;
u_i 2.9(1)	indicates one of the 24 series of Kummer and denotes, e.g., that the first of the six functions and the expression 2.9(1) for it should be taken.

Whenever a solution has at least one analytic continuation which involves logarithms, this has been stated in the table on the following page.

Since the hypergeometric equation is symmetric with respect to a and b, we shall assume in the following table that

(i) if a or b is an integer, then a is an integer;

(ii) if $c-a$ or $c-b$ is an integer, then $c-a$ is an integer;

(iii) if $b-a$ is an integer, then $b-a \geqq 0$.

SOLUTIONS IN THE DEGENERATE CASE

Case	a	b	c	$c-a-b$	Degenerate solution	Second solution
1	$-m$	n.i.	n.i.	n.i.	u_1 rat. 2.9 (1)	u_5 2.9 (18)
2	$m+1$	n.i.	n.i.	n.i.	u_5 2.9 (18)	u_1 2.9 (1)
3	$c+m$	n.i.	n.i.	n.i.	u_1 2.9 (2)	u_5 2.9 (17)
4	$c-m-1$	n.i.	n.i.	n.i.	u_5 2.9 (17)	u_1 2.9 (2)
5	$-m$	n.i.	n.i.	$l+1$	u_1 rat. 2.9 (1)	u_5 2.9 (18) log. 2.3 (2)
6	$m+1$	n.i.	n.i.	$l+1$	u_5 2.9 (18)	u_1 2.9 (1) log. 2.3 (2)
7	$-m$	n.i.	n.i.	$-l$	u_1 rat. 2.9 (1)	u_5 2.9 (18) log. 2.3 (4)
8	$m+1$	n.i.	n.i.	$-l$	u_5 2.9 (18)	u_1 2.9 (1) log. 2.3 (2)
9	$m+1$	$m+l+1$	n.i.	n.i.	u_5 2.9 (18)	u_1 2.9 (1) log. 2.1 (18)
10	$-m$	$l+1$	n.i.	n.i.	u_1 rat. 2.9 (1)	u_5 deg. 2.9 (18)
11	$-m-l$	$-m$	n.i.	n.i.	u_1 rat. 2.9 (1)	u_5 2.9 (18) log. 2.1 (18)
12	$-m$	n.i.	$-m-l$	n.i.	u_1 rat. 2.9 (1)	u_4 deg. 2.9 (15)
13	$-m-l-1$	n.i.	$-m$	n.i.	u_5 rat. 2.9 (17)	u_4 2.9 (14) log. 2.1 (18)
14	$-m$	n.i.	$n+1$	n.i.	u_1 rat. 2.9 (1)	u_4 2.9 (14) log. 2.1 (18)

SOLUTIONS IN THE DEGENERATE CASE (Continued)

Case	a	b	c	$c-a-b$	Degenerate solution	Second solution
15	$m+1$	n.i.	$-n$	n.i.	u_5 2.9(18)	u_3 2.9(9) log. 2.1(18)
16	$m+l+1$	n.i.	$m+1$	n.i.	u_1 2.9(2)	u_3 2.9(9) log. 2.1(18)
17	$m+1$	n.i.	$m+l+2$	n.i.	u_3 rat. 2.9(9)	u_6 deg. 2.9(22)
18	$m+n+1$	$m+n+l+1$	$n+1$	$-2m-n-l-1$	u_1 rat. 2.9(2)	u_2 2.9(5) log. 2.2(4)
19	$m+1$	$m+n+l+2$	$m+n+2$	$-l-m-1$	u_1 rat. 2.9(2)	u_5 rat. 2.9(18)
20	$m+1$	$m+l+1$	$m+n+l+2$	$n-m$	u_2 rat. 2.9(13)	u_1 2.9(1) log. 2.2(4)
21	$-m$	$n+l+1$	$n+1$	$m-l$	u_1 rat. 2.9(1)	u_4 2.9(14) log. 2.2(4)

Case	a	b	c	$c-a-b$	Degenerate solution	Second solution
22	$-m$	$l+1$	$n+l+2$	$n+1-m$	u_1 rat. 2.9 (1)	u_5 rat. 2.9 (17)
23	$-m-l$	$-m$	$n+1$	$n+2m+l+1$	u_1 rat. 2.9 (1)	u_6 2.9 (21) log. 2.2 (4)
24	$-m-l$	$-m$	$-m-l-n$	$m-n$	u_1 rat. 2.9 (1)	u_5 2.9 (18) log. 2.2 (4)
25	$-1-m-l-n$	$-n-l$	$-n$	$2l+m+n+1$	u_5 rat. 2.9 (17)	u_6 2.9 (22) log. 2.2 (4)
26	$-m-n-l-2$	$-l$	$-n-l-1$	$l+m+1$	u_1 rat. 2.9 (1)	u_5 rat. 2.9 (17)
27	$-m-n-1$	$l+1$	$-n$	$m+l+2$	u_5 rat. 2.9 (17)	u_4 2.9 (13) log. 2.2 (4)
28	$-m$	$l+1$	$-m-n-1$	$l-n$	u_1 rat. 2.9 (1)	u_5 rat. 2.9 (18)
29	$m+1$	$m+l+1$	$-n$	$-2m-l-n-2$	u_5 rat. 2.9 (18)	u_2 2.9 (6) log. 2.2 (4)

2.3. The full solution and asymptotic expansion in the general case

2.3.1. Linearly independent solutions of the hypergeometric equation in the non-degenerate case

We may assume now that none of the numbers a, b, $c-a$, $c-b$ is an integer. Then two linearly independent solutions $u_1(z)$, $u_2(z)$ of 2.1(1) can be obtained from any not identically vanishing solution by analytic continuation along a path which encircles one of the points $z = 0, \infty, 1$. If c is not an integer, we may choose

(1) $$u_1(z) = F(a, b; c; z),$$

$$u_2(z) = z^{1-c} F(a-c+1, b-c+1; 2-c; z).$$

If $a-b$ and $c-a-b$ are also non-integers, the analytic continuation of $u_1(z)$, $u_2(z)$ can be carried through by 2.10(1) to 2.10(6). If $a-b$ is an integer, but c is not an integer, formula 2.1(18) gives the analytic continuation of $u_1(z)$ and $u_2(z)$ into the neighborhood of the point at $z = \infty$, and if $c-a-b$ is an integer, we have for $c = a+b+l$, $l = 0, 1, 2, \ldots$

(2) $$F(a, b; a+b+l; z)$$

$$= \frac{\Gamma(l)\,\Gamma(a+b+l)}{\Gamma(a+l)\,\Gamma(b+l)} \sum_{n=0}^{l-1} \frac{(a)_n (b)_n}{(1-l)_n\, n!} (1-z)^n$$

$$+ (1-z)^l (-1)^l \frac{\Gamma(a+b+l)}{\Gamma(a)\,\Gamma(b)}$$

$$\times \sum_{n=0}^{\infty} \frac{(a+l)_n (b+l)_n}{n!\,(n+l)!} [k_n - \log(1-z)] (1-z)^n$$

where

$$k_n = \psi(n+1) + \psi(n+1+l) - \psi(a+n+l) - \psi(b+n+l)$$

and where $\sum_{n=0}^{l-1}$ denotes zero if $l = 0$.

This result can be obtained from 2.10(1) by putting $c = a+b+l+\epsilon$ and by a limiting process $\epsilon \to 0$.

In the same way we obtain for u_2 in this case

(3) $$z^{1-a-b-l} F(1-b-l, 1-a-l; 2-a-b-l; z)$$

$$= \frac{\Gamma(l)\,\Gamma(2-a-b-l)}{\Gamma(1-a)\,\Gamma(1-b)} z^{1-a-b-l} \sum_{n=0}^{l-1} \frac{(1-b-l)_n (1-a-l)_n}{(1-l)_n\, n!}$$

$$\times (1-z)^n + z^{1-a-b-l} (1-z)^l (-1)^l \frac{\Gamma(2-a-b-l)}{\Gamma(1-b-l)\,\Gamma(1-a-l)}$$

$$\times \sum_{n=0}^{\infty} \frac{(1-a)_n (1-b)_n}{n!(n+l)!} [k_n' - \log(1-z)] (1-z)^n$$

where

$$k_n' = \psi(n+1) + \psi(n+1+l) - \psi(1-b+n) - \psi(1-a+n)$$

and where $\sum_{n=0}^{l-1}$ denotes zero if $l = 0$.

Finally, if $c = a + b - l$ where $l = 0, 1, 2, \ldots$, and if c is not an integer, we have for $u_1(z)$:

(4) $F(a, b; a+b-l; z)$

$$= (1-z)^{-l} \frac{\Gamma(l)\,\Gamma(a+b-l)}{\Gamma(a)\,\Gamma(b)} \sum_{n=0}^{l-1} \frac{(b-l)_n (a-l)_n}{n!(1-l)_n} (1-z)^n$$

$$+ (-1)^l \frac{\Gamma(a+b-l)}{\Gamma(a-l)\,\Gamma(b-l)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(n+l)!}$$

$$\times [k_n'' - \log(1-z)] (1-z)^n$$

where

$$k_n'' = \psi(1+l+n) + \psi(1+n) - \psi(a+n) - \psi(b+n)$$

and where $\sum_{n=0}^{l-1}$ denotes zero if $l = 0$.

The corresponding formula for u_2 is obtained by replacing a, b by $1 + l - a$, $1 + l - b$ respectively.

If c is an integer, we may take

(5) $u_1(z) = F(a, b; c; z)$ $\qquad c > 0,$

(6) $u_1(z) = z^{1-c} F(a-c+1, b-c+1; 2-c; z)$ $\qquad c \leq 0,$

(7) $u_2(z) = F(a, b; 1+a+b-c; 1-z)$ $\qquad 1+a+b-c \neq 0, -1, -2, \ldots,$

(8) $u_2(z) = (1-z)^{c-a-b} F(c-a, c-b; 1-a-b+c; 1-z)$

$\qquad 1+a+b-c = 0, -1, -2, \ldots,$

The analytic continuation of $u_1(z)$ and $u_2(z)$ into the neighborhood of the points at $z = 0$, $z = \infty$ or $z = 1$ can be carried through by using the formulas 2.10(1) to 2.10(15) since we may assume throughout this section that none of the numbers a, b, $c-a$, $c-b$ is an integer.

2.3.2. Asymptotic expansions

The behavior of the solutions of the hypergeometric equation for large values of $|z|$ can be fully described by means of the formulas for the analytic continuation into the neighborhood of the point at $z = \infty$. Unless $a - b$ is an integer, every solution $u(z)$ can be put into the form

$$(9) \quad u(z) = \lambda_1 z^{-a} + \lambda_2 z^{-b} + O(z^{-a-1}) + O(z^{-b-1})$$

where λ_1 and λ_2 are constants; if $a - b$ is an integer, z^{-a} or z^{-b} has to be multiplied by a factor $\log z$.

The behavior of $F(a, b; c; z)$ for large values of $|a|$, $|b|$, $|c|$ has been investigated by O. Perron (1916-17) and by G. N. Watson (1918).

If a, b and z are fixed numbers and $|c|$ is large with the restriction $|\arg c| \leq \pi - \epsilon$, $\epsilon > 0$, then for $|z| < 1$ we have

$$(10) \quad F(a,b; c; z) = 1 + \frac{ab}{c} z + \cdots + \frac{(a)_n (b)_n}{(c)_n \, n!} z^n + O(|c^{-n-1}|).$$

With a slightly modified expression for the remainder term this remains valid even if $|z| > 1$, $|\arg(1 - z)| < \pi$, provided that $\operatorname{Re} c \to \infty$. Then, if $|c| \to \infty$ and b is fixed, $\operatorname{Re} c > \operatorname{Re} b$. For a sufficiently large value of n we also have $\operatorname{Re}(b + n) > 0$. We have

$$(11) \quad F(a,b; c; z) - 1 - \frac{ab}{c} z \cdots - \frac{(a)_n (b)_n}{(c)_n \, n!} z^n \equiv \rho_{n+1}(a,b; c; z)$$

$$= \Gamma(c)\,\Gamma(a+n)\, z^{n+1}/[\Gamma(b)\,\Gamma(c-b)\,\Gamma(a)\, n!]$$

$$\times \int_0^1 \int_0^1 t^{b+n} (1-t)^{c-b-1} (1-s)^n (1-stz)^{-a-n-1} \, ds \, dt.$$

We split $a = \alpha + i\alpha'$, $b = \beta + i\beta'$, $c = \gamma + i\gamma'$ into real and imaginary parts. Then, for $0 \leq s$, $t \leq 1$ we have $|(1 - stz)^{-a-n-1}| \leq M^{-\alpha-n-1}$, where M depends on z and denotes either the minimum or the maximum of $|1-stz|$.

This gives

$$|\rho_{n+1}| \leq \frac{|\Gamma(c)\,(a)_{n+1}\, z^{n+1}|}{|\Gamma(b)\,\Gamma(c-b)|\,(n+1)!} M^{-\alpha-n-1} \int_0^1 t^{\beta+n} (1-t)^{\gamma-\beta-1} \, dt$$

$$= |z/M|^{n+1} M^{-\alpha} \frac{|(a)_{n+1}\,(\beta)_{n+1}|}{(n+1)!} \frac{|\Gamma(\beta)|\,|\Gamma(c)|}{|\Gamma(b)|\,|\Gamma(c-b)|}$$

$$\times \frac{\Gamma(\gamma-\beta)}{\Gamma(\gamma)} \frac{\Gamma(\gamma)}{\Gamma(\gamma+n+1)}.$$

From 1.18(5) we obtain an estimate for the last three quotients of gamma functions. The result is

$$(12) \quad |\rho_{n+1}| \leq \mu(n)\, |z|^{n+1} |c|^{-\beta} \gamma^{-\beta-n-1}$$

where $\mu(n)$ depends on n, a, b and on Im z if Re $z > 0$. This proves (10) [with $|\rho_{n+1}|$ instead of $O(|c|^{-n-1})$ for a sufficiently large value of n]. But then (10) is clearly also valid for $n = 1, 2, \ldots$, since each term of the asymptotic series in (10) behaves like $|\rho_{n+1}|$. For more general results see T. M. MacRobert (1923) who has proved (10) for a range of arg c which is $> \pi$.

If a, c and z are fixed numbers, $c \neq 0, -1, -2, \ldots$ and $0 < |z| < 1$, and if $|b| \to \infty$ such that $-3\pi/2 < \arg zb < \frac{1}{2}\pi$, then we have

$$(13)\quad F(a,b;\ c;\ z) = F(a,b;\ c;\ bz/b) = \left[\sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{(bz)^n}{n!}\right] [1 + O(|b|^{-1})]$$

and here the asymptotic formulas for the confluent hypergeometric function of a large argument (cf. Chapter 6 or Whittaker-Watson 1927, 16.3) give

$$(14)\quad F(a,b;\ c;\ z) = e^{-i\pi a}[\Gamma(c)/\Gamma(c-a)]\,(bz)^{-a}[1 + O(|bz|^{-1})]$$
$$+ [\Gamma(c)/\Gamma(a)]\,e^{bz}(bz)^{a-c}[1 + O(|bz|^{-1})].$$

Similarly, if $-\frac{1}{2}\pi < \arg bz < 3\pi/2$, we have

$$(15)\quad F(a,b;\ c;\ z) = e^{i\pi a}[\Gamma(c)/\Gamma(c-a)]\,(bz)^{-a}[1 + O(|bz|^{-1})]$$
$$+ [\Gamma(c)/\Gamma(a)]\,e^{bz}(bz)^{a-c}[1 + O(|bz|^{-1})]$$

In the case where more than one of the parameters tends to infinity, G. N. Watson (1918) has obtained the following results.

Let ξ be defined by $z \pm (z^2 - 1)^{\frac{1}{2}} = e^{\pm\xi}$, and put

$$1 - e^{\xi} = (e^{\xi} - 1)\,e^{\mp i\pi}$$

where the upper or lower sign is taken according as Im $z \gtrless 0$. Then for large $|\lambda|$ we have

$$(16)\quad (\tfrac{1}{2}z - \tfrac{1}{2})^{-a-\lambda} F[a+\lambda,\ a-c+1+\lambda;\ a-b+1+2\lambda;\ 2(1-z)^{-1}]$$
$$= \frac{2^{a+b}\,\Gamma(a-b+1+2\lambda)\,\Gamma(\tfrac{1}{2})\,\lambda^{-\frac{1}{2}}}{\Gamma(a-c+1+\lambda)\,\Gamma(c-b+\lambda)}\,e^{-(a+\lambda)\xi}$$
$$\times (1-e^{-\xi})^{-c+\frac{1}{2}}\,(1+e^{-\xi})^{c-a-b-\frac{1}{2}}\,[1 + O(\lambda^{-1})]$$

where $|\arg \lambda| \le \pi - \delta$, $\delta > 0$ and also

$$(17)\quad F(a+\lambda,\ b-\lambda;\ c;\ \tfrac{1}{2} - \tfrac{1}{2}z)$$
$$= \frac{\Gamma(1-b+\lambda)\,\Gamma(c)}{\Gamma(\tfrac{1}{2})\,\Gamma(c-b+\lambda)}\,2^{a+b-1}\,(1-e^{-\xi})^{-c+\frac{1}{2}}\,(1+e^{-\xi})^{c-a-b-\frac{1}{2}}$$
$$\times \lambda^{-\frac{1}{2}}[e^{(\lambda-b)\xi} + e^{\pm i\pi(c-\frac{1}{2})}\,e^{-(\lambda+a)\xi}]\,[1 + O(|\lambda^{-1}|)]$$

where the upper or lower sign is taken according as Im $z \gtrless 0$ and where $|\lambda|$ is large, $\xi = \zeta + i\eta$,

$$-\tfrac{1}{2}\pi - w_2 + \delta < \arg \lambda < \tfrac{1}{2}\pi + w_1 - \delta \qquad \delta > 0,$$

$$w_2 = \tan^{-1}(\eta/\zeta), \; -w_1 = \tan^{-1}[(\eta - \pi)/\zeta] \qquad \eta \geqq 0,$$

$$w_2 = \tan^{-1}[(\eta + \pi)/\zeta], \; -w_1 = \tan^{-1}(\eta/\zeta) \qquad \eta \leqq 0.$$

Here $\tan^{-1} x$ denotes the principal value of the function, i.e.,

$$-\tfrac{1}{2}\pi < \tan^{-1} x < \tfrac{1}{2}\pi.$$

Other cases where a, b, c (or their moduli) are large have been investigated by M. J. Lighthill (1947), H. Seifert (1947), and T. M. Cherry (1950 a, b). The case where $a = i\rho\nu$, $b = i\nu$, $c = 1$, where ρ, ν are real where ρ is fixed, and where $\nu \to \infty$ has been investigated by A. Sommerfeld (1939).

2.4. Integrals representing or involving hypergeometric functions

Euler's integral 2.1(10) cannot be transformed into itself by an elementary subsitution in such a way that the relation

$$F(a, b; c; z) = F(b, a; c; z)$$

becomes evident, although this is a trivial property of the hypergeometric series. Wirtinger (1902) has given a triple integral for F which makes the symmetry with respect to a and b evident, and for the same purpose A. Erdélyi (1937 b) has derived the double integral

$$(1) \quad F(a, b; c; z) = \frac{\Gamma(c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c-a)\,\Gamma(c-b)}$$

$$\times \int_0^1 \int_0^1 t^{b-1}\,\tau^{a-1}\,(1-t)^{c-b-1}\,(1-\tau)^{c-a-1}\,(1-t\tau z)^{-c}\,dt\,d\tau$$

which can be obtained from 2.1(10) and 1.5(11).

H. Bateman (1909) [cf. also A. Erdélyi (1937 a, b)] has proved that

$$(2) \quad F(a, b; c; z) = \{\Gamma(c)/[\Gamma(s)\,\Gamma(c-s)]\}$$

$$\times \int_0^1 x^{s-1}(1-x)^{c-s-1} F(a, b; s; xz)\,dx$$

$$\operatorname{Re} c > \operatorname{Re} s > 0, \quad z \neq 1, \quad |\arg(1-z)| < \pi.$$

This can be obtained by expanding $F(a, b; s; xz)$ in a series of powers of xz, integrating term by term and applying 1.5(1) and 1.5(5).

By means of fractional integration by parts, the following generalizations of (2) may be obtained (Erdélyi, 1939):

$$(3) \quad F(a, b; c; z) = \Gamma(c)\,[\Gamma(s)\,\Gamma(c-s)]^{-1} \int_0^1 x^{s-1}(1-x)^{c-s-1}(1-xz)^{-a'}$$

$$\times F(a-a', b; s; xz)\, F[a', b-s; c-s; (1-x)\,z/(1-xz)]\,dx$$

$$= \{\Gamma(c)/[\Gamma(s)\,\Gamma(c-s)]\} \int_0^1 x^{s-1}(1-x)^{c-s-1}(1-xz)^{r-a-b}$$

$$\times F(r-a,\, r-b;\, s;\, xz)\, F[a+b-r,\, r-s; c-s; (1-x)\,z/(1-xz)]\, dx.$$

Combining the integral representations 2.1(10) and 2.1(15) with the linear and quadratic transformations of the hypergeometric series we obtain a large number of integral formulas. Starting, e.g., with 2.1(15), substituting $-z$ for z, and applying Mellin's transformation formula, we obtain

$$(4)\quad \frac{\Gamma(a+s)}{\Gamma(a)}\,\frac{\Gamma(b+s)}{\Gamma(b)}\,\frac{\Gamma(c)}{\Gamma(c+s)}\,\Gamma(-s) = \int_0^\infty F(a,\, b;\, c;\, -z)\, z^{-s-1}\, dz$$

$c \neq 0, -1, -2, \ldots,$ and $\operatorname{Re} s < 0$, $\operatorname{Re}(a+s) > 0$, $\operatorname{Re}(b+s) > 0$.

Splitting the right-hand side in (4) into the sum of two integrals extended from 0 to 1 and from 1 to ∞ respectively, applying 2.1(17) to the integrand of the latter integral, and substituting $-z$ and $-z^{-1}$ for z, we obtain:

$$\int_0^\infty F(a,\, b;\, c;\, -z)\, z^{-s-1}\, dz = e^{\pm i\pi s} \int_0^1 F(a,\, b;\, c;\, z)\, z^{-s-1}\, dz$$

$$+\,\Gamma(c)\,\Gamma(b-a)\,[\Gamma(b)\,\Gamma(c-a)]^{-1}\, e^{\pm i\pi(a+s)}$$

$$\times \int_0^1 F(a,\, 1-c+a;\, 1-b+a;\, z)\, z^{a+s-1}\, dz$$

$$+\,\Gamma(c)\,\Gamma(a-b)\,[\Gamma(a)\,\Gamma(c-b)]^{-1}\, e^{\pm i\pi(b+s)}$$

$$\times \int_0^1 F(b,\, 1-c+b,\, 1-a+b,\, z)\, z^{b+s-1}\, dz$$

where throughout we may take either the upper or lower signs. If we now eliminate the third integral by combining the two formulas with the upper and lower sign we obtain, with $s = w - \tfrac{1}{2}a$:

$$(5)\quad \frac{\Gamma(\tfrac{1}{2}a+w)\,\Gamma(\tfrac{1}{2}a-w)}{\Gamma(c-\tfrac{1}{2}a+w)\,\Gamma(1-b+\tfrac{1}{2}a-w)}$$

$$= \frac{\Gamma(a)}{\Gamma(1+a-b)\,\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_n\,(1-c+a)_n}{(1-b+a)_n\, n!}\,\frac{1}{n+w+\tfrac{1}{2}a}$$

$$+ \frac{\Gamma(a)}{\Gamma(c)\,\Gamma(1-b)} \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\, n!}\,\frac{1}{n-w+\tfrac{1}{2}a}$$

$$= \frac{\Gamma(a)}{\Gamma(1+a-b)\,\Gamma(c-a)} \int_0^1 F(a,\, 1-c+a;\, 1-b+a;\, z)$$

$$\times z^{\frac{1}{2}a+w-1}\, dz + \frac{\Gamma(a)}{\Gamma(c)\,\Gamma(1-b)} \int_0^1 F(a,\, b;\, c;\, z)\, z^{\frac{1}{2}a-w-1}\, dz.$$

The first equation in (5) is valid if $\operatorname{Re}(a+b-c)<1$; the second equation is valid if we also have $\operatorname{Re}(\tfrac{1}{2}a \pm w)>0$, $\operatorname{Re}(1-b+a)>0$, and $\operatorname{Re} b<1$. The formula becomes particularly simple if $c-a=1-b$.

We have

$$(6)\quad \int_0^{\frac{1}{2}\pi} (\cos\theta)^{2\mu}(\sin\theta)^{2\nu} e^{i2a\theta}\,d\theta$$

$$= 2^{-2\mu-2\nu-1} e^{i\pi(a-\mu-\frac{1}{2})}[\Gamma(a-\mu-\nu)\Gamma(2\mu+1)/\Gamma(1+a-\nu+\mu)]$$

$$\times F(-2\nu, a-\mu-\nu; 1+a-\nu+\mu; -1)$$

$$+ 2^{-2\mu-2\nu-1} e^{i\pi(\nu+\frac{1}{2})}[\Gamma(a-\mu-\nu)\Gamma(2\nu+1)/\Gamma(1+a-\mu+\nu)]$$

$$\times F(-2\mu, a-\mu-\nu; 1+a-\mu+\nu; -1)$$

$$\operatorname{Re}\mu>-\tfrac{1}{2},\qquad \operatorname{Re}\nu>-\tfrac{1}{2}.$$

In particular, if $a=\nu+\mu+1$, $2\mu=x$, $2\nu=y$, we find from 1.5(13) (with $a=1$, $b=-1$, $v=e^{2i\theta}$) that

$$(7)\quad e^{-\frac{1}{2}i\pi(y+1)}\,2^{x+y+1}\int_0^{\frac{1}{2}\pi}(\cos\theta)^x(\sin\theta)^y e^{i(x+y+2)\theta}\,d\theta$$

$$= 2^{x+y+1}[\Gamma(x+1)\Gamma(y+1)/\Gamma(x+y+2)]$$

$$= (y+1)^{-1}F(-x,1;y+2;-1)+(x+1)^{-1}F(-y,1;x+2;-1)$$

$$\operatorname{Re}x>-1,\qquad \operatorname{Re}y>-1.$$

These formulas may be obtained by introducing $e^{2i\theta}$ as a new variable in the left-hand side of (6) and applying Euler's integral representation 2.1(10) to it.

We may deduce from (6) for $\operatorname{Re}\mu>-\tfrac{1}{2}$, $\operatorname{Re}\nu>-\tfrac{1}{2}$

$$(8)\quad \int_0^{\pi}(\cos\theta)^{2\mu}(\sin\theta)^{2\nu} e^{2ia\theta}\,d\theta$$

$$= e^{i\pi(a-\mu)}4^{-\mu-\nu}\frac{\pi\,\Gamma(2\nu+1)}{\Gamma(1+\mu+\nu-a)\,\Gamma(1+\nu+a-\mu)}$$

$$\times F(-2\mu, a-\mu-\nu; 1+a-\mu+\nu; -1)$$

if we define

$$(\cos\theta)^{2\mu}=e^{-2i\pi\mu}[\sin(\theta-\tfrac{1}{2}\pi)]^{2\mu}\qquad \tfrac{1}{2}\pi\le\theta\le\pi.$$

In particular we have for $2\mu=m=0,1,2,\ldots$ and $\nu=0$

$$F(-m,-\tfrac{1}{2}m-\tfrac{1}{2}a;1-\tfrac{1}{2}m-\tfrac{1}{2}a;-1)$$

$$=(-2)^m(m+a)\csc(a\pi)\int_0^{\pi}(\cos\theta)^m\cos a\theta\,d\theta$$

$$a\neq 0,\pm 1,\pm 2,\ldots.$$

The quadratic transformation 2.1(24) combined with 2.1(10) gives

(9) $F(a, a-b+\frac{1}{2}; b+\frac{1}{2}; z^2)$

$$= \frac{\Gamma(b+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(b)} \int_0^{\pi} (\sin\phi)^{2b-1} (1+2z\cos\phi+z^2)^{-a}\, d\phi$$

$\operatorname{Re} b > 0, \quad |z| < 1.$

If $n = 0, 1, 2, \ldots$, we have

(10) $F(a, n+a; n+1; z^2)$

$$= \frac{z^{-n}\, n!}{2\pi\,(a)_n} \int_0^{2\pi} \cos n\phi\,(1-2z\cos\phi+z^2)^{-a}\, d\phi$$

$\operatorname{Re} a > 0, \quad |z| < 1;$

this may be shown by expanding

$$(1-2z\cos\phi+z^2)^{-a} = (1-ze^{i\phi})^{-a}\,(1-ze^{-i\phi})^{-a}$$

into the product

$$\sum_{l=0}^{\infty} \frac{(a)_l}{l!} z^l e^{i\phi l} \sum_{m=0}^{\infty} \frac{(a)_m}{m!} z^m e^{-i\phi m}.$$

If we multiply these sums and collect the coefficients of $e^{\pm i\phi n}$, we obtain (10). We also have

(11) $$\frac{\pi\, 2^{-\beta}\,\Gamma(1+\beta)}{\Gamma(1+\frac{1}{2}\beta+\frac{1}{2}a-\frac{1}{2}\nu)\,\Gamma(1+\frac{1}{2}\beta-\frac{1}{2}a+\frac{1}{2}\nu)}$$

$$\times F(-\nu, \tfrac{1}{2}a-\tfrac{1}{2}\beta-\tfrac{1}{2}\nu; 1+\tfrac{1}{2}\beta+\tfrac{1}{2}a-\tfrac{1}{2}\nu; a^2/b^2)$$

$$= \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} e^{ia\theta} (\cos\theta)^{\beta} (a^2 e^{i\theta} + b^2 e^{-i\theta})^{\nu}\, d\theta$$

$\operatorname{Re}\beta > -1, \quad |b| > |a|.$

This follows from an expansion of the integrand in a binomial series and term by term integration. If $|a| > |b|$, the corresponding formula may be obtained by substituting $-a$ for a and $-\theta$ for θ. The analytic continuation of the hypergeometric function in (11) into the domain $|a/b| > 1$ does not give the value of the integral on the right-hand side.

2.5. Miscellaneous results

2.5.1. A generating function

If $n = 0, 1, 2, \ldots$, the polynomials $F(-n, a+n; c; z)$ are the Jacobi polynomials (cf. Chapter 10 on orthogonal polynomials). For these we have the generating function

(1) $\sum_{n=0}^{\infty} s^n (c)_n F(-n, a+n; c; z)/n!$

$$= S^{-1}\left(\frac{S+s-1}{2sz}\right)^{c-1} \left(\frac{S+s+1}{2}\right)^{c-a}$$

where

$$S = [1 - 2(1 - 2z)\, s + s^2]^{1/2}$$

and $S \to 1$ as $s \to 0$. The expansion on the left-hand side in (1) is convergent if $|s| < 1$ and $|1 - 2z| < 1$. In the case Re $c > 0$ we can prove (1) by introducing $u = t(1 - tz)(1 - t)^{-1}$ as a new variable in the integral 2.1(10) which gives

$$(2)\quad F(b, a - b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c - b)} \times \int_0^\infty u^{b-1} \left(\frac{1 - u + U}{2}\right)^{c-a} \left(\frac{1 + u - U}{2uz}\right)^{c-1} \frac{du}{U}$$

$$U = [1 + 2u(1 - 2z) + u^2]^{1/2}.$$

Applying the inversion formula of the Mellin transformation we obtain

$$(3)\quad \left(\frac{1 - u + U}{2}\right)^{c-a} \left(\frac{1 + u - U}{2uz}\right)^{c-1} U^{-1} = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} u^{-b} F(b, a - b; c; z)\, \frac{\Gamma(b)\,\Gamma(c - b)}{\Gamma(c)}\, db,$$

where β is a suitably chosen real number. The integrand on the right-hand side of (3) has poles at $b = -n$ $(n = 0, 1, 2, \ldots)$. An application of the theorem of residues can be justified by the results of section 2.3 and gives (1) with $s = -u$. By analytic continuation of both sides of (1) with respect to c it can be shown that the restriction on c can be dropped.

We may consider (3) as a continuous linear generating function for $F(b, a - b; c; z)$. For a bilinear generating function and for many related results see A. Erdélyi (1941).

By the same method which has been used in proving (1) it can also be shown that

$$(1 - s)^{a-c}(1 - s + sz)^{-a} = \sum_{n=0}^{\infty} s^n (c)_n\, F(-n, a; c; z)/n!$$

$$|s| < 1, \qquad |s(1 - z)| < 1.$$

2.5.2. Products of hypergeometric series

Cayley, Orr (1899), and Bailey (1935) have proved a series of identities which have been generalized by J. L. Burchnall and T. W. Chaundy (1948). (For the proof and for more general results compare the latter paper). A result of Cayley and Orr is that if

$$(1 - z)^{a+b-c}\, F(2a, 2b; 2c - 1; z) = \sum_{n=0}^{\infty} A_n z^n,$$

we then have

(4) $$F(a, b; c - \tfrac{1}{2}; z)\, F(c - a, c - b; c + \tfrac{1}{2}; z) = \sum_{n=0}^{\infty} \frac{(c)_n}{(c + \frac{1}{2})_n} A_n z^n .$$

J. L. Burchnall and T. W. Chaundy (1948) have proved the multiplication and duplication formulas:

(5) $$F(2a, 2b; 2c; z) = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n (a)_n (b)_n (c-a)_n (c-b)_n}{n!\,(c+\frac{1}{2})_n (c+n-1)_n (c)_{2n}} z^{2n} \times [F(a+n, b+n; c+2n; z)]^2,$$

(6) $$F(a, b; c; z^2) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c-a)_n (c-b)_n}{n!\,(c+n-1)_n (c)_{2n}} z^{2n} \times F(a+n, b+n; c+2n; z)\, F(a+n, b+n; c+2n; -z),$$

(7) $$[F(a, b; c; z)]^2 = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (a)_n (b)_n (c-a)_n (c-b)_n}{n!\,(c)_n (c)_{2n} (c+n-\frac{1}{2})_n} z^{2n} \times F(2a+2n, 2b+2n; 2c+4n; z).$$

All these formulas are valid if $|z| < 1$ and if the hypergeometric series involved are defined (i.e., if $2c$ and c are different from $0, -1, -2, \ldots$).

Formulas of Burchnall and Chaundy which are of the type of an addition theorem [cf. Burchnall and Chaundy (1940)] follow:

$$F(a, b; c; z + \zeta - z\zeta) = \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n (b)_n (c-a)_n (c-b)_n}{n!\,(c+n-1)_n (c)_{2n}} z^n \zeta^n \times F(a+n, b+n; c+2n; z)\, F(a+n, b+n; c+2n; \zeta)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(a)_n (b)_n}{n!\,(c)_n} z^n \zeta^n F(a+n, b+n; c+n; z+\zeta),$$

$$F(a, b; c; z)\, F(a, b; c; \zeta) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c-a)_n (c-b)_n}{n!\,(c)_n (c)_{2n}} (z\zeta)^n \times F(a+n, b+n; c+2n; z+\zeta-z\zeta).$$

These formulas are valid if $|z|$, $|\zeta|$, and $|z+\zeta-z\zeta|$ or $|z+\zeta|$ are <1, and if $c \neq 0, -1, -2, \ldots$.

Certain bilinear relations have been proved by J. Meixner (1941), for example,

$$(8)\qquad \sum_{n=0}^{\infty} \binom{\lambda}{n} s^n F(-n, b; c; z) F(-n, \beta; \gamma; \zeta)$$

$$= (1+s)^{\lambda} \sum_{n=0}^{\infty} \binom{\lambda}{n} \frac{(z\zeta s)^n}{(1+s)^{2n}} \frac{(b)_n (\beta)_n}{(c)_n (\gamma)_n}$$

$$\times F[n-\lambda, b+n; c+n; sz/(1+s)]\, F[n-\lambda, \beta+n; \gamma+n; \zeta s/(1+s)],$$

$$(9)\qquad \sum_{n=0}^{\infty} \binom{\lambda}{n} s^n F(-n, b; c; z) F(n-\lambda, \beta; \gamma; \zeta)$$

$$= (1+s)^{\lambda} \sum_{n=0}^{\infty} \binom{\lambda}{n} \frac{(-s\zeta z)^n}{(s+1)^{2n}} \frac{(b)_n (\beta)_n}{(c)_n (\gamma)_n}$$

$$\times F[n-\lambda, b+n; c+n; sz/(1+s)]\, F[n-\lambda, \beta+n; \gamma+n; \zeta/(1+s)],$$

$$(10)\qquad \sum_{n=0}^{\infty} \binom{\lambda}{n} s^n F(n-\lambda, b; c; z) F(n-\lambda, \beta; \gamma; \zeta)$$

$$= (1+s)^{\lambda} \sum_{n=0}^{\infty} \binom{\lambda}{n} \frac{(z\zeta s)^n}{(1+s)^{2n}} \frac{(b)_n (\beta)_n}{(c)_n (\gamma)_n}$$

$$\times F[n-\lambda, b+n; c+n; z/(1+s)]\, F[n-\lambda, \beta+n; \gamma+n; \zeta/(1+s)],$$

$$(11)\qquad \sum_{n=0}^{\infty} \binom{\lambda}{n} s^n \frac{(c-b)_n (\gamma-\beta)_n}{(c)_n (\gamma)_n} F(n-\lambda, b; c+n; z)$$

$$\times F(n-\lambda, \beta; \gamma+n; \zeta) = (1+s)^{\lambda} \sum_{n=0}^{\infty} \binom{n}{\lambda} \frac{[s(1-z)(1-\zeta)]^n}{(1+s)^{2n}}$$

$$\times \frac{(b)_n (\beta)_n}{(c)_n (\gamma)_n} F[n-\lambda, b+n; c+n; (s+z)/(1+s)]$$

$$\times F[n-\lambda, \beta+n; \gamma+n; (s+\zeta)/(1+s)].$$

All these formulas are valid if s, z, ζ are such that the fourth argument of the hypergeometric functions involved is different from 1 and from ∞ and if $|s|$ is sufficiently small. If $c, \gamma \to 0, -1, -2, \ldots$, we obtain a valid formula again on multiplying first by $[\Gamma(c)]^{-1}$ or $[\Gamma(\gamma)]^{-1}$ or by both and then let c or γ both tend to $0, -1, -2, \ldots$. As a special case of these formulas we have

$$(12)\quad \sum_{n=0}^{\infty} \binom{\lambda}{n} s^n F(-n, b; -\lambda, z)\, F(-n, \beta; -\lambda; \zeta)$$

$$= (1+s)^{\lambda+b+\beta}(1+s-sz)^{-b}(1+s-s\zeta)^{-\beta}$$

$$\times F\left[b, \beta; -\lambda; \frac{-z\zeta s}{(1+s-sz)(1+s-s\zeta)}\right];$$

since $F(a, b; b; z) = (1-z)^{-a}$, one of the sums can be put in a finite form if $c = \gamma = -\lambda$. If we put $\zeta = 0$ in (8) and (9), we obtain

$$\sum_{n=0}^{\infty} \binom{\lambda}{n} s^n\, F(-n, b; c; z) = (1+s)^{\lambda} F[-\lambda, b; c; sz/(1+s)],$$

$$\sum_{n=0}^{\infty} \binom{\lambda}{n} s^n F(n-\lambda, b; c; z) = (1+s)^{\lambda} F[-\lambda, b; c; z/(1+s)].$$

Finally, if we put $\zeta = 0$ in (12), we obtain (1).

Generalizing Legendre's relation from the theory of elliptic integrals, E. B. Elliot (1904) has proved:

$$(13)\quad F(\tfrac12+\lambda, -\tfrac12-\nu; 1+\lambda+\mu; z)\, F(\tfrac12-\lambda, \tfrac12+\nu; 1+\nu+\mu; 1-z)$$

$$+ F(\tfrac12+\lambda, \tfrac12-\nu; 1+\lambda+\mu; z)\, F(-\tfrac12-\lambda, \tfrac12+\nu; 1+\nu+\mu; 1-z)$$

$$- F(\tfrac12+\lambda, \tfrac12-\nu; 1+\lambda+\mu; z)\, F(\tfrac12-\lambda, \tfrac12+\nu; 1+\nu+\mu; 1-z)$$

$$= \frac{\Gamma(1+\lambda+\mu)\,\Gamma(1+\nu+\mu)}{\Gamma(\lambda+\mu+\nu+3/2)\,\Gamma(\tfrac12+\mu)}.$$

If $\lambda = \mu = \nu = 0$, we obtain Legendre's relation from (13). This result has been generalized again by A. L. Dixon (1905).

2.5.3. Relations involving binomial coefficients and the incomplete beta function

We define the polynomial $\binom{u}{n}$ in u by

$$(14)\quad \binom{u}{n} = (-1)^n \frac{\Gamma(n-u)}{\Gamma(n+1)\,\Gamma(-u)} = \begin{cases} 1 & n = 0 \\ u(u-1)(u-2)\cdots(u-n+1)/n! & n = 1, 2, 3, \ldots. \end{cases}$$

Then we have

$$F(1, -u; -v; z) = \sum_{n=0}^{\infty} z^n \binom{u}{n} \Big/ \binom{v}{n}$$

and, according to 2.1 (14),

$$(15)\quad \sum_{n=0}^{\infty} \binom{u}{n} \Big/ \binom{v}{n} = \frac{\Gamma(-v)\,\Gamma(u-v-1)}{\Gamma(-v-1)\,\Gamma(u-v)} = \frac{v+1}{v-u+1}$$

$$= 1 + \frac{u}{v-u+1} \qquad \text{Re}\,(u-v) > 1.$$

Since

$$\binom{u}{n+m} = \binom{u}{m}\binom{u-m}{n}\, n!\, m!/(n+m)!$$

we also have

$$\sum_{n=0}^{\infty} \binom{u}{n+m} \Big/ \binom{v}{n+m} = \left[\binom{u}{m} \Big/ \binom{v}{m}\right]$$

$$\times \sum_{n=0}^{\infty} \binom{u-m}{n} \Big/ \binom{v-m}{n} = \frac{v-m+1}{v-u+1} \binom{u}{m} \Big/ \binom{v}{m},$$

and combined with (15) this gives

$$(16)\quad \sum_{n=0}^{m-1} \binom{u}{n} \Big/ \binom{v}{n} = \frac{v+1}{v-u+1}\left[1 - \binom{u}{m} \Big/ \binom{v+1}{m}\right].$$

Equations (15) and (16) are called "Lerch's theorem".

If we put $z = -1$ in 2.1 (25) and use 2.1 (14), we obtain

$$F(a, b; 1+a-b; -1) = \frac{2^{-a}\Gamma(1+a-b)\,\Gamma(\tfrac{1}{2})}{\Gamma(1-b+\tfrac{1}{2}a)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}a)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \binom{-a}{n}\binom{-b}{n} \Big/ \binom{b-a-1}{n},$$

and if we put $a = -m$ where m is an integer, we obtain

$$\sum_{n=0}^{m} (-1)^n \binom{m}{n}\binom{u}{n} \Big/ \binom{m-u-1}{n} = \frac{2^m\,\Gamma(1-m+u)\sqrt{\pi}}{\Gamma(1+u-\tfrac{1}{2}m)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}m)}.$$

In a similar way, the linear and quadratic transformations of the hypergeometric series can be used to sum in closed form sums involving binomial coefficients by one or several of the following processes:

(i) giving special values to z,

(ii) putting a or b equal to a negative integer,

(iii) comparing the coefficients of the same power of z in different expressions for a hypergeometric series, e.g., in both sides of 2.1(22) or in an equation which is of a similar type.

So we have that $F(n, -n;\ 1;\ 1) = 0$ due to 2.1(14), and this may be written in the form

$$\sum_{n=1}^{m} \frac{(-1)^{n-1}(m+n-1)!}{n!\,n!\,(m-n)!} = 1/m.$$

Another example is Saalschütz's formula 2.1(30).

The truncated binomial series

$$\sum_{n=0}^{m-1} \binom{a}{n} z^n$$

may be expressed by two hypergeometric series, viz.,

$$F(1, -a;\ 1;\ -z) - z^m \frac{\Gamma(a+1)}{\Gamma(a+1-m)\,m!} F(m-a,\ 1;\ m+1;\ -z)$$

or by

$$\binom{a}{m-1} z^{m-1} F(1-m,\ 1;\ a-m+2;\ -z^{-1})$$

$$a \neq m-2,\ m-3,\ \ldots,\ 0.$$

The incomplete beta function. In mathematical statistics, the functions

$$B_x(p, q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt$$

and

$$I_x(p, q) = B_x(p, q)/B_1(p, q)$$

are found. $B_x(p, q)$ is called the incomplete beta function. We have

$$B_x(p, q) = p^{-1} x^p F(p,\ 1-q;\ p+1;\ x),$$

$$B_1(p, q) = \Gamma(p)\,\Gamma(q)/\Gamma(p+q).$$

The following recurrence relations can be derived from 2.8(31) to 2.8(45) [For an application cf. T. A. Bancroft (1949)]:

$$xI_x(p, q) - I_x(p+1, q) + (1-x)I_x(p+1, q-1) = 0,$$

$$(p+q-px)\,I_x(p, q) - q\,I_x(p, q+1) - p(1-x)\,I_x(p+1, q-1) = 0,$$

$$q\,I_x(p, q+1) + p\,I_x(p+1, q) - (p+q)\,I_x(p, q) = 0.$$

2.5.4. A continued fraction

Gauss (1812) [cf. Gauss (1876)] has found continued fractions for the quotients of certain associated hypergeometric series. A typical example is

$$\frac{F(a, b+1; c+1; z)}{F(a, b; c; z)} = \cfrac{1}{1 - \cfrac{u_1 z}{1 - \cfrac{v_1 z}{1 - \cfrac{u_2 z}{1 - \cfrac{v_2 z}{\cdots}}}}}$$

where

$$u_n = \frac{(a+n-1)(c-b+n-1)}{(c+2n-2)(c+2n-1)}$$

$$v_n = \frac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)} \qquad n = 1, 2, 3, \ldots .$$

When we have $b = 0$, we obtain a continued fraction for $F(a, 1; c; z)$.

2.5.5. Special cases of the hypergeometric function

In many cases the hypergeometric series reduces to an expansion of an elemenatry function. In the degenerate case at least one solution of 2.1(1) always is an elementary function. Results of the type

$$F(a, a+\tfrac{1}{2}; \tfrac{1}{2}; z) = \tfrac{1}{2}(1 - z^{1/2})^{-2a} + \tfrac{1}{2}(1 + z^{1/2})^{-2a}$$

may be obtained from the quadratic transformation 1.5(24) by a limiting process $b \to 0$. Various other cases where $F(a, b; c; z)$ is an elementary function are listed in 2.8(4) to 2.8(17). All of them either may be verified directly or may be deduced from the linear and quadratic transformations.

Other classes of hypergeometric functions which have been particularly investigated or which occur in the theory of other functions are listed in the table on the following page.

SPECIAL HYPERGEOMETRIC FUNCTIONS

Parameters a, b, c	Variable	Name	Chapter
Two of the numbers $1-c$, $\pm(a-b)$, $\pm(c-a-b)$ are equal to each other or one of them equals $\pm\frac{1}{2}$	$\frac{1}{2}-\frac{1}{2}z$	Legendre functions	3
$-n, n+2\nu; \nu+\frac{1}{2}$ $(n=0, 1, 2, \ldots)$	$\frac{1}{2}-\frac{1}{2}z$	Gegenbauer polynomials	10, 11
$-n, \alpha+n; \gamma$ $(n=0, 1, 2, \ldots)$	z	Jacobi polynomials	10, 11
$\frac{1}{2}, \frac{1}{2}, 1$ $-\frac{1}{2}, \frac{1}{2}, 1$	z^2 z^2	Complete elliptic integrals	13
$1/l$, $1/m$, $1/n$ or zero $(l, m, n = 1, 2, 3, \ldots)$ Quotient of two solutions of 2.1(1).	z	Inverse of automorphic functions	14
One of the numbers a, b, $c-a$, $c-b$ is an integer.	z	Degenerate case	2.2
$c-a=1$	z	Incomplete beta function	2.5.3

2.6. Riemann's equation

2.6.1. Reduction to the hypergeometric equation

For proofs of the theorems given in this section see E. G. C. Poole (1936). If a homogeneous linear differential equation of the second order has only three singularities and if these singularities are of the regular type (cf. Poole 1936) then the equation may be written in the form:

$$(1)\quad \frac{d^2u}{dz^2}+\left(\sum_{n=1}^{3}\frac{1-a_n-a'_n}{z-z_n}\right)\frac{du}{dz}+\left[\sum_{n=1}^{3}\frac{a_n a'_n (z_n-z_{n+1})(z_n-z_{n+2})}{z-z_n}\right]\times\frac{u}{(z-z_1)(z-z_2)(z-z_3)}=0$$

Here a_n, a_n', z_n are constants such that $z_4 = z_2$, $z_5 = z_3$ but $z_1 \neq z_2 \neq z_3 \neq z_1$, and

(2) $$\sum_{n=1}^{3} (a_n + a_n') = 1.$$

The singularities are at $z = z_n$ ($n = 1, 2, 3$); the constants a_n, a_n' are called the *exponents* belonging to $z = z_n$. We admit the case where one of the singularities is at infinity, and the coefficients of du/dz and u in (1) are obtained by an appropriate limiting process.

We shall call (1) *Riemann's equation.* The name *"Papperitz equation"* is also common.

The constants $\lambda_n = a_n - a_n'$ are called the *exponent differences.* If none of them is an integer, (1) has two linearly independent solutions $u_1(z)$, $u_2(z)$ for the neighborhood of $z = z_n$ such that

(3) $$\begin{cases} u_1(z) = (z - z_n)^{a_n} \sum_{m=0}^{\infty} v_m (z - z_n)^m \\ u_2(z) = (z - z_n)^{a_n'} \sum_{m=0}^{\infty} v_m' (z - z_n)^m. \end{cases}$$

Of course, v_m, v_m' may depend on z_1, z_2, z_3 but not on z, and $v_0 \neq 0$, $v_0' \neq 0$. If one or several of the exponent differences are integers, then one or several of the series (3) may involve logarithmic terms. Since we shall be able to reduce (1) to the hypergeometric equation, we may refer to the preceding section for the details of the logarithmic cases.

By the symbol

(4) $$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1 & a_2 & a_3 & z \\ a_1' & a_2' & a_3' & \end{Bmatrix}$$

we shall denote the complete set of solutions of (1). Then B. Riemann (1892) has shown

(5) $$\left(\frac{z - z_1}{z - z_2}\right)^{\rho} \left(\frac{z - z_3}{z - z_2}\right)^{\sigma} P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1 & a_2 & a_3 & z \\ a_1' & a_2' & a_3' & \end{Bmatrix}$$

$$= P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1 + \rho & a_2 - \rho - \sigma & a_3 + \sigma & z \\ a_1' + \rho & a_2' - \rho - \sigma & a_3' + \sigma & \end{Bmatrix}$$

and

$$(6)\quad P\begin{Bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \\ a_1 & a_2 & a_3 & \zeta \\ a_1' & a_2' & a_3' & \end{Bmatrix} = P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1 & a_2 & a_3 & z \\ a_1' & a_2' & a_3' & \end{Bmatrix}$$

where

$$(7)\quad \zeta = \frac{Az+B}{Cz+D}, \qquad \zeta_n = \frac{Az_n+B}{Cz_n+D},$$

and A, B, C, D are arbitrary constants such that $AD - CB \neq 0$. From (5) we see that the product of a solution of (1) with a factor

$$\left(\frac{z-z_1}{z-z_2}\right)^{\rho}\left(\frac{z-z_3}{z-z_2}\right)^{\sigma}$$

also satisfies a Riemann equation. Of course we can permute the subscripts 1, 2, 3 in (5). If $z_n = \infty$, then $z - z_n$ in (5) must be replaced by unity.

If z_1, z_2, z_3 are given, we can always find four constants A, B, C, D satisfying $AD - BC \neq 0$ such that ζ_1, ζ_2, ζ_3 are any three arbitrarily assigned numbers. Therefore (6) shows that we can always transform (1) by means of a homographic substitution of the variable into a Riemann equation which has its singularities at the points ζ_1, ζ_2, ζ_3.

Combining (5) and (6) we obtain

$$(8)\quad P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1 & a_2 & a_3 & z \\ a_1' & a_2' & a_3' & \end{Bmatrix} = \left(\frac{z-z_1}{z-z_2}\right)^{a_1}\left(\frac{z-z_3}{z-z_2}\right)^{a_3}$$

$$\times P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a_1+a_2+a_3 & 0 & \dfrac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)} \\ a_1'-a_1 & a_1+a_2'+a_3 & a_3'-a_3 & \end{Bmatrix}.$$

Now the hypergeometric equation 2.1(1) is a special case of Riemann's equation. This may be expressed by saying that

$$(9)\quad P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix}$$

is the set of all solutions of the hypergeometric equation. Therefore (8) gives a reduction of the most general Riemann equation to the special case 2.1(1). Since there are 24 different ways to transform the hypergeometric equation into itself this reduction is not uniquely determined. This may be seen as follows. The hypergeometric equation is characterized among Riemann's equations by the fact that its singularities are at $0, \infty, 1$, that its exponent differences at these points are $1-c$, $a-b$, $c-a-b$ respectively, and that one of the exponents at $z=0$ and at $z=1$ is zero. The six homographic substitutions

$$(10)\quad \zeta = z, \quad 1-z, \quad z/(1-z), \quad z^{-1}, \quad (1-z)^{-1}, \quad 1-z^{-1}$$

lead to the six possible permutations of the singularities. If we multiply the solutions of a Riemann equation with singularities at $z = 0, \infty, 1$, by $z^{\rho}(1-z)^{\tau}$, then we can always choose ρ, τ in such a way that one of the exponents at $z=0$ and at $z=1$ becomes zero. Since we have the choice between two exponents at each of these points, we obtain $2 \cdot 2 \cdot 6 = 24$ transformations of 2.1(1) into itself. This leads to 24 solutions of the type

$$z^{\rho}(1-z)^{\sigma} F(a^*, b^*; c^*; z^*)$$

where z^* is one of the expressions in (10) and a^*, b^*, c^* are linear functions of a, b, c. These solutions are called *Kummer's series;* they are given in 2.9(1) to 2.9(24). These 24 solutions can be arranged in six sets such that the four series belonging to each set represent the same function; the six resulting functions which, in general, are different from each other will be denoted by $u_1, u_2, \ldots, u_6$.

Any three of these are connected by a linear relation with constant coefficients, the resulting 20 linear relations (valid in the half-plane $\operatorname{Re} z > 0$) are given by 2.9(25) to 2.9(44).

2.6.2. Quadratic and cubic transformations

The following relations are the source of the quadratic and cubic transformation of the hypergeometric series:

$$(11)\quad P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \alpha_2 & \alpha_3 & z \\ \tfrac{1}{2} & \alpha_2' & \alpha_3' & \end{matrix}\right\} = P\left\{\begin{matrix} -1 & \infty & 1 & \\ \alpha_3 & 2\alpha_2 & \alpha_3 & z^{1/2} \\ \alpha_3' & 2\alpha_2' & \alpha_3 & \end{matrix}\right\}$$

$$= P\left\{\begin{matrix} -1 & \infty & 1 & \\ \alpha_2 & 2\alpha_3 & \alpha_2 & \dfrac{z^{1/2}}{(z-1)^{1/2}} \\ \alpha_2' & 2\alpha_3' & \alpha_2 & \end{matrix}\right\}$$

where $\alpha_2 + \alpha_2' + \alpha_3 + \alpha_3' = \tfrac{1}{2}$, and

$$(12)\quad P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & 0 & a_3 & z \\ 1/3 & 1/3 & a_3' & \end{matrix}\right\} = P\left\{\begin{matrix} 1 & w & w^2 & \\ a_3 & a_3 & a_3 & z^{1/3} \\ a_3' & a_3' & a_3' & \end{matrix}\right\}$$

where $a_3 + a_3' = 1/3$ and $w = \exp(2\pi i/3)$. Both of these were discovered by B. Riemann and investigated by E. Goursat (1881) [cf. also E. W. Barnes (1908) and G. N. Watson (1909)].

There exist higher transformations in a limited number of cases where all the constants involved are rational numbers. For these see E. Goursat (1881), (1938).

2.7. Conformal representations

[For this section cf. Goursat (1936), (1938)].

2.7.1. Group of the hypergeometric equation

If none of the exponent differences $1-c$, $b-a$, $c-a-b$ is an integer, then

$$(1)\quad \begin{cases} u_1(z) = F(a, b; c; z) \\ u_2(z) = z^{1-c} F(a-c+1, b-c+1; 2-c; z) \end{cases}$$

are two linearly independent solutions of 2.1(1) which are one-valued and regular within the domain $|z-\tfrac{1}{2}| < \tfrac{1}{2}$, but at each of the three points $z = 0, \infty, 1$, at least one of them will have a branch point. We shall consider the effect upon the two solutions (13) of z describing any closed circuit, beginning and ending for example at $z = \tfrac{1}{2}$ and enclosing one or several of the branch points. We can reduce this problem to the following one. What will be the effect of two simple positive circuits $C_{(0)}$ and $C_{(1)}$, beginning and ending at $z = \tfrac{1}{2}$ and enclosing $z = 0$ and $z = 1$? The effect of any loop can be reduced to the effect of a sequence of the loops $C_{(0)}$, $C_{(1)}$, and $C'_{(0)}$, $C'_{(1)}$, where the prime denotes the corresponding negative circuit. Now we can derive from 2.9(25) to 2.9(44) that u_1 and u_2 are affected by $C_{(0)}$, and $C_{(1)}$ in the way indicated by the arrows. We have

$$(2)\quad C_{(0)}\begin{cases} u_1 \to u_1 \\ u_2 \to e^{-2i\pi c}\, u_2, \end{cases}$$

$$(3)\quad C_{(1)}\begin{cases} u_1 \to B_{11}\, u_1 + B_{12}\, u_2 \\ u_2 \to B_{21}\, u_1 + B_{22}\, u_2 \end{cases}$$

where

$$B_{11} = 1 - 2ie^{i\pi(c-a-b)} \frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi c)},$$

$$B_{12} = -2i\pi\, e^{i\pi(c-a-b)} \frac{\Gamma(c)\,\Gamma(c-1)}{\Gamma(c-a)\,\Gamma(c-b)\,\Gamma(b)\,\Gamma(a)},$$

$$B_{21} = 2i\pi\, e^{i\pi(c-a-b)} \frac{\Gamma(2-c)\,\Gamma(1-c)}{\Gamma(1-a)\,\Gamma(1-b)\,\Gamma(1+a-c)\,\Gamma(1+b-c)},$$

$$B_{22} = 1 + 2ie^{i\pi(c-a-b)} \frac{\sin\pi(c-a)\sin\pi(c-b)}{\sin(\pi c)}.$$

The proof of (2) is obvious. To prove (3) we start with the formulas [cf. 2.10(1)]

$$F(a, b; c; z) = \lambda_{11} F(a, b; a+b-c+1; 1-z) + \lambda_{12}(1-z)^{c-a-b} F(c-a, c-b; c-a-b+1; 1-z),$$

$$\begin{aligned} z^{1-c} F(a-c+1, b-c+1; 2-c; z) &= \lambda_{21} z^{1-c} F(a-c+1, b-c+1; a+b-c+1; 1-z) \\ &+ \lambda_{22}(1-z)^{c-a-b} z^{1-c} F(1-a, 1-b; c-a-b+1; 1-z) \\ &= \lambda_{21} F(a, b; a+b-c+1; 1-z) \\ &+ \lambda_{22}(1-z)^{c-a-b} F(c-a, c-b; c-a-b+1; 1-z). \end{aligned}$$

This gives

$$\begin{cases} \lambda_{22} u_1 - \lambda_{12} u_2 = (\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}) F(a, b; a+b-c+1; 1-z), \\ \lambda_{21} u_1 - \lambda_{11} u_2 = (\lambda_{12}\lambda_{21} - \lambda_{11}\lambda_{22})(1-z)^{c-a-b} \\ \quad \times F(c-a, c-b; c-a-b+1; 1-z), \end{cases}$$

and from these equations we have

$$C_{(1)} \begin{cases} \lambda_{22} u_1 - \lambda_{12} u_2 \to \lambda_{22} u_1 - \lambda_{12} u_2 \\ \lambda_{21} u_1 - \lambda_{11} u_2 \to \exp[2\pi i(c-a-b)]\,(\lambda_{21} u_1 - \lambda_{11} u_2). \end{cases}$$

Here

$$\begin{cases} \lambda_{11} = \dfrac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, & \lambda_{12} = \dfrac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}, \\ \lambda_{21} = \dfrac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma(1-a)\,\Gamma(1-b)}, & \lambda_{22} = \dfrac{\Gamma(2-c)\,\Gamma(a+b-c)}{\Gamma(a-c+1)\,\Gamma(b-c+1)}. \end{cases}$$

The rest of the proof of (3) amounts to a repeated application of the fundamental relations which hold for the gamma function.

The effect upon u_1, u_2, of any closed circuit beginning and ending at a fixed point, say at $z = \frac{1}{2}$, may be described by a linear substitution; the whole set of these linear substitutions forms a group, the group of the hypergeometric equation. All substitutions of the group may be obtained by a composition of those given by (2) and (3). Usually the set of homographic substitutions for $u_1(z)/u_2(z)$ is also called the group of the hypergeometric equation. If c is different from $0, \pm 1, \pm 2, \ldots$, then (2) and (3) always have a meaning. Obviously, either u_1, or u_2 is merely multiplied by a constant factor after any closed circuit if at least one of the numbers a, b, $c - a$, $c - b$, is an integer. This leads to the degenerate case which has been investigated in section 2.2. If c is an integer, it may be necessary to modify (2) and (3). To do this we may use 2.1(18), 2.1(14), 2.2(4), and also 2.10(7) to 2.10(15). For instance, in the non-degenerate case $a = b = \frac{1}{2}$, $c = 1$, all of the three exponent differences $1 - c$, $b - a$, $c - a - b$ equal zero, and in this case we may take

$$\begin{cases} u_1 = F(\tfrac{1}{2}, \tfrac{1}{2};\ 1;\ z) \\ u_2 = i\,F(\tfrac{1}{2}, \tfrac{1}{2};\ 1;\ 1 - z). \end{cases}$$

Then we obtain

$$(4) \qquad C_{(0)} \begin{cases} u_1 \to u_1 \\ u_2 \to 2u_1 + u_2, \end{cases}$$

$$(5) \qquad C_{(1)} \begin{cases} u_1 \to u_1 - 2u_2 \\ u_2 \to u_2. \end{cases}$$

This result may be derived from

$$(6) \qquad \tfrac{1}{2}\pi\, F(\tfrac{1}{2}, \tfrac{1}{2};\ 1;\ z) + \tfrac{1}{2}\log(1 - z)\, F(\tfrac{1}{2}, \tfrac{1}{2};\ 1;\ 1 - z)$$

$$= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{n!\,n!} [\psi(n + 1) - \psi(n + \tfrac{1}{2})]\,(1 - z)^n$$

which follows from 2.1(15) by the method used in section 2.1.4.

B. Riemann has shown [cf. Poole (1936)] that associated hypergeometric equations have the same group of homographic substitutions and from this it can be deduced that there exists a linear relation between any three associated hypergeometric series where the coefficients are rational functions of the variable.

2.7.2. Schwarz's function

[cf. Kampé de Fériet (1937), Poole (1936)]. From now on we shall denote the exponent differences by

(7) $1 - c = \lambda, \quad b - a = \mu, \quad c - a - b = \nu.$

If u is a solution of the hypergeometric equation, then

$$y(z) = z^{1/2 - 1/2\lambda}(1 - z)^{1/2 - 1/2\nu}\, u(z)$$

satisfies

(8) $$\frac{d^2y}{dz^2} + I(\lambda, \mu, \nu; z)\, y = 0$$

where

(9) $$I(\lambda, \mu, \nu; z) = \frac{1 - \lambda^2}{4z^2} + \frac{1 - \nu^2}{4(1 - z)^2} + \frac{1 - \lambda^2 + \mu^2 - \nu^2}{4z(1 - z)}.$$

Defining the *Schwarzian derivative* $\{w, z\}$ of a function by

$$\{w, z\} = \frac{d^3w}{dz^3}\Big/\frac{dw}{dz} - \frac{3}{2}\left(\frac{d^2w}{dz^2}\Big/\frac{dw}{dz}\right)^2$$

and putting

$$w(z) = y_1(z)/y_2(z)$$

where y_1 and y_2 are two linearly independent solutions of (8), we have

(10) $\{w, z\} = 2I(\lambda, \mu, \nu; z).$

If ζ is a function of z we have *Cayley's identity*;

(11) $$\{w, z\} = \{w, \zeta\}\left(\frac{d\zeta}{dz}\right)^2 + \{\zeta, z\}.$$

Also

(12) $$\left\{\frac{Ax + B}{Cx + D}, x\right\} = \left\{x, \frac{Ax + B}{Cx + D}\right\} = 0$$

where A, B, C, D, are constants such that $AD - BC \neq 0$. Therefore, if $\zeta = (Aw + B)/(Cw + D)$, we have

$$\{w, z\} = \left\{\frac{Aw + B}{Cw + D}, z\right\}.$$

This shows that if $w(z)$ is a solution of (10), then $(Aw + B)/(Cw + D)$ also satisfies (10), and it can be proved that all the solutions of (10) are of this type. Therefore we know all the solutions of (10) if we know two linearly independent solutions of (8). On the other hand, if $w(z)$ is a solution of (10), then $w(z)$ cannot be a constant unless $\lambda^2 = \mu^2 = \nu^2 = 1$. We shall exclude this case, and then we have that

$$(13)\quad y_1(z) = w\left(\frac{dw}{dz}\right)^{-\frac{1}{2}}, \quad y_2(z) = \left(\frac{dw}{dz}\right)^{-\frac{1}{2}}$$

are two linearly independent solutions of (8).

We shall denote by $s\ (\mu, \nu, \lambda; z)$ the whole set of solutions of (10), and we will call s the *general Schwarz function.* A particular Schwarz function will be denoted by $S(\mu, \nu, \lambda; z)$. It may be shown that the function $s(\lambda, \mu, \nu; z)$ is meromorphic in the neighborhood of any point $z \neq 0$, ∞, 1, and that the correspondence between w and s is locally one to one (schlicht) since ds/dz or, if s has a pole, ds^{-1}/dz is different from zero at all points $z \neq 0, \infty, 1$. This follows from

$$\frac{ds}{dz} = \left(y_1 \frac{dy_2}{dz} - y_2 \frac{dy_1}{dz}\right) \Big/ y_2^2$$

where the numerator is a constant.

If in particular λ, μ, ν are real, we may use the theorems on associated hypergeometric series (cf. sections 2.1.2 and 2.7.1) to show that the functions

$$s(l \pm \lambda, m \pm \mu, n \pm \nu; z) \qquad l, m, n = 0, \pm 1, \pm 2, \ldots$$

arise from hypergeometric functions with the same group if $l + m + n$ is even. Among these functions there is one set which we will call *reduced* and for which

$$0 \le \lambda, \mu, \nu < 1, \quad 0 \le \mu + \nu, \quad \nu + \lambda, \quad \lambda + \mu \le 1.$$

It has been shown by H. A. Schwarz (1873) that a reduced function $\tau = S(\lambda, \mu, \nu; z)$ maps the upper half-plane $\operatorname{Im} z \ge 0$ upon a triangle Δ_0 in the τ-plane bounded by three circular arcs (some of which may be segments of straight lines) which do not overlap and which enclose interior angles $\lambda\pi$, $\mu\pi$, $\nu\pi$, at the points corresponding to $z = 0, \infty, 1$. Here the word *interior* refers to that domain bounded by the triangle which contains for example the point corresponding to $z = i$.

If follows from Schwarz's principle of symmetry that the complete system of branches of $\tau = S(\lambda, \mu, \nu; z)$ maps the z-plane upon a Riemann surface spread over the τ-plane which consists of Δ_0 and of all the triangles obtained from Δ_0 by the following construction. If a circle (or a straight line as a limiting case of a circle) is given, we can map the τ-plane upon itself by a substitution

$$\tau' = \frac{a_{11}\,\bar{\tau} + a_{12}}{a_{21}\,\bar{\tau} + a_{22}}$$

where τ' is the point corresponding to τ and where $\bar{\tau}$ is the conjugate complex value of τ, such that $\tau = \tau'$ for the points of the circle. We shall call that an *inversion with respect to the circle.* Carrying out the inversions with respect to the circles bounding Δ_0, this will be mapped upon three new triangles Δ_1, Δ_2, Δ_3, which again are bounded by arcs of circles, and we obtain new triangles from Δ_1, Δ_2, Δ_3, by carrying out the inversions with respect to these boundaries and so on. If any two of the triangles thus obtained do not overlap, the function $\tau = S(\lambda, \mu, \nu; z)$ has a meromorphic one-valued inverse function

(14) $z = \phi(\lambda, \mu, \nu; \tau)$

which is called an *automorphic function.* As a necessary condition for the existence of a function ϕ with this property we have that λ, μ, ν, must be either zero or equal to the reciprocal values of integers.

There are 15 reduced sets of values for λ, ν, μ, for which $s(\lambda, \mu, \nu; z)$ is algebraic. They have been listed by H. A. Schwarz (1873). His list (in which Number 1 actually contains an infinite number of cases) is given in the table below. Here n, p, are non-negative integers and $2p \leq n$.

SCHWARZ'S TABLE

Number	λ	μ	ν
1	1/2	1/2	p/n
2	1/2	1/3	1/3
3	2/3	1/3	1/3
4	1/2	1/3	1/4
5	2/3	1/4	1/4
6	1/2	1/3	1/5
7	2/5	1/3	1/3

Number	λ	μ	ν
8	2/3	1/5	1/5
9	1/2	2/5	1/5
10	3/5	1/3	1/5
11	2/5	2/5	2/5
12	2/3	1/3	1/5
13	4/5	1/5	1/5
14	1/2	2/5	1/3
15	3/5	2/5	1/3

In the cases 1 (for $p = 1$), 2, 4, 6, the inverse function $\phi(\lambda, \mu, \nu; \tau)$ is a rational function. In particular, we may take

$$\phi(\tfrac{1}{2}, \tfrac{1}{2}, 1/n; \tau) = \left(\frac{\tau^n - 1}{\tau^n + 1}\right)^2.$$

In the case $\lambda = \mu = \nu = 0$ we obtain a particular automorphic function if we take

$$\tau = S(0, 0, 0, z) = i \; \frac{F(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - z)}{F(\tfrac{1}{2}, \tfrac{1}{2}; 1; z)} \, .$$

The corresponding inverse function $z(\tau)$ is usually denoted by $\kappa^2(\tau)$ and is called the *elliptic modular function*. This function is the subject of a large literature. Consult Chapter 14 and Klein and Fricke (1890, 1892) and for general automorphic functions Fricke and Klein (1897). An explicit expression for $\kappa^2(\tau)$ is $\kappa^2(\tau) = (\theta_2/\theta_3)^4$ where

$$\theta_2 = 2 \sum_{n=0}^{\infty} (-1)^n \, e^{i\pi\tau(n+\frac{1}{2})^2} = e^{i\pi/4} \, \tau^{-\frac{1}{2}} [1 + 2 \sum_{n=1}^{\infty} (-1)^n \, e^{-i\pi n^2/\tau}]$$

$$\theta_3 = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi\tau n^2} = e^{i\pi/4} \, \tau^{-\frac{1}{2}} (1 + 2 \sum_{n=1}^{\infty} e^{-i\pi n^2/\tau})$$

are functions of τ which are regular if and only if $\operatorname{Im} \tau > 0$. (cf. Whittaker-Watson 1927, sections 21.7, 22.3).

2.7.3. Uniformization

If we introduce τ by

$$z = \kappa^2(\tau)$$

(cf. 2.7.2) as a new variable, $F(a, b; c; z)$ becomes a one-valued function of τ which is defined and regular in the half-plane $\operatorname{Im} \tau > 0$. Wirtinger (1902, 1903), has proved the formula.

$$(15) \quad \tfrac{1}{2} \Gamma(b) \, \Gamma(c - b) \, F[a, b; c; \kappa^2(\tau)] = \pi^{2b} \Gamma(c) [\theta_3(0, \tau)]^{4b} \int_0^{\frac{1}{2}} \Phi(u, \tau) \, du$$

where

$$\Phi(u, \tau) = \left[\frac{\theta_1(u, \tau)}{\theta_1(0, \tau)} \right]^{2b-1} \left[\frac{\theta_2(u, \tau)}{\theta_2(0, \tau)} \right]^{2(c-b)-1}$$

$$\times \left[\frac{\theta_3(u, \tau)}{\theta_3(0, \tau)} \right]^{1-2a} \left[\frac{\theta_4(u, \tau)}{\theta_4(0, \tau)} \right]^{1-2(c-a)}$$

The functions $\theta_i(u, \tau)$ $(i = 1, 2, 3, 4)$ are the four Jacobian theta functions (cf. Chapter 13). The integral in (15) must be replaced by a contour integral if the conditions $\operatorname{Re} c > \operatorname{Re} b > 0$ are not fulfilled. (cf. 2.1.3).

2.7.4. Zeros

Let $u(a, b; c; z)$ be a one-valued branch of a solution of 2.1(1) which is defined in the half-plane $z \geq 0$ with the possible exception of the points 0, ∞, 1. Then the equation

(16) $u(a, b; c; z) = \Lambda$

is satisfied only by a finite number of values of z if Λ is any given constant. This follows from the fact that the solution of 2.1(1) can be expanded in the neighborhood of the singular points 0, ∞, 1, in the way shown by 2.10(1) to 2.10(5). From these we can deduce that the behavior of u at these points is determined by a single term of the type

$$v_0(z) = c_0(z - z_0)^{d_0} \quad \text{or} \quad v_0(z) = c_0(z - z_0)^{d_0} \log(z - z_0)$$

where $z_0 = 0, \infty, 1$ and $z - z_0$ is to be replaced by z^{-1} if $z_0 = \infty$. This is to say that u/v_0 approaches a finite value not equal to zero if $z \to z_0$. Therefore (16) can have only a finite number of solutions in a sufficiently small neighborhood of 0, ∞, 1 (if we confine ourselves to the domain $\operatorname{Im} z \geq 0$ and therefore to a one-valued branch of v_0). In the remaining parts of the upper half-plane u is regular and therefore (16) can be satisfied there only at a finite number of points.

In the case where a, b, c, are real, the number of zeros [i.e., the number of solutions of (16) with $\Lambda = 0$] of a one-valued branch of $u(a, b; c; z)$ has been determined by Hurwitz (1907), Van Vleck (1901), (1902), and Herglotz (1917). The methods used by the latter authors are closely connected with the results given in 2.7(1) and 2.7(2).

In the case of the polynomials

$$u = F(-n, a + n; \gamma; z) \qquad n = 0, 1, 2, \ldots$$

where a, γ are real and $\gamma > 0$, $a + 1 - \gamma > 0$ all the zeros are real and lie in the interval $0 < z < 1$. (See Chapter 10 on Jacobi polynomials.)

SECOND PART: FORMULAS

2.8. The hypergeometric series

(1) $$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n \qquad c \neq 0, -1, -2, \ldots,$$

(2) $$\begin{cases} (a)_0 = 1 \\ (a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1) \cdots (a+n-1) \qquad n = 1, 2, 3 \ldots. \end{cases}$$

If $c = -m - l$ where $m, l = 0, 1, 2, \ldots$, then we have

(3) $$F(-m, b, -m-l; z) = \sum_{n=0}^{m} \frac{(-m)_n (b)_n}{(-m-l)_n \, n!} z^n.$$

Some elementary functions which can be expressed by hypergeometric series follow (cf. sections 2.2.1 and 2.5.5).

(4) $$(1+z)^a = F(-a, b; b; -z)$$

(5) $$\tfrac{1}{2}(1+z^{1/2})^{-2a} + \tfrac{1}{2}(1-z^{1/2})^{-2a} = F(a, a+\tfrac{1}{2}; \tfrac{1}{2}; z)$$

(6) $$[\tfrac{1}{2} + (1-z)^{1/2}/2]^{1-2a} = F(a-\tfrac{1}{2}, a; 2a; z) = (1-z)^{1/2} F(a, a+\tfrac{1}{2}; 2a; z)$$

(7) $$(1-z)^{-2a-1}(1+z) = F(2a, a+1; a; z)$$

The truncated binomial series follow:

(8) $$1 + \binom{a}{1} z + \cdots + \binom{a}{m} z^m = \binom{a}{m} z^m \, F(-m, 1; a-m+1; -z^{-1})$$

(9) $$\sum_{n=m+1}^{\infty} \binom{a}{n} z^n = z^{m+1} \frac{\Gamma(a+1)}{\Gamma(a-m)\,(m+1)!} F(m+1-a, 1; m+2; -z)$$

(10) $$e^{-az} = (2\cosh z)^{-a} \tanh z \, F[1+\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a; 1+a; (\cosh z)^{-2}]$$

(11) $$\begin{aligned} \cos az &= F[\tfrac{1}{2}a, -\tfrac{1}{2}a; \tfrac{1}{2}; (\sin z)^2] \\ &= \cos z \, F[\tfrac{1}{2}+\tfrac{1}{2}a, \tfrac{1}{2}-\tfrac{1}{2}a; \tfrac{1}{2}; (\sin z)^2] \\ &= (\cos z)^a \, F[-\tfrac{1}{2}a, \tfrac{1}{2}-\tfrac{1}{2}a; \tfrac{1}{2}; -(\tan z)^2] \end{aligned}$$

(12) $$\begin{aligned} \sin az &= a \sin z \, F[\tfrac{1}{2}+\tfrac{1}{2}a, \tfrac{1}{2}-\tfrac{1}{2}a; 3/2; (\sin z)^2] \\ &= a \sin z \cos z \, F[1+\tfrac{1}{2}a, 1-\tfrac{1}{2}a; 3/2; (\sin z)^2] \end{aligned}$$

(13) $\sin^{-1} z = z\, F(\tfrac{1}{2}, \tfrac{1}{2}; 3/2; z^2)$

(14) $\tan^{-1} z = z\, F(\tfrac{1}{2}, 1; 3/2; -z^2)$

(15) $\log(z+1) = z\, F(1, 1; 2; -z)$

(16) $\log \dfrac{1+z}{1-z} = 2z\, F(\tfrac{1}{2}, 1; 3/2; z^2)$

(17) $\dfrac{d^n}{dz^n} [z^{n+c-1} (1-z)^{b-c}] = (c)_n\, z^{c-1} (1-z)^{b-c-n} F(-n, b; c; z)$

ELEMENTARY RELATIONS

(18) $F(a, b; c; z) = F(b, a; c; z)$

(19) $\lim\limits_{c \to -n} [\Gamma(c)]^{-1} F(a, b; c; z)$

$$= \frac{(a)_{n+1} (b)_{n+1}}{(n+1)\,!} z^{n+1} F(a+n+1, b+n+1; n+2; z)$$

(20) $\dfrac{d^n}{dz^n} F(a, b; c; z) = \dfrac{(a)_n (b)_n}{(c)_n} F(a+n, b+n; c+n; z)$

(21) $(a)_n\, z^{a-1} F(a+n, b; c; z) = \dfrac{d^n}{dz^n} [z^{a+n-1} F(a, b; c; z)]$

(22) $(c-n)_n\, z^{c-1-n} F(a, b; c-n; z) = \dfrac{d^n}{dz^n} [z^{c-1} F(a, b; c; z)]$

(23) $(c-a)_n\, z^{c-a-1} (1-z)^{a+b-c-n} F(a-n, b; c; z)$

$$= \frac{d^n}{dz^n} [z^{c-a+n-1} (1-z)^{a+b-c} F(a, b; c; z)]$$

(24) $\dfrac{(c-a)_n (c-b)_n}{(c)_n} (1-z)^{a+b-c-n} F(a, b; c+n; z)$

$$= \frac{d^n}{dz^n} [(1-z)^{a+b-c} F(a, b; c; z)]$$

(25) $\dfrac{(-1)^n (a)_n (c-b)_n}{(c)_n} (1-z)^{a-1} F(a+n, b; c+n; z)$

$$= \frac{d^n}{dz^n} [(1-z)^{a+n-1} F(a, b; c; z)]$$

(26) $(c-n)_n\, z^{c-1-n}(1-z)^{b-c} F(a-n,\ b;\ c-n;\ z)$

$$= \frac{d^n}{dz^n}[z^{c-1}(1-z)^{b-c+n} F(a, b; c; z)]$$

(27) $(c-n)_n\, z^{c-1-n}(1-z)^{a+b-c-n} F(a-n,\ b-n;\ c-n;\ z)$

$$= \frac{d^n}{dz^n}[z^{c-1}(1-z)^{a+b-c} F(a, b; c; z)] .$$

The relations between contiguous hypergeometric series in the case where two parameters are constants are given below:

(28) $(c-a)\, F(a-1,\ b;\ c;\ z) + (2a - c - az + bz)\, F(a, b; c; z)$
$+\, a(z-1)\, F(a+1,\ b;\ c;\ z) = 0$

(29) $(c-b)\, F(a,\ b-1;\ c;\ z) + (2b - c - bz + az)\, F(a, b; c; z)$
$+\, b(z-1)\, F(a,\ b+1;\ c;\ z) = 0$

(30) $c(c-1)(z-1)\, F(a, b; c-1; z) + c[c - 1 - (2c - a - b - 1)\, z]$
$\times\, F(a, b; c; z) + (c-a)(c-b)\, z\, F(a, b; c+1; z) = 0.$

The fifteen relations of Gauss between contiguous functions [F denotes $F(a, b; c; z)$, and $F(a \pm 1)$, $F(b \pm 1)$, and $F(c \pm 1)$ stands for $F(a \pm 1, b; c; z)$, $F(a, b \pm 1, c; z)$, and $F(a, b; c \pm 1; z)$ respectively].

(31) $[c - 2a - (b-a)\, z]\, F + a(1-z)\, F(a+1) - (c-a)\, F(a-1) = 0$

(32) $(b-a)\, F + a\, F(a+1) - b\, F(b+1) = 0$

(33) $(c-a-b)\, F + a(1-z)\, F(a+1) - (c-b)\, F(b-1) = 0$

(34) $c[a - (c-b)\, z]\, F - ac(1-z)\, F(a+1) + (c-a)(c-b)\, z\, F(c+1) = 0$

(35) $(c-a-1)\, F + a\, F(a+1) - (c-1)\, F(c-1) = 0$

(36) $(c-a-b)\, F - (c-a)\, F(a-1) + b(1-z)\, F(b+1) = 0$

(37) $(b-a)(1-z)\, F - (c-a)\, F(a-1) + (c-b)\, F(b-1) = 0$

(38) $c(1-z)\, F - c\, F(a-1) + (c-b)\, z\, F(c+1) = 0$

(39) $[a - 1 - (c-b-1)\, z]\, F + (c-a)\, F(a-1) - (c-1)(1-z)\, F(c-1) = 0$

(40) $[c - 2b + (b-a)\, z]\, F + b(1-z)\, F(b+1) - (c-b)\, F(b-1) = 0$

(41) $c\,[b - (c-a)\, z]\, F - bc(1-z)\, F(b+1) + (c-a)(c-b)z\, F(c+1) = 0$

(42) $(c-b-1)\, F + b\, F(b+1) - (c-1)\, F(c-1) = 0$

(43) $c\,(1-z)\,F - c\,F(b-1) + (c-a)\,F(c+1) = 0$

(44) $[b-1-(c-a-1)\,z]\,F + (c-b)\,F(b-1) - (c-1)(1-z)\,F(c-1) = 0$

(45) $c\,[c-1-(2c-a-b-1)\,z]\,F + (c-a)\,(c-b)\,z\,F(c+1)$
$\qquad - c(c-1)(1-z)\,F(c-1) = 0$

VALUES FOR SPECIAL z

(46) $$F(a,\ b;\ c;\ 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$
$c \neq 0, -1, -2, \ldots, \quad \mathrm{Re}\ c > \mathrm{Re}(a+b)$

(47) $$F(a,\ b;\ 1+a-b;\ -1) = 2^{-a}\,\frac{\Gamma(1+a-b)\,\Gamma(\tfrac{1}{2})}{\Gamma(1-b+\tfrac{1}{2}a)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}a)}$$
$1+a-b \neq 0, -1, -2, \ldots$

(48) $$(a+1)\,F(-a,\ 1;\ b+2;\ -1) + (b+1)\,F(-b,\ 1;\ a+2;\ -1) = 2^{a+b+1}\,\frac{\Gamma(a+2)\,\Gamma(b+2)}{\Gamma(a+b+2)}$$
$a,\ b \neq -2, -3, -4, \ldots$

(49) $F(1,\ a;\ a+1;\ -1) = 2a\,[\,\psi(\tfrac{1}{2}+\tfrac{1}{2}a) - \psi(\tfrac{1}{2}a)\,]$

(50) $$F(2a,\ 2b;\ a+b+\tfrac{1}{2};\ \tfrac{1}{2}) = \frac{\Gamma(a+b+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2})}{\Gamma(a+\tfrac{1}{2})\,\Gamma(b+\tfrac{1}{2})}$$
$a+b+\tfrac{1}{2} \neq 0, -1, -2, \ldots$

(51) $$F(a,\ 1-a;\ b;\ \tfrac{1}{2}) = 2^{1-b}\,\frac{\Gamma(b)\,\Gamma(\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}b)\,\Gamma(\tfrac{1}{2}b-\tfrac{1}{2}a+\tfrac{1}{2})}$$
$b \neq 0, -1, -2, \ldots$

(52) $F(2a,\ 2b;\ a+b+1;\ \tfrac{1}{2})$
$$= \pi^{1/2}\,(a-b)^{-1}\,\Gamma(a+b+1)\,\{[\Gamma(a)\,\Gamma(b+\tfrac{1}{2})]^{-1} - [\Gamma(a+\tfrac{1}{2})\,\Gamma(b)]^{-1}\}$$
$a+b+1 \neq 0, -1, -2, \ldots$

(53) $$F(-a,\ -a+\tfrac{1}{2};\ 2a+3/2;\ -1/3) = \left(\frac{8}{9}\right)^{2a}\,\frac{\Gamma(4/3)\,\Gamma(2a+3/2)}{\Gamma(3/2)\,\Gamma(2a+4/3)}$$
$2a+3/2 \neq 0, -1, -2, \ldots$

(54) $$F(3a,\ 3a+1/2;\ 3a+5/6;\ 1/9) = \left(\frac{3}{4}\right)^{3a}\,\frac{\Gamma(2a+5/6)\,\Gamma(1/2)}{\Gamma(a+1/2)\,\Gamma(a+5/6)}$$
$2a+5/6 \neq 0, -1, -2, \ldots$

(55) $F(a+1/3,\, 3a;\; 2a+2/3;\; e^{i\pi/3})$

$$= 2\pi\, e^{\frac{1}{6} i\pi a}\, 3^{-\frac{1}{2}(3a+1)} \frac{\Gamma(2a+2/3)}{\Gamma(a+1/3)\,\Gamma(a+2/3)\,\Gamma(2/3)}$$

(56) $F(a+1/3,\, 3a;\; 2a+2/3;\; e^{-i\pi/3})$

$$= 2\pi\, e^{-\frac{1}{6} i\pi a}\, 3^{-\frac{1}{2}(3a+1)} \frac{\Gamma(2a+2/3)}{\Gamma(a+1/3)\,\Gamma(a+2/3)\,\Gamma(2/3)}$$

$$2a+2/3 \neq 0, -1, -2, \ldots$$

2.9. Kummer's series and the relations between them

Kummer's 24 solutions of the hypergeometric equation are given below:

(1) $u_1 = F(a, b;\; c;\; z)$

(2) $\quad = (1-z)^{c-a-b} F(c-a,\, c-b;\; c;\; z)$

(3) $\quad = (1-z)^{-a} F[a,\, c-b;\; c;\; z/(z-1)]$

(4) $\quad = (1-z)^{-b} F[c-a,\, b;\; c;\; z/(z-1)]$

(5) $u_2 = F(a, b;\; a+b+1-c;\; 1-z)$

(6) $\quad = z^{1-c} F(a+1-c,\, b+1-c;\; a+b+1-c;\; 1-z)$

(7) $\quad = z^{-a} F(a,\, a+1-c;\; a+b+1-c;\; 1-z^{-1})$

(8) $\quad = z^{-b} F(b+1-c,\, b;\; a+b+1-c;\; 1-z^{-1})$

(9) $u_3 = (-z)^{-a} F(a,\, a+1-c;\; a+1-b;\; z^{-1})$

(10) $\quad = (-z)^{b-c}(1-z)^{c-a-b} F(1-b,\, c-b;\; a+1-b;\; z^{-1})$

(11) $\quad = (1-z)^{-a} F[a,\, c-b;\; a+1-b;\; (1-z)^{-1}]$

(12) $\quad = (-z)^{1-c}(1-z)^{c-a-1} F[a+1-c,\, 1-b;\; a+1-b;\; (1-z)^{-1}]$

(13) $u_4 = (-z)^{-b} F(b+1-c,\, b;\; b+1-a;\; z^{-1})$

(14) $\quad = (-z)^{a-c}(1-z)^{c-a-b} F(1-a,\, c-a;\; b+1-a;\; z^{-1})$

(15) $\quad = (1-z)^{-b} F[b,\, c-a;\; b+1-a;\; (1-z)^{-1}]$

(16) $\quad = (-z)^{1-c}(1-z)^{c-b-1} F[b+1-c,\, 1-a;\; b+1-a;\; (1-z)^{-1}]$

(17) $u_5 = z^{1-c} F(a+1-c,\, b+1-c;\; 2-c;\; z)$

(18) $\quad = z^{1-c}(1-z)^{c-a-b} F(1-a,\, 1-b;\; 2-c;\; z)$

(19) $\quad = z^{1-c}(1-z)^{c-a-1} F[a+1-c,\, 1-b;\; 2-c;\; z/(z-1)]$

(20) $\quad = z^{1-c}(1-z)^{c-b-1} F[b+1-c,\ 1-a;\ 2-c;\ z/(z-1)]$

(21) $u_6 = (1-z)^{c-a-b} F(c-a,\ c-b;\ c+1-a-b;\ 1-z)$

(22) $\quad = z^{1-c}(1-z)^{c-a-b} F(1-a,\ 1-b;\ c+1-a-b;\ 1-z)$

(23) $\quad = z^{a-c}(1-z)^{c-a-b} F(c-a,\ 1-a;\ c+1-a-b;\ 1-z^{-1})$

(24) $\quad = z^{b-c}(1-z)^{c-a-b} F(c-b,\ 1-b;\ c+1-a-b;\ 1-z^{-1})$

Any three of the functions $u_1, \ldots, u_6$ are connected by a linear relation with constant coefficients. This gives 20 relations, viz., :

(25) $$e^{i\pi b}\frac{\Gamma(b)\,\Gamma(a+1-c)}{\Gamma(a+b+1-c)}u_2 = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}u_1 + e^{i\pi(b+1-c)}\frac{\Gamma(a+1-c)\,\Gamma(c-b)}{\Gamma(a+1-b)}u_3$$

(26) $$e^{i\pi a}\frac{\Gamma(a)\,\Gamma(b+1-c)}{\Gamma(a+b+1-c)}u_2 = \frac{\Gamma(a)\,\Gamma(c-a)}{\Gamma(c)}u_1 + e^{i\pi(a+1-c)}\frac{\Gamma(b+1-c)\,\Gamma(c-a)}{\Gamma(b+1-a)}u_4$$

(27) $$e^{i\pi(c-b)}\frac{\Gamma(c-b)\,\Gamma(1-a)}{\Gamma(c+1-a-b)}u_6 = \frac{\Gamma(b)\,\Gamma(c-b)}{\Gamma(c)}u_1 + e^{i\pi(1-b)}\frac{\Gamma(1-a)\,\Gamma(b)}{\Gamma(b+1-a)}u_4$$

(28) $$e^{i\pi(b+1-c)}\frac{\Gamma(b+1-c)\,\Gamma(a)}{\Gamma(a+b+1-c)}u_2 = \frac{\Gamma(b+1-c)\,\Gamma(1-b)}{\Gamma(2-c)}u_5 + e^{i\pi(b+1-c)}\frac{\Gamma(a)\,\Gamma(1-b)}{\Gamma(a+1-b)}u_3$$

(29) $$e^{i\pi(c-a)}\frac{\Gamma(c-a)\,\Gamma(1-b)}{\Gamma(c+1-a-b)}u_6 = \frac{\Gamma(a)\,\Gamma(c-a)}{\Gamma(c)}u_1 + e^{i\pi(1-a)}\frac{\Gamma(1-b)\,\Gamma(a)}{\Gamma(a+1-b)}u_3$$

(30) $$e^{i\pi(a+1-c)}\frac{\Gamma(a+1-c)\,\Gamma(b)}{\Gamma(a+b+1-c)}u_2 = \frac{\Gamma(a+1-c)\,\Gamma(1-a)}{\Gamma(2-c)}u_5 + e^{i\pi(a+1-c)}\frac{\Gamma(b)\,\Gamma(1-a)}{\Gamma(b+1-a)}u_4$$

(31) $$e^{i\pi(1-b)} \frac{\Gamma(1-b)\,\Gamma(c-a)}{\Gamma(c+1-a-b)} u_6 = \frac{\Gamma(1-b)\,\Gamma(b+1-c)}{\Gamma(2-c)} u_5 + e^{i\pi(1-b)} \frac{\Gamma(c-a)\,\Gamma(b+1-c)}{\Gamma(b+1-a)} u_4$$

(32) $$e^{i\pi(1-a)} \frac{\Gamma(1-a)\,\Gamma(c-b)}{\Gamma(c+1-a-b)} u_6 = \frac{\Gamma(1-a)\,\Gamma(a+1-c)}{\Gamma(2-c)} u_5 + e^{i\pi(1-a)} \frac{\Gamma(c-b)\,\Gamma(a+1-c)}{\Gamma(a+1-b)} u_3$$

(33) $$u_1 = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} u_2 + \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)} u_6$$

(34) $$= \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(c-a)\,\Gamma(b)} u_3 + \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(c-b)\,\Gamma(a)} u_4$$

(35) $$u_2 = \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(a+1-c)\,\Gamma(b+1-c)} u_1 + \frac{\Gamma(a+b+1-c)\,\Gamma(c-1)}{\Gamma(a)\,\Gamma(b)} u_5$$

(36) $$= \frac{\Gamma(a+b+1-c)\,\Gamma(b-a)}{\Gamma(b+1-c)\,\Gamma(c)} e^{-i\pi a} u_3 + \frac{\Gamma(a+b+1-c)\,\Gamma(a-b)}{\Gamma(a+1-c)\,\Gamma(a)} e^{-i\pi b} u_4$$

(37) $$u_3 = \frac{\Gamma(1-c)\,\Gamma(a+1-b)}{\Gamma(1-b)\,\Gamma(a+1-c)} u_1 - \frac{\Gamma(c)\,\Gamma(1-c)\,\Gamma(a+1-b)}{\Gamma(2-c)\,\Gamma(c-b)\,\Gamma(a)} e^{i\pi(c-1)} u_5$$

(38) $$= \frac{\Gamma(c+1-a-b)\,\Gamma(a+b-c)\,\Gamma(a+1-b)}{\Gamma(1-b)\,\Gamma(c-b)\,\Gamma(a+b+1-c)} e^{i\pi a} u_2 - \frac{\Gamma(a+b-c)\,\Gamma(a+1-b)}{\Gamma(a+1-c)\,\Gamma(a)} e^{i\pi(c-b)} u_6$$

(39) $$u_4 = \frac{\Gamma(1-c)\,\Gamma(1+b-a)}{\Gamma(1-a)\,\Gamma(1+b-c)} u_1 - \frac{\Gamma(c)\,\Gamma(1-c)\,\Gamma(b+1-a)}{\Gamma(2-c)\,\Gamma(c-a)\,\Gamma(b)} e^{i\pi(c-1)} u_5$$

(40) $$= \frac{\Gamma(c+1-a-b)\,\Gamma(a+b-c)\,\Gamma(b+1-a)}{\Gamma(1-a)\,\Gamma(c-a)\,\Gamma(a+b+1-c)} e^{i\pi b} u_2 - \frac{\Gamma(a+b-c)\,\Gamma(b+1-a)}{\Gamma(b+1-c)\,\Gamma(b)} e^{i\pi(c-a)} u_6$$

(41) $$u_5 = \frac{\Gamma(2-c)\,\Gamma(c-a-b)}{\Gamma(1-a)\,\Gamma(1-b)}\,u_2 + \frac{\Gamma(2-c)\,\Gamma(a+b-c)}{\Gamma(a+1-c)\,\Gamma(b+1-c)}\,u_6$$

(42) $$= \frac{\Gamma(2-c)\,\Gamma(b-a)}{\Gamma(1-a)\,\Gamma(b+1-c)}\,e^{i\pi(1-c)}\,u_3 + \frac{\Gamma(2-c)\,\Gamma(a-b)}{\Gamma(1-b)\,\Gamma(a+1-c)}\,e^{i\pi(1-c)}\,u_4$$

(43) $$u_6 = \frac{\Gamma(c+1-a-b)\,\Gamma(1-c)}{\Gamma(1-a)\,\Gamma(1-b)}\,u_1 + \frac{\Gamma(c+1-a-b)\,\Gamma(c-1)}{\Gamma(c-a)\,\Gamma(c-b)}\,u_5$$

(44) $$= \frac{\Gamma(c+1-a-b)\,\Gamma(b-a)}{\Gamma(1-a)\,\Gamma(c-a)}\,e^{-i\pi(c-b)}\,u_3 + \frac{\Gamma(c+1-a-b)\,\Gamma(a-b)}{\Gamma(1-b)\,\Gamma(c-b)}\,e^{-i\pi(c-a)}\,u_4$$

These relations hold for all values of a, b, c, for which the gamma factors of the numerators are finite and for all values of z for which the series involved converge and $\operatorname{Im} z > 0$.

The first eight of these relations connect three of the functions u_1, $u_2, \ldots, u_6$ which are not defined in the same domain. The last twelve express a function which is defined in one domain D by two other functions which are both defined in one and the same domain D' where $D = D'$. These formulas can be used for analytic continuation.

If $\operatorname{Im} z < 0$ the signs of the arguments of the exponential functions in (25) to (44) must be changed.

2.10. Analytic continuation

The fundamental formulas for the analytic continuation of the hypergeometric series are given next. General case is where $1-c$, $b-a$, and $c-b-a$ are not integers. [Cf. also 2.9(1) to 2.9(4) and 2.9(33) to 2.9(44).]

(1) $$F(a, b; c; z) = A_1\, F(a, b; a+b-c+1; 1-z) + A_2\,(1-z)^{c-a-b}\, F(c-a, c-b; c-a-b+1; 1-z) \qquad |\arg(1-z)| < \pi$$

(2) $$F(a, b; c; z) = B_1\,(-z)^{-a}\, F(a, 1-c+a; 1-b+a; z^{-1}) + B_2\,(-z)^{-b}\, F(b, 1-c+b; 1-a+b; z^{-1}) \qquad |\arg(-z)| < \pi$$

(3) $$F(a, b; c; z) = B_1 (1-z)^{-a} F[a, c-b; a-b+1; (1-z)^{-1}] + B_2 (1-z)^{-b} F[b, c-a; b-a+1; (1-z)^{-1}] \qquad |\arg(1-z)| < \pi$$

(4) $$F(a, b; c; z) = A_1 z^{-a} F(a, a+1-c; a+b+1-c; 1-z^{-1}] + A_2 z^{a-c} (1-z)^{c-a-b} F(c-a, 1-a; c+1-a-b; 1-z^{-1}] \qquad |\arg z| < \pi$$

The coefficients A_1, A_2, B_1, and B_2 are given below:

(5) $$\begin{cases} A_1 = \dfrac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, & A_2 = \dfrac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)} \\[2ex] B_1 = \dfrac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}, & B_2 = \dfrac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)} \end{cases}$$

(6) $$F(a, b; c; z) = (1-z)^{-a} F[a, c-b; c; z/(z-1)] = (1-z)^{-b} F[b, c-a; c; z/(z-1)]$$

Analytic continuation of $F(a, b; c; z)$ in the logarithmic cases follows next. The letters l, m, n, denote non-negative integers.

(7) $$F(a, a+m; c; z)\,\Gamma(a+m)/\Gamma(c) = \frac{(-z)^{-a-m}}{\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_{n+m}\,(1-c+a)_{n+m}}{n!\,(n+m)!} z^{-n} [\log(-z) + h_n] + (-z)^{-a} \sum_{n=0}^{m-1} \frac{\Gamma(m-n)\,(a)_n}{\Gamma(c-a-n)\,n!} z^{-n}$$

$-\pi < \arg(-z) < \pi$, $\quad c-a$ not an integer

(8) $$h_n = \psi(1+m+n) + \psi(1+n) - \psi(a+m+n) - \psi(c-a-m-n) = \psi(1+m+n) + \psi(1+n) - \psi(a+m+n) - \psi(1-c+a+n+m) + \pi \operatorname{ctn} \pi(c-a)$$

if $\sum_{n=0}^{m-1}$ is to be interpreted as zero when $m = 0$.

(9) $$F(a, a+m; a+m+l+1; z)\,\Gamma(a+m)/\Gamma(a+m+l+1) = (-1)^{m+l+1} (-z)^{-a-m} \sum_{n=l+1}^{\infty} \frac{(a)_{n+m}\,(n-l-1)!}{(n+m)!\,n!} z^{-n} + (-z)^{-a} \sum_{n=0}^{m-1} \frac{(m-n-1)!\,(a)_n}{(m+l-n)!\,n!} z^{-n}$$

$$+\frac{(-z)^{-a-m}}{(l+m)!}\sum_{n=0}^{l}\frac{(a)_{n+m}(-m-l)_{n+m}}{(n+m)!\,n!}z^{-n}[\log(-z)+h_n']$$

$$-\pi<\arg(-z)<\pi$$

(10) $h_n' = \psi(1+m+n)+\psi(1+n)-\psi(a+m+n)-\psi(l+1-n)$

(11) $F(n+1,\, n+m+1;\, n+m+l+2;\, z)$

$$=\frac{(n+m+l+1)!\,(-1)^{1+m}}{l!\,n!\,(n+m)!\,(m+l)!}\frac{d^{n+m}}{dz^{n+m}}\left\{(1-z)^{m+l}\frac{d^l}{dz^l}[z^{-1}\log(1-z)]\right\}$$

(12) $F(a,\, b;\, a+b+m;\, z)/\Gamma(a+b+m)$

$$=\frac{\Gamma(m)}{\Gamma(a+m)\,\Gamma(b+m)}\sum_{n=0}^{m-1}\frac{(a)_n(b)_n}{(1-m)_n\,n!}(1-z)^n$$

$$+\frac{(1-z)^m(-1)^m}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a+m)_n(b+m)_n}{(n+m)!\,n!}[h_n''-\log(1-z)](1-z)^n$$

$$-\pi<\arg(1-z)<\pi,\qquad a,\, b,\neq 0,\,-1,\,-2,\,\ldots$$

(13) $h_n'' = \psi(n+1)+\psi(n+m+1)-\psi(a+n+m)-\psi(b+n+m)$

and $\sum_{n=0}^{m-1}$ is to be interpreted as zero when $m=0$.

(14) $F(a,\, b;\, a+b-m;\, z)/\Gamma(a+b-m)$

$$=\frac{\Gamma(m)\,(1-z)^{-m}}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{m-1}\frac{(a-m)_n(b-m)_n}{(1-m)_n\,n!}(1-z)^n$$

$$+\frac{(-1)^m}{\Gamma(a-m)\,\Gamma(b-m)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n+m)!\,n!}[\overline{h}_n-\log(1-z)](1-z)^n$$

$$-\pi<\arg(1-z)<\pi,\qquad a,\, b,\neq 0,\,-1,\,-2,\,\ldots$$

(15) $\overline{h}_n = \psi(1+n)+\psi(1+n+m)-\psi(a+n)+\psi(b+n)$

and $\sum_{n=0}^{m-1}$ is to be interpreted as zero when $m=0$.

For the *degenerate case* (one of the numbers a, b, $c-a$, $c-b$ is an integer) see section 2.2.2.

2.11. Quadratic and higher transformations

All quadratic transformations can be derived from the linear transformations in section 2.9 and the special transformations (cf. section 2.1.5 for the range of validity).

(1) $F(a,\, b;\, a-b+1;\, z)$

$$=(1-z)^{-a}F[\tfrac{1}{2}a,\, -b+(a+1)/2;\, 1+a-b;\, -4z(1-z)^{-1}]$$

(2) $F(2a, 2b; a+b+\tfrac{1}{2}; z) = F[a, b; a+b+\tfrac{1}{2}; 4z(1-z)]$

(3) $$F(2a, 2b; a+b+\tfrac{1}{2}; \tfrac{1}{2}+\tfrac{1}{2}z) = \frac{\Gamma(a+b+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})} F(a, b; \tfrac{1}{2}; z^2)$$
$$- z\,\frac{\Gamma(a+b+\frac{1}{2})\,\Gamma(-\frac{1}{2})}{\Gamma(a)\,\Gamma(b)} F(a+\tfrac{1}{2}, b+\tfrac{1}{2}; 3/2; z^2)$$

(4) $F(a, b; 2b; z) = (1-\tfrac{1}{2}z)^{-a} F\{\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a; b+\tfrac{1}{2}; [z/(2-z)]^2\}$

(5) $F[a, b; 2b; 4z(1+z)^{-2}] = (1+z)^{2a} F(a, a+\tfrac{1}{2}-b; b+\tfrac{1}{2}; z^2)$

(6) $F(a, a+\tfrac{1}{2}; b; 2z-z^2) = (1-\tfrac{1}{2}z)^{-2a} F[2a, 2a-b+1; b; z/(2-z)]$

Goursat's table of quadratic transformations. The square roots are defined in such a way that their value becomes real and positive if z is real and $0 \le z < 1$. All formulas are valid in a neighborhood of $z = 0$.

(7) $$\frac{2\,\Gamma(\frac{1}{2})\,\Gamma(a+b+\frac{1}{2})}{\Gamma(a+\frac{1}{2})\,\Gamma(b+\frac{1}{2})} F(a, b; \tfrac{1}{2}; z)$$
$$= F[2a, 2b; a+b+\tfrac{1}{2}; \tfrac{1}{2}(1+z^{1/2})]$$
$$+ F[2a, 2b; a+b+\tfrac{1}{2}; \tfrac{1}{2}(1-z^{1/2})]$$

(8) $$\frac{2\,\Gamma(\frac{1}{2})\,\Gamma(a+1-b)}{\Gamma(a+\frac{1}{2})\,\Gamma(1-b)} (1+z)^a F(a, b; \tfrac{1}{2}; -z)$$
$$= F[2a, 1-2b; a+1-b; \tfrac{1}{2}+\tfrac{1}{2}z^{1/2}(1+z)^{-1/2}]$$
$$+ F[2a, 1-2b; a+1-b; \tfrac{1}{2}-\tfrac{1}{2}z^{1/2}(1+z)^{-1/2}]$$

(9) $$\frac{2\,\Gamma(-\frac{1}{2})\,\Gamma(a+b-\frac{1}{2})}{\Gamma(a-\frac{1}{2})\,\Gamma(b-\frac{1}{2})} z^{1/2} F(a, b; 3/2; z)$$
$$= F(2a-1, 2b-1; a+b-\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{2}z^{1/2})$$
$$- F(2a-1, 2b-1; a+b-\tfrac{1}{2}; \tfrac{1}{2}+\tfrac{1}{2}z^{1/2})$$

(10) $F(a, b; a+b+\tfrac{1}{2}; z) = F[2a, 2b; a+b+\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{2}(1-z)^{1/2}]$

(11) $$F(a, b; a+b+\tfrac{1}{2}; z)$$
$$= [\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{1/2}]^{-2a} F\left[2a, a-b+\tfrac{1}{2}; a+b+\tfrac{1}{2}; \frac{(1-z)^{1/2}-1}{(1-z)^{1/2}+1}\right]$$

(12) $$F(a, b; a+b+\tfrac{1}{2}; -z)$$
$$= [(1+z)^{1/2}+z^{1/2}]^{-2a} F[2a, a+b; 2a+2b; 2(z+z^2)^{1/2}-2z]$$

(13) $$F(a, b; a+b-\tfrac{1}{2}; z)$$
$$= (1-z)^{-1/2} F[2a-1, 2b-1; a+b-\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{2}(1-z)^{1/2}]$$

(14) $F(a, b; a+b-\tfrac{1}{2}; z) = (1-z)^{-\frac{1}{2}}[\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{\frac{1}{2}}]^{1-2a}$
$$\times F\left[2a-1, a-b+\tfrac{1}{2}; a+b-\tfrac{1}{2}; \frac{(1-z)^{\frac{1}{2}}-1}{(1-z)^{\frac{1}{2}}+1}\right]$$

(15) $F(a, b; a+b-\tfrac{1}{2}; -z) = (1+z)^{-\frac{1}{2}}[(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{1-2a}$
$$\times F[2a-1, a+b-1; 2a+2b-2; 2(z+z^2)^{\frac{1}{2}}-2z]$$

(16) $F(a, a+\tfrac{1}{2}; c; z)$
$$= (1-z)^{-a}F[2a, 2c-2a-1; c; \tfrac{1}{2}-\tfrac{1}{2}(1-z)^{-\frac{1}{2}}]$$

(17) $F(a, a+\tfrac{1}{2}; c; z)$
$$= (1+z^{\frac{1}{2}})^{-2a}F[2a, c-\tfrac{1}{2}; 2c-1; 2z^{\frac{1}{2}}(1+z^{\frac{1}{2}})^{-1}]$$

(18) $F[a, b; (a+b+1)/2; z] = F[\tfrac{1}{2}a, \tfrac{1}{2}b; (a+b+1)/2; 4z(1-z)]$

(19) $F[a, b; (a+b+1)/2; z]$
$$= (1-2z)F[\tfrac{1}{2}+\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}b; (a+b+1)/2; 4z(1-z)]$$

(20) $F[a, b; (a+b+1)/2; z] = (1-2z)^{-a}$
$$= F[\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a; (a+b+1)/2; 4z(z-1)(2z-1)^{-2}]$$

(21) $F[a, b; (a+b+1)/2; -z] = [(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{-2a}$
$$= F\{a, \tfrac{1}{2}a+\tfrac{1}{2}b; a+b; 4z^{\frac{1}{2}}(z+1)^{\frac{1}{2}}[(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{-2}\}$$

(22) $F(a, 1-a; c; z)$
$$= (1-z)^{c-1}F[\tfrac{1}{2}c-\tfrac{1}{2}a, (c+a-1)/2; c; 4z(1-z)]$$

(23) $$= (1-z)^{c-1}(1-2z)F[\tfrac{1}{2}c+\tfrac{1}{2}a, (c+1-a)/2; c; 4z(1-z)]$$

(24) $F(a, 1-a; c; z) = (1-z)^{c-1}(1-2z)^{a-c}$
$$\times F[\tfrac{1}{2}c-\tfrac{1}{2}a, (c+1-a)/2; c; 4z(z-1)(1-2z)^{-2}]$$

(25) $F(a, 1-a; c; -z) = (1+z)^{c-1}[(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{2-2a-2c}$
$$\times F\{c+a-1, c-\tfrac{1}{2}; 2c-1; 4z^{\frac{1}{2}}(1+z)^{\frac{1}{2}}[(1+z)^{\frac{1}{2}}+z^{\frac{1}{2}}]^{-2}\}$$

(26) $F(a, b; 2b; z)$
$$= (1-z)^{-\frac{1}{2}a}F[\tfrac{1}{2}a, b-\tfrac{1}{2}a; b+\tfrac{1}{2}; (z^2/4)(z-1)^{-1}]$$

(27) $$= (1-\tfrac{1}{2}z)(1-z)^{-\frac{1}{2}a-\frac{1}{2}}F[b+\tfrac{1}{2}-\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a; b+\tfrac{1}{2}; z^2/(4z-4)]$$

(28) $$= (1-\tfrac{1}{2}z)^{-a}F[\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a; b+\tfrac{1}{2}; z^2(2-z)^{-2}]$$

(29) $$= (1-z)^{b-a}(1-\tfrac{1}{2}z)^{a-2b}F[b-\tfrac{1}{2}a, b+\tfrac{1}{2}-\tfrac{1}{2}a; b+\tfrac{1}{2}; z^2/(2-z)^{-2}]$$

(30) $F(a, b; 2b; z) = (1-z)^{-\frac{1}{2}a}$
$\times F\{a, 2b-a; b+\tfrac{1}{2}; (-\tfrac{1}{4})(1-z)^{-\frac{1}{2}}[1-(1-z)^{\frac{1}{2}}]^2\}$

(31) $F(a, b; 2b; z) = [\tfrac{1}{2}+\tfrac{1}{2}(1-z)^{\frac{1}{2}}]^{-2a}$
$$\times F\left\{a, a-b+\tfrac{1}{2}; b+\tfrac{1}{2}; \left[\frac{1-(1-z)^{\frac{1}{2}}}{1+(1-z)^{\frac{1}{2}}}\right]^2\right\}$$

(32) $F(a, b; a-b+1; z) = (1-z)^{-a}$
$\times F[\tfrac{1}{2}a, (a+1-2b)/2; a-b+1; -4z(1-z)^{-2}]$

(33) $F(a, b; a-b+1; z) = (1+z)(1-z)^{-a-1}$
$\times F[\tfrac{1}{2}+\tfrac{1}{2}a, \tfrac{1}{2}a+1-b; a-b+1; -4z(1-z)^{-2}]$

(34) $F(a, b; a-b+1; z) = (1+z)^{-a}$
$\times F[\tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}; a-b-1; \;\; 4z(1+z)^{-2}]$

(35) $F(a, b; a-b+1; z) = (1-z)^{1-2b}(1+z)^{2b-a-1}$
$\times F[(a+1-2b)/2, (a-2b+2)/2; a+1-b; 4z(1+z)^{-2}]$

(36) $F(a, b; a-b+1; z) = (1+z^{\frac{1}{2}})^{-2a}$
$\times F[a, a-b+\tfrac{1}{2}; 2a-2b+1; 4z^{\frac{1}{2}}(1+z^{\frac{1}{2}})^{-2}]$

Cubic transformations. These can be reduced to (37), (38), or (39) by using 2.10(1-6) and 2.11(1-6).

(37) $$2\pi(1-z^3)^a(-z)^{-3a}\left[\frac{F(a, a+1/3; 2/3; z^{-3})}{\Gamma(2/3)\,\Gamma(a+2/3)} + \frac{e^{i\pi/3}}{z}\;\frac{F(a+1/3, a+2/3; 4/3; z^{-3})}{\Gamma(4/3)\,\Gamma(a)}\right] = 2\pi(1-z^3)^a e^{-i\pi a}$$
$$\times\left[\frac{F(a, a+1/3; 2/3; z^3)}{\Gamma(2/3)\,\Gamma(a+2/3)} + ze^{-i\pi/3}\,\frac{F(a+1/3, a+2/3; 4/3; z^3)}{\Gamma(4/3)\,\Gamma(a)}\right]$$

(38) $= 3^{(3a+1)/2}e^{\pm i\pi a/2}\,\Gamma(a+1/3)(-w)^{-2a}(1-w)^a$
$\times[\Gamma(2a+2/3)]^{-1}F(a+1/3, 3a; 2a+2/3; w^{-1})$
$w = \epsilon(1-z)/(z-\epsilon^2), \qquad \epsilon = e^{2i\pi/3}$

The $\pm$ sign stands according as $\operatorname{Im}(-w) \gtrless 0$.
We also have:

$|\arg(-z)| < \pi/3, \quad |\arg(1-z^3)| < \pi, \quad |\arg(-w)| < \pi,$
$|\arg(1-w)| < \pi; \quad |w| > 1, \quad \operatorname{Re} w > 1/2.$

(39) $F[3a/2, (3a-1)/2; a+\tfrac{1}{2}; -z^{2/3}]$

$$= (1+z)^{1-3a} F[a-1/3, a; 2a; 2z(3+z^2)(1+z)^{-3}].$$

E. Goursat (1881) gives an extensive list of cubic transformations. Of these, the rational ones only will be repeated here:

(40) $F(3a, 3a+\tfrac{1}{2}; 4a+2/3; z) = (1-9z/8)^{-2a}$

$$\times F[a, a+\tfrac{1}{2}; a+5/6; -27z^2(1-z)(9z-8)^{-2}]$$

(41) $F(3a, 3a+\tfrac{1}{2}; 2a+5/6; z) = (1-9z)^{-2a}$

$$\times F[a, a+\tfrac{1}{2}; 2a+5/6; -27z(1-z)^2(1-9z)^{-2}]$$

(42) $F(3a, a+1/6; 4a+2/3; z) = (1-z/4)^{-3a}$

$$\times F[a, a+1/3; 2a+5/6; -27z^2(z-4)^{-3}]$$

(43) $F(3a, 1/3-a; 2a+5/6; z) = (1-4z)^{-3a}$

$$\times F[a, a+1/3; 2a+5/6; 27z(4z-1)^{-3}]$$

(44) $F(3a, 1/3-a; 1/2; z) = (1-z)^{-a}$

$$\times F[a, 1/6-a; 1/2; (z/27)(9-8z)^2(1-z)^{-1}]$$

(45) $F(3a, a+1/6; 1/2; z) = (1-z)^{-2a}$

$$\times F[a, 1/6-a; 1/2; -(z/27)(z-9)^2(1-z)^{-2}]$$

(46) $F(3a+1/2, 5/6-a; 3/2; z) = (1-8z/9)(1-4z/3)^{-3a-3/2}$

$$\times F[a+1/2, a+5/6; 3/2; z(9-8z)^2(4z-3)^{-3}]$$

(47) $F(3a, +1/2, a+2/3; 3/2; z) = (1-z/9)(1+z/3)^{-3a-3/2}$

$$\times F[a+1/2, a+5/6; 3/2; z(z-9)^2(z+3)^{-3}].$$

2.12. Integrals

(1) $$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

$\operatorname{Re} c > \operatorname{Re} b > 0, \quad |\arg(1-z)| < \pi$

(2) $F(a, b; c; z)$

$$= \frac{i\Gamma(c)\exp[i\pi(b-c)]}{2\Gamma(b)\,\Gamma(c-b)\sin\pi(c-b)} \int_0^{(1+)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

$\operatorname{Re} b > 0, \quad |\arg(1-z)| < \pi, \quad c-b \neq 1, 2, 3, \ldots$

(3) $F(a, b; c; z)$

$$= \frac{-i\Gamma(c)\exp(-i\pi b)}{2\Gamma(b)\,\Gamma(c-b)\sin(\pi b)} \int_1^{(0+)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

$\operatorname{Re} c > \operatorname{Re} b, \quad |\arg(-z)| < \pi, \quad b \neq 1, 2, 3, \ldots$

(4) $$F(a, b; c; z) = \frac{-\Gamma(c)\exp(-i\pi c)}{4\Gamma(b)\,\Gamma(c-b)\,\sin(\pi b)\,\sin\pi(c-b)} \times \int^{(1+,0+,1-,0-)} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

$b,\ c-b \neq 1, 2, 3, \ldots$

(5) $$F(a, b; c; 1-z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^\infty s^{b-1}(1+s)^{a-c}(1+sz)^{-a}\,ds$$

$\operatorname{Re} c > \operatorname{Re} b > 0, \quad |\arg z| < \pi$

(6) $$F(a, b; c; z^{-1}) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_1^\infty (s-1)^{c-b-1}\, s^{a-c}(s-z^{-1})^{-a}\,ds$$

$1 + \operatorname{Re} a > \operatorname{Re} c > \operatorname{Re} b, \quad |\arg(z-1)| < \pi$

(7) $$F(a, b; c; z) = \frac{2\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^{\pi/2} \frac{(\sin t)^{2b-1}(\cos t)^{2c-2b-1}}{(1-z\sin^2 t)^a}\,dt$$

(8) $$F(a, b; c; z) = \frac{2^{1-c}\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^{\pi} \frac{(\sin t)^{2b-1}(1+\cos t)^{c-2b}}{(1-\tfrac{1}{2}z+\tfrac{1}{2}z\cos t)^a}\,dt$$

$$= \frac{2^{1-c}\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^{\pi} \frac{(\sin t)^{2c-2b-1}(1-\cos t)^{2b-c}}{(1-\tfrac{1}{2}z+\tfrac{1}{2}z\cos t)^a}\,dt$$

(9) $$F(a, b; c; z) = \frac{2\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^{\infty} \frac{(\cosh v)^{2a-2c+1}(\sinh v)^{2c-2b-1}}{[(\cosh v)^2 - z]^a}\,dv$$

(10) $$F(a, b; c; z) = \frac{2^{b-a}\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^{\infty} \frac{(\sinh t)^{2a-2c+1}(\cosh t - 1)^{2c-a-b-1}}{(\tfrac{1}{2} - z + \tfrac{1}{2}\cosh t)^a}\,dt$$

(11) $$F(a, b; c; z) = \frac{2^{b-a}\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^{\infty} \frac{(\sinh t)^{2a-2b-1}(\cosh t + 1)^{a+b-2c+1}}{(\tfrac{1}{2} - z + \tfrac{1}{2}\cosh t)^a}\,dt$$

(12) $$F(a, b; c; z) = \frac{2\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^{\infty} \frac{(\sinh v)^{2b-1}(\cosh v)^{2a-2c+1}}{[(\cosh v)^2 - z(\sinh v)^2]^a}\,dv$$

(13) $F(a, b; c; z)$

$$= \frac{2^{c-b}\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^\infty \frac{(\sinh t)^{2b-1}\,(\cosh t + 1)^{a-c-b+1}}{[1 + z + (1-z)\cosh t]^{a}}\,dt$$

(14) $F(a, b; c; z)$

$$= \frac{2^{c-b}\,\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^\infty \frac{(\sinh t)^{2a-2c+1}\,(\cosh t - 1)^{b+c-a-1}}{[(1+z) + (1-z)\cosh t]^{a}}\,dt$$

(15) $F(a, b; c; z)$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^\infty e^{-bt}\,(1-e^{-t})^{c-b-1}\,(1-ze^{-t})^{-a}\,dt.$$

The conditions for the validity of (7) to (15) are

$\mathrm{Re}\, c > \mathrm{Re}\, b > 0.$

For other integrals see 2.1(15), 2.4(1) to 2.4(10), 2.1(34), 2.1(35) and 3.7 where the Legendre functions may be expressed by special hypergeometric functions. For integrals which lead to hypergeometric functions see also Chap. 7 (Sonine-Schafheitlin and related integrals).

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CHAPTER III

LEGENDRE FUNCTIONS

3.1. Introduction

The expression $(a^2 - 2ar\cos\gamma + r^2)^{-1/2}$ represents the potential at a point P of a source situated at A when r and a are the distances respectively of P and A from a point O, and γ is the angle subtended by PA at O. The expansion of this expression in ascending powers of r is of the form

$$(1)\quad (1/a)\sum_{n=0}^{\infty} P_n(\cos\gamma)\,(r/a)^n \qquad 0 < r < a,$$

where the coefficients P_n $(\cos\gamma)$ depend on $\cos\gamma$ only (they are independent of a and r) and can be shown to be polynomials of degree n in $\cos\gamma$. They were introduced in 1784 by Legendre and are known as *Legendre polynomials.*

Legendre polynomials and related functions occur for example when Laplace's equation $\Delta V = 0$, the wave equation, or the diffusion equation are discussed in spherical polar coordinates r, θ, ϕ, defined by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

In these coordinates

$$\Delta V = r^{-2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + r^{-2}(\sin\theta)^{-1}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + (r\sin\theta)^{-2}\frac{\partial^2 V}{\partial\phi^2},$$

and if a solution of $\Delta V = 0$ is of the form $V = R(r)\,T(\theta)\,F(\phi)$, where R, T, F depend respectively on r, θ, ϕ only, T must satisfy the ordinary differential equation

$$(2)\quad \frac{d^2T}{d\theta^2} + \cot\theta\frac{dT}{d\theta} + [\nu(\nu+1) - (\mu\csc\theta)^2]\,T = 0$$

in which μ and ν are separation constants. The substitution $\zeta = \cos\theta$

reduces (2) to Legendre's equation of degree ν and order μ

$$(3)\quad (1-\zeta^2)\frac{d^2T}{d\zeta^2} - 2\zeta\frac{dT}{d\zeta} + [\nu(\nu+1) - \mu^2(1-\zeta^2)^{-1}]\,T = 0.$$

The same differential equation arises in potential problems in spheroidal and toroidal coordinates; see sections 3.13 and 3.14.

In spherical polar coordinates $\zeta = \cos\theta$ is real and between -1 and 1; and if the potential is to be one-valued and continuous with continuous partial derivatives on the surface r = constant of a sphere, it can be shown that μ and ν must be integers. However, (3) arises in other connections when the restriction of ζ to the interval $(-1, 1)$ and of μ, ν to integer values is inappropriate, and therefore we shall investigate the solutions of (3) for unrestricted real or complex values of ζ, μ, ν.

The Legendre equation also arises in the theory of hypergeometric functions. In that theory it is found that whenever Gauss' hypergeometric series admits of a quadratic transformation, the hypergeometric differential equation may be reduced to (3).

An entirely different approach is given by the theory of orthogonal polynomials. Legendre polynomials $P_n(\zeta)$ are the orthogonal polynomials associated with the weight function unity on the interval $(-1, 1)$: hence their occurrence in interpolation theory and in mechanical quadrature. The orthogonal polynomials with the weight function $(1-\zeta^2)^{\alpha}$ on the interval $-1 \le \zeta \le 1$ may also be expressed in terms of Legendre functions (cf. Chapter 10).

In the present chapter, the differential equation (3) will be the basis for the study of Legendre functions.

3.2. The solutions of Legendre's differential equation

The Legendre functions are solutions of Legendre's differential equation

$$(1)\quad (1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + [\nu(\nu+1) - \mu^2(1-z^2)^{-1}]\,w = 0$$

z, ν, μ, unrestricted.

Under the substitution $w = (z^2-1)^{\frac{1}{2}\mu}\,v$, (1) becomes

$$(2)\quad (1-z^2)\frac{d^2v}{dz^2} - 2(\mu+1)\,z\,\frac{dv}{dz} + (\nu-\mu)(\nu+\mu+1)\,v = 0.$$

And, with $\zeta = \frac{1}{2} - \frac{1}{2}z$ as the independent variable, this differential equation becomes

$$\zeta(1-\zeta)\frac{d^2v}{d\zeta^2} + (\mu+1)(1-2\zeta)\frac{dv}{d\zeta} + (\nu-\mu)(\nu+\mu+1)\,v = 0.$$

This is Gauss' equation 2.1(1) with $a = \mu - \nu$, $b = \mu + \nu + 1$, and $c = \mu + 1$.

Hence by 2.3(1) it follows that the function

$$(3)\quad w = P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} F(-\nu, \nu+1; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2}z) \qquad |1-z| < 2$$

is a solution of (1).

If we set $\zeta = z^2$, (2) becomes

$$(4)\quad 4\zeta(1-\zeta)\frac{d^2 v}{d\zeta^2} + [2-(4\mu+6)\zeta]\frac{dv}{d\zeta} - (\mu-\nu)(\mu+\nu+1)\,v = 0,$$

which is also of hypergeometric type with $a = \frac{1}{2}(\mu+\nu+1)$, $b = \frac{1}{2}(\mu-\nu)$, $c = \frac{1}{2}$. Hence equation (1) by 2.9(9) has a solution

$$(5)\quad w = Q_\nu^\mu(z) = e^{\mu i\pi}\, 2^{-\nu-1}\, \pi^{\frac{1}{2}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)} z^{-\nu-\mu-1}(z^2-1)^{\frac{1}{2}\mu}$$

$$\times F(\tfrac{1}{2}\nu + \tfrac{1}{2}\mu + 1, \tfrac{1}{2}\nu + \tfrac{1}{2}\mu + \tfrac{1}{2}; \nu + 3/2; z^{-2}) \qquad |z| > 1.$$

The functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are known as the Legendre functions of the first and second kind, respectively. They are one-valued and regular in the z-plane supposed cut along the real axis from 1 to $-\infty$. We assume

$$(6)\quad |\arg(z \pm 1)| < \pi, \qquad |\arg z| < \pi$$

$$\text{and}\quad (z^2-1)^{\frac{1}{2}\mu} = (z-1)^{\frac{1}{2}\mu}(z+1)^{\frac{1}{2}\mu}.$$

Legendre's differential equation remains unchanged if μ is replaced by $-\mu$, z by $-z$, and ν by $-\nu-1$. Therefore

$$P_\nu^{\pm\mu}(\pm z), \quad Q_\nu^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad Q_{-\nu-1}^{\pm\mu}(\pm z)$$

are solutions of (1) (cf. also section 3.3.1).

Applying 2.1(23) to (3) and (5) we obtain

$$(7)\quad \Gamma(1-\mu)\,P_\nu^\mu(z) = 2^\mu (z^2-1)^{-\frac{1}{2}\mu} F(1-\mu+\nu, -\mu-\nu; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2}z)$$

$$(8)\quad \Gamma(\nu+3/2)\,Q_\nu^\mu(z) = e^{i\mu\pi}\, 2^{-\nu-1}\, \pi^{\frac{1}{2}}\, \Gamma(\nu+\mu+1)\, z^{-1-\nu+\mu}(z^2-1)^{-\frac{1}{2}\mu}$$

$$\times F(\tfrac{1}{2} + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu, 1 + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu; \nu + 3/2; z^{-2}).$$

(If ν or μ or both are positive integers, see section 3.6.)

By means of the transformation formulas of the hypergeometric function, given in section 2.10, (3) and (5) are expressible in several ways in the forms

(9) $P_\nu^\mu(z) = A_1\ F(a_1, b_1; c_1, \zeta) + A_2\ F(a_2, b_2; c_2; \zeta) \qquad |\zeta| < 1,$

(10) $e^{-i\mu\pi}\, Q_\nu^\mu(z) = A_3\ F(a_3, b_3; c_3; \zeta) + A_4\ F(a_4, b_4; c_4; \zeta) \qquad |\zeta| < 1$

where ζ is a function of z and depends on the choice of the transformation. The various expansions (9) and (10) are shown in (14) to (49). The last column in each table indicates the way in which the expansion in question has been derived. For example (15) results when the transformation 2.10(1) is applied to (14). The tables contain 36 different hypergeometric series, each being a solution of (1). If the transformation 2.1(23) is applied to each of these series, we find 36 other hypergeometric series. These constitute Olbricht's 72 solutions of the differential equation (1) (Olbricht, 1888, p. 1). In all these formulas

(11) $(z^2-1)^\alpha = (z-1)^\alpha (z+1)^\alpha, \qquad |\arg(z \pm 1)| < \pi, \quad |\arg z| < \pi,$

and therefore

(12) $-z-1 = e^{\mp i\pi}(z+1), \quad -z+1 = e^{\mp i\pi}(z-1),$

$1 - z^2 = e^{\mp i\pi}(z^2-1)$

where the upper or lower sign is to be taken according as Im $z \gtrless 0$.

From (1) it is found that the Wronskian

$$W\{P_\nu^\mu(z), Q_\nu^\mu(z)\} = P_\nu^\mu(z)\,\frac{d}{dz}\,Q_\nu^\mu(z) - Q_\nu^\mu(z)\,\frac{d}{dz}\,P_\nu^\mu(z)$$

must be of the form $c/(1-z^2)$, where the constant c may be evaluated by putting $z = 0$. Using (22) and (40) we thus obtain

(13) $W\{P_\nu^\mu(z), Q_\nu^\mu(z)\}$

$$= \frac{e^{i\mu\pi}\, 2^{2\mu}\, \Gamma(1 + \tfrac{1}{2}\mu + \tfrac{1}{2}\nu)\, \Gamma(\tfrac{1}{2} + \tfrac{1}{2}\mu + \tfrac{1}{2}\nu)}{(1-z^2)\, \Gamma(1 + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu)\, \Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu)}\,.$$

There follows a table of expansions of the form given by equations (9) and (10).

EXPANSIONS FOR $P_\nu^\mu(z)$

	A_1 / A_2
(14)	$(z+1)^{\frac{1}{2}\mu}(z-1)^{-\frac{1}{2}\mu}/\Gamma(1-\mu)$
	0
(15)	$\Gamma(-\mu)(z+1)^{\frac{1}{2}\mu}(z-1)^{-\frac{1}{2}\mu}/[\Gamma(1+\nu-\mu)\Gamma(-\nu-\mu)]$
	$-\pi^{-1}\sin(\nu\pi)\Gamma(\mu)(z-1)^{\frac{1}{2}\mu}(z+1)^{-\frac{1}{2}\mu}e^{\mp i\mu\pi}$
(16)	$2^{-\nu}(z+1)^{\frac{1}{2}\mu+\nu}(z-1)^{-\frac{1}{2}\mu}/\Gamma(1-\mu)$
	0
(17)	$-2^{\nu+1}\Gamma(-\mu)e^{\pm i\pi\nu}(z+1)^{\frac{1}{2}\mu}(z-1)^{-\frac{1}{2}\mu-\nu-1} \times[\Gamma(1+\nu-\mu)\Gamma(-\nu-\mu)]^{-1}$
	$\pi^{-1}2^{\nu+1}\sin(\nu\pi)\Gamma(\mu)e^{\pm i\pi(\nu-\mu)}(z-1)^{\frac{1}{2}\mu-\nu-1}(z+1)^{-\frac{1}{2}\mu}$
(18)	$2^{\nu+1}\Gamma(-1-2\nu)(z+1)^{\frac{1}{2}\mu-\nu-1}(z-1)^{-\frac{1}{2}\mu}/[\Gamma(-\nu)\Gamma(-\nu-\mu)]$
	$2^{-\nu}\Gamma(1+2\nu)(z+1)^{\frac{1}{2}\mu+\nu}(z-1)^{-\frac{1}{2}\mu}/[\Gamma(1+\nu)\Gamma(1+\nu-\mu)]$

EXPANSIONS FOR $P_\nu^\mu(z)$

	ζ	a_1 / a_2	b_1 / b_2	c_1 / c_2	Remarks
(14)	$\frac{1-z}{2}$	$-\nu$	$1+\nu$	$1-\mu$	
		...	...	...	
(15)	$\frac{1+z}{2}$	$-\nu$	$1+\nu$	$1+\mu$	(14), 2.10(1)
		$-\nu$	$1+\nu$	$1-\mu$	The upper or lower sign according as Im $z \gtrless 0$
(16)	$\frac{z-1}{z+1}$	$-\nu$	$-\nu-\mu$	$1-\mu$	(14), 2.10(6)
		...	...	...	
(17)	$\frac{z+1}{z-1}$	$1+\nu$	$1+\nu+\mu$	$1+\mu$	(15), 2.10(6)
		$1+\nu$	$1+\nu-\mu$	$1-\mu$	The upper or lower sign according as Im $z \gtrless 0$
(18)	$\frac{2}{1+z}$	$1+\nu$	$1+\nu-\mu$	$2+2\nu$	(16), 2.10(1)
		$-\nu$	$-\nu-\mu$	-2ν	

EXPANSIONS FOR $P_\nu^\mu(z)$

	A_1 / A_2
(19)	$2^{-\nu}\,\Gamma(1+2\nu)\,(z+1)^{\frac{1}{2}\mu}\,(z-1)^{-\frac{1}{2}\mu+\nu}/[\Gamma(1+\nu)\,\Gamma(1+\nu-\mu)]$
	$2^{\nu+1}\,\Gamma(-1-2\nu)\,(z+1)^{\frac{1}{2}\mu}\,(z-1)^{-\frac{1}{2}\mu-\nu-1}/[\Gamma(-\nu)\,\Gamma(-\nu-\mu)]$
(20)	$2^{\mu}\,(z^2-1)^{-\frac{1}{2}\mu}/\Gamma(1-\mu)$
	0
(21)	$2^{-\nu-1}\,\pi^{-\frac{1}{2}}\,\Gamma(-\tfrac{1}{2}-\nu)\,(z^2-1)^{-\frac{1}{2}\nu-\frac{1}{2}}/\Gamma(-\nu-\mu)$
	$2^{\nu}\,\pi^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{2}+\nu)\,(z^2-1)^{\frac{1}{2}\nu}/\Gamma(1+\nu-\mu)$
(22)	$2^{\mu}\,\pi^{\frac{1}{2}}\,(z^2-1)^{-\frac{1}{2}\mu}/[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)\,\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)]$
	$-\pi^{\frac{1}{2}}\,2^{\mu+1}\,z(z^2-1)^{-\frac{1}{2}\mu}/[\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)\,\Gamma(-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)]$
(23)	$2^{-\nu-1}\,\pi^{-\frac{1}{2}}\,\Gamma(-\tfrac{1}{2}-\nu)\,z^{-\nu+\mu-1}\,(z^2-1)^{-\frac{1}{2}\mu}/\Gamma(-\nu-\mu)$
	$2^{\nu}\,\pi^{-\frac{1}{2}}\,\Gamma(\tfrac{1}{2}+\nu)\,z^{\nu+\mu}\,(z^2-1)^{-\frac{1}{2}\mu}/\Gamma(1+\nu-\mu)$

EXPANSIONS FOR $P_\nu^\mu(z)$

	ζ	a_1 / a_2	b_1 / b_2	c_1 / c_2	Remarks
(19)	$\frac{2}{1-z}$	$-\nu$	$-\nu+\mu$	-2ν	(14), 2.10(2)
		$1+\nu$	$1+\nu+\mu$	$2+2\nu$	
(20)	$1-z^2$	$\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu$	$-\frac{1}{2}\nu-\frac{1}{2}\mu$	$1-\mu$	(15), 2.11(2)
		...	...	...	Re $z>0$
(21)	$\frac{1}{1-z^2}$	$\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu$	$\nu+3/2$	(20), 2.10(2)
		$-\frac{1}{2}\nu+\frac{1}{2}\mu$	$-\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}-\nu$	
(22)	z^2	$-\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}$	(20), 2.10(1)
		$\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu$	$1+\frac{1}{2}\nu-\frac{1}{2}\mu$	$3/2$	
(23)	$\frac{1}{z^2}$	$\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu$	$1+\frac{1}{2}\nu-\frac{1}{2}\mu$	$\nu+3/2$	(21), 2.10(6)
		$-\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}-\nu$	

EXPANSIONS FOR $P_\nu^\mu(z)$

	A_1 / A_2
(24)	$2^\mu (z^2-1)^{-\frac{1}{2}\mu} z^{\nu+\mu}/\Gamma(1-\mu)$
	0
(25)	$\pi^{\frac{1}{2}} 2^\mu e^{\mp i\pi\frac{1}{2}(\mu+\nu)} (z^2-1)^{\frac{1}{2}\nu}/[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)\,\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)]$
	$-\pi^{\frac{1}{2}} 2^{\mu+1} e^{\mp i\pi\frac{1}{2}(\mu+\nu-1)} z(z^2-1)^{\frac{1}{2}\nu-\frac{1}{2}} \times[\Gamma(-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)]^{-1}$
(26)	$(2\pi)^{-\frac{1}{2}} (z^2-1)^{-\frac{1}{4}} \Gamma(-\tfrac{1}{2}-\nu)[z-(z^2-1)^{\frac{1}{2}}]^{\nu+\frac{1}{2}}/\Gamma(-\nu-\mu)$
	$(2\pi)^{-\frac{1}{2}} (z^2-1)^{-\frac{1}{4}} \Gamma(\tfrac{1}{2}+\nu)[z-(z^2-1)^{\frac{1}{2}}]^{-\nu-\frac{1}{2}}/\Gamma(1+\nu-\mu)$
(27)	$\pi^{-\frac{1}{2}} 2^\mu \Gamma(-\tfrac{1}{2}-\nu)(z^2-1)^{\frac{1}{2}\mu}[z+(z^2-1)^{\frac{1}{2}}]^{-\nu-\mu-1}/\Gamma(-\nu-\mu)$
	$\pi^{-\frac{1}{2}} 2^\mu \Gamma(\tfrac{1}{2}+\nu)(z^2-1)^{\frac{1}{2}\mu}[z+(z^2-1)^{\frac{1}{2}}]^{\nu-\mu}/\Gamma(1+\nu-\mu)$
(28)	$2^\mu (z^2-1)^{-\frac{1}{2}\mu}[z+(z^2-1)^{\frac{1}{2}}]^{\nu+\mu}/\Gamma(1-\mu)$
	0

EXPANSIONS FOR $P_\nu^\mu(z)$

	ζ	a_1 / a_2	b_1 / b_2	c_1 / c_2	Remarks
(24)	$1-\frac{1}{z^2}$	$-\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu$	$1-\mu$	(20), 2.10(6)
		$\cdots$	$\cdots$	$\cdots$	$\mathrm{Re}\, z>0$
(25)	$\frac{z^2}{z^2-1}$	$-\frac{1}{2}\nu-\frac{1}{2}\mu$	$-\frac{1}{2}\nu+\frac{1}{2}\mu$	$\frac{1}{2}$	(24), 2.10(2)
		$\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}\mu$	$3/2$	The upper or lower sign according as $\mathrm{Im}\, z \gtrless 0$
(26)	$\frac{-z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}}$	$\frac{1}{2}+\mu$	$\frac{1}{2}-\mu$	$3/2+\nu$	(23), 2.11(16)
		$\frac{1}{2}+\mu$	$\frac{1}{2}-\mu$	$\frac{1}{2}-\nu$	
(27)	$\frac{z-(z^2-1)^{\frac{1}{2}}}{z+(z^2-1)^{\frac{1}{2}}}$	$\frac{1}{2}+\mu$	$1+\nu+\mu$	$\nu+3/2$	(26), 2.10(6)
		$\frac{1}{2}+\mu$	$-\nu+\mu$	$\frac{1}{2}-\nu$	
(28)	$\frac{2(z^2-1)^{\frac{1}{2}}}{z+(z^2-1)^{\frac{1}{2}}}$	$-\nu-\mu$	$\frac{1}{2}-\mu$	$1-2\mu$	(24), 2.11(17)
		$\cdots$	$\cdots$	$\cdots$	$\mathrm{Re}\, z>0$

EXPANSIONS FOR $P_\nu^\mu(z)$

	A_1 / A_2
(29)	$2^\mu(z^2-1)^{-\frac{1}{2}\mu}[z-(z^2-1)^{\frac{1}{2}}]^{\nu+\mu}/\Gamma(1-\mu)$
	0
(30)	$(2\pi)^{-\frac{1}{2}}\Gamma(\frac{1}{2}+\nu)\,e^{\mp i\pi(\mu-\frac{1}{2})}(z^2-1)^{-\frac{1}{4}}[z+(z^2-1)^{\frac{1}{2}}]^{-\nu-\frac{1}{2}} \times [\Gamma(\nu-\mu+1)]^{-1}$
	$(2\pi)^{-\frac{1}{2}}\Gamma(-\frac{1}{2}-\nu)\,e^{\mp i\pi(\mu-\frac{1}{2})}(z^2-1)^{-\frac{1}{4}}[z+(z^2-1)^{\frac{1}{2}}]^{\nu+\frac{1}{2}} \times [\Gamma(-\nu-\mu)]^{-1}$
(31)	$\pi^{-\frac{1}{2}}2^{-\mu}\Gamma(\frac{1}{2}+\nu)(z^2-1)^{-\frac{1}{2}\mu}[z-(z^2-1)^{\frac{1}{2}}]^{\nu+\mu}/\Gamma(\nu-\mu+1)$
	$\pi^{-\frac{1}{2}}2^{-\mu}\Gamma(-\frac{1}{2}-\nu)(z^2-1)^{-\frac{1}{2}\mu}[z+(z^2-1)^{\frac{1}{2}}]^{\nu-\mu+1}/\Gamma(-\nu-\mu)$

EXPANSIONS FOR $e^{-i\mu\pi}Q_\nu^\mu(z)$

	A_3 / A_4
(32)	$\Gamma(1+\nu+\mu)\,\Gamma(-\mu)\,(z-1)^{\frac{1}{2}\mu}(z+1)^{-\frac{1}{2}\mu}/[2\,\Gamma(1+\nu-\mu)]$
	$\frac{1}{2}\Gamma(\mu)\,(z+1)^{\frac{1}{2}\mu}(z-1)^{-\frac{1}{2}\mu}$

EXPANSIONS FOR $P_\nu^\mu(z)$

	ζ	a_1 / a_2	b_1 / b_2	c_1 / c_2	Remarks
(29)	$\dfrac{2(z^2-1)^{1/2}}{-z+(z^2-1)^{1/2}}$	$-\nu-\mu$	$1/2-\mu$	$1-2\mu$	(28), 2.10(6)
		...	...	...	
(30)	$\dfrac{z+(z^2-1)^{1/2}}{2(z^2-1)^{1/2}}$	$1/2-\mu$	$1/2+\mu$	$1/2-\nu$	(28), 2.10(2)
		$1/2-\mu$	$1/2+\mu$	$\nu+3/2$	The upper or lower sign according as Im $z \gtrless 0$
(31)	$\dfrac{z+(z^2-1)^{1/2}}{z-(z^2-1)^{1/2}}$	$-\nu-\mu$	$1/2-\mu$	$1/2-\nu$	(29), 2.10(1)
		$1+\nu-\mu$	$1/2-\mu$	$\nu+3/2$	The h.g. series converge nowhere in the cut plane

EXPANSIONS FOR $e^{-i\mu\pi}Q_\nu^\mu(z)$

	ζ	a_3 / a_4	b_3 / b_4	c_3 / c_4	Remarks
(32)	$\dfrac{1-z}{2}$	$-\nu$	$1+\nu$	$1+\mu$	(37), 2.10(2)
		$-\nu$	$1+\nu$	$1-\mu$	

EXPANSIONS FOR $e^{-i\mu\pi} Q_\nu^\mu(z)$

	A_3 / A_4
(33)	$-e^{\mp i\pi\nu}\,\Gamma(1+\nu+\mu)\,\Gamma(-\mu)\,(z+1)^{\frac{1}{2}\mu}\,(z-1)^{-\frac{1}{2}\mu}/[2\Gamma(1+\nu-\mu)]$
	$-\frac{1}{2}\,e^{\mp i\pi\nu}\,\Gamma(\mu)\,(z-1)^{\frac{1}{2}\mu}\,(z+1)^{-\frac{1}{2}\mu}$
(34)	$2^{-1-\nu}\,\Gamma(\mu)\,(z+1)^{\nu+\frac{1}{2}\mu}\,(z-1)^{-\frac{1}{2}\mu}$
	$2^{-1-\nu}\,\Gamma(1+\nu+\mu)\,\Gamma(-\mu)\,(z+1)^{-\frac{1}{2}\mu+\nu}\,(z-1)^{\frac{1}{2}\mu}/\Gamma(1+\nu-\mu)$
(35)	$2^{\nu}\,\Gamma(-\mu)\,\Gamma(1+\nu+\mu)\,(z-1)^{-\frac{1}{2}\mu-\nu-1}\,(z+1)^{\frac{1}{2}\mu}/\Gamma(1+\nu-\mu)$
	$2^{\nu}\,\Gamma(\mu)\,(z+1)^{-\frac{1}{2}\mu}\,(z-1)^{\frac{1}{2}\mu-\nu-1}$
(36)	$2^{\nu}\,\Gamma(1+\nu)\,\Gamma(1+\nu+\mu)\,(z+1)^{\frac{1}{2}\mu-\nu-1}\,(z-1)^{-\frac{1}{2}\mu}/\Gamma(2+2\nu)$
	0
(37)	$2^{\nu}\,\Gamma(1+\nu)\,\Gamma(1+\nu+\mu)\,(z+1)^{\frac{1}{2}\mu}\,(z-1)^{-\frac{1}{2}\mu-\nu-1}/\Gamma(2+2\nu)$
	0

EXPANSIONS FOR $e^{-i\mu\pi} Q_{\nu}^{\mu}(z)$

	ζ	a_3 / a_4	b_3 / b_4	c_3 / c_4	Remarks
(33)	$\frac{1+z}{2}$	$-\nu$	$1+\nu$	$1+\mu$	(36), 2.10(2)
		$-\nu$	$1+\nu$	$1-\mu$	The upper or lower sign according as Im $z \gtrless 0$
(34)	$\frac{z-1}{z+1}$	$-\nu$	$-\nu-\mu$	$1-\mu$	(36), 2.10(1)
		$-\nu$	$-\nu+\mu$	$1+\mu$	
(35)	$\frac{z+1}{z-1}$	$1+\iota$	$1+\nu+\mu$	$1+\mu$	(36), 2.10(3)
		$1+\nu$	$1+\nu-\mu$	$1-\mu$	
(36)	$\frac{2}{1+z}$	$1+\nu-\mu$	$1+\nu$	$2+2\nu$	(41), 2.11(17)
		...	...	...	
(37)	$\frac{2}{1-z}$	$1+\nu+\mu$	$1+\nu$	$2+2\nu$	(36), 2.10(6)
		...	...	...	

EXPANSIONS FOR $e^{-i\mu\pi} Q_{\nu}^{\mu}(z)$

	A_3 / A_4
(38)	$2^{-1+\mu}\,\Gamma(\mu)\,(z^2-1)^{-\frac{1}{2}\mu}$
	$2^{-1-\mu}\,\Gamma(1+\nu+\mu)\,\Gamma(-\mu)\,(z^2-1)^{\frac{1}{2}\mu}/\Gamma(1+\nu-\mu)$
(39)	$2^{-1-\nu}\,\pi^{\frac{1}{2}}\,\Gamma(1+\nu+\mu)\,(z^2-1)^{-\frac{1}{2}-\frac{1}{2}\nu}/\Gamma(\nu+3/2)$
	0
(40)	$\pi^{\frac{1}{2}}\,2^{\mu-1}\,\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)\,e^{\pm i\pi\frac{1}{2}(\mu-\nu-1)}\,(z^2-1)^{-\frac{1}{2}\mu}\times[\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)]^{-1}$
	$\pi^{\frac{1}{2}}\,2^{\mu}\,\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu)\,e^{\pm i\pi\frac{1}{2}(\mu-\nu)}\,z\,(z^2-1)^{-\frac{1}{2}\mu}\times[\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)]^{-1}$
(41)	$2^{-1-\nu}\,\pi^{\frac{1}{2}}\,\Gamma(1+\nu+\mu)\,z^{-1-\nu-\mu}\,(z^2-1)^{\frac{1}{2}\mu}/\Gamma(\nu+3/2)$
	0
(42)	$2^{\mu-1}\,\Gamma(\mu)\,z^{\nu+\mu}\,(z^2-1)^{-\frac{1}{2}\mu}$
	$2^{-1-\mu}\,\Gamma(1+\nu+\mu)\,\Gamma(-\mu)\,z^{\nu-\mu}\,(z^2-1)^{\frac{1}{2}\mu}/\Gamma(1+\nu-\mu)$

EXPANSIONS FOR $e^{-i\mu\pi}\, Q_\nu^\mu(z)$

	ζ	a_3 / a_4	b_3 / b_4	c_3 / c_4	Remarks
(38)	$1-z^2$	$\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu$	$-\frac{1}{2}\nu-\frac{1}{2}\mu$	$1-u$	(39), 2.10(2)
		$\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu$	$-\frac{1}{2}\nu+\frac{1}{2}\mu$	$1+\mu$	Re $z>0$
(39)	$\frac{1}{1-z^2}$	$\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu$	$\nu+3/2$	(41), 2.10(6)
		...	...	...	
(40)	z^2	$-\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}$	(41), 2.10(2)
		$\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu$	$1+\frac{1}{2}\nu-\frac{1}{2}\mu$	$3/2$	The upper or lower sign according as Im $z \gtrless 0$
(41)	$\frac{1}{z^2}$	$1+\frac{1}{2}\nu+\frac{1}{2}\mu$	$\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu$	$\nu+3/2$	
		...	...	...	
(42)	$1-\frac{1}{z^2}$	$-\frac{1}{2}\nu-\frac{1}{2}\mu$	$\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu$	$1-\mu$	(38), 2.10(6)
		$-\frac{1}{2}\nu+\frac{1}{2}\mu$	$\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}\mu$	$1+\mu$	

EXPANSIONS FOR $e^{-i\mu\pi} Q_\nu^\mu(z)$

	A_3
	A_4
(43)	$\pi^{1/2}\ 2^{\mu-1}\ \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)\ e^{\mp i\pi(\nu+\frac{1}{2})}\ (z^2-1)^{\frac{1}{2}\nu}/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)$
	$\pi^{1/2}\ 2\mu\ \Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)\ e^{\mp i\pi(\nu-\frac{1}{2})}\ z(z^2-1)^{\frac{1}{2}\nu-\frac{1}{2}}/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)$
(44)	$(\tfrac{1}{2}\pi)^{1/2}\ \Gamma(1+\nu+\mu)\ (z^2-1)^{-1/4}\ [z-(z^2-1)^{1/2}]^{\nu+\frac{1}{2}}/\Gamma(\nu+3/2)$
	0
(45)	$\pi^{1/2}\ 2^{\mu}\ \Gamma(1+\nu+\mu)\ (z^2-1)^{\frac{1}{2}\mu}[z+(z^2-1)^{1/2}]^{-1-\nu-\mu}/\Gamma(\nu+3/2)$
	0
(46)	$2^{\mu-1}\ \Gamma(\mu)\ (z^2-1)^{-\frac{1}{2}\mu}\ [z+(z^2-1)^{1/2}]^{\nu+\mu}$
	$2^{-1-\mu}\ \Gamma(1+\nu+\mu)\ \Gamma(-\mu)\ (z^2-1)^{\frac{1}{2}\mu}\ [z+(z^2-1)^{1/2}]^{\nu-\mu}/\Gamma(1+\nu-\mu]$
(47)	$2^{-1+\mu}\ \Gamma(\mu)\ (z^2-1)^{-\frac{1}{2}\mu}\ [z-(z^2-1)^{1/2}]^{\nu+\mu}$
	$2^{-1-\mu}\ \Gamma(1+\nu+\mu)\ \Gamma(-\mu)\ (z^2-1)^{\frac{1}{2}\mu}\ [z-(z^2-1)]^{\nu-\mu}/\Gamma(1+\nu-\mu)$

EXPANSIONS FOR $e^{-i\mu\pi} Q_\nu^\mu(z)$

	ζ	a_3 / a_4	b_3 / b_4	c_3 / c_4	Remarks
(43)	$\frac{z^2}{z^2-1}$	$-\frac{1}{2}\nu - \frac{1}{2}\mu$	$-\frac{1}{2}\nu + \frac{1}{2}\mu$	$\frac{1}{2}$	(41), 2.10(3)
		$\frac{1}{2} - \frac{1}{2}\nu - \frac{1}{2}\mu$	$\frac{1}{2} - \frac{1}{2}\nu + \frac{1}{2}\mu$	$3/2$	The upper or lower sign according as Im $z \gtrless 0$
(44)	$\frac{-z+(z^2-1)^{1/2}}{2(z^2-1)^{1/2}}$	$\frac{1}{2}+\mu$	$\frac{1}{2}-\mu$	$\nu+3/2$	(41), 2.11(16)
		...	...	...	
(45)	$\frac{z-(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}$	$\mu+\frac{1}{2}$	$1+\nu+\mu$	$\nu+3/2$	(44), 2.10(6)
		...	...	...	
(46)	$\frac{2(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}$	$-\nu-\mu$	$\frac{1}{2}-\mu$	$1-2\mu$	(44), 2.10(3)
		$-\nu+\mu$	$\frac{1}{2}+\mu$	$1+2\mu$	Re $z>0$
(47)	$\frac{2(z^2-1)^{1/2}}{-z+(z^2-1)^{1/2}}$	$-\nu-\mu$	$\frac{1}{2}-\mu$	$1-2\mu$	(44), 2.10(2)
		$-\nu+\mu$	$\frac{1}{2}+\mu$	$1+2\mu$	

EXPANSIONS FOR $e^{-i\mu\pi} Q_\nu^\mu(z)$

	A_3
	A_4
(48)	$(\tfrac{1}{2}\pi)^{\frac{1}{2}} \Gamma(\tfrac{1}{2}+\nu)(z^2-1)^{-\frac{1}{4}}[z-(z^2-1)^{\frac{1}{2}}]^{\frac{1}{2}+\nu}/\Gamma(1+\nu-\mu)$
	$\pi^{-\frac{1}{2}} 2^{-1-\nu}\Gamma(1+\nu+\mu)\Gamma(-\tfrac{1}{2}-\nu)\cos(\mu\pi)(z^2-1)^{-\frac{1}{2}-\frac{1}{2}\nu}$
(49)	$2^{\mu}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}+\nu)e^{\mp i\pi(\frac{1}{2}+\mu)}(z^2-1)^{\frac{1}{2}\mu}[z+(z^2-1)^{\frac{1}{2}}]^{\mu-\nu}$ $\times[\Gamma(1+\nu-\mu)]^{-1}$
	$-2^{-\mu}\pi^{-\frac{1}{2}}\cos(\mu\pi)\Gamma(1+\nu+\mu)\Gamma(-\tfrac{1}{2}-\nu)e^{\mp i\pi(\nu-\mu)}$ $\times(z^2-1)^{-\frac{1}{2}\mu}[z+(z^2-1)^{\frac{1}{2}}]^{1+\nu-\mu}$

EXPANSIONS FOR $e^{-i\mu\pi} Q_\nu^\mu(z)$

	ζ	a_3 / a_4	b_3 / b_4	c_3 / c_4	Remarks
(48)	$\dfrac{z+(z^2-1)^{1/2}}{2(z^2-1)^{1/2}}$	$1/2+\mu$	$1/2-\mu$	$1/2-\nu$	(44), 2.10(1)
		$1+\nu-\mu$	$1+\nu+\mu$	$\nu+3/2$	
(49)	$\dfrac{z+(z^2-1)^{1/2}}{z-(z^2-1)^{1/2}}$	$1/2+\mu$	$\mu-\nu$	$1/2-\nu$	(48), 2.10(6) The upper or lower sign according as Im $z \gtrless 0$
		$1/2-\mu$	$1+\nu-\mu$	$\nu+3/2$	The h.g. series converge nowhere in the cut plane

3.3.1. Relations between Legendre functions

From 3.2(3) we have

(1) $P_\nu^\mu(z) = P_{-\nu-1}^\mu(z).$

From 3.2(5) and 3.2(8) we have

(2) $e^{i\mu\pi}\,\Gamma(\nu+\mu+1)\,Q_\nu^{-\mu}(z) = e^{-i\mu\pi}\,\Gamma(\nu-\mu+1)\,Q_\nu^\mu(z).$

From 3.2(5) it follows by 3.2(23) that

(3) $Q_\nu^\mu(z)\sin[\pi(\nu+\mu)] - Q_{-\nu-1}^\mu(z)\sin[\pi(\nu-\mu)]$

$= \pi\, e^{i\mu\pi}\cos(\nu\pi)\, P_\nu^\mu(z).$

From 3.2(32) and 3.2(3) we obtain

(4) $Q_\nu^\mu(z)\sin(\mu\pi) = \tfrac{1}{2}\pi\, e^{\mu\pi i}\left[P_\nu^\mu(z) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(z)\right],$

and hence

(5) $P_\nu^{-\mu}(z) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\,[P_\nu^\mu(z) - (2/\pi)\, e^{-i\mu\pi}\sin(\mu\pi)\, Q_\nu^\mu(z)],$

(6) $P_\nu^\mu(z) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\, P_\nu^{-\mu}(z) + (2/\pi)\, e^{-i\mu\pi}\sin(\mu\pi)\, Q_\nu^\mu(z),$

and thus, if $\mu = m$ $(m = 1, 2, 3, \dots)$,

(7) $P_\nu^m(z) = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}\, P_\nu^{-m}(z).$

From (5) and (3)

(8) $P_\nu^{-\mu}(z) = \frac{e^{-i\mu\pi}\,\Gamma(\nu-\mu+1)}{\pi\cos(\nu\pi)\,\Gamma(\nu+\mu+1)}\,\sin[\pi(\nu-\mu)]\,[Q_\nu^\mu(z) - Q_{-\nu-1}^\mu(z)]$

or with 1.2(6)

(9) $Q_{-\nu-1}^\mu(z) - Q_\nu^\mu(z) = e^{i\mu\pi}\cos(\nu\pi)\,\Gamma(\nu+\mu+1)\,\Gamma(\mu-\nu)\, P_\nu^{-\mu}(z).$

From 3.2(15), 3.2(3), and 3.2(32)

(10) $P_\nu^\mu(-z) = e^{\mp\nu\pi i} P_\nu^\mu(z) - (2/\pi)\, e^{-i\mu\pi}\sin[\pi(\nu+\mu)]\, Q_\nu^\mu(z)$

or

(11) $Q_\nu^\mu(z)\, e^{-i\mu\pi}\sin[\pi(\nu+\mu)] = \tfrac{1}{2}\pi\,[e^{\mp i\nu\pi} P_\nu^\mu(z) - P_\nu^\mu(-z)].$

Hence, replacing z by $-z$

(12) $Q_\nu^\mu(-z) = -e^{\pm i\nu\pi}\, Q_\nu^\mu(z).$

In (10), (11), (12) the upper or lower sign is to be taken according as $\operatorname{Im} z \gtrless 0$.

If we replace z by $z(z^2-1)^{-½}$, ν by $-\mu-½$ and μ by $-\nu-½$ in 3.2(3) and compare with 3.2(44) we obtain Whipple's formula

(13) $$Q_\nu^\mu(z) = e^{i\mu\pi}(½\pi)^{½}\,\Gamma(\nu+\mu+1)\,(z^2-1)^{-¼}\,P_{-\mu-½}^{-\nu-½}[z(z^2-1)^{-½}],$$

$$\operatorname{Re} z > 0,$$

which is equivalent to

(14) $$\Gamma(-\nu-\mu)\,P_\nu^\mu(z) = ie^{i\nu\pi}(½\pi)^{-½}(z^2-1)^{-¼}\,Q_{-\mu-½}^{-\nu-½}[z(z^2-1)^{-½}]$$

$$\operatorname{Re} z > 0.$$

As z varies from a point on the real axis at which $z > 1$ to a point on the imaginary axis, $z(z^2-1)^{-½}$ varies from a point on the real axis to a point on the cut between 0 and 1. As the Legendre function of the second kind becomes discontinuous on the cut, we must introduce the restriction $\operatorname{Re} z > 0$.

3.3.2. Some further relations with hypergeometric series

From 3.2(3) and (11)

(15) $$Q_\nu^\mu(z)\,e^{-i\mu\pi}\sin[\pi(\nu+\mu)]\,\Gamma(1-\mu)$$
$$= ½\pi\left[e^{\mp i\nu\pi}\left(\frac{z+1}{z-1}\right)^{½\mu}F(-\nu,\nu+1;1-\mu;½-½z)\right.$$
$$\left.-\left(\frac{z-1}{z+1}\right)^{½\mu}F(-\nu,\nu+1;1-\mu;½+½z)\right],$$

and hence by means of (2)

(16) $$2\,Q_\nu^\mu(z)\,e^{-i\mu\pi}\,\Gamma(1+\mu) = \Gamma(1+\nu+\mu)\left(\frac{z+1}{z-1}\right)^{½\mu}\Gamma(\mu-\nu)$$
$$\times\left[F(-\nu,\nu+1;1+\mu;½+½z)\right.$$
$$\left.-e^{\mp i\nu\pi}\left(\frac{z-1}{z+1}\right)^{\mu}F(-\nu,\nu+1;1+\mu;½-½z)\right]$$

with the upper or lower sign chosen according as $\operatorname{Im} z \gtrless 0$.

From (6), 3.2(3), and (16)

(17) $$\pi\,\Gamma(1+\mu)\,P_\nu^\mu(z) = \Gamma(\nu+\mu+1)\,\Gamma(\mu-\nu)\left(\frac{z+1}{z-1}\right)^{½\mu}\sin(\mu\pi)$$

(7) $P^{\mu}_{-\nu-1}(x) = P^{\mu}_{\nu}(x)$,

(8) $e^{-\frac{1}{2}i\mu\pi} Q^{\mu}_{\nu}(x+i0) - e^{\frac{1}{2}i\mu\pi} Q^{\mu}_{\nu}(x-i0) = i\pi e^{i\mu\pi} P^{\mu}_{\nu}(x)$

(9) $e^{-i\mu\pi} Q^{\mu}_{\nu}(x \pm i0) = e^{\pm\frac{1}{2}i\mu\pi} [Q^{\mu}_{\nu}(x) \mp i(\pi/2) P^{\mu}_{\nu}(x)]$,

(10) $$Q^{\mu}_{\nu}(x) = \frac{\Gamma(1+\nu+\mu)\Gamma(-\mu)}{2\Gamma(1+\nu-\mu)} \left(\frac{1-x}{1+x}\right)^{\frac{1}{2}\mu} F(-\nu, \nu+1; \mu+1; \tfrac{1}{2}-\tfrac{1}{2}x)$$
$$+ \tfrac{1}{2}\Gamma(\mu)\cos(\mu\pi)\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}\mu} F(-\nu, \nu+1; 1-\mu; \tfrac{1}{2}-\tfrac{1}{2}x),$$

(11) $$(1-x^2)^{\frac{1}{2}\mu} 2^{-\mu} \pi^{-\frac{1}{2}} P^{\mu}_{\nu}(x) = \frac{F(-\frac{1}{2}\mu-\frac{1}{2}\nu, \frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu; \frac{1}{2}; x^2)}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)}$$
$$- \frac{2xF(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu, 1+\frac{1}{2}\nu-\frac{1}{2}\mu; 3/2; x^2)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu)},$$

(12) $$(1-x^2)^{\frac{1}{2}\mu} 2^{-\mu} \pi^{-3/2} Q^{\mu}_{\nu}(x)$$
$$= \frac{\cot[\frac{1}{2}\pi(\nu+\mu)]\, z\, F(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu, \frac{1}{2}\nu-\frac{1}{2}\mu+1; 3/2; x^2)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu)}$$
$$- \frac{\frac{1}{2}\tan[\frac{1}{2}\pi(\nu+\mu)]\, F(-\frac{1}{2}\nu-\frac{1}{2}\mu, \frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu; \frac{1}{2}; x^2)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu)},$$

(13) $$2Q^{\mu}_{\nu}(x)\sin(\mu\pi) = \pi\left[P^{\mu}_{\nu}(x)\cos(\mu\pi) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P^{-\mu}_{\nu}(x)\right],$$

(14) $P^{\mu}_{\nu}(-x) = P^{\mu}_{\nu}(x)\cos[\pi(\nu+\mu)] - (2/\pi) Q^{\mu}_{\nu}(x)\sin[\pi(\nu+\mu)]$
$0 < x < 1$,

(15) $Q^{\mu}_{\nu}(-x) = -Q^{\mu}_{\nu}(x)\cos[\pi(\nu+\mu)] - \frac{1}{2}\pi P^{\mu}_{\nu}(x)\sin[\pi(\nu+\mu)]$,
$0 < x < 1$,

(16) $\sin[\pi(\nu-\mu)] Q^{\mu}_{-\nu-1}(x)$
$= \sin[\pi(\nu+u)] Q^{\mu}_{\nu}(x) - \pi\cos(\nu\pi)\cos(\mu\pi) P^{\mu}_{\nu}(x)$,

(17) $\Gamma(\nu+\mu+1) P^{-\mu}_{\nu}(x)$
$= \Gamma(\nu-\mu+1)[P^{\mu}_{\nu}(x)\cos(\mu\pi) - (2/\pi)\sin(\mu\pi) Q^{\mu}_{\nu}(x)]$,

(18) $\Gamma(\nu+\mu+1) Q^{-\mu}_{\nu}(x)$
$= \Gamma(\nu-\mu+1)[Q^{\mu}_{\nu}(x)\cos(\mu\pi) + \frac{1}{2}\pi P^{\mu}_{\nu}(x)\sin(\mu\pi)]$.

Equation	Proved from
(3)	3.2(3) and 3.2(12)
(4)	3.2(3) and 3.2(12)
(5)	(3) and (4)
(6)	(1), (3) and (4)
(7)	(6)
(8)	(6) and 3.2(32)
(9)	(8) and (2)
(10)	(2) and 3.2(32)
(11)	3.2(22), (1), and 3.2(12)
(12)	3.2(40), (2), and 3.2(12)
(13)	3.3(4), (1), and (9)
(14)	3.3(10), (7), and (9)
(15)	3.3(12), (2), and (3)
(16)	3.3(3), (5), and (2)
(17)	(13)
(18)	3.3(2), (9), and (2)

For integer m and n we have from (14) and (15)

(19) $\mathrm{P}_n^m(-x) = (-1)^{m+n}\,\mathrm{P}_n^m(x); \qquad \mathrm{Q}_n^m(-x) = (-1)^{m+n+1}\,\mathrm{Q}_n^m(x).$

For $x = 0$ we find from (11) and (12)

(20) $\mathrm{P}_\nu^\mu(0) = 2^\mu \pi^{-1/2} \cos[\tfrac{1}{2}\pi(\nu+\mu)]\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu),$

(21) $\mathrm{Q}_\nu^\mu(0) = -\,2^{\mu-1} \pi^{1/2} \sin[\tfrac{1}{2}\pi(\nu+\mu)]\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu).$

As $\left.\dfrac{d}{dz}F(a, b; c; z)\right|_{z=0} = \dfrac{ab}{c}$ we have from (11) and (12)

(22) $\left(\dfrac{d\,\mathrm{Q}_\nu^\mu(x)}{dx}\right)_{x=0} = 2^\mu \pi^{1/2} \cos[\tfrac{1}{2}\pi(\nu+\mu)]\,\Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu),$

(23) $\left(\dfrac{d\,\mathrm{P}_\nu^\mu(x)}{dx}\right)_{x=0} = 2^{\mu+1} \pi^{-1/2} \sin[\tfrac{1}{2}\pi(\nu+\mu)]\,\Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu).$

Furthermore, from equations (20) to (23)

(24) $$\left[P_\nu^\mu(x)\frac{d}{dx}Q_\nu^\mu(x) - Q_\nu^\mu(x)\frac{d}{dx}P_\nu^\mu(x)\right]_{x=0}$$

$$= 2^{2\mu}\,\Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)/[\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)].$$

Since we have from section 3.2 that

$$P_\nu^\mu(x)\frac{d}{dx}Q_\nu^\mu(x)-Q_\nu^\mu(x)\frac{d}{dx}P_\nu^\mu(x)=C(1-x^2)^{-1},$$

the constant C may be determined by putting $x=0$ and using (24). Then we obtain

$$(25)\quad (1-x^2)\left[P_\nu^\mu(x)\frac{d}{dx}Q_\nu^\mu(x)-Q_\nu^\mu(x)\frac{d}{dx}P_\nu^\mu(x)\right]$$
$$=\frac{2^{2\mu}\Gamma(1+\tfrac12\mu+\tfrac12\nu)\Gamma(\tfrac12+\tfrac12\mu+\tfrac12\nu)}{\Gamma(1+\tfrac12\nu-\tfrac12\mu)\Gamma(\tfrac12+\tfrac12\nu-\tfrac12\mu)}.$$

3.5. Trigonometric expansions for $P_\nu^\mu(\cos\theta)$ and $Q_\nu^\mu(\cos\theta)$

We put $z=\cos\theta\pm i0$, $(z^2-1)^{1/2}=e^{\pm i\frac12\pi}\sin\theta$ in 3.2(45) and obtain

$$(1)\quad e^{-i\mu\pi}Q_\nu^\mu(\cos\theta\pm i0)\Gamma(\nu+3/2)=\pi^{1/2}2^\mu\Gamma(\nu+\mu+1)e^{\pm\frac12 i\mu\pi}$$
$$\times(\sin\theta)^\mu e^{\mp i\theta(1+\nu+\mu)}F(\tfrac12+\mu,1+\nu+\mu;\nu+3/2;e^{\mp i2\theta}),$$

and hence by means of 3.4(8) and 2.1(2)

$$(2)\quad P_\nu^\mu(\cos\theta)=\pi^{-1/2}2^{\mu+1}(\sin\theta)^\mu\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}$$
$$\times\sum_{l=0}^{\infty}\frac{(\tfrac12+\mu)_l(1+\nu+\mu)_l}{l!(\nu+3/2)_l}\sin[(2l+\nu+\mu+1)\theta].$$

Similarly from 3.4(2)

$$(3)\quad Q_\nu^\mu(\cos\theta)=\pi^{1/2}2^\mu(\sin\theta)^\mu\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}$$
$$\times\sum_{l=0}^{\infty}\frac{(\tfrac12+\mu)_l(1+\nu+\mu)_l}{l!(\nu+3/2)_l}\cos[(2l+\nu+\mu+1)\theta].$$

Both series are convergent for $0<\theta<\pi$. In the same manner we obtain from 3.2(44)

$$(4)\quad e^{-i\mu\pi}Q_\nu^\mu(\cos\theta\pm i0)\Gamma(\nu+3/2)=\pi^{1/2}(2\sin\theta)^{-1/2}e^{\mp i[\frac14\pi+(\nu+\frac12)\theta]}$$
$$\times F\left(\tfrac12+\mu,\tfrac12-\mu;\nu+3/2;\frac{\pm ie^{\mp i\theta}}{2\sin\theta}\right)\Gamma(\nu+\mu+1)$$

and hence by means of 3.4(8), 3.4(2), and 2.1(2)

(5) $$\Gamma(\nu+3/2)\,\mathrm{P}_\nu^\mu(\cos\theta) = 2^{1/2}\,(\pi\sin\theta)^{-1/2}\,\Gamma(\nu+\mu+1)$$
$$\times\sum_{l=0}^{\infty}(-1)^l\frac{(\tfrac12+\mu)_l\,(\tfrac12-\mu)_l}{l!\,(2\sin\theta)^l\,(\nu+3/2)_l}$$
$$\times\sin\left[(\nu+l+\tfrac12)\,\theta+(\tfrac12\mu+\tfrac14)\,\pi+\tfrac12 l\pi\right],$$

(6) $$\Gamma(\nu+3/2)\,\mathrm{Q}_\nu^\mu(\cos\theta) = \pi^{1/2}\,(2\sin\theta)^{-1/2}\,\Gamma(\nu+\mu+1)$$
$$\times\sum_{l=0}^{\infty}(-1)^l\frac{(\tfrac12+\mu)_l\,(\tfrac12-\mu)_l}{l!\,(2\sin\theta)^l\,(\nu+3/2)_l}$$
$$\times\cos\left[(\nu+l+\tfrac12)\,\theta+(\tfrac12\mu+\tfrac14)\,\pi+\tfrac12 l\pi\right].$$

As can be seen from (4), the expansions (5) and (6) are convergent if $\pi/6<\theta<5\pi/6$. From 3.4(5) and 3.2(20), 3.2(7), and 3.2(3), respectively

(7) $$\Gamma(1-\mu)\,\mathrm{P}_\nu^\mu(\cos\theta)$$
$$=(\tfrac12\sin\theta)^{-\mu}\,F[\tfrac12+\tfrac12\nu-\tfrac12\mu;-\tfrac12\nu-\tfrac12\mu;1-\mu;(\sin\theta)^2],\qquad 0<\theta<\pi/2,$$

(8) $$\Gamma(1-\mu)\,\mathrm{P}_\nu^\mu(\cos\theta)$$
$$=(\tfrac12\sin\theta)^{-\mu}\,F[1+\nu-\mu,-\nu-\mu;1-\mu;(\sin\tfrac12\theta)^2],\qquad 0<\theta<\pi,$$

(9) $$\Gamma(1-\mu)\,\mathrm{P}_\nu^\mu(\cos\theta)$$
$$=(\cot\tfrac12\theta)^{\mu}\,F[-\nu,\nu+1;1-\mu;(\sin\tfrac12\theta)^2],\qquad 0<\theta<\pi.$$

A formula for $\mathrm{P}_\nu^{-\mu}(\cos\theta)$ suitable when θ is small (MacDonald, 1914, p. 220) is

(10) $$\mathrm{P}_\nu^{-\mu}(\cos\theta)=[(\nu+\tfrac12)\cos\tfrac12\theta]^{-\mu}$$
$$\times\{J_\mu(a)+(\sin\tfrac12\theta)^2\,[(1/6)\,a\,J_{\mu+3}(a)-J_{\mu+2}(a)+(2a)^{-1}J_{\mu+1}(a)]$$
$$+O[(\sin\tfrac12\theta)^4]\}$$

Here $a=(2\nu+1)\sin\tfrac12\theta$ and $J_\lambda(a)$ denotes the Bessel function (see Ch. 7).

This expression can be obtained by writing 3.4(6) in the form

$$\mathrm{P}_\nu^{-\mu}(x)=\left(\frac{1-x}{1+x}\right)^{\frac12\mu}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\nu+n+1)}{\Gamma(\nu+n+1)\,\Gamma(\mu+n+1)}$$
$$\times\frac{(\tfrac12-\tfrac12x)^n}{n!}$$

and expressing $\Gamma(\nu+n+1)/\Gamma(\nu-n+1)$ in terms of powers of $\nu+\tfrac12$.

3.6.1. Special values of μ and ν

If μ is a positive integer, $\mu = m$ ($m = 1, 2, 3, \ldots$), we have from 3.3(7) and 3.2(7)

(1) $\Gamma(\nu - m + 1)\, m!\, P_\nu^m(z)$

$$= 2^{-m}\,\Gamma(\nu + m + 1)\,(z^2 - 1)^{\frac{1}{2}m}\, F(1 + m + \nu,\, m - \nu;\, 1 + m;\, \tfrac{1}{2} - \tfrac{1}{2}z),$$

and from 3.4(5) and (1)

(2) $\Gamma(\nu - m + 1)\, m!\, \mathrm{P}_\nu^m(x)$

$$= (-2)^{-m}\,\Gamma(\nu + m + 1)\,(1 - x^2)^{\frac{1}{2}m}\, F(1 + m + \nu,\, m - \nu;\, 1 + m;\, \tfrac{1}{2} - \tfrac{1}{2}x).$$

If ν is an integer, 3.3(1) shows that it is no restriction to assume that ν is a non-negative integer, $\nu = n$, $n = 0, 1, 2, \ldots$. We have to distinguish three cases:

(i) If μ is not a positive integer, the hypergeometric series in 3.2(3) is a polynomial of degree n in z.

(ii) If $\mu = m$, $m = 1, 2, 3, \ldots$, and $n \geq m$, (1) and (2) are valid, and the hypergeometric series involved are polynomials of degree $n - m$ in z.

(iii) If $\mu = m$, $m = 1, 2, 3, \ldots$, and $m > n$, then $P_\nu^\mu(z)$ and $\mathrm{P}_\nu^\mu(x)$ vanish identically. However $\Gamma(\nu - \mu + 1)\, P_\nu^\mu(z)$ and $\Gamma(\nu - \mu + 1)\, \mathrm{P}_\nu^\mu(x)$ approach finite limits as $\mu \to m$, $\nu \to n$.

It is customary to write $P_\nu^0(z) = P_\nu(z)$ etc. Often $P_\nu(z)$ and $Q_\nu(z)$ are called Legendre functions, $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ associated Legendre functions.

From 3.2(7) we have

(3) $P_\nu(z) = F(1 + \nu, -\nu; 1; \tfrac{1}{2} - \tfrac{1}{2}z)$.

If we differentiate (3) m times with respect to z ($m = 1, 2, 3, \ldots$) and consider 2.1(7), 1.20(5), 3.2(7), and 3.3(7) it follows that

(4) $$P_\nu^m(z) = (z^2 - 1)^{\frac{1}{2}m}\,\frac{d^m P_\nu(z)}{dz^m} \qquad m = 1, 2, 3, \ldots,$$

and hence, with 3.3(11), and 3.2(8)

(5) $$Q_\nu^m(z) = (z^2 - 1)^{\frac{1}{2}m}\,\frac{d^m Q_\nu(z)}{dz^m} \qquad m = 1, 2, 3, \ldots .$$

From (4), (5), 3.4(2) and 3.4(5)

(6) $$\mathrm{P}_\nu^m(x) = (-1)^m\,(1 - x^2)^{\frac{1}{2}m}\,\frac{d^m \mathrm{P}_\nu(x)}{dx^m},$$

(7) $$\mathrm{Q}_\nu^m(x) = (-1)^m (1-x^2)^{\frac{1}{2}m} \frac{d^m \mathrm{Q}_\nu(x)}{dx^m}$$

$$m = 1, 2, 3, \ldots, \qquad -1 < x < 1.$$

Again from (3), 2.1(7), and 3.2(7)

(8) $$P_\nu^{-m}(z) = (z^2-1)^{-\frac{1}{2}m} \int_1^z \cdots \int_1^z P_\nu(z)\,(dz)^m,$$

(9) $$Q_\nu^{-m}(z) = (-1)^m (z^2-1)^{-\frac{1}{2}m} \int_z^\infty \cdots \int_z^\infty Q_\nu(z)\,(dz)^m,$$

(10) $$\mathrm{P}_\nu^{-m}(x) = (-1)^m (1-x^2)^{-\frac{1}{2}m} \int_1^x \cdots \int_1^x \mathrm{P}_\nu(x)\,(dx)^m.$$

Furthermore if μ is a positive integer, $\mu = m$ $(m = 1, 2, 3, \ldots)$ the formula 3.2(32) becomes an undetermined form which can be evaluated by applying the usual rules. It results (Hobson, 1931, p. 205) in

(11) $$\frac{2\pi}{\sin(\nu\pi)} Q_\nu^m(z)$$

$$= \frac{\pi}{\sin(\nu\pi)} P_\nu^m(z)\left[\log\left(\frac{z+1}{z-1}\right) - 2\gamma - \psi(\nu+m+1) - \psi(\nu-m+1)\right]$$

$$- e^{im\pi}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}m} \sum_{r=0}^{m-1} \Gamma(r-\nu)\Gamma(r+\nu+1)\Gamma(m-r)\frac{\cos(r\pi)}{r!}(\tfrac{1}{2}-\tfrac{1}{2}z)^r$$

$$-\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}m} \sum_{l=1}^{\infty} \frac{\Gamma(m+l-\nu)\,\Gamma(m+l+\nu+1)}{l!\,(m+l)!}\sigma(l)\,(\tfrac{1}{2}-\tfrac{1}{2}z)^{m+l}$$

$$-\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}m} \sum_{r=0}^{\infty} \frac{\Gamma(r-\nu)\Gamma(r+\nu+1)}{r!\,(r+m)!}\sigma(m+r)\,(\tfrac{1}{2}-\tfrac{1}{2}z)^r$$

where

$$\sigma(l) = 1 + \tfrac{1}{2} + \cdots + \frac{1}{l} = \psi(l+1) - \psi(1) = \psi(l+1) + \gamma,$$

$$\sigma(0) = 0.$$

If μ is a negative integer use 3.3(2) and (11). When $m = 0$, (11) becomes

$$Q_\nu(z) = \tfrac{1}{2} P_\nu(z)\left[\log\left(\frac{z+1}{z-1}\right) - 2\gamma - 2\psi(\nu+1)\right]$$

$$-\pi^{-1}\sin(\nu\pi)\sum_{l=1}^{\infty}\Gamma(-\nu+l)\Gamma(\nu+l+1)\sigma(l)\,(\tfrac{1}{2}-\tfrac{1}{2}z)^l/(l!)^2.$$

In case if ν is also a positive integer, $\nu = n$ $(n = 1, 2, 3, \ldots)$ we have

$$Q_n(z) = \tfrac{1}{2} P_n(z)\left[\log\left(\frac{z+1}{z-1}\right) - 2\sigma(n)\right]$$

$$+\sum_{l=0}^{n}(-1)^{l}(n+l)!\,\psi(l)\,(\tfrac{1}{2}-\tfrac{1}{2}z)^{l}/[(l!)^{2}(n-l)!]$$

(see also 3.6.2).

From 3.2(16), 3.2(20), 3.2(26), 3.2(36), 3.2(37) and 3.2(44), respectively it is evident that the expressions for $P^{-\nu}_{\nu+n}$, $P^{\nu+2n+1}_{\nu}$, $P^{\pm(n+\frac{1}{2})}_{\nu}$, $Q^{\nu+n+1}_{\nu}$, $Q^{-\nu-n-1}_{\nu}$, $Q^{\pm(n+\frac{1}{2})}_{\nu}$ in case $n = 0, 1, 2, \ldots$ reduce to a finite number of terms. In particular we have

$$(12)\quad\begin{cases} P^{\frac{1}{2}}_{\nu}(z) = (2\pi)^{-\frac{1}{2}}(z^2-1)^{-\frac{1}{4}}\{[z+(z^2-1)^{\frac{1}{2}}]^{\nu+\frac{1}{2}}+[z+(z^2-1)^{\frac{1}{2}}]^{-\nu-\frac{1}{2}}\} \\ Q^{\frac{1}{2}}_{\nu}(z) = i\,(\pi/2)^{\frac{1}{2}}(z^2-1)^{-\frac{1}{4}}[z+(z^2-1)^{\frac{1}{2}}]^{-\nu-\frac{1}{2}}, \end{cases}$$

$$(13)\quad\begin{cases} P^{-\frac{1}{2}}_{\nu}(z) = (2/\pi)^{\frac{1}{2}}\dfrac{(z^2-1)^{-\frac{1}{4}}}{2\nu+1}\{[z+(z^2-1)^{\frac{1}{2}}]^{\nu+\frac{1}{2}}-[z+(z^2-1)^{\frac{1}{2}}]^{-\nu-\frac{1}{2}}\} \\ Q^{-\frac{1}{2}}_{\nu}(z) = i(2\pi)^{\frac{1}{2}}\dfrac{(z^2-1)^{-\frac{1}{4}}}{2\nu+1}[z+(z^2-1)^{\frac{1}{2}}]^{-\nu-\frac{1}{2}}, \end{cases}$$

and from 3.2(16)

$$(14)\quad\begin{cases} P^{-\nu}_{\nu}(z) = 2^{-\nu}(z^2-1)^{\frac{1}{2}\nu}/\Gamma(\nu+1), \\ \mathrm{P}^{-\nu}_{\nu}(\cos\theta) = 2^{-\nu}(\sin\theta)^{\nu}/\Gamma(\nu+1). \end{cases}$$

From equations (11) to (14) a number of other formulas can be derived by applying 3.3(13) and 3.3(14).

3.6.2. Legendre polynomials

A particularly important case of the Legendre function is that in which $\mu = 0$ and ν is an integer (cf. also section 10.10). We may assume ν to be non-negative. From 3.2(22) we have for $n = 0, 1, 2, \ldots$,

$$(15)\quad P_{2n}(z) = \frac{\pi^{\frac{1}{2}}}{n!\,\Gamma(\frac{1}{2}-n)}F(-n, n+\tfrac{1}{2}; \tfrac{1}{2}; z^2)$$

$$= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}F(-n, n+\tfrac{1}{2}; \tfrac{1}{2}; z^2),$$

$$P_{2n+1}(z) = \frac{-2\pi^{\frac{1}{2}}z}{n!\,\Gamma(-\frac{1}{2}-n)}F(-n, n+3/2; 3/2; z^2)$$

$$= \frac{(-1)^n (2n+1)!}{2^{2n} (n!)^2} \, z \, F(-n, n+3/2; 3/2; z^2),$$

or, in both cases

$$(16) \quad P_n(z) = \frac{(2n)!}{2^n (n!)^2} \times \left[z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} z^{n-4} - \cdots \right],$$

which may be written as

$$(17) \quad P_n(z) = (2^n n!)^{-1} \frac{d^n}{dz^n} (z^2 - 1)^n.$$

This is Rodrigues' formula.

Thus $P_n(z)$ is a polynomial of degree n in z which has the same parity as n.

$$P_n(-z) = (-1)^n P_n(z).$$

These polynomials are known as Legendre polynomials. They form an orthogonal system for the interval $(-1, 1)$, and all their roots are real, simple and between -1 and 1 (cf. also chapter 10).

From 3.5 (2) and 3.5 (3) we have

$$(18) \quad \mathrm{P}_n(\cos\theta) = \frac{2^{2n+2} (n!)^2}{\pi (2n+1)!} \left[\sin(n+1)\theta + \frac{n+1}{2n+3} \sin(n+3)\theta + \frac{1 \cdot 3}{2!} \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \sin(n+5)\theta + \frac{1 \cdot 3 \cdot 5}{3!} \frac{(n+1)(n+2)(n+3)}{(2n+3)(2n+5)(2n+7)} \sin(n+7)\theta + \cdots \right]$$

and

$$(19) \quad \mathrm{Q}_n(\cos\theta) = \frac{2^{2n+1} (n!)^2}{(2n+1)!} \left[\cos(n+1)\theta + \frac{n+1}{2n+3} \cos(n+3)\theta + \frac{1 \cdot 3}{2!} \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \cos(n+5)\theta + \frac{1 \cdot 3 \cdot 5}{3!} \frac{(n+1)(n+2)(n+3)}{(2n+3)(2n+5)(2n+7)} \cos(n+7)\theta + \cdots \right]$$

$$0 < \theta < \pi.$$

From 3.2(40)

$$Q_0(z) = \mp \tfrac{1}{2}i\pi + z\,F(\tfrac{1}{2},\ 1;\ 3/2;\ z^2) = \mp \tfrac{1}{2}i\pi + \tfrac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

the upper or lower sign being taken according as $\operatorname{Im} z \gtrless 0$. Since $1 - z = (z-1)\,e^{\mp i\pi}$, we have

$$(20)\quad Q_0(z) = \tfrac{1}{2}\log\left(\frac{z+1}{z-1}\right),$$

and from 3.4(12)

$$(21)\quad \mathrm{Q}_0(x) = x\,F(\tfrac{1}{2},\ 1;\ 3/2;\ x^2) = \tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right).$$

Equation 3.2(13) gives for $\mu = 0$ and $\nu = n$

$$W\{P_n(z),\ Q_n(z)\} = [P_n(z)]^2\,\frac{d}{dz}\left[\frac{Q_n(z)}{P_n(z)}\right] = -(z^2-1)^{-1}$$

hence

$$(22)\quad Q_n(z) = P_n(z)\int_z^\infty (t^2-1)^{-1}\,[P_n(t)]^{-2}\,dt,$$

where the path of integration does not cross the cut. As $P_0(t) = 1$ [cf. 3.6(3)], (20) and (22), the latter for $n = 0$, are in agreement.

Now, $P_n(t)$ is a polynomial of degree n with n distinct zeros $t_1, t_2, \ldots, t_n$ say, and as $P_n(1) = (-1)^n P_n(-1) = 1$, no zero is equal to ± 1. By decomposition into partial fractions we have

$$(t^2-1)^{-1}\,[P_n(t)]^{-2}\,dt = \tfrac{1}{2}(t-1)^{-1} - \tfrac{1}{2}(t+1)^{-1} + \sum_{l=1}^{n} b_l(t-t_l)^{-2}$$

and hence from (22)

$$(23)\quad Q_n(z) = \tfrac{1}{2}P_n(z)\log\left(\frac{z+1}{z-1}\right) + P_n(z)\sum_{l=1}^{n}\frac{b_l}{(z-z_l)},$$

or

$$(24)\quad Q_n(z) = \tfrac{1}{2}P_n(z)\log\left(\frac{z+1}{z-1}\right) - W_{n-1}(z),$$

where $W_{n-1}(z)$ is a polynomial of degree $n-1$. So, for instance,

$$Q_1(z) = \frac{1}{2}\,z\log\left(\frac{z+1}{z-1}\right) - 1,$$

$$(25)\quad Q_2(z) = \frac{1}{2}P_2(z)\log\left(\frac{z+1}{z-1}\right) - \frac{3}{2}z,$$

$$Q_3(z) = \frac{1}{2}P_3(z)\log\left(\frac{z+1}{z-1}\right) - \frac{5}{2}z^2 + \frac{2}{3}.$$

From (24) it is evident that $Q_n(z)$ has logarithmic branch points at $z = \pm 1$, but there is no branch point at infinity so that any branch of the function is one-valued and regular in the z-plane supposed cut along the real axis from -1 to 1.

From (24) and 3.4 (2)

$$(26)\quad \mathrm{Q}_n(x) = \tfrac{1}{2}\,\mathrm{P}_n(x) \log\left(\frac{1+x}{1-x}\right) - W_{n-1}(x).$$

Substituting (24) in Legendre's equation 3.2 (1) with $\mu = 0$, we find that W_{n-1} $(n = 1, 2, 3, \ldots)$ satisfies the equation

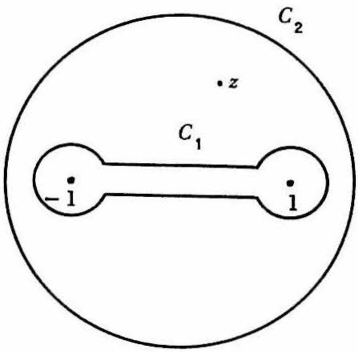
$$(27)\quad (1-z^2)\frac{d^2 W_{n-1}}{dz^2} - 2z\frac{dW_{n-1}}{dz} + n(n+1)W_{n-1} = 2\frac{dP_n}{dz}$$

and hence it can be shown that (Hobson, 1931, p. 54)

$$(28)\quad W_{n-1}(z) = \sum_{m=0}^{[n/2-\frac{1}{2}]} \frac{(2n-4m-1)}{(n-m)(2m+1)} P_{n-2m-1}(z).$$

This is Christoffel's formula.

Next, let z be any fixed point of the w plane which does not lie on the real axis between -1 and 1, and apply Cauchy's theorem to the domain bounded by the contours C_1 and C_2 of the figure. Then we have



$$2\pi i\, Q_n(z) = \int_{C_2} Q_n(w)\,(w-z)^{-1}\,dw - \int_{C_1} Q_n(w)\,(w-z)^{-1}\,dw.$$

If the radius of C_2 increases indefinitely, $\int_{C_2} \to 0$ by virtue of 3.2 (5); and the contribution of the circular arcs to $\int_{C_1}$ vanishes with their radius. Thus we have

$$2\pi i\, Q_n(z) = \int_{-1}^{1} [Q_n(v - i0) - Q_n(v + i0)]\,(z-v)^{-1}\,dv,$$

and since the expression in the braces is $\pi i P_n(v)$, we have Neumann's integral representation

(29) $Q_n(z) = \frac{1}{2} \int_{-1}^{1} (z - v)^{-1} P_n(v)\, dv = (-1)^{n+1} Q_n(-z).$

Writing (29) in the form

(30) $Q_n(z) = \frac{1}{2} P_n(z) \int_{-1}^{1} (z - v)^{-1}\, dv - \frac{1}{2} \int_{-1}^{1} (z - v)^{-1} [P_n(z) - P_n(v)]\, dv,$

we see by comparison with (24) that

(31) $W_{n-1}(z) = \frac{1}{2} \int_{-1}^{1} (z - v)^{-1} [P_n(z) - P_n(v)]\, dv.$

Generalizations of Neumann's formula (29) are(Gormley, 1934, p. 149)

(32) $Q_\nu^\mu(z) = \frac{1}{2} e^{i\mu\pi} (z^2 - 1)^{\mu/2} \int_{-1}^{1} (1 - v^2)^{-\mu/2} (z - v)^{-1} P_\nu^\mu(v)\, dv,$

$\nu + \mu = 0, 1, 2, \ldots$, $\operatorname{Re} \nu > -1$, z not on the real axis between -1 and 1; and (Wrinch 1930, p. 1037)

$$P_n(z)\, Q_m(z) = \frac{1}{2} \int_{-1}^{1} (z - v)^{-1} P_n(v)\, P_m(v)\, dv,$$

$n \le m$, z not on the real axis between -1 and 1, n, m integers.

For the Legendre polynomials we have the generating function

(33) $$(1 - 2hz + h^2)^{-1/2} = \begin{cases} \sum_{n=0}^{\infty} h^n P_n(z) \text{ for } |h| < \min|z \pm (z^2 - 1)^{1/2}| \\ \sum_{n=0}^{\infty} h^{-n-1} P_n(z) \text{ for } |h| > \max|z \pm (z^2 - 1)^{1/2}|. \end{cases}$$

This may easily be found by expanding (33) in a series respectively of ascending and descending powers of h, and using (16). (For a generalization cf. section 3.15).

On the other hand if $z = \cosh(u + iv)$ (u, v real),

$$\sum_{n=0}^{\infty} h^n Q_n(z)$$

converges for $|h| < e^{\pm u}$. Inserting here the expression (30) for $Q_n(z)$ and considering (33), we obtain

$$\sum_{n=0}^{\infty} Q_n(z)\, h^n = \frac{1}{2} \int_{-1}^{1} (z - v)^{-1} (1 - 2hv + h^2)^{-1/2} dv,$$

or

(34) $$\sum_{n=0}^{\infty} Q_n(z)\, h^n = (1 - 2hz + h^2)^{-1/2} \log \left[\frac{z - h + (1 - 2hz + h^2)^{1/2}}{(z^2 - 1)^{1/2}} \right],$$

and from 3.4(2)

$$(35)\quad \sum_{n=0}^{\infty} Q_n(x)\, h^n = (1-2hx+h^2)^{-1/2} \log\left[\frac{x-h+(1-2hx+h^2)^{1/2}}{(1-x^2)^{1/2}}\right]$$

For further results in case of integral values of μ and ν compare Chapters 10 and 11, and for results on associated Legendre functions with the sum of the degree and the order equal to a positive integer see section 3.15 and MacRobert, 1943, p. 1; 1947, p. 332.

3.7. Integral representations

From 3.2(7) and 2.12(10) it follows at once that

$$(1)\quad P_\nu^\mu(z) = \frac{2^{-\nu}(z^2-1)^{-\mu/2}}{\Gamma(-\mu-\nu)\,\Gamma(\nu+1)} \int_0^\infty (z+\cosh t)^{\mu-\nu-1} (\sinh t)^{2\nu+1}\, dt$$

$$\operatorname{Re}(-\mu) > \operatorname{Re}\nu > -1,$$

z not on the real axis between -1 and ∞.

Similarly from 3.2(45). and 2.12(14)

$$(2)\quad \Gamma(\nu-\mu+1)\,\Gamma(\mu+\tfrac{1}{2})\, Q_\nu^\mu(z) = e^{i\mu\pi}\pi^{1/2}\, 2^{-\mu}\,\Gamma(\nu+\mu+1)(z^2-1)^{\mu/2}$$
$$\times \int_0^\infty [z+(z^2-1)^{1/2}\cosh t]^{-\nu-\mu-1} (\sinh t)^{2\mu}\, dt,$$

$$(3)\quad \Gamma(\nu-\mu+1)\,\Gamma(\mu+\tfrac{1}{2})\, Q_\nu^\mu(\cosh a) = e^{i\mu\pi}\pi^{1/2}\, 2^{-\mu}\,\Gamma(\nu+\mu+1)(\sinh a)^{\mu}$$
$$\times \int_0^\infty (\cosh a + \sinh a\cosh t)^{-\nu-\mu-1} (\sinh t)^{2\mu}\, dt,$$

both formulas valid for $\operatorname{Re}(\nu \pm \mu + 1) > 0$.

Setting $e^v = \cosh a + \sinh a \cosh t$ in (3) and using 3.3(2) we obtain

$$(4)\quad Q_\nu^\mu(\cosh a) = (\tfrac{1}{2}\pi)^{1/2}\, e^{\mu\pi i}\frac{(\sinh a)^\mu}{\Gamma(\frac{1}{2}-\mu)} \int_a^\infty e^{-(\nu+\frac{1}{2})v} (\cosh v - \cosh a)^{-\mu-\frac{1}{2}}\, dv$$

$$a > 0, \quad \operatorname{Re}(\nu+\mu+1) > 0, \quad \operatorname{Re}\mu < \tfrac{1}{2}.$$

Furthermore it follows from 3.2(36) and 2.12(8) that when z is not on the real axis between -1 and 1,

$$(5)\quad Q_\nu^\mu(z) = e^{\mu\pi i}\, 2^{-\nu-1}\,\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+1)}(z^2-1)^{-\mu/2}$$
$$\times \int_0^\pi (z+\cos t)^{\mu-\nu-1} (\sin t)^{2\nu+1}\, dt$$
$$\operatorname{Re}\nu > -1, \quad \operatorname{Re}(\nu+\mu+1) > 0.$$

From 3.2(28) and 2.12(8) or from (5) and 3.3(14) we have

$$(6)\quad P_\nu^\mu(z) = \frac{\pi^{-1/2}\, 2^\mu\,(z^2-1)^{-\mu/2}}{\Gamma(\frac{1}{2}-\mu)} \int_0^\pi [z+(z^2-1)^{1/2}\cos t]^{\nu+\mu} (\sin t)^{-2\mu}\, dt$$

$$\operatorname{Re}\mu < \tfrac{1}{2};$$

From (6)

(7) $$P_\nu^\mu(\cosh a) = \frac{\pi^{-1/2}\, 2^\mu (\sinh a)^{-\mu}}{\Gamma(\tfrac{1}{2} - \mu)} \times \int_0^\pi (\sin t)^{-2\mu} (\cosh a + \sinh a \cos t)^{\nu+\mu}\, dt$$

$\operatorname{Re} \mu < \tfrac{1}{2}$;

and hence with the substitution $\cosh a + \sinh a \cos t = e^v$,

(8) $$P_\nu^\mu(\cosh a) = (\tfrac{1}{2}\pi)^{-1/2} \frac{(\sinh a)^\mu}{\Gamma(\tfrac{1}{2} - \mu)} \times \int_0^a (\cosh a - \cosh v)^{-\mu-1/2} \cosh[(\nu + \tfrac{1}{2}) v]\, dv$$

$\operatorname{Re} \mu < \tfrac{1}{2}$.

Another integral representation may be obtained by considering $\int e^{(\nu+1/2)v} (\cosh a - \cosh v)^{-\mu-1/2}\, dv$ taken around the rectangle with vertices $(\pm c, 0)$ and $(\pm c, i\pi)$, and having indentations at the points $(\pm a, 0)$. Making $c \to \infty$ and using (8) we obtain an integral formula for $P_\nu^\mu(\cosh a)$; in this we change ν into $-\nu - 1$ and then add the two expressions using 3.3 (1). The result is

(9) $$P_\nu^\mu(\cosh a) = (\tfrac{1}{2}\pi)^{-1/2} \frac{(\sinh a)^\mu}{\Gamma(\tfrac{1}{2}-\mu)} \left\{ \int_a^\infty \frac{\sin(\mu\pi) \cosh[(\nu + \tfrac{1}{2}) t]}{(\cosh t - \cosh a)^{\mu+1/2}}\, dt - \int_0^\infty \frac{\sin(\nu\pi) \cosh[(\nu + \tfrac{1}{2}) t]}{(\cosh t + \cosh a)^{\mu+1/2}}\, dt \right\}$$

$\operatorname{Re} \mu < \tfrac{1}{2}, \quad \operatorname{Re}(\nu + \mu + 1) > 0, \quad \operatorname{Re}(\mu - \nu) > 0.$

On substituting for $I_{\nu+1/2}(t)$ in 7.8 (6) from 7.3 (31) and changing the order of integration, it is found that

(10) $$Q_\nu^\mu(z) = e^{\mu\pi i} (2\pi)^{-1/2} (z^2 - 1)^{\mu/2}\, \Gamma(\mu + \tfrac{1}{2}) \times \{ \int_0^\pi (z - \cos t)^{-\mu-1/2} \cos[(\nu + \tfrac{1}{2}) t]\, dt - \cos(\nu\pi) \int_0^\infty (z + \cosh t)^{-\mu-1/2} e^{-(\nu+1/2) t} dt \}$$

$\operatorname{Re} \mu > -\tfrac{1}{2}, \quad \operatorname{Re}(\nu + \mu + 1) > 0$

where z is not a point on the real axis between 1 and $-\infty$. Hence, with 3.3 (9) we have

(11) $$P_\nu^{-\mu}(z) = (\tfrac{1}{2}\pi)^{-1/2} \frac{\Gamma(\mu + \tfrac{1}{2}) (z^2 - 1)^{\mu/2}}{\Gamma(\nu + \mu + 1)\, \Gamma(\mu - \nu)} \times \int_0^\infty (z + \cosh t)^{-\mu-1/2} \cosh[(\nu + \tfrac{1}{2}) t]\, dt$$

$\operatorname{Re}(\mu - \nu) > 0, \quad \operatorname{Re}(\mu + \nu + 1) > 0$

where z is not a point on the real axis between -1 and $-\infty$.

Applying Whipple's transformation 3.3(13) to (11) we obtain

$$(12)\quad Q_\nu^\mu(z) = e^{i\mu\pi}\frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\int_0^\infty [z+(z^2-1)^{1/2}\cosh t]^{-\nu-1}\cosh(\mu t)dt$$

$$\operatorname{Re}(\nu\pm\mu) > -1, \quad \nu \neq -1,-2,-3,\ldots.$$

Applying 3.3(14) to (10)

$$(13)\quad P_\nu^\mu(z) = \frac{\Gamma(-\nu)}{\pi\Gamma(-\nu-\mu)}\{\int_0^\pi [z-(z^2-1)^{1/2}\cos t]^\nu \cos(\mu t)\,dt$$

$$+\sin(\mu\pi)\int_0^\infty [z+(z^2-1)^{1/2}\cosh t]^\nu e^{\mu t}dt\}$$

$$\operatorname{Re}(\nu+\mu)<0, \quad \operatorname{Re} z>0, \quad \operatorname{Re}\nu<0.$$

Hence, if μ is an integer $\mu = m$ $(m = 0, 1, 2, \ldots)$ [cf. (1.2.3)], we have

$$(14)\quad P_\nu^m(z) = \frac{\Gamma(\nu+m+1)}{\pi\Gamma(\nu+1)}\int_0^\pi [z+(z^2-1)^{1/2}\cos t]^\nu \cos(mt)\,dt$$

$$\operatorname{Re} z>0.$$

This equation may be written

$$(15)\quad P_\nu^m(z) = \frac{\Gamma(\nu+m+1)}{2\pi\Gamma(\nu+1)}\int_{-\pi}^\pi [z+(z^2-1)^{1/2}\cos t]^\nu e^{imt}dt$$

$$\operatorname{Re} z>0.$$

Hence, substituting $t = \Phi - \psi$ we obtain

$$(16)\quad P_\nu^m(z)\cos(m\psi) = \frac{\Gamma(\nu+m+1)}{2\pi\Gamma(\nu+1)}$$

$$\times\int_0^{2\pi}[z+(z^2-1)^{1/2}\cos(\Phi-\psi)]^\nu\cos(m\Phi)\,d\Phi \qquad \operatorname{Re} z>0,$$

$$(17)\quad P_\nu^m(z)\sin(m\psi) = \frac{\Gamma(\nu+m+1)}{2\pi\Gamma(\nu+1)}$$

$$\times\int_0^{2\pi}[z+(z^2-1)^{1/2}\cos(\Phi-\psi)]^\nu\sin(m\Phi)\,d\Phi \qquad \operatorname{Re} z>0.$$

In case $\psi = 0$, (16) may be extended to unrestricted values of m and $\operatorname{Re}\nu > -1$ (Erdélyi, 1941, p. 351).

In case $m = 0$ a generalization of (16) is (Whittaker-Watson 1927, p. 329)

$$(18)\quad P_\nu[zz' - (z^2-1)^{1/2}(z'^2-1)^{1/2}\cos\psi]$$

$$= (2\pi)^{-1}\int_0^{2\pi}[z+(z^2-1)^{1/2}\cos(\psi-\Phi)]^\nu$$

$$\times[z'+(z'^2-1)^{1/2}\cos\Phi]^{-\nu-1}\,d\Phi \qquad \operatorname{Re} z>0, \quad \operatorname{Re} z'>0.$$

Other expressions for $P_\nu^\mu(z)$ may be derived from the previous results by means of 3.3 (1).

By means of 3.4 (5), 3.4 (8), and 3.4 (2) similar expressions for $\mathrm{P}_\nu^\mu(x)$ and $\mathrm{Q}_\nu^\mu(x)$ may be deduced. From 3.4 (8) and (2) with $z = \cos\theta$ we obtain

$$(19)\quad \Gamma(\mu+\tfrac{1}{2})\,\Gamma(\nu-\mu+1)\,\mathrm{P}_\nu^\mu(\cos\theta) = i\pi^{-\frac{1}{2}}\,2^{-\mu}\,\Gamma(\nu+\mu+1)\,(\sin\theta)^\mu$$
$$\times\Big[\int_0^\infty(\cos\theta+i\sin\theta\cosh t)^{-\nu-\mu-1}(\sinh t)^{2\mu}\,dt$$
$$-\int_0^\infty(\cos\theta-i\sin\theta\cosh t)^{-\nu-\mu-1}(\sinh t)^{2\mu}\,dt\Big]$$
$$\operatorname{Re}\mu > -\tfrac{1}{2},\qquad \operatorname{Re}(\nu\pm\mu+1)>0.$$

And from 3.4 (2) and (2)

$$(20)\quad \Gamma(\mu+\tfrac{1}{2})\,\Gamma(\nu-\mu+1)\,\mathrm{Q}_\nu^\mu(\cos\theta) = \pi^{\frac{1}{2}}\,2^{-\mu-1}\,\Gamma(\nu+\mu+1)\,(\sin\theta)^\mu$$
$$\times\Big[\int_0^\infty(\cos\theta+i\sin\theta\cosh t)^{-\nu-\mu-1}(\sinh t)^{2\mu}dt$$
$$+\int_0^\infty(\cos\theta-i\sin\theta\cosh t)^{-\nu-\mu-1}(\sinh t)^{2\mu}\,dt\Big],$$
$$\operatorname{Re}\mu > -\tfrac{1}{2},\qquad \operatorname{Re}(\nu\pm\mu+1)>0.$$

From 3.4 (8) and (12), and 3.4 (2) and (12) respectively, we have

$$(21)\quad \mathrm{P}_\nu^\mu(\cos\theta) = \frac{i\,\Gamma(\nu+1)}{\pi\,\Gamma(\nu-\mu+1)}\Big[e^{-i\frac{1}{2}\mu\pi}\int_0^\infty(\cos\theta+i\sin\theta\cosh t)^{-\nu-1}$$
$$\times\cosh(\mu t)\,dt - e^{\frac{1}{2}i\mu\pi}\int_0^\infty(\cos\theta-i\sin\theta\cosh t)^{-\nu-1}$$
$$\times\cosh(\mu t)\,dt\Big]$$

$$(22)\quad \mathrm{Q}_\nu^\mu(\cos\theta) = \frac{\Gamma(\nu+1)}{2\Gamma(\nu-\mu+1)}\Big[e^{-\frac{1}{2}i\mu\pi}\int_0^\infty(\cos\theta+i\sin\theta\cosh t)^{-\nu-1}$$
$$\times\cosh(\mu t)\,dt + e^{\frac{1}{2}i\mu\pi}\int_0^\infty(\cos\theta-i\sin\theta\cosh t)^{-\nu-1}$$
$$\times\cosh(\mu t)\,dt\Big],$$

both formulas being valid for $\operatorname{Re}(\nu\pm\mu) > -1$, $\nu \neq -1, -2, -3, \ldots$.

From 3.4 (5) and (6)

$$(23)\quad \mathrm{P}_\nu^\mu(\cos\theta)$$
$$=\frac{\pi^{-\frac{1}{2}}2^\mu(\sin\theta)^{-\mu}}{\Gamma(\frac{1}{2}-\mu)}\int_0^\pi(\cos\theta+i\sin\theta\cos t)^{\nu+\mu}(\sin t)^{-2\mu}\,dt$$
$$\operatorname{Re}\mu<\tfrac{1}{2}$$

and from 3.4 (5) and (13)

$$(24)\quad \mathrm{P}_\nu^\mu(\cos\theta)$$
$$=\frac{e^{\frac{1}{2}i\mu\pi}\,\Gamma(-\nu)}{\Gamma(-\nu-\mu)}\Big[\int_0^\pi(\cos\theta-i\sin\theta\cosh t)^\nu\cos(\mu t)\,dt$$

$$+ \sin(\mu\pi) \int_0^\infty (\cos\theta + i\sin\theta\cosh t)^\nu e^{\mu t} dt]$$

$$0 < \theta < \pi/2, \quad \mathrm{Re}(-\nu - \mu) > 0.$$

From 3.4(5), (16), and (17)

$$(25) \quad \mathrm{P}_\nu^m(\cos\theta)\cos(m\psi) = i^m \frac{\Gamma(\nu+m+1)}{2\pi\,\Gamma(\nu+1)} \times \int_0^{2\pi} [\cos\theta + i\sin\theta\cos(\Phi-\psi)]^\nu \cos(m\Phi)\, d\Phi,$$

$$(26) \quad \mathrm{P}_\nu^m(\cos\theta)\sin(m\psi) = i^m \frac{\Gamma(\nu+m+1)}{2\pi\,\Gamma(\nu+1)} \times \int_0^{2\pi} [\cos\theta + i\sin\theta\cos(\Phi-\psi)]^\nu \sin(m\Phi)\, d\Phi \qquad 0 < \theta < \pi/2.$$

In case $\pi/2 < \theta < \pi$ the expression on the right-hand side of (25) and (26) can be evaluated by means of 3.4(14).

With the substitution $\cos\theta + i\sin\theta\cos t = e^{iv}$ we find, from (23),

$$(27) \quad \mathrm{P}_\nu^\mu(\cos\theta) = (\tfrac{1}{2}\pi)^{-\frac{1}{2}} \frac{(\sin\theta)^\mu}{\Gamma(\frac{1}{2}-\mu)} \times \int_0^\theta (\cos v - \cos\theta)^{-\mu-\frac{1}{2}} \cos[(\nu+\tfrac{1}{2})v]\, dv$$

$$0 < \theta < \pi, \quad \mathrm{Re}\,\mu < \tfrac{1}{2}.$$

This is the Mehler-Dirichlet formula.

Furthermore, (Copson, 1945, p. 81)

$$(28) \quad \mathrm{Q}_n(\cos\theta) = \tfrac{1}{4} i^{n+1} \int_{-\pi}^{\pi} |\sin t|^n (\sin\theta + i\cos\theta\sin t)^{-n-1}\, dt.$$

The formula

$$(29) \quad \Gamma(\mu)\, P_\nu^{-\mu}(z) = (z^2-1)^{-\mu/2} \int_1^z P_\nu(t)\,(z-t)^{\mu-1}\, dt,$$

valid for $\mathrm{Re}\,\mu > 0$, z not on the real axis between -1 and 1, may easily be proved from 3.6(3) and 2.1(7). The integral $\int_1^z (t-1)^n (z-t)^{\mu-1}\, dt$, which occurs in the proof, may be evaluated by the substitution

$$t = v(z-1) + 1$$

and 1.5(1) as

$$(z-1)^{n+\mu}\, \Gamma(n+1)\, \Gamma(\mu)/\Gamma(n+1+\mu).$$

In the same manner

$$(30) \quad \Gamma(\mu)\, \mathrm{P}_\nu^{-\mu}(x) = (1-x^2)^{-\mu/2} \int_x^1 P_\nu(t)\,(t-x)^{\mu-1}\, dt \qquad \mathrm{Re}\,\mu > 0,$$

by means of 3.6(3), 1.5(13), and 3.4(5).

Similarly we have

(31) $\Gamma(\mu)\, Q_\nu^{-\mu}(z) = e^{-\mu\pi i}\,(z^2-1)^{-\mu/2}\int_z^\infty Q_\nu(t)\,(t-z)^{\mu-1}\,dt$

$|z|>1, \quad \operatorname{Re}\mu>0, \quad \operatorname{Re}(\nu-\mu+1)>0,$

by means of 3.2(5), 1.5(2) and the substitution $t = vz + z$.

Furthermore from 3.2(45) and 2.12(15)

(32) $\Gamma(\tfrac12-\mu)\, Q_\nu^\mu(z) = e^{i\mu\pi}\,2^\mu\,\pi^{1/2}\,(z^2-1)^{\mu/2}\,[z+(z^2-1)^{1/2}]^{-\nu-1/2}$
$\times\int_0^\infty e^{-(\nu+\mu+1)t}\{(1-e^{-t})\,[z+(z^2-1)^{1/2}-ze^{-t}+(z^2-1)\,e^{-t}]\}dt$

$\operatorname{Re}\mu<\tfrac12, \quad \operatorname{Re}(\nu+\mu+1)>0.$

The corresponding expressions for $P_\nu^\mu(z)$, $\mathrm{Q}_\nu^\mu(\cos\theta)$, and $\mathrm{P}_\nu^\mu(\cos\theta)$ follow by means of 3.3(9), 3.4(2), and 3.4(8) respectively.

The formula

(33) $$P_\nu^\mu(z) = \frac{2^\mu\,\Gamma(1-2\mu)\,(z^2-1)^{-\mu/2}}{\Gamma(1-\mu)\,\Gamma(-\mu-\nu)\,\Gamma(\nu-\mu+1)}$$
$$\times\int_0^\infty (1+2tz+t^2)^{\mu-1/2}\,t^{-1-\nu-\mu}\,dt$$

$\operatorname{Re}(\mu+\nu)<0, \quad \operatorname{Re}(\mu-\nu)<1, \quad |\arg(z\pm1)|<\pi$

may be proved by writing $1+2tz+t^2 = (1+t)^2\,[1-2t\,(1+t)^{-2}\,(1-z)]$, expanding the integrand in a series, integrating term by term, and using 1.5(12), 2.1(2), and 3.2(7). For representations of the Legendre functions as loop integrals see Hobson, 1931, pp. 183-200, 236-243, 266.

3.8. Relations between contiguous Legendre functions

The recurrence formulas for the Legendre functions may be derived by applying Gauss' relations between contiguous hypergeometric functions, 2.8. So, we have from 3.2(14) and 2.8(30)

(1) $P_\nu^{\mu+2}(z) + 2(\mu+1)\,z\,(z^2-1)^{-1/2}\,P_\nu^{\mu+1}(z)$
$= (\nu-\mu)\,(\nu+\mu+1)\,P_\nu^\mu(z)$

from 3.2(28) and 2.8(28)

(2) $(2\nu+1)\,z\,P_\nu^\mu(z) = (\nu-\mu+1)\,P_{\nu+1}^\mu(z) + (\nu+\mu)\,P_{\nu-1}^\mu(z)$

and from 3.2(24) and 3.2(4)

(3) $P_{\nu-1}^\mu(z) - P_{\nu+1}^\mu(z) = -(2\nu+1)\,(z^2-1)^{1/2}\,P_\nu^{\mu-1}(z).$

The following formulas may be derived from (1) to (3)

(4) $(\nu-\mu)\,(\nu-\mu+1)\,P_{\nu+1}^\mu(z) - (\nu+\mu)\,(\nu+\mu+1)\,P_{\nu-1}^\mu(z)$
$= (2\nu+1)\,(z^2-1)^{1/2}\,P_\nu^{\mu+1}(z),$

(5) $P^{\mu}_{\nu-1}(z) - zP^{\mu}_{\nu}(z) = -(\nu-\mu+1)(z^2-1)^{1/2} P^{\mu-1}_{\nu}(z),$

(6) $zP^{\mu}_{\nu}(z) - P^{\mu}_{\nu+1}(z) = -(\nu+\mu)(z^2-1)^{1/2} P^{\mu-1}_{\nu}(z),$

(7) $(\nu-\mu)zP^{\mu}_{\nu}(z) - (\nu+\mu)P^{\mu}_{\nu-1}(z) = (z^2-1)^{1/2} P^{\mu+1}_{\nu}(z),$

(8) $(\nu-\mu+1)P^{\mu}_{\nu+1}(z) - (\nu+\mu+1)zP^{\mu}_{\nu}(z) = (z^2-1)^{1/2} P^{\mu+1}_{\nu}(z).$

Differentiating 3.2(7) and using 2.1(7)

(9) $$\frac{dP^{\mu}_{\nu}(z)}{dz} = (\nu+\mu)(\nu-\mu+1)(z^2-1)^{-1/2} P^{\mu-1}_{\nu}(z) - \frac{\mu z}{z^2-1} P^{\mu}_{\nu}(z).$$

Eliminating $P^{\mu-1}_{\nu}(z)$ by means of (6) it follows that

(10) $$(z^2-1)\frac{dP^{\mu}_{\nu}(z)}{dz} = (\nu-\mu+1)P^{\mu}_{\nu+1}(z) - (\nu+1)zP^{\mu}_{\nu}(z)$$
$$= \nu zP^{\mu}_{\nu}(z) - (\nu+\mu)P^{\mu}_{\nu-1}(z).$$

It may easily be shown that the formulas (1) to (10) are valid for $Q^{\mu}_{\nu}(z)$.

With $P^{\mu}_{\nu}(x+i0) = e^{-\frac{1}{2}i\mu\pi}\,\mathrm{P}^{\mu}_{\nu}(x)$ we have the following recurrence relations for Legendre functions on the cut:

(11) $\mathrm{P}^{\mu+2}_{\nu}(x) + 2(\mu+1)x(1-x^2)^{-1/2}\mathrm{P}^{\mu+1}_{\nu}(x) + (\nu-\mu)(\nu+\mu+1)\mathrm{P}^{\mu}_{\nu}(x) = 0,$

(12) $(2\nu+1)x\,\mathrm{P}^{\mu}_{\nu}(x) = (\nu-\mu+1)\mathrm{P}^{\mu}_{\nu+1}(x) + (\nu+\mu)\mathrm{P}^{\mu}_{\nu-1}(x),$

(13) $\mathrm{P}^{\mu}_{\nu-1}(x) - \mathrm{P}^{\mu}_{\nu+1}(x) = (2\nu+1)(1-x^2)^{1/2}\mathrm{P}^{\mu-1}_{\nu}(x),$

(14) $(\nu-\mu)(\nu-\mu+1)\mathrm{P}^{\mu}_{\nu+1}(x) - (\nu+\mu)(\nu+\mu+1)\mathrm{P}^{\mu}_{\nu-1}(x)$
$= (2\nu+1)(1-x^2)^{1/2}\mathrm{P}^{\mu+1}_{\nu}(x),$

(15) $\mathrm{P}^{\mu}_{\nu-1}(x) - x\mathrm{P}^{\mu}_{\nu}(x) = (\nu-\mu+1)(1-x^2)^{1/2}\mathrm{P}^{\mu-1}_{\nu}(x),$

(16) $x\mathrm{P}^{\mu}_{\nu}(x) - \mathrm{P}^{\mu}_{\nu+1}(x) = (\nu+\mu)(1-x^2)^{1/2}\mathrm{P}^{\mu-1}_{\nu}(x),$

(17) $(\nu-\mu)x\,\mathrm{P}^{\mu}_{\nu}(x) - (\nu+\mu)\mathrm{P}^{\mu}_{\nu-1}(x) = (1-x^2)^{1/2}\mathrm{P}^{\mu+1}_{\nu}(x),$

(18) $(\nu-\mu+1)\mathrm{P}^{\mu}_{\nu+1}(x) - (\nu+\mu+1)x\,\mathrm{P}^{\mu}_{\nu}(x) = (1-x^2)^{1/2}\mathrm{P}^{\mu+1}_{\nu}(x),$

(19) $$(1-x^2)\frac{d\mathrm{P}^{\mu}_{\nu}(x)}{dx} = (\nu+1)x\mathrm{P}^{\mu}_{\nu}(x) - (\nu-\mu+1)\mathrm{P}^{\mu}_{\nu+1}(x)$$
$$= -\nu x\,\mathrm{P}^{\mu}_{\nu}(x) + (\nu+\mu)\mathrm{P}^{\mu}_{\nu-1}(x).$$

Here again it can be shown, that the formulas (11) to (19) are valid for the $Q_\nu^\mu(x)$.

From (2) we have Christoffel's first and second summation formula

$$(20)\quad (\zeta - z) \sum_{m=0}^{n} (2m+1) P_m(z) P_m(\zeta) = (n+1)[P_{n+1}(\zeta) P_n(z) - P_n(\zeta) P_{n+1}(z)],$$

$$(21)\quad (\zeta - z) \sum_{m=0}^{n} (2m+1) P_m(z) Q_m(\zeta) = 1 - (n+1)[P_{n+1}(z) Q_n(\zeta) - P_n(z) Q_{n+1}(\zeta)]$$

3.9.1. Asymptotic expansions

For large positive Re c, the hypergeometric series $F(a, b; c; z)$ is an asymptotic expansion in c, even when the series does not converge, provided only that z is not a real number > 1. Hence, for fixed z and ν, and Re $\mu \to \infty$, 3.3(17), 3.3(16), 3.2(3), and 3.3(15) are asymptotic expansions of, respectively, $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$, $P_\nu^{-\mu}(z)$, and $Q_\nu^{-\mu}(z)$. The first, second, and fourth expansions hold for all z save for points on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$, and 3.2(3) holds for all z not on the real axis between $-\infty$ and -1.

For fixed z and μ, and Re $\nu \to \infty$, 3.3(21), 3.2(44), 3.3(21) together with 3.3(1) and 3.3(22) are asymptotic expansions of respectively $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$, $P_{-\nu}^\mu(z)$, and $Q_{-\nu}^\mu(z)$. The first, third and fourth expansions hold for all z save for points on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$, and 3.2(44) holds for all z not on the real axis between $-\infty$ and $+1$.

The expressions 3.5(5) and 3.5(6) are asymptotic expansions in ν of $P_\nu^\mu(\cos\theta)$ and $Q_\nu^\mu(\cos\theta)$, respectively, valid if $\epsilon \le \theta \le \pi - \epsilon$, $\epsilon > 0$.

Thus we have

$$(1)\quad Q_\nu^\mu(\cos\theta) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)} \left(\frac{\pi}{2\sin\theta}\right)^{\frac12} \times \{\cos[(\nu+\tfrac12)\theta + \pi/4 + \tfrac12\mu\pi] + O(\nu^{-1})\},$$

and

$$(2)\quad P_\nu^\mu(\cos\theta) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)} (\tfrac12\pi\sin\theta)^{-\frac12} \times \{\cos[(\nu+\tfrac12)\theta - \pi/4 + \tfrac12\mu\pi] + O(\nu^{-1})\},$$

$$\epsilon \le \theta \le \pi - \epsilon, \quad \epsilon > 0.$$

For small values of θ see 3.5(10). Formula 3.5(9) gives an asymptotic expansion in μ of $P_\nu^{-\mu}(\cos\theta)$.

3.9.2. Behavior of the Legendre functions near the singular points

BEHAVIOR AT 1

	Function	Restrictions	Leading term
(3)	$P_\nu^\mu(z)$	$\mu \neq 1, 2, 3, \ldots,$	$2^{\mu/2}(z-1)^{-\mu/2}/\Gamma(1-u)$
(4)	$P_\nu^m(z)$	$m = 0, 1, 2, \ldots,$	$\dfrac{2^{-\frac{1}{2}m}\Gamma(\nu+m+1)(z-1)^{\frac{1}{2}m}}{m!\,\Gamma(\nu-m+1)}$
(5)	$Q_\nu^\mu(z)\,e^{-i\mu\pi}$	$\operatorname{Re}\mu > 0$	$2^{\mu/2-1}\Gamma(\mu)(z-1)^{-\mu/2}$
(6)	$Q_\nu^\mu(z)\,e^{-i\mu\pi}$	$\operatorname{Re}\mu < 0$	$\dfrac{2^{-\mu/2-1}\Gamma(-\mu)\Gamma(\nu+\mu+1)(z-1)^{\mu/2}}{\Gamma(\nu-\mu+1)}$
(7)	$Q_\nu(z)$	$\nu \neq -1, -2, -3, \ldots,$	$-\frac{1}{2}\log(\frac{1}{2}z-\frac{1}{2})-\gamma-\psi(\nu+1)$
(8)	$\mathrm{P}_\nu^\mu(x)$	$\mu \neq 1, 2, 3, \ldots,$	$2^{\mu/2}(1-x)^{-\mu/2}/\Gamma(1-\mu)$
(9)	$\mathrm{P}_\nu^m(x)$	$m = 0, 1, 2, \ldots,$	$\dfrac{(-1)^m 2^{-\frac{1}{2}m}\Gamma(\nu+m+1)(1-x)^{\frac{1}{2}m}}{m!\,\Gamma(\nu-m+1)}$
(10)	$\mathrm{Q}_\nu^\mu(x)$	$\operatorname{Re}\mu > 0$	$2^{\mu/2-1}\Gamma(\mu)\cos(\mu\pi)(1-x)^{-\mu/2}$
(11)	$\mathrm{Q}_\nu^\mu(x)$	$\operatorname{Re}\mu < 0$	$\dfrac{2^{-\mu/2-1}\Gamma(-\mu)\Gamma(\nu+\mu+1)(1-x)^{\mu/2}}{\Gamma(\nu-\mu+1)}$
(12)	$\mathrm{Q}_\nu(x)$	$\nu \neq -1, -2, -3, \ldots,$	$-\frac{1}{2}\log(\frac{1}{2}-\frac{1}{2}x)-\gamma-\psi(\nu+1)$

BEHAVIOR AT −1

	Function	Restrictions	Leading term
(13)	$P_\nu^\mu(x)$	$\mathrm{Re}\,\mu > 0$	$-2^{\mu/2}\sin(\pi\nu)\,\Gamma(\mu)\,\pi^{-1}\,(1+x)^{-\mu/2}$
(14)	$P_\nu^\mu(x)$	$\mathrm{Re}\,\mu < 0$	$\dfrac{2^{-\mu/2}\,\Gamma(-\mu)\,(1+x)^{\mu/2}}{\Gamma(1+\nu-\mu)\,\Gamma(-\nu-\mu)}$
(15)	$P_\nu(x)$	None	$\pi^{-1}\sin(\nu\pi)\,[\log(\tfrac{1}{2}+\tfrac{1}{2}x)+\gamma+2\psi(\nu+1)+\pi\cot(\nu\pi)]$
(16)	$Q_\nu^\mu(x)$	$\mathrm{Re}\,\mu > 0$	$-2^{\mu/2-1}\,\Gamma(\mu)\cos(\nu\pi)\,(1+x)^{-\mu/2}$
(17)	$Q_\nu^\mu(x)$	$\mathrm{Re}\,\mu < 0$	$\dfrac{-2^{-\mu/2-1}\cos[\pi(\nu+\mu)]\,\Gamma(-\mu)\,\Gamma(\nu+\mu+1)\,(1+x)^{\mu/2}}{\Gamma(\nu-\mu+1)}$
(18)	$Q_\nu(x)$	$\nu \neq -1, -2, -3, \ldots,$	$\tfrac{1}{2}\cos(\nu\pi)[\log(\tfrac{1}{2}+\tfrac{1}{2}x)+\gamma+2\psi(\nu+1)-\pi\tan(\nu\pi)]$

BEHAVIOR AT ∞

	Function	Restrictions	Leading term
(19)	$P_\nu^\mu(z)$	$\mathrm{Re}\,\nu > -\tfrac{1}{2}$	$2^\nu\pi^{-1/2}\,\Gamma(\nu+\tfrac{1}{2})\,z^\nu/\Gamma(1+\nu-\mu)$
(20)	$P_\nu^\mu(z)$	$\mathrm{Re}\,\nu < -\tfrac{1}{2}$	$\pi^{-1/2}\,2^{-\nu-1}\,\Gamma(-\nu-\tfrac{1}{2})\,z^{-\nu-1}/\Gamma(-\nu-\mu)$
(21)	$Q_\nu^\mu(z)$		$e^{i\mu\pi}\,2^{-\nu-1}\,\pi^{1/2}\,\Gamma(\nu+\mu+1)\,z^{-\nu-1}/\Gamma(\nu+3/2)$

The behavior of Legendre functions in the neighborhood of one of the singular points, 1, − 1, or ∞, may be investigated by means of the expansions of sections 3.2, 3.4, and 3.6. The results, together with the restrictions that have to be imposed, are shown in the tables on the preceding pages. Information about the source of these results follows this paragraph. Equation (3), for instance, shows that in the neighborhood of $z = 1$, the function $P^{\mu}(z)$ is equal to $[2^{\mu/2}(z-1)^{-\mu/2}/\Gamma(1-\mu)]$ + terms of higher order in $z - 1$ provided that μ is not a positive integer. This result is derived from 3.2 (3).

Equation	Proved from
(3)	3.2 (3)
(4)	3.6 (1)
(5)	3.2 (32)
(6)	3.2 (32)
(7)	3.2 (36) and 2.10 (14)
(8)	3.4 (6)
(9)	3.6 (2)
(10)	3.4 (10)
(11)	3.4 (10)
(12)	(7), 3.4 (2), and 3.2 (12)
(13)	3.4 (14), (8), and (10)
(14)	3.4 (14), (8), and (11)
(15)	3.4 (14), (8), and (12)
(16)	3.4 (15), (10), and (8)
(17)	3.4 (15), (11), and (8)
(18)	3.4 (15), (12), and (8)
(19)	3.2 (18)
(20)	3.2 (18)
(21)	3.2 (5)

3.10. Expansions in terms of Legendre functions

Some of the integral representations of Legendre functions are of the form of Fourier coefficients and may be used to sum certain Fourier series.

For a fixed θ, $0 < \theta < \pi$, let

$$f(v) = \begin{cases} (\cos v - \cos\theta)^{-\mu-\frac{1}{2}} & 0 \leq v < \theta \quad \text{or} \quad 2\pi - \theta < v \leq 2\pi, \\ 0 & \theta < v < 2\pi - \theta. \end{cases}$$

The Fourier series of $f(v)$ may be formed from 3.7(27). This establishes the expansion

$$(1)\quad \Gamma(\tfrac{1}{2}-\mu)\left[\mathrm{P}^{\mu}_{-\frac{1}{2}}(\cos\theta)+2\sum_{n=1}^{\infty}\mathrm{P}^{\mu}_{n-\frac{1}{2}}(\cos\theta)\cos(nv)\right]$$
$$=\begin{cases}(2\pi)^{\frac{1}{2}}(\sin\theta)^{\mu}(\cos v-\cos\theta)^{-\mu-\frac{1}{2}} & 0\le v<\theta,\\ 0 & \theta<v<\pi,\end{cases}$$
$$0<\theta<\pi,\quad \mathrm{Re}\,\mu<\tfrac{1}{2}.$$

Hence, replacing v by $\pi-\theta$ and θ by $\pi-v$,

$$\Gamma(\tfrac{1}{2}-\mu)\left[P^{\mu}_{-\frac{1}{2}}(-\cos v)+2\sum_{n=1}^{\infty}(-1)^{n}P^{\mu}_{n-\frac{1}{2}}(-\cos v)\cos(n\theta)\right]$$
$$=\begin{cases}(2\pi)^{\frac{1}{2}}(\sin v)^{n}(\cos v-\cos\theta)^{-\mu-\frac{1}{2}} & v<\theta<\pi,\\ 0 & 0<\theta<v,\end{cases}$$
$$0<v<\pi,\quad \mathrm{Re}\,\mu<\tfrac{1}{2}.$$

In a similar manner, from 3.7(27)

$$(2)\quad \Gamma(\tfrac{1}{2}-\mu)\sum_{n=0}^{\infty}\mathrm{P}^{\mu}_{n}(\cos\theta)\cos(n+\tfrac{1}{2})v$$
$$=\begin{cases}(\tfrac{1}{2}\pi)^{\frac{1}{2}}(\sin\theta)^{\mu}(\cos v-\cos\theta)^{-\mu-\frac{1}{2}} & 0\leqq v<\theta,\\ 0 & \theta<v<\pi,\end{cases}$$
$$0<\theta<\pi,\quad \mathrm{Re}\,\mu<\tfrac{1}{2}.$$

If we expand $(z-\cos v)^{-\mu-\frac{1}{2}}$ into a Fourier series (z fixed and not on the real axis between -1 and 1) and use 3.7(10), we obtain:

$$(3)\quad Q^{\mu}_{-\frac{1}{2}}(z)+2\sum_{n=1}^{\infty}Q^{\mu}_{n-\frac{1}{2}}(z)\cos(nv)$$
$$=e^{i\mu\pi}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\Gamma(\mu+\tfrac{1}{2})(z^{2}-1)^{\mu/2}(z-\cos v)^{-\mu-\frac{1}{2}},\quad \mathrm{Re}\,\mu>-\tfrac{1}{2},$$

z not on the real axis between -1 and 1.

Furthermore from 3.7(16)

$$(4)\quad P_{\nu}(z)+2\sum_{m=1}^{\infty}\frac{\Gamma(\nu+1)}{\Gamma(\nu+m+1)}P^{m}_{\nu}(z)\cos[m(v-\phi)]$$
$$=[z+(z^{2}-1)^{\frac{1}{2}}\cos(v-\phi)]^{\nu}\qquad \mathrm{Re}\,z>0.$$

Hence, putting $\phi = 0$, changing ν into $-\nu - 1$ and using 3.3(1) and 1.2(3)

(5) $$P_\nu(z) + 2 \sum_{m=1}^{\infty} (-1)^m \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + 1)} P_\nu^m(z) \cos(mv)$$
$$= [z + (z^2 - 1)^{1/2} \cos v]^{-\nu-1} \qquad \operatorname{Re} z > 0$$

Dougall's expansion

(6) $$\mathrm{P}_\nu^{-\mu}(\cos\theta) = \frac{\sin(\nu\pi)}{\pi} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\nu - n} - \frac{1}{\nu + n + 1}\right) \mathrm{P}_n^{-\mu}(\cos\theta)$$
$$-\pi < \theta < \pi, \quad \mu \geq 0$$

may be proved as follows. We start with the formula

(7) $$\cos[(\nu + \tfrac{1}{2})v] = \frac{\sin(\nu\pi)}{\pi} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\nu - n} - \frac{1}{\nu + n + 1}\right) \cos[(n + \tfrac{1}{2})v]$$
$$-\pi < v < \pi,$$

which may easily be established by evaluating the contour integral

$$\int [(z - v)\sin(\pi z)]^{-1} \cos[(z + \tfrac{1}{2})v]\, dz$$

taken around a circle with the origin as center and $(N + \tfrac{1}{2})\pi$ as radius. (N is a finite integer.) Applying Cauchy's theorem and making $N \to \infty$ we get (7). Substituting (7) in 3.7(27), integrating term by term and replacing μ by $-\mu$ we find (6).

Again from 3.7(27)

$$\mathrm{P}_\nu^{-\mu}(\cos\theta)\, \mathrm{P}_\nu^{-\lambda}(\cos\theta')$$
$$= (\tfrac{1}{2}\pi)^{-1} \frac{(\sin\theta)^{-\mu}}{\Gamma(\mu + \tfrac{1}{2})} \int_0^\theta \cos[(\nu + \tfrac{1}{2})v] (\cos v - \cos\theta)^{\mu - 1/2}\, dv$$
$$\times \frac{(\sin\theta')^{-\lambda}}{\Gamma(\lambda + \tfrac{1}{2})} \int_0^{\theta'} \cos[(\nu + \tfrac{1}{2})\phi] (\cos\phi - \cos\theta')^{\lambda - 1/2}\, d\phi.$$

Using (7) again and integrating term by term we find

(8) $$\mathrm{P}_\nu^{-\mu}(\cos\theta)\, \mathrm{P}_\nu^{-\lambda}(\cos\theta') = \frac{\sin(\nu\pi)}{\pi}$$
$$\times \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\nu - n} - \frac{1}{\nu + n + 1}\right) \mathrm{P}_n^{-\mu}(\cos\theta)\, \mathrm{P}_n^{-\lambda}(\cos\theta')$$
$$-\pi < \theta + \theta' < \pi, \quad -\pi < \theta - \theta' < \pi, \quad \mu \geq 0, \quad \lambda \geq 0.$$

Furthermore, putting $\mu = 0$ and $\lambda = m$ in (8), (m is a positive integer) differentiating m times with respect to x and using 3.6(6),

(9) $\mathrm{P}_\nu^m(x)\,\mathrm{P}_\nu^{-m}(x')$

$$= \frac{\sin(\nu\pi)}{\pi} \sum_{n=m}^{\infty} (-1)^m \left(\frac{1}{\nu-n} - \frac{1}{\nu+n+1}\right) \mathrm{P}_n^m(x)\,\mathrm{P}_n^{-m}(x')$$

$$0<\theta<\pi, \quad 0<\theta'<\pi, \quad \theta+\theta'<\pi, \quad x=\cos\theta, \quad x'=\cos\theta'.$$

(For similar expansions cf. MacRobert, 1934.)

From the asymptotic expansions 3.3(21) and 3.2(44) for $P_n(z)$ and $Q_n(z)$ respectively [cf. 3.9(1)] we find that the right-hand side of 3.8(21) tends to 1 when n tends to infinity, provided, that

$$|z+(z^2-1)^{1/2}| < |\zeta+(\zeta^2-1)^{1/2}|,$$

and we obtain Heine's formula

(10) $$(\zeta-z)^{-1} = \sum_{m=0}^{\infty} (2m+1)\, P_m(z)\, Q_m(\zeta).$$

For numerous other formulas see Dougall, 1919; Darling, 1923; Prasad, 1930, pp. 64-67; 159; 1931; Shabde, 1931, 1932, 1933; Banerjee, 1932; Mac Robert, 1934, 1935, 1936.

3.11. The addition theorems

The relation

(1) $$P_\nu[z z' - (z^2-1)^{1/2}(z'^2-1)^{1/2}\cos\psi] = P_\nu(z)\,P_\nu(z')$$

$$+ 2\sum_{m=1}^{\infty} (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(z)\,P_\nu^m(z')\cos(m\psi)$$

$$\mathrm{Re}\, z>0, \quad \mathrm{Re}\, z'>0, \quad |\arg(z-1)|<\pi, \quad |\arg(z'-1)|<\pi$$

may be proved as follows. We have from 3.10(4), 3.10(5) and Parseval's theorem (Titchmarsh, 1932, p. 421) the series

$$2P_\nu(z)\,P_\nu(z') + 4\sum_{m=1}^{\infty} (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m)} P_\nu^m(z)\,P_\nu^m(z')\cos(m\psi)$$

converging to

$$(1/\pi)\int_{-\pi}^{\pi} [z+(z^2-1)^{1/2}\cos(\Phi-\psi)]^\nu [z'+(z'^2-1)^{1/2}\cos\Phi]^{-\nu-1}\,d\Phi,$$

and this latter expression is equal to

$$2\,P_\nu[z z' - (z^2-1)^{1/2}(z'^2-1)^{1/2}\cos\psi]$$

by 3.7(18) which establishes (1).

Furthermore (Hobson, 1931, p. 371) we have by virtue of 3.4(14)

(2) $\mathrm{P}_\nu(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\psi)$

$$= \mathrm{P}_\nu(\cos\theta)\,\mathrm{P}_\nu(\cos\theta')$$

$$+ 2\sum_{m=1}^{\infty} (-1)^m\,\mathrm{P}_\nu^{-m}(\cos\theta)\,\mathrm{P}_\nu^m(\cos\theta')\cos(m\psi)$$

$$= \mathrm{P}_\nu(\cos\theta)\,\mathrm{P}_\nu(\cos\theta')$$

$$+ 2\sum_{m=1}^{\infty} \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}\,\mathrm{P}_\nu^m(\cos\theta)\,\mathrm{P}_\nu^m(\cos\theta')\cos(m\psi)$$

$$0 \le \theta < \pi, \quad 0 \le \theta < \pi, \quad \theta+\theta' < \pi, \quad \psi \text{ real.}$$

Hence, with 3.4(14)

(3) $\mathrm{Q}_\nu(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\psi)$

$$= \mathrm{P}_\nu(\cos\theta')\,\mathrm{Q}_\nu(\cos\theta)$$

$$+ 2\sum_{m=1}^{\infty} (-1)^m\,\mathrm{P}_\nu^{-m}(\cos\theta')\,\mathrm{Q}_\nu^m(\cos\theta)\cos(m\psi)$$

$$= \mathrm{P}_\nu(\cos\theta')\,\mathrm{Q}_\nu(\cos\theta)$$

$$+ 2\sum_{m=1}^{\infty} \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}\,\mathrm{P}_\nu^m(\cos\theta')\,\mathrm{Q}_\nu^m(\cos\theta)\cos(m\psi)$$

$$0 < \theta' < \pi/2, \quad 0 < \theta' < \pi, \quad 0 < \theta+\theta' < \pi, \quad \psi \text{ real.}$$

From (1) and 3.3(11)

(4) $Q_\nu[t\,t' - (t^2-1)^{1/2}(t'^2-1)^{1/2}\cos\psi]$

$$= Q_\nu(t)\,P_\nu(t') + 2\sum_{m=1}^{\infty} (-1)^m\,Q_\nu^m(t)\,P_\nu^{-m}(t')\cos(m\psi)$$

$$t,\ t' \text{ real}, \quad 1 < t' < t, \quad \nu \ne -1, -2, -3, \ldots, \quad \psi \text{ real.}$$

(For similar expansions cf. Cowling, 1940, p. 222.)

3.12. Integrals involving Legendre functions

If $w_\nu^\mu(z)$ and $w_\sigma^\rho(z)$ denote any solutions of Legendre's differential equation 3.2(1) with the parameters ν, μ and σ, ρ, respectively, then it follows from 3.2(1), 3.8(10), and 3.8(19) that

(1) $\int_a^b [(\nu-\sigma)(\nu+\sigma+1) + (\rho^2-\mu^2)(1-z^2)^{-1}]\, w_\nu^\mu\, w_\sigma^\rho\, dz$

$$= \left[(1-z^2)\left(w_\nu^\mu \frac{d}{dz} w_\sigma^\rho - w_\sigma^\rho \frac{d}{dz} w_\nu^\mu\right)\right]_a^b$$

$$= [z(\nu-\sigma)\, w_\nu^\mu\, w_\sigma^\rho + (\sigma+\rho)\, w_\nu^\mu\, w_{\sigma-1}^\rho - (\nu+\mu)\, w_{\nu-1}^\mu\, w_\sigma^\rho\,]_a^b$$

When $\mu = \rho = 0$, we have from (1) and 3.8(7)

(2) $\int_a^b w_\nu \, w_\sigma \, dz = [(\nu - \sigma)(\nu + \sigma + 1)]^{-1} [(z^2 - 1)^{1/2} (w_\sigma w_\nu^1 - w_\nu w_\sigma^1)]_a^b$.

If w_ν and w_σ denote two Legendre functions on the cut, we find from (1) and 3.8(17)

(3) $\int_a^b \mathrm{w}_\nu \, \mathrm{w}_\sigma \, dx = [(\nu - \sigma)(\nu + \sigma + 1)]^{-1} [(1 - x^2)^{1/2} (\mathrm{w}_\sigma \mathrm{w}_\nu^1 - \mathrm{w}_\nu \mathrm{w}_\sigma^1)]_a^b$.

The following results can be easily proved from (2) and (3) by the aid of theformulas of section 3.9.2.

(4) $\int_1^\infty P_\nu(x) \, Q_\sigma(x) \, dx = [(\sigma - \nu)(\sigma + \nu + 1)]^{-1}$

$\mathrm{Re}\,\sigma > \mathrm{Re}\,\nu > 0,$

(5) $\int_1^\infty Q_\nu(x) \, Q_\sigma(x) \, dx = [(\sigma - \nu)(\sigma + \nu + 1)]^{-1} [\psi(\sigma + 1) - \psi(\nu + 1)]$

$\mathrm{Re}(\sigma + \nu) > -1, \quad \sigma + \nu + 1 \neq 0, \quad \nu, \sigma \neq -1, -2, -3, \ldots,$

(6) $\int_1^\infty [Q_\nu(x)]^2 \, dx = (2\nu + 1)^{-1} \, \psi'(\nu + 1)$ $\qquad \mathrm{Re}\,\nu > -\tfrac{1}{2},$

(7) $\int_{-1}^1 \mathrm{P}_\nu(x) \, \mathrm{P}_\sigma(x) \, dx = 2\pi^{-2} [(\sigma - \nu)(\sigma + \nu + 1)]^{-1}$

$\times \{2 \sin(\pi\nu) \sin(\pi\sigma) [\psi(\nu + 1) - \psi(\sigma + 1)] + \pi \sin(\pi\sigma - \pi\nu)\}$

$\sigma + \nu + 1 \neq 0,$

in particular for $\nu = n$, $\sigma = m$ (n, m integers)

(8) $\int_{-1}^1 \mathrm{P}_n(x) \, \mathrm{P}_m(x) \, dx = 0,$

(9) $\int_{-1}^1 [\mathrm{P}_\nu(x)]^2 \, dx = \pi^{-2} (\nu + \tfrac{1}{2})^{-1} [\pi^2 - 2(\sin \pi\nu)^2 + \psi'(\nu + 1)],$

(10) $\int_{-1}^1 [\mathrm{P}_n(x)]^2 \, dx = (n + \tfrac{1}{2})^{-1}$ $\qquad n = 0, 1, 2, \ldots,$

(11) $\int_{-1}^1 \mathrm{Q}_\nu(x) \, \mathrm{Q}_\sigma(x) \, dx = [(\sigma - \nu)(\sigma + \nu + 1)]^{-1}$

$\times \{[\psi(\nu + 1) - \psi(\sigma + 1)] [1 + \cos(\sigma\pi) \cos(\nu\pi)] - \tfrac{1}{2}\pi \sin(\nu\pi - \sigma\pi)\}$

$\sigma + \nu + 1 \neq 0, \quad \nu, \sigma \neq -1, -2, -3, \ldots,$

(12) $\int_{-1}^1 [\mathrm{Q}_\nu(x)]^2 \, dx = (2\nu + 1)^{-1} \{\tfrac{1}{2}\pi^2 - \psi'(\nu + 1) [1 + (\cos \nu\pi)^2]\}$

$\nu \neq -1, -2, -3, \ldots,$

(13) $\int_{-1}^1 \mathrm{P}_\nu(x) \, \mathrm{Q}_\sigma(x) \, dx = [(\nu - \sigma)(\nu + \sigma + 1)]^{-1}$

$\times [1 - \cos(\sigma\pi - \nu\pi) - 2\pi^{-1} \sin(\pi\nu) \cos(\pi\sigma) [\psi(\nu + 1) - \psi(\sigma + 1)]\}$

$\mathrm{Re}\,\nu > 0, \quad \mathrm{Re}\,\sigma > 0, \quad \sigma \neq \nu,$

(14) $\int_{-1}^{1} P_\nu(x)\, Q_\nu(x)\, dx = -\pi^{-1}(2\nu+1)^{-1}\sin(2\nu\pi)\,\psi'(\nu+1)$ $\qquad \operatorname{Re}\nu > 0,$

(15) $\int_0^1 P_\nu(x)\, P_\sigma(x)\, dx = 2\pi^{-1}[(\sigma-\nu)(\sigma+\nu+1)]^{-1}$

$\times[A\sin(\tfrac{1}{2}\sigma\pi)\cos(\tfrac{1}{2}\nu\pi) - A^{-1}\sin(\tfrac{1}{2}\nu\pi)\cos(\tfrac{1}{2}\sigma\pi)],$

(16) $\int_0^1 Q_\nu(x)\, Q_\sigma(x)\, dx = [(\sigma-\nu)(\sigma+\nu+1)]^{-1}\{\psi(\nu+1) - \psi(\sigma+1)$

$- \tfrac{1}{2}\pi[(A - A^{-1})\sin(\tfrac{1}{2}\sigma\pi + \tfrac{1}{2}\nu\pi) - (A + A^{-1})\sin(\tfrac{1}{2}\sigma\pi - \tfrac{1}{2}\nu\pi)]\}$

$\operatorname{Re}\nu > 0, \quad \operatorname{Re}\sigma > 0,$

(17) $\int_0^1 P_\nu(x)\, Q_\sigma(x)\, dx = [(\sigma-\nu)(\sigma+\nu+1)]^{-1}$

$\times[A^{-1}\cos(\tfrac{1}{2}\nu\pi - \tfrac{1}{2}\sigma\pi) - 1]$ $\qquad \operatorname{Re}\sigma > 0, \quad \operatorname{Re}\nu > 0.$

In (15) to (17)

$$A = \frac{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu)\,\Gamma(1+\tfrac{1}{2}\sigma)}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\sigma)\,\Gamma(1+\tfrac{1}{2}\nu)}.$$

If in (1) $\mu = \rho = m$, $\nu = n$, $\sigma = l$ (l, m, n, positive integers) we obtain from 3.9(8), 3.9(10), and 3.4(19)

(18) $$\int_{-1}^{1} Q_n^m(x)\, P_l^m(x)\, dx = (-1)^m \frac{1-(-1)^{l+n}}{(l-n)(l+n+1)} \frac{(n+m)!}{(n-m)!}.$$

Likewise from (1) if m, n, l, k, are non-negative integers we obtain

(19) $\int_{-1}^{1} P_n^m(x)\, P_l^m(x)\, dx = 0$ $\qquad l \neq n,$

(20) $\int_{-1}^{1} P_n^m(x)\, P_n^k(x)\,(1-x^2)^{-1}\, dx = 0$ $\qquad k \neq m,$

(21) $\int_{-1}^{1} [P_n^m(x)]^2\, dx = (n+\tfrac{1}{2})^{-1}(n+m)!/(n-m)!,$

and

(22) $\int_{-1}^{1} (1-x^2)^{-1}[P_n^m(x)]^2\, dx = (n+m)!/[m\,(n-m)!].$

Furthermore we have

(23) $$\int_0^1 P_\nu(x)\, x^\sigma\, dx = \frac{\pi^{\frac{1}{2}}\, 2^{-\sigma-1}\,\Gamma(1+\sigma)}{\Gamma(1+\tfrac{1}{2}\sigma-\tfrac{1}{2}\nu)\,\Gamma(\tfrac{1}{2}\sigma+\tfrac{1}{2}\nu+3/2)} \qquad \operatorname{Re}\sigma > -1.$$

This can be proved by substituting 3.2(3) and 2.1(2) and integrating term by term. Using 1.5(1) and 2.1(23) we first find

$$\int_0^1 P_\nu(x)\, x^\sigma\, dx = (\sigma+1)^{-1} F(-\nu, \nu+1; \sigma+2; \tfrac{1}{2})$$

$$= 2^{-\sigma-1}(\sigma+1)^{-1} F(\sigma+\nu+2, \sigma-\nu+1; \sigma+2; \tfrac{1}{2})$$

and with 2.8(50) the result (23) follows at once.

Barnes (1908, pp. 183 ff.) has proved that

(24) $$\int_0^1 x^{\sigma}(1-x^2)^{-\mu/2}\,\mathrm{P}_\nu^\mu(x)\,dx = \frac{2^{\mu-1}\,\Gamma(\tfrac12+\tfrac12\sigma)\,\Gamma(1+\tfrac12\sigma)}{\Gamma(1+\tfrac12\sigma-\tfrac12\nu-\tfrac12\mu)\,\Gamma(\tfrac12\sigma+\tfrac12\nu-\tfrac12\mu+3/2)}$$

$\operatorname{Re}\mu < 1, \quad \operatorname{Re}\sigma > -1,$

(25) $$(-1)^m\, 2^{m+1}\,\Gamma(1-m+\nu)\int_0^1 x^{\sigma}(1-x^2)^{m/2}\,\mathrm{P}_\nu^m(x)\,dx = \frac{\Gamma(\tfrac12+\tfrac12\sigma)\,\Gamma(1+\tfrac12\sigma)\,\Gamma(1+m+\nu)}{\Gamma(1+\tfrac12\sigma+\tfrac12 m-\tfrac12\nu)\,\Gamma(3/2+\tfrac12\sigma+\tfrac12 m+\tfrac12\nu)}$$

$\operatorname{Re}\sigma > -1,\quad m$ a positive integer,

(26) $$\int_0^1 (1-x^2)^{-1}\,[\mathrm{P}_\nu^\mu(x)]^2\,dx = -\Gamma(1+\mu+\nu)/[2\mu\,\Gamma(1-\mu+\nu)]$$

$\operatorname{Re}\mu < 0, \quad \nu+\mu$ a positive integer.

Other integrals involving Legendre and trigonometric functions are (Mac Robert, 1940, p. 95, 96; 1947, p. 366, 367):

(27) $$\int_0^\pi (\sin t)^{\alpha-1}\,\mathrm{P}_\nu^{-\mu}(\cos t)\,dt = \frac{2^{-\mu}\,\pi\,\Gamma(\tfrac12\alpha+\tfrac12\mu)\,\Gamma(\tfrac12\alpha-\tfrac12\mu)}{\Gamma(\tfrac12+\tfrac12\alpha+\tfrac12\nu)\,\Gamma(\tfrac12\alpha-\tfrac12\nu)\,\Gamma(\tfrac12\mu+\tfrac12\nu+1)\,\Gamma(\tfrac12\mu-\tfrac12\nu+\tfrac12)}$$

$\operatorname{Re}(\alpha\pm\mu) > 0,$

(28) $$\int_0^\infty (\sinh t)^{\alpha-1}\,P_\nu^{-\mu}(\cosh t)\,dt = \frac{2^{-1-\mu}\,\Gamma(\tfrac12\alpha+\tfrac12\mu)\,\Gamma(\tfrac12\nu-\tfrac12\alpha+1)\,\Gamma(\tfrac12-\tfrac12\alpha-\tfrac12\nu)}{\Gamma(\tfrac12\mu+\tfrac12\nu+1)\,\Gamma(\tfrac12+\tfrac12\mu-\tfrac12\nu)\,\Gamma(1+\tfrac12\mu-\tfrac12\alpha)}$$

$\operatorname{Re}(\alpha+\mu) > 0, \quad \operatorname{Re}(\nu-\alpha+2) > 0, \quad \operatorname{Re}(1-\alpha-\nu) > 0,$

(29) $$\Gamma(1+\tfrac12\nu-\tfrac12\mu)\,\Gamma(\tfrac12+\tfrac12\nu+\tfrac12\alpha)\int_0^\infty (\sinh t)^{\alpha-1}\,Q_\nu^\mu(\cosh t)\,dt$$

$$= e^{i\mu\pi}\,2^{\mu-\alpha}\,\Gamma(\tfrac12+\tfrac12\nu+\tfrac12\mu)\,\Gamma(1+\tfrac12\nu-\tfrac12\alpha)\,\Gamma(\tfrac12\alpha+\tfrac12\mu)\,\Gamma(\tfrac12\alpha-\tfrac12\mu)$$

$\operatorname{Re}(\alpha\pm\mu) > 0, \quad \operatorname{Re}(\nu-\alpha+2) > 0.$

Furthermore (Shabde, 1945, p. 51) we have

(30) $$[\Gamma(\nu+1)\,\Gamma(\mu+1)]^2\int_{-1}^1 \mathrm{P}_\nu(x)\,\mathrm{P}_\mu(x)\,(1+x)^{\nu+\mu}\,dx$$

$$= 2^{\nu+\mu+1}\,[\Gamma(\nu+\mu+1)]^4/\Gamma(2\nu+2\mu+2) \qquad \operatorname{Re}(\nu+\mu+1) > 0.$$

For further integrals involving Legendre functions see Chapter 7 and Bailey, 1931, p. 187; Banerjee, 1940, p. 25; Barnes, 1908, pp. 179-204; B. N. Bose, 1944, p. 125; S. K. Bose, 1946, p. 177; Dhar and Shabde, 1932, p. 177; Mac Robert, 1939, p. 203; 1940, p. 95; 1947, p. 366, 367;

Meijer, 1939, p. 930; Prasad, 1930, p. 33; Shabde, 1934, p. 41; Sircar, 1927, p. 244. For integrals with respect to their degree see MacRobert 1934, 1935.

3.13. The ring or toroidal functions

The ring or toroidal functions arise when Laplace's equation $\Delta V = 0$ is transformed into toroidal coordinates (η, θ, ϕ)

$$(1)\qquad x = \frac{c \sinh \eta \cos \phi}{\cosh \eta - \cos \theta}, \quad y = \frac{c \sinh \eta \sin \phi}{\cosh \eta - \cos \theta}, \quad z = \frac{c \sin \theta}{\cosh \eta - \cos \theta}.$$

With the substitution $s = \cosh \eta$ and $V = (\cosh \eta - \cos \theta)\, v(s, \theta, \phi)$ the equation $\Delta V = 0$ becomes

$$(2)\qquad \frac{\partial}{\partial s}\left[(s^2 - 1)\frac{\partial v}{\partial s}\right] + \frac{\partial^2 v}{\partial \theta^2} + v/4 + (s^2 - 1)^{-1}\frac{\partial^2 v}{\partial \phi^2} = 0.$$

With $v = v_1(s)\, v_2(\theta)\, v_3(\phi)$, this leads to the following differential equation for v_1

$$(3)\qquad (1 - s^2)\frac{d^2 v_1}{ds^2} - 2s\frac{dv_1}{ds} + [(\nu - \tfrac{1}{2})(\nu + \tfrac{1}{2}) - (1 - s^2)^{-1} \mu^2]\, v_1 = 0,$$

ν and μ being separation parameters. According to 3.2(1) solutions of (3) are

$$(4)\qquad v_1 = \begin{cases} P^{\mu}_{\nu-1/2}(s) \\ Q^{\mu}_{\nu-1/2}(s) \end{cases} = \begin{cases} P^{\mu}_{\nu-1/2}(\cosh \eta) \\ Q^{\mu}_{\nu-1/2}(\cosh \eta). \end{cases}$$

The behavior for large values of η follows from 3.2(28) and 3.2 (45):

$$(5)\qquad \Gamma(1 - \mu)\, P^{\mu}_{\nu-1/2}(\cosh \eta) = 2^{2\mu}(1 - e^{-2\eta})^{-\mu} e^{-(\nu + 1/2)\eta} \times F(\tfrac{1}{2} - \mu, \tfrac{1}{2} + \nu - \mu; 1 - 2\mu; 1 - e^{-2\eta}),$$

$$(6)\qquad \Gamma(1 + \nu)\, Q^{\mu}_{\nu-1/2}(\cosh \eta) = \pi^{1/2} e^{i\mu\pi}\, \Gamma(\tfrac{1}{2} + \nu + \mu)(1 - e^{-2\eta})^{\mu} e^{-(\nu + 1/2)\eta} \times F(\tfrac{1}{2} + \mu, \tfrac{1}{2} + \nu + \mu; 1 + \nu; e^{-2\eta}).$$

Special cases of (4) are

$$(7)\qquad P_{-1/2}(\cosh \eta) = (\tfrac{1}{2}\pi \cosh \eta/2)^{-1} K(\tanh \eta/2),$$

$$(8)\qquad Q_{-1/2}(\cosh \eta) = 2e^{-\eta/2} K(e^{-\eta}),$$

$$(9)\qquad P_{1/2}(\cosh \eta) = (\pi/2)^{-1} e^{\eta/2} E[(1 - e^{-2\eta})^{1/2}].$$

K and E are the complete elliptic integrals of the first and second kind respectively (cf. also Darling, 1923; Lowry, 1926; Airey, 1935).

All other properties and representations of the ring functions follow from the formulas of the earlier sections of this chapter. (For an expansion theorem involving toroidal functions cf. Banerjee, 1938; 1942.)

3.14. The conical functions

The differential equation

(1) $$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} - [p^2 + \tfrac{1}{4} + (1-z^2)^{-1}\mu^2]\, w = 0$$

(p is a real parameter) is a special case of 3.2(1) with $\nu = -\tfrac{1}{2} + ip$.
Solutions of (1) are

(2) $$P^{\mu}_{-\frac{1}{2}+ip}(z) \quad \text{and} \quad Q^{\mu}_{-\frac{1}{2}+ip}(z).$$

The conical functions are solutions for a real argument x numerically less than unity.

(3) $$\mathrm{P}^{\mu}_{-\frac{1}{2}+ip}(\cos\theta) \quad \text{and} \quad \mathrm{Q}^{\mu}_{-\frac{1}{2}+ip}(\cos\theta).$$

The principal properties of these functions can be obtained from the general results regarding $\mathrm{P}^{\mu}_{\nu}(\cos\theta)$ and $\mathrm{Q}^{\mu}_{\nu}(\cos\theta)$. For instance from 3.5(7) and 3.5(8)

(4) $$\mathrm{P}_{-\frac{1}{2}+ip}(\cos\theta) = 1 + \frac{p^2 + (\frac{1}{2})^2}{2^2}(\sin\theta)^2 + \frac{[p^2 + (\frac{1}{2})^2][(p^2 + (3/2)^2]}{2^2 \cdot 4^2}(\sin\theta)^4 + \cdots \qquad 0 \le \theta < \tfrac{1}{2}\pi,$$

(5) $$\mathrm{P}_{-\frac{1}{2}+ip}(\cos\theta) = 1 + \frac{4p^2 + 1^2}{2^2}(\sin\theta/2)^2 + \frac{(4p^2+1^2)(4p^2+3^2)}{2^2 \cdot 4^2}(\sin\theta/2)^4 + \cdots \qquad 0 \le \theta < \pi,$$

and it is seen that conical functions of the first kind are positive for real p. A special case of (5)is

$$\mathrm{P}_{-\frac{1}{2}}(\cos\theta) = (\tfrac{1}{2}\pi)^{-1} K(\sin\theta/2).$$

K is the complete elliptic integral of the first kind. (cf. also Darling, 1923; Lowry, 1926; Airey, 1935.)

Formulas similar to those of Neumann, 3.6(29) and Heine, 3.10(9) respectively, have been given by Mehler, 1881, p. 193:

(6) $$\pi P_{-\frac{1}{2}+ip}(-x) = \cosh(p\pi)\int_1^{\infty}(v-x)^{-1} P_{-\frac{1}{2}+ip}(v)\,dv \qquad x<1,$$

(7) $$(y-x)^{-1} = \pi\int_0^{\infty}\frac{p\tanh p\pi}{\cos p\pi} P_{-\frac{1}{2}+ip}(y)\, P_{-\frac{1}{2}+ip}(-x)\,dp.$$

These formulas are a special case of the following inversion formulas (Mehler, 1881, p. 192; Fock, 1943).

(8) $f(t) = t \tanh(\pi t) \int_1^\infty P_{-\frac{1}{2}+it}(x)\, g(x)dx,$

(9) $g(x) = \int_0^\infty P_{-\frac{1}{2}+it}(x)\, f(t)\, dt.$

All other properties and representations of the conical functions follow from the formulas of the earlier sections of this paper. See also Mehler, 1881 and Neumann 1881. For an expansion theorem involving conical functions see Banerjee, 1938, p. 30.

3.15. Gegenbauer functions

3.15.1. Gegenbauer polynomials

Gegenbauer's polynomial $C_n^\nu(z)$ for integral value of n is defined to be the coefficient of h^n in the expansion of $(1 - 2hz + h^2)^{-\nu}$ in powers of h (cf. also section 10.9):

(1) $(1 - 2hz + h^2)^{-\nu} = \sum_{n=0}^\infty C_n^\nu(z)\, h^n \qquad |h| < |z \pm (z^2 - 1)^{\frac{1}{2}}|.$

Since

$$(1 - 2hz + h^2)^{-\nu} = (1-h)^{-2\nu}\,[1 + 2h(1-h)^{-2}\,(1-z)]^{-\nu}$$

$$= \sum_{s=0}^\infty (-2)^s\,\Gamma(s+\nu)\,(1-z)^s\, h^s(1-h)^{-2s-2\nu}/[s!\,\Gamma(\nu)]$$

and

$$h^s(1-h)^{-2s-2\nu} = \sum_{m=0}^\infty \Gamma(m+2s+2\nu)\, h^{m+s}/[m!\,\Gamma(2s+2\nu)],$$

the coefficient of h^n in (1) is found to be

(2) $C_n^\nu(z) = \sum_{l=0}^n \dfrac{(-1)^l\Gamma(\nu+l)\,\Gamma(n+2\nu+l)\,(\frac{1}{2}-\frac{1}{2}z)^l}{l!\,\Gamma(\nu)\,\Gamma(2l+2\nu)\,(n-l)!},$

and by means of 1.2(3), 1.20(5), and 2.1(2) we find that

(3) $C_n^\nu(z) = \dfrac{\Gamma(n+2\nu)}{\Gamma(n+1)\,\Gamma(2\nu)} F(n+2\nu, -n; \nu+\frac{1}{2}; \frac{1}{2}-\frac{1}{2}z)$

$n = 0, 1, 2, \ldots.$

From (3) and 3.2(7)

(4) $C_n^\nu(z) = 2^{\nu-\frac{1}{2}}\,\Gamma(n+2\nu)\,\Gamma(\nu+\frac{1}{2})\,(z^2-1)^{\frac{1}{4}-\frac{1}{2}\nu}$

$\times P^{\frac{1}{2}-\nu}_{n+\nu-\frac{1}{2}}(z)/[\Gamma(2\nu)\,\Gamma(n+1)]$

From (4) and 3.2 (22)

(5) $C_{2n}^{\nu}(z) = (-1)^n \Gamma(\nu+n) F(-n, n+\nu; \tfrac{1}{2}; z^2)/[n!\,\Gamma(\nu)]$,

(6) $C_{2n+1}^{\nu}(z) = (-1)^n 2\Gamma(\nu+n+1)\, z F(-n, n+\nu+1; 3/2; z^2)/[n!\,\Gamma(\nu)]$.

From (3), (5), and (6)

(7) $C_n^{\nu}(1) = (-1)^n C_n^{\nu}(-1) = \Gamma(n+2\nu)/[n!\,\Gamma(2\nu)]$.

From 3.2 (23)

(8) $C_n^{\nu}(z) = (2z)^n \Gamma(\nu+n) F(-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n; 1-\nu-n; z^{-2})/[n!\,\Gamma(\nu)]$,

(9) $C_n^{\nu}(z) = [\Gamma(\nu)]^{-1} \sum_{m=0}^{\leq \frac{1}{2}n} (-1)^m \Gamma(\nu+n-m)(2z)^{n-2m}/[m!\,(n-2m)!]$.

Hence it can be shown that

(10) $C_n^{\nu}(z) = (1-z^2)^{\frac{1}{2}-\nu} \dfrac{d^n}{dz^n}[(1-z^2)^{n+\nu-\frac{1}{2}}]$

$\times (-2)^n \Gamma(\nu+n)\Gamma(2\nu+n)/[n!\,\Gamma(\nu)\Gamma(2\nu+2n)]$.

From (9)

(11) $\dfrac{d^n}{dz^n}[C_n^{\nu}(z)] = 2^n \Gamma(\nu+n)/\Gamma(\nu)$.

A trigonometric expansion of $C_n^{\nu}(\cos\phi)$ may be obtained as follows. We write

(12) $(1-2h\cos\phi+h^2)^{-\nu} = (1-he^{i\phi})^{-\nu}(1-he^{-i\phi})^{-\nu}$

and expand the left-hand side and each of the two factors on the right-hand side in powers of h. Thus we have

$$\sum_{n=0}^{\infty} C_n^{\nu}(\cos\phi)\, h^n\, [\Gamma(\nu)]^2$$

$$= \{\sum_{s=0}^{\infty} \Gamma(s+\nu)\, h^s e^{is\phi}/s!\}\{\sum_{l=0}^{\infty} \Gamma(l+\nu)\, h^l e^{-il\phi}/l!\}$$

and comparing coefficients of h^n on both sides

$$[\Gamma(\nu)]^2 C_n^{\nu}(\cos\phi)$$

$$= \sum_{m=0}^{n} \Gamma(m+\nu)\Gamma(n-m+\nu)\, e^{-i(n-2m)\phi}/[m!\,(n-m)!]$$

or

(13) $\tfrac{1}{2}[\Gamma(\nu)]^2 C_n^{\nu}(\cos\phi)$

$$= \sum_{m=0}^{\leq \frac{1}{2}n} \Gamma(m+\nu)\Gamma(n-m+\nu)\cos[(n-2m)\phi]/[m!\,(n-m)!].$$

When n is even, only half of the last term (corresponding to $m = \frac{1}{2} n$) must be taken.

From (13)

$$(14) \quad \lim_{\nu \to 0} \Gamma(\nu)\, C_n^{\nu}(\cos\phi) = 2n^{-1} \cos(n\phi) \qquad n = 1, 2, 3, \ldots .$$

From the identity

$$(1 - 2h \cos\phi + h^2)^{-1} = (2i \sin\phi)^{-1} \times [(1 - he^{i\phi})^{-1} e^{i\phi} - (1 - he^{-i\phi})^{-1} e^{-i\phi}]$$

we obtain the result

$$(15) \quad C_n^1(\cos\phi) = \sin[(n+1)\phi]/\sin\phi .$$

To establish the orthogonality relations of Gegenbauer's polynomials,

$$(16) \quad \int_{-1}^{1} C_n^{\nu}(x)\, C_r^{\nu}(x)\, (1 - x^2)^{\nu - \frac{1}{2}}\, dx = 0 \qquad n \neq r,$$

$$(17) \quad \int_{-1}^{1} [C_n^{\nu}(x)]^2 (1 - x^2)^{\nu - \frac{1}{2}}\, dx = \frac{2^{1-2\nu}\, \pi\, \Gamma(n + 2\nu)}{n!\, (\nu + n)\, [\Gamma(\nu)]^2},$$

we write the integral on the left-hand side by means of (10) in the form

$$\frac{(-2)^n\, \Gamma(\nu + n)\, \Gamma(2\nu + n)}{n!\, \Gamma(\nu)\, \Gamma(2\nu + 2n)} \int_{-1}^{1} C_r^{\nu}(x)\, \frac{d^n}{dx^n} [(1 - x^2)^{n + \nu - \frac{1}{2}}]\, dx.$$

Integrating n times by parts and using (11) we obtain (16) and (17).

From (16) and (17) with $C_0^{\nu}(x) = 1$ we have

$$(18) \quad \int_0^{\pi} C_n^{\nu}(\cos\phi)\, (\sin\phi)^{2\nu}\, d\phi = \begin{cases} 0 & n = 1, 2, 3, \ldots, \\ 2^{-2\nu}\, \pi\, \Gamma(2\nu + 1)\, [\Gamma(1 + \nu)]^{-2} & n = 0. \end{cases}$$

The addition theorem for the $C_n^{\nu}(z)$ has been given by Gegenbauer (Gegenbauer, 1893, p. 942)

$$(19) \quad [\Gamma(\nu)]^2\, C_n^{\nu}[z z_1 - (z^2 - 1)^{\frac{1}{2}} (z_1^2 - 1)^{\frac{1}{2}} \cos\phi]$$
$$= \Gamma(2\nu - 1) \Big[\sum_{l=0}^{n} (-1)^l 4^l\, \Gamma(n - l + 1)\, [\Gamma(\nu + l)]^2\, (2\nu + 2l - 1)$$
$$\times [\Gamma(n + 2\nu + l)]^{-1} (z^2 - 1)^{\frac{1}{2} l} (z_1^2 - 1)^{\frac{1}{2} l}\, C_{n-l}^{\nu + l}(z)\, C_{n-l}^{\nu + l}(z_1)$$
$$\times C_l^{\nu - \frac{1}{2}}(\cos\phi) \Big].$$

From (18) and (19)

$$(20) \quad \int_0^{\pi} C_n^{\nu}(\cos\psi \cos\psi' + \sin\psi \sin\psi' \cos\phi)\, (\sin\phi)^{2\nu - 1}\, d\phi$$
$$= 2^{2\nu - 1}\, n!\, [\Gamma(\nu)]^2\, C_n^{\nu}(\cos\psi)\, C_n^{\nu}(\cos\psi') / \Gamma(2\nu + n) \qquad \mathrm{Re}\, \nu > 0.$$

(For further integral formulas cf. Watson, 1944, pp 367-369.)

3.15.2 Gegenbauer functions

From 2.1(1) it is to be seen that (3) with n replaced by α (α arbitrary) is a solution of

(21) $(z^2-1)\,w'' + (2\nu+1)\,zw' - \alpha(\alpha+2\nu)\,w = 0.$

We therefore define Gegenbauer's function for arbitrary (possibly complex) values of α by (3) or (4) with n replaced by α.

From (4), 3.7(6), 3.7(8), and 3.7(34), respectively

(22) $$C^{\nu}_{\alpha}(z) = \pi^{-1/2}\,\Gamma(\alpha+2\nu)\,\Gamma(\nu+\tfrac{1}{2})/[\Gamma(\nu)\,\Gamma(2\nu)\,\Gamma(\alpha+1)] \times \int_0^{\pi} [z+(z^2-1)^{1/2}\cos t]^{\alpha}(\sin t)^{2\nu-1}\,dt \qquad \operatorname{Re}\nu > 0,$$

(23) $$C^{\nu}_{\alpha}(\cos\phi) = 2^{\nu}\,\pi^{-1/2}\,\Gamma(\alpha+2\nu)\,\Gamma(\nu+\tfrac{1}{2})/[\Gamma(\nu)\,\Gamma(2\nu)\,\Gamma(\alpha+1)] \times (\sin\phi)^{1-2\nu}\int_0^{\phi}\cos[(\nu+\alpha)\,v]\,(\cos v-\cos\phi)^{\nu-1}\,dv$$
$$\operatorname{Re}\nu > 0, \quad 0<\phi<\pi,$$

(24) $$C^{\nu}_{\alpha}(z) = -\pi^{-1}\sin(\alpha\pi)\int_0^{\infty}(1+2tz+t^2)^{-\nu}\,t^{-\alpha-1}\,dt$$
$$-2<\operatorname{Re}\nu<\operatorname{Re}\alpha<0, \qquad |\arg(z\pm 1)|<\pi.$$

(For further integral representations cf. Dinghas, 1950.)

The Mellin inversion of this last relation is

(25) $$(1+2tz+t^2)^{-\nu} = \tfrac{1}{2}\,i\int_{c-i\infty}^{c+i\infty}(\sin\alpha\pi)^{-1}\,t^{\alpha}\,C^{\nu}_{\alpha}(z)\,d\alpha$$
$$-2<\operatorname{Re}\nu<c<0.$$

From (4) and section 3.8

(26) $(\alpha+2)\,C^{\nu}_{\alpha+2}(z) = 2(\nu+\alpha+1)\,z\,C^{\nu}_{\alpha+1}(z) - (2\nu+\alpha)\,C^{\nu}_{\alpha}(z),$

(27) $\alpha\,C^{\nu}_{\alpha}(z) = 2\nu[z\,C^{\nu+1}_{\alpha-1}(z) - C^{\nu+1}_{\alpha-2}(z)],$

(28) $(\alpha+2\nu)\,C^{\nu}_{\alpha}(z) = 2\nu[C^{\nu+1}_{\alpha}(z) - z\,C^{\nu+1}_{\alpha-1}(z)],$

(29) $\alpha\,C^{\nu}_{\alpha}(z) = (\alpha-1+2\nu)\,z\,C^{\nu}_{\alpha-1}(z) - 2\nu(1-z^2)\,C^{\nu-1}_{\alpha-2}(z),$

(30) $\dfrac{d}{dz}\,C^{\nu}_{\alpha}(z) = 2\nu\,C^{\nu+1}_{\alpha-1}(z).$

From 3.3(1) and (4)

(31) $C^{\nu}_{\alpha}(z) = -\sin(\alpha\pi)\,C^{\nu}_{-\alpha-2\nu}(z)/[\sin\pi(\alpha+2\nu)].$

A second solution of Gegenbauer's differential equation (21) is easily found to be

(32) $D_\alpha^\nu(z) = 2^{-1-\alpha} \dfrac{\Gamma(\nu)\,\Gamma(2\nu+\alpha)}{\Gamma(\nu+\alpha+1)} F(\nu+\tfrac{1}{2}\alpha, \nu+\tfrac{1}{2}+\tfrac{1}{2}\alpha; \nu+\alpha+1; z^2).$

The $D_\alpha^\nu(z)$ satisfy the same recurrence relations as the $C_\alpha^\nu(z)$.

A relation between $D_n^\nu(z)$ and $C_n^\nu(z)$ analogous to Christoffel's relation between $Q_n(z)$ and $P_n(z)$ [cf. 3.6(24), 3.6(28)] has been given by Watson (1938)

(33) $$D_n^\nu(z) = \Gamma(2\nu)\, C_n^\nu(z) \int_z^\infty (t^2-1)^{-\nu-\frac{1}{2}}\, dt$$

$$- \Gamma(2\nu)\,(z^2-1)^{\frac{1}{2}-\nu} \sum_{m=0}^{[\frac{1}{2}n-\frac{1}{2}]} (\nu+n-2m-1)$$

$$\times \frac{(1-\nu)_m\,(2\nu+n-m)_m}{(n-m)_{m+1}\,(\nu)_{m+1}}\, C_{n-2m-1}^\nu(z) \qquad \operatorname{Re}\nu > 0.$$

3.16. Some other notations

The factor $e^{i\mu\pi}$ in the definition of $Q_\nu^\mu(z)$ 3.2(5) is often omitted. (MacRobert).

In the definition of $Q_\nu^\mu(z)$ given by Barnes, the factor $e^{i\mu\pi}$ in 3.2(8) is replaced by

$$\frac{\sin[\pi(\nu+\mu)]}{\sin(\nu\pi)};$$

furthermore the factor $e^{i\mu\pi}$ on the left-hand side of 3.4(2) is omitted.

Ferrers' associated Legendre function (Mac Robert, 1947, p. 307) is denoted by $T_\nu^\mu(x)$ and is identical with $\mathrm{P}_\nu^\mu(x)$ $(-1 < x < 1)$.

A different notation for Gegenbauer's functions is used by Chu and Stratton, 1941. Instead of (4) and (32) respectively we find

$$C_\alpha^\nu(z) = (z^2-1)^{-\frac{1}{2}\nu} P_{\alpha+\nu}^\nu(z),$$

$$D_\alpha^\nu(z) = (z^2-1)^{-\frac{1}{2}\nu} Q_{\alpha+\nu}^\nu(z).$$

They satisfy the following differential equation

$$(z^2-1)\, w'' + 2(\nu+1)\, zw - \alpha(\alpha+2\nu+1) = 0.$$

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CHAPTER IV

THE GENERALIZED HYPERGEOMETRIC SERIES

4.1. Introduction

Gauss' hypergeometric series $F(a, b; c; z)$ has been generalized by the introduction of p parameters of the nature of a, b, and of q parameters of the nature of c. The ensuing series

$$(1) \quad {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; z \\ \rho_1, \ldots, \rho_q \end{matrix}\right] = {}_pF_q(a_r; \rho_t; z)$$

$$= \sum_{n=1}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(\rho_1)_n \cdots (\rho_q)_n} \frac{z^n}{n!}$$

is known as the generalized hypergeometric series. Gauss' series in the present notation is

$${}_2F_1(a, b; c; z) \equiv F\left[\begin{matrix} a, b; z \\ c \end{matrix}\right]$$

Here

$$(2) \quad (a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1) = \Gamma(a+n)/\Gamma(a)$$

$$n = 1, 2, \ldots,$$

and z is a complex variable. In general (that is, excepting certain integer values of the parameters for which the series terminates or fails to make sense) ${}_pF_q$ converges for all finite z if $p \leq q$, converges for $|z| < 1$ if $p = q + 1$, and diverges for all $z \neq 0$ if $p > q + 1$.

The numbers $a_1, \ldots, a_p$ are called the numerator-parameters and $\rho_1, \ldots, \rho_q$, are referred to as denominator-parameters.

The series ${}_pF_q$ is not the only generalization of Gauss' series. The hypergeometric equation is a linear differential equation of Fuchsian type, but, while the series (1) also satisfies a linear differential equation, this is not of Fuchsian type. Thus the differential equation can be used as the basis for another generalization in which one considers the generalization to be the solution of an equation of Fuchsian type of higher order. In this connection L. Pochhammer (1870) has investigated the most general homogeneous linear differential equation of the n-th order which has singularities at $a_1, a_2, \ldots, a_n, \infty$ and is such that the general solution in the neighborhood of every singularity a_ν ($\nu = 1, 2, \ldots, n$) is of the form

$$\sum_{m=0}^{\infty} c_m (z - a_\nu)^m + z^\rho \sum_{m=0}^{\infty} c'_m (z - a_\nu)^m$$

where $c_0, c_1, \ldots, c_{n-2}$ and c'_0 are arbitrary constants. Similarly, the expansion for large z of the general solution is assumed in the form

$$z^\sigma \sum_{m=0}^{\infty} g_m z^{-m} + z^\tau \sum_{m=0}^{\infty} g'_m z^{-m},$$

where $g_0, \ldots, g_{n-2}$ and g'_0 are arbitary constants. It can be shown that the differential equation is determined by these postulates.

The point of departure for another generalization is Schwarz's s-function (cf. 2.7.2) which maps a half-plane on a triangle formed of three circular arcs. Koppenfels (1937, 1939) has studied the function which maps the area bounded by four circular arcs on a half-plane, assuming that two of the four angles of the curvilinear quadrilateral are $\pi/2$ and the other two either $\pi/2$ and $3\pi/2$ or $3\pi/2$ and $3\pi/2$.

E. M. Wright (1935, 1940) has investigated the asymptotic behavior of the sum

$$\sum_{n=0}^{\infty} \frac{\Gamma(a_1 + \beta_1 n) \cdots \Gamma(a_p + \beta_p n)}{\Gamma(\rho_1 + \mu_1 n) \cdots \Gamma(\rho_q + \mu_q n)} \frac{z}{n!}$$

for large $|z|$. Here the β_r and the μ_t are real, positive and

$$1 + \sum_{t=1}^{q} \mu_t - \sum_{r=1}^{p} \beta_r > 0.$$

If all the μ_t and β_r are equal to unity, this reduces to a multiple of ${}_pF_q$.

Heine's hypergeometric series are discussed in section 4.8. For further generalizations see Chapter 5.

The following conventions will be observed throughout this chapter:

The values of the q parameters $\rho_1, \ldots, \rho_q$ in ${}_pF_q$ are always different from $0, -1, -2, \ldots$.

If the variable z in ${}_{q+1}F_q$ is not equal to unity, we assume tacitly $|z| < 1$.

If the variable z of ${}_{q+1}F_q$ is unity, we always assume

$$\operatorname{Re}\left\{\sum_{t=1}^{q} \rho_t - \sum_{r=1}^{q+1} a_r\right\} > 0.$$

If the variable z in a ${}_pF_q$ is unity it will be omitted.

4.2. Differential equations

The function ${}_pF_q$ is defined by 4.1(1). The series was introduced by Clausen (1828) in the case $p = 3$, $q = 2$; the notation is that of Pochhammer as modified by Barnes. If one of the α_r is a non-positive integer, the series terminates; the cases in which one of the ρ_t is a non-positive integer are excluded.

If $p = q + 1$ and

(1) $$s = \mathrm{Re}(\rho_1 + \cdots + \rho_q - \alpha_1 - \cdots - \alpha_{q+1}),$$

then the series in 4.1(1) converges for all $|z| = 1$ if $s > 1$; it converges for all $|z| = 1$, $z \neq 1$ if $1 \geq s \geq 0$ and is divergent if $s \leq 0$. The proof is the same as the one given in 2.1.1 for ${}_2F_1$.

Let δ denote the operator zd/dz. Then $u = {}_pF_q$ satisfies

(2) $$\{\delta(\delta + \rho_1 - 1) \cdots (\delta + \rho_q - 1) - z(\delta + \alpha_1) \cdots (\delta + \alpha_p)\}\, u = 0.$$

which is equivalent to the general equation of the type

(3) $$\sum_{n=1}^{q} z^{n-1}(a_n z - b_n) D^n v + a_0 v + z^q D^{q+1} v = 0 \qquad q \geq p,$$

or

(4) $$\sum_{n=1}^{q} z^{n-1}(a_n z - b_n) D v + a_0 v + z^q (1 - z) D^{q+1} v = 0 \qquad p = q + 1,$$

where a_n, b_n, are constants, $a_n \neq 0$ and $D = d/dz$. For (3), $z = 0$ and $z = \infty$ are singular points, of which $z = 0$ is of regular type (cf. Poole 1936, Chapter 5); (4) is of Fuchsian type (cf. Poole 1936, p. 77) with regular singularities at $0, \infty, 1$. For a set of linearly independent solutions for the neighborhood of $z = 0$ and $z = \infty$ see section 5.4. There and in the greater part of the literature, only the "general" case is investigated where none of the ρ_t and none of the differences $\rho_r - \rho_0$, $\alpha_r - \alpha_s$ $(r \neq s)$ is an integer.

In the case $p \leq q$, E. W. Barnes (1906) gave the asymptotic expansions for the solutions of (2) in the "general" case (see above). L. Pochhammer (1893b) considered various forms of (2) and (3) and gave a complete set of linearly independent solutions in the form of multiple integrals. In particular the cases, $q = 3$ and $q = 4$ $(p \leq q)$ were investigated by L. Pochhammer (1893 a, 1895, 1898).

In the case $p = q + 1$ Pochhammer (1888 b) investigated (2) and (4) in the "general" case and gave multiple integrals which are solutions in the neighborhood of $z = 0, \infty, 1$. He also showed that there exist p linearly independent solutions which are one-valued in the neighborhood

$z = 1$. In the case $p = 3$, $q = 2$ this had been done already by E. Goursat (1883, 1884). Thomae (1869) gave the connection between complete sets of linearly independent solutions at $z = 0$ and $z = \infty$ for $p = 3$, $q = 2$. This has been done for all values of p ($= q + 1$) by F. C. Smith (1938, 1939), who also investigated various special cases where some of the solutions of (2) and (4) involve logarithms (some or all of the differences $\rho_t - \rho_s$ may be integers, but the $\rho_t - a$ and the ρ_t themselves are not integers, or some or all of the $a_r - a_s$ are integers, but the $a_r - \rho_t$ and the a_r themselves are not integers).

If $v(z)$ satisfies (2) and if it can be obtained from an analytic function $w(t)$, such that $\lim_{t \to 0} w(t) = 0$, by a Laplace transformation

$$v(z) = \int_0^\infty e^{-zt} w(t)\, dt,$$

then $w(t)$ satisfies

$$(5) \quad \{(-1)^{p+1}\, \theta(\theta + 1 - a_1) \cdots (\theta + 1 - a_p) + (-1)^{q+1}\, t(\theta + 1)(\theta + 2 - \rho_1) \cdots (\theta + 2 - \rho_q)\}\, v = 0$$

where $\theta = t\partial/\partial t$.

For other results on differential equations satisfied by functions involving generalized hypergeometric series see Chaundy (1943).

4.3. Identities and recurrence relations

T. Clausen (1828) proved that

$$(1) \quad [F(a, b; a + b + \tfrac{1}{2}; z)]^2 = {}_3F_2\left[\begin{matrix} 2a,\ a + b,\ 2b;\ z \\ a + b + \tfrac{1}{2};\ a + 2b; \end{matrix}\right]$$

and he also showed that this is the only case in which the square of $F(a, b; c; z)$ can be expressed in terms of a ${}_3F_2$ of argument z. More generally, E. Goursat (1883) proved that $F(a, b; c; z)\, F(a', b'; c'; z)$ is a ${}_pF_q$ of argument z only if either

$$a' = a + 1 - c, \quad b' = b + 1 - c, \quad c' = 2 - c$$

$$c - a - b = n + \tfrac{1}{2}, \quad n = 0, \pm 1, \pm 2, \ldots,$$

or if $a - a'$, $b - b'$, $c - c'$ are integers and $c + c' - a - a' - b - b' = n$ where $n = 0, 1, 2, \ldots$.

W. N. Bailey (1928) gave new proofs for this and for similar results by Orr, Preece (1924), and Ramanujan and supplemented their results by some new formulas. We have:

$$(2) \quad {}_0F_1(\rho; z)\, {}_0F_1(\sigma; z) = {}_2F_3(\tfrac{1}{2}\rho + \tfrac{1}{2}\sigma, \tfrac{1}{2}\rho + \tfrac{1}{2}\sigma - \tfrac{1}{2}; \rho, \sigma, \rho + \sigma - 1; 4z)$$

(3) $\quad {}_0F_1(\rho; z)\, {}_0F_1(\rho; -z) = {}_0F_3(\rho, \tfrac{1}{2}\rho, \tfrac{1}{2}\rho + \tfrac{1}{2}; -z^2/4).$

(4) $\quad {}_2F_0(\alpha, \beta; z)\, {}_2F_0(\alpha, \beta; -z)$

$$= {}_4F_1[\alpha, \beta, \tfrac{1}{2}(\alpha+\beta), \tfrac{1}{2}(\alpha+\beta+1); \alpha+\beta; 4z^2]$$

(5) $\quad {}_1F_1(\alpha; \rho; z)\, {}_1F_1(\alpha; \rho; -z)$

$$= {}_2F_3(\alpha, \rho-\alpha; \rho, \tfrac{1}{2}\rho, \tfrac{1}{2}(\rho+1); z^2/4)$$

(6) $\quad {}_1F_1(\alpha; 2\alpha; z)\, {}_1F_1(\beta; 2\beta; -z)$

$$= {}_2F_3[\tfrac{1}{2}(\alpha+\beta), \tfrac{1}{2}(\alpha+\beta+1); \alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2}, \alpha+\beta; z^2/4]$$

(7) $\quad {}_0F_2(\rho_1, \rho_2; z)\, {}_0F_2(\rho_1, \rho_2; -z)$

$$= {}_3F_8\left[\begin{matrix} 1/3(\rho_1+\rho_2-1), 1/3(\rho_1+\rho_2), 1/3(\rho_1+\rho_2+1); -(3/4)^3 z^2 \\ \rho_1, \rho_2, \tfrac{1}{2}\rho_1, \tfrac{1}{2}\rho_2, \tfrac{1}{2}\rho_1+\tfrac{1}{2}, \tfrac{1}{2}\rho_2+\tfrac{1}{2}, \lambda, \mu \end{matrix}\right]$$

where $\lambda = \tfrac{1}{2}(\rho_1+\rho_2-1)$, $\quad \mu = \tfrac{1}{2}(\rho_1+\rho_2)$.

(8) $\quad {}_2F_1(\alpha, \beta; \alpha+\beta-\tfrac{1}{2}; z)\, {}_2F_1(\alpha, \beta; \alpha+\beta+\tfrac{1}{2}; z)$

$$= {}_3F_2(2\alpha, 2\beta, \alpha+\beta; 2\alpha+2\beta-1, \alpha+\beta+\tfrac{1}{2}; z)$$

(9) $\quad {}_2F_1(\alpha, \beta; \alpha+\beta-\tfrac{1}{2}; z)\, {}_2F_1(\alpha-1, \beta; \alpha+\beta-\tfrac{1}{2}; z)$

$$= {}_3F_2(2\alpha-1, 2\beta, \alpha+\beta-1; 2\alpha+2\beta-2, \alpha+\beta-\tfrac{1}{2}; z)$$

The series in (4) converge only if α or β is a non-positive integer, but (4) is formally valid for all values of α, β, in the sense that a formal multiplication of the power series on the left-hand side gives the right-hand side. The other formulas hold at least for $|z| < 1$.

For the related formulas by Cayley and Orr [cf. 2.5 (4)] see Bailey (1935). The following formula is due to H. B. C. Darling (1932) (cf. also Bailey 1935):

(10) $\quad {}_3F_2\left[\begin{matrix}\alpha, \beta, \gamma; z \\ \delta, \epsilon\end{matrix}\right] {}_3F_2\left[\begin{matrix}1-\alpha, 1-\beta, 1-\gamma; z \\ 2-\delta, 2-\epsilon\end{matrix}\right]$

$$= \frac{\epsilon-1}{\epsilon-\delta}\, {}_3F_2\left[\begin{matrix}\alpha+1-\delta, \beta+1-\delta, \gamma+1-\delta; z \\ 2-\delta, \epsilon+1-\delta\end{matrix}\right]$$

$$\times\, {}_3F_2\left[\begin{matrix}\delta-\alpha, \delta-\beta, \delta-\gamma; z \\ \delta, \delta+1-\epsilon\end{matrix}\right] + \text{a product obtained}$$

by interchanging δ and ϵ.

For other relations connected with the identities of Cayley and Orr see section 2.5.2 and J. L. Burchnall, T. W. Chaundy (1948). A large number of expansions of hypergeometric functions in series of other hypergeometric functions has been given by T. W. Chaundy (1942, 1943). The simplest cases are:

$$(11)\quad F(A, B; C; z) = \sum_{r=0}^{\infty} (-1)^r \frac{(a)_r (b)_r}{(c)_r\, r!} \times {}_4F_3\left[\begin{matrix} A, B, c, -r; 1 \\ a, b, C \end{matrix}\right] z^r\, {}_2F_1(a+r, b+r; c+r; z)$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{(a)_r (b)_r}{(c+r-1)_r\, r!} \times {}_4F_3\left[\begin{matrix} A, B, c+r-1, -r; 1 \\ a, b, C \end{matrix}\right] z^r\, {}_2F_1(a+r, b+r; c+2r; z)$$

$$(12)\quad {}_0F_1(c; pz)\, {}_0F_1(c'; qz) = \sum_{n=0}^{\infty} \frac{(pz)^n}{n!\,(c)_n}\, {}_2F_1(1-c-n, -n; c'; p/q)$$

$$(13)\quad {}_1F_1(a; c; pz)\, {}_1F_1(a'; c'; qz) = \sum_{n=0}^{\infty} \frac{(a)_n (pz)^n}{n!\,(c)_n}\, {}_3F_2\left[\begin{matrix} -a', 1-c-n, -n; -q/p \\ c', 1-a-n \end{matrix}\right]$$

$$(14)\quad {}_2F_1(a, b; c; pz)\, {}_2F_1(a', b'; c'; qz) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (pz)^n}{n!\,(c)_n}\, {}_4F_3\left[\begin{matrix} a', b', 1-c-n, -n; q/p \\ c', 1-a-n, 1-b-n \end{matrix}\right],$$

$$(15)\quad {}_2F_0(a, b; pz)\, {}_2F_0(a', b'; qz) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (pz)^n}{n!}\, {}_3F_2\left[\begin{matrix} a', b', -n; -q/p \\ 1-a-n, 1-b-n \end{matrix}\right]$$

The series in (15) do not converge unless they terminate; but the coefficients of corresponding powers of z on both sides are equal.

Two functions ${}_pF_q$ are called *contiguous* if their arguments and parameters have the same value with the exception of one parameter for which the values are supposed to differ by ± 1. There exist $2p + q$ linearly independent linear relations between a fixed ${}_pF_q$ and its $2(p + q)$ contiguous functions. The coefficients of these relations are linear functions of the variable and polynomials of the parameters. These relations have been given by E. Rainville (1945) for the cases where all of the denominator-parameters are different from $0, -1, -2, \dots$ and if $p \leqq q+1$.

4.4. Generalized hypergeometric series with unit argument in the case $p = q + 1$

Standard types of generalized hypergeometric series. When the parameters in ${}_{q+1}F_q(\alpha_r; \rho_t; z)$ are such that

(1) $$\alpha_1 + \alpha_2 + \cdots + \alpha_{q+1} = -1 + \rho_1 + \cdots + \rho_q$$

the series is called *Saalschützian.* If we have

(2) $$1 + \alpha_1 = \rho_1 + \alpha_2 = \cdots = \rho_q + \alpha_{q+1}$$

the series is called *well-poised*. If all but one of the equations in (2) are satisfied, the series is called *nearly-poised.* Then the parameters can always be arranged, (without changing the series itself) in such a way that the breakdown in the equality of sums of pairs of parameters occurs with the first or with the last pair; accordingly, ${}_{q+1}F_q$ is called *nearly-poised of the first or second* kind, respectively.

Saalschütz's theorem

(3) $${}_3F_2\left[\begin{matrix} a,\ b,\ -n; \\ c,\ 1+a+b-c-n \end{matrix}\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} \qquad n = 0, 1, 2, \dots,$$

which has been proved in section 2.1.5 and stated in 2.1 (30) sums every terminating Saalschützian ${}_3F_2$. For a non-terminating Saalschützian ${}_3F_2$ we have (cf. Saalschütz 1891, Bailey 1935, p. 21)

(4) $$\begin{aligned} {}_3F_2 &\left[\begin{matrix} a,\ b,\ c+f-a-b-1; \\ c,\ f \end{matrix}\right] \\ &= \frac{\Gamma(c)\,\Gamma(f)\,\Gamma(c-a-b)\,\Gamma(f-a-b)}{\Gamma(c-a)\,\Gamma(c-b)\,\Gamma(f-a)\,\Gamma(f-b)} \\ &+ (a+b-c)^{-1} \frac{\Gamma(c)\,\Gamma(f)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+f-a-b)} \\ &\times {}_3F_2\left[\begin{matrix} c-a,\ c-b,\ 1; \\ c-a-b+1,\ c+f-a-b \end{matrix}\right]. \end{aligned}$$

Again, this is a special case of one of a large number of linear relations between three series ${}_3F_2$; for a full account of these see Bailey (1935, Chapter 3).

Every well-poised ${}_3F_2$ of unit argument can be summed in terms of gamma functions. The result is

$$(5)\quad {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c\end{matrix}\right] = \frac{\Gamma(1+\tfrac12 a)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1-b-c+\tfrac12 a)}{\Gamma(1+a)\,\Gamma(1-b+\tfrac12 a)\,\Gamma(1-c+\tfrac12 a)\,\Gamma(1+a-b-c)}$$

This is *Dixon's theorem* (cf. Dixon 1903, Watson 1924, Bailey 1937 and MacRobert 1939). Other cases where a ${}_3F_2$ of unit argument can be evaluated are:

$$(6)\quad {}_3F_2\left[\begin{matrix} a,\ b,\ c; \\ \tfrac12(a+b+1),\ 2c\end{matrix}\right] = \frac{\Gamma(\tfrac12)\,\Gamma(c+\tfrac12)\,\Gamma(\tfrac12 a+\tfrac12 b+\tfrac12)\,\Gamma(c+\tfrac12-\tfrac12 a-\tfrac12 b)}{\Gamma(\tfrac12 a+\tfrac12)/\Gamma(\tfrac12 b+\tfrac12)\,\Gamma(c+\tfrac12-\tfrac12 a)\,\Gamma(c+\tfrac12-\tfrac12 b)}$$

(*Watson's theorem,* cf. Whipple 1925) and

$$(7)\quad {}_3F_2\left[\begin{matrix} a,\ 1-a,\ c; \\ f,\ 2c+1-f\end{matrix}\right] = \frac{\pi\,\Gamma(f)\,\Gamma(2c+1-f)\,2^{1-2c}}{\Gamma(c+\tfrac12 a+\tfrac12-\tfrac12 f)\,\Gamma(\tfrac12 a+\tfrac12 f)\,\Gamma(c+\tfrac12+\tfrac12 b-\tfrac12 f)\,\Gamma(\tfrac12 b+\tfrac12 f)}$$

(*Whipple's theorem,* cf. Whipple 1925).

Dougall's theorem sums a terminating well-poised ${}_7F_6$ of unit argument with a special value of the second parameter:

$$(8)\quad {}_7F_6\left[\begin{matrix} a,\ 1+\tfrac12 a,\ b,\ c,\ d,\ e,\ -n; \\ \tfrac12 a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a+n\end{matrix}\right] = \frac{(1+a)_n\ (1+a-b-c)_n\,(1+a-b-d)_n\,(1+a-c-d)_n}{(1+a-b)_n\,(1+a-c)_n\,(1+a-d)_n\,(1+a-b-c-d)_n},$$

where $1+2a = b+c+d+e-n$ and $n = 0, 1, 2, \ldots$ (Dougall 1907).

There exists a large number of transformations of series ${}_{q+1}F_q$ with unit element, i.e., formulas which express one such series by one or more others, in many cases with varying values of q. A simple example is:

$$(9)\quad {}_4F_3\left[\begin{matrix} -n,\ b,\ c,\ d; \\ 1-n-b,\ 1-n-c,\ w\end{matrix}\right] = \frac{(w-d)_n}{(w)_n} \times {}_5F_4\left[\begin{matrix} d,\ 1-n-b-c,\ -\tfrac12 n,\ \tfrac12-\tfrac12 n,\ 1-n-w; \\ 1-n-b,\ 1-n-c,\ \tfrac12(1+d-w-n),\ 1+\tfrac12(d-w-n)\end{matrix}\right]$$

which transforms a terminating nearly-poised ${}_4F_3$ of the second kind into a ${}_5F_4$. For this and for results of a more difficult type see Bailey (1935 Chapters 3 to 6), Whipple (1935, 1937), MacRobert (1939), Mitra (1942), Bose (1944).

4.5. Transformations of ${}_{q+1}F_q$ and values for arguments other than unity

Except for the connection between generalized hypergeometric series of argument z and z^{-1} which are solutions of the same differential equation (cf. Thomae 1869, F. C. Smith 1938, 1939) no linear transformations of ${}_{q+1}F_q$ seem to be known in the general case if $q > 1$. In certain cases, quadratic and cubic transformations exist for ${}_3F_2$, e.g.,

$$(1)\quad {}_3F_2\left[\begin{matrix} a,\ b,\ c;\ z \\ 1+a-b,\ 1+a-c\end{matrix}\right] = (1-z)^{-a}\ {}_3F_2\left[\begin{matrix} 1+a-b-c,\ \tfrac{1}{2}a,\ \tfrac{1}{2}a+\tfrac{1}{2};\ -4z(1-z)^{-2} \\ 1+a-b,\ 1+a-c\end{matrix}\right]$$

(Whipple 1927) and

$$(2)\quad {}_3F_2\left[\begin{matrix} a,\ 2b-a-1,\ 2-2b+a;\ z/4 \\ b,\ a-b+3/2\end{matrix}\right] = (1-z)^{-a}\ {}_3F_2\left[\begin{matrix} a/3,\ a/3+1/3,\ a/3+2/3;\ -27z/[4(1-z)^3] \\ b,\ a-b+3/2\end{matrix}\right]$$

which have been proved by Bailey (1929). Bailey also gave a linear transformation of a nearly-poised ${}_3F_2$ of the first kind:

$$(3)\quad (1-z)^{2a-1}\ {}_3F_2\left[\begin{matrix} 2a-1,\ a+\tfrac{1}{2},\ a-b-\tfrac{1}{2};\ z \\ a-\tfrac{1}{2},\ a+b+\tfrac{1}{2}\end{matrix}\right] = (1-z)^{2b-1}\ {}_3F_2\left[\begin{matrix} 2b-1,\ b+\tfrac{1}{2},\ b-a-\tfrac{1}{2};\ z \\ b-\tfrac{1}{2},\ a+b+\tfrac{1}{2}\end{matrix}\right].$$

Burchnall (1948) has proved that a well-poised ${}_3F_2$ of argument x can be expressed as a sum of series ${}_3F_2$ of argument $-(x-1)^2/(4x)$.

There are also a few cases known where a ${}_{q+1}F_q$ can be evaluated for an argument $\neq 1$, e.g.,

$$(4)\quad {}_4F_3\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c;\ -1 \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c\end{matrix}\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)}$$

(cf. Bailey 1935, p. 28, and also Bailey 1929).

Some special cases: The formulas developed in section 4.4 and in the present section contain a large number of results which cannot easily be proved directly and which have found much attention in the literature.

For instance, 4.4(5) gives (with $a = b = c = -n$):

$$(5)\qquad \sum_{r=0}^{n} (-1)^r \left[\binom{n}{r}\right]^3 = \frac{\cos \tfrac{1}{2}\pi n\, \Gamma(\tfrac{1}{2})\, \Gamma(\tfrac{1}{2} + \tfrac{1}{2}n)\, \Gamma(1 + 3n/2)\, 2^n}{\pi\, \Gamma(1 + \tfrac{1}{2}n)\, \Gamma(1 + \tfrac{1}{2}n)\, n\,!}$$

If we substitute $b = 1 + a + n$ in 4.4(8) and let $n \to \infty$ we obtain

$$(6)\qquad {}_5F_4\left[\begin{matrix} a,\ 1 + \tfrac{1}{2}a,\ c,\ d,\ e; \\ \tfrac{1}{2}a,\ 1 + a - c,\ 1 + a - d,\ 1 + a - e \end{matrix}\right]$$
$$= \frac{\Gamma(1 + a - c)\, \Gamma(1 + a - d)\, \Gamma(1 + a - e)\, \Gamma(1 + a - c - d - e)}{\Gamma(1 + a)\, \Gamma(1 + a - d - e)\, \Gamma(1 + a - c - e)\, \Gamma(1 + a - c - d)}$$

A special case of this is the Dougall-Ramanujan formula

$$(7)\qquad 1 + 2 \sum_{n=1}^{\infty} \frac{(-x)_n\, (-y)_n\, (-z)_n}{(1 + x)_n\, (1 + y)_n\, (1 + z)_n}$$
$$= \frac{\Gamma(x + 1)\, \Gamma(y + 1)\, \Gamma(z + 1)\, \Gamma(x + y + z + 1)}{\Gamma(y + z + 1)\, \Gamma(x + z + 1)\, \Gamma(x + y + 1)}$$

which is valid for $\operatorname{Re}(x + y + z + 1) > 0$. Another consequence of (6) (with $a = 1$, $c = 1 - x$, $d = e = 1$) is

$$(8)\qquad 1 - 3\,\frac{x - 1}{x + 1} + 5\,\frac{(x - 1)(x - 2)}{(x + 1)(x + 2)} - \cdots = 0.$$

The series on the left-hand side converges if $\operatorname{Re} x > 1$. A large number of other special cases is given by Bailey (1935, Examples pp. 96, 97) and Hardy (1923).

The truncated hypergeometric series

$$y_n(a, b, c, z) = \sum_{r=0}^{n} \frac{(a)_r\, (b)_r}{r\,!\ (c)_r}\, z^r$$

can be expressed in two different ways in terms of ${}_3F_2$:

$$y_n = {}_3F_2\left[\begin{matrix} -n,\ a,\ b;\ z \\ -n,\ c \end{matrix}\right] = \lim_{\epsilon \to 0}\ {}_3F_2\left[\begin{matrix} \epsilon^2 - n,\ a,\ b;\ z \\ \epsilon - n,\ c \end{matrix}\right]$$

and

$$y_n = \frac{(a)_n\, (b)_n}{(c)_n\ m\,!}\, z^n\ {}_3F_2\left[\begin{matrix} 1,\ -n,\ 1 - n - c;\ z^{-1} \\ 1 - n - a,\ 1 - n - b \end{matrix}\right]$$

If $z = 1$ we have

$$y_n(a, b, c, 1) = \frac{\Gamma(a + n + 1)\, \Gamma(b + n + 1)}{n\,!\, \Gamma(a + b + n + 1)}\ {}_3F_2\left[\begin{matrix} a,\ b,\ c + n; \\ c,\ a + b + n + 1 \end{matrix}\right]$$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+b-c)}{\Gamma(1-c)\,\Gamma(1-c+a+b)}$$

$$\times \left\{ 1 - \frac{(a)_{n+1}\,(b)_{n+1}}{(n+1)!\,(c-1)_{n+1}}\, {}_3F_2 \left[\begin{matrix} 1-a,\ 1-b,\ n+1; \\ 2-c,\ n+2 \end{matrix} \right] \right\}$$

$$= \frac{\Gamma(a+n+1)\,\Gamma(b+n+1)}{n!\,\Gamma(a+b-c+1)\,\Gamma(c+n+1)}\, {}_3F_2 \left[\begin{matrix} c-a,\ c-b;\ c+n; \\ c,\ c+n+1 \end{matrix} \right]$$

(cf. Bailey 1935, pp. 93, 94). Corresponding formulas exist for the truncated ${}_3F_2$ and a special truncated ${}_7F_6$ with unit argument (cf. Bailey 1935, pp. 94, 95).

4.6. Integrals

The analogue of the Eulerian integral representations in section 2.1.3, in the case of the general ${}_pF_q$ has been given by Pochhammer (1893b), Erdélyi (1937), [cf. also 5.2(2)] and by Pevnyi (1940) for integral relations of the type given in 2.4(2). The extension of Barnes' integral representation 2.1(15) to the case of a general ${}_pF_q$ can be derived from the results in sections 5.3 and 5.6.

Many definite integrals can be expressed by one or several series ${}_pF_q$. In particular, the Laplace transform of a ${}_pF_q$, $p \le q$, is

$$s \int_0^\infty e^{-st}\, {}_pF_q \left[\begin{matrix} a_1, \dots, a_p; t \\ \rho_1, \dots, \rho_q \end{matrix} \right] dt = {}_{p+1}F_q \left[\begin{matrix} 1, a_1, \dots, a_p; s^{-1} \\ \rho_1, \dots, \rho_q \end{matrix} \right]$$

A formula by Shanker (1946) gives the Hankel-transform of a ${}_2F_2$:

$$\int_0^\infty t^{\rho-\frac12}\, {}_2F_2\,[\rho,\ \tfrac12(\rho+m+1);\ \tfrac12(\rho-m+1),\ \tfrac12(\rho+\nu+1);\ \tfrac12 t^2]$$

$$\times J_\nu(zt)\,(zt)^{\frac12}\, dt = (-1)^m\, z^{\rho-\frac12}$$

$$\times {}_2F_2\,[\rho,\ \tfrac12(\rho+m+1);\ \tfrac12(\rho-m+1);\ \tfrac12(\rho+\nu+1);\ -\tfrac12 z^2],$$

where

$m = 0, 1, 2, \dots,$ and $\operatorname{Re}(\rho+\nu+1) > 0$, $\operatorname{Re}(\rho-m+1) > 0$, $\operatorname{Re}\rho > \tfrac12$.

For other special results see section 4.7 and Erdélyi (1938).

4.7. Various special results

H. Bateman (1933, 1934) and Pasternack (1939) investigated the polynomials in z,

$$F_n(z) = {}_3F_2\,(-n,\ n+1,\ \tfrac12+\tfrac12 z;\ 1,\ 1;\ 1) \qquad n = 0, 1, 2, \dots,$$

and Bateman (1936) studied the polynomials $Z_n(z)$ and $J_n^{u,v}(z)$ defined by

$$Z_n(z) = {}_2F_2(-n, n+1, 1; 1; z) \qquad n = 0, 1, 2, \ldots,$$

$$z^{-u} J_n^{u,v}(z) = \frac{\Gamma(v+n+1+\frac{1}{2}u)}{n!\,\Gamma(u+1)\,\Gamma(v+1+\frac{1}{2}u)}\, {}_1F_2(-n; u+1, v+1+\tfrac{1}{2}u; z^2).$$

Bateman's results have been generalized by Pasternack (1937, 1939) and by S. O. Rice (1939). The latter introduced

$$H_n(\xi, p, v) = {}_3F_2(-n, n+1, \xi; 1, p; v)$$

where $n = 0, 1, 2, \ldots$ and ξ, p, v, are complex variables but $p \neq -n-1, -n-2, \ldots$. Rice found that

$$H_n(\xi, p, v) = \frac{\Gamma(p)}{\Gamma(\xi)\,\Gamma(p-\xi)} \int_0^1 t^{\xi-1}(1-t)^{p-\xi-1} P_n(1-2vt)\,dt$$

$$\operatorname{Re} p > \operatorname{Re} \xi > 0, \qquad P_n(z) = {}_2F_1(-n, n+1; 1; \tfrac{1}{2} - \tfrac{1}{2}z)$$

and that

$$H_n(\xi, p, v)\,\Gamma(p-q)\,\Gamma(q)\,\Gamma(\xi)\,\Gamma(p-\xi)$$

$$= \frac{\Gamma(p)}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)\,\Gamma(q-s)\,\Gamma(\xi-s)\,\Gamma(p-q-\xi+s)\,H_n(s, q, v)\,ds$$

$$0 < \operatorname{Re}\sigma < \operatorname{Re} q, \qquad 0 < \operatorname{Re}(\xi-\sigma) < \operatorname{Re}(q-p).$$

The generating function of the H_n is

$$(1-t)^{-1}\, {}_2F_1[\xi, \tfrac{1}{2}; p; -4vt(1-t)^{-2}]$$

$$= \sum_{n=0}^{\infty} t^n H_n(\xi, p, v).$$

If $Q_n(z)$ is defined by 3.6(24), then

$$\sum_{n=0}^{\infty} (2n+1)\, Q_n(z)\, H_n(\xi, p, v) = (s-1)^{-1}\, {}_2F_1[\xi, 1; p; 2v/(1-s)]$$

If $n \to \infty$, an asymptotic expression for $H_n(\xi, p, 1)$ is

$$\frac{\Gamma(p)\, n^{-2\xi}}{\Gamma(p-\xi)\,\Gamma(1-\xi)} + (-1)^n \frac{\Gamma(p)\, n^{2\xi-2p}}{\Gamma(\xi-p+1)\,\Gamma(\xi)}.$$

Rainville (1945) gave recurrence relations for the $J_n^{u,v}$ and showed that $H_n(\xi, p, v)$ satisfies the four-term recurrence relation

$$n(2n-3)(p+n-1)H_n$$
$$=(2n-1)[(n-2)(p-n+1)+2(n-1)(2n-3)-2(2n-3)(\xi+n-1)v]$$
$$\times H_{n-1}-(2n-3)[2(n-1)^2-n(p-n+1)+2(2n-1)(\xi-n+1)v]\,H_{n-2}$$
$$-(n-2)(2n-1)(p-n+1)\,H_{n-3}.$$

M. C. Fasenmyer (1947) proved that

$$v\frac{\partial}{\partial v}[H_n(\xi,p,v)+H_{n-1}(\xi,p,v)]$$
$$=n[H_n(\xi,p,v)-H_{n-1}(\xi,p,v)]$$

and studied the polynomials

$$f_n(a_i;b_j;z)={}_{p+2}F_{q+2}\left[\begin{matrix}-n,\ n+1,\ a_1,\ldots,a_p;\ z\\ \tfrac{1}{2},\ 1,\ b_1,\ldots,b_q\end{matrix}\right]$$
$$i=1,\ldots,p,\qquad j=1,\ldots,q$$

which can be generated by

$$(1-t)^{-1}G\left[\frac{-4zt}{(1-t)^2}\right]=\sum_{n=0}^{\infty}f_n(z)\,t^n,$$

where

$$G(y)={}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\ y\\ b_1,\ldots,b_q\end{matrix}\right]$$

For recurrence relations, series and integrals involving the f_n see Fasenmyer (1947) and also Chaundy (1943). A particularly simple result is an integral representation for Bateman's $Z_n(z)$:

$$Z_n(z^2)=\pi^{-1/2}\int_0^{\infty}e^{-t^2/4}\,L_n(tz)\,L_n(-tz)\,dt$$

where $L_n(z)$ denotes the n-th Laguerre polynomial (cf. Chapter 10), and

$$f_n(\xi;p;v)=i(4\pi)^{-1/2}\int_{\infty}^{(0+)}(-t)^{-1/2}e^{-t}H_n(\xi,p,-v/t)\,dt,$$
$$H_n(\xi,p,v)=\pi^{-1/2}\int_0^{\infty}t^{-1/2}e^{-t}f_n(\xi;p;vt)\,dt.$$

Erdélyi (1938) has proved the expansion

$$\frac{e^{-z/2}}{t-z}=\sum_{n=0}^{\infty}A_n(\lambda,\mu+n,t)\,z^n\,{}_1F_1(\lambda+n+1;\,2\mu+2n;\,z)$$

where

$$A_n(\lambda,\mu,t)=\sum_{r=0}^{n}c_{\lambda,\mu,r}\,t^{r-n-1}$$

and

$$c_{\lambda,\mu,r} = \frac{(-1)^r}{2^r r!} \, {}_2F_1[-r, 1-\lambda; 2(1-\mu); 2],$$

which is valid for $|t| > 1$, $|z| < 1$, $\mu \neq 0, \pm 1, \pm 2, \ldots$.

Krall and Frink (1949) have investigated a class of polynomials $y_n(a, z)$. According to Rainville these can be written in the form

$$y_n(a, z) = {}_2F_0(-n, a+n-1; -z)$$

where $n = 0, 1, 2, \ldots$, and $a - 1 \neq 0, -1, -2, \ldots$. The $y_n(a, z)$ are orthogonal polynomials on the unit circle associated with the weight function

$$(a-1) \, {}_1F_1(1; a-1; -z^{-1}).$$

They can be expressed in terms of Whittaker's function $W_{\kappa,\mu}(z)$ [cf. 6.9(5)] in the form

$$y_n(a, z) = e^{1/(2z)} z^{1-\frac{1}{2}a} W_{1-\frac{1}{2}a,\, n-\frac{1}{2}+\frac{1}{2}a}(z^{-1}).$$

For recurrence relations, a differential equation, and other results for $y_n(a, z)$ see Krall and Frink (1949).

4.8. Basic hypergeometric series

The following account of the theory of basic generalized hypergeometric series follows closely Chapter 8 of Bailey (1935). Let q be a parameter which in general shall be restricted to the domain $|q| < 1$. We define

(1) $\quad (a)_{q,n} = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) \qquad n = 1, 2, \ldots,$

(2) $\quad (a)_{q,0} = 1$

for all a and q. [The notation $[a]_n$ for $(a)_{q,n}$ (where the q is not shown explicitly) is even more common in the literature.]

Then

(3) $$ {}_r\Phi_s \begin{bmatrix} a_1, a_2, \ldots, a_r; z \\ \rho_1, \ldots, \rho_s \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a_1)_{q,n}(a_2)_{q,n}\cdots(a_r)_{q,n}}{(q)_{q,n}(\rho_1)_{q,n}\cdots(\rho_s)_{q,n}} z^n $$

is a function of z and of $r + s + 1$ parameters $a_1, \ldots, a_r; \rho_1, \ldots, \rho_s; q$ which reduces to

$$ {}_rF_s \begin{bmatrix} a_1, a_2, \ldots, a_r; z \\ \rho_1, \ldots, \rho_s \end{bmatrix} $$

if $r = s + 1$ and $q \to 1$. The function ${}_2\Phi_1$ was first investigated by E. Heine (1878, p. 97-125). ${}_r\Phi_s$ is called a *basic hypergeometric series*; it seems that the case $r = s + 1$ is the only one which has been dealt with. For basic hypergeometric series of two variables see section 5.14.

The simplest case is that of ${}_1\Phi_0(a; z)$ which is a generalization of the binomial series. It may be shown that

$$(4) \quad {}_1\Phi_0(a; z) \equiv \sum_{n=0}^{\infty} \frac{(a)_{q,n}}{(q)_{q,n}} z^n = \prod_{n=0}^{\infty} \frac{1 - aq^n z}{1 - q^n z}$$

and therefore

$$(5) \quad {}_1\Phi_0(a; z)\ {}_1\Phi_0(b; az) = {}_1\Phi_0(ab; z)$$

(cf. Bailey 1935, p. 66). Other elementary cases are:

$$(6) \quad \frac{z}{1-q}\ {}_2\Phi_1(q, q; q^2; z) = \sum_{n=1}^{\infty} \frac{z^n}{1-q^n}$$

$$(7) \quad {}_2\Phi_1(q, -1; -q; z) = 1 + 2\sum_{n=0}^{\infty} \frac{z^n}{1+q^n}$$

$$(8) \quad \frac{z}{1-q^{1/2}}\ {}_2\Phi_1(q, q^{1/2}; q^{3/2}; z) = \sum_{n=1}^{\infty} \frac{z^n}{1-q^{n-1/2}}$$

If we divide (8) by $z^{1/2}$ and replace q, z, by q^2, $q \exp(2ix)$ where x is real, the imaginary part of the series becomes

$$\sum_{n=0}^{\infty} \frac{q^{n+1/2} \sin(2n+1)x}{1-q^{2n+1}} = \frac{Kk}{2\pi}\, sn\left(\frac{2Kx}{\pi}\right)$$

$$= (\sin x)\, q^{1/2} \prod_{n=1}^{\infty} \frac{(1 - 2q^{2n}\cos 2x + q^{4n})(1+q^{2n-1})^2(1-q^{4n})^2}{(1 - 2q^{2n-1}\cos 2x + q^{4n-2})}$$

where $\operatorname{sn} u$ denotes Jacobi's elliptic function of modulus k, and where k, K, and q are connected by

$$(9) \quad q = \exp(-\pi K'/K),$$

$$(10) \quad K = \tfrac{1}{2}\pi F(\tfrac{1}{2}, \tfrac{1}{2}; 1, k^2), \qquad K' = \tfrac{1}{2}\pi F(\tfrac{1}{2}, \tfrac{1}{2}; 1, 1-k^2).$$

This result is part of the theory of elliptic theta-functions, see Chapter 11.

Many of the results in the theory of generalized hypergeometric series have analogues in the theory of basic hypergeometric series. This holds in particular for the theorems on well-poised series. The analogue of Dougall's theorem 4.4(8) is

$$(11)\quad {}_8\Phi_7\left[\begin{matrix} a,\ qa^{1/2},\ -qa^{1/2},\ b,\ c,\ d,\ e,\ q^{-n};\ q \\ a^{1/2},\ -a^{1/2},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{N+1} \end{matrix}\right]$$

$$= \frac{(aq)_{q,N}\,(aq/cd)_{q,N}\,(aq/bd)_{q,N}\,(aq/bc)_{q,N}}{(aq/b)_{q,N}\,(aq/c)_{q,N}\,(aq/d)_{q,N}\,(aq/bcd)_{q,N}}$$

where $bcde = a^2 q^{N+1}$ and $N = 1, 2, 3, \ldots$. The effect of the presence of the four elements $qa^{1/2}$, $-qa^{1/2}$, $a^{1/2}$, $-a^{1/2}$ in the function on the left is merely the insertion of the factor $(1 - aq^{2n})/(1 - a)$ in the general term of the series. Another important result (which is the analogue of a theorem due to Whipple) was proved by Watson (1929):

$$(12)\quad {}_8\Phi_7\left[\begin{matrix} a,\ qa^{1/2},\ -qa^{1/2},\ c,\ d,\ e,\ f,\ g;\ a^2 q^2/(cdefg) \\ a^{1/2},\ -a^{1/2},\ aq/c,\ aq/d,\ aq/e,\ aq/f,\ aq/g \end{matrix}\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-aq^n)\,(1-aq^n/fg)\,(1-aq^n/ge)\,(1-ag^n/ef)}{(1-aq^n/e)\,(1-aq^n/f)\,(1-aq^n/g)\,(1-aq^n/efg)}\right]$$

$$\times\ {}_4\Phi_3\left[\begin{matrix} aq/cd,\ e,\ f,\ g;\ q \\ efg/a,\ aq/c,\ aq/d \end{matrix}\right].$$

For more general results see W. N. Bailey (1935, 1936, 1947 a, b, 1948 a, b). A discussion of the state of the theory is also given by Bailey (1947 a, 1948 b).

Among the consequences of (11) there are the basic analogues of the theorems of Saalschütz 4.4 (3), Gauss 2.1 (14) and of 2.9 (2) viz.

$${}_3\Phi_2\left[\begin{matrix} b,\ c,\ q^{-N};\ q \\ d,\ bcq^{1-N}/d \end{matrix}\right] = \frac{(d/b)_{q,N}\,(d/c)_{q,N}}{(d)_{q,N}\,(d/bc)_{q,N}},$$

$${}_2\Phi_1\left[\begin{matrix} b,\ c;\ d/bc \\ d \end{matrix}\right] = \prod_{n=0}^{\infty}\frac{(1-dq^n/b)\,(1-dq^n/c)}{(1-dq^n)\,(1-dq^n/bc)},$$

$${}_2\Phi_1\left[\begin{matrix} a,\ b;\ z \\ c \end{matrix}\right] = {}_1\Phi_0\,[ab/c;\ z]\ {}_2\Phi_1\left[\begin{matrix} c/a,\ c/b;\ abz/c \\ c \end{matrix}\right].$$

A large number of identities can be derived from (12). Some of these are *Euler's identity*

$$1 + \sum_{n=1}^{\infty}(-1)^n\,[q^{\frac{1}{2}n(3n-1)} + q^{\frac{1}{2}n(3n+1)}] = \prod_{n=1}^{\infty}(1-q^n),$$

the *Rogers-Ramanujan identities*

$$1 + \sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)\,(1-q^2)\cdots(1-q^n)}$$

$$= \prod_{n=0}^{\infty} (1 - q^{1+5n})^{-1} (1 - q^{4+5n})^{-1},$$

$$1 + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

$$= \prod_{n=0}^{\infty} (1 - q^{2+5n})^{-1} (1 - q^{3+5n})^{-1},$$

and a result due to Gauss

$$1 + \sum_{n=1}^{\infty} q^{\frac{1}{2}n(n+1)} = \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 - q^{2n-1}} \right).$$

For the proofs and for other results see Bailey (1935).

The basic analogue of Kummer's formula 2.8 (47) has been proved by Daum (1942). The result is

$${}_2\Phi_1 \left[\begin{matrix} a, \ b; -q/b \\ aq/b \end{matrix} \right]$$

$$= \frac{\Omega(aq/b)\, \Omega(qa^{\frac{1}{2}})\, \Omega(-qa^{\frac{1}{2}})\, \Omega(-q/b)}{\Omega(aq)\, \Omega(-q)\, \Omega(qa^{\frac{1}{2}}/b)\, \Omega(-qa^{\frac{1}{2}}/b)} \cdot$$

where

$$\Omega(z) = \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - zq^n}.$$

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CHAPTER V

FURTHER GENERALIZATIONS OF THE HYPERGEOMETRIC FUNCTION

5.1. Various generalizations

The classical hypergeometric series

$$(1) \quad F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{1\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \cdots$$

has inspired the investigation of many functions and series. In this chapter we shall be concerned with those generalizations which are commonly regarded as hypergeometric functions. Other generalizations, for instance, Mathieu and Lamé functions, will be found in other chapters.

The generalized hypergeometric series

$$(2) \quad {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!},$$

where

$$(3) \quad (a)_\nu = \frac{\Gamma(a+\nu)}{\Gamma(a)}, \quad (a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1),$$

and where it is assumed that the parameters are such that at least one of the definitions (3) makes sense, has already been introduced in Chapter 4. Here we shall regard (2) as a formal power series and will pay no attention to questions of convergence.

It is usual to define the most general (formal) hypergeometric series as a formal power series for which the ratio of two consecutive coefficients is a (fixed) rational function of the index. More precisely,

$$(4) \quad y = \sum_{n=0}^{\infty} c_n x^n, \quad \frac{c_{n+1}}{c_n} = \frac{P(n)}{Q(n+1)},$$

where P and Q are polynomials; we assume that

$$Q(n+1) = (n+1)\, Q_1(n+1)$$

where Q_1 is a polynomial such that $P(n)$ and $Q_1(n+1)$ have no common factors. The respective degrees of P and Q_1 are p and q. The factorization of P and Q_1 in linear factors at once leads to the expression of (4) in terms of the symbol ${}_pF_q$, so that essentially (2) is the most general (formal) hypergeometric series.

The differential equation satisfied by (4) can be written in terms of the differential operator

$$(5) \quad \delta \equiv x \frac{d}{dx}.$$

We have $\delta x^n = nx^n$ and hence for any polynomial operator with constant coefficients $f(\delta)\, x^n = f(n)\, x^n$. Thus (4) satisfies the differential equation

$$(6) \quad \{xP(\delta) - Q(\delta)\}\, y = 0 \quad \text{or} \quad \{P(\delta) - \frac{d}{dx} Q_1(\delta)\}\, y = 0,$$

which is of order $\max(p, q+1)$ and has singularities at 0 and ∞ if $p \neq q+1$, and at $0, \infty$, and at a third, finite, point if $p = q+1$.

In connection with this generalized hypergeometric function there are two distinct problems: (i) to interpret (2) when $p > q+1$ and the series is divergent for every $x \neq 0$, and (ii) to find fundamental systems of solutions of (6) for the neighborhood of every singularity.

A further generalization introduces basic hypergeometric functions. For these see Chapter 4.

Hypergeometric functions of two or more variables are similarly defined. For these see section 5.7.

MACROBERT'S E-FUNCTION

5.2. Definition of the E-function

MacRobert's E-function arose from an attempt to give a meaning to the symbol ${}_pF_q$ when $p > q+1$. For $p \leq q+1$ we have

$$(1) \quad E(p; a_r : q; \rho_s : x) = \frac{\Gamma(a_1) \cdots (a_p)}{\Gamma(\rho_1) \cdots (\rho_q)} \times {}_pF_q(a_1, \ldots, a_p; \rho_1, \ldots, \rho_q; -1/x)$$

where $x \neq 0$ if $p < q$ and $|x| > 1$ if $p = q+1$.

For $p \geq q+1$ we have

$$(2) \quad E(p; a_r : q; \rho_s : x) = \sum_{r=1}^{p} \frac{\prod_{s=1}^{p}{}' \Gamma(a_s - a_r)}{\prod_{t=1}^{q} \Gamma(\rho_t - a_r)} \Gamma(a_r)\, x^{a_r}$$

$$\times {}_{q+1}F_{p-1}[a_r, a_r - \rho_1 + 1, \ldots, a_r - \rho_q + 1;$$
$$a_r - a_1, \ldots, *, \ldots, a_r - a_p + 1; (-1)^{p+q} x]$$

where $|x| < 1$ if $p = q + 1$. The prime in Π' indicates the omission of the factor $\Gamma(a_r - a_r)$; the asterisk in F the omission of the parameter $a_r - a_r +$ 1; an empty product is to be interpreted as 1; and zero or negative integer values of the a_r are tacitly excluded. Similar conventions will be observed throughout this chapter. The asymptotic expansion, for $x \to \infty$, $-\frac{1}{2}(p - q + 1)\pi < \arg x < \frac{1}{2}(p - q + 1)\pi$, of (2) is given by the right-hand side of (1). Originally (MacRobert, 1937-1938), E was defined by the multiple integral

$$(3) \quad E(p; a_r: q; \rho_s: x) = \frac{\Gamma(a_{q+1})}{\Gamma(\rho_1 - a_1)\Gamma(\rho_2 - a_2)\cdots\Gamma(\rho_q - a_q)}$$

$$\times \prod_{\mu=1}^{q} \int_0^\infty \lambda_\mu^{\rho_\mu - a_\mu - 1} (1 + \lambda_\mu)^{-\rho_\mu} \, d\lambda_\mu$$

$$\times \prod_{\nu=2}^{p-q-1} \int_0^\infty e^{-\lambda_{q+\nu}} \lambda_{q+\nu}^{a_{q+\nu}-1} \, d\lambda_{q+\nu}$$

$$\times \int_0^\infty e^{-\lambda_p} \lambda_p^{a_p - 1} \left[1 + \frac{\lambda_{q+2}\lambda_{q+3}\cdots\lambda_p}{(1+\lambda_1)(1+\lambda_2)\cdots(1+\lambda_q)x} \right]^{-a_{q+1}} d\lambda_p$$

where $|\arg x| < \pi$, $p \geq q + 1$ and the a_r and ρ_s are such that the integrals are convergent. The equivalence of (2) and (3) can be proved (Mac Robert 1937-1938).

Other related functions are the P-function which arises when (2) is written as

$$(4) \quad E(p; a_r: q; \rho_s: x) = \sum_{r=1}^{p} P(a_r; p - 1; a_s: q; \rho_t: x)$$

and two further functions, denoted by Q and H, which are multiples of P and E respectively (MacRobert 1937-1938). An alternative notation for $E(p; a_r: q; \rho_s: x)$ is $E(a_1, \ldots, a_p: \rho_1, \ldots, \rho_q: x)$.

Several particular cases of the E-function arise from (1) by means of the formulas developed or quoted in Chapter 4. From (2) and (3) some further interesting particular cases arise among which the most important are the expression of the modified Bessel function of the third kind,

$$(5) \quad (2\pi x)^{\frac{1}{2}} K_\nu(x) = e^{-x} \cos(\nu\pi) E(\tfrac{1}{2} + \nu, \tfrac{1}{2} - \nu: : 2x)$$

[MacRobert 1937-1938 (12)], and that of Whittaker's W-function

(6) $$\Gamma(\tfrac{1}{2}-k-m)\,\Gamma(\tfrac{1}{2}-k+m)\,x^{-k}\,e^{\frac{1}{2}x}\,W_{k,m}(x)$$

$$= E(\tfrac{1}{2}-k-m,\ \tfrac{1}{2}-k+m: \, : x)$$

[MacRobert 1941(25)]. It may be noted that $W_{k,m}(ix)\,W_{k,m}(-ix)$ may also be expressed in terms of the E-function [MacRobert 1941 (15′)] as can other combinations. The E-function itself is a particular case of Meijer's G-function [cf. 5.6(2)].

5.2.1. Recurrence relations

A basic system of recurrence formulas has been given by MacRobert [1941, equations (20) to (24)]. The three most important formulas are

(7) $$a_1\, x\, E(a_1, \dots, a_p: \rho_1, \dots, \rho_q: x)$$

$$= x\, E(a_1+1, a_2, \dots, a_p: \rho_1, \dots, \rho_q: x)$$

$$+ E(a_1+1, a_2+1, \dots, a_p+1: \rho_1+1, \dots, \rho_q+1: x),$$

(8) $$\frac{d}{dx} E(a_1, \dots, a_p: \rho_1, \dots, \rho_q: x)$$

$$= x^{-2} E(a_1+1, \dots, a_p+1: \rho_1+1, \dots, \rho_q+1: x),$$

and

(9) $$(\rho_1-1)\, x\, E(a_1, \dots, a_p: \rho_1, \dots, \rho_q: x)$$

$$= x\, E(a_1, \dots, a_p: \rho_1-1, \rho_2, \dots, \rho_q: x)$$

$$+ E(a_1+1, \dots, a_p+1: \rho_1+1, \dots, \rho_q+1: x).$$

5.2.2. Integrals

A few integrals with hypergeometric functions may be evaluated in terms of the E-function. A typical example [MacRobert 1937-1938 (21)] is

(10) $$\int_0^\infty e^{-x\lambda}\,\lambda^{\gamma-1}\,{}_2F_1(a,\beta;\delta;-\lambda)\,d\lambda$$

$$= \frac{\Gamma(\delta)\,x^{-\gamma}}{\Gamma(a)\,\Gamma(\beta)}\,E(a,\beta,\gamma:\delta:x) \qquad \operatorname{Re} x > 0, \quad \operatorname{Re}\gamma > 0,$$

which may be proved by substituting

$${}_2F_1(a,\beta;\delta;-\lambda)$$

$$= \frac{\Gamma(\delta)}{\Gamma(\beta)\,\Gamma(\delta-\beta)} \int_0^\infty \mu^{\delta-\beta-1}(1+\mu)^{-\delta}\left[1+\frac{\lambda}{(1+\mu)}\right]^{-a} d\mu$$

on the left-hand side and using (3).

The integral

$$\int_0^1 e^{ix\xi} \,{}_2F_1(a, \beta; \gamma; \xi) d\xi$$

may be evaluated by complex integration; we integrate around a contour which is a rectangle with vertices at 0, 1, $1 + ik$, $ik\ (k > 0)$ and which has indentations at 0 and 1 and then make the radii of the indentations tend to zero and $k \to \infty$, using (10). [MacRobert 1937-1938 (23); there are also a few other integrals in this paper.]

Integrals with the E-function are discussed in Mac Robert 1941. The most important are the integrals of the Laplace or Euler type, but there are some more general integrals too. Typical samples are

$$(11)\quad \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{\xi}\, \xi^{-\sigma} E(a_1, \ldots, a_p : \rho_1, \ldots, \rho_q : \xi x)\, d\xi$$

$$= E(a_1, \ldots, a_p : \rho_1, \ldots, \rho_q, \sigma : x) \qquad -\pi \le \arg \xi \le \pi,$$

$$(12)\quad \int_0^\infty e^{-\lambda} \lambda^{\beta-1} E(a_1, \ldots, a_p : \rho_1, \ldots, \rho_q : x/\lambda)\, d\lambda$$

$$= E(a_1, \ldots, a_p, \beta : \rho_1, \ldots, \rho_q : x) \qquad \operatorname{Re} \beta > 0,$$

$$(13)\quad \int_{-\infty}^{(0+)} e^{\xi} \xi^{\beta-1} E(a_1, \ldots, a_p : \rho_1, \ldots, \rho_q : x/\xi)\, d\xi$$

$$= e^{i\beta\pi} E(a_1, \ldots, a_p, \beta : \rho_1, \ldots, \rho_q : xe^{-i\pi})$$

$$- e^{-i\beta\pi} E(a_1, \ldots, a_p, \beta : \rho_1, \ldots, \rho_q : xe^{i\pi})$$

If $p = q + 1$, this holds provided that $\xi = x$ inside the loop; if $p < q$ or $p = q$ and $x > 1$, the right-hand side reduces to

$$2i \sin(\beta\pi)\, E(a_1, \ldots, a_p, \beta : \rho_1, \ldots, \rho_q : -x).$$

$$(14)\quad \int_0^\infty \lambda^{\sigma-\beta-1} (1+\lambda)^{-\sigma} E[a_1, \ldots, a_p : \rho_1, \ldots, \rho_q : (1+\lambda)x]$$

$$= \Gamma(\sigma-\beta)\, E(a_1, \ldots, a_p, \beta : \rho_1, \ldots, \rho_q, \sigma : x)$$

$$\operatorname{Re}(\beta) > 0, \quad \operatorname{Re}(\sigma - \beta) > 0.$$

The proof of these formulas is based on the definition of the E-function; (1) or (2) and (3) are used according to the values of p and q.

MEIJER'S G-FUNCTION

5.3. Definition of the G-function

Meijer's G-function provides an interpretation of the symbol ${}_pF_q$

when $p > q + 1$; this interpretation is in complete agreement with the one given by MacRobert's E-function. In addition, all significant particular solutions of a hypergeometric differential equation 5.1(6) may be expressed in terms of the G-function.

Originally (Meijer 1936), the G-function was defined in a manner resembling 5.2(2). Later (Meijer 1941c, 1946), this definition was replaced by one in terms of Mellin-Barnes type integrals (for which see section 1.19). The latter definition has the advantage that it allows a greater freedom in the relative values of p and q. Here we shall complete Meijer's definition so as to include all values of p and q without placing any (non-trivial) restriction on m and n.

We define

(1) $$G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\, x^s\, ds$$

where an empty product is interpreted as 1, $0 \le m \le q$, $0 \le n \le p$, and the parameters are such that no pole of $\Gamma(b_j - s)$, $j = 1, \ldots, m$, coincides with any pole of $\Gamma(1 - a_k + s)$, $k = 1, \ldots, n$. These assumptions will be retained throughout. Whenever there is no danger of misunderstanding we shall write more briefly

$$G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_r\\ b_s\end{matrix}\right),\quad G^{mn}_{pq}(x) \quad \text{or simply} \quad G(x).$$

There are three different paths L of integration:

(2) L runs from $-i\infty$ to $+i\infty$ so that all poles of $\Gamma(b_j - s)$, $j = 1, \ldots, m$, are to the right, and all the poles of $\Gamma(1 - a_k + s)$, $k = 1, \ldots, n$, to the left, of L. The integral converges if $p + q < 2(m + n)$ and $|\arg x| < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$.

(3) L is a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma(b_j - s)$, $j = 1, \ldots, m$, once in the negative direction, but none of the poles of $\Gamma(1 - a_k + s)$, $k = 1, \ldots, n$. The integral converges if $q \ge 1$ and either $p < q$ or $p = q$ and $|x| < 1$.

(4) L is a loop starting and ending at $-\infty$ and encircling all poles of $\Gamma(1 - a_k + s)$, $k = 1, \ldots, n$, once in the positive direction, but none of the poles of $\Gamma(b_j - s)$, $j = 1, \ldots, m$. The integral converges if $p \ge 1$ and either $p > q$ or $p = q$ and $|x| > 1$.

We shall always assume that the values of the parameters and of the variable x are such that at least one of the three definitions (2), (3), (4) makes sense. In cases when more than one of these definitions make sense, they lead to the same result so that there will be no ambiguity involved.

The G-function is an analytic function of x; it is symmetric in the parameters $a_1, \ldots, a_n$, likewise in $a_{n+1}, \ldots, a_p$, in $b_1, \ldots, b_m$, and in $b_{m+1}, \ldots, b_q$.

With (3) the integral can be evaluated as a sum of residues. If no two b_j, $j = 1, \ldots, m$, differ by an integer, all poles are of the first order and

$$(5)\quad G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right)$$

$$= \sum_{h=1}^{m} \frac{\prod_{j=1}^{m}{}' \Gamma(b_j - b_h) \prod_{j=1}^{n} \Gamma(1 + b_h - a_j)}{\prod_{j=m+1}^{q} \Gamma(1 + b_h - b_j) \prod_{j=n+1}^{p} \Gamma(a_j - b_h)} x^{b_h}$$

$$\times {}_pF_{q-1}[1 + b_h - a_1, \ldots, 1 + b_h - a_p;$$

$$1 + b_h - b_1, \ldots, *, \ldots, 1 + b_h - b_q; (-1)^{p-m-n} x]$$

$$p < q \quad \text{or} \quad p = q \quad \text{and} \quad |x| < 1.$$

Similarly if no two a_k, $k = 1, \ldots, n$, differ by an integer, we have from (4)

$$(6)\quad G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right)$$

$$= \sum_{h=1}^{n} \frac{\prod_{j=1}^{n}{}' \Gamma(a_h - a_j) \prod_{j=1}^{m} \Gamma(b_j - a_h + 1)}{\prod_{j=n+1}^{p} \Gamma(a_j - a_h + 1) \prod_{j=m+1}^{q} \Gamma(a_h - b_j)} x^{a_h - 1}$$

$$\times {}_qF_{p-1}\left[\begin{matrix} 1 + b_1 - a_h, \ldots, 1 + b_q - a_h; \\ 1 + a_1 - a_h, \ldots, *, \ldots, 1 + a_p - a_h; (-1)^{q-m-n} x^{-1} \end{matrix}\right]$$

$$q < p \quad \text{or} \quad q = p \quad \text{and} \quad |x| > 1.$$

In these expansions conventions corresponding to those of 5.2 (2) hold.

5.3.1. Simple identities

If one of the a_j, $j = 1, \ldots, n$, is equal to one of the b_j, $j = m+1, \ldots, q$ (or one of the b_j, $j = 1, \ldots, m$, equals one of the a_j, $j = n+1, \ldots, p$), then the G-function reduces to one of lower order: p, q, and n (and m) decrease by unity.

$$(7)\quad G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_{q-1}, a_1\end{matrix}\right) = G^{m,n-1}_{p-1,q-1}\left(x\,\middle|\,\begin{matrix}a_2, \ldots, a_p\\ b_1, \ldots, b_{q-1}\end{matrix}\right)\qquad n, p, q \geq 1$$

is such a reduction formula, and all others are similar.

Obvious changes of the variable in the integral (1) give

$$(8)\quad x^{\sigma} G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_r\\ b_s\end{matrix}\right) = G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_r+\sigma\\ b_s+\sigma\end{matrix}\right),$$

$$(9)\quad G^{mn}_{pq}\left(x^{-1}\,\middle|\,\begin{matrix}a_r\\ b_s\end{matrix}\right) = G^{nm}_{qp}\left(x\,\middle|\,\begin{matrix}1-b_s\\ 1-a_r\end{matrix}\right),$$

$$(10)\quad G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_r\\ b_s\end{matrix}\right)$$
$$= (2\pi)^{\frac{1}{2}p+\frac{1}{2}q-m-n}\, 2^{\frac{1}{2}p-\frac{1}{2}q+1-a_1-\cdots-a_p+b_1+\cdots+b_q}$$
$$\times G^{2m,2n}_{2p,2q}\left(2^{2p-2q}x\,\middle|\,\begin{matrix}\frac{1}{2}a_r, \frac{1}{2}a_r+\frac{1}{2}\\ \frac{1}{2}b_s, \frac{1}{2}b_s+\frac{1}{2}\end{matrix}\right).$$

In (10) the duplication formula of the gamma function has been used, and there is a corresponding formula which uses the multiplication theorem 1.2(11) of the gamma function.

The most important of these formulas is (9) because it enables us to transform a G-function with $p > q$ to one with $p < q$. In this way in all discussions $p \leq q$ may be assumed without loss of generality. Samples of other fairly obvious relations are

$$(11)\quad (1-a_1+b_1)\, G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right)$$
$$= G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_1-1, a_2, \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix}\right) + G^{mn}_{pq}\left(x\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_p\\ b_1+1, b_2, \ldots, b_q\end{matrix}\right)\qquad m, n \geq 1,$$

$$(12)\quad (a_p - a_1)\, G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

$$= G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_1 - 1, a_2, a_3, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

$$+ G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_{p-1}, a_p - 1 \\ b_1, \ldots b_q \end{matrix}\right) \qquad 1 \le n \le p-1,$$

$$(13)\quad x\frac{d}{dx}\, G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_1 - 1, a_2, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

$$+ (a_1 - 1)\, G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) \qquad n \ge 1,$$

$$(14)\quad G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = \frac{1}{2\pi i}\left[e^{\pi i b_{m+1}}\, G^{m+1,n}_{pq}\left(x e^{-\pi i} \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right)\right.$$

$$\left. - e^{-\pi i b_{m+1}}\, G^{m+1,n}_{pq}\left(x e^{\pi i} \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right)\right] \qquad m \le q-1,$$

$$G^{mn}_{pq}\left(x \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right) = \frac{1}{2\pi i}\left[e^{\pi i a_{n+1}}\, G^{m,n+1}_{pq}\left(x e^{-\pi i} \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right)\right.$$

$$\left. - e^{-\pi i a_{n+1}}\, G^{m,n+1}_{pq}\left(x e^{\pi i} \,\middle|\, \begin{matrix} a_r \\ b_s \end{matrix}\right)\right] \qquad n \le p-1.$$

5.4. Differential equations

From 5.3(1) it is seen that $G(x)$ satisfies the linear differential equation

$$(1)\quad [(-1)^{p-m-n}\, x \prod_{j=1}^{p} (\delta - a_j + 1) - \prod_{j=1}^{q} (\delta - b_j)]\, y = 0 \qquad \delta \equiv x\frac{d}{dx}.$$

Clearly, every differential equation of the form 5.2(6) may be reduced to this form by a change of variable. Equation (1) is of degree $\max(p, q)$ and on account of 5.3(9) we may assume $p \le q$. The solutions of (1) have been investigated by Meijer (1946).

If $p < q$, the only singularities of (1) are $x = 0, \infty$; $x = 0$ is a regular singularity, $x = \infty$ an irregular one. A fundamental system of q linearly independent solutions of (1) for the neighborhood of $x = 0$ is

$$(2)\quad G^{1,p}_{pq}\left(xe^{(p-m-n-1)\pi i}\,\middle|\,\begin{matrix} a_1, \ldots, a_p \\ b_h, b_1, \ldots, b_{h-1}, b_{h+1}, \ldots, b_q\end{matrix}\right) h = 1, \ldots, q.$$

For the neighborhood of the irregular singularity, x must be restricted to a sector, and Meijer determines two integers, k, g, so that

$$(3)\quad |\arg x + (q - m - n - 2k + 1)\,\pi| < (\tfrac{1}{2}q - \tfrac{1}{2}p + 1)\,\pi$$

and

$$(4)\quad |\arg x + (q - m - n - 2h)\,\pi| < (q - p + \epsilon)\,\pi$$

for $h = g, g + 1, \ldots, g + q - p - 1$, where $\epsilon = \tfrac{1}{2}$ if $q = p + 1$, and $\epsilon = 1$ if $q \geq p + 2$. Then, if x is in the sector determined by (3) and (4), the p functions

$$(5)\quad G^{q,1}_{pq}\left(xe^{(q-m-n-2k+1)\pi}\,\middle|\,\begin{matrix} a_h, a_1, \ldots, a_{h-1}, a_{h+1}, \ldots, a_p \\ b_1, \ldots, b_q\end{matrix}\right)$$

$$h = 1, \ldots, p,$$

and the $q - p$ functions

$$(6)\quad G^{q,0}_{pq}\left(xe^{(q-m-n-2h)\pi}\,\middle|\,\begin{matrix} a_r \\ b_s\end{matrix}\right) \qquad h = g, \quad g + 1, \ldots, g + q - p + 1$$

form a fundamental system of solutions of (1).

If $p = q$, the point $x = \infty$ is also a singularity of the regular type and, with the conditions (3), a fundamental system is given by (5). In this case $x = (-1)^{p-m-n}$ is also a regular singularity, but no fundamental system for the neighborhood of this point has been given in the literature.

If $p > q$, 5.3(9) may be used in order to reduce the differential equation to the case already discussed; $x = 0$ and $x = \infty$ interchange their role.

In any case, it is clear from (1) that for fixed p, q, $a_1, \ldots, a_p$, $b_1, \ldots, b_q$, all $(p + 1)(q + 1)$ functions

$$(7)\quad G^{mn}_{pq}\,[(-1)^{m+n}\,x] \qquad 0 \leq m \leq q, \quad 0 \leq n \leq p$$

satisfy the same differential equation.

5.4.1. Asymptotic expansions

We shall assume $p \leq q$; the results for $p > q$ can be obtained from 5.3(9) by interchanging the role of $x = 0$ and $x = \infty$. Certain integer values of combinations of parameters are excluded, and so are a few other exceptional cases, for example, those in which the coefficient of the dominant term of the asymptotic expansion is zero.

The point $x = 0$ is a singular point of the differential equation (1), and the behavior of $G(x)$ in the neighborhood of this point follows from 5.3(5). We have

(8) $G_{pq}^{mn}(x) = O(|x|^{\beta})$ as $x \to 0$

where $p \leq q$ and $\beta = \max \operatorname{Re} b_h$ for $h = 1, 2, \ldots, m$.

The point $x = \infty$ is a singular point of irregular type of (1), and accordingly the behavior of $G(x)$ as $x \to \infty$ is much more involved. The investigation of this behavior was commenced by Barnes (1907, and other papers), continued by other authors (among them MacRobert, 1937-1938), and completed by C. S. Meijer (1946). The detailed results are too involved to be included here, and only a brief summary (Meijer 1946, §18) will be given.

As $x \to \infty$, the G-function is of the order of some power of x if $p < q$ and if

(9) $n \geq 1, \quad m + n > \tfrac{1}{2}p + \tfrac{1}{2}q,$ and $|\arg x| < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi;$

or if

(10) $q = p + 1,$ k is some integer, and

$$|\arg x - (m + n - p + 2k - 1)\pi| < \tfrac{1}{2}\pi.$$

As $x \to \infty$, $p < q$, the G-function vanishes exponentially if

(11) $m > \tfrac{1}{2}p + \tfrac{1}{2}q, \quad n = 0, \quad |\arg x| < (m - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi.$

When $x \to \infty$ and $p < q$, the G-function becomes exponentially infinite in the regions described below.

(12) If $q \geq p + 2$, we must have either

(i) $m + n > \tfrac{1}{2}p + \tfrac{1}{2}q$ and $|\arg x| > (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$

or

(ii) $m + n \leq \tfrac{1}{2}p + \tfrac{1}{2}q$ and no restriction on $\arg x$.

(13) If $q = p + 1$, let k be an integer, and let

$$|\arg x - (m + n - p + 2k)\pi| < \tfrac{1}{2}\pi.$$

In this case we must have either

(i) $m + n \geq p + 1$ and $k \geq 0$ or $k \leq p - m - n$

or

(ii) $m + n \leq p$ and no restriction on the integer k.

When $p = q$, the behavior of $G(x)$ as $x \to \infty$ follows from 5.3(6); it is that of a power of x.

For more complete and more detailed results see Meijer 1946 §18.

5.5. Series and integrals

Of the more involved functional relations for the G-function the most important ones are series and integrals. Comparatively few series of G-functions have been investigated. The number of known integrals is, however, very considerable. Only a few samples will be given here; for more details refer to the original papers, mostly by C. S. Meijer.

5.5.1. Series of G-functions

A first group of series expansions in terms of G-functions consists of the *four multiplication theorems* (Meijer 1941 c),

$$G_{pq}^{mn}\left(\lambda x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

(1) $$= \lambda^{b_1} \sum_{r=0}^{\infty} \frac{1}{r!} (1-\lambda)^r \, G_{pq}^{mn}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1+r, b_2, \ldots, b_q \end{matrix}\right)$$

$$|\lambda - 1| < 1, \quad m \geq 1,$$

(2) $$= \lambda^{b_q} \sum_{r=0}^{\infty} \frac{1}{r!} (\lambda-1)^r \, G_{pq}^{mn}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{q-1}, b_q+r \end{matrix}\right)$$

$$m < q, \quad |\lambda - 1| < 1,$$

(3) $$= \lambda^{a_1-1} \sum_{r=0}^{\infty} \frac{1}{r!} \left(1-\frac{1}{\lambda}\right)^r G_{pq}^{mn}\left(x \,\middle|\, \begin{matrix} a_1-r, a_2, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

$$n \geq 1, \quad \mathrm{Re}\,\lambda > \tfrac{1}{2},$$

(4) $$= \lambda^{a_p-1} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{1}{\lambda}-1\right)^r G_{pq}^{mn}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_{p-1}, a_p-r \\ b_1, \ldots, b_q \end{matrix}\right)$$

$$n < p, \quad \mathrm{Re}\,\lambda > \tfrac{1}{2},$$

which express $G(\lambda x)$ as an infinite series of $G(x)$. If $p < q$ and $m = 1$, the restriction $|\lambda - 1| < 1$ may be omitted in (1) and a similar remark applies to (3) when $n = 1$, $p > q$.

The second set of important series consists of the so-called expansion formulas (Meijer 1946). These serve to express a G-function as a finite combination of G-functions with the same p, q, but changed m, n and are very useful in the investigation of the solutions of the differential equation 5.4(1). For instance, the fourth expansion formula (Meijer 1946, Theorem 5) expresses a G_{pq}^{mn} as a linear combination of ν functions of the type G_{pq}^{kl}, μ functions of the type $G_{pq}^{k,l-1}$, and another $k - \mu - \nu$ functions of the same type, under suitable restrictions on $k, l, m, n, p, q, \mu, \nu$.

5.5.2. Integrals with G-functions

The most important integrals are those which express the properties of the G-function under integral transformations. The Euler transform is

$$(5)\quad \int_0^1 y^{-\alpha}(1-y)^{\alpha-\beta-1}\, G_{pq}^{mn}\left(xy\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dy$$

$$= \Gamma(\alpha-\beta)\; G_{p+1,q+1}^{m,n+1}\left(x\,\middle|\,\begin{matrix}\alpha,\ a_1,\ldots,a_p\\ b_1,\ldots,b_q,\ \beta\end{matrix}\right)$$

and may be proved from 5.3(1) when

$$(6)\quad p+q<2(m+n),\qquad |\arg x|<(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi,$$

$$(7)\quad \operatorname{Re}\beta<\operatorname{Re}\alpha<\operatorname{Re}b_h+1 \qquad h=1,\ldots,m;$$

or from 5.3(5), for $p<q$ (or for $p=q$ and $|x|<1$), under the conditions (7). With α equal to one of the numbers $b_{n+1},\ldots,b_q$ or β equal to one of the numbers $a_1,\ldots,a_n$, (5) becomes an important functional equation of G_{pq}^{mn}.

The behavior of G_{pq}^{mn} under the Laplace transformation follows from

$$(8)\quad \int_0^\infty e^{-y}\, y^{-\alpha}\, G_{pq}^{mn}\left(xy\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dy$$

$$= G_{p+1,q}^{m,n+1}\left(x\,\middle|\,\begin{matrix}\alpha,\ a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)$$

valid for example under the conditions (6) and (7), the part referring to β being deleted from the latter.

The behavior of the G-function under the Mellin transformation can be read off 5.3(1) with conditions 5.3(2). Certain integrals of products of G-functions may be evaluated by means of the *product theorem* of the Mellin or of the Laplace transformation. Some such integrals were given by Meijer (1936). Meijer also gave generalizations of (5).

The Hankel transform is

$$(9)\quad \int_0^\infty y^{-\alpha}\, J_\nu(2y^{1/2})\, G_{pq}^{mn}\left(xy\,\middle|\,\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dy$$

$$= G_{p+2,q}^{m,n+1}\left(x\,\middle|\,\begin{matrix}\alpha-\tfrac{1}{2}\nu,\ a_1,\ldots,a_p,\ \alpha+\tfrac{1}{2}\nu\\ b_1,\ldots,b_q\end{matrix}\right)$$

valid for example under the conditions (6) and

(10) $\operatorname{Re}(-a + \tfrac{1}{2}\nu + b_h) > -1 \qquad h = 1, \ldots, m$

$\operatorname{Re}(-a + a_j) < \tfrac{1}{4} \qquad j = 1, \ldots, n.$

In all these formulas the introduction of loop integrals enables us to relax the conditions on a [or on β in (5)]. Other conditions of validity should be possible, but apart from a number of cases in the papers by C. S. Meijer these have not been given explicitly.

A last integral containing (modified) Bessel functions is

$$(11)\quad \int_0^\infty y^{-a} K_\nu(2y^{1/2})\, G_{pq}^{mn}\left(xy \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) dy$$

$$= \tfrac{1}{2}\, G_{p+2,q}^{m,n+2}\left(x \,\middle|\, \begin{matrix} a - \tfrac{1}{2}\nu,\ a + \tfrac{1}{2}\nu,\ a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

valid for example under the conditions (6) and

(12) $\operatorname{Re}(-a \pm \tfrac{1}{2}\nu + b_h) > -1 \qquad h = 1, \ldots, m.$

Better conditions have been given by Meijer [1936, equation (58)] in a particular case.

5.6. Particular cases of the G-function

Clearly,

$$(1)\quad {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$$

$$= \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)}\, G_{p,q+1}^{1,p}\left(x \,\middle|\, \begin{matrix} 1 - a_1, \ldots, 1 - a_p \\ 0,\ 1 - b_1, \ldots, 1 - b_q \end{matrix}\right)$$

$$= \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)}\, G_{q+1,p}^{p,1}\left(-\frac{1}{x} \,\middle|\, \begin{matrix} 1, b_1, \ldots, b_q \\ a_1, \ldots, a_p \end{matrix}\right)$$

and by comparison of 5.2(2) and 5.3(5)

$$(2)\quad E(p; a_r: q; \beta_s : x)$$

$$= G_{q+1,p}^{p,1}\left(x \,\middle|\, \begin{matrix} 1, \beta_1, \ldots, \beta_q \\ a_1, \ldots, a_p \end{matrix}\right)$$

The importance of the G-function derives largely from the possibility of expressing by means of the G-symbol a great many of the special functions appearing in applied mathematics, so that each of the formulas developed for the G-function becomes a master or key formula from which a very large number of relations can be deduced for Bessel, Legendre, Whittaker functions, their combinations and other related functions. The material for the following list of special G-functions has been obtained mainly from several papers by C. S. Meijer.

(3) $G_{02}^{10}(x \mid a, b) = x^{\frac{1}{2}(a+b)} J_{a-b}(2x^{\frac{1}{2}})$

(4) $G_{02}^{20}(x \mid a, b) = 2x^{\frac{1}{2}(a+b)} K_{a-b}(2x^{\frac{1}{2}})$

(5) $G_{12}^{20}\left(x \,\middle|\, \begin{matrix} \frac{1}{2} \\ b, -b \end{matrix}\right) = \pi^{-\frac{1}{2}} e^{-\frac{1}{2}x} K_b(\tfrac{1}{2}x)$

(6) $G_{12}^{20}\left(x \,\middle|\, \begin{matrix} a \\ b, c \end{matrix}\right) = x^{\frac{1}{2}(b+c-1)} e^{-\frac{1}{2}x} W_{k,m}(x)$

$k = \tfrac{1}{2}(1 + b + c) - a, \quad m = \tfrac{1}{2}b - \tfrac{1}{2}c$

(7) $G_{12}^{21}\left(x \,\middle|\, \begin{matrix} \frac{1}{2} \\ b, -b \end{matrix}\right) = \dfrac{\pi^{\frac{1}{2}}}{\cos b\pi} e^{\frac{1}{2}x} K_b(\tfrac{1}{2}x)$

(8) $G_{12}^{21}\left(x \,\middle|\, \begin{matrix} a \\ b, c \end{matrix}\right) = \Gamma(b - a + 1)\,\Gamma(c - a + 1)\, x^{\frac{1}{2}(b+c-1)} e^{\frac{1}{2}x} W_{k,m}(x)$

$k = a - \tfrac{1}{2}(b + c + 1), \quad m = \tfrac{1}{2}b - \tfrac{1}{2}c$

(9) $G_{04}^{10}(x \mid a, b, 2b - a, 2b - a + \tfrac{1}{2}) = \pi^{-\frac{1}{2}} x^b$

$\times I_{2(a-b)}(2^{3/2} x^{1/4})\, J_{2(a-b)}(2^{3/2} x^{1/4})$

(10) $G_{04}^{10}(x \mid a + \tfrac{1}{2}, a, b, 2a - b) = \pi^{-\frac{1}{2}}[\sin(a - b)\pi]^{-1}\, 2^{-5/2}$

$\times x^a [J_{2(a-b)}(2^{3/2} x^{1/4})\, I_{2(b-a)}(2^{3/2} x^{1/4}) - I_{2(a-b)}(2^{3/2} x^{1/4})$

$\times J_{2(b-a)}(2^{3/2} x^{1/4})]$

(11) $G_{04}^{20}(x \mid a, a + \tfrac{1}{2}, b, b + \tfrac{1}{2}) = x^{\frac{1}{2}(a+b)} J_{2(a-b)}(4x^{1/4})$

(12) $G_{04}^{20}(x \mid a, -a, 0, \tfrac{1}{2}) = -\pi^{\frac{1}{2}}(\sin 2a\pi)^{-1}$

$\times [J_{2a}(ze^{\pi i/4})\, J_{2a}(ze^{-\pi i/4}) - J_{-2a}(ze^{\pi i/4})\, J_{-2a}(ze^{-\pi i/4})]$

$z = 2^{3/2} x^{1/4}$

(13) $G_{04}^{20}(x \mid 0, \tfrac{1}{2}, a, -a) = \pi^{1/2} i^{-1} (\sin 2a\pi)^{-1}$

$\times [e^{2a\pi i} J_{2a}(ze^{-\pi i/4}) J_{-2a}(ze^{\pi i/4}) - e^{-2a\pi i} J_{2a}(ze^{\pi i/4})$

$\times J_{-2a}(ze^{-\pi i/4})] \qquad z = 2^{3/2} x^{1/4}$

(14) $G_{04}^{30}(x \mid 3a - \tfrac{1}{2}, a, -a - \tfrac{1}{2}, a - \tfrac{1}{2}) = 2\pi^{1/2} (\cos 2a\pi)^{-1}$

$\times x^{a-1/2} K_{4a}(2^{3/2} x^{1/4}) [J_{4a}(2^{3/2} x^{1/4}) + J_{-4a}(2^{3/2} x^{1/4})]$

(15) $G_{04}^{30}(x \mid 0, a - \tfrac{1}{2}, -a - \tfrac{1}{2}, -\tfrac{1}{2}) = 4\pi^{1/2} x^{-1/2}$

$\times K_{2a}(2^{3/2} x^{1/4}) [J_{2a}(2^{3/2} x^{1/4}) \cos a\pi - Y_{2a}(2^{3/2} x^{1/4}) \sin a\pi]$

(16) $G_{04}^{30}(x \mid -\tfrac{1}{2}, a - \tfrac{1}{2}, -a - \tfrac{1}{2}, 0) = -4\pi^{1/2} x^{-1/2}$

$\times K_{2a}(2^{3/2} x^{1/4}) [J_{2a}(2^{3/2} x^{1/4}) \sin a\pi + Y_{2a}(2^{3/2} x^{1/4}) \cos a\pi]$

(17) $G_{04}^{30}(x \mid a, b + \tfrac{1}{2}, b, 2b - a) = \pi^{1/2} 2^{1/2} x^{b} K_{2(a-b)}(2^{3/2} x^{1/4})$

$\times J_{2(a-b)}(2^{3/2} x^{1/4})$

(18) $G_{04}^{40}(x \mid a, a + \tfrac{1}{2}, b, b + \tfrac{1}{2}) = 4\pi x^{1/2(a+b)} K_{2(a-b)}(4x^{1/4})$

(19) $G_{04}^{40}(x \mid a, a + \tfrac{1}{2}, b, 2a - b) = 2^{3} \pi^{1/2} x^{a}$

$\times K_{2(b-a)}(2^{3/2} x^{1/4} e^{\pi i/4}) K_{2(b-a)}(2^{3/2} x^{1/4} e^{-\pi i/4})$

(20) $G_{13}^{11}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2} \\ a, 0, -a \end{matrix}\right) = \pi^{1/2} J_a^2(x^{1/2})$

(21) $G_{13}^{11}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2} \\ 0, a, -a \end{matrix}\right) = \pi^{1/2} J_a(x^{1/2}) J_{-a}(x^{1/2})$

(22) $G_{13}^{11}\left(x \,\middle|\, \begin{matrix} a \\ a, b, a - \tfrac{1}{2} \end{matrix}\right) = x^{1/2 a + 1/2 b - 1/4} \mathbf{H}_{a-b-1/2}(2x^{1/2})$

(23) $G_{13}^{20}\left(x \,\middle|\, \begin{matrix} a - \tfrac{1}{2} \\ a, b, a - \tfrac{1}{2} \end{matrix}\right) = x^{1/2(a+b)} Y_{b-a}(2x^{1/2})$

(24) $G_{13}^{20}\left(x \,\middle|\, \begin{matrix} a + \tfrac{1}{2} \\ b, a, 2a - b \end{matrix}\right) = -\pi^{1/2} x^{a} J_{b-a}(x^{1/2}) Y_{b-a}(x^{1/2})$

(25) $G_{13}^{20}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2} \\ a, -a, 0 \end{matrix}\right) = \pi^{1/2} 2^{-1} (\sin a\pi)^{-1} [J_{-a}^2(x^{1/2}) - J_a^2(x^{1/2})]$

(26) $G^{21}_{13}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2} \\ a,\ 0,\ -a \end{matrix}\right) = 2\pi^{1/2}\ I_a(x^{1/2})\, K_a(x^{1/2})$

(27) $G^{21}_{13}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2} \\ a,\ -a,\ 0 \end{matrix}\right) = \pi^{3/2}\ (\sin 2a\pi)^{-1}\ [I^2_{-a}(x^{1/2}) - I^2_a(x^{1/2})]$

(28) $G^{21}_{13}\left(x \,\middle|\, \begin{matrix} a+\tfrac{1}{2} \\ a+\tfrac{1}{2},\ b,\ a \end{matrix}\right) = \dfrac{\pi\, x^{\frac{1}{2}(a+b)}}{\cos(a-b)\,\pi}\ [I_{b-a}(2x^{1/2})\ \mathbf{L}_{a-b}(2x^{1/2})]$

(29) $G^{21}_{13}\left(x \,\middle|\, \begin{matrix} a+\tfrac{1}{2} \\ a,\ a+\tfrac{1}{2},\ b \end{matrix}\right) = \pi x^{\frac{1}{2}(a+b)}\ [I_{a-b}(2x^{1/2}) - \mathbf{L}_{a-b}(2x^{1/2})]$

(30) $G^{30}_{13}\left(x \,\middle|\, \begin{matrix} a+\tfrac{1}{2} \\ a+b,\ a-b,\ 0 \end{matrix}\right) = 2\pi^{-1/2}\ x^a\ K^2_b(x^{1/2})$

(31) $G^{31}_{13}\left(x \,\middle|\, \begin{matrix} a+\tfrac{1}{2} \\ a+\tfrac{1}{2},\ -a,\ a \end{matrix}\right) = \dfrac{\pi^2}{\cos 2a\pi}\ [\mathbf{H}_{2a}(2x^{1/2}) - Y_{2a}(2x^{1/2})]$

(32) $G^{31}_{13}\left(x \,\middle|\, \begin{matrix} a \\ a,\ b,\ -b \end{matrix}\right) = 2^{-2a+2}\ \Gamma(1-a-b)\ \Gamma(1-a+b)\ S_{2a-1,2b}(2x^{1/2})$

(33) $G^{31}_{13}\left(x \,\middle|\, \begin{matrix} a+\tfrac{1}{2} \\ b,\ 2a-b,\ a \end{matrix}\right) = \pi^{5/2}\ 2^{-1}\ [\cos(b-a)\pi]^{-1}$

$\times\, x^a\ H^{(1)}_{b-a}(x^{1/2})\ H^{(2)}_{b-a}(x^{1/2})$

(34) $G^{12}_{22}\left(x \,\middle|\, \begin{matrix} a-1,\ -b \\ -c_1,\ -c_2 \end{matrix}\right) = \dfrac{\Gamma(a+c_1)\ \Gamma(a+c_2)}{\Gamma(a+b)}$

$\times\, x^{a-1}\ {}_2F_1(a+c_1,\ a+c_2;\ a+b;\ -x)$

(35) $G^{12}_{24}\left(x \,\middle|\, \begin{matrix} a+\tfrac{1}{2},\ a \\ b+a,\ a-c,\ a+c,\ a-b \end{matrix}\right) = \pi^{1/2}\ x^a\ J_{b+c}(x^{1/2})\ J_{b-c}(x^{1/2})$

(36) $G^{22}_{24}\left(x \,\middle|\, \begin{matrix} a,\ a+\tfrac{1}{2} \\ b,\ c,\ 2a-c,\ 2a-b \end{matrix}\right) = 2\pi^{1/2}\ x^a\ I_{b+c-2a}(x^{1/2})\, K_{b-c}(x^{1/2})$

(37) $G^{30}_{24}\left(x \,\middle|\, \begin{matrix} 0,\ \tfrac{1}{2} \\ a,\ b,\ -b,\ -a \end{matrix}\right) = i\ 2^{-2}\ \pi^{1/2}$

$\times\, [H^{(1)}_{a-b}(x^{1/2})\ H^{(1)}_{a+b}(x^{1/2}) - H^{(2)}_{a-b}(x^{1/2})\ H^{(2)}_{a+b}(x^{1/2})]$

(38) $G^{40}_{24}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2}+a,\ \tfrac{1}{2}-a \\ 0,\ \tfrac{1}{2},\ b,\ -b \end{matrix}\right) = \pi^{1/2}\ x^{-1/2}\ W_{a,b}(2x^{1/2})\ W_{-a,b}(2x^{1/2})$

(39) $G^{40}_{24}\left(x \,\middle|\, \begin{matrix} a,\ a+\tfrac{1}{2} \\ a+b,\ a+c,\ a-c,\ a-b \end{matrix}\right) = 2\pi^{-1/2}\, x^{a}\, K_{b+c}(x^{1/2})\, K_{b-c}(x^{1/2})$

(40) $G^{41}_{24}\left(x \,\middle|\, \begin{matrix} 0,\ \tfrac{1}{2} \\ a,\ b,\ -b,\ -a \end{matrix}\right) = \dfrac{-2^{-2}\,\pi^{5/2}}{i \sin a\pi \sin b\pi}$

$\times\, [e^{-b\pi i} H^{(1)}_{a-b}(x^{1/2})\, H^{(2)}_{a+b}(x^{1/2}) - e^{b\pi i} H^{(1)}_{a+b}(x^{1/2})\, H^{(2)}_{a-b}(x^{1/2})]$

(41) $G^{41}_{24}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2},\ 0 \\ a,\ b,\ -b,\ -a \end{matrix}\right) = \dfrac{2^{-2}\,\pi^{5/2}}{\cos a\pi \cos b\pi}$

$\times\, [e^{-b\pi i} H^{(1)}_{a-b}(x^{1/2})\, H^{(2)}_{a+b}(x^{1/2}) + e^{b\pi i} H^{(1)}_{a+b}(x^{1/2})\, H^{(2)}_{a-b}(x^{1/2})]$

(42) $G^{41}_{24}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2}+a,\ \tfrac{1}{2}-a \\ 0,\ \tfrac{1}{2},\ b,\ -b \end{matrix}\right) = x^{-1/2}\,\pi^{1/2}\,\Gamma(\tfrac{1}{2}+b-a)\,\Gamma(\tfrac{1}{2}-b-a)$

$\times\, W_{a,b}(2ix^{1/2})\, W_{a,b}(-2ix^{1/2})$

(43) $G^{14}_{44}\left(x \,\middle|\, \begin{matrix} a-1,\ -c_1,\ -c_2,\ -c_3 \\ -b_1,\ -b_2,\ -b_3,\ -b_4 \end{matrix}\right) = \dfrac{\prod_{h=1}^{4} \Gamma(a+b_h)}{\prod_{h=1}^{3} \Gamma(a+c_h)}\, x^{a-1}$

$\times\, {}_4F_3(a+b_1,\ a+b_2,\ a+b_3,\ a+b_4;\ a+c_1,\ a+c_2,\ a+c_3;\ -x)$

The following combinations of special functions are among those that may be expressed in terms of the G-function.

(44) $x^{\mu} J_{\nu}(x) = 2^{\mu} G^{10}_{02}(\tfrac{1}{4}x^2 \mid \tfrac{1}{2}\nu+\tfrac{1}{2}\mu,\ \tfrac{1}{2}\mu-\tfrac{1}{2}\nu)$

(45) $x^{\mu} J_{\nu}(x) = 4^{\mu} G^{20}_{04}(4^{-4}x^4 \mid \tfrac{1}{4}\nu+\tfrac{1}{4}\mu,\ \tfrac{1}{4}\nu+\tfrac{1}{4}\mu+\tfrac{1}{2},\ \tfrac{1}{4}\mu-\tfrac{1}{4}\nu,\ \tfrac{1}{2}+\tfrac{1}{4}\mu-\tfrac{1}{4}\nu)$

(46) $x^{\mu} Y_{\nu}(x) = 2^{\mu} G^{20}_{13}\left(\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2}\mu-\tfrac{1}{2}\nu-\tfrac{1}{2} \\ \tfrac{1}{2}\mu-\tfrac{1}{2}\nu,\ \tfrac{1}{2}\mu+\tfrac{1}{2}\nu,\ \tfrac{1}{2}\mu-\tfrac{1}{2}\nu-\tfrac{1}{2} \end{matrix}\right)$

(47) $x^{\mu} K_{\nu}(x) = 2^{\mu-1} G^{20}_{02}(\tfrac{1}{4}x^2 \mid \tfrac{1}{2}\mu+\tfrac{1}{2}\nu,\ \tfrac{1}{2}\mu-\tfrac{1}{2}\nu)$

(48) $x^{\mu} K_{\nu}(x) = 4^{\mu-1}\,\pi^{-1}$

$\times\, G^{40}_{04}(4^{-4}x^4 \mid \tfrac{1}{4}\nu+\tfrac{1}{4}\mu,\ \tfrac{1}{2}+\tfrac{1}{4}\nu+\tfrac{1}{4}\mu,\ -\tfrac{1}{4}\nu+\tfrac{1}{4}\mu,\ \tfrac{1}{2}-\tfrac{1}{4}\nu+\tfrac{1}{4}\mu)$

(49) $e^{-x} K_{\nu}(x) = \pi^{1/2}\, G^{20}_{12}\left(2x \,\middle|\, \begin{matrix} \tfrac{1}{2} \\ \nu,\ -\nu \end{matrix}\right)$

(50) $e^{x} K_{\nu}(x) = \pi^{-1/2} \cos \nu\pi \, G_{12}^{21}\left(2x \,\middle|\, \begin{matrix} 1/2 \\ \nu, -\nu \end{matrix}\right)$

(51) $x^{\mu} \mathbf{H}_{\nu}(x) = 2^{\mu} G_{13}^{11}\left(\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} 1/2 + 1/2\nu + 1/2\mu \\ 1/2 + 1/2\nu + 1/2\mu,\ 1/2\mu - 1/2\nu,\ 1/2\mu + 1/2\nu \end{matrix}\right)$

(52) $\mathbf{H}_{\nu}(x) - Y_{\nu}(x) = \pi^{-2} \cos \nu\pi \, G_{13}^{31}\left(\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} 1/2 + 1/2\nu \\ 1/2 + 1/2\nu,\ -1/2\nu,\ 1/2\nu \end{matrix}\right)$

(53) $x^{\mu} [I_{\nu}(x) - \mathbf{L}_{\nu}(x)] = \pi^{-1} 2^{\mu} G_{13}^{21}\left(\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} 1/2\mu + 1/2\nu + 1/2 \\ 1/2\mu + 1/2\nu,\ 1/2\mu + 1/2\nu + 1/2,\ 1/2\mu - 1/2\nu \end{matrix}\right)$

(54) $x^{\mu} [I_{-\nu}(x) - \mathbf{L}_{\nu}(x)]$

$$= \pi^{-1} 2^{\mu} \cos \nu\pi \, G_{13}^{21}\left(\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} 1/2 + 1/2\nu + 1/2\mu \\ 1/2 + 1/2\nu + 1/2\mu,\ 1/2\mu - 1/2\nu,\ 1/2\mu + 1/2\nu \end{matrix}\right)$$

(55) $S_{\mu,\nu}(x) = 2^{\mu-1} \dfrac{1}{\Gamma(1/2 - 1/2\mu - 1/2\nu)\,\Gamma(1/2 - 1/2\mu + 1/2\nu)}$

$$\times G_{13}^{31}\left(\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} 1/2 + 1/2\mu \\ 1/2 + 1/2\mu,\ 1/2\nu,\ -1/2\nu \end{matrix}\right)$$

(56) $J_{\nu}^{2}(x) = \pi^{-1/2} G_{13}^{11}\left(x^2 \,\middle|\, \begin{matrix} 1/2 \\ \nu, 0, -\nu \end{matrix}\right)$

(57) $J_{\nu}(x) J_{-\nu}(x) = \pi^{-1/2} G_{13}^{11}\left(x^2 \,\middle|\, \begin{matrix} 1/2 \\ 0, \nu, -\nu \end{matrix}\right)$

(58) $x^{\sigma} J_{\mu}(x) J_{\nu}(x)$

$$= \pi^{-1/2} G_{24}^{12}\left[x^2 \,\middle|\, \begin{matrix} 1/2 + 1/2\sigma,\ 1/2\sigma \\ 1/2(\mu + \nu + \sigma),\ 1/2(\nu + \sigma - \mu),\ 1/2(\mu + \sigma - \nu),\ 1/2(\sigma - \mu - \nu) \end{matrix}\right]$$

(59) $x^{\mu} I_{\nu}(x) J_{\nu}(x) = \pi^{1/2} 2^{3\mu/2} G_{04}^{10}\left(\frac{x^4}{64} \,\middle|\, 1/4\mu + 1/2\nu,\ 1/4\mu - 1/2\nu,\ 1/4\mu,\ 1/4\mu + 1/2\right)$

(60) $x^{\mu} J_{\nu}(x) Y_{\nu}(x) = -\pi^{-1/2} G_{13}^{20}\left(x^2 \,\middle|\, \begin{matrix} 1/2 + 1/2\mu \\ \nu + 1/2\mu,\ 1/2\mu,\ 1/2\mu - \nu \end{matrix}\right)$

(61) $I_{\nu}(x) K_{\nu}(x) = 2^{-1} \pi^{-1/2} G_{13}^{21}\left(x^2 \,\middle|\, \begin{matrix} 1/2 \\ \nu, 0, -\nu \end{matrix}\right)$

(62) $$x^{\mu} K_{\nu}(x)\, J_{\nu}(x) = \pi^{-1/2}\, 2^{3\mu/2-1/2} \times G^{30}_{04}\left(\frac{1}{64}x^4 \,\middle|\, \tfrac{1}{4}\mu+\tfrac{1}{2}\nu,\ \tfrac{1}{4}\mu+\tfrac{1}{2},\ \tfrac{1}{4}\mu,\ \tfrac{1}{4}\mu-\tfrac{1}{2}\nu\right)$$

(63) $$x^{\sigma} I_{\nu}(x)\, K_{\mu}(x) = 2^{-1}\pi^{-1/2} \times G^{22}_{24}\left[x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2}\sigma,\ \tfrac{1}{2}\sigma+\tfrac{1}{2} \\ \tfrac{1}{2}(\nu+\mu+\sigma),\ \tfrac{1}{2}(\nu+\sigma-\mu),\ \tfrac{1}{2}(\mu+\sigma-\nu),\ \tfrac{1}{2}(\sigma-\nu-\mu)\end{matrix}\right]$$

(64) $$x^{\mu} H^{(1)}_{\nu}(x)\, H^{(2)}_{\nu}(x) = \pi^{-5/2}\, 2\cos\nu\pi \times G^{31}_{13}\left(x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2}+\tfrac{1}{2}\mu \\ \tfrac{1}{2}\mu+\nu,\ \tfrac{1}{2}\mu-\nu,\ \tfrac{1}{2}\mu\end{matrix}\right)$$

(65) $$x^{\mu} K^2_{\nu}(x) = 2^{-1}\pi^{1/2}\, G^{30}_{13}\left(x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2}+\tfrac{1}{2}\mu \\ \nu+\tfrac{1}{2}\mu,\ -\nu+\tfrac{1}{2}\mu,\ 0\end{matrix}\right)$$

(66) $$x^{\sigma} K_{\nu}(x)\, K_{\mu}(x) = 2^{-1}\pi^{1/2} \times G^{40}_{24}\left[x^2 \,\middle|\, \begin{matrix} \tfrac{1}{2}\sigma,\ \tfrac{1}{2}\sigma+\tfrac{1}{2} \\ \tfrac{1}{2}(\nu+\mu+\sigma),\ \tfrac{1}{2}(\nu+\sigma-\mu),\ \tfrac{1}{2}(\mu+\sigma-\nu),\ \tfrac{1}{2}(\sigma-\nu-\mu)\end{matrix}\right]$$

(67) $$x^{2\mu} K_{2\nu}(xe^{\pi i/4})\, K_{2\nu}(xe^{-\pi i/4}) = 2^{3\mu-3}\pi^{-1/2} \times G^{40}_{04}\left(\frac{1}{64}x^4 \,\middle|\, \tfrac{1}{2}\mu,\ \tfrac{1}{2}\mu+\tfrac{1}{2},\ \tfrac{1}{2}\mu+\nu,\ \tfrac{1}{2}\mu-\nu\right)$$

(68) $$x^{l} e^{-\frac{1}{2}x}\, W_{k,m}(x) = G^{20}_{12}\left(x \,\middle|\, \begin{matrix} l-k+1 \\ m+l+\tfrac{1}{2},\ l-m+\tfrac{1}{2}\end{matrix}\right)$$

(69) $$x^{l} e^{\frac{1}{2}x}\, W_{k,m}(x) = \frac{1}{\Gamma(\tfrac{1}{2}+m-k)\,\Gamma(\tfrac{1}{2}-m-k)} \times G^{21}_{12}\left(x \,\middle|\, \begin{matrix} k+l+1 \\ l-m+\tfrac{1}{2},\ m+l+\tfrac{1}{2}\end{matrix}\right)$$

(70) $$e^{-\frac{1}{2}x}\, W_{k,m}(x) = \pi^{-1/2} x^{1/2}\, 2^{k-1/2} \times G^{40}_{24}\left(2^{-2}x^2 \,\middle|\, \begin{matrix} \tfrac{1}{4}-\tfrac{1}{2}k,\ \tfrac{3}{4}-\tfrac{1}{2}k \\ \tfrac{1}{2}+\tfrac{1}{2}m,\ \tfrac{1}{2}-\tfrac{1}{2}m,\ \tfrac{1}{2}m,\ -\tfrac{1}{2}m\end{matrix}\right)$$

(71) $$e^{x}\, W_{k,m}(2x) = \frac{x^{1/2}\, 2^{-(k+1)}\, \pi^{-3/2}}{\Gamma(\tfrac{1}{2}+m-k)\,\Gamma(\tfrac{1}{2}-m-k)} \times G^{42}_{24}\left(x^2 \,\middle|\, \begin{matrix} \tfrac{1}{4}+\tfrac{1}{2}k,\ \tfrac{3}{4}+\tfrac{1}{2}k \\ \tfrac{1}{2}m,\ \tfrac{1}{2}+\tfrac{1}{2}m,\ -\tfrac{1}{2}m,\ \tfrac{1}{2}-\tfrac{1}{2}m\end{matrix}\right)$$

$$(72)\quad x^{l} W_{k,m}(2ix)\, W_{k,m}(-2ix) = \frac{x\pi^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}+m-k)\,\Gamma(\frac{1}{2}-m-k)}$$

$$\times\, G^{41}_{24}\left(x^2 \,\middle|\, \begin{matrix} \frac{1}{2}+\frac{1}{2}l+k,\ \frac{1}{2}+\frac{1}{2}l-k \\ \frac{1}{2}l,\ \frac{1}{2}+\frac{1}{2}l,\ \frac{1}{2}l+m,\ \frac{1}{2}l-m \end{matrix}\right)$$

$$(73)\quad x^{l} W_{k,m}(x)\, W_{-k,m}(x) = \pi^{-\frac{1}{2}}$$

$$\times\, G^{40}_{24}\left(\tfrac{1}{4}x^2 \,\middle|\, \begin{matrix} \frac{1}{2}l+k+1,\ \frac{1}{2}l-k+1 \\ \frac{1}{2}l+\frac{1}{2},\ \frac{1}{2}l+1,\ \frac{1}{2}l+m+\frac{1}{2},\ \frac{1}{2}l-m+\frac{1}{2} \end{matrix}\right)$$

$$(74)\quad {}_2F_1(a;\, b,\, c;\, -x) = \frac{\Gamma(c)\, x}{\Gamma(a)\,\Gamma(b)}\; G^{12}_{22}\left(x \,\middle|\, \begin{matrix} -1,\ -c \\ -a,\ -b \end{matrix}\right)$$

$$(75)\quad {}_4F_3(a,\, b,\, c,\, d;\, e,\, f,\, l;\, -z) = \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(l)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d)}\; z$$

$$\times\, G^{14}_{44}\left(z \,\middle|\, \begin{matrix} -1,\ -e,\ -f,\ -l \\ -a,\ -b,\ -c,\ -d \end{matrix}\right)$$

Other similar formulas are contained in Meijer's papers, in particular formulas for Legendre functions and various generalized hypergeometric functions.

HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

5.7. Hypergeometric series in two variables

The great success of the theory of hypergeometric series in one variable has stimulated the developement of a corresponding theory in two and more variables. Appell has defined, in 1880, four series, F_1 to F_4 [cf. equations (6) to (9) *infra*] which are all analogous to Gauss' $F(\alpha, \beta; \gamma; x)$. Picard has pointed out that one of these series is intimately related to a function studied by Pochhammer in 1870, and Picard and Goursat also constructed a theory of Appell's series which is analogous to Riemann's theory of Gauss' hypergeometric series. P. Humbert has studied confluent hypergeometric series in two variables. An exposition of the results of the French school together with references to the original literature are to be found in the monograph by Appell and Kampé de Fériet (1926), which is the standard work on the subject. This work also contains an extensive bibliography of all relevant papers up to 1926; the list of references given in the present chapter is largely supplementary to Appell and Kampé de Fériet's bibliography.

Horn (1889) gave the following general definition: the double power series

$$(1) \quad \sum_{m,n=0}^{\infty} A_{mn} x^m y^n$$

is a hypergeometric series if the two quotients

$$(2) \quad \frac{A_{m+1,n}}{A_{mn}} = f(m, n) \quad \text{and} \quad \frac{A_{m,n+1}}{A_{mn}} = g(m, n)$$

are rational functions of m and n. Horn has investigated the convergence of hypergeometric series of two variables and established the systems of partial differential equations which they satisfy.

For the special hypergeometric series investigated by the authors already named, by Mellin, and by several others, A_{mn} is a gamma product, that is to say it is of the form

$$(3) \quad \gamma_{mn} = \prod_i \Gamma(a_i + u_i m + v_i n)/\Gamma(a_i)$$

where the a_i are arbitrary (real or complex) constants, and the u_i and v_i are arbitrary integers which may be positive, negative, or zero. The question, then, arises, whether this type of series is the most general one compatible with Horn's definition. Clearly, f and g must satisfy

$$(4) \quad f(m, n)\, g(m+1, n) = f(m, n+1)\, g(m, n)$$

for all $m, n = 0, 1, 2, \ldots$, and hence identically in m and n, since each of the two sides is $A_{m+1,n+1}/A_{mn}$; and it is easy to see that any pair of rational functions of m, n satisfying (4) generates a hypergeometric series.

Birkeland (1927) stated that every rational solution of the functional equation (4) can be decomposed into linear factors, and this would seem to lead to gamma products for A_{mn}. Ore noted (1929) that the Birkeland theorem is not entirely general and gave (1930) a thorough analysis of the rational solutions of (4). From Ore's result it can be deduced that the most general form of A_{mn} is of the form

$$A_{mn} = R(m, n)\, \gamma_{mn}\, a^m b^n$$

where R is a fixed rational function of m and n, a and b are constants, and γ_{mn} is a gamma product. This is equivalent to saying that the most general hypergeometric series of two variables results from the operation of a rational differential operator,

$$R\left(x \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}\right)$$

on a hypergeometric series of the type

$$\Sigma\ \gamma_{mn}\,(ax)^m\,(by)^n.$$

It would seem, therefore, sufficient to investigate series of the Horn-Birkeland type.

5.7.1. Horn's list

Horn puts

$$(5)\quad f(m,n)=\frac{F(m,n)}{F'(m,n)},\qquad g(m,n)=\frac{G(m,n)}{G'(m,n)}$$

where F, F', G, G' are polynomials in m, n of respective degrees p, p', q, q'. F' is assumed to have a factor $m+1$, and G' a factor $n+1$; F and F' have no common factor except, possibly, $m+1$; and G and G' no common factor except possibly $n+1$. The highest of the four numbers p, p', q, q', is the *order* of the hypergeometric series. Horn investigated in particular hypergeometric series of *order two* and found that, apart from certain series which are either expressible in terms of one variable or are products of two hypergeometric series, each in one variable, there are essentially 34 distinct convergent series of order two (Horn 1931, corrections in Borngässer 1933).

There are 14 complete series for which $p=p'=q=q'=2$:

$$(6)\quad F_1(\alpha,\beta,\beta',\gamma,x,y)=\sum\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}\,m!\,n!}\,x^m y^n,$$

$$(7)\quad F_2(\alpha,\beta,\beta',\gamma,\gamma',x,y)=\sum\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n\,m!\,n!}\,x^m y^n,$$

$$(8)\quad F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y)=\sum\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}\,m!\,n!}\,x^m y^n,$$

$$(9)\quad F_4(\alpha,\beta,\gamma,\gamma',x,y)=\sum\frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n\,m!\,n!}\,x^m y^n,$$

$$(10)\quad G_1(\alpha,\beta,\beta',x,y)=\sum\frac{(\alpha)_{m+n}(\beta)_{n-m}(\beta')_{m-n}}{m!\,n!}\,x^m y^n,$$

$$(11)\quad G_2(\alpha,\alpha',\beta,\beta',x,y)=\sum\frac{(\alpha)_m(\alpha')_n(\beta)_{n-m}(\beta')_{m-n}}{m!\,n!}\,x^m y^n,$$

(12) $G_3(\alpha, \alpha', x, y) = \sum \frac{(\alpha)_{2n-m} (\alpha')_{2m-n}}{m!\, n!} x^m y^n,$

(13) $H_1(\alpha, \beta, \gamma, \delta, x, y) = \sum \frac{(\alpha)_{m-n} (\beta)_{m+n} (\gamma)_n}{(\delta)_m\, m!\, n!} x^m y^n,$

(14) $H_2(\alpha, \beta, \gamma, \delta, \epsilon, x, y) = \sum \frac{(\alpha)_{m-n} (\beta)_m (\gamma)_n (\delta)_n}{(\epsilon)_m\, m!\, n!} x^m y^n$

(15) $H_3(\alpha, \beta, \gamma, x, y) = \sum \frac{(\alpha)_{2m+n} (\beta)_n}{(\gamma)_{m+n}\, m!\, n!} x^m y^n,$

(16) $H_4(\alpha, \beta, \gamma, \delta, x, y) = \sum \frac{(\alpha)_{2m+n} (\beta)_n}{(\gamma)_m (\delta)_n\, m!\, n!} x^m y^n,$

(17) $H_5(\alpha, \beta, \gamma, x, y) = \sum \frac{(\alpha)_{2m+n} (\beta)_{n-m}}{(\gamma)_n\, m!\, n!} x^m y^n,$

(18) $H_6(\alpha, \beta, \gamma, x, y) = \sum \frac{(\alpha)_{2m-n} (\beta)_{n-m} (\gamma)_n}{m!\, n!} x^m y^n,$

(19) $H_7(\alpha, \beta, \gamma, \delta, x, y) = \sum \frac{(\alpha)_{2m-n} (\beta)_n (\gamma)_n}{(\delta)_m\, m!\, n!} x^m y^n,$

and there are 20 *confluent* series which are limiting forms of the complete ones and for which $p \le p' = 2$, $q \le q' = 2$ and p, q not both $= 2$:

(20) $\Phi_1(\alpha, \beta, \gamma, x, y) = \sum \frac{(\alpha)_{m+n} (\beta)_m}{(\gamma)_{m+n}\, m!\, n!} x^m y^n \qquad |x| < 1,$

(21) $\Phi_2(\beta, \beta', \gamma, x, y) = \sum \frac{(\beta)_m (\beta')_n}{(\gamma)_{m+n}\, m!\, n!} x^m y^n,$

(22) $\Phi_3(\beta, \gamma, x, y) = \sum \frac{(\beta)_m}{(\gamma)_{m+n}\, m!\, n!} x^m y^n,$

(23) $\Psi_1(\alpha, \beta, \gamma, \gamma', x, y) = \sum \frac{(\alpha)_{m+n} (\beta)_m}{(\gamma)_m (\gamma')_n\, m!\, n!} x^m y^n \qquad |x| < 1,$

(24) $\Psi_2(\alpha, \gamma, \gamma', x, y) = \sum \frac{(\alpha)_{m+n}}{(\gamma)_m (\gamma')_n\, m!\, n!} x^m y^n,$

(25) $\Xi_1(\alpha, \alpha', \beta, \gamma, x, y) = \sum \frac{(\alpha)_m (\alpha')_n (\beta)_m}{(\gamma)_{m+n}\, m!\, n!} x^m y^n,$ $|x| < 1,$

(26) $\Xi_2(\alpha, \beta, \gamma, x, y) = \sum \frac{(\alpha)_m (\beta)_n}{(\gamma)_{m+n}\, m!\, n!} x^m y^n,$ $|x| < 1,$

(27) $\Gamma_1(\alpha, \beta, \beta', x, y) = \sum \frac{(\alpha)_m (\beta)_{n-m} (\beta')_{m-n}}{m!\, n!} x^m y^n$ $|x| < 1,$

(28) $\Gamma_2(\beta, \beta', x, y) = \sum \frac{(\beta)_{n-m} (\beta')_{m-n}}{m!\, n!} x^m y^n,$

(29) $H_1(\alpha, \beta, \delta, x, y) = \sum \frac{(\alpha)_{m-n} (\beta)_{m+n}}{(\delta)_m\, m!\, n!} x^m y^n$ $|x| < 1,$

(30) $H_2(\alpha, \beta, \gamma, \delta, x, y) = \sum \frac{(\alpha)_{m-n} (\beta)_m (\gamma)_n}{(\delta)_m\, m!\, n!} x^m y^n$ $|x| < 1,$

(31) $H_3(\alpha, \beta, \delta, x, y) = \sum \frac{(\alpha)_{m-n} (\beta)_m}{(\delta)_m\, m!\, n!} x^m y^n$ $|x| < 1,$

(32) $H_4(\alpha, \gamma, \delta, x, y) = \sum \frac{(\alpha)_{m-n} (\gamma)_n}{(\delta)_m\, m!\, n!} x^m y^n,$

(33) $H_5(\alpha, \delta, x, y) = \sum \frac{(\alpha)_{m-n}}{(\delta)_m\, m!\, n!} x^m y^n,$

(34) $H_6(\alpha, \gamma, x, y) = \sum \frac{(\alpha)_{2m+n}}{(\gamma)_{m+n}\, m!\, n!} x^m y^n$ $|x| < 1/4,$

(35) $H_7(\alpha, \gamma, \delta, x, y) = \sum \frac{(\alpha)_{2m+n}}{(\gamma)_m (\delta)_n\, m!\, n!} x^m y^n$ $|x| < 1/4,$

(36) $H_8(\alpha, \beta, x, y) = \sum \frac{(\alpha)_{2m-n} (\beta)_{n-m}}{m!\, n!} x^m y^n$ $|x| < 1/4,$

(37) $H_9(\alpha, \beta, \delta, x, y) = \sum \frac{(\alpha)_{2m-n} (\beta)_n}{(\delta)_m\, m!\, n!} x^m y^n$ $|x| < 1/4,$

(38) $H_{10}(\alpha, \delta, x, y) = \sum \frac{(\alpha)_{2m-n}}{(\delta)_m\, m!\, n!} x^m y^n$ $|x| < 1/4,$

(39) $H_{11}(\alpha, \beta, \gamma, \delta; x, y) = \sum \frac{(\alpha)_{m-n} (\beta)_n (\gamma)_n}{(\delta)_m \, m! \, n!} x^m y^n \qquad |y| < 1.$

In all these double series m and n run from 0 to ∞.

5.7.2. Convergence of the series

The positive quantities r, s are called the associated radii of convergence of the double power series $\Sigma A_{mn} x^m y^n$ if the power series is absolutely convergent for $|x| < r$, $|y| < s$, and divergent when $|x| > r$, $|y| > s$. We put max $r = R$, max $s = S$. In the *absolute plane* (r, s), the points representing associated radii of convergence lie on a curve C which is entirely in the rectangle $0 < r < R$, $0 < s < S$, and divides this rectangle into two parts of which the one containing $r = s = 0$ is the two-dimensional representation of the domain of convergence of the double power series.

Investigating the convergence of (1), Horn defines

(40) $\Phi(\mu, \nu) = \lim f(\mu t, \nu t), \quad \Psi(\mu, \nu) = \lim g(\mu t, \nu t) \qquad t \to \infty.$

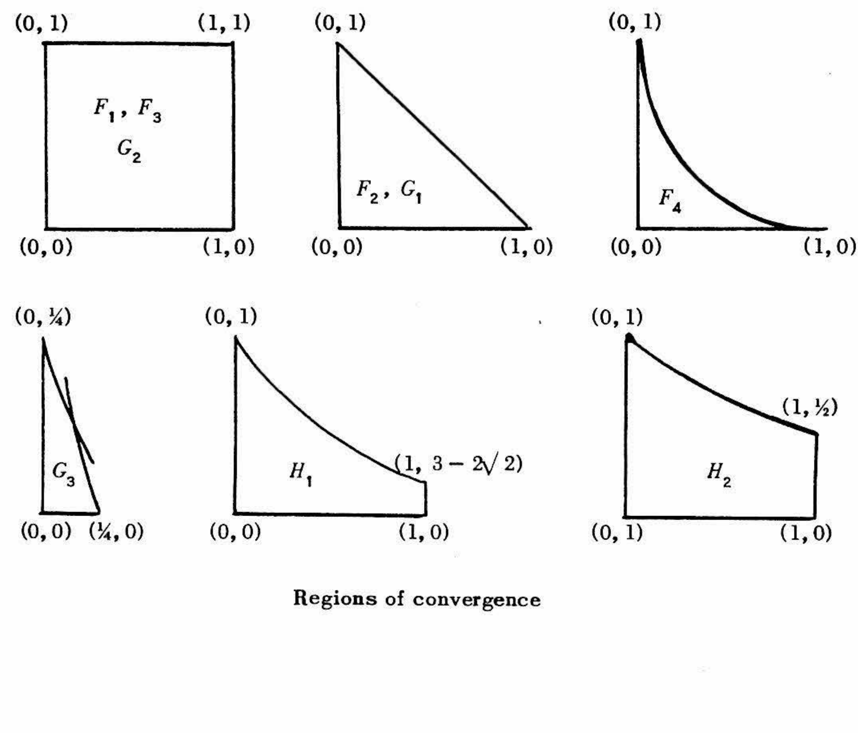
and shows that $R = |\Phi(1, 0)|^{-1}$, $S = |\Psi(0, 1)|^{-1}$, and that C has the parametric representation $r = |\Phi(\mu, \nu)|^{-1}$, $s = |\Psi(\mu, \nu)|^{-1}$, where μ, $\nu > 0$.

The application of this to the complete series of the second order gives the following table:

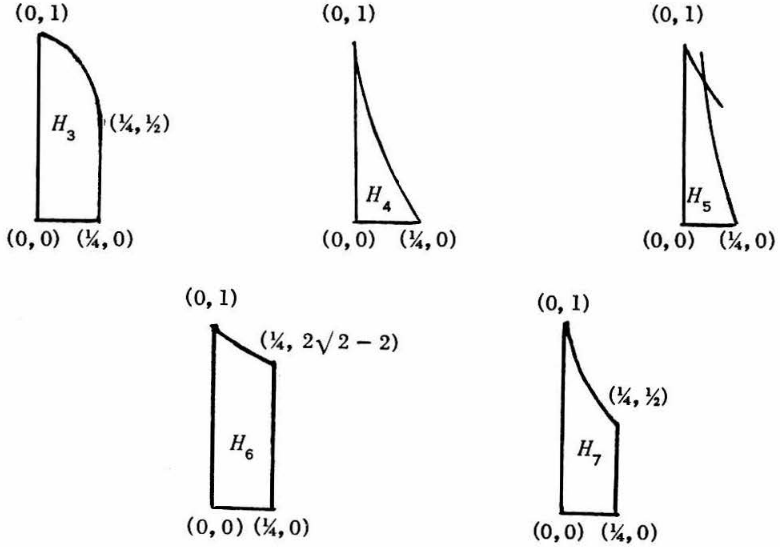
Series		$\Phi(\mu, \nu)$	$\Psi(\mu, \nu)$	Cartesian equation of C
(41)	F_1	1	1	
(42)	F_2	$\frac{\mu+\nu}{\mu}$	$\frac{\mu+\nu}{\nu}$	$r + s = 1$
(43)	F_3	$\frac{\mu}{\mu+\nu}$	$\frac{\nu}{\mu+\nu}$	
(44)	F_4	$\left(\frac{\mu+\nu}{\mu}\right)^2$	$\left(\frac{\mu+\nu}{\nu}\right)^2$	$r^{1/2} + s^{1/2} = 1$
(45)	G_1	$\frac{-(\mu+\nu)}{\mu}$	$\frac{-(\mu+\nu)}{\nu}$	$r + s = 1$
(46)	G_2	-1	-1	
(47)	G_3	$\frac{(2\mu-\nu)^2}{\mu(2\nu-\mu)}$	$\frac{(2\nu-\mu)^2}{\nu(2\mu-\nu)}$	$27r^2 s^2 + 18rs \pm 4(r-s) - 1 = 0$
(48)	H_1	$\frac{\mu^2-\nu^2}{\mu^2}$	$\frac{\mu+\nu}{\mu-\nu}$	$4rs = (s-1)^2$

Series		$\Phi(\mu,\nu)$	$\Psi(\mu,\nu)$	Cartesian equation of C
(49)	H_2	$\dfrac{\mu-\nu}{\mu}$	$\dfrac{\nu}{\mu-\nu}$	$-r+s^{-1}=1$
(50)	H_3	$\dfrac{(2\mu+\nu)^2}{\mu(\mu+\nu)}$	$\dfrac{2\mu+\nu}{\mu+\nu}$	$r+(s-\frac{1}{2})^2=\frac{1}{4}$
(51)	H_4	$\dfrac{(2\mu+\nu)^2}{\mu^2}$	$\dfrac{2\mu+\nu}{\nu}$	$4r=(s-1)^2$
(52)	H_5	$\dfrac{(2\mu+\nu)^2}{\mu(\nu-\mu)}$	$\dfrac{(2\mu+\nu)(\nu-\mu)}{\nu^2}$	$1+16r^2-36rs\pm(8r-s+27rs^2)=0$
(53)	H_6	$\dfrac{(2\mu-\nu)^2}{\mu(\nu-\mu)}$	$\dfrac{\nu-\mu}{2\mu-\nu}$	$s^2r+s-1=0$
(54)	H_7	$\dfrac{(2\mu-\nu)^2}{\mu^2}$	$\dfrac{\nu}{2\mu-\nu}$	$4r=(s^{-1}-1)^2$

The domains of convergence of the various series can be represented as follows:



Regions of convergence



Regions of convergence

In the case of confluent series either Φ or Ψ vanishes identically, the region of convergence simplifies considerably, and any inequalities which may be necessary to secure convergence are recorded in (20) to (39).

5.8. Integral representations

Basically, hypergeometric functions of two variables, as the corresponding functions of one variable, can be represented either by the Euler-Laplace type or by the Mellin-Barnes type of definite integrals. The latter type invariably, and the former type with an elementary integrand mostly, leads to double integrals. Since double integrals are somewhat untractable and not very well suited for the integration of differential equations, it is natural to seek for single integrals representing the functions. Such representations can be found in every case, but the integrand in most cases contains a hypergeometric function of one variable, or even a product of such functions.

The large number of functions in Horn's list makes it impossible to give a complete list of integral representations here; integral representations will be given for Appell's functions only, but it should be noted that all of Horn's functions have similar representations and that many of these have been given in the papers quoted in the list of references. Integral representations are useful in connection with the analytic continuation of hypergeometric series in two variables, their transformation theory, and also for the integrationof hypergeometric systems of partial differential equations.

5.8.1. Double integrals of Euler's type

The integral representations

$$(1)\quad F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta-\beta')}$$

$$\times \iint_{\left(\substack{u\geq 0,\ v\geq 0\\ u+v\leq 1}\right)} u^{\beta-1}\, v^{\beta'-1}\,(1-u-v)^{\gamma-\beta-\beta'-1}\,(1-ux-vy)^{-\alpha}\,du\,dv$$

$$\operatorname{Re}\beta > 0,\quad \operatorname{Re}\beta' > 0,\quad \operatorname{Re}(\gamma-\beta-\beta') > 0,$$

$$(2)\quad F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = \frac{\Gamma(\gamma)\,\Gamma(\gamma')}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta)\,\Gamma(\gamma'-\beta')}$$

$$\times \int_0^1\int_0^1 u^{\beta-1}\, v^{\beta'-1}\,(1-u)^{\gamma-\beta-1}\,(1-v)^{\gamma'-\beta'-1}\,(1-ux-vy)^{-\alpha}\,du\,dv$$

$$\operatorname{Re}\beta > 0,\quad \operatorname{Re}\beta' > 0,\quad \operatorname{Re}(\gamma-\beta) > 0,\quad \operatorname{Re}(\gamma'-\beta') > 0,$$

and

$$(3)\quad F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\beta')\,\Gamma(\gamma-\beta-\beta')}$$

$$\times \iint_{\left(\substack{u\geq 0,\ v\geq 0\\ u+v\leq 1}\right)} u^{\beta-1}\, v^{\beta'-1}\,(1-u-v)^{-\gamma-\beta-\beta'-1}$$

$$\times (1-ux)^{-\alpha}(1-vy)^{-\alpha'}\,du\,dv$$

$$\operatorname{Re}\beta > 0,\quad \operatorname{Re}\beta' > 0,\quad \operatorname{Re}(\gamma-\beta-\beta') > 0$$

are easily obtained from the series by using either Euler's integral of the first kind for the beta-function, or the corresponding double integral (Appell and Kampe'de Fériet 1926, Chapter II).

The function F_4 is much more difficult to handle and does not seem to possess a very simple integral representation. Of the various double integrals proposed, perhaps the simplest is Burchnall and Chaundy's [1940, equation (68)]

$$(4)\quad F_4(\alpha, \beta, \gamma, \gamma'; x(1-y), y(1-x)] = \frac{\Gamma(\gamma)\,\Gamma(\gamma')}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma'-\beta)}$$

$$\times \int_0^1\int_0^1 u^{\alpha-1}\, v^{\beta-1}\,(1-u)^{\gamma-\alpha-1}\,(1-v)^{\gamma'-\beta-1}$$

$$\times (1-ux)^{\alpha-\gamma-\gamma'+1} (1-vy)^{\beta-\gamma-\gamma'+1} (1-ux-vy)^{\gamma+\gamma'-\alpha-\beta-1} du\,dv$$

$$\mathrm{Re}\,(\alpha) > 0, \quad \mathrm{Re}\,(\beta) > 0, \quad \mathrm{Re}\,(\gamma-\alpha) > 0, \quad \mathrm{Re}\,(\gamma'-\beta) > 0.$$

In all these integral representations it is assumed that $|x|$ and $|y|$ are small enough to make both series and integrals converge.

5.8.2. Single integrals of Euler's type

Picard has pointed out that F_1 can be represented by a single integral in the form

(5) $F_1(\alpha, \beta, \beta', \gamma; x, y)$

$$\times \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1} (1-u)^{\gamma-\alpha-1} (1-ux)^{-\beta} (1-uy)^{-\beta'}\, du$$

$$\mathrm{Re}\,\alpha > 0, \quad \mathrm{Re}\,(\gamma-\alpha) > 0.$$

This representation has the great advantage that it can easily be converted into a contour integral valid for negative Re α and Re $(\gamma-\alpha)$ and that it is the best tool for the complete integration of the system of partial differential equations connected with F_1. Equation (5) sets up a *factorization*, into a function of x and a function of y, of F_1 by means of Euler's transformation, and the great usefulness of (5) is due to this property.

There are corresponding relations for F_2 and F_3 (Erdélyi, 1948)

$$(6) \quad F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = \frac{\Gamma(\rho)\,\Gamma(\rho')\,\Gamma(1-\alpha)}{(2\pi i)^2}$$

$$\times \int (-t)^{\rho} (t-1)^{-\rho'} F(\rho, \beta; \gamma; x/t)\, F[\rho', \beta'; \gamma'; y/(1-t)]\, dt$$

where $\rho + \rho' = \alpha + 1$, and the contour of integration is a Pochhammer double loop $(1+, 0+, 1-, 0-)$ such that $|t| > |x|$ and $|1-t| > |y|$ along it; and

$$(7) \quad F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = (2\pi i)^{-2}\,\Gamma(1-\rho)\,\Gamma(1-\rho')\,\Gamma(\gamma)$$

$$\times \int^{(1+,0+,1-,0-)} (-t)^{\rho-1} (t-1)^{\rho'-1}$$

$$\times F(\alpha, \beta; \rho; tx)\, F[\alpha', \beta'; \rho'; (1-t)\,y]\, dt$$

where $\rho + \rho' = \gamma$. In either case a special choice of ρ reduces one of the hypergeometric functions in the integrand to an elementary function, but with general values of the parameters it is impossible to reduce the whole integrand to elementary functions.

The function F_4 proves again more difficult to handle than its fellows: no single integral is known which factorizes F_4. On the other hand, the integral representation [Erdélyi 1941, equation (3)]

$$(8)\quad F_4(\alpha, \beta, \gamma, \gamma'; x, y) = (2\pi i)^{-2}\, \Gamma(\gamma)\, \Gamma(\gamma')\, \Gamma(2-\gamma-\gamma') \times \int^{(1+,0+,1-,0-)} (-t)^{-\gamma} (t-1)^{-\gamma'} \times F[\alpha, \beta; \gamma+\gamma'+1; x/t + y/(1-t)]\, dt,$$

in which $|x/t + y/(1-t)| < 1$ along the contour, is reasonably simple and useful for the integration of the system of partial differential equations associated with F_4.

5.8.3. Mellin-Barnes type double integrals

Following Appell and Kampé de Fériet, the four integral representations can be summed up as

$$(9)\quad \Phi(x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\, \Gamma(\beta)\, (2\pi i)^2} \times \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \Psi(s, t)\, \Gamma(-s)\, \Gamma(-t)\, (-x)^s (-y)^t\, ds\, dt$$

where the contours of integration are indented in the usual manner (cf. section 2.1.3). Φ and Ψ are given in the four cases as

	$\Phi(x, y)$	$\Psi(s, t)$
(10)	$F_1(\alpha, \beta, \beta', \gamma; x, y)$	$\dfrac{\Gamma(\alpha+s+t)\, \Gamma(\beta+s)\, \Gamma(\beta'+t)}{\Gamma(\beta')\, \Gamma(\gamma+s+t)}$
(11)	$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y)$	$\dfrac{\Gamma(\alpha+s+t)\, \Gamma(\beta+s)\, \Gamma(\beta'+t)\, \Gamma(\gamma')}{\Gamma(\beta')\, \Gamma(\gamma+s)\, \Gamma(\gamma'+t)}$
(12)	$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y)$	$\dfrac{\Gamma(\alpha+s)\, \Gamma(\alpha'+t)\, \Gamma(\beta+s)\, \Gamma(\beta'+t)}{\Gamma(\alpha')\, \Gamma(\beta')\, \Gamma(\gamma+s+t)}$
(13)	$F_4(\alpha, \beta, \gamma, \gamma'; x, y)$	$\dfrac{\Gamma(\alpha+s+t)\, \Gamma(\beta+s+t)\, \Gamma(\gamma')}{\Gamma(\gamma+s)\, \Gamma(\gamma'+t)}$

Integrals of this type were used in Mellin's investigation of the hypergeometric functions.

5.9. Systems of partial differential equations

The series $\Sigma A_{mn} x^m y^n$, where $A_{m+1,n}/A_{mn} = F(m, n)/F'(m, n)$,

$A_{m,n+1}/A_{mn} = G(m, n)/G'(m, n)$ and F, F', G, G', are polynomials as in 5.7(5), satisfies a system of linear partial differential equations which can be written in terms of the differential operators

(1) $\delta \equiv x \dfrac{\partial}{\partial x}$ and $\delta' \equiv y \dfrac{\partial}{\partial y}$

as

(2) $[F'(\delta, \delta')\, x^{-1} - F(\delta, \delta')]\, z = 0,$

$[G'(\delta, \delta')\, y^{-1} - G(\delta, \delta')]\, z = 0.$

In what follows we shall restrict ourselves to hypergeometric functions of the second order, in which case we find two partial differential equations of the second order.

The two equations are certainly compatible (since the hypergeometric series satisfies both of them), and from the general theory of such systems (cf. for instance, Appell and Kampé de Fériet 1926, Chapter III) it follows that they have at most four, and possibly less, linearly independent solutions in common. A closer investigation shows that the systems of partial differential equations associated with the eight series

(3) $F_1, G_1, G_2, \Phi_1, \Phi_2, \Phi_3, \Gamma_1$ and Γ_3

of Horn's list have only three linearly independent solutions, while all the other 26 systems have four independent solutions each. However, in the case of the systems associated with

(4) $G_3, H_3, H_6, \mathrm{H}_6, \mathrm{H}_8$

one of the solutions is a comparatively trivial elementary function of the form $x^\rho y^\sigma$, with the following values of ρ and σ:

	Series	ρ	σ
(5)	G_3	$-\frac{1}{3}(\alpha + 2\alpha')$	$-\frac{1}{3}(2\alpha + \alpha')$
(6)	H_3, H_6	$\gamma - \alpha - 1$	$\alpha - 2\gamma + 2$
(7)	H_6, H_8	$-\alpha - \beta$	$-\alpha - 2\beta.$

In the following list of partial differential equations z is the unknown function of x and y,

(8) $p = \dfrac{\partial z}{\partial x}, \quad q = \dfrac{\partial z}{\partial y}, \quad r = \dfrac{\partial^2 z}{\partial x^2}, \quad s = \dfrac{\partial^2 z}{\partial x\, \partial y}, \quad t = \dfrac{\partial^2 z}{\partial y^2}.$

(9) $$\left.\begin{aligned} &x(1-x)\, r + y(1-x)\, s + [\gamma - (\alpha+\beta+1)\, x]\, p - \beta y q - \alpha\beta z = 0 \\ &y(1-y)\, t + x(1-y)\, s + [\gamma - (\alpha+\beta'+1)\, y]\, q - \beta' x p - \alpha\beta' z = 0 \end{aligned}\right\} F_1,$$

$$(10)\quad \left.\begin{aligned} &x(1-x)\,r - xys + [\gamma - (\alpha+\beta+1)\,x]\,p - \beta y q - \alpha\beta z = 0 \\ &y(1-y)\,t - xys + [\gamma' - (\alpha+\beta'+1)\,y]\,q - \beta' xp - \alpha\beta' z = 0 \end{aligned}\right\} F_2,$$

$$(11)\quad \left.\begin{aligned} &x(1-x)\,r + ys + [\gamma - (\alpha+\beta+1)\,x]\,p - \alpha\beta z = 0 \\ &y(1-y)\,t + xs + [\gamma - (\alpha'+\beta'+1)\,y]\,q - \alpha'\beta' z = 0 \end{aligned}\right\} F_3,$$

$$(12)\quad \left.\begin{aligned} &x(1-x)\,r - y^2\,t - 2xys + [\gamma - (\alpha+\beta+1)\,x]\,p \\ &\qquad - (\alpha+\beta+1)\,yq - \alpha\beta z = 0 \\ &y(1-y)\,t - x^2\,r - 2xys + [\gamma' - (\alpha+\beta+1)\,y]\,q \\ &\qquad - (\alpha+\beta+1)\,xp - \alpha\beta z = 0 \end{aligned}\right\} F_4,$$

$$(13)\quad \left.\begin{aligned} &x(1+x)\,r - ys - y^2\,t + [1-\beta+(\alpha+\beta'+1)\,x]\,p \\ &\qquad + (\beta'-\alpha-1)\,yq + \alpha\beta' z = 0 \\ &y(y+1)\,t - xs - x^2\,r + [1-\beta' + (\alpha+\beta+1)\,y]\,q \\ &\qquad + (\beta-\alpha-1)\,xp + \alpha\beta z = 0 \end{aligned}\right\} G_1,$$

$$(14)\quad \left.\begin{aligned} &x(1+x)\,r - y(1+x)\,s + [1-\beta+(\alpha+\beta'+1)\,x]\,p \\ &\qquad - \alpha yq + \alpha\beta' z = 0 \\ &y(1+y)\,t - x(1+y)\,s + [1-\beta'+(\alpha'+\beta+1)\,y]\,q \\ &\qquad - \alpha' xp + \alpha'\beta z = 0 \end{aligned}\right\} G_2,$$

$$(15)\quad \left.\begin{aligned} &x(1+4x)\,r - (4x+2)\,ys + y^2\,t + [1-\alpha+(4\alpha'+6)\,x]\,p \\ &\qquad - 2\alpha' yq + \alpha'(\alpha'+1)\,z = 0 \\ &y(1+4y)\,t - x(4y+2)\,s + x^2\,r + [1-\alpha'+(4\alpha+6)\,y]\,q \\ &\qquad - 2\alpha xp + \alpha(\alpha+1)\,z = 0 \end{aligned}\right\} G_3,$$

$$(16)\quad \left.\begin{aligned} &x(1-x)\,r + y^2\,t + [\delta - (\alpha+\beta+1)\,x]\,p - (\alpha-\beta-1)\,yq - \alpha\beta z = 0 \\ &-y(1+y)\,t + x(1-y)\,s + [\alpha - 1 - (\beta+\gamma+1)\,y]q - \gamma xp - \beta\gamma z = 0 \end{aligned}\right\} H_1,$$

$$(17)\quad \left.\begin{aligned} &x(x-1)\,r - xys + [(\alpha+\beta+1)\,x - \epsilon]p - \beta yq + \alpha\beta z = 0 \\ &y(y+1)\,t - xs + [1-\alpha+(\gamma+\delta+1)\,y]\,q + \gamma\delta z = 0 \end{aligned}\right\} H_2,$$

$$(18)\quad \left.\begin{aligned} &x(1-4x)\,r + y(1-4x)\,s - y^2\,t + [\gamma - (4\alpha+6)\,x]\,p \\ &\qquad - 2(\alpha+1)\,yq - \alpha(\alpha+1)\,z = 0 \\ &y(1-y)\,t + x(1-2y)\,s + [\gamma - (\alpha+\beta+1)\,y]\,q \\ &\qquad - 2\beta xp - \alpha\beta z = 0 \end{aligned}\right\} H_3,$$

$$(19)\quad \left.\begin{aligned} & x(1-4x)\,r - 4xys - y^2\,t + [\gamma - (4\alpha+4)\,x]\,p \\ & \qquad - (3\alpha+2)\,yq - \alpha(\alpha+1)\,z = 0 \\ & y(1-y)\,t - 2xys + [\delta - (\alpha+\beta)\,y]\,q - 2\beta xp - \alpha\beta z = 0 \end{aligned}\right\}\ H_4,$$

$$(20)\quad \left.\begin{aligned} & x(1+4x)\,r - y(1-4x)\,s + y^2\,t + [1-\gamma+4(\alpha+1)\,x]\,p \\ & \qquad + (3\alpha+2)\,yq - \alpha(\alpha+1)\,z = 0 \\ & y(1-y)\,t - xys + 2x^2\,r^2 + [\gamma - (\alpha+\beta+1)\,y]\,q \\ & \qquad + (2+\alpha-2\beta)\,xp - \alpha\beta z = 0 \end{aligned}\right\}\ H_5,$$

$$(21)\quad \left.\begin{aligned} & x(1+4x)\,r - (1+4x)\,ys + y^2\,t + [1-\beta+(4\alpha+6)\,x]\,p \\ & \qquad - 2\alpha yq + \alpha(\alpha+1)\,z = 0 \\ & y(1+y)\,t - x(2+y)\,s + [1-\alpha+(\beta+\gamma+1)\,y]\,q \\ & \qquad - \gamma xp + \beta\gamma z = 0 \end{aligned}\right\}\ H_6,$$

$$(22)\quad \left.\begin{aligned} & x(1-4x)\,r + 4xys - y^2\,t + [\delta - (4\alpha+6)\,x]\,p \\ & \qquad + 2\alpha yq - \alpha(\alpha+1)\,z = 0 \\ & y(1+y)\,t - 3xys + [1-\alpha+(\beta+\gamma+1)\,y]\,q - \gamma xp + \beta\gamma z = 0 \end{aligned}\right\}\ H_7,$$

$$(23)\quad \left.\begin{aligned} & x(1-x)\,r + y(1-x)\,s + [\gamma - (\alpha+\beta+1)\,x]\,p - \beta yq - \alpha\beta z = 0 \\ & yt + xs + (\gamma - y)\,q - xp - \alpha z = 0 \end{aligned}\right\}\ \Phi_1,$$

$$(24)\quad \left.\begin{aligned} & xr + ys + (\gamma - x)\,p - \beta z = 0 \\ & yt + xs + (\gamma - y)\,q - \beta' z = 0 \end{aligned}\right\}\ \Phi_2$$

$$(25)\quad \left.\begin{aligned} & xr + ys + (\gamma - x)\,p - \beta z = 0 \\ & yt + xs + \gamma q - z = 0 \end{aligned}\right\}\ \Phi_3$$

$$(26)\quad \left.\begin{aligned} & x(1-x)\,r - xys + [\gamma - (\alpha+\beta+1)\,x]\,p - \beta yq - \alpha\beta z = 0 \\ & yt + (\gamma' - y)\,q - xp - \alpha z = 0 \end{aligned}\right\}\ \Psi_1,$$

$$(27)\quad \left.\begin{aligned} & xr + (\gamma - x)\,p - yq - \alpha z = 0 \\ & yt + (\gamma' - y)\,q - xp - \alpha z = 0 \end{aligned}\right\}\ \Psi_2,$$

$$(28)\quad \left.\begin{aligned} & x(1-x)\,r + ys + [\gamma - (\alpha+\beta+1)\,x]\,p - \alpha\beta z = 0 \\ & yt + xs + (\gamma - y)\,q - \alpha' z = 0 \end{aligned}\right\}\ \Xi_1,$$

$$(29)\quad \left.\begin{aligned} & x(1-x)\,r + ys + [\gamma - (\alpha+\beta+1)\,x]\,p - \alpha\beta z = 0 \\ & yt + xs + \gamma q - z = 0 \end{aligned}\right\}\ \Xi_2,$$

$$(30)\quad \left.\begin{aligned} & x(x+1)\,r - y(x+1)\,s + [1-\beta+(\alpha+\beta'+1)\,x]\,p \\ & \qquad - \alpha yq + \alpha\beta' z = 0 \\ & yt - \alpha s + (1-\beta'+y)\,q - xp + \beta z = 0 \end{aligned}\right\}\ \Gamma_1,$$

(31) $\left.\begin{aligned} & xr - ys + (x - \beta + 1)\, p - yq + \beta' z = 0 \\ & yt - xs + (y - \beta' + 1)\, q - xp + \beta z = 0 \end{aligned}\right\} \Gamma_2,$

(32) $\left.\begin{aligned} & x(1-x)\, r + y^2 t + [\delta - (\alpha + \beta + 1)\, x]\, p \\ & \qquad + (\beta - \alpha + 1)\, yq - \alpha\beta z = 0 \\ & yt - xs + (1 - \alpha + y)\, q + xp + \beta z = 0 \end{aligned}\right\} \mathrm{H}_1,$

(33) $\left.\begin{aligned} & x(1-x)\, r + xyt + [\delta - (\alpha + \beta + 1)\, x]\, p + \beta yq - \alpha\beta z = 0 \\ & yt - xs + (1 - \alpha + y)\, q + \gamma z = 0 \end{aligned}\right\} \mathrm{H}_2,$

(34) $\left.\begin{aligned} & x(1-x)\, r + xyt + [\delta - (\alpha + \beta + 1)\, x]\, p + \beta yq - \alpha\beta z = 0 \\ & yt - xs + (1 - \alpha + y)\, q + z = 0 \end{aligned}\right\} \mathrm{H}_3,$

(35) $\left.\begin{aligned} & xr + (\delta - x)\, p + yq - \alpha z = 0 \\ & -yt + xs - (1 - \alpha + y)\, q - \gamma z = 0 \end{aligned}\right\} \mathrm{H}_4,$

(36) $\left.\begin{aligned} & xr + (\delta - x)\, p + yq - \alpha z = 0 \\ & yt - xs + (1 - \alpha + y)\, q + z = 0 \end{aligned}\right\} \mathrm{H}_5,$

(37) $\left.\begin{aligned} & x(1-4x)\, r + y(1-4x)\, s - y^2 t + [\gamma - (4\alpha + 6)\, x]\, p \\ & \qquad - (2\alpha + 2)\, yq - \alpha(\alpha + 1)\, z = 0 \\ & yt + xs + (\gamma - y)\, q - 2xp - \alpha z = 0 \end{aligned}\right\} \mathrm{H}_6,$

(38) $\left.\begin{aligned} & x(1-4x) - 4xys - y^2 t + [\gamma - 4(\alpha + 1)\, x]\, p \\ & \qquad - (3\alpha + 2)\, yq - \alpha(\alpha + 1)\, z = 0 \\ & yt + (\delta - y)\, q - 2xp - \alpha z = 0 \end{aligned}\right\} \mathrm{H}_7,$

(39) $\left.\begin{aligned} & x(1+4x)\, r - y(1+4x)\, s + y^2 t + [1 - \beta + (4\alpha + 6)\, x]p \\ & \qquad - 2\alpha yq + \alpha(\alpha + 1)\, z = 0 \\ & yt - 2xs + (1 - \alpha + y)\, q - xp + \beta z = 0 \end{aligned}\right\} \mathrm{H}_8,$

(40) $\left.\begin{aligned} & x(1-4x)\, r + 4xys - y^2 t + [\delta - (4\alpha + 6)\, x]\, p \\ & \qquad + 2\alpha yq - \alpha(\alpha + 1)\, z = 0 \\ & yt - 2xs + (1 - \alpha + y)\, q + \beta z = 0 \end{aligned}\right\} \mathrm{H}_9,$

(41) $\left.\begin{aligned} & x(1-4x)\, r + 4xys - y^2 t + [\delta - (4\alpha + 6)\, x]p \\ & \qquad + 2\alpha yq - \alpha(\alpha + 1)\, z = 0 \\ & yt - 2xs + (1 - \alpha)\, q + z = 0 \end{aligned}\right\} \mathrm{H}_{10},$

(42) $\left.\begin{aligned} & xr + (\delta - x)\, p + yq - \alpha z = 0 \\ & y(y+1)\, t - xs + [1 - \alpha + (\beta + \gamma + 1)\, y]\, q + \beta\gamma z = 0 \end{aligned}\right\} \mathrm{H}_{11}.$

The monograph by Appell and Kampé de Fériet gives an account of the attempts to investigate some of these systems of partial differential equations; since its publication more work has been done, in particular by Horn, Borngässer, Burchnall, and Erdélyi. Some solutions are known for each system, but a complete set of all relevant fundamental solutions is not known except for the systems of F_1, G_2, (Erdélyi, unpublished), F_4 (Burchnall 1939, Erdélyi 1941), and Φ_1, Φ_2, Γ_1 (Erdélyi 1939, 1940).

The difficulties in dealing with these systems of partial differential equations have two sources. One is the unsatisfactory state of the general analytic theory of systems of partial differential equations; in particular our very scant knowledge of the behavior of solutions in the neighborhood of points at which more than two singular curves of the system intersect, or at which two singular curves are at contact. The other is the large number of apparently distinct systems. This second difficulty can be diminished considerably by using a result of the transformation theory (cf. section 5.11) which suggests that with the possible exception of the systems of

(43) $F_4, H_1, H_5, \mathrm{H}_1,$

every system connected with a hypergeometric series of the second order can be reduced to the system of F_2 or to a particular or limiting case thereof.

5.9.1. Ince's investigation

Ince (1942) investigated the system

(44) $ar + bc_1 s + dp + e_1 q + fz = 0$

$$a_1 t + b_1 cs + d_1 q + ep + f_1 z = 0$$

in which a, b, c, d, e, f, are polynomials in x, and a_1, b_1, c_1, d_1, e_1, f_1, are polynomials in y. He made certain assumptions regarding the $a, b, \ldots, e_1, f_1$, which ensure that (44) has four linearly independent solutions, that it is symmetric in x and y (with a suitable interchange of constant parameters), and that the singular curves are determined by the coefficients of the second partial derivatives in (44); under these assumptions he proved that (44) can be reduced to the system of F_2 or to a particular or limiting case of that system.

5.10. Reduction formulas

Under exceptional circumstances, hypergeometric functions of two variables can be expressed in terms of simpler functions, notably in terms of hypergeometric functions of one variable or in terms of elementary functions. In such cases we speak of *reducible* hypergeometric functions and of *reduction formulas.* The exceptional circumstances arise either if the parameters in a hypergeometric series satisfy one or several relations, or if the two variables are connected by a relation. In the latter case the relation is usually the equation of a singular curve of the system of partial differential equations associated with the series in question.

Certain trivial reduction formulas are obvious: if $\beta' = 0$ in F_1, F_2 or F_3 or Φ_2, if $y = 0$ in any of the series, the hypergeometric series of two variables can be expressed in terms of series of one variable: such trivial reductions are disregarded in the sequel.

REDUCIBILITY FOR PARTICULAR VALUES OF PARAMETERS

The following reduction formulas can be proved either by expanding in infinite series and comparing coefficients, or by manipulating integral representations.

(1) $$F_1(\alpha, \beta, \beta', \beta+\beta'; x, y) = (1-y)^{-\alpha} \times F[\alpha, \beta; \beta+\beta'; (x-y)/(1-y)]$$

(2) $$F_2[\alpha, \beta, \beta', \beta, \gamma'; x, y) = (1-x)^{-\alpha} F[\alpha, \beta'; \gamma'; y/(1-x)]$$

(3) $$F_2(\alpha, \beta, \beta', \alpha, \alpha; x, y) = (1-x)^{-\beta}(1-y)^{-\beta'} \times F\{\beta, \beta'; \alpha; xy/[(1-x)(1-y)]\}$$

(4) $$F_3(\alpha, \gamma-\alpha, \beta, \gamma-\beta, \gamma; x, y) = (1-y)^{\alpha+\beta-\gamma} F(\alpha, \beta, \gamma; x+y-xy)$$

(5) $$F_4[\alpha, \gamma+\gamma'-\alpha-1, \gamma, \gamma'; x(1-y), y(1-x)] = F(\alpha, \gamma+\gamma'-\alpha-1; \gamma; x)\, F(\alpha, \gamma+\gamma'-\alpha-1; \gamma'; y)$$

(6) $$F_4\{\alpha, \beta, \alpha, \beta; -x/[(1-x)(1-y)], -y/[(1-x)(1-y)]\} = (1-xy)^{-1}(1-x)^{\beta}(1-y)^{\alpha}$$

(7) $$F_4\{\alpha, \beta, \beta, \beta; -x/[(1-x)(1-y)], -y/[(1-x)(1-y)]\} = (1-x)^{\alpha}(1-y)^{\alpha} F(\alpha, 1+\alpha-\beta; \beta; xy)$$

(8) $$F_4\{\alpha, \beta, 1+\alpha-\beta, \beta; -x/[(1-x)(1-y)], -y/[(1-x)(1-y)]\} = (1-y)^{\alpha} F(\alpha, \beta; 1+\alpha-\beta; -x(1-y)/(1-x)]$$

(9) $$H_4[\gamma+\beta-1, \beta, \gamma, 2\beta; \tfrac{1}{4}(1-x^2)(1-y^2)(1+xy)^{-2}, 2xy(1+xy)^{-1}] = (1+xy)^{\gamma+\beta-1} F(\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma-\tfrac{1}{2}, \tfrac{1}{2}\beta+\tfrac{1}{2}\gamma; \gamma; 1-x^2) \times F(\tfrac{1}{2}\beta+\tfrac{1}{2}\gamma-\tfrac{1}{2}, \tfrac{1}{2}\beta+\tfrac{1}{2}\gamma; \beta+\tfrac{1}{2}; y^2)$$

For (1), (2), and (4) see Appell and Kampé de Fériet (1926) Chapter I; for (3), (5), (6), (7), and (8) see Bailey (1935) Chapter IX and examples; and for (9) see Erdélyi (1948) p. 384. A few other reducibile cases are also found in the literature quoted.

REDUCIBILITY FOR SPECIAL VALUES OF THE VARIABLES

The methods are, broadly, speaking, the same as in the previous case, but the known results are far less numerous. Appell and Kampé de Fériet's monograph gives only

$$(10)\quad F_1(a, \beta, \beta', \gamma; x, 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma - a - \beta')}{\Gamma(\gamma - a)\,\Gamma(\gamma - \beta')} F(a, \beta; \gamma - \beta'; x)$$

$$(11)\quad F_1(a, \beta, \beta', \gamma; x, x) = F(a, \beta + \beta'; \gamma; x),$$

both immediate consequences of 5.8(5).

5.11. Transformations

Although there is essentially only one hypergeometric series of the second order in one variable (namely Gauss' series), its transformation theory is quite extensive (cf. section 2.9 to 2.11). With the considerable number of hypergeometric series of the second order in two variables, the complete set of transformations would run into the hundreds, and only a few examples can be given here. The best means for deriving these (and other) transformations is the integral representations of the functions concerned where a change of variables of integration, or a deformation of the contour of integration will often yield the desired results. With integral representations, such as 5.8(6), in whose integrands hypergeometric functions of one variable occur, the known transformation theory of these functions can be used with good effect.

First we have transformations of a series into a series of the same type:

$$F_1(a, \beta, \beta', \gamma; x, y)$$

$$(1)\qquad = (1 - x)^{-\beta}(1 - y)^{-\beta'} F_1\left(\gamma - a, \beta, \beta', \gamma; \frac{x}{x - 1}, \frac{y}{y - 1}\right)$$

$$(2)\qquad = (1 - x)^{-a} F_1\left(a, \gamma - \beta - \beta', \beta', \gamma; \frac{x}{x - 1}, \frac{y - x}{1 - x}\right)$$

$$(3)\qquad = (1 - y)^{-a} F_1\left(a, \beta, \gamma - \beta - \beta', \gamma; \frac{y - x}{y - 1}, \frac{y}{y - 1}\right)$$

$$(4) \qquad = (1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'}\, F_1\left(\gamma-\alpha,\ \gamma-\beta-\beta',\ \beta',\ \gamma; x,\ \frac{x-y}{1-y}\right)$$

$$(5) \qquad = (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'}\, F_1\left(\gamma-\alpha,\ \beta, \gamma-\beta-\beta', \gamma; \frac{x-y}{x-1},\ y\right)$$

$$F_2(\alpha,\ \beta,\ \beta';\ \gamma,\ \gamma';\ x,\ y)$$

$$(6) \qquad = (1-x)^{-\alpha}F_2\left(\alpha,\ \gamma-\beta,\ \beta',\ \gamma,\ \gamma';\ \frac{x}{x-1},\ \frac{y}{1-x}\right)$$

$$(7) \qquad = (1-y)^{-\alpha}F_2\left(\alpha,\ \beta,\ \gamma'-\beta',\ \gamma,\ \gamma';\ \frac{x}{1-y},\ \frac{y}{y-1}\right)$$

$$(8) \qquad = (1-x-y)^{-\alpha}F_2\left(\alpha,\ \gamma-\beta,\ \gamma'-\beta',\ \gamma,\ \gamma';\ \frac{x}{x+y-1},\ \frac{y}{x+y-1}\right).$$

All these correspond to Euler's transformation of the ordinary hypergeometric series, 2.9(4). No simple transformations of this type seem to be known for any of the other complete hypergeometric series of two variables.

There are also transformations of a series into a series of the same type which represent analytic continuations, such as,

$$(9) \quad F_4(\alpha,\ \beta,\ \gamma,\ \gamma';\ x,\ y)$$

$$= \frac{\Gamma(\gamma')\,\Gamma(\beta-\alpha)}{\Gamma(\gamma'-\alpha)\,\Gamma(\beta)}(-y)^{-\alpha}F_4(\alpha, \alpha+1-\gamma',\ \gamma, \alpha+1-\beta;\ x/y,\ 1/y)$$

$$+\frac{\Gamma(\gamma')\,\Gamma(\alpha-\beta)}{\Gamma(\gamma'-\beta)\,\Gamma(\alpha)}(-y)^{\beta}\, F_4(\beta+1-\gamma',\ \beta, \gamma, \beta+1-\alpha;\ x/y,\ 1/y)\,;$$

and lastly, for special values of the parameters, there are quadratic and higher transformations. All these transformations occur in the monograph of Appell and Kampé de Fériet (Chapters I and II).

Secondly, we have transformations of a hypergeometric series of two variables into another type of such series. There are two kinds of such transformations: the one provides an analytic continuation of a series in terms of series of another type, the other is a kind of reduction formula showing that with certain special values of the parameters the series is expressible in terms of a simpler series (i.e., with a smaller number of parameters). Perhaps the best known example of analytic continuation in terms of *another* hypergeometric series is the transformation

(10) $F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y)$

$$= \sum \frac{\Gamma(\gamma)\,\Gamma(\rho-\lambda)\,\Gamma(\sigma-\mu)}{\Gamma(\rho)\,\Gamma(\sigma)\,\Gamma(\gamma-\lambda-\mu)} (-x)^{-\lambda} (1-y)^{-\mu}$$

$$\times F_2(\lambda+\mu+1-\gamma, \lambda, \mu, \lambda+1-\rho, \mu+1-\sigma; 1/x, 1/y)$$

where the sum consists of four terms in which λ, μ, ρ, σ, are respectively, α, α', β, β'; α, β', β, α'; β, α', α, β'; and β, β', α, α'. The best known example for the reduction is

(11) $F_3(\alpha, \alpha', \beta, \beta', \alpha+\alpha'; x, y) = (1-y)^{-\beta'}$

$$\times F_1[\alpha, \beta, \beta', \alpha+\alpha'; x, y/(y-1)].$$

The following two tables give a condensed account of the various known transformations of complete series, i.e., those denoted by F, G, H, in Horn's list. The actual formulas, and similar transformations of confluent series, will be found in Appell and Kampé de Fériet (1926), Bailey (1935) Chapter IX and examples, Burchnall and Chaundy (1940, 1941) (where the transformations are not explicitly stated, but occur as degenerate cases of expansions), and Erdélyi (1948).

ANALYTIC CONTINUATIONS

Series	Variables	Continued in terms of	With variables
F_3	x, y	F_2	$1/x, 1/y$
F_3	x, y	H_2	$1/x, -y$
H_2	x, y	F_2	$1/x, -y$

TRANSFORMATION OR REDUCTION FORMULAS FOR SPECIAL VALUES OF THE PARAMETERS

Series	Restriction	Transforms into	With restriction
F_1	$\beta'+\gamma=\alpha+1$	F_4	$\beta=\gamma'$
F_2	$\alpha=\gamma'$	F_1	none
F_2	$\gamma+\gamma'=\alpha+1$	G_2	none
F_2	$\gamma+\gamma'=\alpha+1$	H_2	$\alpha+\gamma=\epsilon$
F_2	$2\beta=\gamma$	H_4	none

TRANSFORMATION OR REDUCTION FORMULAS FOR SPECIAL VALUES OF THE PARAMETERS (Continued):

Series	Restriction	Transforms into	With restriction
F_2	$\gamma = 2\beta$ and $\gamma' = 2\beta'$	F_4	$\gamma + \gamma' = \alpha + 1$
F_2	$\gamma' = 2\beta'$ and $\beta + \beta' = \alpha + \frac{1}{2}$	F_4	$\beta + \gamma' = \alpha + 1$
F_2	$\beta = \beta'$ and $\gamma + \gamma' = \alpha + 1$	F_4	$\gamma + \gamma' = \alpha + 1$
F_2	$\beta = \beta'$ and $\gamma + \gamma' = \alpha + 1$	G_1	none
F_2	$\gamma + \gamma' = \alpha + 1 = 2 - \beta = 2 - \beta'$	G_3	none
F_3	$\alpha + \alpha' = \gamma$	F_1	none
F_3	$\alpha + \beta = 1$ and $\alpha + \alpha' = \gamma$	H_3	none
F_4	$\beta = \alpha + \frac{1}{2}$	H_4	$\delta = 2\beta$
F_4	$\beta + \gamma' = \alpha + 1$	H_4	$\beta + \gamma = \alpha + 1$
H_2	$\epsilon = 2\beta$	H_7	none
H_2	$\beta = \gamma$ and $\alpha + \delta = \epsilon$	H_1	$\alpha + \delta = \epsilon$
H_2	$\gamma + \delta = 1,\ \delta + \epsilon = \alpha$	H_6	none

The first entry in the second table, for instance, indicates that if $\beta' + \gamma = \alpha + 1$ in the series $F_1(\alpha, \beta, \beta', \gamma; x, y)$, then this series can be expressed in terms of a series $F_4(\alpha, \beta, \gamma, \gamma'; x, y)$ where the new $\alpha, \beta, \gamma, \gamma'$ (of F_4) depend on the parameters of F_1 in such a manner that $\beta = \gamma'$ in F_4; and the x, y in F_4 depend on the variables of F_1.

These tables show that the series F_1, F_3, G_1, G_2, G_3, H_2, H_3, H_4, H_6, and H_7, with arbitrary values of their parameters, can be expressed in terms of F_2, and clearly the confluent series which are their limiting cases can be expressed in terms of limiting cases of F_2. We then have the result that all hypergeometric series of the second order in two variables, with the possible exception of F_4, H_1, H_5, and H_1, can be expressed in terms of F_2 or its special or limiting forms, and this in its turn leads to the corresponding theorem on the systems of partial differential equations associated with these series (cf. also section 5.9). It is not known at present if, although it seems likely that, F_4, H_1, and H_5 are independent of F_2 and also of each other when their parameters are arbitrary.

5.12. Symbolic forms and expansions

Burchnall and Chaundy (1940, 1941) introduce the operators

$$(1)\quad \nabla(h) \equiv \frac{\Gamma(h)\,\Gamma(\delta+\delta'+h)}{\Gamma(\delta+h)\,\Gamma(\delta'+h)}, \qquad \Delta(h) \equiv \frac{\Gamma(\delta+h)\,\Gamma(\delta'+h)}{\Gamma(h)\,\Gamma(\delta+\delta'+h)},$$

$$\delta \equiv x\frac{\partial}{\partial x}, \qquad \delta' \equiv y\frac{\partial}{\partial y},$$

by means of which they write

$$(2)\quad F_2(a, \beta, \beta', \gamma, \gamma'; x, y) = \nabla(a)\, F(a, \beta; \gamma; x)\, F(a\, \beta'; \gamma'; y),$$

$$(3)\quad F_3(a, a', \beta, \beta', \gamma; x, y) = \Delta(\gamma)\, F(a, \beta; \gamma; x)\, F(a', \beta'; \gamma; y),$$

$$(4)\quad F_1(a, \beta, \beta'; \gamma; x, y) = \nabla(a)\,\Delta(\gamma)\, F(a, \beta; \gamma; x)\, F(a, \beta'; \gamma; y),$$

$$(5)\quad F_4(a, \beta, \gamma, \gamma'; x, y) = \nabla(a)\,\nabla(\beta)\, F(a, \beta; \gamma; x)\, F(a, \beta; \gamma'; y),$$

thus factorizing Appell's functions by means of the operators Δ and ∇; they also obtain transformations of Appell's functions such as

$$(6)\quad F_1(a, \beta, \beta', \gamma; x, y) = \nabla(a)\, F_3(a, a, \beta, \beta', \gamma; x, y),$$

$$(7)\quad F_1(a, \beta, \beta', \gamma; x, y) = \Delta(\gamma)\, F_2(a, \beta, \beta', \gamma, \gamma; x, y),$$

$$(8)\quad F_4(a, \beta, \gamma, \gamma'; x, y) = \nabla(\beta)\, F_2(a, \beta, \beta, \gamma, \gamma'; x, y),$$

and some others.

These symbolic forms are used to obtain a large number of expansions of Appell's functions in terms of each other, of Appell's functions in terms of products of ordinary hypergeometric functions, or vice versa. To give an example, by Gauss' formula for $F(a, \beta; \gamma; 1)$, 2.8(46), we have symbolically

$$\nabla(h) = \sum_{r=0}^{\infty} \frac{(-\delta)_r\,(-\delta')_r}{(h)_r\; r!}.$$

Now,

$$(-\delta)_r\; F(a, \beta; \gamma; x) = (-)^r\,\frac{(a)_r\,(\beta)_r}{(\gamma)_r}\, x^r F(a+r, \beta+r; \gamma+r; x),$$

and hence (2) suggests the expansion

$$(9)\quad F_2(a, \beta, \beta', \gamma, \gamma'; x, y)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(\beta)_r(\beta')_r}{r!\;(\gamma)_r(\gamma')_r}\, x^r y^r F(a+r, \beta+r; \gamma+r; x)\, F(a+r, \beta'+r; \gamma'+r; y)$$

By inversion of (2) in the form

$$F(a, \beta; \gamma; x) F(a, \beta'; \gamma'; y) = \Delta(a) F_2(a, \beta, \beta'; \gamma, \gamma'; x, y)$$

and a corresponding expansion of $\Delta(a)$, the companion to (9),

$$(10)\quad F(a, \beta; \gamma; x) F(a, \beta'; \gamma'; y) = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r (\beta)_r (\beta')_r}{r!(\gamma)_r (\gamma')_r} x^r y^r \times F_2(a+r, \beta+r, \beta'+r, \gamma+r, \gamma'+r; x, y)$$

is obtained. These expansions can be proved without symbolic methods by comparing coefficients of equal powers of x and y on both sides.

By these methods Burchnall and Chaundy obtained 15 pairs of expansions involving Appell's functions and ordinary hypergeometric functions, as well as a further considerable number of expansions involving hypergeometric series of higher order, and Humbert's confluent hypergeometric series, Φ, Ψ, and Ξ. The method also yields useful integral representations [such as 5.8(4)] and integral formulas.

There are many other expansions involving Appell's functions and ordinary hypergeometric functions. A particularly important one is the bilinear generating function of Jacobi polynomials,

$$\sum_{n=0}^{\infty} \frac{n!(a+\beta+1)_n}{(a+1)_n (\beta+1)_n} (2n+a+\beta+1) t^n \times P_n^{(a,\beta)}(\cos 2\phi) P_n^{(a,\beta)}(\cos 2\psi) = \frac{(a+\beta+1)(1-t)}{(1+t)^{a+\beta+2}} \times F_4(\tfrac{1}{2}a+\tfrac{1}{2}\beta+1, \tfrac{1}{2}a+\tfrac{1}{2}\beta+3/2, a+1, \beta+1, a^2/k^2, b^2/k^2)$$

where $a = \sin\phi \sin\psi$, $b = \cos\phi \cos\psi$, $k = \tfrac{1}{2}(t^{-1/2} + t^{1/2})$ (Bailey 1935, p. 102 example 19).

5.13. Special cases

Generalizing Jacobi's polynomials to two variables, Appell has studied the following families of polynomials (m and n are non-negative integers):

$$(1)\quad \mathfrak{F}_{mn} = [(\gamma)_m (\gamma')_n]^{-1} x^{1-\gamma} y^{1-\gamma'} (1-x-y)^{\gamma+\gamma'-a} \times \frac{\partial^{m+n}}{\partial x^m \partial y^n} [x^{\gamma+m-1} y^{\gamma'+n-1} (1-x-y)^{a+m+n-\gamma-\gamma'}]$$

$$= (1-x-y)^{m+n} F_2\left(\gamma+\gamma'-a-m-n, -m, -n, \gamma, \gamma'; \frac{x}{x+y-1}, \frac{y}{x+y-1}\right)$$

(2) $$F_{mn} = \frac{x^{1-\gamma}\, y^{1-\gamma'}}{(\gamma)_m\, (\gamma')_n} \frac{\partial^{m+n}}{\partial x^m\, \partial y^n} [x^{\gamma+m-1} y^{\gamma'+n-1} (1-x-y)^{m+n}]$$

$$= F_2(-m-n,\ \gamma+m,\ \gamma+n,\ \gamma,\ \gamma';\ x,\ y),$$

(3) $$E_{mn} = F_2(\gamma+\gamma'+m+n,\ -m,\ -n,\ \gamma,\ \gamma';\ x,\ y).$$

The latter two families form a biorthogonal system for the domain $x,\ y \geq 0$, $x + y \leq 1$ with the weight function $x^{\gamma-1}\, y^{\gamma'-1}$.

The series

(4) $$x^m\, y^n\, F_3(-\tfrac{1}{2}m,\ -\tfrac{1}{2}n,\ \tfrac{1}{2}-\tfrac{1}{2}m,\ \tfrac{1}{2}-\tfrac{1}{2}n,\ -m-n-\tfrac{1}{2}s+\tfrac{1}{2};\ x^{-2},\ y^{-2})$$

(which can be expressed also in terms of F_2) and

(5) $$x^m\, y^n\, F_3[-\tfrac{1}{2}m,\ -\tfrac{1}{2}n,\ \tfrac{1}{2}-\tfrac{1}{2}m,\ \tfrac{1}{2}-\tfrac{1}{2}n,\ \tfrac{1}{2}s+\tfrac{1}{2};\ (x^2+y^2-1)\, x^{-2},$$
$$(x^2+y^2-1)\, y^{-2}]$$

occur in the investigation of hyperspherical harmonics.

All these special cases are investigated in the monograph by Appell and Kampé de Fériet.

5.14. Further series

Hypergeometric series of higher order than two, in two variables, have been studied by Mellin, Birkeland, Kampé de Fériet (cf. Appell and Kampé de Fériet 1926 Chapter IX), and by Burchnall and Chaundy (1941), Burchnall (1942), and Chaundy (1942). Hypergeometric series in three variables have been investigated by Horn (1889); series in n variables by Lauricella (Appell and Kampé de Fériet Chapter VII) and by Erdélyi (1937, 1939a). Particular cases of Lauricella's series occur in the investigation of hyperspherical harmonics.

An extension of basic hypergeometric series to two variables has been given by Jackson (1942, 1944).

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CHAPTER VI

CONFLUENT HYPERGEOMETRIC FUNCTIONS

6.1. Orientation

If we put $z = x/b$ in Gauss' hypergeometric series

$$F(a, b; c; z) = 1 + \frac{ab}{1 \cdot c} z + \frac{a(a+1)\, b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \cdots,$$

in which we assume that neither a nor c is zero or a negative integer we obtain a power series in x whose radius of convergence is $|b|$ and which defines an analytic function with singularities at $x = 0$, b, and ∞. As $b \to \infty$, the limiting case will define an entire function whose singularity at $x = \infty$ is a *confluence* of two singularities of $F(a, b; c; x/b)$ In this manner we are lead to Kummer's series

$$(1) \quad 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2!} + \cdots.$$

In the notation of generalized hypergeometric series 4.1(1) this is ${}_1F_1(a; c; x)$, but in this chapter and in the following two chapters (1) will be denoted by Humbert's symbol

$$\Phi(a, c; x).$$

Sometimes the notation $M(a, c, x)$ is also used.

The series (1) satisfies the differential equation

$$(2) \quad x\frac{d^2y}{dx^2} + (c - x)\frac{dy}{dx} - ay = 0.$$

The substitution

$$(3) \quad y = x^{-c/2} e^{x/2} z, \qquad a = \tfrac{1}{2} - \kappa + \mu, \qquad c = 1 + 2\mu$$

reduces (2) to Whittaker's standard form

$$(4) \quad \frac{d^2z}{dx^2} + \left(-\tfrac{1}{4} + \frac{\kappa}{x} + \frac{\tfrac{1}{4} - \mu^2}{x^2}\right)z = 0.$$

Either of the two equations will be called a *confluent hypergeometric equation*, and any solution of either of them a *confluent hypergeometric function*; a and c (or κ and μ) will be called the *parameters*, x the *variable*.

In this chapter, the investigation of confluent hypergeometric functions will be based on (2), but some of the more significant results will also be given in Whittaker's notation.

6.2. Differential equations

Equation 6.1(2) is a homogeneous linear differential equation of the second order whose coefficients are linear functions of the independent variable; and it can be shown that every such differential equation can be integrated in terms of confluent hypergeometric functions. Let

$$(1)\quad (a_0 x + b_0)\frac{d^2y}{dx^2} + (a_1 x + b_1)\frac{dy}{dx} + (a_2 x + b_2)\, y = 0$$

be the differential equation. If $a_0 = a_1 = a_2 = 0$, the equation has constant coefficients and can be integrated by elementary functions. This case will be excluded. The transformation

$$(2)\quad y = e^{hx} z, \qquad x = \lambda \xi + \mu, \qquad (\lambda \neq 0)$$

carries (1) into

$$(3)\quad (\alpha_0 \xi + \beta_0)\frac{d^2 z}{d\xi^2} + (\alpha_1 \xi + \beta_1)\frac{dz}{d\xi} + (\alpha_2 \xi + \beta_2)\, z = 0,$$

where

$$(4)\quad \alpha_0 = a_0/\lambda, \qquad \alpha_1 = A'(h), \qquad \alpha_2 = \lambda A(h),$$

$$\beta_0 = (a_0\mu + b_0)/\lambda^2, \qquad \beta_1 = [\mu A'(h) + B'(h)]/\lambda,$$

$$\beta_2 = \mu A(h) + B(h),$$

$$A(h) = a_0 h^2 + a_1 h + a_2, \qquad A'(h) = 2a_0 h + a_1,$$

$$B(h) = b_0 h^2 + b_1 h + b_2, \qquad B'(h) = 2b_0 h + b_1.$$

If we can determine λ, μ, and h, so that

$$a_0\mu + b_0 = 0, \qquad a_0 + \lambda A'(h) = 0, \qquad A(h) = 0,$$

then (3) reduces to 6.1(2). In other cases a change of the independent variable will reduce (3) to a confluent hypergeometric equation or to the Bessel equation. The results are set out in the table on the following page, where $\mathfrak{Z}(a, c, x)$ stands for any solution of 6.1(2), and $\mathfrak{C}_\nu(x)$ for any solution of the Bessel equation

$$(5)\quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)\, y = 0.$$

Reduction of $(a_0 x + b_0) y'' + (a_1 x + b_1) y' + (a_2 x + b_2) y = 0.$

$y = e^{hx} z,\quad x = \lambda \xi + \mu,\quad A(h) = a_0 h^2 + a_1 h + a_2,\quad A'(h) = \frac{dA}{dh},\quad B(h) = b_0 h^2 + b_1 h + b_2,\quad B'(h) = \frac{dB}{dh}.$

	Assumptions	h	λ	μ	z	Parameters
(6)	$a_0 \neq 0$, $D^2 \equiv a_1^2 - 4a_0 a_2 \neq 0$	$\frac{-a_1 + D}{2a_0}$	$-\frac{a_0}{A'(h)}$	$-\frac{b_0}{a_0}$	$\mathfrak{F}(a, c, \xi)$	$a = B(h)/A'(h)$, $c = (a_0 b_1 - a_1 b_0) a_0^{-2}$
(7)	$a_0 = 0, \quad a_1 \neq 0$	$-\frac{a_2}{a_1}$	1	$-\frac{B'(h)}{a_1}$	$\mathfrak{F}(a, \tfrac{1}{2}, k\xi^2)$	$a = B(h)/(2a_1)$, $k = -a_1/(2b_0)$
(8)	$a_0 \neq 0, \quad a_1^2 = 4a_0 a_2$	$-\frac{a_1}{2a_0}$	a_0	$-\frac{b_0}{a_0}$	$\xi^{\alpha} \mathcal{C}_{2\alpha}(\beta \xi^{1/2})$	$\alpha = \tfrac{1}{2} - B'(h)/(2a_0)$, $\beta = 2[B(h)]^{1/2}$
(9)	$a_0 = a_1 = 0, \quad a_2 \neq 0$	$-\frac{b_1}{2b_0}$	1	$\frac{4b_0 b_2 - b_1^2}{4a_2 b_0}$	$\xi^{1/2} \mathcal{C}_{1/3}(k\xi^{3/2})$	$k = \frac{2}{3}(a_2/b_0)^{1/2}$

The reduction of (1) to the confluent hypergeometric equation is not unique, for 6.1 (2) can be transformed into an equation of the same form in several ways. If we put

(10) $x = \lambda\xi, \quad y = x^{\rho}\, e^{hx}\, z,$

the differential equation 6.1 (2) tranforms into

(11) $$\xi \frac{d^2\eta}{d\xi^2} + [c + 2\rho - (1-2h)\,\lambda\xi]\,\frac{d\eta}{d\xi}$$

$$+ [\rho(\rho + c - 1)/\xi - \lambda(a - hc + \rho - 2h\rho) + \lambda^2 h(h-1)\,\xi]\,\eta = 0,$$

and this is the differential equation of $\mathfrak{F}(\alpha, \gamma, \xi)$ if

(12) $$\rho = 0 \text{ or } 1 - c, \quad h = 0 \text{ or } 1, \quad \lambda(1-2h) = 1,$$

$$\alpha = \lambda(a - hc) + \rho, \quad \gamma = c + 2\rho.$$

Thus, counting the identity transformation, 6.1 (2) can be transformed in four different ways into an equation of the same form.

The confluent hypergeometric equation is a homogeneous linear differential equation of the second order with a singular point of regular type at $x = 0$, and with coefficients which are regular for all $x \neq 0$ (including $x = \infty$). Every such equation can be shown (Tricomi 1948) to be of the form

(13) $$\frac{d^2y}{dx^2} + \left(a + \frac{b}{x}\right)\frac{dy}{dx} + \left(\alpha + \frac{\beta}{x} + \frac{\gamma}{x^2}\right) y = 0\,,$$

and it is easy to see that [apart from the trivial case when (13) can be integrated in terms of elementary functions] the integration of (13) leads to confluent hypergeometric functions or to Bessel functions. We have to distinguish between two cases of (13).

If $a^2 \neq 4\alpha$, we have

(14) $y = x^{-b/2}\, c^{-ax/2}\, w(\kappa, \mu, \xi),$

where

(15) $$\kappa = (\beta - \tfrac{1}{2}ab)\,(a^2 - 4\alpha)^{-\frac{1}{2}}, \quad \mu = \tfrac{1}{2}[(b-1)^2 - 4\gamma]^{\frac{1}{2}},$$

$$\xi = (a^2 - 4\alpha)^{\frac{1}{2}}\, x,$$

and $w(\kappa, \mu, x)$ is any solution of 6.1 (4); and if $a^2 = 4\alpha$, we have

(16) $y = x^{\frac{1}{2} - b/2}\, e^{-ax/2}\, \mathcal{C}_{\nu}(\xi),$

where

(17) $\nu = [(b-1)^2 - 4\gamma]^{\frac{1}{2}}, \quad \xi = 2[(\beta - \tfrac{1}{2}ab)\, x]^{\frac{1}{2}}.$

The equation

(18) $$X^2 \frac{d^2y}{dX^2} + (AX^{\rho} + B)\, X\frac{dy}{dX} + (DX^{2\rho} + GX^{\rho} + K)\, y = 0.$$

in which $\rho \neq 0$ is any number, is reduced to (13) by the substitution

(19) $x = X^{\nu}, \quad a = A/\nu, \quad b = (B + \nu - 1)/\nu,$

$\alpha = D\nu^{-2}, \quad \beta = G\nu^{-2}, \quad \gamma = K\nu^{-2}$

and hence can be reduced to the confluent hypergeometric equation. A more general form can be obtained from (18) by putting

(20) $$y = Y \exp\left[\int \phi(X) \frac{dX}{X}\right].$$

Y satisfies the equation

(21) $$X^2 \frac{d^2 z}{dX^2} + (AX^{\rho} + B + 2\phi) X \frac{dz}{dX} + \left[DX^{2\rho} + GX^{\rho} + K + (AX^{\rho} + B - 1)\phi + \phi^2 + X\frac{d\phi}{dX}\right] Y = 0.$$

For instance, with $\phi(X) = hX^{\sigma}$ it follows that

(22) $$X^2 \frac{d^2 z}{dX^2} + (AX^{\rho} + BX^{\sigma} + C) X \frac{dz}{dX} + (DX^{2\rho} + EX^{\rho+\sigma} + FX^{2\sigma} + GX^{\rho} + HX^{\sigma} + K) z = 0$$

can be integrated in terms of confluent hypergeometric functions provided that

(23) $E = \frac{1}{2}AB, \quad F = \frac{1}{4}B^2, \quad H = \frac{1}{2}B(C + \sigma - 1).$

At this point it is worth mentioning that Bessel functions themselves are special confluent hypergeometric functions (cf. section 6.9.1) so that in all non-trivial cases the differential equations of this section can be integrated in terms of confluent hypergeometric functions.

The principal linear differential equations of the second order which can be integrated in terms of confluent hypergeometric functions are (1), (13), (18), and (22), the last under conditions (23).

6.3. The general solution of the confluent equation near the origin

We shall investigate the confluent hypergeometric equation in the form

(1) $$x\frac{d^2y}{dx^2} + (c - x)\frac{dy}{dx} - ay = 0.$$

One solution of this equation is

(2) $y_1 = \Phi(a, c; x),$

and the transformations 6.2(10), 6.2(12) give three further solutions, viz.,

(3) $y_2 = x^{1-c}\,\Phi(a - c + 1, 2 - c; x)$,

(4) $y_3 = e^x\,\Phi(c - a, c; -x)$,

(5) $y_4 = x^{1-c}\,e^x\,\Phi(1 - a, 2 - c; -x)$.

From their behavior at the origin it follows that y_1 and y_2 are linearly independent if c is not an integer so that (in this case) the general solution of (1) may be written as

(6) $y = Ay_1 + By_2$

where A and B are arbitrary constants. The exceptional case of an integer c will be discussed later (cf. section 6.7.1).

On the other hand, both y_1 and y_3 are solutions of (1), both are regular at the origin, and have the value unity there. Since (with a non-integer c) the differential equation cannot have more than one such solution, we must have $y_1 = y_3$ or

(7) $\Phi(a, c; x) = e^x\,\Phi(c - a, c; -x)$.

This is known as *Kummer's transformation.* Similarly $y_2 = y_4$ by the Kummer transformation.

Kummer's transformation is a limiting case of Euler's transformation

$$F(a, b; c; z) = (1 - z)^{-b}\,F[c - a, b; c; z/(z - 1)]$$

of the hypergeometric series [cf. 2.1(22)], and is a very important relation. Our proof assumes that c is not an integer, but the transformation holds, by continuity, for positive integer c.

If y_p and y_q are any two solutions of (1), their Wronskian,

(8) $W_{pq} = y_p \dfrac{d}{dx} y_q - y_q \dfrac{d}{dx} y_p$

must be of the form $K_{pq}\,e^x x^{-c}$, where the constant K_{pq} can be evaluated by using the first two terms of the series expansions of the y's. We find

(9) $W_{12} = W_{34} = W_{14} = -W_{23} = (1 - c)\,x^{-c}\,e^x$.

All the other Wronskians of the four solutions vanish identically.

6.4. Elementary relations for the Φ function

As in the theory of Gauss' hypergeometric series, the four functions

(1) $\Phi(a+) \equiv \Phi(a + 1, c; x)$, $\quad \Phi(a-) \equiv \Phi(a - 1, c; x)$,

$\Phi(c+) \equiv \Phi(a, c + 1; x)$, $\quad \Phi(c-) \equiv \Phi(a, c - 1; x)$

are said to be *contiguous* to $\Phi \equiv \Phi(a, c; x)$. The function Φ and any two functions contiguous to it are linearly connected. The six formulas describing these connections can be derived from Gauss' relations between the contiguous functions (cf. section 2.1.2) and can also be verified by comparing coefficients of like powers of x. The formulas are as follows:

(2) $(c-a)\,\Phi(a-) + (2a-c+x)\,\Phi - a\,\Phi(a+) = 0,$

(3) $c(c-1)\,\Phi(c-) - c(c-1+x)\Phi + (c-a)\,x\,\Phi(c+) = 0,$

(4) $(a-c+1)\,\Phi - a\,\Phi(a+) + (c-1)\,\Phi(c-) = 0,$

(5) $c\,\Phi - c\,\Phi(a-) - x\,\Phi(c+) = 0,$

(6) $c(a+x)\,\Phi - (c-a)\,x\,\Phi(c+) - ac\,\Phi(a+) = 0,$

(7) $(a-1+x)\,\Phi + (c-a)\,\Phi(a-) - (c-1)\,\Phi(c-) = 0.$

These relations are not all independent. From two suitably chosen ones, e.g., from (2) and (4), all the others follow by simple operations.

Any function $\Phi(a+m, c+n; x)$, m, n, integers, is said to be *associated* with $\Phi(a, c; x)$. By repeated application of the relations between contiguous functions, it is easy to prove that *any three associated functions are connected by a homogeneous linear relation* whose coefficients are polynomials in x.

By means of term-by-term differentiation, we have

(8) $$\frac{d}{dx}\Phi(a, c; x) = \frac{a}{c}\,\Phi(a+1, c+1; x),$$

or $\Phi' = (a/c)\,\Phi(a+, c+)$. By repeated application, the derivative of any order of Φ is an associated function, and hence *any three derivatives are connected by a homogeneous linear relation* whose coefficients are polynomials in x. The differential equation 6.1(2) is the simplest example of such a relation. Combining (8) with the relations between contiguous functions, we have

(9) $$\Phi' = \frac{a}{x}\,[\Phi(a+) - \Phi] = \left(\frac{a}{c} - 1\right)\Phi(c+) + \Phi = \frac{1-c}{x}\,[\Phi - \Phi(c-1)]$$

Further useful formulas are:

(10) $$\frac{d^n}{dx^n}\Phi(a, c; x) = \frac{(a)_n}{(c)_n}\,\Phi(a+n, c+n; x)$$

(11) $$\frac{d^n}{dx^n}[x^{a+n-1}\,\Phi(a, c; x)] = (a)_n\,x^{a-1}\,\Phi(a+n, c; x)$$

$$(12)\ \frac{d^n}{dx^n}\,[x^{c-1}\,\Phi(a, c; x)] = (-1)^n\,(1-c)_n\,x^{c-1-n}\,\Phi(a, c-n; x)$$

$$(13)\ \frac{d^n}{dx^n}\,[e^{-x}\,\Phi(a, c; x)] = (-1)^n\,\frac{(c-a)_n}{(c)_n}\,e^{-x}\,\Phi(a, c+n; x)$$

$$(14)\ \frac{d^n}{dx^n}\,[e^{-x}\,x^{c-a+n-1}\,\Phi(a, c; x)] = (c-a)_n\,e^{-x}\,x^{c-a-1}\,\Phi(a-n, c; x).$$

Here

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad n = 1, 2, 3, \ldots .$$

6.5. Basic integral representations

It is known that homogeneous linear differential equations whose coefficients are linear functions of the independent variable can be integrated by Laplace integrals. The integral representation

$$(1)\quad \Phi(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 e^{xu}\,u^{a-1}\,(1-u)^{c-a-1}\,du \qquad \operatorname{Re} c > \operatorname{Re} a > 0$$

can be verified at once by expanding e^{xu} in powers of x. This representation suggests the integration of 6.1 (2) by an integral of the form

$$y = \int_C e^{-xt}\,t^{a-1}\,(1+t)^{c-a-1}\,dt,$$

which is also suggested by Laplace's method. Substituting this in 6.1 (2) we use the identity

$$\left[x\,\frac{d^2}{dx^2} + (c-x)\,\frac{d}{dx} - a\right]\,[e^{-xt}\,t^{a-1}\,(1+t)^{c-a-1}]$$

$$= -\frac{d}{dt}[e^{-xt}\,t^a\,(1+t)^{c-a}]$$

to show that the integral satisfies 6.1(2) provided that C does not pass through any singular point of the integrand, and that the initial and final values of $e^{-xt}\,t^a\,(1+t)^{c-a}$ are finite and equal to one another. If $\operatorname{Re} c > \operatorname{Re} a > 0$, one possible choice for C is the interval $(0,-1)$, thus showing that (1) satisfies 6.1(2). If $\operatorname{Re} a > 0$, another possible choice is a ray from the origin. We put

$$(2)\quad \Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt}\,t^{a-1}\,(1+t)^{c-a-1}\,dt \qquad \operatorname{Re} a > 0.$$

This equation defines a solution of 6.1(2) in the half-plane $\operatorname{Re} x > 0$.

The domain of definition can be extended by rotating the path of integration. Thus,

(3) $$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{i\phi}} e^{-xt} t^{a-1} (1+t)^{c-a-1} \, dt$$

$$\text{Re } a > 0, \quad -\pi < \phi < \pi, \quad -\tfrac{1}{2}\pi < \phi + \arg x < \tfrac{1}{2}\pi.$$

Here t^{a-1} and $(1+t)^{c-a-1}$ are assumed to have their principal values. The condition Re $a > 0$, and also the condition Re $c >$ Re $a > 0$ for Φ, will be removed later, in 6.11(2) and 6.11(1), when we introduce contour integrals.

Another type of integral representation uses Mellin-Barnes integrals (cf. section 1.19). The formula

(4) $$\Phi(a, c; x) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(-s)\,\Gamma(a+s)}{\Gamma(c+s)} (-x)^s \, ds$$

$$-\tfrac{1}{2}\pi < \arg(-x) < \tfrac{1}{2}\pi, \quad 0 > \gamma > -\text{Re } a, \quad c \neq 0, 1, 2, \ldots,$$

can be verified by evaluating the integral as the sum of residues of the integrand at the poles of $\Gamma(-s)$.

The corresponding representation of Ψ is obtained by substituting

$$\Gamma(a-c+1)\,(1+t)^{c-a-1} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(-s)\,\Gamma(a-c+s+1)\, t^s \, ds$$

$$0 > \gamma > \text{Re}(c-a)$$

in (2). Interchanging the order of integration is permissible if $\gamma + \text{Re}\, a > 0$, and gives

$$\Gamma(a)\,\Gamma(a-c+1)\,\Psi(a, c; x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} ds\, \Gamma(-s)\,\Gamma(a-c+s+1) \int_0^\infty e^{-xt} t^{a+s-1} \, dt.$$

Evaluating the last integral,

$$\Psi(a, c; x) = \frac{x^{-a}}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(-s)\,\Gamma(a+s)\,\Gamma(a-c+1+s)}{\Gamma(a)\,\Gamma(a-c+1)} x^{-s} \, ds,$$

or, with a slight change of notation,

(5) $$\Psi(a, c; x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(a+s)\,\Gamma(-s)\,\Gamma(1-c-s)}{\Gamma(a)\,\Gamma(a-c+1)} x^s \, ds$$

$$-\text{Re } a < \gamma < \min(0, 1 - \text{Re } c), \quad -\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}.$$

In the derivation, more stringent assumptions had to be made. The extension to the conditions stated in (5) follows from the theory of analytic continuation. The conditions imposed upon the parameters in (4) and (5) can be relaxed still further if the path of integration is suitably deformed. In fact, (4) is valid (with any γ) whenever a is not zero or a negative integer, provided that the contour of integration is indented, if necessary, so as to separate the poles of $\Gamma(-s)$ from the poles of $\Gamma(a+s)$. Similarly, (5) is valid (with any γ) as long as neither a nor $a-c+1$ is zero or a negative integer, provided that the path of integration separates the poles of $\Gamma(a+s)$ from the poles of $\Gamma(-s)\,\Gamma(1-c-s)$. The conditions on $\arg x$ cannot be relaxed.

It can be verified that (4) and (5) satisfy 6.1(2) (Whittaker and Watson, 1927, section 16.4).

From (5) we have

(6) $$\Psi(a, c; x) = x^{1-c}\,\Psi(a-c+1, 2-c; x).$$

If c is not an integer, the poles of $\Gamma(-s)\,\Gamma(1-c-s)$ are all simple. The evaluation of (5) as the sum of residues of the integrand at these poles leads to the important relation

(7) $$\Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,\Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)}\,x^{1-c}\,\Phi(a-c+1, 2-c; x)$$

between the Φ function and the Ψ function.

6.6. Elementary relations for the Ψ function

The Ψ function was introduced by Tricomi (1927) who denotes it by G; it is related to F_1 (Meixner, 1933), to $E(\alpha, \beta :: x)$ (MacRobert 1941), and to ${}_2F_0(\alpha, \beta; x)$ (Erdélyi, 1939) by the relations

(1) $$E(\alpha, \beta :: x) = \Gamma(\alpha)\,\Gamma(\beta)\,x^{\alpha}\,\Psi(\alpha, \alpha-\beta+1; x),$$

(2) $$F_1(a, \gamma, x) = e^{i\pi a}\,\frac{\Gamma(\gamma)}{\Gamma(\gamma-a)}\,\Psi(a, \gamma; x),$$

(3) $${}_2F_0(\alpha, \beta; -1/x) = x^{\alpha}\,\Psi(\alpha, \alpha-\beta+1; x).$$

$\Psi(a, c; x)$ is a many-valued function of x, and we shall usually consider its principal branch in the plane cut along the negative real axis, this branch being determined by 6.5(3) with $-\tfrac{1}{2}\pi \le \phi \le \tfrac{1}{2}\pi$.

A number of elementary relations for the Ψ function follows directly from the corresponding relations of section 6.4 for Φ.

(4) $$\Psi(a-) - (2a-c+x)\,\Psi + a(a-c+1)\,\Psi(a+) = 0$$

(5) $$(c-a-1)\,\Psi(c-) - (c-1+x)\,\Psi + x\,\Psi(c+) = 0$$

(6) $\Psi - a\,\Psi(a+) - \Psi(c-) = 0$

(7) $(c-a)\,\Psi - x\,\Psi(c+) + \Psi(a-) = 0$

(8) $(a+x)\,\Psi + a(c-a-1)\,\Psi(a+) - x\,\Psi(c+) = 0$

(9) $(a-1+x)\,\Psi - \Psi(a-) + (a-c+1)\,\Psi(c-) = 0$

(10) $$\Psi' = a\,\Psi(a+,\ c+) = \Psi - \Psi(c+)$$
$$= (a/x)\,[(a-c+1)\,\Psi(a+) - \Psi] = (1/x)\,[(a-c+x)\,\Psi - \Psi(a-)]$$

(11) $$\frac{d^n}{dx^n}\,\Psi(a,\ c;\ x) = (-1)^n\,(a)_n\,\Psi(a+n,\ c+n;\ x)$$

(12) $$\frac{d^n}{dx^n}\,[x^{c-1}\,\Psi(a,\ c;\ x)] = (-1)^n\,(a-c+1)_n\,x^{c-n-1}\,\Psi(a,\ c-n;\ x)$$

(13) $$\frac{d^n}{dx^n}\,[x^{a+n-1}\,\Psi(a,\ c;\ x)] = (a)_n\,(a-c+1)_n\,x^{a-1}\,\Psi(a+n,\ c;\ x)$$

(14) $$\frac{d^n}{dx^n}\,[e^{-x}\,\Psi(a,\ c;\ x)] = (-1)^n\,e^{-x}\,\Psi(a,\ c+n;\ x)$$

(15) $$\frac{d^n}{dx^n}\,[e^{-x}\,x^{c-a+n-1}\,\Psi(a,\ c;\ x)]$$
$$= (-1)^n\,e^{-x}\,x^{c-a-1}\,\Psi(a-n,\ c;\ x).$$

6.7. Fundamental systems of solutions of the confluent equation

Four solutions, y_1 to y_4, of the confluent equation are listed in section 6.3. From the work of section 6.5 in conjunction with section 6.3 it follows that

(1) $y_5 = \Psi(a,\ c;\ x),$

(2) $y_6 = x^{1-c}\,\Psi(a-c+1,\ 2-c;\ x),$

(3) $y_7 = e^x\,\Psi(c-a,\ c;\ -x),$

(4) $y_8 = e^x\,x^{1-c}\,\Psi(1-a,\ 2-c;\ -x)$

are four further solutions. The relationship of the four solutions $y_1, \ldots, y_4$ has been investigated in section 6.3; in particular we know that $y_1 = y_3$ and $y_2 = y_4$ form a fundamental system of solutions of 6.1(2) provided that c is not an integer. There remains a discussion of the four solutions $y_5, \ldots, y_8$. In this section we shall assume that c is not an integer.

The exceptional case will be dealt with in the next section.

From 6.5(6) we see that

(5) $y_5 = y_6$, and $y_8 = -e^{-i\epsilon\pi c} y_7$

where $\epsilon = \operatorname{sgn}(\operatorname{Im} x) = 1$ if $\operatorname{Im} x > 0$, $= -1$ if $\operatorname{Im} x < 0$. This significance of ϵ will be retained in the present section. The factor $e^{i\epsilon\pi(1-c)}$ is caused by our conventions in determining x^{1-c}. Thus, there remain four, in general distinct, solutions, $y_1 = y_3$, $y_2 = y_4$, $y_5 = y_6$ and $y_7 = e^{i\pi\epsilon(c-1)} y_8$. All of these are defined when c is not an integer. Their Wronskians, cf. 6.3(8), are all of the form $W_{pq} = K_{pq}\, e^x x^{-c}$ with

(6) $K_{12} = 1 - c, \quad K_{15} = -\dfrac{\Gamma(c)}{\Gamma(a)}$,

$$K_{57} = e^{i\pi\epsilon(c-a)}, \quad K_{17} = \frac{\Gamma(c)}{\Gamma(c-a)}\, e^{i\pi\epsilon c},$$

$$K_{25} = -\frac{\Gamma(2-c)}{\Gamma(a-c+1)}, \quad K_{27} = -\frac{\Gamma(2-c)}{\Gamma(1-a)}.$$

In general, that is if c, a, $c-a$, are non-integer, any two of the four solutions are distinct and form a fundamental system. However, if n denotes a non-negative integer and $a = -n$, W_{15} vanishes identically, so that y_1 and y_5 are constant multiples of each other; 6.5(7) shows that this is indeed the case. Similarly, if $a = 1+n$, y_2 and y_7 coincide; if $c - a = -n$, y_1 and y_7 coincide; and if $c - a = 1 + n$, y_2 and y_5 are constant multiples of each other. If c is an integer, either y_1 or y_2 fails to be defined and cannot be used. W_{57} has been derived, from W_{12} and 6.5(7), under the assumption that c is not an integer; however, on account of continuity it holds for integer c too. Since it never vanishes identically in x, it follows that y_5 and y_7 form a fundamental system of solutions of 6.1(2) under all circumstances.

The expression of Ψ in terms of Φ is given by 6.5(7). The converse expression of Φ in terms of Ψ is obtained by writing down an expression similar to 6.5(7) for y_7, and then eliminating one of the two Φ-functions. The result is

(7) $$\Phi(a, c; x) = \frac{\Gamma(c)}{\Gamma(c-a)}\, e^{i\epsilon a\pi}\, \Psi(a, c; x) + \frac{\Gamma(c)}{\Gamma(a)}\, e^{i\pi(a-c)\epsilon}\, e^x\, \Psi(c-a, c; -x),$$

with a companion formula for y_2. By means of (7) and 6.5(7), the following relations between the solutions of the confluent equation may be established:

$$(8)\quad y_5 = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} y_1 + \frac{\Gamma(c-1)}{\Gamma(a)} y_2,$$

$$(9)\quad y_7 = \frac{\Gamma(1-c)}{\Gamma(1-a)} y_1 - \frac{\Gamma(c-1)}{\Gamma(c-a)} e^{i\pi c\epsilon} y_2,$$

$$(10)\quad y_1 = \frac{\Gamma(c)}{\Gamma(c-a)} e^{i\pi a\epsilon} y_5 + \frac{\Gamma(c)}{\Gamma(a)} e^{i\pi(a-c)\epsilon} y_7,$$

$$(11)\quad y_2 = e^{\epsilon(a-c)\pi i}\left[-\frac{\Gamma(2-c)}{\Gamma(1-a)} y_5 + \frac{\Gamma(2-c)}{\Gamma(a-c+1)} y_7\right].$$

6.7.1. The logarithmic case

The function $\Phi(a; c; x)$ is an entire function of x, and also an entire function of a. Considered as a function of c, Φ has poles, and fails to be defined, at $c = 0, -1, -2, \ldots$. However, we have the relation

$$(12)\quad \lim_{c\to 1-n} \frac{\Phi(a, c; x)}{\Gamma(c)} = \frac{(a)_n}{n!} x^n \Phi(a+n, 1+n; x) \qquad n = 1, 2, 3, \ldots,$$

showing that $\Phi^* = \Phi(a, c; x)/\Gamma(c)$ is an entire function of both parameters as well as of the variable. In other respects too this function shows a simple behavior, for instance some of the differentiation formulas of section 6.4 become simpler when expressed in terms of Φ^*.

As a consequence of this situation, the Φ function furnishes only one solution of the confluent equation when c is an integer. If $c = 1$, y_1 and y_2 are identical. If $c = 2, 3, \ldots, y_2$ does not exist, and although the equation $y_2/\Gamma(2-c)$ tends to a finite limit as c approaches one of the integers > 1, (12) shows that this limit is a numerical multiple of y_1 and does not provide a new solution. For $c = 0, -1, -2, \ldots$, the situation is similar, except that y_1 and y_2 interchange their roles. Whenever c is an integer, y_1 or y_2 provide one solution, and the second solution will contain logarithmic terms. This second solution can be determined by the familiar method of Frobenius.

Another approach to the exceptional case uses the Ψ function as its point of departure. The integral representation 6.5(3) defines the Ψ function for all values of c and shows that this function always satisfies the confluent equation. If c is not an integer, the expansion of Ψ in ascending powers of x is given by 6.5(7). If c is an integer it will be sufficient to assume $c = 1 + n$ where $n = 0, 1, 2, \ldots$. For $c = 1 + n$, both terms on the right-hand side of 6.5(7) become infinite, and the expansion in ascending powers of x can be obtained either by making $c \to n+1$ in 6.5(7), or more directly from 6.5(5). We shall briefly describe the second method.

If $c = n+1$, the integrand of 6.5(5) has simple poles at $s = -n, -n+1, \ldots, -1$ and double poles at $s = 0, 1, 2, \ldots$, (if $n = 0$, there are only double poles). The residue of $\Gamma(a+s)\,\Gamma(-s)\,\Gamma(-n-s)\,x^{-s}$ at the simple pole $s = r-n$, $r = 0, 1, \ldots, n-1$ is

$$(-)^{r-1}\,\Gamma(a-n+r)\,\Gamma(n-r)\,x^{r-n}/r\,!\,;$$

the residue at the double pole $s = r$, $r = 0, 1, 2, \ldots$, is

$$\frac{(-1)^n\,\Gamma(a+r)}{r\,!\,(n+r)\,!}\,[\log x + \psi(a+r) - \psi(1+r) - \psi(1+n+r)]\,x^r$$

where $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$ (cf. section 1.7). Evaluating the integral 6.5(5) as the sum of residues of the integrand at the poles which lie to the right of the path of integration, we obtain

$$(13)\quad \Psi(a, n+1; x) = \frac{(-1)^{n-1}}{n\,!\,\Gamma(a-n)}\Bigg\{\Phi(a, n+1; x)\log x$$

$$+\sum_{r=0}^{\infty}\frac{(a)_r}{(n+1)_r}\,[\psi(a+r) - \psi(1+r) - \psi(1+n+r)]\,\frac{x^r}{r\,!}\Bigg\}$$

$$+\frac{(n-1)\,!}{\Gamma(a)}\sum_{r=0}^{n-1}\frac{(a-n)_r}{(1-n)_r}\,\frac{x^{r-n}}{r\,!} \qquad n = 0, 1, 2, \ldots.$$

The last sum is to be omitted if $n = 0$. The corresponding expansion of $\Psi(a, 1-n; x)$ can be obtained from (13) and 6.5(6).

Some formulas simplify in the logarithmic case. As an example we shall show that

$$(14)\quad f(c) = \left(\frac{\partial}{\partial a} + 2\,\frac{\partial}{\partial c}\right)\Phi(a, c; x)$$

can be expressed in terms of confluent hypergeometric functions when c is an integer.

Since

$$\frac{\partial}{\partial a}\,(a)_n = (a)_n\;[\psi(a+n) - \psi(a)]$$

where $\psi(z)$ is the logarithmic derivative of the gamma function, term-by-term differentiation of 6.1(1) gives

$$f(c) = \sum_{r=0}^{\infty}\frac{(a)_r}{(c)_r}\,[\psi(a+r) - \psi(a) - 2\,\psi(c+n) + 2\,\psi(c)]\frac{x^r}{r\,!}\,.$$

By comparison with (13), with $n = 0$,

(15) $f(1) = [2\,\psi(1) - \psi(a) - \log x]\,\Phi(a,\ 1;\ x) - \Gamma(a)\,\Psi(a,\ 1;\ x).$

The corresponding result for $f(1+n)$ with $n = 0, 1, 2, \ldots$, may be written down by means of 6.4(13).

6.8. Further properties of the Ψ function

Like the Φ function, the Ψ function is a limiting case of Gauss' hypergeometric function, since

(1) $$\lim_{c \to \infty} F(a,\ b;\ c;\ 1 - c/x) = x^a\,\Psi(a,\ a-b+1;\ x)$$

Erdélyi 1939a). The proof is based on 6.5(7) and the expansion of $F(1 - c/x)$ in ascending powers of x.

The behavior of Ψ for small x can be investigated by means of 6.5(7), 6.7(13), and the corresponding formula for c equal to zero or a negative integer. The results are summarized in the table

$\Psi(a,\ c;\ x)$ for small x

	c	Ψ
(2)	$\mathrm{Re}\,c > 1$	$x^{1-c}\,\Gamma(c-1)/\Gamma(a) + R$
(3)	$\mathrm{Re}\,c < 1$	$\Gamma(1-c)/\Gamma(a-c+1) + R$
(4)	$\mathrm{Re}\,c = 1, \quad c \neq 1$	$\dfrac{\Gamma(1-c)}{\Gamma(a-c+1)} + \dfrac{\Gamma(c-1)}{\Gamma(a)}\,x^{1-c} + R$
(5)	$c = 1$	$-[\Gamma(a)]^{-1}\log x + R$

The order of the remainder is seen from the following table

$R = O(u)$

	c	u
(6)	$\mathrm{Re}\,c \geq 2, \quad c \neq 2$	$\lvert x\rvert^{\mathrm{Re}\,c-2}$
(7)	$c = 2$	$\lvert\log x\rvert$
(8)	$1 < \mathrm{Re}\,c < 2$	1
(9)	$\mathrm{Re}\,c = 1, \quad c \neq 1$	$\lvert x\rvert$
(10)	$c = 1$	$\lvert x \log x\rvert$

	c	u
(11)	$0 < \operatorname{Re} c < 1$	$\|x\|^{1-\operatorname{Re} c}$
(12)	$\operatorname{Re} c \leq 0, \quad c \neq 0$	$\|x\|$
(13)	$c = 0$	$\|x \log x\|$

According to the convention of section 6.6, the negative real axis is a branch cut of the Ψ function. We denote by $f(-\xi + i\,0)$ the limit of $f(-\xi + i\eta)$ as $\eta \to 0$ through positive values, and define $f(-\xi - i\,0)$ similarly. From 6.5(7),

(14) $\Psi(a, c; -\xi \pm i0)$

$$= \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \Phi(a, c; -\xi)$$

$$- \frac{\Gamma(c-1)}{\Gamma(a)} e^{\mp i\pi c} \xi^{1-c} \Phi(a-c+1, 2-c; -\xi)$$

$$= e^{-\xi}\left[\frac{\Gamma(1-c)}{\Gamma(a-c+1)} \Phi(c-a, c; -\xi)\right.$$

$$\left. - \frac{\Gamma(c-1)}{\Gamma(a)} e^{\mp i\pi c} \xi^{1-c} \Phi(1-a, 2-c; -\xi)\right]$$

where $\xi > 0$, and either all upper or all lower signs have to be taken. In particular,

(15) $\Delta \equiv \Psi(a, c; -\xi + i\,0) - \Psi(a, c; -\xi - i\,0)$

$$= - \frac{2\pi i}{\Gamma(a)\,\Gamma(2-c)} \xi^{1-c} \Phi(a-c+1, 2-c; -\xi).$$

Since the derivation of this formula is based on 6.5(7), integer c are excluded in the first place. By continuity, the formula remains valid for $c = 1, 0, -1, -2, \dots$. For $c = 2, 3, 4, \dots$, the right-hand side appears in an indeterminate form, and when it is written as

$$- \frac{2\pi i}{\Gamma(a)\Gamma(a-c+1)} \sum_{r=0}^{\infty} \frac{\Gamma(a-c+r+1)}{\Gamma(2-c+r)} (-1)^r \xi^{1-c+r},$$

c can be made to approach $1 + n$ ($n = 1, 2, \dots$,) with the result that

(16) $$\Delta = (-1)^{n-1} \frac{2\pi i}{\Gamma(a-n)} \Phi(a, 1+n; -\xi)$$

$$\xi > 0, \qquad c = 1 + n, \qquad n = 0, 1, 2, \dots .$$

$\Psi(a, c; x)$ is one-valued (i) if $a = 0, -1, -2, \dots$, when Ψ is a polynomial in x and according to 6.5(7) a multiple of Φ, and (ii) if $c = n + 1$, $n = 1, 2, \ldots$, and a is one of the integers $1, 2, \ldots, n$, when 6.7(12) shows that Ψ is a polynomial in x^{-1}.

The behavior of Ψ when x encircles the origin can be understood from the formula

$$(17)\quad \Psi(a, c; xe^{2m\pi i}) = e^{-2mc\pi i}\Psi(a, c; x) + (1 - e^{-2mc\pi i})\frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a, c; x) \qquad m = 0, \pm 1, \pm 2, \ldots .$$

6.9. Whittaker functions

For some purposes it may be convenient to use Whittaker's notation. Whittaker writes the confluent equation in the standard form 6.1(4). From 6.2(13), with

$$a = -1, \quad b = c, \quad \alpha = \gamma = 0, \quad \beta = -a,$$

we see that

$$\mathfrak{Z}(a, c; x) = e^{x/2} x^{-c/2} w(\kappa, \mu, x)$$

with $\kappa = -a + \frac{1}{2}c$, $\mu = \frac{1}{2}c - \frac{1}{2}$. Two solutions of the Whittaker equation are the *Whittaker functions*

$$(1)\quad M_{\kappa,\mu}(x) = e^{-x/2} x^{c/2}\Phi(a, c; x) \equiv z_1,$$

$$(2)\quad W_{\kappa,\mu}(x) = e^{-x/2} x^{c/2}\Psi(a, c; x) \equiv z_5 \qquad a = \tfrac{1}{2} - \kappa + \mu, \quad c = 2\mu + 1.$$

Conversely,

$$(3)\quad \Phi(a, c; x) = e^{x/2} x^{-\frac{1}{2}-\mu} M_{\kappa,\mu}(x)$$

$$(4)\quad \Psi(a, c; x) = e^{x/2} x^{-\frac{1}{2}-\mu} W_{\kappa,\mu}(x) \qquad \kappa = \tfrac{1}{2}c - a, \quad \mu = \tfrac{1}{2}c - \tfrac{1}{2}.$$

We also have, in the notation of 6.6(3),

$$(5)\quad W_{\kappa,\mu}(x) = e^{-x/2} x^{\kappa}\, {}_2F_0(\tfrac{1}{2} - \kappa + \mu, \tfrac{1}{2} - \kappa - \mu; -1/x).$$

Further solutions of the Whittaker equation 6.1(14) can be derived from sections 6.3 and 6.7. Indicating corresponding solutions of 6.1(2) and 6.1(4) by the same subscript

$$(6)\quad z_2 = M_{\kappa,-\mu}(x), \quad z_3 = M_{-\kappa,\mu}(-x), \quad z_4 = M_{-\kappa,-\mu}(-x),$$

$$z_6 = W_{\kappa,-\mu}(x), \quad z_7 = W_{-\kappa,\mu}(-x), \quad z_8 = W_{-\kappa,-\mu}(-x).$$

Kummer's transformation 6.3(7) becomes

$$(7)\quad M_{\kappa,\mu}(x) = e^{i\epsilon\pi(\frac{1}{2}+\mu)} M_{-\kappa,\mu}(-x)$$

where $\epsilon = 1$ if $\operatorname{Im}(x) > 0$ and $\epsilon = -1$ if $\operatorname{Im}(x) < 0$, and the transformation 6.5 (6) becomes

(8) $W_{\kappa,\mu}(x) = W_{\kappa,-\mu}(x)$.

Whittaker defined M by (1), while in the definition of W he used an integral representation equivalent to 6.5 (2).

Buchholz (1943 and other papers) uses the notation

$$m_{\nu}^{(p)}(z) = (2z/\pi)^{-\frac{1}{2}} M_{\nu,\frac{1}{2}p}(z)$$

$$w_{\nu}^{(p)}(z) = (2z/\pi)^{-\frac{1}{2}} W_{\nu,\frac{1}{2}p}(z)$$

6.9.1. Bessel functions

If $\kappa = 0$, the Whittaker equation 6.1 (4) can easily be reduced to the Bessel equation 6.2 (5). The result may be indicated as

$$w(0, \mu, x) = x^{\frac{1}{2}} \mathcal{C}_{\mu}(\tfrac{1}{2}ix).$$

In the notation of 6.1 (2) this corresponds to $c = 2a$. The connection between Bessel functions and confluent hypergeometric functions can be summarized in the following formulas in which the standard notation is used for Bessel functions (cf. Watson, 1922).

(9) $$J_{\nu}(x) = \frac{1}{\Gamma(\nu+1)} (\tfrac{1}{2}x)^{\nu} e^{-ix} \Phi(\tfrac{1}{2}+\nu, 1+2\nu; 2ix)$$

(10) $$I_{\nu}(x) = \frac{1}{\Gamma(\nu+1)} (\tfrac{1}{2}x)^{\nu} e^{-x} \Phi(\tfrac{1}{2}+\nu, 1+2\nu; 2x)$$

(11) $$M_{0,\mu}(2ix) = (2ix)^{\mu+\frac{1}{2}} \Lambda_{\mu}(x) = \Gamma(\mu+1)\, i^{\mu+\frac{1}{2}}\, 2^{2\mu+\frac{1}{2}}\, x^{\frac{1}{2}} J_{\mu}(x)$$

(12) $$H_{\nu}^{(1)}(x) = (\tfrac{1}{2}\pi x)^{-\frac{1}{2}} e^{i(x-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} {}_2F_0[\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu; 1/(2ix)]$$
$$= -2i\pi^{-\frac{1}{2}} e^{i(x-\nu\pi)} (2x)^{\nu} \Psi(\tfrac{1}{2}+\nu, 1+2\nu; -2ix)$$
$$= (\tfrac{1}{2}\pi x)^{-\frac{1}{2}} e^{-i(\frac{1}{2}\nu+\frac{1}{4})\pi} W_{0,\nu}(-2ix)$$

(For $H_{\nu}^{(2)}$, change i into $-i$)

(13) $$K_{\nu}(x) = \pi^{\frac{1}{2}} e^{-x} (2x)^{\nu} \Psi(\tfrac{1}{2}+\nu, 1+2\nu; 2x)$$

(14) $$W_{0,\mu}(x) = \pi^{-\frac{1}{2}} x^{\frac{1}{2}} K_{\mu}(\tfrac{1}{2}x)$$

(15) $$Y_{\nu}(x) = -\pi^{-\frac{1}{2}} (2x)^{\nu} [e^{i(\nu\pi-x)} \Psi(\tfrac{1}{2}+\nu, 1+2\nu; 2ix)$$
$$+ e^{i(x-\nu\pi)} \Psi(\tfrac{1}{2}+\nu, 1+2\nu; -2ix)]$$

Bessel functions occur also as limiting cases of the confluent hypergeometric functions. Performing the limiting process term by term, we find

(16) $$\lim_{a \to \infty} \Phi(a, c; -x/a) = \Gamma(c)\, x^{\frac{1}{2}-\frac{1}{2}c} J_{c-1}(2x^{\frac{1}{2}}).$$

The corresponding result for the Ψ function can be obtained from 6.5(7) for non-integral c. From Stirling's formula, or still better, from 1.18(4),

$$\lim_{a \to \infty} \frac{\Gamma(1+a-c)}{a^{1-c}\,\Gamma(a)} = 1,$$

and this result, in conjunction with (16) and 6.5(7), shows that

(17) $$\lim_{a \to \infty} [\Gamma(a-c+1)\, \Psi(a, c; -x/a)]$$
$$= \pi x^{\frac{1}{2}-\frac{1}{2}c} \operatorname{cosec}(\pi c)\, [J_{c-1}(2x^{\frac{1}{2}}) + e^{i\epsilon c\pi} J_{1-c}(2x^{\frac{1}{2}})]$$
$$= -i\pi e^{i\pi c} x^{\frac{1}{2}-\frac{1}{2}c} H^{(1)}_{c-1}(2x^{\frac{1}{2}}) \qquad \operatorname{Im} x > 0,$$
$$= i\pi c^{-i\pi c} x^{\frac{1}{2}-\frac{1}{2}c} H^{(2)}_{c-1}(2x^{\frac{1}{2}}) \qquad \operatorname{Im} x < 0.$$

Similar is the proof of

(18) $$\lim_{a \to \infty} \Phi(a, c; x/a) = \Gamma(c)\, x^{\frac{1}{2}-\frac{1}{2}c} I_{c-1}(2x^{\frac{1}{2}}),$$

(19) $$\lim_{a \to \infty} [\Gamma(a-c+1)\, \Psi(a, c; x/a)] = 2x^{\frac{1}{2}-\frac{1}{2}c} K_{c-1}(2x^{\frac{1}{2}}).$$

6.9.2. Other particular confluent hypergeometric functions

The relation

(20) $$\Phi(a, a; x) = e^{x}$$

is obvious, and many other special functions can be expressed in terms of the confluent hypergeometric functions. The first group of such functions consists of *incomplete gamma functions and* related functions. By 6.5(2) and 6.5(6), and by 6.5(7) and 6.3(7),

(21) $$\Gamma(a, x) \equiv \int_x^\infty e^{-t} t^{a-1}\, dt = e^{-x}\, \Psi(1-a, 1-a; x),$$

(22) $$\gamma(a, x) \equiv \Gamma(a) - \Gamma(a, x) = a^{-1} x^{a}\, \Phi(a, a+1; -x).$$

For the *error functions*,

(23) $$\operatorname{Erf}(x) \equiv \int_0^x e^{-t^2}\, dt = \tfrac{1}{2}\gamma(\tfrac{1}{2}, x^2) = x\, \Phi(1/2, 3/2; -x^2),$$

(24) $$\operatorname{Erfc}(x) \equiv \int_x^\infty e^{-t^2}\, dt = \tfrac{1}{2}\Gamma(\tfrac{1}{2}, x^2) = e^{-x^2}\, \Psi(\tfrac{1}{2}, \tfrac{1}{2}; x^2).$$

For the *exponential integral*, the *logarithmic integral*, *integral sine* and *cosine* and the *Fresnel integrals* we have

(25) $-\operatorname{Ei}(-x) \equiv \int_x^\infty e^{-t} t^{-1}\, dt = e^{-x}\, \Psi(1, 1; x)$,

(26) $\operatorname{li}(x) \equiv \int_0^x \frac{dt}{\log t} = \operatorname{Ei}(\log x) = -x\, \Psi(1, 1; -\log x)$,

(27) $\operatorname{Si}(x) \equiv \int_0^x t^{-1} \sin t\, dt = \tfrac{1}{2}\pi - \int_x^\infty t^{-1} \sin t\, dt$

$= \tfrac{1}{2}\pi - \tfrac{1}{2} i e^{-ix}\, \Psi(1, 1; ix) + \tfrac{1}{2} i e^{ix}\, \Psi(1, 1; -ix)$,

(28) $\operatorname{Ci}(x) \equiv -\int_x^\infty t^{-1} \cos t\, dt = -\tfrac{1}{2} e^{-ix}\, \Psi(1, 1; ix) - \tfrac{1}{2} e^{ix}\, \Psi(1, 1; -ix)$,

(29) $C(x) \equiv 2^{-1/2}\, \pi^{-1/2} \int_0^x t^{-1/2} \cos t\, dt$

$= \pi^{-1/2}\, 2^{-1/2}\, x^{1/2}\, [\Phi(1/2, 3/2; -xe^{i\pi/2}) + \Phi(1/2, 3/2; -xe^{-i\pi/2})]$,

(30) $S(x) \equiv 2^{-1/2}\, \pi^{-1/2} \int_0^x t^{-1/2} \sin t\, dt$

$= \pi^{-1/2}\, 2^{-1/2}\, x^{1/2}\, i\, [\Phi(1/2, 3/2; -xe^{i\pi/2}) - \Phi(1/2, 3/2; -xe^{-i\pi/2})]$.

The *parabolic cylinder functions* of Chapter VIII are also confluent hypergeometric functions:

(31) $D_\nu(x) = 2^{\frac{1}{2}\nu}\, e^{-\frac{1}{4}x^2}\, \Psi(-\tfrac{1}{2}\nu, \tfrac{1}{2}; \tfrac{1}{2}x^2)$

$= 2^{\frac{1}{2}\nu - \frac{1}{2}}\, e^{-\frac{1}{4}x^2}\, x\, \Psi(\tfrac{1}{2} - \tfrac{1}{2}\nu, 3/2; \tfrac{1}{2}x^2)$.

Special cases of these are the *Hermite polynomials:*

(32) $H_n(2^{-1/2} x) = 2^{\frac{1}{2}n}\, e^{\frac{1}{4}x^2}\, D_n(x) = 2^{n-\frac{1}{2}}\, x\, \Psi(\tfrac{1}{2} - \tfrac{1}{2}n, 3/2; \tfrac{1}{2}x^2)$

where n is a non-negative integer.

Again $\Psi(a, 0; x) = x\, \Psi(a + 1, 2; x)$ is connected with *Bateman's function* $k_\nu(\tfrac{1}{2}x)$ with $\nu = \tfrac{1}{2} - \tfrac{1}{2}a$. Originally Bateman put

(33) $k_\nu(x) = 2\pi^{-1} \int_0^{\frac{1}{2}\pi} \cos(x \tan\theta - \nu\theta)\, d\theta$

for real values of x and ν. Then

(34) $\Gamma(\nu + 1)\, k_{2\nu}(x) = e^{-x}\, \Psi(-\nu, 0; 2x)$ for $x > 0$,

and it is useful to consider this as the definition of the k-function in the cut plane. With Bateman's original definition, $k_\nu(-x) = k_{-\nu}(x)$. This formula no longer holds when the definition (34) is adopted. Instead we have

(35) $k_{2\nu}(-\xi \pm i0) = k_{-2\nu}(\xi) - e^{\pm\nu\pi i} e^{\xi}\, \Phi(-\nu, 0; -2\xi) \qquad \xi > 0.$

If a is zero or a negative integer, Φ and Ψ are polynomials in x, related to the generalized *Laguerre polynomials* of Chapter X,

$$(36)\quad L_n^{a}(x) = \frac{(a+1)_n}{n!}\,\phi(-n,\ a+1;\ x) = \frac{(-1)^n}{n!}\,\Psi(-n,\ a+1;\ x)$$

$$n = 0,\ 1,\ 2,\ \ldots .$$

The so-called *Laguerre functions*, for unrestricted values of ν, are an alternative notation for confluent hypergeometric functions (Pinney 1946)

$$(37)\quad L_\nu^{(\alpha)}(x) = \frac{1}{\Gamma(\nu+1)}\,\Phi(-\nu,\ a+1,\ x)$$

If $c - a - 1$ is a non-negative integer, $x^{c-1}\,\Psi(a,\ c;\ x)$ is a polynomial and

$$(38)\quad \Psi(a,\ a+n+1;\ x) = x^{-a-n}\,\Psi(-n,\ 1-a-n;\ x)$$

$$= (a)_n\, x^{-a-n}\,\Phi(-n,\ 1-a-n;\ x) = (a)_n\, x^{-a-n}\, L_n^{(-a-n)}(x)$$

$$n = 0,\ 1,\ 2,\ \ldots .$$

It can be proved (Erdélyi 1937f) that the confluent equation has a solution which is a finite combination of elementary functions if and only if either a or $c - a$ is an integer.

Of recent importance are the *Poisson-Charlier polynomials* (Szegö 1939) arising in the calculus of probability. They can be expressed by means of the confluent hypergeometric function as

$$(39)\quad p_n(x) = a^{-\frac{1}{2}n}\,(n!)^{-\frac{1}{2}}\,(x-n+1)_n\,\Phi(-n,\ x-n+1;\ a).$$

The so-called *Toronto functions* (Heatley 1943) are defined by

$$(40)\quad T(m,\ n,\ x) = x^{2n-m+1}\,e^{-x^2}\,\frac{\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(1+n)}\,\phi(\tfrac{1}{2}m+\tfrac{1}{2},\ n+1;\ x).$$

The following table gives information about some of the more important particular and limiting cases of confluent hypergeometric functions.

SPECIAL CASES OF THE CONFLUENT HYPERGEOMETRIC FUNCTION

Parameters	Functions
$a = 1,\ 2,\ \ldots,$	Incomplete gamma functions and their particular cases
$a = 0,\ -1,\ -2,\ \ldots,$	Laguerre polynomials
$a \to \infty$	Bessel functions
$c = 0,\quad c = 2$	Bateman's k-function
$c = \frac{1}{2},\quad c = 3/2$	Parabolic cylinder functions
$c = a$	Elementary functions, incomplete gamma functions

SPECIAL CASES OF THE CONFLUENT HYPERGEOMETRIC FUNCTION
(Continued)

Parameters	Functions
$c - a = 1, 2, 3, \ldots,$	Φ Laguerre polynomials, Ψ Incomplete gamma functions
$c = 2a$	Bessel functions
$c = a = 1$	Exponential integral and related functions
$a = 1/2, \quad c = 3/2$	Error functions and related functions

6.10. Laplace transforms and confluent hypergeometric functions

We shall use the notation

(1) $$L\{F(t); s\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$$

for the Laplace transform of $F(t)$, and shall occasionally write more briefly $L\{F\}$. From the theory of the Laplace transformation we quote the product theorem,

(2) $$L\{\int_0^t F_1(u) F_2(t-u)\, du\} = L\{F_1(t)\} L\{F_2(t)\},$$

valid, for instance, if $L\{F_1\}$ and $L\{F_2\}$ converge absolutely, and the complex inversion formula

(3) $$F(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f(s)\, e^{st}\, ds$$

valid, e.g., if $L\{F\}$ is absolutely convergent for Re $s = \gamma$, and $F(x)$ is of bounded variation and continuous in some neighborhood of t. The infinite integral is in general a Cauchy Principal Value, that is $\lim \int_{\gamma - iA}^{\gamma + iA}$ as $A \to \infty$. However, if $f(s)$ is absolutely integrable on Re $s = \gamma$, we may write simply $\int_{\gamma - i\infty}^{\gamma + i\infty}$ in (3).

There are many pairs of Laplace transforms in which confluent hypergeometric functions occur. We may rewrite 6.5(2) as

(4) $$L\{t^{a-1}(1+t)^{c-a-1}\} = \Gamma(a)\, \Psi(a, c; x) \qquad \text{Re } a > 0, \quad \text{Re } s > 0.$$

We also have, for Re $b > 0$, and Re $s >$ Max(0, Re k),

(5) $$L\{t^{b-1} \Phi(a, c; kt)\} = \Gamma(b)\, s^{-b} F(a, b: c; ks^{-1}) \qquad |s| > |k|,$$
$$= \Gamma(b)(s-k)^{-b} F[c-a, b; c; k/(k-s)] \qquad |s-k| > |k|.$$

The first form can be obtained either by term by term integration of the power series for Φ, or from 6.5(1) and Euler's integral for Gauss' series; the second form follows by Euler's transformation of the first form. If k is negative real, we may make $s \to 0$ in (5) provided that $\operatorname{Re} a > \operatorname{Re} b$. With $b = c$, (5) assumes the simpler form

$$(6) \quad L\{t^{c-1}\,\Phi(a,\,c;\,t)\} = \Gamma(c)\,s^{-c}\,(1-s^{-1})^{-a}$$

$$\operatorname{Re} c > 0, \quad \operatorname{Re} s > 1.$$

From (5), 6.5(7), and 2.9(33)

$$(7) \quad L\{t^{b-1}\,\Psi(a,\,c;\,t)\}$$

$$= \frac{\Gamma(b)\,\Gamma(b-c+1)}{\Gamma(a+b-c+1)}\,F(b,\,b-c+1;\,a+b-c+1;\,1-s^{-1})$$

$$\operatorname{Re} b > 0, \quad \operatorname{Re} c < \operatorname{Re} b + 1, \quad |1-s| < 1,$$

$$= \frac{\Gamma(b)\,\Gamma(b-c+1)}{\Gamma(a+b-c+1)}\,s^{-b}\,F(a,\,b;\,a+b-c+1;\,1-s^{-1}) \quad \operatorname{Re} s > \tfrac{1}{2}.$$

The result can be written in several equivalent forms, and can be extended by analytic continuation to the half-plane $\operatorname{Re} s > 0$. If $\operatorname{Re} a > \operatorname{Re} b$, we may make $s \to +0$ in (7).

Another Laplace transform pair is given by Weber's first exponential integral

$$(8) \quad L\{t^{a-\frac{1}{2}\nu-1}\,J_\nu(2t^{\frac{1}{2}})\} = \frac{\Gamma(a)}{\Gamma(\nu+1)}\,s^{-a}\,\Phi(a,\,\nu+1;\,-s^{-1})$$

$$\operatorname{Re} a > 0, \quad \operatorname{Re} s > 0.$$

Similarly

$$(9) \quad L\{t^{a-\frac{1}{2}\nu-1}\,K_\nu(2t^{\frac{1}{2}})\} = \tfrac{1}{2}\,\Gamma(a)\,\Gamma(a-\nu)\,s^{-a}\,\Psi(a,\,\nu+1;\,s^{-1})$$

$$\operatorname{Re} a > 0, \quad \operatorname{Re} a > \operatorname{Re} \nu, \quad \operatorname{Re} s > 0.$$

The relation

$$(10) \quad L\{t^{c-1}\,F(a,\,b;\,c;\,-t)\} = \Gamma(c)\,s^{a-c}\,\Psi(a,\,a-b+1;\,s)$$

$$\operatorname{Re} c > 0, \quad \operatorname{Re} s > 0$$

is equivalent to an integral representation of $W_{\kappa,\mu}$ given by C. S. Meijer.

Another transform pair is

$$(11) \quad L\{e^{-t^2}\,t^{2c-2}\,\Phi(a,\,c;\,t^2)\} = 2^{1-2c}\,\Gamma(2c-1)\,\Psi(c-\tfrac{1}{2},\,a+\tfrac{1}{2};\,\tfrac{1}{4}s^2)$$

$$\operatorname{Re} c > \tfrac{1}{2}, \quad \operatorname{Re} s > 0.$$

In Whittaker's notation, the principal formulas read:

$$(12) \quad L\{e^{\frac{1}{2}t}\,t^{a}M_{\kappa,\mu}(t)\} = \Gamma(a+\mu+3/2)\,s^{-a-\mu-3/2}$$

$$\times\,F(a+\mu+3/2,\,\mu-\kappa+1/2;\,2\mu+1;\,s^{-1})$$

$$\operatorname{Re}(a+\mu+3/2) > 0, \quad \operatorname{Re} s > 0,$$

(13) $L\{e^{\lambda t} t^{\mu-\frac{1}{2}} M_{\kappa,\mu}(t)\} = \Gamma(2\mu+1)(s-\lambda-\tfrac{1}{2})^{\kappa-\mu-\frac{1}{2}}(s-\lambda+\tfrac{1}{2})^{-\kappa-\mu-\frac{1}{2}}$

$\operatorname{Re}\mu > -\tfrac{1}{2}, \quad \operatorname{Re}s > \operatorname{Re}\lambda - \tfrac{1}{2},$

(14) $L\{e^{\frac{1}{2}t} t^{a} W_{\kappa,\mu}(t)\}$

$$= \frac{\Gamma(a+\mu+3/2)\,\Gamma(a-\mu+3/2)}{\Gamma(a-\kappa+2)}\, s^{-a-\mu-3/2}$$

$$\times F(a+\mu+3/2,\ \mu-\kappa+1/2;\ a-\kappa+2;\ 1-s^{-1})$$

$\operatorname{Re}(a \pm \mu + 3/2) > 0, \quad \operatorname{Re}s > 0.$

From the product theorem, (2), and (6) we have the *integral addition theorem* for Φ:

(15) $$\int_0^t \frac{u^{c-1}}{\Gamma(c)}\,\Phi(a, c; u)\,\frac{(t-u)^{c'-1}}{\Gamma(c')}\,\Phi(a', c'; t-u)\,du$$

$$= \frac{t^{c+c'-1}}{\Gamma(c+c')}\,\Phi(a+a', c+c'; t) \qquad \operatorname{Re}c > 0, \quad \operatorname{Re}c' > 0.$$

The particular case $a' = 0$, or

(16) $$\int_0^t u^{\gamma-1}(t-u)^{c-\gamma-1}\,\Phi(a, \gamma; u)\,du$$

$$= \frac{\Gamma(\gamma)\,\Gamma(c-\gamma)}{\Gamma(c)}\,\Phi(a, c; t) \qquad \operatorname{Re}c > \operatorname{Re}\gamma > 0$$

may be mentioned separately.

6.11. Integral representations

In this section the basic integral representations will be generalized by the introduction of contour integrals, and further integral representations will be listed.

6.11.1. The Φ function

The integral representation 6.5(1) is based on Euler's integral of the first kind, 1.5(1), for the beta function. The restrictions imposed upon the parameters can be removed, partly or entirely, by the introduction of complex integrals, as in section 1.6. Using 1.6(6), 1.6(7), or 1.6(8), instead of 1.5(1), we arrive at the integral representations

(1) $$\Phi(a, c; x) = (2\pi i)^{-2} e^{-i\pi c}\,\Gamma(1-a)\,\Gamma(c)\,\Gamma(1+a-c)$$

$$\times \int^{(1+,0+,1-,0-)} e^{xt}\, t^{a-1}(1-t)^{c-a-1}\,dt,$$

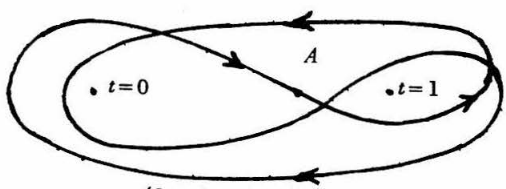
(2) $$\Phi(a, c; x) = (2\pi i)^{-1}\,\Gamma(c)\,\Gamma(a-c+1)/\Gamma(a) \times \int_0^{(1+)} e^{xt}\,t^{a-1}\,(t-1)^{c-a-1}\,dt \qquad \text{Re } a > 0,$$

and

(3) $$\Phi(a, c; x) = -(2\pi i)^{-1}\,\Gamma(c)\,\Gamma(1-a)/\Gamma(c-a) \times \int_1^{(0+)} e^{xt}(-t)^{a-1}\,(1-t)^{c-a-1}\,dt \qquad \text{Re}(c-a) > 0.$$

In (1), the contour of integration is a double loop starting at a point A between 0 and 1 on the real t axis, with $\arg t = \arg(1-t) = 0$ at A, encircling first $t = 1$ in the positive sense, then $t = 0$ in the positive sense, then $t = 1$ in the negative sense, and finally $t = 0$ in the negative sense, returning to A.

In (2), the contour is a loop starting (and ending) at $t = 0$ and encircling 1 once in the positive sense. Similarly in (3), with 0 and 1 interchanging roles. All powers have their principal values in (2) and (3).



(1+, 0+, 1−, 0−)

The Φ function can also be represented in terms of Bessel functions. From 6.10(16) with $\gamma = 2a$, and 6.9(10),

(4) $$\Gamma(a)\,\Gamma(c-2a)\,\Phi(a, c; x) = \pi^{1/2}\,\Gamma(c)\,x^{1/2-a} \times \int_0^1 e^{1/2xt}\,t^{a-1/2}\,(1-t)^{c-2a-1}\,I_{a-1/2}\,(\tfrac{1}{2}xt)\,dt$$

$$\text{Re } c > 2, \qquad \text{Re } a > 0,$$

and from 6.10(8), with a slight change in notation,

(5) $$\Gamma(c-a)\,\Phi(a, c; x) = \Gamma(c)\,e^{x}\,x^{1/2-1/2c} \times \int_0^{\infty} e^{-t}\,t^{1/2c-a-1/2}\,J_{c-1}\,[2(xt)^{1/2}]\,dt$$

$$\text{Re } c > \text{Re } a > 0, \qquad \text{Re } x > 0.$$

Further integral representations may be obtained by using the complex inversion formula, 6.10(3), in connection with the Laplace transforms 6.10(5), with $k = 1$, and 6.10(6).

(6) $$\Phi(a, c; t) = (2\pi i)^{-1}\,\Gamma(b)\,t^{1-b} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}s^{-b}\,F(a, b; c; s^{-1})\,ds$$

$$\text{Re } b > 0, \qquad \gamma > 1,$$

(7) $\Phi(a, c; t) = (2\pi i)^{-1} \Gamma(c)\, t^{1-c} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} s^{-c} (1 - s^{-1})^{-a}\, ds$

$\mathrm{Re}\, c > 0, \quad \gamma > 1.$

If $b = n + 1$, $n = 0, 1, \dots$, in (6), the integrand is a one-valued function of s, and the path of integration may be replaced by a closed contour, for instance, by a circle $|s| = \rho > 1$.

(8) $\Phi(a, c; t) = (2\pi i)^{-1}\, n!\, t^{-n} \int_C e^{st} s^{-n-1} F(a, n+1; c; s^{-1})\, ds$

$n = 0, 1, 2, \dots .$

6.11.2. The Ψ function

A similar discussion of the Ψ function leads to the following integral representations:

(9) $\Psi(a, c; x) = (2\pi i)^{-1} e^{-a\pi i} \Gamma(1-a) \int_{\infty e^{i\phi}}^{(0+)} e^{-xt} t^{a-1} (1+t)^{c-a-1} dt$

$-\tfrac{1}{2}\pi < \phi + \arg x < \tfrac{1}{2}\pi$, $\quad \arg t = \phi$ at the beginning of the loop,

(10) $\Gamma(a)\Gamma(a - c + 1)\Psi(a, c; x)$

$= 2x^{\frac{1}{2} - \frac{1}{2}c} \int_0^\infty e^{-t} t^{a - \frac{1}{2}c - \frac{1}{2}} K_{c-1}[2(xt)^{\frac{1}{2}}]\, dt$

$\mathrm{Re}\, a > 0, \quad \mathrm{Re}(a - c) > -1,$

(11) $\Gamma(b)\Psi(a, c; x)$

$= x^{a-b} \int_0^\infty e^{-xt} t^{b-1} F(a, a - c + 1; b; -t)\, dt$

$\mathrm{Re}\, b > 0, \quad \mathrm{Re}\, x > 0,$

(12) $\Gamma(a + b - c + 1)\Psi(a, c; t) = (2\pi i)^{-1}\Gamma(b)\Gamma(b - c + 1)\, t^{1-b}$

$\times \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} s^{-b} F(a, b; a + b - c + 1; 1 - s^{-1})\, ds$

$\mathrm{Re}\, b > 0, \quad \mathrm{Re}(b - c) > -1, \quad \gamma > \tfrac{1}{2}.$

For $x > 0$, the following integral representation can be derived from 6.5(2). We assume $\mathrm{Re}\, a > 0$, $\mathrm{Re}\, c < 1$, $x > 0$. Then it is permissible to deform the path of integration into the segment of the real axis from $t = 0$ to $t = -\tfrac{1}{2}$ and the ray from $t = -\tfrac{1}{2}$ to $t = -\tfrac{1}{2} + i\infty$. Along $(0, -\tfrac{1}{2})$ we put $t = ue^{i\pi}$; along $(-\tfrac{1}{2}, -\tfrac{1}{2} + i\infty)$ we put

$$t = \tfrac{1}{2} e^{i(\pi - \theta)} \sec\theta, \quad 1 + t = \tfrac{1}{2} e^{i\theta} \sec\theta \; (0 \le \theta \le \tfrac{1}{2}\pi).$$

Thus we find

$$e^{-i\pi a}\Gamma(a)\Psi(a, c; x) = \int_0^{\frac{1}{2}} e^{-ux} u^{a-1} (1-u)^{c-a-1}\, du$$
$$- i\, 2^{1-c} \int_0^{\frac{1}{2}\pi} (\cos\theta)^{-c} \exp[\tfrac{1}{2}x - \tfrac{1}{2} i x \tan\theta + (c - 2a) i\theta]\, d\theta.$$

A corresponding formula can be obtained when the path of integration is deformed into the segment $(0, -\tfrac{1}{2})$, along which now $t = ue^{-i\pi}$, and the ray $(-\tfrac{1}{2}, -\tfrac{1}{2} - i\infty)$, along which $t = \tfrac{1}{2} e^{i(\theta-\pi)} \sec\theta$. Subtracting the two formulas,

$$(13)\quad \pi\,\Psi(a, c; x) = 2^{1-c}\,\Gamma(1-a)\, e^{\frac{1}{2}x} \times \int_0^{\frac{1}{2}\pi} (\cos\theta)^{-c} \cos[\tfrac{1}{2}x\tan\theta + (2a-c)\,\theta]\, d\theta$$

$$x > 0, \quad \operatorname{Re} c < 1, \quad a \neq 1, 2, \ldots .$$

The condition $\operatorname{Re} a > 0$, used in the derivation, can be discarded by analytic continuation. However, a new restriction on a must be introduced in order to avoid the poles of the gamma function. Formula (13) corresponds to the integral representation 6.9(29) of Bateman's k function.

Two further integral representations, due to C. S. Meijer (1938, 1941), are

$$(14)\quad \Psi(a, c; x) = x^{a-\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2}x\cosh 2t}\, P_\nu^\mu(\cosh t)\,(2\sinh t)^{1-\mu}\, dt$$

$$\mu = c - 2a + \tfrac{1}{2}, \quad \nu = c - 3/2, \quad \operatorname{Re} x > 0, \quad \operatorname{Re}\lambda < 1,$$

and

$$(15)\quad \Psi(a, c; x) = \pi^{-\frac{1}{2}}\, 2^{a-\frac{1}{2}c+1}\, e^{\frac{1}{2}x}\, x^{\frac{1}{2}-\frac{1}{2}c} \times \int_0^\infty e^{-\frac{1}{2}(\sinh t)^2}\, D_{c-2a}[(2x)^{\frac{1}{2}}\cosh t]\cosh[(c-1)\,t]\, dt$$

$$\operatorname{Re} x > 0.$$

6.11.3. Whittaker functions

The basic integral representations for Whittaker's confluent hypergeometric functions are:

$$(16)\quad \Gamma(\tfrac{1}{2}-\kappa+\mu)\,\Gamma(\tfrac{1}{2}+\kappa+\mu)\, M_{\kappa,\mu}(x) = 2^{-2\mu}\,\Gamma(2\mu+1)\, x^{\mu+\frac{1}{2}} \int_{-1}^{1} e^{\frac{1}{2}xt}(1-t)^{-\frac{1}{2}+\kappa+\mu}(1+t)^{-\frac{1}{2}-\kappa+\mu}\, dt$$

$$\operatorname{Re}(\mu \pm \kappa) > -\tfrac{1}{2},$$

$$(17)\quad \Gamma(\tfrac{1}{2}+\kappa+\mu)\, M_{\kappa,\mu}(x) = \Gamma(2\mu+1) e^{\frac{1}{2}x} x^{\frac{1}{2}} \int_0^\infty e^{-t}\, t^{\kappa-\frac{1}{2}}\, J_{2\mu}[2(tx)^{\frac{1}{2}}]\, dt$$

$$\operatorname{Re}(\mu+\kappa) > -\tfrac{1}{2},$$

$$(18)\quad \Gamma(\tfrac{1}{2}-\kappa+\mu)\, W_{\kappa,\mu}(x) = e^{-\frac{1}{2}x}\, x^{\mu+\frac{1}{2}} \int_0^\infty e^{-tx}\, t^{-\frac{1}{2}-\kappa+\mu}(1+t)^{-\frac{1}{2}+\kappa+\mu}\, dt$$

$$\operatorname{Re}(\mu-\kappa) > -\tfrac{1}{2}, \quad \operatorname{Re} x > 0,$$

and

(19) $$W_{\kappa,\mu}(x) = (2\pi i)^{-1} e^{-\frac{1}{2}x} x^{\mu} \times \int_{-\infty i}^{\infty i} \frac{\Gamma(s)\,\Gamma(\frac{1}{2}-\kappa-\mu-s)\,\Gamma(\frac{1}{2}-\kappa+\mu-s)}{\Gamma(\frac{1}{2}-\kappa-\mu)\,\Gamma(\frac{1}{2}-\kappa+\mu)} x^{s}\, ds \qquad -\frac{3}{2}\pi < \arg x < \frac{3}{2}\pi,$$

where neither $\frac{1}{2}+\kappa+\mu$ nor $\frac{1}{2}+\kappa-\mu$ is a positive integer, and the path of integration separates the poles of $\Gamma(s)$ from the poles of

$$\Gamma(\tfrac{1}{2}-\kappa-\mu-s)\,\Gamma(\tfrac{1}{2}-\kappa+\mu-s).$$

6.12. Expansions in terms of Laguerre polynomials and Bessel functions

Beside the power series expansions, there are other useful expansions of confluent hypergeometric functions.

We put $\phi = 0$, $c = a+1$, $t = u/(1-u)$, in 6.11 (9) and obtain

(1) $$\Psi(a, a+1; x) = (2\pi i)^{-1} e^{i\pi a}\,\Gamma(1-a) \times \int_{1}^{(0+)} e^{-xu/(1-u)} u^{a-1} (1-u)^{-a-1}\, du.$$

Now,

(2) $$(1-u)^{-a-1} e^{-xu/(1-u)} = \sum_{n=0}^{\infty} L_n^{(a)}(x)\, u^n \qquad |u| < 1$$

is the well-known generating function of generalized Laguerre polynomials [Szegö 1939 (5.1.9)]. Since this series converges only inside the unit circle $|u| < 1$, we write $\int_1^{(0+)} = \lim \int_v^{(0+)}$, $v \to 1-$, and substitute (2). Since

$$\int_v^{(0+)} u^{a+n-1}\, du = 2ie^{i\pi a}\sin(a\pi)\, v^{n+a}/(n+a),$$

we thus obtain

$$\Gamma(a)\,\Psi(a, a+1; x) = \lim_{v\to 1-} v^{a} \sum_{n=0}^{\infty} (n+a)^{-1} v^{n} L_n^{(a)}(x)$$

and by Abel's theorem on the continuity of power series,

(3) $$\Gamma(a)\,\Psi(a, a+1; x) = \sum_{n=0}^{\infty} (n+a)^{-1} L_n^{(a)}(x)$$

whenever the infinite series converges. From the known estimate of generalized Laguerre polynomials (Szegö 1939 Theorem 7.6.4) it is seen that the series is convergent in any finite positive interval of x if $a < \frac{1}{2}$. On the other hand, using the transformation 6.5 (6),

(4) $\Gamma(a-\alpha)\,\Psi(a,\,\alpha+1;\,x) = x^{-\alpha}\sum_{n=0}^{\infty}(n+a-\alpha)^{-1}\,L_n^{(-\alpha)}(x)$

where the series converges (for positive x) if $a > -\tfrac{1}{2}$. Outside the positive real axis both series are divergent. The expansions (2) and (4) are due to Tricomi. Another expansion in terms of Laguerre polynomials

(5) $\Phi(a,\,c;\,xy(x-1)^{-1}] = (1-x)^{a}\sum_{n=0}^{\infty}\dfrac{(a)_n}{(c)_n}\,L_n^{(c-1)}(y)\,x^{n}$ $\qquad |x| < 1, \quad y > 0$

is a generalization of (2) and, with $x = \tfrac{1}{2}$, it provides an expansion of Φ on the negative real axis in terms of Laguerre polynomials. It is due to A. Erdélyi [1937 a, equation (5, 7)].

Tricomi (1947, 1949) has given two expansions of Φ in terms of Bessel functions. These expansions are useful for the investigation of the behavior of Φ when the parameters are large. The first expansion is

(6) $\Phi(-a,\,\alpha+1;\,x) = \Gamma(\alpha+1)\,(ax)^{-\frac{1}{2}\alpha}e^{hx}\sum_{m=0}^{\infty}A_m(h)\,(x/a)^{\frac{1}{2}m}\,J_{\alpha+m}[2(ax)^{\frac{1}{2}}]$

where a is real, $h \geq 0$, and the coefficients A_m are determined by the generating function

(7) $\sum_{m=0}^{\infty}A_m(h)\,z^{m} = e^{az}\,[1+(h-1)\,z]^{a}\,(1+hz)^{-a-\alpha-1}$.

From the asymptotic representation of Bessel functions for large order it is seen that the infinite series in (6) converges as

$$\sum_{m=0}^{\infty}A_m(h)\,x^{m}/\Gamma(\alpha+m+1)$$

and hence is absolutely and uniformly convergent in every bounded region of the complex x plane. From (7) we have

(8) $A_0 = 1, \quad A_1 = -(\alpha+1)\,h, \quad A_2 = (h-\tfrac{1}{2})\,a + \tfrac{1}{2}(\alpha+1)\,(\alpha+2)\,h^{2}$

and the recurrence relation

(9) $(m+1)\,A_{m+1} = [(1-2h)\,m - h(\alpha+1)]\,A_m - [(1-2h)\,a$

$\qquad + h(h-1)\,(\alpha+m)]\,A_{m-1} + h(h-1)\,a\,A_{m-2} \qquad m = 2, 3, 4, \ldots,$

for the computation of the later A. Also,

(10) $A_m(0) = (-1)^{m}\,L_m^{(a-m)}(a) \quad A_m(1) = L_m^{-(a+m+\alpha+1)}(-a),$

and if $a = n = 0, 1, 2, \ldots$, also

$$A_m(h) = h^{m-n} \sum_{k=0}^{n} \binom{n}{k} (h-1)^{n-k} L_m^{-(k+m+\alpha+1)}(-a/h).$$

The A are polynomials in a, α, h, and the degree of A_m in a is $[\frac{1}{2}m]$ if $h \neq \frac{1}{2}$, and $[m/3]$ if $h = \frac{1}{2}$ ($[b]$ is the largest integer $\leq b$). The most suitable choice for h is $\frac{1}{2}$.

Tricomi's second expansion is

(11) $$e^{-\frac{1}{2}x}\,\Phi(a, \alpha+1; x) = \Gamma(\alpha+1)\,(\kappa x)^{-\frac{1}{2}\alpha}$$
$$\times \sum_{n=0}^{\infty} A_n(\kappa, \tfrac{1}{2}+\tfrac{1}{2}\alpha)\,(\tfrac{1}{4}\kappa^{-1}x)^{\frac{1}{2}n}\,J_{\alpha+n}(2\kappa^{\frac{1}{2}}x^{\frac{1}{2}})$$

where $\kappa = \frac{1}{2} + \frac{1}{2}\alpha - a$ is Whittaker's parameter, and the $A_m(\kappa, \lambda)$ are coefficients in the expansion

(12) $$e^{2\kappa z}(1-z)^{\kappa-\lambda}(1+z)^{-\kappa-\lambda} = \sum_{n=0}^{\infty} A_n(\kappa, \lambda)\,z^n \qquad |z| < 1.$$

Tricomi (1950) proved that, for reasons similar to those advanced in connection with (6), the infinite series in (11) is convergent in the entire x-plane. Moreover, (11) can be used for the approximate computation of the Φ function for large κ. From (12) we have

(13) $$A_0(\kappa, \lambda) = 1, \quad A_1(\kappa, \lambda) = 0, \quad A_2(\kappa, \lambda) = \lambda,$$
$$A_3(\kappa, \lambda) = -\frac{2}{3}\kappa, \quad A_4(\kappa, \lambda) = \tfrac{1}{2}(\lambda)_2,$$
$$A_5(\kappa, \lambda) = -2\left(\frac{1}{3}\lambda + \frac{1}{5}\right)\kappa, \quad A_6(\kappa, \lambda) = \frac{1}{3!}(\lambda)_3 + \frac{2}{9}\kappa^2,$$
$$(n+1)A_{n+1}(\kappa, \lambda) = (n+2\lambda-1)A_{n-1}(\kappa, \lambda) - 2\kappa A_{n-2}(\kappa, \lambda)$$
$$n = 2, 3, \ldots,$$

and A_n is a polynomial of degree $[n/3]$ in κ. Also

(14) $$A_n(-\kappa, \lambda) = (-1)^n A_n(\kappa, \lambda)$$

For further details about these coefficients see Tricomi (1949).

6.13. Asymptotic behavior

The asymptotic behavior of confluent hypergeometric functions is different according as the large quantity is the variable, one of the parameters, or two or all three of these quantities. A complete set of results is not yet available.

The investigations are based either on integral representations, or directly on the differential equation, or else on suitable expansions in infinite series.

6.13.1. Behavior for large $|x|$

The behavior of confluent hypergeometric functions as $x \to \infty$ can be investigated by means of the basic integral representations. If in 6.5(3) we put

$$(1+t)^{c-a-1} = \sum_{n=0}^{N} (-1)^n (a-c+1)_n \frac{t^n}{n!} + R_N$$

and estimate the integral involving R_N, it can be shown that

(1) $$\Psi(a, c; x) = \sum_{n=0}^{N} (-1)^n \frac{(a)_n (a-c+1)_n}{n!} x^{-a-n} + O(|x|^{-a-N-1})$$
$$N = 0, 1, 2, \ldots, \quad x \to \infty, \quad -\frac{3}{2}\pi < \arg x < \frac{3}{2}\pi.$$

The same result may be obtained from 6.5(5) by shifting the path of integration to the left, evaluating the residue whenever we pass a pole of $\Gamma(a+s)$, and estimating the remaining integral. Note that this result is in agreement with 6.6(3), showing that the (divergent) formal power series ${}_2F_0$ is the asymptotic expansion in an appropriate sector of the analytic function defined by 6.6(3).

The behavior of Φ can be inferred from 6.7(7).

(2) $$\Phi(a, c; x) = \frac{\Gamma(c)}{\Gamma(c-a)} (e^{i\pi\epsilon}/x)^a \sum_{n=0}^{M} \frac{(a)_n (a-c+1)_n}{n!} (-x)^{-n}$$
$$+ O(|x|^{-a-M-1}) + \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c} \sum_{n=0}^{N} \frac{(c-a)_n (1-a)_n}{n!} x^{-n}$$
$$+ O(|e^x x^{a-c-N-1}|) \qquad M, N = 0, 1, 2, \ldots,$$
$$\epsilon = 1 \text{ if } \operatorname{Im} x > 0, \quad \epsilon = -1 \text{ if } \operatorname{Im} x < 0, \quad x \to \infty, \quad -\pi < \arg x < \pi.$$

In particular, as $\operatorname{Re} x \to \infty$,

(3) $$\Phi(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)} e^x x^{a-c} [1 + O(|x|^{-1})],$$

and as $\operatorname{Re} x \to -\infty$,

$$\Phi(a, c; x) = \frac{\Gamma(c)}{\Gamma(c-a)} (-x)^{-a} [1 + O(|x|^{-1}].$$

6.13.2. Large parameters

If $c \to \infty$, while a and x are bounded, 6.1(1) describes the behavior of Φ. In particular,

(4) $$\Phi(a, c; x) = 1 + O(|c|^{-1}) \qquad a, x \text{ bounded}, \quad c \to \infty.$$

If $c \to \infty$ while $c - a$ and x are bounded, 6.3(7) provides such a description. In particular,

(5) $\Phi(a, c; x) = e^x [1 + O(|c|^{-1})]$

$$c - a, \quad x \text{ bounded}, \quad c \to \infty.$$

The behavior of Ψ in this case can be investigated by means of 6.5(7), 1.18(3), and 1.18(4).

(6) $\Psi(a, c; x) = (-c)^{-a} [1 + O(|c|^{-1})]$

$$+ (2\pi)^{½} x^{1-c} \exp[x - c + (c - 3/2) \log c] [1 + O(|c|^{-1})]$$

$$a, x \text{ bounded}, \quad c \to \infty, \quad |\arg c| \leq \pi - \epsilon, \quad |\arg(-c)| \leq \pi - \epsilon, \quad \epsilon > 0.$$

The corresponding result for $c \to \infty$ when $c - a$ and x remain bounded follows from 6.5(6).

The case $a \to \infty$ is more difficult. Perron (1921) based his investigations of this case on integral representations of the Laplace type and used the method of steepest descent. Among later investigators we mention F. Tricomi (1947), who showed that under certain restrictions his expansion 6.12(11) is an asymptotic expansion, and W. C. Taylor (1939) who used E. R. Langer's method of asymptotic representation of solutions of differential equations. Taylor's results are described in 6.13.3. If $|x|$ is bounded, and bounded away from zero, they simplify considerably. In the formulas below, c is bounded, x is bounded and also bounded away from zero, $a \to \infty$. It will be more convenient to express the formulas in terms of

(7) $\kappa = ½c - a.$

It is assumed that $|\arg x - \arg \kappa| \leq \pi$. We then have the following table in which ϵ is any positive number:

	$\arg \kappa$ between	$\kappa^{¼-\kappa} x^{¼+½c} e^{\kappa-½x} \Psi(a, c; x) =$
(8)	$-\pi + \epsilon, \quad \pi - \epsilon$	$2^{½} \cos[\kappa\pi - 2(\kappa x)^{½} - ¼\pi] \cdot [1 + O(\lvert\kappa\rvert^{-½})]$
(9)	$\epsilon, \quad 3\pi - \epsilon$	$½(1+i) \exp[-i\kappa\pi + 2i(\kappa x)^{½}] \cdot [1 + O(\lvert\kappa\rvert^{-½})]$
(10)	$2\pi + \epsilon, \quad 4\pi - \epsilon$	$i\, 2^{½} \cos[\kappa\pi - 2(\kappa x)^{½} + ¼\pi] \cdot [1 + O(\lvert\kappa\rvert^{-½})]$
(11)	$3\pi + \epsilon, \quad 6\pi - \epsilon$	$-½(1-i) \exp[i\kappa\pi - 2i(\kappa x)^{½}] \cdot [1 + O(\lvert\kappa\rvert^{-½})]$

The asymptotic form of Φ, under the same circumstances is obtained from 6.7(7):

(12) $\Phi(a, c; x) = 2^{½} \Gamma(c) e^{½x} \kappa^{¾-½c} x^{¾-½c} \{c_1 e^{2i(\kappa x)^{½}} + c_2 e^{-2i(\kappa x)^{½}}$

$$+ (\kappa x)^{-½} O[\exp|\mathrm{Im}(2\kappa^{½} x^{½})|]\}$$

and with s an integer we have

(13) $c_1 = (2\pi)^{-\frac{1}{2}} e^{i\pi(s-\frac{1}{4})(2c-1)}$
$$(2s-2)\pi + \epsilon \leq \arg(\kappa x)^{\frac{1}{2}} \leq (2s+1)\pi - \epsilon,$$

(14) $c_2 = (2\pi)^{-\frac{1}{2}} e^{i\pi(s+\frac{1}{4})(2c-1)}$
$$(2s-1)\pi + \epsilon \leq \arg(\kappa x)^{\frac{1}{2}} \leq (2s+2)\pi - \epsilon.$$

For Φ Taylor also gives an asymptotic form uniformly valid in the neighborhood of $x = 0$. If κx is bounded,

(15) $$\Phi(a, c; x) = \Gamma(c)\,(\kappa x)^{\frac{1}{2}-\frac{1}{2}c}\, e^{\frac{1}{2}x}\, J_{c-1}[2(\kappa x)^{\frac{1}{2}}] + O(|\kappa|^{-1})$$
$$c,\ \kappa x \text{ bounded},\ \kappa \to \infty.$$

The case where a, c, and $c - a \to \infty$ simultaneously, has not been investigated fully. It is, however, known that if

(16) $$a = \nu A + \alpha, \qquad c - a = \nu B + \beta$$

where α, β, are fixed, possibly complex, numbers, A, B, are fixed positive numbers, and $\nu \to \infty$ through positive values, and if the abbreviations $t = -A/(A+B)$, $u = A(1+t) = AB/(A+B)$ are used, then

(17) $$\Phi(a, c; x) = \frac{(2\pi)^{\frac{1}{2}}\,\Gamma(c)}{(u\nu)^{\frac{1}{2}}\,\Gamma(a)\,\Gamma(c-a)}\, e^{-tx}(-t)^{\alpha}(1+t)^{c-a}[1 + O(\nu^{-1})].$$

See also section 6.13.3.

6.13.3. Variable and parameters large

If a is bounded and c and $x \to \infty$ in such a manner that $|x| < |c|$, the behavior of Φ may be investigated by means of 6.1(1). We put $x = c\xi$, $0 \leq |\xi| \leq 1 - \epsilon\,(\epsilon > 0)$, and use 1.18(4). Then we have as $c \to \infty$

$$\frac{1}{(c)_n} = \frac{\Gamma(c)}{\Gamma(c+n)} = c^{-n}[1 - \tfrac{1}{2}n(n-1)\,c^{-1} + O(|c|^{-2})]$$

and

(18) $$\Phi(a, c; c\xi) = (1-\xi)^{-a}\left[1 - \frac{a(a+1)}{2c}\left(\frac{\xi}{1-\xi}\right)^2 + O(|c|^{-2})\right]$$
$$a \text{ bounded}, \quad |\xi| \leq 1 - \epsilon, \quad \epsilon > 0.$$

The corresponding formula for bounded $c - a$, and c, $x \to \infty$ may be obtained by means of Kummer's transformation, the formula for Ψ by means of 6.5(7).

When a and x increase indefinitely, the behavior of confluent hypergeometric functions is more complicated. W. C. Taylor (1939) has applied E. R. Langer's method to derive the asymptotic forms of confluent hypergeometric functions from the differential equation. We shall use again,

(19) $$\kappa = \tfrac{1}{2}c - a.$$

Taylor introduces the auxiliary variable

(20) $\xi = i\{\tfrac{1}{2}x^{1/2}(x-4\kappa)^{1/2} - \kappa \log[(x^{1/2}+(x-4\kappa)^{1/2})^2/(4\kappa)]\}$,

where the arguments of x, κ, and $x-4\kappa$ are all zero when these quantities are positive, and for other values the arguments are obtained by analytic continuation in such a manner that $|\arg x - \arg \kappa| \leq \pi$ throughout the process. First Taylor investigates the asymptotic behavior of Ψ under the assumption that $\xi \to \infty$ and that there are constants r and N such that $0 < r \leq 1$, $N > 0$, and $|x| > N|\kappa|^{-1+2r}$ as $\kappa \to \infty$. His results for this case are:

	$\arg \xi$ between	$\kappa^{-\kappa} x^{\frac{1}{2}c-\frac{1}{4}}(x-4\kappa)^{\frac{1}{4}} e^{\kappa-\frac{1}{2}x}\Psi(a, c; x) =$
(21)	$-2\pi+\epsilon, \quad -\epsilon$	$(e^{i\xi} - ie^{-i\xi})[1+O(\|\kappa\|^{-r})+O(\|\xi\|^{-1}]$
(22)	$-\pi+\epsilon, \quad 2\pi-\epsilon$	$e^{i\xi}[1+O(\|\kappa\|^{-r})+O(\|\xi\|^{-1})]$
(23)	$\pi+\epsilon, \quad 3\pi-\epsilon$	$(e^{i\xi}+ie^{-i\xi})[1+O(\|\kappa\|^{-r})+O(\|\xi\|^{-1})]$
(24)	$2\pi+\epsilon, \quad 5\pi-\epsilon$	$ie^{-i\xi}[1+O(\|\kappa\|^{-r})+O(\|\xi\|^{-1})]$

For bounded ξ, or $x - 4\kappa = O(|\kappa|^{1/3})$, Taylor has the asymptotic form

(25) $$\kappa^{-\kappa} x^{\frac{1}{2}c-\frac{1}{4}}(x-4\kappa)^{\frac{1}{4}} e^{\kappa-\frac{1}{2}x}\Psi(a, c; x)$$
$$=[2\pi\xi/(3i)]^{\frac{1}{2}}[e^{i\pi/6} J_{-1/3}(\xi) - e^{-i\pi/6} J_{1/3}(\xi)] + O(|\kappa|^{-5/6})$$
$$x - 4\kappa = O(|\kappa|^{1/3}), \qquad \kappa \to \infty.$$

For Φ, Taylor has

(26) $$(\kappa x)^{\frac{1}{2}c-\frac{1}{2}} e^{-\frac{1}{2}x}\Phi(a, c; x)$$
$$=\Gamma(c) J_{c-1}[2(\kappa x)^{\frac{1}{2}}] + \kappa^{-\frac{3}{4}} x^{5/4} O[\exp|\mathrm{Im}(2\kappa^{\frac{1}{2}} x^{\frac{1}{2}})|]$$
$$c, \arg x, \arg \kappa \text{ bounded}, \quad x = O(|\kappa|^{1/3-\epsilon}), \qquad \kappa \to \infty.$$

If κx is large, the Bessel function can be expressed in terms of elementary functions

$$J_c(2\kappa^{\frac{1}{2}} x^{\frac{1}{2}}) = 2^{-\frac{1}{2}}\kappa^{-\frac{1}{4}} x^{-\frac{1}{4}}\{c_1 e^{2i(\kappa x)^{\frac{1}{2}}} + c_2 e^{-2i(\kappa x)^{\frac{1}{2}}}$$
$$+ (\kappa x)^{-\frac{1}{2}} O[\exp|\mathrm{Im}(2\kappa^{\frac{1}{2}} x^{\frac{1}{2}})|]\}$$

where c_1 and c_2 are as in (13) and (14).

Another result for large a and large x is that Tricomi's expansion 6.12(10) provides an asymptotic representation if $\kappa \to \infty$ and $x = O(|\kappa|^{\rho})$ with some $\rho < 1/3$ (Tricomi 1949).

The case when both parameters and the variable are large has not been investigated systematically. Erdélyi (1938 d) applies the method of steepest descents to the integral representations of the Laplace type to discuss the behavior of confluent hypergeometric functions when

(27) $x = \nu X + \xi, \quad a = \nu A + \alpha, \quad c - a = \nu B + \beta$ $\qquad A, B, X$ real,

and either

(28) $A > 0, \quad X > 0$

or

(29) $A > 0, \quad B > 0.$

The real numbers A, B, X, and the possibly complex numbers α, β, ξ, are fixed while $\nu \to \infty$ through positive real values. The quadratic equation in t,

(30) $Xt(t+1) - A(t+1) - Bt = 0$

has two distinct real roots in both cases considered. In the case (28), let t_1 be the (only) positive root; in the case (29) let t_2 be the (only) root between -1 and 0 of (30). Also put

(31) $u_h = A(1+t_h)^2 + Bt_h^2 = (1+t_h)(A + Xt_h^2)$ $\qquad h = 1, 2$

which is positive in both cases. Then we have

(32) $\Gamma(a)\,\Psi(a, c; x) = (2\pi/u_1)^{1/2}\, e^{-t_1 x}\, t_1^{a}(1+t_1)^{c-a}\,[1 + O(\nu^{-1})]$

$A > 0, \quad X > 0, \quad \nu \to \infty$

and

(33) $\Gamma(a)\,\Gamma(c-a)\,\Phi(a, c; x)$

$= \Gamma(c)\,(2\pi/u_2)^{1/2}\, e^{-t_2 x}\,(-t_2)^{a}\,(1+t_2)^{c-a}\,[1 + O(\nu^{-1})]$

$A > 0, \quad B > 0, \quad \nu \to \infty$

which can be proved from 6.5(2) and 6.5(1), respectively.

6.14. Multiplication theorems

Taylor's expansion

$$f(\lambda x) = \sum_{n=0}^{\infty} \frac{(\lambda - 1)^n x^n}{n!} \frac{d^n}{dx^n} f(x)$$

on the one hand, and the expansion

$$\lambda f(\lambda x) = \sum_{n=0}^{\infty} \frac{1}{n!} (1 - \lambda^{-1})^n \frac{d^n}{dx^n} [x^n f(x)],$$

which is a particular case of Lagrange's expansion (E. T. Whittaker and G. N. Watson 1927, section 7.32), on the other hand are sources of the "multiplication theorems" for functions for which either all $d^n f(x)/dx^n$ or all $d^n [x^n f(x)]/dx^n$ are known. Using the formulas of section 6.4 and section 6.6, we thus arrive at the following multiplication formulas:

(1) $\Phi(a, c; \lambda x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!\,(c)_n} (\lambda - 1)^n x^n \Phi(a+n, c+n; x),$

(2) $\Phi(a, c; \lambda x) = \lambda^{1-c} \sum_{n=0}^{\infty} \frac{(1-c)_n}{n!} (1-\lambda)^n \Phi(a, c-n; x),$

(3) $\Phi(a, c; \lambda x) = \lambda^{-a} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (1-\lambda^{-1})^n \Phi(a+n, c; x) \qquad \mathrm{Re}\,\lambda > \tfrac{1}{2},$

(4) $\Psi(a, c; \lambda x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} (1-\lambda)^n x^n \Psi(a+n, c+n; x) \qquad |\lambda - 1| < 1,$

(5) $\Psi(a, c; \lambda x) = \lambda^{1-c} \sum_{n=0}^{\infty} \frac{(a-c+1)_n}{n!} (1-\lambda)^n \Psi(a, c-n; x) \qquad |\lambda - 1| < 1,$

(6) $\Psi(a, c; \lambda x) = \lambda^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (a-c+1)_n}{n!} (1-\lambda^{-1})^n \Psi(a+n, c; x) \qquad |\lambda - 1| < 1, \quad \mathrm{Re}\,\lambda > \tfrac{1}{2}.$

All these formulas can be re-written as addition formulas by putting $\lambda = 1 + y/x$, $\lambda x = x + y$. A further multiplication formula,

(7) $\Phi(a, c; \lambda x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(g+n)_n\, n!} (-x)^n F(-n, g+n; c; \lambda) \times \Phi(a+n, g+2n+1; x),$

was given by Erdélyi (1936 c). Here g is an arbitrary parameter except that it must not be a negative odd integer. The Gauss series F appearing in (7) is a Jacobi polynomial, and becomes an ultraspherical polynomial when $g = 2c - 1$, and a Legendre polynomial when $g = c = 1$.

6.15. Series and integral formulas

A vast number of relationships involving infinite series or integrals of confluent hypergeometric functions is to be found in papers of the last 20 years. No unified theory exists, and a full presentation of the results is impracticable. Only samples of the more interesting results and some references to the literature will be given. The references are far from complete, and further papers will be found, in particular, in English and Indian periodicals.

6.15.1. Series

Many of the series of confluent hypergeometric functions which were investigated have one of the following three forms:

(1) $\sum_{n=0}^{\infty} a_n \Phi(a-n, c; x)$

(2) $\sum_{n=0}^{\infty} \beta_n x^n \Phi(a+n, c+2n; x)$

(3) $\sum_{n=0}^{\infty} \gamma_n x^n \Phi(a, c+n; x)$

A few of the results are shown in the table below

	Coefficients	Sum
(4)	$a_n=(c-c')_n/n!$ $\mathrm{Re}(2c'-c)>\frac{1}{2}$	$\frac{\Gamma(c)}{\Gamma(c')} x^{c'-c} \Phi(a, c'; x)$
(5)	$a_n=t^n(c-c')_n/n!$ $\mathrm{Re}\, c > \mathrm{Re}\, c' > 0$, $\|t\|<1$, $\|\arg x\|<\frac{3}{4}\pi$	$\Gamma(c)[\Gamma(c')\Gamma(c-c')]^{-1}(1-t)^{c'-c} x^{1-c}$ $\times \int_0^x u^{c'-1}(x-u)^{c-c'-1} \Phi(a, c'; u)$ $\times \exp[-(x-u)t/(1-t)]\, du$
(6)	$\beta_n = A_n(t)$, $\|x\|<\|t\|$	$(t-x)^{-1} e^{\frac{1}{2}x}$
(7)	$\beta_n=[(-\frac{1}{2})^n/n!]$ $\times F(-n, a-c-n+1; 2-c; 2)$	$e^{\frac{1}{2}x}$
(8)	$\gamma_n = \frac{\Gamma(\nu+n)\, t^n}{n!\,\Gamma(c+n)}$ $\mathrm{Re}\, c > \mathrm{Re}\, \nu > 0.$	$\frac{x^{1-c}}{\Gamma(c-\nu)} \int_0^x e^{ut} u^{\nu-1}(x-u)^{c-\nu-1}$ $\times \Phi(a, c-\nu; x-u)\, du$

For these and other, related, results see Erdélyi (1936b, c, 1937a, c). In (6),

(9) $A_n(t) = \sum_{m=0}^{n} (-\frac{1}{2}t)^m F(-m, a-c-n+1; 2-c)/m!\,.$

For other series see section 6.15.3.

6.15.2 Integrals

Indefinite integrals with confluent hypergeometric functions follow from the differentiation formulas of sections 6.4 and 6.6. Many definite integrals can be derived from the formulas of section 6.10.

If $k=-1$ and $\mathrm{Re}\, a > \mathrm{Re}\, b > 0$ in 6.10(5) or if $\mathrm{Re}\, a > \mathrm{Re}\, b > 0$ and $\mathrm{Re}\, c > \mathrm{Re}\, b + 1$ in 6.10(7), we may make $s \to 0$. Thus

$$(10)\quad \int_0^\infty t^{b-1}\,\Phi(a, c; -t)\,dt = \frac{\Gamma(b)\,\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)} \qquad 0 < \operatorname{Re} b < \operatorname{Re} a,$$

$$(11)\quad \int_0^\infty t^{b-1}\,\Psi(a, c; t)\,dt = \frac{\Gamma(b)\,\Gamma(a-b)\,\Gamma(b-c+1)}{\Gamma(a)\,\Gamma(a-c+1)}$$

$$0 < \operatorname{Re} b < \operatorname{Re} a, \qquad \operatorname{Re} c < \operatorname{Re} b + 1.$$

These are the Mellin-inversions of 6.5(4) and 6.5(5).

Other integral formulas are

$$(12)\quad \int_0^\infty \cos(2xy)\,\Phi(a, c; -y^2)\,dy$$

$$= \tfrac{1}{2}\pi^{1/2}\,\frac{\Gamma(c)}{\Gamma(a)}\,x^{2a-1}\,e^{-x^2}\,\Psi(c-\tfrac{1}{2}, a+\tfrac{1}{2}; x^2)$$

$$(13)\quad \Gamma(a)\int_0^\infty y^{\frac{1}{2}c-\frac{1}{2}}\,J_{c-1}[2(xy)^{1/2}]\,\Phi(a, c; -2y^{1/2})\,\Psi(a, c; 2y^{1/2})\,dy$$

$$= 2^{-c}\,\Gamma(c)\,x^{a-\frac{1}{2}c-\frac{1}{2}}\,[1+(1+x)^{1/2}]^{c-2a}\,(1+x)^{-1/2}$$

$$\operatorname{Re} c > 2, \qquad \operatorname{Re}(c-2a) < \tfrac{1}{2}$$

[which is the Hankel inversion of 6.15(19)], the reciprocity formula

$$(14)\quad \Gamma(a)\int_0^\infty t^{a'-1}\,(1+t)^{c'-a'-1}\,\Psi(a, c; tx)\,dt$$

$$= \Gamma(a')\int_0^\infty t^{a-1}\,(1+t)^{c-a-1}\,\Psi(a', c'; tx)\,dt$$

$$\operatorname{Re} a > 0, \quad \operatorname{Re} a > \operatorname{Re} c' - 1, \quad \operatorname{Re} a' > 0, \quad \operatorname{Re} a' > \operatorname{Re} c - 1,$$

Magnus's addition theorem (1946)

$$(15)\quad (2\pi i)^{-1}\int_{-i\infty}^{+i\infty} \Gamma(-a)\,\Gamma(c-a)\,\Psi(a, c; x)\,\Psi(c-a, c; y)\,da$$

$$= \Gamma(c)\,\Psi(c, 2c; x+y),$$

and the formula

$$(16)\quad \int_0^\infty e^{-x}\,x^{c+n-1}\,(x+y)^{-1}\,\Phi(a, c; x)\,dx$$

$$= (-1)^n\,\Gamma(c)\,\Gamma(1-a)\,y^{c+n-1}\,\Psi(c-a, c; y)$$

$$-\operatorname{Re} c < n < 1 - \operatorname{Re} a, \qquad n = 0, 1, 2, \ldots, \qquad |\arg y| < \pi,$$

which is closely related to the Stieltjes inversion of 6.8(15), and also to some results by Meixner (1933). Integrals with respect to the parameters have also been evaluated by Erdélyi (1941) and Buchholz (1947, 1948, 1949).

Another type of integral arises in the investigation of the zeros of confluent hypergeometric functions. Tsvetkoff (1941) proved that for any two zeros ξ, η of Φ.

$$\int_0^1 [\kappa/x - \tfrac{1}{4}(\xi+\eta)]\,e^{-\frac{1}{2}(\xi+\eta)x}\,x^c\,\Phi(a, c; \xi x)\,\Phi(a, c; \eta x)\,dx = 0$$

$$\xi \neq \eta, \qquad \operatorname{Re} c > 0,$$

$$= (a/\xi)\, e^{-\xi}\, [\Phi(a+1,\ c;\ \xi)]^2 \qquad \xi = \eta, \quad \operatorname{Re} c > 0,$$

and for any two zeros ξ, η, of Ψ

$$\int_1^\infty [\kappa/x - \tfrac{1}{4}(\xi+\eta)]\, e^{-\frac{1}{2}(\xi+\eta)x}\, x^c\, \Psi(a,\ c;\ \xi x)\, \Psi(a, c; \eta x)\, dx = 0 \qquad \xi \neq \eta,$$

$$= -\,\xi^{-1}\, e^{-\xi}\, [\Psi(a-1,\ c;\ \xi)]^2 \qquad \xi = \eta.$$

We also mention here an inversion formula with the kernel

$$N(k,\ x) = e^{-ix/2}\, \Phi(\tfrac{1}{2}c + ik,\ c;\ ix) \qquad c > 0.$$

From

$$f(x) = \int_{-\infty}^{\infty} N(k,\ x)\, g(k)\, dk$$

if follows, under certain assumptions that

$$g(k) = \frac{\Gamma(\tfrac{1}{2}c+ik)\,\Gamma(\tfrac{1}{2}c-ik)}{2\pi\,[\Gamma(c)]^2}\, e^{k\pi} \int_0^\infty y^{c-1}\, N(k,\ y)\, f(y)\, dy.$$

6.15.3. Products of confluent hypergeometric functions

The investigation of products of confluent hypergeometric functions often involves generalizations of the hypergeometric series (cf. Section 5). In this section we note some of the cases in which such generalizations do not occur.

Some of the more important integral representations are:

(17) $$e^{-\frac{1}{2}x-\frac{1}{2}y}\, \Phi(a,\ c;\ x)\, \Phi(a',\ c;\ x)$$
$$= (2\pi i)^{-1}\, \Gamma(c) \int_L e^s (s - \tfrac{1}{2}x + \tfrac{1}{2}y)^{-a} (s + \tfrac{1}{2}x - \tfrac{1}{2}y)^{-a'}$$
$$\times (s + \tfrac{1}{2}x + \tfrac{1}{2}y)^{a+a'-c}\, F\{a, a'; c; 4xy\,[4s^2 - (x-y)^2]^{-1}\}\, ds$$

where L is a loop starting and ending at $-\infty$, and encircling all singularities of the integrand, i.e., the four points $s = \pm\tfrac{1}{2}x \pm \tfrac{1}{2}y$, in the positive direction,

(18) $$\Gamma(a)\, \Gamma(c-a)\, \Phi(a,\ c;\ x)\, \Phi(a,\ c;\ -x)$$
$$= [\Gamma(c)]^2\, x^{1-c} \int_{-\infty}^{\infty} I_{c-1}(x \operatorname{sech} t)\, e^{(c-2a)t} \operatorname{sech} t\, dt$$
$$\operatorname{Re} a > 0, \quad \operatorname{Re}(c-a) > 0,$$

(19) $$\Gamma(a)\, \Phi(a,\ c;\ -x)\, \Psi(a,\ c;\ x)$$
$$= \Gamma(c)\, x^{1-c} \int_0^\infty J_{c-1}(x \sinh t)\, (\tanh \tfrac{1}{2}t)^{2a-c}\, dt$$
$$\operatorname{Re} a > 0, \quad \operatorname{Re} x > 0,$$

(20) $$\pi\, \Psi(a,\ c;\ x)\, \Psi(c-a,\ c;\ x)$$
$$= 2x^{1-c}\, e^x \int_0^{\frac{1}{2}\pi} K_{c-1}(x \sec t) \cos[(c-2a)\,t] \sec t\, dt$$
$$\operatorname{Re} x > 0,$$

(21) $\Gamma(\gamma)\,\Psi(a, c; x)\,\Psi(a', c; y)$

$$= \int_0^\infty e^{-t} t^{\gamma-1} (x+t)^{-a} (y+t)^{-a'}$$

$$\times F[a, a'; \gamma; t(x+y+t)(x+t)^{-1}(y+t)^{-1}]\, dt$$

$$\gamma = a + a' - c + 1, \quad \operatorname{Re}\gamma > 0, \quad xy \neq 0.$$

We may add the following integral formulas:

(22) $\int_0^\infty e^{-st} t^{c-1}\, \Phi(a, c; t)\, \Phi(a', c; \lambda t)\, dt$

$$= \Gamma(c)(s-1)^{-a}(s-\lambda)^{-a'} s^{a+a'-c} F[a, a'; c; \lambda(s-1)^{-1}(s-\lambda)^{-1}]$$

$$\operatorname{Re} c > 0, \quad \operatorname{Re} s > \operatorname{Re}\lambda + 1,$$

(23) $\Gamma(a+a') \int_0^\infty y^{\frac{1}{2}c+\frac{1}{2}c'-1} J_{c+c'-2}[2(xy)^{\frac{1}{2}}]\, \Psi(a, c; y)\, \Phi(a', c'; -y)\, dy$

$$= \Gamma(c')\, x^{\frac{1}{2}c+\frac{1}{2}c'-1}\, \Psi(c'-a', c+c'-a-a'; x)\, \Phi(a', a+a'; -x)$$

$$\operatorname{Re} c' > 0, \quad 1 < \operatorname{Re}(c+c') < 2\operatorname{Re}(a+a') + \tfrac{1}{2},$$

and

(24) $\Gamma(\gamma) \int_0^\infty e^{-t} t^{\rho}\, \Phi(a, c; t)\, \Psi(a', c'; \lambda t)\, dt$

$$= C\,\Gamma(c)\,\Gamma(\beta)\,\lambda^{\sigma} F(c-a, \beta; \gamma; 1-\lambda^{-1}),$$

where either

$$\rho = c-1, \quad \sigma = -c, \quad \beta = c - c' + 1, \quad \gamma = c - a + a' - c + 1,$$
$$C = \Gamma(a'-a)/\Gamma(a'),$$

or

$$\rho = c + c' - 2, \quad \sigma = 1 - c - c', \quad \beta = c + c' - 1,$$
$$\gamma = a' - a + c, \quad C = \Gamma(a' - a - c' + 1)/\Gamma(a' - c' + 1).$$

These, and many more, integrals can be found in papers by Erdélyi and especially C. S. Meijer. Erdélyi (1936d) has also evaluated the integral

$$\int_\infty^{(0+)} e^{-pz} z^q M_{\kappa_1, \mu_1}(a_1 z) \cdots M_{\kappa_n, \mu_n}(a_n z)\, dz$$

in terms of Lauricella's hypergeometric function of n variables.

A few infinite series involving products of confluent hypergeometric functions are:

(25) $$\sum_{n=0}^\infty \frac{(c-a)_n (c'-a')_n}{(c)_n (c')_n\, n!} \Phi(a, -c+n; x)\, \Phi(a', c'+n; y)\, z^n$$

$$= e^z \sum_{n=0}^\infty \frac{(a)_n (a')_n}{(c)_n (c')_n} \Phi(a+n, c+n; x-z)\, \Phi(a'+n, c'+n; y-z),$$

(26) $$\sum_{n=0}^{\infty} \frac{(a)_n (c'-a')_n}{(c)_n (c')_n n!} \Phi(a+n, c+n; x-z) \Phi(a', c'+n; y) z^n$$

$$= \sum_{n=0}^{\infty} \frac{(c-a)_n (a')_n}{(c)_n (c')_n n!} \Phi(a, c+n; x) \Phi(a'+n, c'+n; y-z) z^n,$$

(27) $$\sum_{n=0}^{\infty} \frac{(h)_n n!}{(c)_n (c')_n} L_n^{(c-1)}(x) L_n^{(c'-1)}(y) z^n$$

$$= (1-z)^{-h} \sum_{n=0}^{\infty} \frac{(h)_n}{(c)_n (c')_n n!} \Phi[h+n, c+n; xz(z-1)^{-1}]$$

$$\times \Phi[h+n, c'+n; yz(z-1)^{-1}] [xyz(1-z)^{-2}]^n \qquad |z| < 1,$$

(28) $$[\Gamma(c-\lambda)]^2 \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+n)}{n!} \Phi(a-n, c; x) \Phi(a-n, c; y)$$

$$= [\Gamma(c)]^2 (xy)^{1-c} \int_0^{\min(x,y)} e^t t^{\lambda-1} [(x-t)(y-t)]^{c-\lambda-1}$$

$$\times \Phi(a, c-\lambda; x-t) \Phi(a, c-\lambda; y-t)\, dt$$

(29) $$\sum_{n=0}^{\infty} \frac{(a)_n (a')_n}{(c)_n (c')_n n!} \Phi(a+a'-c, c+2n; x) x^{2n}$$

$$= \Phi(a, c; x) \Phi(a', c; x).$$

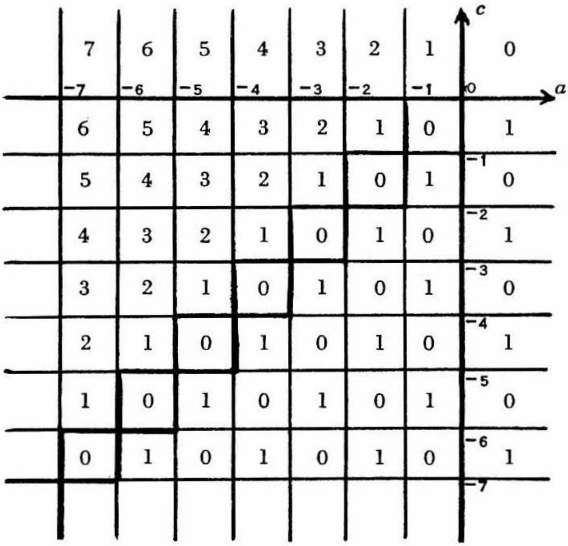
These series are from papers by Erdélyi who discussed other series too. See also Burchnall and Chaundy (1940, 1941).

6.16. Real zeros for real a, c

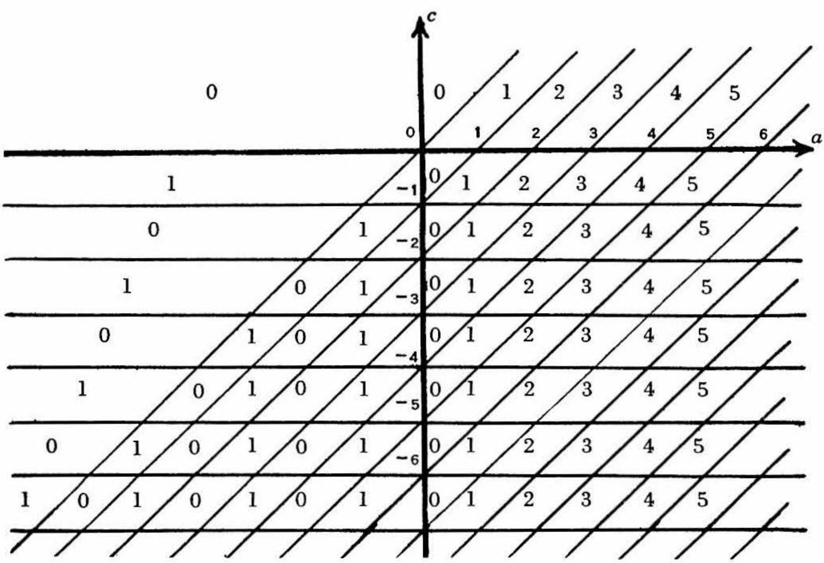
On account of 6.9(1), 6.9(2), the zeros of $M_{\kappa,\mu}$ coincide with those of the Φ function and the zeros of $W_{\kappa,\mu}$ with those of the Ψ function, except possibly for $x = 0, \infty$.

If a and c are real, Φ has only a finite number of real zeros and Ψ only a finite number of positive zeros, since there is only a finite number of zeros in any finite interval and, by virtue of 6.13(1) and 6.9(3), no zero for sufficiently large $|x|$. From Whittaker's differential equation 6.1(4) it can be proved that, for real a and c, every confluent hypergeometric function has at most a finite number of zeros.

A more detailed investigation of the number of real zeros of $\Phi(a, c; x)$ when a and c are real (Kienast 1921) is based on the circumstance that under suitable assumptions either the functions $\Phi(a+1-j, c; x)$ $(j = 0, 1, \ldots, m+1)$ or the functions $\Phi(a+j, c; x)$ $(j = 0, 1, \ldots, n)$ form a Sturmian chain. Kienast's results can conveniently be represented by diagrams in which the (real) a, c plane is divided into compartments inside each of which Φ has a given number of positive, or negative, zeros. In the following diagrams vertical boundary lines belong to the compartments to their right, oblique boundary lines to the compartments to their left. Along the horizontal lines $c = 0, -1, -2, \ldots$, the function Φ is not defined.

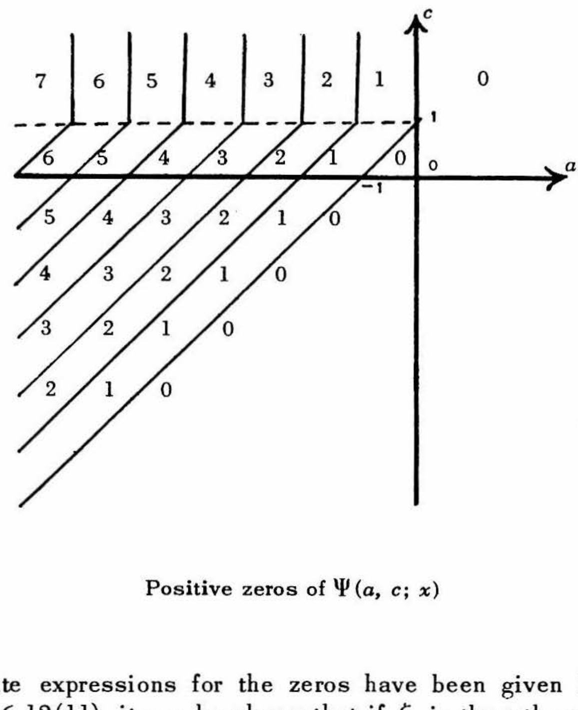


Positive zeros of $\Phi(a, c; x)$



Negative zeros of $\Phi(a, c; x)$

The positive zeros of $\Psi(a, c; x)$ can be investigated similarly; for negative real x, however, Ψ is in general complex and different from zero [cf. 6.8(14)]. Equations 6.5(2) and 6.5(6) show that Ψ cannot have positive zeros if a and c are real and either $a > 0$ or $a - c + 1 > 0$. If $-n < a < 1 - n$, $n = 1, 2, \ldots$, the functions $\Psi(a + j, c; x)$, $j = 0, 1, \ldots, n$, form a Sturmian chain; all these functions are positive for large positive x, and their signs as $x \to 0$ are governed by 6.8(2) to 6.8(5). The information derived therefrom is represented in the diagram below. The results coincide with those by A. Milne (1915) and G. E. Tsvetkoff (1941a).



Positive zeros of $\Psi(a, c; x)$

Approximate expressions for the zeros have been given by Tricomi (1947). From 6.12(11), it can be shown that if ξ_r is the r-th positive zero of $\Phi(a, c; x)$, and $j_{c-1,r}$ the r-th positive zero of $J_{c-1}(x)$, then for large κ,

(1) $$\xi_r = j^2_{c-1,r}(4\kappa)^{-1}\{1 + 1/3[2c(c-2) + j^2_{c-1,r}](4\kappa)^{-2}\} + O(\kappa^{-4}).$$

H. Schmidt (1938) found a similar result and showed that the series of which (1) gives the first two terms is convergent for sufficiently large $|a|$. The r-th positive zero can be approximated by

(2) $$\pi^2(r + \tfrac{1}{2}c - \tfrac{3}{4})^2/(2c - 4a).$$

Further details about the zeros are contained in the papers by Tricomi and Tsvetkoff quoted above.

The complex zeros (for real a and c) have been investigated by Tsvetkoff (1941 b) and by Tricomi (1950 a).

6.17. Descriptive properties for real a, c, x

The results of section 6.16 together with differentiation formulas such as 6.4 (10) and 6.6 (11) give information about the number and approximate position of the zeros, turning points, and points of inflexion of the confluent hypergeometric functions when a, c, x, and hence also Φ and Ψ are real. Moreover, the Sonine-Pólya theorem (Szegö 1939) gives results on the magnitude of successive maxima and minima. Writing 6.1 (2) in the self-adjoint form

$$\frac{d}{dx}\left(x^c e^{-x}\frac{dy}{dx}\right) - ax^{c-1}e^{-x}y = 0, \tag{1}$$

an application of that theorem shows that the turning values, or rather their absolute values, form an increasing or decreasing sequence according as

$$-ax^{c-1}e^{-x}.\, x^c e^{-x} = -ax^{2c-1}e^{-2x} \tag{2}$$

is a decreasing or increasing function of x. Hence the successive maxima of $|y|$ are increasing if

$$a > 0, \quad x < c - \tfrac{1}{2} \quad \text{or} \quad a < 0, \quad x > c - \tfrac{1}{2}, \tag{3}$$

and are decreasing if

$$a > 0, \quad x > c - \tfrac{1}{2} \quad \text{or} \quad a < 0, \quad x < c - \tfrac{1}{2}. \tag{4}$$

For Whittaker's functions $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$ let us call $\mathfrak{J}$ the interval between 0 and $2(\mu^2 - \tfrac{1}{4})/\kappa$ and let us apply the Sonine-Pólya theorem to 6.1 (4). The successive relative maxima of $|z|$ are increasing if

$$\begin{cases} \kappa > 0 \text{ and } x \text{ is outside of } \mathfrak{J} \\ \text{or} \\ \text{if } \kappa < 0 \text{ and } x \text{ is inside } \mathfrak{J}, \end{cases} \tag{5}$$

and are decreasing if

$$\begin{cases} \kappa > 0 \text{ and } x \text{ is inside } \mathfrak{J} \\ \text{or} \\ \text{if } \kappa < 0 \text{ and } x \text{ is outside } \mathfrak{J}. \end{cases} \tag{6}$$

The sizes of the later turning values can be approximated by means of the asymptotic representations developed in section 6.13.

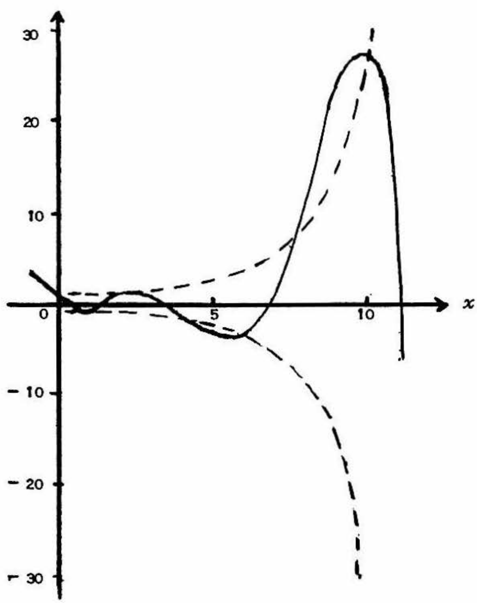
As an example, we shall investigate

$$y = \Phi(-4{\cdot}5, 1; x) \tag{7}$$

for real x. In Whittaker's notation, $\kappa = 5$, $\mu = 0$. By 6.4(10),

(8) $y' = -4\cdot5\,\Phi(-3\cdot5, 2; x)$.

Clearly $y(0) = 1$, $y'(0) = -4\cdot5$, and since $\Gamma(-4\cdot5) < 0$, we have from 6.13(3) that $y(-\infty) = \infty$, $y'(-\infty) = \infty$, $y(\infty) = -\infty$, $y'(\infty) = -\infty$. From the diagrams of section 6.16, y has five positive, and no negative zeros. From 6.16(2) the zeros of y can be approximated by 0.3, 1.5, 3.7, 6.9, 10.6, from 6.16(2) and (8) the turning points by $x = 0.9, 2.8, 5.8, 9.9$. Moreover, at all these points (2) is satisfied and so the maxima of $|y|$ form an increasing sequence. A rough graph of y based on this information is given below:



$\Phi(-4\cdot5, 1; x)$

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GREEK LETTERS

MISCELLANEOUS NOTATIONS

$\arg z$ argument (or phase) of z (complex)

$\mathrm{Im}\, z$ imaginary part of z (complex)

$\mathrm{Re}\, z$ real part of z (complex)

γ Euler-Mascheroni constant (see p. 1)

$(a)_n = \Gamma(a+n)/\Gamma(a)$

$(a)_{q,n}$ or $[a]_n = \prod_{\nu=0}^{n-1} (1 - aq^\nu)$

$⨍$ Cauchy principal value of an integral