

# Geometric Algebra for Computer Graphics

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John Vince

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# **Geometric Algebra for Computer Graphics**

 Springer

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# Preface

In December 2006 I posted my manuscript *Vector Analysis for Computer Graphics* to Springer and looked forward to a short rest before embarking upon another book. But whilst surfing the Internet, and probably before my manuscript had reached its destination, I discovered a strange topic called *geometric algebra*. Advocates of geometric algebra (GA) were claiming that a revolution was coming and that the cross product was dead. I couldn't believe my eyes. I had just written a book about vectors extolling the power and benefits of the cross product, and now moves were afoot to have it banished! I continued to investigate GA and was amazed that a Google search revealed over 2 million entries. I started to read up the subject and discovered that GA was a Clifford algebra which had a natural affinity with geometry. It appeared that Prof. David Hestenes [14] had invented *geometric calculus* and successfully applied it to classical and quantum mechanics, electrodynamics, projective and conformal geometry. Chris Doran, Anthony and Joan Lasenby at Cambridge University had continued this research and were a driving force behind its understanding, dissemination and application to computer graphics. It seems that if I had been attending SIGGRAPH regularly, I would have been aware of these developments, but alas that was not the case, and I had a lot of catching up to do.

As I started reading various technical papers, especially by Hestenes, Doran and the Lasenbys, I realized the importance of the subject and the need to understand it. Slowly I was drawn into a world of complex numbers, antisymmetric operators, non-commutative products, conformal space, null vectors and the promise of elegance in CGI algorithms. I would be able to divide, rotate and reflect vectors with an ease never before known.

As I was finding it so difficult to understand GA, probably other people would also be finding it difficult, and then I realized the title of my next book: *Geometric Algebra for Computer Graphics*. But how could I write about a subject of which I knew nothing? This was a real challenge and became the driving force that has kept me working day and night for the past year. I took every opportunity to read about the subject: in bed, on planes, trains and boats; whilst waiting at the dentist and even waiting whilst my car was being serviced!

Before embarking on my summer vacation this year (2007) I bought a copy of Doran & Lasenby's excellent book *Geometric Algebra for Physicists* and took it, and my embryonic manuscript, with me to the south of France. My wife and I stayed at the Hotel Horizon in Cabris, overlooking Grasse and Cannes on the Côtés d'Azur. Previous guests have included authors, philosophers and musicians such as Leonard Bernstein, Jean-Paul Sartre, Simone de Beauvoir, Gregory Peck and Antoine de St. Exupéry whose names have been carved into table tops in the

bar. Now that I have spent a few days at Hotel Horizon studying bivectors, trivectors and multi-vector products, I am looking forward to seeing my name cut into a table top when I return next year!

This book is a linear narrative of how I came to understand geometric algebra. For example, when I started writing the manuscript, *conformal geometry* were no more than two words, about which, I knew I would eventually have to master and write a chapter. The conformal model has been the most challenging topic I have ever had to describe. To say that I understand conformal geometry would be an overstatement. I understand the action of the algebra but I do not have a complete picture in my mind of 5D Minkowski space which is the backdrop for the conformal model. I admire the authors who have written so confidently about the conformal model, not only for their mathematical skills but their visual skills to visualize what is happening at a geometric level.

When I first started to read about GA I was aware of the complex features of the algebra, in that certain elements had imaginary qualities. Initially, I thought that this would be a major stumbling block, but having now completed the manuscript, the imaginary side of GA is a red herring. If one accepts that some algebraic elements square to  $-1$ , that is all there is to it. Consequently, do not be put off by this aspect of the algebra.

Another, stumbling block that retarded my progress in the early days was the representation at a programming level of bivectors, trivectors, quadvectors, etc. I recall spending many days walking my dog Monty trying to resolve this problem. Monty, a Westie, whose knowledge of Clifford algebra was only slightly less than my own, made no contribution whatsoever, but this daily mental and physical exercise eventually made the penny drop and I realized that bivectors, trivectors, quadvectors, etc., were just names recording a numerical value within the algebra. Why had I found it so difficult? Why had this not been explicitly described by other authors? If only someone had told me, I could have avoided this unnecessary mental anguish. But, in retrospect, the mental pain of learning about GA single-handed, has provided me with some degree of confidence when talking about the subject. In fact, in September 2007, I organized a one-day Workshop on GA in London where Dr. Hugh Vincent, Dr. Chris Doran, Dr. Joan Lasenby and me gave presentations to an audience from the computer animation and computer games sectors. It was extremely successful.

I have structured this book such that the first six chapters provide the reader with some essential background material covering complex algebra, vector algebra, quaternion algebra and geometric conventions. These can be skipped if you are already familiar with these topics. Chapter 7 goes into the history of geometric algebra, but I was already prepared for this as I had read Michael Crowe's fantastic book *A History of Vector Analysis*. In fact, this book is so good I have read it at least four times! Chapter 8 describes the geometric product, which was introduced by Clifford and is central to GA. Chapter 9 explores how GA handles reflections and rotations. Chapter 10 shows how GA is used to solve various problems in 2D and 3D geometry. Chapter 11 describes the conformal model. Chapter 12 is a short review of some typical applications of GA and Chapter 13 identifies important programming tools for GA. Finally, chapter 14 draws the book to a conclusion.

I am not a mathematician, just a humble consumer of mathematics, and whenever I read a book about mathematics I need to see examples, which is why I have included so many in this book. It is so tempting to write:

“It is obvious that Eq. (12.56) is the required rotor”

for very often it is not obvious that this equation is a rotor, or even how it is used in practice. Therefore, whenever I have introduced an equation, I have shown its derivation and its application.

I would like to thank Dr. Hugh Vincent for reading through an early manuscript and offering some constructive feedback. I would also like to thank Dr. Chris Doran for taking the time to read the manuscript and advising me on numerous inconsistencies, and Dr. Joan Lasenby for her responsive, supportive emails when I had lost my way in untangling conformal null vectors. Once again I would like to acknowledge Chris Doran and Anthony Lasenby's excellent book *Geometric Algebra for Physicists*. I could not have written this book without their book. I also must not forget to thank Helen Desmond and Beverley Ford, General Manager of Springer, UK, for their continuous support, memorable lunches and transforming my manuscript into such a beautiful book.

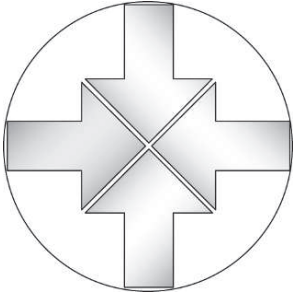
Although I have done my best to ensure that the book is error free, if there are any inconsistencies, I apologize, as they are entirely my fault.

Finally, I must remind the reader that this book is intended only as a gentle introduction to GA. Hopefully, it will provide a bridge that will ease the understanding of technical papers and books about GA, where the subject is covered at a more formal and rigorous level.

Ringwood, UK

John Vince

*This book is affectionately dedicated to my family: Annie, Samantha, Anthony, Genny, Peter,  
Megan, Mia and Monty.*



# 1 Introduction

## 1.1 Aims and objectives of this book

The aim of this book is to provide the reader with a gentle introduction to the embryonic subject of geometric algebra (GA). The GA books that currently exist are either directed at physicists or assume that their readers possess a formal understanding of mathematics. To my knowledge, this is the first book that introduces GA without overwhelming the reader with the formalism of linear algebra that supports the subject. The real objective of the book is to make the reader familiar with the concepts of GA. Hopefully on completing the book readers will be able to read more advanced books and technical papers.

## 1.2 Mathematics for CGI software

Anyone who has written software for computer animation or computer games will know the wide range of mathematical tools needed to implement the algorithms for resolving 2D and 3D geometric problems. Perhaps one of the most important mathematical tools is the matrix transform, where it is difficult to imagine how one could get by without using

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (1.1)$$

to rotate a point about the origin. Although matrices exploit the ability to represent a transform as an array of numbers, the origin of these numbers is linear algebra. Matrix notation simply introduces a degree of elegance that permits the solution to a problem to be addressed at a higher symbolic level, without becoming bogged down in the longhand notation of algebra. Even writing

down an array of numbers eventually becomes tedious, and a further substitution can be made by giving names to the transforms such as

$$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.2)$$

which permits us to write their product as  $P = ST$ . At this stage we have basically created another algebra with its own axioms and embellishments such as  $\llbracket P \rrbracket^{-1}$  and  $\llbracket P \rrbracket^T$ .

When the algebra of matrices is combined with the algebra of vectors, quaternions, analytic geometry, barycentric coordinates, etc., one realizes the wide range of algebraic notation employed in CGI. Fortunately, this notation is relatively easy to understand and master and has even been incorporated at a hardware level in graphics cards. GA reveals that matrices, determinants, complex numbers, quaternions and vectors are all closely related, which must have an impact upon the design of CGI algorithms.

As you will discover, the notation of GA is rather elegant, even though the underlying algebra is fussy. But we mortals should not be concerned with any inherent fussiness, just in the same way that matrix multiplication or inversion does not prevent us from using matrices. Any complexity associated with GA is readily hidden inside software so that programmers can develop solutions using high-level controls and commands.

### 1.3 The book's structure

This book is designed to be read in a linear fashion. Chapters 2 to 5 review elementary, complex, vector and quaternion algebra. These chapters are very concise and have been included to provide a unified reference source when some of their features are discussed in later chapters. Those readers already familiar with these topics should consider starting at chapter 6 where geometric conventions such as clockwise and anticlockwise traditions, left and right-handed axial systems are reviewed.

Chapter 7 introduces the reader to the reasons why GA has surfaced in the 21st century rather than the 19th century when it was discovered. Researching this material was very enlightening and brought home the existence of politics in mathematical and scientific progress. GA could have easily become established at the end of the 19th century, but influential mathematicians and scientists of the day decided between them the direction vector analysis would take in future years. Fortunately, enlightened people such as Clifford and Hestenes know a good idea when they see one, and their personal tenacity and dedication have ensured that Grassmann's original ideas have prevailed.

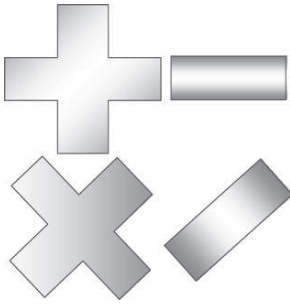
Chapter 8 covers the geometric product, which combine the inner and outer products into a single non-commutative new vector product. Discovering this for the first time is something I will always remember, as its simplicity and structure make one wonder why it took so long to come to become part of everyday vector analysis. Initially, I was cautious of the outer product portion of the geometric product as it possesses imaginary qualities, and I thought that this would be a dominant feature of GA. However, eventually you will discover that elements that square to  $-1$  are so natural you will wonder what all the fuss is about.

Chapter 9 applies GA to calculating reflections and rotations, which is where the power and elegance of the notation emerges. Chapter 10 applies GA to a variety of simple geometric problems encountered in computer graphics. It is far from exhaustive, but illustrates alternative approaches to geometric problem solving.

Chapter 11 addresses the conformal model developed by David Hestenes *et al.* Its use of 5D Minkowski space is a recent development and has natural applications to quantum physics and electrodynamics, but is also being applied to computer graphics. This chapter introduces the reader to the basic concepts, and there is a wide range of technical literature awaiting those readers possessing the appropriate mathematical skills.

Chapter 12 reviews how GA is being compared to existing approaches to algorithm design and the programming implications of GA. Chapter 13 identifies programming tools for GA, and chapter 14 summarizes the book's aims and objectives.





# 2 Elementary Algebra

## 2.1 Introduction

The evolution of algebra can be traced back to a treatise called *Al-Kitab al-Jabr wa-l-Muqabala* written by the Persian mathematician, Muhammed ibn Mūsā al-Khowārizmī [circa 780–850]. An English translation of the title is *The Compendious Book on Calculation by Completion and Balancing*. It is highly probable that the word *algebra* descended from the middle words of the title *al-Jabr*.

It took a few centuries of experimentation before today's notation emerged. For instance, Robert Recorde [1] [1510–1558] was an academic at Oxford University and seems to be the first person to employ the word 'algebra' in his book *Pathway to Knowledge*. He also introduced the equality sign '=' in 1557 in his book *Whetstone of Witte*.

The German mathematician, Johannes Widman [2] [1462–1498], is credited with using the symbols '+' and '-' for the first time in his book on arithmetic in 1489. The English mathematician, William Oughtred [3] [1574–1660] introduced the '×' symbol to represent multiplication, not to mention his invention of an early form of the slide rule.

Today, algebra is a major branch of mathematics and allows us to write mathematical statements in the form:

$$\langle \text{LHS} \rangle = \langle \text{RHS} \rangle$$

where the left-hand-side (LHS) and right-hand-side (RHS) expressions are manipulated to resolve the value of some variable. For example, given

$$ax^2 + bx + c = 0 \tag{2.1}$$

a solution for  $x$  is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.2}$$

which is derived by applying the axioms of elementary algebra.

Let us remind ourselves of these rules.

## 2.2 Numbers, variables and arithmetic operators

To begin with, we acknowledge the existence of different sets of numbers such as natural numbers  $\mathbb{N}$ , integers  $\mathbb{Z}$ , reals  $\mathbb{R}$ , rationals  $\mathbb{Q}$ , irrationals, etc., and the role of *variables* such as  $x, y, z, \dots$  etc. to stand in for numbers whose values are not stated explicitly, such as

$$x^2 = 16 \quad (2.3)$$

which implies that

$$x = \pm 4. \quad (2.4)$$

Next, we introduce the binary operators  $+$ ,  $-$ ,  $\times$  and  $/$  to represent addition, subtraction, multiplication and division respectively.

## 2.3 Closure

*Closure* is a property that relates the result of a binary operation to the original operands. For example, if we add an integer to an integer the result is an integer, which secures closure for this operation. However, if we divide an integer by an integer, the result is not necessarily an integer, and closure fails. For instance,

$$3/4 = 0.75 \quad (2.5)$$

for although the numerator and denominator are integers, the result is a real quantity. This is not a problem that should worry us — simply something of which we should be aware.

## 2.4 Identity element

The *identity element* is a useful feature of any algebra and helps simplify algebraic expressions or invoke an analytical strategy. So let's find the identity elements for the four binary operators.

If  $\lambda$  (lambda) is the identity element and  $\Omega$  (omega) is a binary operator then the following two rules must be satisfied:

$$x\Omega\lambda = x \quad (2.6)$$

and

$$\lambda\Omega x = x. \quad (2.7)$$

In the case of addition  $\lambda = 0$  because

$$x + 0 = x \quad (2.8)$$

and

$$0 + x = x. \quad (2.9)$$

But in the case of subtraction, if  $\lambda = 0$

$$x - 0 = x \quad (2.10)$$

but

$$0 - x = -x \quad (2.11)$$

which fails.

In the case of multiplication,  $\lambda = 1$  because

$$x \times 1 = x \quad (2.12)$$

and

$$1 \times x = x. \quad (2.13)$$

But in the case of division, if  $\lambda = 1$

$$x/1 = x \quad (2.14)$$

but

$$1/x \neq x \quad (2.15)$$

which fails. Similarly, if  $\lambda = 0$  then

$$x/0 = \infty \quad (2.16)$$

which fails, and

$$0/x = 0 \quad (2.17)$$

which also fails.

## 2.5 Inverse element

The *inverse element* is a powerful analytical tool for solving equations and inverting functions. In the case of addition, the inverse of  $x$  is  $-x$ , whilst the inverse of  $-x$  is  $x$  which ensures that

$$x + (-x) = 0 \quad (2.18)$$

and

$$-x + x = 0. \quad (2.19)$$

In the case of multiplication, the inverse of  $x$  is  $1/x$ , and the inverse of  $1/x$  is  $x$ , which ensures that

$$x \left( \frac{1}{x} \right) = 1 \quad (2.20)$$

and

$$\left( \frac{1}{x} \right) x = 1. \quad (2.21)$$

The inverse element is related to the identity element as follows:

$$x \Omega x^{-1} = \lambda \quad (2.22)$$

where  $\lambda$  is the identity element. For addition  $\lambda = 0$  and for multiplication  $\lambda = 1$ .

## 2.6 The associative law

When we add or multiply three or more numbers together the final result is independent of their order, which doesn't seem surprising. However, it is not true for subtraction or division. In general, the *associative law* is summarized as:

$$a + (b + c) = (a + b) + c \quad (2.23)$$

or

$$a(bc) = (ab)c. \quad (2.24)$$

We take for granted this last axiom where it is possible to associate different pairs of products, but it is extremely useful as its existence permits division within the algebra. Later on, we will discover that GA is also associative, which permits us to divide by vectors.

To see why subtraction and division fail, consider the following simple examples:

$$9 - (8 - 2) \neq (9 - 8) - 2 \quad (2.25)$$

and

$$9/(9/3) \neq (9/9)/3. \quad (2.26)$$

## 2.7 The commutative law

When we add or multiply two numbers the final result is independent of their sequence. This, too, may not seem surprising, but matrices and vectors *anticommute* when multiplied. The *commutative law* is summarized as:

$$a + b = b + a \quad (2.27)$$

and

$$ab = ba. \quad (2.28)$$

To see why subtraction and division fail, consider the following simple examples:

$$6 - 2 \neq 2 - 6 \quad (2.29)$$

and

$$9/3 \neq 3/9. \quad (2.30)$$

## 2.8 The distributive law

The *distributive law* of multiplication over addition is best summarized as:

$$a(b + c) = ab + ac \quad (2.31)$$

and

$$(b + c)a = ba + ca. \quad (2.32)$$

## 2.9 Summary

The above laws are the axioms for elementary algebra and are expressed formally as follows:

$$\text{Given } u, u_1, u_2, u_3 \in \mathbb{R}: \quad (2.33)$$

### Closure

For all  $u_1$  and  $u_2$

$$\text{addition } u_1 + u_2 \in \mathbb{R} \quad (2.34)$$

$$\text{multiplication } u_1 u_2 \in \mathbb{R}. \quad (2.35)$$

### Identity

For each  $u$  there is an identity element 0 and 1 such that

$$\text{addition } u + 0 = 0 + u = u \quad (2.36)$$

$$\text{multiplication } 1u = u. \quad (2.37)$$

### Inverse

For each  $u$  there is an inverse element  $-u$  and  $1/u$  such that

$$\text{addition } u + (-u) = -u + u = 0 \quad (2.38)$$

$$\text{multiplication } u \left( \frac{1}{u} \right) = \left( \frac{1}{u} \right) u = 1 \quad (u \neq 0). \quad (2.39)$$

### Associativity

For all  $u_1, u_2$  and  $u_3$

$$\text{addition } (u_1 + u_2) + u_3 = u_1 + (u_2 + u_3) \quad (2.40)$$

$$\text{multiplication } u_1(u_2 u_3) = (u_1 u_2)u_3. \quad (2.41)$$

### Commutativity

For all  $u_1$  and  $u_2$

$$\text{addition } u_1 + u_2 = u_2 + u_1 \quad (2.42)$$

$$\text{multiplication } u_1 u_2 = u_2 u_1. \quad (2.43)$$

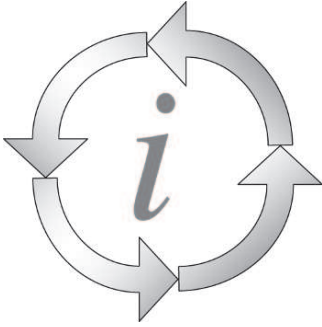
**Distributivity**

For all  $u_1, u_2$  and  $u_3$

$$u_1(u_2 + u_3) = u_1u_2 + u_1u_3 \quad (2.44)$$

$$(u_1 + u_2)u_3 = u_1u_3 + u_2u_3. \quad (2.45)$$

This chapter on elementary algebra reminds us that a group of axioms are behind any algebra and later on we will discover that GA has its own set of axioms, some of which are familiar and others not so familiar. It is also important to know which aspects of an algebra are closed or not, so that we know what to expect when computing products. Finally, an algebra that is associative for products supports division, which is very useful for solving algebraic problems. The next chapter explores the algebra of complex numbers.



# 3 Complex Algebra

## 3.1 Introduction

On the whole, elementary algebra is relatively consistent. However, division does cause problems when the divisor is zero, or when both numerator and denominator equal zero. We must also be careful with square-roots, especially when using negative numbers, because, by definition, the square of a real number is a positive real number. This, however, did not prevent mathematicians from finding a way around such an inconvenience — for in the 16th century, Girolamo Cardano [1501–1576] and Rafael Bombelli [1526–1572] were trying to make sense of this irreducible object. It was Bombelli who demonstrated that  $\sqrt{-1}$  could be embraced by elementary algebra so long as such objects were left undisturbed. Eventually, in 1777 the brilliant German mathematician, Leonhard Euler [1707–1783], introduced the symbol  $i$  to stand in for  $\sqrt{-1}$  which permitted expressions such as  $2 + \sqrt{-3}$  to be expressed as  $2 + i\sqrt{3}$ . This gave birth to complex numbers which have the form:

$$a + ib \quad \text{or} \quad a + bi. \tag{3.1}$$

The role of the ‘+’ sign in this context is rather strange as it is impossible to coalesce the two terms into a single value. Perhaps the conjunction ‘&’ would be more appropriate ( $a \& ib$ ) or just a simple comma ( $a, b$ ), where the second element implies a multiplication by  $i$ , but we are where we are, and the ‘+’ sign is probably here to stay, and does not appear to cause any problems.

It is also rather unfortunate that  $i$  or  $\sqrt{-1}$  are referred to as *imaginary* quantities, as there is nothing imaginary about them —  $i$  just has the useful feature that  $i^2 = -1$ . We have René Descartes to blame for this.

## 3.2 Complex numbers

A complex number takes the form

$$z = a + ib \tag{3.2}$$

where

$$\begin{aligned} a &\text{ is the real part,} \\ b &\text{ is the imaginary part, and} \\ i &= \sqrt{-1}. \end{aligned}$$

Either part may be zero, which implies that the set of real numbers  $\mathbb{R}$  is a subset of complex numbers  $\mathbb{C}$ .

The symbol  $i$  is regarded as if it were a normal algebraic quantity, but has the property that  $i^2 = -1$ . It is also worth pointing out the following pattern:

$$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1 \quad i^5 = i \quad \dots \quad (3.3)$$

which hints at  $i$ 's ability to perform a rotation — but more of this later.

$i$  also commutes with scalars, which permits us to write  $ai$  or  $ia$ .

### 3.3 Complex arithmetic

Complex numbers obey all the laws of elementary algebra, as illustrated by the following examples.

*Addition/subtraction*

$$(a_1 + ib_1) \pm (a_2 + ib_2) = (a_1 \pm a_2) + i(b_1 \pm b_2). \quad (3.4)$$

For example

$$(4 + i3) + (2 - i5) = 6 - i2. \quad (3.5)$$

*Multiplication*

$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 + i^2b_1b_2 + ia_1b_2 + ib_1a_2). \quad (3.6)$$

But as  $i^2 = -1$

$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2). \quad (3.7)$$

For example

$$(2 + i3)(3 - i2) = 12 + i5. \quad (3.8)$$

*Division* presents a slight problem as it is not immediately obvious how to expand

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2}. \quad (3.9)$$

Fortunately, help is at hand in the form of the *conjugate* of a complex number, which involves a simple sign reversal. For example, the conjugate of  $z = a + ib$  is  $z^* = a - ib$ , which, when multiplied together, produces a real quantity:

$$zz^* = (a + ib)(a - ib) = a^2 + b^2 \quad (3.10)$$

the imaginary part vanishes into thin air!



Therefore, we can write Eq. (3.9) as

$$\frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} = \frac{a_1 + ib_1}{a_2 + ib_2} \cdot \frac{a_2 - ib_2}{a_2 - ib_2} \quad (3.11)$$

which effectively multiplies the equation by 1 but at the same time ensures that the denominator dissolves into a real quantity:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + ib_1}{a_2 + ib_2} \cdot \frac{a_2 - ib_2}{a_2 - ib_2} \\ \frac{z_1}{z_2} &= \frac{(a_1a_2 + b_1b_2) + i(a_2b_1 - a_1b_2)}{a_2^2 + b_2^2} \\ \frac{z_1}{z_2} &= \left( \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} \right) + i \left( \frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2} \right) \end{aligned} \quad (3.12)$$

which is a complex number. For example

$$\begin{aligned} \frac{(4 + i2)}{(3 + i2)} &= \frac{(4 + i2)}{(3 + i2)} \cdot \frac{(3 - i2)}{(3 - i2)} \\ &= \frac{16 - i2}{9 + 4} \\ \frac{(4 + i2)}{(3 + i2)} &= \left( \frac{16}{13} \right) - i \left( \frac{2}{13} \right). \end{aligned} \quad (3.13)$$

Now let's form the product of two complex numbers using the conjugate for one of the numbers:

$$z_1 = a_1 + ib_1 \quad (3.14)$$

$$z_2 = a_2 + ib_2 \quad (3.15)$$

$$z_2^* = a_2 - ib_2 \quad (3.16)$$

$$z_1 z_2^* = (a_1 + ib_1)(a_2 - ib_2)$$

$$z_1 z_2^* = (a_1a_2 + b_1b_2) + i(a_2b_1 - a_1b_2). \quad (3.17)$$

The scalar part of Eq. (3.17) ( $a_1a_2 + b_1b_2$ ) is the familiar scalar product of two vectors, but the imaginary part ( $a_2b_1 - a_1b_2$ ) has the form of a determinant:

$$\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} \quad (3.18)$$

which is an area. In fact, it is the area of a parallelogram formed by the two vectors  $z_1$  and  $z_2$  as shown in Fig. 3.1:

$$z_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \quad z_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}. \quad (3.19)$$

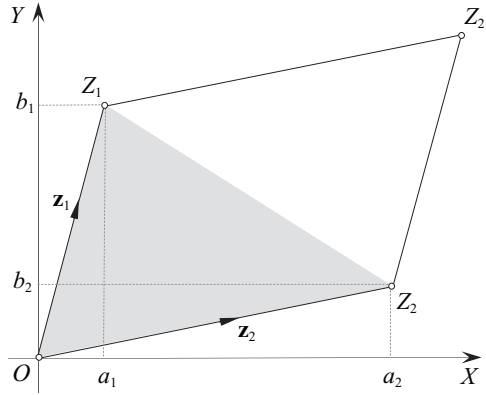


FIGURE 3.1.

The area of the shaded triangle  $\Delta OZ_2Z_1$  is half the area of the parallelogram  $(OZ_2Z_{21}Z_1)$ :

$$\text{area } \Delta OZ_2Z_1 = a_2b_1 - \frac{1}{2}a_2b_2 - \frac{1}{2}a_1b_1 - \frac{1}{2}(a_2 - a_1)(b_1 - b_2) \tag{3.20}$$

$$\text{area } (OZ_2Z_{21}Z_1) = 2 \times \Delta OZ_2Z_1 = 2a_2b_1 - a_2b_2 - a_1b_1 - a_2b_1 + a_2b_2 + a_1b_1 - a_1b_2 \tag{3.21}$$

$$\text{area } (OZ_2Z_{21}Z_1) = a_2b_1 - a_1b_2. \tag{3.22}$$

Therefore, the magnitude of the imaginary part of Eq. (3.17) is the area of the parallelogram formed by  $z_1$  and  $z_2$ .

Note that by reversing the vectors, the area becomes

$$\text{area } (OZ_1Z_{21}Z_2) = a_1b_2 - a_2b_1 \tag{3.23}$$

which reverses the sign of the area.

Before moving on, let's summarize in tabular form the rules for multiplying scalar and imaginary quantities (Table 3.1). This is a trivial task for complex arithmetic, but when we come to GA such tabulations are non-trivial and very important in understanding how the algebra functions.

TABLE 3.1

$\times$	$\beta$ (scalar)	$i$
$\alpha$ (scalar)	$\alpha\beta$	$i\alpha$
$i$	$i\beta$	$-1$

The axioms associated with complex numbers are as follows:

Given  $z, z_1, z_2, z_3 \in \mathbb{C}$ :

**Closure**

For all  $z_1$  and  $z_2$

$$\text{addition } z_1 + z_2 \in \mathbb{C} \quad (3.24)$$

$$\text{multiplication } z_1 z_2 \in \mathbb{C}. \quad (3.25)$$

**Identity**

For each  $z$  there is an identity element 0 and 1 such that:

$$\text{addition } z + 0 = 0 + z = z \quad (0 = 0 + 0i) \quad (3.26)$$

$$\text{multiplication } z(1) = (1)z = z \quad (1 = 1 + 0i). \quad (3.27)$$

**Inverse**

For each  $z$  there is an inverse element  $-z$  and  $1/z$  such that:

$$\text{addition } z + (-z) = -z + z = 0 \quad (3.28)$$

$$\text{multiplication } z \left( \frac{1}{z} \right) = \left( \frac{1}{z} \right) z = 1 \quad (z \neq 0). \quad (3.29)$$

**Associativity**

For all  $z_1, z_2$  and  $z_3$

$$\text{addition } z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \quad (3.30)$$

$$\text{multiplication } z_1(z_2 z_3) = (z_1 z_2) z_3. \quad (3.31)$$

**Commutativity**

For all  $z_1$  and  $z_2$

$$\text{addition } z_1 + z_2 = z_2 + z_1 \quad (3.32)$$

$$\text{multiplication } z_1 z_2 = z_2 z_1. \quad (3.33)$$

**Distributivity**

For all  $z_1, z_2$  and  $z_3$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \quad (3.34)$$

$$(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3. \quad (3.35)$$

### 3.4 The complex plane

The English mathematician, John Wallis [1616–1703], published his *De algebra tractatus* in 1685 in which he described a method of representing complex numbers as points on a plane. Unfortunately, his description was rather obtuse and failed to influence mathematicians of the day. Paul Nahin's book *An Imaginary Tale: The Story of  $\sqrt{-1}$*  [4] records that the Norwegian surveyor, Caspar Wessel [1745–1818], was another person to discover the geometric interpretation of complex numbers. His paper entitled “*On the Analytic Representation of Direction: An Attempt*” was announced in 1797 and in 1799 was published in a local Danish journal with a small international circulation, which ensured that his idea remained hidden for almost 100 years! Wessel's paper was discovered in 1895 by an antiquarian and its importance recognized by the Danish mathematician, Christian Juel [1855–1935], but it was too late — the Swiss-born writer, Jean-Robert Argand [1768–1822], had also thought of the same idea in 1806, and it is Argand's name that is associated with the *complex plane* rather than Wessel's.

The complex plane, or *Argand diagram*, uses two orthogonal axes to locate a complex number: the horizontal axis is the real axis whilst the vertical axis is the imaginary axis. Figure 3.2 illustrates this concept and shows how the complex number  $z = a + ib$  is viewed as a 2D point with coordinates  $(a, b)$  or as a vector  $\mathbf{z}$  with components  $[a \ b]^T$ .

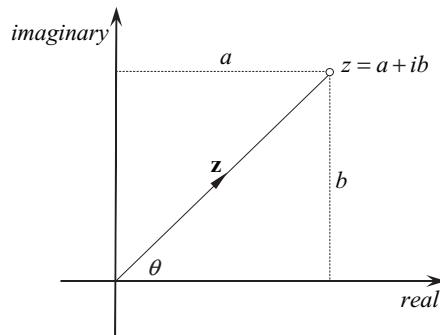


FIGURE 3.2.

The length of the position vector  $\mathbf{z}$  pointing to  $z$  is called the *modulus* of the complex number and written  $|z|$ , which makes

$$\|\mathbf{z}\| = |z| = \sqrt{a^2 + b^2} \quad (3.36)$$

and allows Eq. (3.10) to be written as

$$zz^* = |z|^2. \quad (3.37)$$

The polar angle  $\theta$  between the real axis and the position vector is identified using the inverse-tangent function ( $\tan^{-1}$ ). For the 1st and 4th quadrants, when  $a > 0$

$$\theta = \tan^{-1}(b/a) \quad (3.38)$$

and for the 2nd and 3rd quadrants, when  $a < 0$

$$\theta = 180^\circ + \tan^{-1}(b/a) \quad (3.39)$$

$$z = a + ib \equiv \left(\sqrt{a^2 + b^2}, \theta\right) = (|z|, \theta). \quad (3.40)$$

The angle  $\theta$  is also called the *argument* of the complex number.

### 3.5 $i$ as a rotor

Once Wessel had discovered a geometric interpretation for complex numbers, the rotational qualities of  $i$  became apparent. To illustrate this, let's trace the path of a complex number as it is repeatedly multiplied by  $i$ . We start with  $a + ib$ . Multiplying this by  $i$  we obtain  $-b + ia$ . Multiplying again we obtain  $-a - ib$ . Multiplying again we obtain  $b - ia$ , and finally  $a + ib$ . After four multiplications we return to the original complex number. When these intermediate numbers are located on an Argand diagram (Fig. 3.3) we see that each multiplication by  $i$  results in an anticlockwise rotation of  $90^\circ$ .

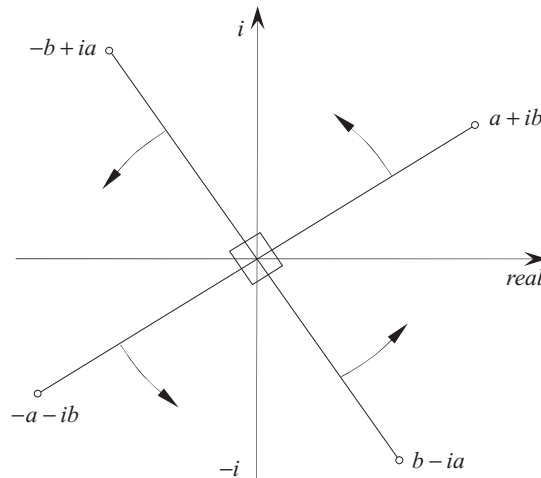


FIGURE 3.3.

Multiplying a complex number by  $-i$  performs a clockwise rotation. For example

$$-i(a + ib) = -ia + b = b - ia \quad (3.41)$$

which is confirmed by Fig. 3.3.

### 3.6 The product of two complex numbers

Referring to Fig. 3.2 we can see that

$$\frac{a}{|z|} = \cos \theta \quad \frac{b}{|z|} = \sin \theta \quad (3.42)$$

therefore we can write  $z$  in Cartesian form as

$$\begin{aligned} z &= a + ib = |z| \cos \theta + i|z| \sin \theta \\ z &= |z|(\cos \theta + i \sin \theta). \end{aligned} \quad (3.43)$$

Using this notation, let us examine the geometric interpretation of multiplying two complex numbers together. We start by defining two complex numbers in Cartesian form:

$$z_1 = r_1(\cos \theta + i \sin \theta) \quad (3.44)$$

$$z_2 = r_2(\cos \alpha + i \sin \alpha) \quad (3.45)$$

where  $r_1$  and  $r_2$  are their moduli.

Therefore,

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta + i \sin \theta)(\cos \alpha + i \sin \alpha) \\ &= r_1 r_2 (\cos \theta \cos \alpha - \sin \theta \sin \alpha) + i(\sin \theta \cos \alpha + \cos \theta \sin \alpha) \\ z_1 z_2 &= r_1 r_2 (\cos(\theta + \alpha) + i \sin(\theta + \alpha)) \end{aligned} \quad (3.46)$$

where we see that the resulting complex number has a modulus equal to the product of the moduli of the original numbers, whilst the new argument is the sum of the original arguments.

For completeness, let's calculate the product  $z_1 z_2^*$ :

$$\begin{aligned} z_1 z_2^* &= r_1 r_2 (\cos \theta + i \sin \theta)(\cos \alpha - i \sin \alpha) \\ &= r_1 r_2 (\cos \theta \cos \alpha + \sin \theta \sin \alpha) + i(\sin \theta \cos \alpha - \cos \theta \sin \alpha) \\ z_1 z_2^* &= r_1 r_2 (\cos(\theta - \alpha) + i \sin(\theta - \alpha)). \end{aligned} \quad (3.47)$$

If  $\beta$  is the angle between  $z_1$  and  $z_2$ , then  $\beta = \theta - \alpha$ , and

$$z_1 z_2^* = r_1 r_2 (\cos \beta + i \sin \beta).$$

Expanding, we obtain

$$z_1 z_2^* = r_1 r_2 \cos \beta + i r_1 r_2 \sin \beta. \quad (3.48)$$

We observe that the scalar part of Eq. (3.48) is the scalar product of two equivalent vectors, and the imaginary part is the magnitude of the corresponding vector product, and represents the area of the parallelogram formed by  $z_1$  and  $z_2$ .

### 3.7 Powers of complex numbers

Equation (3.47) can be used to illustrate what happens when a complex number is raised to some power. To begin, we let  $r = r_1 = r_2$  and  $\alpha = \theta$ , therefore

$$z^2 = r^2(\cos 2\theta + i \sin 2\theta) \quad (3.49)$$

where the square term emerges as a  $2\theta$  term. This is no accident, and Abraham De Moivre [1667–1754] proved that

$$z^n = r^n(\cos n\theta + i \sin n\theta). \quad (3.50)$$

### 3.8 $e, i, \sin$ and $\cos$

The Scottish mathematician and astronomer, David Gregory [1659–1708], published his first major book in 1684, *Exercitatio Geometrica de Dimensione Figurarum*, in which he extended his uncle's work on the method of quadratures by infinite series [5]. After his death, some unpublished papers on infinite series were discovered, and the same infinite series were later discovered in 1715 by the English mathematician, Brook Taylor [1685–1731]. Finally, in 1742, the Scottish professor of mathematics, Colin Maclaurin [1698–1746], described in his *Treatise of Fluxions* a notation that is continued in today's textbooks.

Basically, a Taylor or Maclaurin series consists of an infinite number of terms which normally converges as the number of terms is increased. For example, the base of the natural system of logarithms is defined as

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \quad (3.51)$$

but it can also be represented as an infinite series

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\right), \quad (3.52)$$

and  $e^x$  has the form

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}\right). \quad (3.53)$$

Similarly, it can be shown that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots, \quad (3.54)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots, \quad (3.55)$$

where  $x$  is in radians.

But what happens if we make  $x$  a complex number in Eq. (3.53)?

$$e^{ix} = \lim_{n \rightarrow \infty} \left(1 + \frac{ix}{1!} + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \dots + \frac{i^n x^n}{n!}\right) \quad (3.56)$$

which simplifies to

$$e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} \dots \quad (3.57)$$

Collecting up real and imaginary terms:

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots + i \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots \right) \quad (3.58)$$

which we recognize as the sin and cos functions, therefore,

$$e^{ix} = \cos x + i \sin x. \quad (3.59)$$

When  $x = \pi$  we obtain

$$\begin{aligned} e^{i\pi} &= \cos \pi + i \sin \pi \\ e^{i\pi} &= -1 \end{aligned} \quad (3.60)$$

which is the beautiful relationship discovered by Euler. But apart from its elegance, it seems to possess rotational properties; for if we multiply a complex number by  $e^{i\pi}$  its sign is reversed, i.e. rotated by  $180^\circ$  or  $\pi$  radians. Similarly,

$$\begin{aligned} e^{i\frac{\pi}{2}} &= \cos(\pi/2) + i \sin(\pi/2) \\ e^{i\frac{\pi}{2}} &= i \end{aligned} \quad (3.61)$$

which is equivalent to a rotation of  $90^\circ$ .

Equation (3.42) shows that a complex number can be represented as

$$z = r(\cos \theta + i \sin \theta) \quad (3.62)$$

and as

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (3.63)$$

$$z = r e^{i\theta}. \quad (3.64)$$

This is interpreted as the real quantity  $r$  rotated through an angle  $\theta$ .

We can rotate  $z$  through an angle  $\phi$  to  $z'$  using:

$$z' = z e^{i\phi} = r e^{i\theta} e^{i\phi}$$

and

$$z' = r e^{i(\theta+\phi)}. \quad (3.65)$$

These relationships form an important part of GA and should be understood before proceeding.

To conclude this section, let's evaluate  $i^i$ . Starting with Eq. (3.61) we have

$$i^i = \left( e^{i\frac{\pi}{2}} \right)^i = e^{i^2\frac{\pi}{2}} \quad (3.66)$$

therefore

$$i^i = e^{-\frac{\pi}{2}} = 0.2078795 \dots, \quad (3.67)$$

which is a real number!



### 3.9 Logarithm of a complex number

Now that we have discovered that complex numbers are intimately related to exponential  $e$ , it must be possible to find the natural logarithm of a complex number. Now since

$$i = e^{i\frac{\pi}{2}} \quad (3.68)$$

$$\ln i = i\frac{\pi}{2}. \quad (3.69)$$

However, this is the principal logarithm because if

$$z = r(\cos \theta + i \sin \theta) \quad (3.70)$$

then

$$\begin{aligned} \ln z &= \ln r + \ln(\cos \theta + i \sin \theta) \\ &= \ln r + \ln e^{i\theta} \end{aligned}$$

and

$$\ln z = \ln r + i\theta. \quad (3.71)$$

But  $\theta$  has an infinite number of forms:

$$\theta + 2n\pi \quad (3.72)$$

therefore a complex number has infinitely many logarithms, differing by integer multiples of  $i2\pi$ .

As an example, to find the principal logarithm of  $z = 2 + i2$ , we begin by finding the modulus of  $z$ :

$$|z| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}. \quad (3.73)$$

Next, we find the argument of  $z$ :

$$\theta = \arcsin \frac{2}{2\sqrt{2}} = 45^\circ = \frac{\pi}{4}. \quad (3.74)$$

Therefore,

$$z = 2\sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \quad (3.75)$$

and

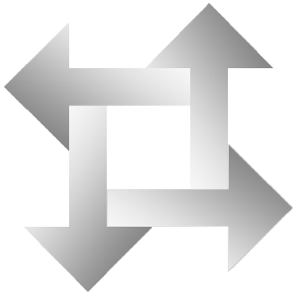
$$\begin{aligned} \ln z &= \ln 2\sqrt{2} + \ln \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \\ \ln z &= \ln 2\sqrt{2} + \ln e^{i\frac{\pi}{4}} \\ \ln z &= \ln 2\sqrt{2} + i \left( \frac{\pi}{4} \pm 2n\pi \right). \end{aligned} \quad (3.76)$$

The principal natural logarithm of  $z$  is

$$\ln 2\sqrt{2} + i\frac{\pi}{4} \approx 1.04 + i0.7854. \quad (3.77)$$

### 3.10 Summary

The algebra of complex numbers is closed for addition, subtraction, multiplication and division, which is why it is such a powerful framework for solving problems. However, in spite of this strength, it can still instil a sense of unease when we are faced with an answer that has real and imaginary parts. One just has to accept that certain mathematical problems cannot have a pure real solution, and the only way to express some sort of answer is to write it in two portions — real and imaginary. Geometric algebra is also riddled with similar uncomfortable imaginary concepts, and this chapter is an attempt to prepare the reader for these surprises.



# 4 Vector Algebra

## 4.1 Introduction

The evolution of vector analysis is an amazing story of intrigue, discovery, jealousy, prior art, coincidence and human weaknesses, all of which are covered in Michael Crowe's excellent book *A History of Vector Analysis* [6]. No one mathematician can claim that he or she discovered vectors — although the word *vector* was coined by the Irish physicist and mathematician, William Rowan Hamilton [1805–1865]. However, long before Hamilton was born, the German diplomat and natural philosopher, Gottfried Wilhelm Leibniz [1646–1716], had written to Christian Huygens in 1679:

*I have discovered certain elements of a new characteristic which is entirely different from algebra and which will have great advantages in representing to the mind, exactly and in a way faithful to its nature, even without figures, everything which depends on sense perception.* [7]

Although Leibniz had developed a system for annotating line segments with letters, he failed to show how such geometric elements could be manipulated arithmetically. We had to wait for the German mathematician, August Ferdinand Möbius [1790–1868], to publish his *Der barycentrische Calcul* in 1827, which contained the foundations of vectors in the form of *barycentric coordinates* [8]. In 1843 Möbius published *Die Elemente der Mechanik des Himmels* where he showed how directed line segments could be added and subtracted.

Meanwhile, in Ireland, Hamilton was looking for a 3D equivalent of complex numbers. Hamilton had started his search in 1830 and on October 16, 1843, he invented *quaternions*, which contained two products that became central to vector analysis and eventually to GA.

It was not easy for Hamilton to convince fellow mathematicians to embrace his new algebra. In fact, 47 years after his discovery, the third edition of Hamilton's *Treatise on Quaternions* (1890) contained a preface by Peter Tait who wrote about his surprise of “how little progress has recently been made with the development of Quaternions.” [9] Tait openly blamed the mathematician, Josiah Willard Gibbs [1839–1903], for “retarding” the development of quaternions in favor of his own newly developed field of vector analysis.

Gibbs' book *Elements of Vector Analysis* [10] was printed in two parts in 1881 and 1884, which laid the foundations for a notation that is used today. The symbols  $i, j$  and  $k$  of Hamilton were dropped and replaced by the unit Cartesian vectors  $\mathbf{i}, \mathbf{j}$  and  $\mathbf{k}$ , which are free from any imaginary connotation. Hamilton's quaternion products were renamed as the *direct product* and the *skew product*. Today we know them as the scalar (dot) product and the vector (cross) product respectively. With the support of scientists such as Oliver Heaviside, who applied vector analysis to the study of electromagnetic theory, vectors became the preferred system of the day and quaternions faded from the scene.

## 4.2 Vector quantities and their graphical representation

Elementary scalar algebra is used to solve most of our everyday problems, but when it comes to manipulating forces, velocities, magnetic fields, etc., where both magnitude and direction are mutually important, vectors provide an intuitive mathematical object for their encoding. Directed line segments, in turn, provide a simple, yet effective, way of visualizing vectors, where magnitude is represented by the line's length and direction by the line's orientation and arrow. Figure 4.1 shows the vector  $\mathbf{a}$  in  $\mathbb{R}^2$  formed from two points  $P$  and  $Q$ , where

$$P = (x_p, y_p) \text{ and } Q = (x_q, y_q) \quad (4.1)$$

and

$$\mathbf{a} = \begin{bmatrix} x_q - x_p \\ y_q - y_p \end{bmatrix} = \begin{bmatrix} x_a \\ y_a \end{bmatrix}. \quad (4.2)$$

The vector is directed from  $P$  to  $Q$ .

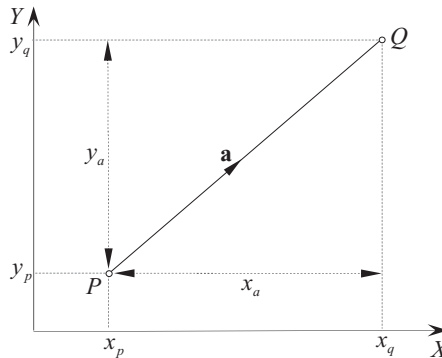


FIGURE 4.1.

Consequently, given the vector  $\mathbf{a} = [x_a \ y_a]^T$ , its magnitude is

$$\|\mathbf{a}\| = \sqrt{x_a^2 + y_a^2}. \quad (4.3)$$

Similarly, the magnitude of the vector  $\mathbf{a} = [x_a \ y_a \ z_a]^T$  in  $\mathbb{R}^3$  is

$$\|\mathbf{a}\| = \sqrt{x_a^2 + y_a^2 + z_a^2}. \quad (4.4)$$

A vector's orientation is defined in terms of the cosines of the angles it forms with the Cartesian axes as shown in Fig. 4.2:

$$\cos \alpha = \frac{x_a}{\|\mathbf{a}\|} \quad \cos \beta = \frac{y_a}{\|\mathbf{a}\|} \quad \cos \chi = \frac{z_a}{\|\mathbf{a}\|}. \quad (4.5)$$

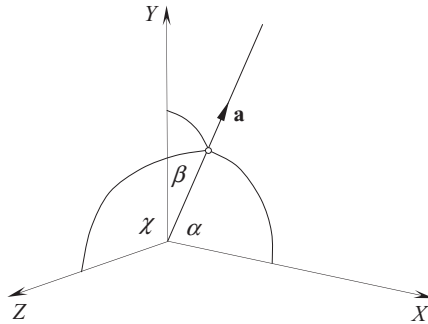


FIGURE 4.2.

The addition and subtraction of vectors is extremely simple as illustrated in Fig. 4.3, which shows the vector operations  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{a} - \mathbf{b}$ :

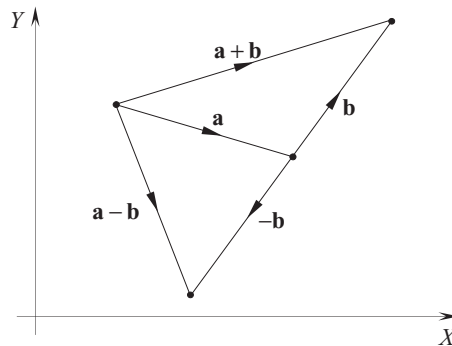


FIGURE 4.3.

Readers wishing to increase their knowledge beyond the contents of this chapter are recommended to read *Vector Spaces* of the Open University's Linear Algebra Block [11], which provided the background for the following sections. Readers wanting to see how vector analysis is employed in computer graphics are directed to the author's book *Vector Analysis for Computer Graphics* [12].

### 4.3 Vector spaces

Vectors are mathematical objects that may or may not have a geometrical significance, and although in this book we are interested in their application to geometry, they are fundamentally an algebraic object devoid of any real-world geometric attributes.

We begin with the term *vector space* which is nothing more than a set of elements called vectors. The two most familiar Euclidean spaces are  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , where  $\mathbb{R}^2$  is the set of ordered pairs of real numbers, and  $\mathbb{R}^3$  is the set of ordered triples of real numbers. Generalizing this definition,  $\mathbb{R}^n$  is the set of all ordered  $n$ -tuples, where an ordered  $n$ -tuple is a sequence of real numbers such as  $(u_1, u_2, \dots, u_n)$ , which permits us to define a vector as

$$\mathbf{u} = (u_1, u_2, \dots, u_n) \quad (4.6)$$

where  $n$  is a positive integer and describes the *dimension* of the vector space. When  $n > 3$  it is impossible to visualize the vector space, which is a limitation imposed by our brains rather than a limit to the physical existence of such objects.

With such a general description of a vector space one can propose all sorts of strange spaces that have as their elements: integers, complex numbers, matrices or even polynomials. So to prevent the creation of useless spaces, the definition of a vector space is subject to specific algebraic axioms, which we have already come across in our analysis of elementary algebra and complex numbers: closure, identity, inverse, associativity, commutativity and distributivity. Thus, a real vector space comprises a set of  $V$  elements (i.e. vectors) that are subject to vector addition and scalar multiplication, such that the following axioms hold:

$$\text{Given } \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{0} \in V \quad \text{and} \quad \lambda, \varepsilon \in \mathbb{R}: \quad (4.7)$$

### Closure

For all  $\mathbf{v}_1, \mathbf{v}_2$  and  $\lambda$

$$\text{addition} \quad \mathbf{v}_1 + \mathbf{v}_2 \in V \quad (4.8)$$

$$\text{multiplication} \quad \lambda \mathbf{v} \in V. \quad (4.9)$$

### Identity

For each  $\mathbf{v}$  there is an identity element  $\mathbf{0}$  and  $1$  such that:

$$\text{addition} \quad \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v} \quad (4.10)$$

$$\text{multiplication} \quad 1\mathbf{v} = \mathbf{v}. \quad (4.11)$$

### Inverse

For each  $\mathbf{v}$  there is an inverse element  $-\mathbf{v}$  such that:

$$\text{addition} \quad \mathbf{v} + (-\mathbf{v}) = -\mathbf{v} + \mathbf{v} = \mathbf{0}. \quad (4.12)$$

### Associativity

For all  $\mathbf{v}_1, \mathbf{v}_2$  and  $\mathbf{v}_3$

$$\text{addition} \quad (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3). \quad (4.13)$$

For all  $\mathbf{v}$ ,  $\lambda$  and  $\varepsilon$

$$\text{multiplication } \lambda(\varepsilon\mathbf{v}) = (\lambda\varepsilon)\mathbf{v}. \quad (4.14)$$

### Commutativity

For all  $\mathbf{v}_1$  and  $\mathbf{v}_2$

$$\text{addition } \mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1. \quad (4.15)$$

### Distributivity

For all  $\mathbf{v}$ ,  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\lambda$  and  $\varepsilon$

$$\lambda(\mathbf{v}_1 + \mathbf{v}_2) = \lambda\mathbf{v}_1 + \lambda\mathbf{v}_2 \quad (4.16)$$

$$(\lambda + \varepsilon)\mathbf{v} = \lambda\mathbf{v} + \varepsilon\mathbf{v}. \quad (4.17)$$

## 4.4 Linear combinations

The algebra of vector analysis employs a rich vocabulary of terms such as vector space, dimension, ordered pairs, triples and  $n$ -tuples, direction cosines, linear combination, linearly dependent, spanning sets, orthonormal basis, scalar and vector product. Geometric algebra possesses an even richer vocabulary and fortunately builds upon the vocabulary of vector algebra, which is why it is important to understand these terms. So let's continue by explaining the rest of the terms starting with linear combination.

Given two vectors  $\mathbf{a}$  and  $\mathbf{b}$  it is possible to construct a third vector  $\mathbf{c}$  such that

$$\mathbf{c} = \lambda\mathbf{a} + \varepsilon\mathbf{b} \quad (4.18)$$

where  $\lambda, \varepsilon \in \mathbb{R}$ .

The vector  $\mathbf{c}$  is then said to be a *linear combination* of  $\mathbf{a}$  and  $\mathbf{b}$ .

As an example, let's discover whether the vector  $(8, 11)$  is a linear combination of the vectors  $(1, 1)$  and  $(2, 3)$ .

If this is true, then

$$(8, 11) = \lambda(1, 1) + \varepsilon(2, 3) \quad (4.19)$$

and

$$8 = \lambda + 2\varepsilon \quad (4.20)$$

$$11 = \lambda + 3\varepsilon. \quad (4.21)$$

Subtracting Eq. (4.20) from (4.21)

$$\varepsilon = 3. \quad (4.22)$$

Substituting  $\varepsilon$  in Eq. (4.20)

$$8 = \lambda + 6 \quad (4.23)$$

and

$$\lambda = 2. \quad (4.24)$$

Therefore,

$$(8, 11) = 2(1, 1) + 3(2, 3) \quad (4.25)$$

which confirms that  $(8, 11)$  is a linear combination of the vectors  $(1, 1)$  and  $(2, 3)$ .

Now let's consider a second example where there is no linear relationship.

We pose the question: Is the vector  $(6, -2)$  a linear combination of the vectors  $(4, 4)$  and  $(2, 2)$ ? If it is, then

$$(6, -2) = \lambda(4, 4) + \varepsilon(2, 2) \quad (4.26)$$

and

$$6 = 4\lambda + 2\varepsilon \quad (4.27)$$

$$-2 = 4\lambda + 2\varepsilon. \quad (4.28)$$

Equations (4.27) and (4.28) are irreconcilable as it is impossible to find values of  $\lambda$  and  $\varepsilon$  that satisfy them, therefore the vector  $(6, -2)$  cannot be expressed as a linear combination of the vectors  $(4, 4)$  and  $(2, 2)$ . This condition is expressed formally as:

$$\text{Given } \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t \in V \quad \text{and} \quad \lambda_1, \lambda_2, \dots, \lambda_t \in \mathbb{R}. \quad (4.29)$$

Then any vector of the form  $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \dots + \lambda_t\mathbf{v}_t$ , is called a linear combination of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t$  and  $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \dots + \lambda_t\mathbf{v}_t \in V$ .

## 4.5 Spanning sets

Accepting the fact that a vector can be a linear combination of a number of other vectors, the idea of spanning sets arises when we pose the question: What is the set of all possible linear combinations of these vectors? For example, could there be two vectors, which when linearly combined in every possible combination, covers the Euclidean plane  $\mathbb{R}^2$ ? Well, the answer to this question is “yes” — there are two such vectors:  $(1, 0)$  and  $(0, 1)$ .

The vector  $(1, 0)$  aligns with the  $x$ -axis, whilst  $(0, 1)$  aligns with the  $y$ -axis. Now if we combine these as follows:

$$\mathbf{v} = \lambda(1, 0) + \varepsilon(0, 1) \quad (4.30)$$

where  $\lambda, \varepsilon \in \mathbb{R}$ , then all possible values of  $\mathbf{v}$  will cover  $\mathbb{R}^2$ , or  $\{(1, 0), (0, 1)\}$  spans  $\mathbb{R}^2$ . Alternatively,  $\mathbb{R}^2$  is spanned by  $\{(1, 0), (0, 1)\}$ .

Obviously, different collections of vectors create different spans, and the problem arises as to whether another vector is a member of a span.

Using the previous example, say we know that the set  $S$  is spanned by the vectors  $\{(1, 1), (2, 3)\}$  i.e.

$$S = \{(1, 1), (2, 3)\} \quad (4.31)$$

and we need to know whether  $(8, 11)$  is a member of  $\langle S \rangle$ .

We proceed as follows by writing

$$(8, 11) = \lambda(1, 1) + \varepsilon(2, 3) \quad (4.32)$$

which we know reveals that  $\lambda = 2$ ,  $\varepsilon = 3$ , and  $(8, 11) \in \langle S \rangle$ .



As another example, consider proving that  $\{(1, 3), (2, -4)\}$  is a spanning set for  $\mathbb{R}^2$ .

We proceed by noting that a vector  $(x, y) \in \mathbb{R}^2$  given that  $x, y \in \mathbb{R}$ . Therefore, we can write

$$(x, y) = \lambda(1, 3) + \varepsilon(2, -4) = (\lambda + 2\varepsilon, 3\lambda - 4\varepsilon). \quad (4.33)$$

Equating corresponding coordinates

$$x = \lambda + 2\varepsilon \quad (4.34)$$

$$y = 3\lambda - 4\varepsilon \quad (4.35)$$

which makes

$$\lambda = \frac{1}{5}(2x + y) \quad \varepsilon = \frac{1}{10}(3x - y) \quad (4.36)$$

therefore,

$$(x, y) = \frac{1}{5}(2x + y)(1, 3) + \frac{1}{10}(3x - y)(2, -4). \quad (4.37)$$

Which confirms that  $\{(1, 3), (2, -4)\}$  is a spanning set for  $\mathbb{R}^2$ . Or

$$\langle\langle(1, 3), (2, -4)\rangle\rangle = \mathbb{R}^2. \quad (4.38)$$

This relationship is expressed formally as:

$$\text{Let } \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t \in V \quad \text{and} \quad \lambda_1, \lambda_2, \dots, \lambda_t \in \mathbb{R}. \quad (4.39)$$

If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t\}$  then the span of  $S$ ,  $\langle S \rangle$ , is the set of all possible linear combinations

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \dots + \lambda_t \mathbf{v}_t, \quad (4.40)$$

or

$$\langle S \rangle = \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \dots + \lambda_t \mathbf{v}_t : \lambda_1, \lambda_2, \dots, \lambda_t \in \mathbb{R}\}. \quad (4.41)$$

## 4.6 Linear independence and dependence

A consequence of spanning sets is the idea of linear independence and dependence. For example, let's show that the following sets span  $\mathbb{R}^2$ :

$$\{(1, 0), (1, 1)\} \text{ and } \{(1, 0), (1, 1), (2, -2)\}. \quad (4.42)$$

We proceed by noting that given  $x, y \in \mathbb{R}$ , a vector

$$(x, y) \in \mathbb{R}^2. \quad (4.43)$$

Therefore, we can write

$$(x, y) = \lambda(1, 0) + \varepsilon(1, 1) = (\lambda + \varepsilon, \varepsilon). \quad (4.44)$$

Equating corresponding coordinates

$$x = \lambda + \varepsilon \quad (4.45)$$

$$y = \varepsilon \quad (4.46)$$

where

$$\lambda = x - y \quad \varepsilon = y \quad (4.47)$$

and

$$(x, y) = (x - y)(1, 0) + y(1, 1). \quad (4.48)$$

Which confirms that  $\{(1, 0), (1, 1)\}$  spans  $\mathbb{R}^2$ .

Similarly, we can write

$$(x, y) = \lambda(1, 0) + \varepsilon(1, 1) + \eta(2, -2). \quad (4.49)$$

Equating corresponding coordinates

$$x = \lambda + \varepsilon + 2\eta \quad (4.50)$$

$$y = \varepsilon - 2\eta \quad (4.51)$$

which creates a pair of simultaneous equations in three unknowns, which has an infinite number of solutions, one of which is

$$\lambda = x - y \quad \varepsilon = y \quad \eta = 0 \quad (4.52)$$

which implies that

$$(x, y) = (x - y)(1, 0) + y(1, 1) + 0(2, -2) \quad (4.53)$$

and confirms that  $\{(1, 0), (1, 1)\}$  spans  $\mathbb{R}^2$ .

Another solution is

$$\lambda = x + y \quad \varepsilon = 0 \quad \eta = -\frac{1}{2}y \quad (4.54)$$

which implies

$$(x, y) = (x + y)(1, 0) + 0(1, 1) - \frac{1}{2}y(2, -2) \quad (4.55)$$

and confirms that  $\{(1, 0), (2, -2)\}$  also spans  $\mathbb{R}^2$ .

Either way, one of the original vectors is redundant, which poses the question: What is the smallest number of vectors that spans a space? This is called the *minimal spanning set*.

The answer to this question is to discover whether one of the vectors is linearly dependent upon the other vectors. In the above example, if we can show that the vector  $(2, -2)$  is linearly related to  $\{(1, 0), (1, 1)\}$  then it is redundant.

We proceed as follows:

$$(2, -2) = \lambda(1, 0) + \varepsilon(1, 1) = (\lambda + \varepsilon, \varepsilon). \quad (4.56)$$

Equating corresponding coordinates

$$\lambda + \varepsilon = 2 \quad (4.57)$$

$$\varepsilon = -2 \quad (4.58)$$

which makes

$$\lambda = 4 \quad \varepsilon = -2 \quad (4.59)$$

and

$$(2, -2) = 4(1, 0) - 2(1, 1) \quad (4.60)$$

which can be substituted in Eq. (4.49):

$$(x, y) = \lambda(1, 0) + \varepsilon(1, 1) + \eta(4(1, 0) - 2(1, 1)). \quad (4.61)$$

Simplifying

$$(x, y) = (\lambda + 4\eta)(1, 0) + (\varepsilon - 2\eta)(1, 1). \quad (4.62)$$

Therefore,  $\{(1, 0), (1, 1)\}$  spans  $\mathbb{R}^2$ .

To identify whether a spanning set is minimal or not, we must establish whether any member of the set is a linear combination of the others. Formally, this is defined as follows:

Given 
$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t \in V \quad \text{and} \quad \lambda_1, \lambda_2, \dots, \lambda_t \in \mathbb{R}. \quad (4.63)$$

$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_t\}$  is *linearly dependent* if  $\lambda_1, \lambda_2, \dots, \lambda_t$  exist that are not all zero, such that

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \dots + \lambda_t \mathbf{v}_t = 0 \quad (4.64)$$

otherwise it is *linearly independent*.

Consequently, for a set of vectors to be a minimal spanning set for a vector space  $V$ , they must be linearly independent. Furthermore, such a set is called a *basis* for the vector space  $V$ .

## 4.7 Standard bases

Because a vector space can have an infinite number of vector sets as its basis, it is tempting to enquire whether any preferred set exists? To which the answer is “yes”. For  $\mathbb{R}^2$  they are  $\{(1, 0), (0, 1)\}$  and for  $\mathbb{R}^3$  they are  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ . In general, the standard basis for  $\mathbb{R}^n$  is the set of  $n$  vectors

$$\{(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1)\}. \quad (4.65)$$

## 4.8 Orthogonal bases

Orthogonal means at right-angles to, and an *orthogonal basis* implies that its vectors are at right-angles. Thus if the basis vectors  $\mathbf{v}_1 = (x_1, y_1)$  and  $\mathbf{v}_2 = (x_2, y_2)$  are orthogonal if

$$x_1 x_2 + y_1 y_2 = 0. \quad (4.66)$$

For example, the standard base for  $\mathbb{R}^3$  is  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  which are orthogonal:

$$(1 \times 0 \times 0 + 0 \times 1 \times 0 + 0 \times 0 \times 1) = 0. \quad (4.67)$$

Readers familiar with the scalar (dot) product will understand why this is so. We examine this product in section 4.10.

## 4.9 Dimension

When we use the word '*dimension*' we normally associate it with physical space, which is apparently, three-dimensional. In vector notation this corresponds to  $\mathbb{R}^3$ . As vector spaces may have any number of dimensions, in this context, dimension means the number of vectors used in the basis for the vector space.

## 4.10 Subspaces

Very often sets contain smaller subsets. For example, the set of integers  $\mathbb{N}$  contains two subsets  $\mathbb{N}_O$  and  $\mathbb{N}_E$  which are the odd and even integers respectively. The same concept arises in vector spaces, where a vector space  $V$  contains a subset  $S$ , which is called a *subspace* if it satisfies all the axioms associated with  $V$ . In general, these axioms are:

- $\mathbf{0} \in S$
- $S$  is closed under vector addition
- $S$  is closed under scalar multiplication

For example, the first axiom  $\mathbf{0} \in S$ , ensures that any proper subset of  $\mathbb{R}^3$  must include

- a zero-dimensional subset  $\{\mathbf{0}\}$
- a one-dimensional subset  $y = ax$  (a line through the origin)
- a two-dimensional subset  $z = ax + by$  (a plane through the origin).

For example, to show that the set of vectors

$$S = \{(x, 4x) : x \in R\} \quad (4.68)$$

is a subspace of  $\mathbb{R}^3$ , we must demonstrate that the above axioms apply to the set  $S$ .

### 1st Axiom

When  $x = 0$  then  $(x, 4x) = (0, 0)$  and  $\mathbf{0} \in S$ .

### 2nd Axiom

Let  $\mathbf{v}_1 = (x_1, 4x_1)$  and  $\mathbf{v}_2 = (x_2, 4x_2)$ , which belong to  $S$ .

Therefore,

$$\begin{aligned} \mathbf{v}_1 + \mathbf{v}_2 &= (x_1, 4x_1) + (x_2, 4x_2) \\ \mathbf{v}_1 + \mathbf{v}_2 &= (x_1 + x_2, 4(x_1 + x_2)) \end{aligned} \quad (4.69)$$

which is closed.

**3rd Axiom**

Let

$$\mathbf{v} = (x, 4x) \in S \quad \text{and} \quad \lambda \in \mathbb{R}. \quad (4.70)$$

Therefore,

$$\lambda \mathbf{v} = \lambda(x, 4x) = (\lambda x, 4(\lambda x)) \quad (4.71)$$

which is closed.

As all three axioms hold,

$$S = \{(x, 4x) : x \in \mathbb{R}\} \quad (4.72)$$

is a subspace of  $\mathbb{R}^3$ .

To demonstrate where a subset is not a subspace of  $\mathbb{R}^3$ , consider the set

$$S = \{(x, y, x + y + 3) : x, y \in \mathbb{R}\}. \quad (4.73)$$

This is obviously a plane and is defined by

$$z = x + y + 3. \quad (4.74)$$

When  $x = y = 0, z = 3$ , and it fails the first axiom, and there is no need to investigate the other two axioms.

**4.11 Scalar product**

There are two products associated with vectors: the scalar and the vector product. As the name suggests the scalar product results in a scalar, and no chance of closing the algebra. By definition, the scalar product for two vectors in  $\mathbb{R}^2$  is

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos \theta = x_1 x_2 + y_1 y_2 \quad (4.75)$$

where

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad (4.76)$$

and

$\theta$  is the angle between  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

Similarly, for two vectors in  $\mathbb{R}^3$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos \theta = x_1 x_2 + y_1 y_2 + z_1 z_2 \quad (4.77)$$

where

$$\mathbf{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad (4.78)$$

and

$\theta$  is the angle between  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

In general, for two vectors in  $\mathbb{R}^n$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos \theta = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \dots + \alpha_n \beta_n \quad (4.79)$$

where

$$\mathbf{v}_1 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \quad (4.80)$$

and

$\theta$  is the angle between  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

The product is also known as the *dot product* to reflect the ‘ $\cdot$ ’ used as the operator.

As an example, given two vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \quad (4.81)$$

then

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 1 \times 2 + 2 \times 2 + 3 \times 4 = 18 \quad (4.82)$$

and

$$\|\mathbf{v}_1\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \quad (4.83)$$

$$\|\mathbf{v}_2\| = \sqrt{2^2 + 2^2 + 4^2} = \sqrt{24}. \quad (4.84)$$

Therefore, the angle between  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is

$$\theta = \cos^{-1} \left( \frac{18}{\sqrt{14}\sqrt{24}} \right) \approx 10.9^\circ. \quad (4.85)$$

Before proceeding with the vector product, we need to examine two laws associated with the dot product: the commutative and distributive laws and we also need to confirm the role of scalars.

### Commutative Law $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

As

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta \quad (4.86)$$

the commutative law of scalar multiplication permits us to write Eq. (4.86) as

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{b}\| \|\mathbf{a}\| \cos \theta = \mathbf{b} \cdot \mathbf{a} \quad (4.87)$$

therefore,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}. \quad (4.88)$$

**Distributive Law**  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ 

Given

$$\mathbf{a} = x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k} \quad (4.89)$$

$$\mathbf{b} = x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k} \quad (4.90)$$

$$\mathbf{c} = x_c \mathbf{i} + y_c \mathbf{j} + z_c \mathbf{k} \quad (4.91)$$

then

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= (x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}) \cdot ((x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k}) + (x_c \mathbf{i} + y_c \mathbf{j} + z_c \mathbf{k})) \\ &= (x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k}) \cdot ((x_b + x_c) \mathbf{i} + (y_b + y_c) \mathbf{j} + (z_b + z_c) \mathbf{k}) \\ &= x_a(x_b + x_c) + y_a(y_b + y_c) + z_a(z_b + z_c) \\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= (x_a x_b + x_a x_c + y_a y_b) + (y_a y_c + z_a z_b + z_a z_c) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \end{aligned} \quad (4.92)$$

therefore,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}. \quad (4.93)$$

**4.12 Vector product**

As the name suggests, the vector product results in a vector, which ensures closure, but it only exists in  $\mathbb{R}^3$ . By definition, the vector product for two vectors is

$$\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3 \quad (4.94)$$

where

$$\|\mathbf{v}_3\| = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \sin \theta \quad (4.95)$$

$\theta$  is the angle between  $\mathbf{v}_1$  and  $\mathbf{v}_2$

and

$\mathbf{v}_3$  is orthogonal to  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

$\mathbf{v}_3$  is given by

$$\mathbf{v}_3 = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \mathbf{i} + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \mathbf{k}. \quad (4.96)$$

As an example, let's find the vector orthogonal to  $\mathbf{v}_1 \times \mathbf{v}_2$  where

$$\mathbf{v}_1 = [1 \ 0 \ 0]^T \text{ and } \mathbf{v}_2 = [0 \ 1 \ 0]^T. \quad (4.97)$$

$$\mathbf{v}_3 = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} \quad (4.98)$$

$$\mathbf{v}_3 = \mathbf{k} \quad (4.99)$$

which is correct.

The vector product is also known as the cross product to reflect the ‘ $\times$ ’ symbol used as the operator.

### Commutative Law $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$

The commutative law does not hold for the vector product.

Given

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix} \\ &= \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} \mathbf{i} + \begin{vmatrix} z_a & x_a \\ z_b & x_b \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix} \mathbf{k} \\ \mathbf{a} \times \mathbf{b} &= (y_a z_b - y_b z_a) \mathbf{i} + (z_a x_b - z_b x_a) \mathbf{j} + (x_a y_b - x_b y_a) \mathbf{k}. \end{aligned} \quad (4.100)$$

Whereas

$$\begin{aligned}\mathbf{b} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_b & y_b & z_b \\ x_a & y_a & z_a \end{vmatrix} \\ &= \begin{vmatrix} y_b & z_b \\ y_a & z_a \end{vmatrix} \mathbf{i} + \begin{vmatrix} z_b & x_b \\ z_a & x_a \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix} \mathbf{k} \\ &= (y_b z_a - y_a z_b) \mathbf{i} + (z_b x_a - z_a x_b) \mathbf{j} + (x_b y_a - x_a y_b) \mathbf{k} \\ &= -(y_a z_b - y_b z_a) \mathbf{i} + (z_a x_b - z_b x_a) \mathbf{j} + (x_a y_b - x_b y_a) \mathbf{k} \\ \mathbf{b} \times \mathbf{a} &= -\mathbf{a} \times \mathbf{b} \end{aligned} \quad (4.101)$$

therefore,

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}. \quad (4.102)$$

### Distributive Law $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

Given

$$\mathbf{a} = x_a \mathbf{i} + y_a \mathbf{j} + z_a \mathbf{k} \quad (4.103)$$

$$\mathbf{b} = x_b \mathbf{i} + y_b \mathbf{j} + z_b \mathbf{k} \quad (4.104)$$

$$\mathbf{c} = x_c \mathbf{i} + y_c \mathbf{j} + z_c \mathbf{k} \quad (4.105)$$

and

$$\mathbf{d} = \mathbf{b} + \mathbf{c} = x_d \mathbf{i} + y_d \mathbf{j} + z_d \mathbf{k} \quad (4.106)$$



then

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_a & y_a & z_a \\ x_d & y_d & z_d \end{vmatrix} \quad (4.107)$$

$$\mathbf{a} \times \mathbf{d} = \begin{vmatrix} y_a & z_a \\ y_d & z_d \end{vmatrix} \mathbf{i} + \begin{vmatrix} z_a & x_a \\ z_d & x_d \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_a & y_a \\ x_d & y_d \end{vmatrix} \mathbf{k}. \quad (4.108)$$

Substituting  $\mathbf{b} + \mathbf{c}$  for  $\mathbf{d}$

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \begin{vmatrix} y_a & z_a \\ y_b + y_c & z_b + z_c \end{vmatrix} \mathbf{i} + \begin{vmatrix} z_a & x_a \\ z_b + z_c & x_b + x_c \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_a & y_a \\ x_b + x_c & y_b + y_c \end{vmatrix} \mathbf{k} \\ &= (y_a(z_b + z_c) - z_a(y_b + y_c))\mathbf{i} \\ &\quad + (z_a(x_b + x_c) - x_a(z_b + z_c))\mathbf{j} \\ &\quad + (x_a(y_b + y_c) - y_a(x_b + x_c))\mathbf{k} \\ &= (y_a z_b - y_b z_a)\mathbf{i} + (y_a z_c - y_c z_a)\mathbf{i} \\ &\quad + (z_a x_b - z_b x_a)\mathbf{j} + (z_a x_c - z_c x_a)\mathbf{j} \\ &\quad + (x_a y_b - x_b y_a)\mathbf{k} + (x_a y_c - x_c y_a)\mathbf{k} \\ \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}. \end{aligned} \quad (4.109)$$

### 4.13 Summary

In this chapter we have reviewed vectors and the axioms of vector algebra. Perhaps the most important points to acknowledge are that the scalar product converts two vectors into a scalar; the cross product anticommutes, and only works in  $\mathbb{R}^3$ . The next chapter reviews quaternions and their associated algebra.



# 5 Quaternion Algebra

## 5.1 Introduction

Quaternions are the result of one man's determination to find the 3D equivalent of complex numbers. Sir William Rowan Hamilton was the man, and in 1843 he revealed to the world his discovery which had taken him over a decade to resolve.

Knowing that a complex number in  $\mathbb{R}^2$  has the form

$$z = a + ib \quad (5.1)$$

it is reasonable to presume that a complex number in  $\mathbb{R}^3$  should take the form

$$z = a + ib + jc \quad (5.2)$$

where  $i$  and  $j$  are unit imaginaries:  $i^2 = j^2 = -1$ . However, when two such objects are multiplied together we have

$$z_1 z_2 = (a_1 + ib_1 + jc_1)(a_2 + ib_2 + jc_2) \quad (5.3)$$

which expands to

$$z_1 z_2 = a_1 a_2 + ia_1 b_2 + ja_1 c_2 + ib_1 a_2 + i^2 b_1 b_2 + ij b_1 c_2 + jc_1 a_2 + jic_1 b_2 + j^2 c_1 c_2. \quad (5.4)$$

Substituting  $i^2 = j^2 = -1$  into Eq. (5.4) and collecting up like terms we obtain

$$z_1 z_2 = (a_1 a_2 - b_1 b_2 - c_1 c_2) + i(a_1 b_2 + b_1 a_2) + j(a_1 c_2 + c_1 a_2) + ij b_1 c_2 + jic_1 b_2 \quad (5.5)$$

which leaves the terms  $ij$  and  $ji$  undefined. These stumped Hamilton for many years, but his tenacity won the day, and he eventually came up with an incredible idea which involved extending the triple into a 4-tuple:

$$z = a + ib + jc + kd. \quad (5.6)$$

When two such objects are multiplied together we have

$$z_1 z_2 = (a_1 + ib_1 + jc_1 + kd_1)(a_2 + ib_2 + jc_2 + kd_2) \quad (5.7)$$

which expands to

$$\begin{aligned}
 z_1 z_2 &= a_1 a_2 + i a_1 b_2 + j a_1 c_2 + k a_1 d_2 \\
 &\quad + i b_1 a_2 + i^2 b_1 b_2 + i j b_1 c_2 + i k b_1 d_2 \\
 &\quad + j c_1 a_2 + j i c_1 b_2 + j^2 c_1 c_2 + j k c_1 d_2 \\
 &\quad + k d_1 a_2 + k i d_1 b_2 + k j d_1 c_2 + k^2 d_1 d_2.
 \end{aligned} \tag{5.8}$$

Substituting  $i^2 = j^2 = k^2 = -1$  in Eq. (5.8) and collecting up like terms we obtain

$$\begin{aligned}
 z_1 z_2 &= a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\
 &\quad + i(a_1 b_2 + b_1 a_2) + j(a_1 c_2 + c_1 a_2) + k(a_1 d_2 + d_1 a_2) \\
 &\quad + i j b_1 c_2 + i k b_1 d_2 + j i c_1 b_2 + j k c_1 d_2 + k i d_1 b_2 + k j d_1 c_2.
 \end{aligned} \tag{5.9}$$

But this, too, has some undefined terms:  $ij, ik, ji, jk, ki, kj$ . However, Hamilton was a genius and he resolved the problem by proposing the following rules:

$$ij = k \quad jk = i \quad ki = j \quad ji = -k \quad kj = -i \quad ik = -j \tag{5.10}$$

which when substituted into Eq. (5.9) produces

$$\begin{aligned}
 z_1 z_2 &= a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\
 &\quad + i(a_1 b_2 + b_1 a_2) + j(a_1 c_2 + c_1 a_2) + k(a_1 d_2 + d_1 a_2) \\
 &\quad + k b_1 c_2 - j b_1 d_2 - k c_1 b_2 + i c_1 d_2 + j d_1 b_2 - i d_1 c_2.
 \end{aligned} \tag{5.11}$$

Collecting up like terms we obtain

$$\begin{aligned}
 z_1 z_2 &= a_1 a_2 - (b_1 b_2 + c_1 c_2 + d_1 d_2) \\
 &\quad + i(a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2) \\
 &\quad + j(a_1 c_2 + c_1 a_2 + d_1 b_2 - b_1 d_2) \\
 &\quad + k(a_1 d_2 + d_1 a_2 + b_1 c_2 - c_1 b_2).
 \end{aligned} \tag{5.12}$$

Although this does not have any undefined terms it can be tidied up as follows:

$$\begin{aligned}
 z_1 z_2 &= a_1 a_2 - (b_1 b_2 + c_1 c_2 + d_1 d_2) \\
 &\quad + a_1 (i b_2 + j c_2 + k d_2) + a_2 (i b_1 + j c_1 + k d_1) \\
 &\quad + i(c_1 d_2 - d_1 c_2) + j(d_1 b_2 - b_1 d_2) + k(b_1 c_2 - c_1 b_2)
 \end{aligned} \tag{5.13}$$

The last step is to write the original object as the sum of a scalar and a vector starting with:

$$z_1 = s_1 + \mathbf{v}_1 \quad z_2 = s_2 + \mathbf{v}_2 \tag{5.14}$$

and the following symmetry emerges:

$$z_1 z_2 = s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 + s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2. \tag{5.15}$$

Hamilton called this object a ‘*quaternion*’ and gave the name ‘*vector*’ to the imaginary portion.

The product  $\mathbf{v}_1 \cdot \mathbf{v}_2$  is equivalent to

$$b_1b_2 + c_1c_2 + d_1d_2 \tag{5.16}$$

and became the *scalar* or *dot product*, whilst  $\mathbf{v}_1 \times \mathbf{v}_2$ , which is equivalent to

$$i(c_1d_2 - d_1c_2) + j(d_1b_2 - b_1d_2) + k(b_1c_2 - c_1b_2) \tag{5.17}$$

became the *vector* or *cross product* and led to the definitions:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos \theta \tag{5.18}$$

and

$$\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3 \tag{5.19}$$

where

$$\mathbf{v}_3 = i(c_1d_2 - d_1c_2) + j(d_1b_2 - b_1d_2) + k(b_1c_2 - c_1b_2) \tag{5.20}$$

and

$$\|\mathbf{v}_3\| = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \sin \theta \tag{5.21}$$

where

$\theta$  is the angle between  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

Strictly speaking, the  $i, j$  and  $k$  are unit imaginaries which obey Hamilton's rules where

$$i^2 = j^2 = k^2 = ijk = -1 \tag{5.22}$$

$$ij = k \quad jk = i \quad ki = j \quad ji = -k \quad kj = -i \quad ik = -j. \tag{5.23}$$

However, when vector algebra became the preferred system over quaternion algebra, the  $i, j$  and  $k$  terms became the Cartesian unit vectors  $\mathbf{i}, \mathbf{j}$  and  $\mathbf{k}$ .

One very important feature of quaternion algebra is its anticommuting rules. Maintaining order between the unit imaginaries is vital for the algebra to remain consistent, which is also a feature of GA.

## 5.2 Adding quaternions

Two quaternions  $q_1$  and  $q_2$

$$q_1 = s_1 + ix_1 + jy_1 + kz_1 \tag{5.24}$$

$$q_2 = s_2 + ix_2 + jy_2 + kz_2 \tag{5.25}$$

are equal if, and only if, their corresponding terms are equal. Furthermore, like vectors, they can be added or subtracted as follows:

$$q_1 \pm q_2 = [(s_1 \pm s_2) + i(x_1 \pm x_2) + j(y_1 \pm y_2) + k(z_1 \pm z_2)]. \tag{5.26}$$

For example, given two quaternions

$$q_1 = 1 + i2 + j3 + k4 \quad (5.27)$$

$$q_2 = 2 - i + j5 - k2 \quad (5.28)$$

their sum is given by

$$q_1 + q_2 = 3 + i + j8 + k2. \quad (5.29)$$

### 5.3 The quaternion product

Given two quaternions

$$q_1 = s_1 + \mathbf{v}_1 = s_1 + ix_1 + jy_1 + kz_1 \quad (5.30)$$

$$q_2 = s_2 + \mathbf{v}_2 = s_2 + ix_2 + jy_2 + kz_2 \quad (5.31)$$

their product is given by

$$q_1q_2 = s_1s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2 + s_1\mathbf{v}_2 + s_2\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 \quad (5.32)$$

which is still a quaternion and ensures closure. However, the quaternion product anticommutes, which we can prove by computing  $q_2q_1$ :

$$q_2q_1 = s_2s_1 - \mathbf{v}_2 \cdot \mathbf{v}_1 + s_2\mathbf{v}_1 + s_1\mathbf{v}_2 + \mathbf{v}_2 \times \mathbf{v}_1. \quad (5.33)$$

The pure scalar terms  $s_2s_1$ ,  $\mathbf{v}_2 \cdot \mathbf{v}_1$  and the products  $s_2\mathbf{v}_1$  and  $s_1\mathbf{v}_2$  commute, but the cross product  $\mathbf{v}_2 \times \mathbf{v}_1$  anticommutes, therefore  $q_1q_2 \neq q_2q_1$ .

For example, given the quaternions

$$q_1 = 1 + i2 + j3 + k4 \quad (5.34)$$

$$q_2 = 2 - i + j5 - k2 \quad (5.35)$$

their product  $q_1q_2$  is

$$q_1q_2 = (1 + i2 + j3 + k4)(2 - i + j5 - k2) \quad (5.36)$$

$$= [1 \times 2 - (2 \times (-1) + 3 \times 5 + 4 \times (-2))$$

$$+ 1(-i + j5 - k2) + 2(i2 + j3 + k4)$$

$$+ i(3 \times (-2) - 4 \times 5) + j(4 \times (-1) - (-2) \times 2) + k(2 \times 5 - (-1) \times 3)]$$

$$= -3 + i3 + j11 + k6 - i26 + k13$$

$$q_1q_2 = -3 - i23 + j11 + k19 \quad (5.37)$$

which is a quaternion.

Whereas the product  $q_2q_1$  is

$$\begin{aligned}
 q_2q_1 &= (2 - i + j5 - k2)(1 + i2 + j3 + k4) \\
 &= [2 - ((-1) \times 2 + 5 \times 3 + (-2) \times 4) \\
 &\quad + 2(i2 + j3 + k4) + 1(-i + j5 - k2) \\
 &\quad + i(5 \times 4 - 3 \times (-2)) + j((-2) \times 2 - 4 \times (-1)) + k((-1) \times 3 - 2 \times 5)] \\
 q_2q_1 &= -3 + i29 + j11 - k7
 \end{aligned} \tag{5.38}$$

which is also a quaternion, but  $q_2q_1 \neq q_1q_2$ .

### 5.4 The magnitude of a quaternion

Given the quaternion

$$q = s + ix + jy + kz \tag{5.39}$$

its magnitude is defined as

$$\|q\| = \sqrt{s^2 + x^2 + y^2 + z^2}. \tag{5.40}$$

For example, given the quaternion

$$q = 1 + i2 + j3 + k4 \tag{5.41}$$

$$\|q\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}. \tag{5.42}$$

### 5.5 The unit quaternion

Like vectors, quaternions have a unit form where the magnitude equals unity. For example, the magnitude of the quaternion

$$q = 1 + i2 + j3 + k4 \tag{5.43}$$

is

$$\|q\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30} \tag{5.44}$$

therefore, the unit quaternion  $\hat{q}$  equals

$$\hat{q} = \frac{1}{\sqrt{30}}(1 + i2 + j3 + k4). \tag{5.45}$$

### 5.6 The pure quaternion

Hamilton named a quaternion with a zero scalar term a *pure quaternion*. For example,

$$q_1 = ix_1 + jy_1 + kz_1 \text{ and } q_2 = ix_2 + jy_2 + kz_2 \tag{5.46}$$

are pure quaternions. Let's see what happen when we multiply them together:

$$q_1 q_2 = (ix_1 + jy_1 + kz_1)(ix_2 + jy_2 + kz_2)$$

$$q_1 q_2 = [-(x_1 x_2 + y_1 y_2 + z_1 z_2) + i(y_1 z_2 - y_2 z_1) + j(z_1 x_2 - z_2 x_1) + k(x_1 y_2 - x_2 y_1)] \quad (5.47)$$

which is no longer a pure quaternion, as a negative scalar term has emerged. Thus the algebra of pure quaternions is not closed.

## 5.7 The conjugate of a quaternion

Given the quaternion

$$q = s + \mathbf{v}$$

$$q = s + ix + jy + kz \quad (5.48)$$

by definition, its conjugate is

$$\bar{q} = s - \mathbf{v} = s - (ix + jy + kz). \quad (5.49)$$

For example, the quaternion

$$q = 1 + i2 + j3 + k4 \quad (5.50)$$

its conjugate is

$$\bar{q} = 1 - i2 - j3 - k4. \quad (5.51)$$

## 5.8 The inverse quaternion

Given the quaternion

$$q = s + ix + jy + kz \quad (5.52)$$

the inverse quaternion  $q^{-1}$  is

$$q^{-1} = \frac{s - ix - jy - kz}{\|q\|^2} \quad (5.53)$$

because this satisfies the product

$$qq^{-1} = \frac{(s + ix + jy + kz)(s - ix - jy - kz)}{\|q\|^2} = 1. \quad (5.54)$$

We can show that this is true by expanding the product as follows:

$$qq^{-1} = \frac{\left( s^2 - isx - jsy - ksz + isx + x^2 - ijxy - ikxz + \right)}{\|q\|^2}$$

$$= \frac{s^2 + x^2 + y^2 + z^2 - ijxy - ikxz - jixy - jkyz - kixz - kjyz}{\|q\|^2}$$

$$qq^{-1} = \frac{s^2 + x^2 + y^2 + z^2}{\|q\|^2} = 1 \quad (5.55)$$

and confirms that the inverse quaternion  $q^{-1}$  is

$$q^{-1} = \frac{\bar{q}}{\|q\|^2}. \quad (5.56)$$

Because the unit imaginaries do not commute, we need to discover whether

$$qq^{-1} = q^{-1}q. \quad (5.57)$$

Expanding this product

$$\begin{aligned} q^{-1}q &= \frac{(s - ix - jy - kz)(s + ix + jy + kz)}{\|q\|^2} \\ &= \frac{(s^2 + isx + jsy + ksz - isx + x^2 - ijxy - ikxz - jsy - jixy + y^2 - jkyz - ksz - kixz - kjyz + z^2)}{\|q\|^2} \\ &= \frac{s^2 + x^2 + y^2 + z^2 - ijxy - ikxz - jixy - jkyz - kixz - kjyz}{\|q\|^2} \\ q^{-1}q &= \frac{s^2 + x^2 + y^2 + z^2}{\|q\|^2} = 1 \end{aligned}$$

therefore,

$$qq^{-1} = q^{-1}q. \quad (5.58)$$

## 5.9 Quaternion algebra

The axioms associated with quaternions are as follows:

Given  $q, q_1, q_2, q_3 \in \mathbb{C}$ : (5.59)

### Closure

For all  $q_1$  and  $q_2$

$$\text{addition } q_1 + q_2 \in \mathbb{C} \quad (5.60)$$

$$\text{multiplication } q_1q_2 \in \mathbb{C}. \quad (5.61)$$

### Identity

For each  $q$  there is an identity element  $\mathbf{0}$  and  $\mathbf{1}$  such that:

$$\text{addition } q + \mathbf{0} = \mathbf{0} + q = q \quad (\mathbf{0} = 0 + i0 + j0 + k0) \quad (5.62)$$

$$\text{multiplication } q(\mathbf{1}) = (\mathbf{1})q = q \quad (\mathbf{1} = 1 + i0 + j0 + k0). \quad (5.63)$$



## Inverse

For each  $q$  there is an inverse element  $-q$  and  $q^{-1}$  such that:

$$\text{addition } q + (-q) = -q + q = 0 \quad (5.64)$$

$$\text{multiplication } qq^{-1} = q^{-1}q = 1 \quad (q \neq 0). \quad (5.65)$$

## Associativity

For all  $q_1, q_2$  and  $q_3$

$$\text{addition } q_1 + (q_2 + q_3) = (q_1 + q_2) + q_3 \quad (5.66)$$

$$\text{multiplication } q_1(q_2q_3) = (q_1q_2)q_3. \quad (5.67)$$

## Commutativity

For all  $q_1$  and  $q_2$

$$\text{addition } q_1 + q_2 = q_2 + q_1 \quad (5.68)$$

$$\text{multiplication } q_1q_2 \neq q_2q_1. \quad (5.69)$$

## Distributivity

For all  $q_1, q_2$  and  $q_3$

$$q_1(q_2 + q_3) = q_1q_2 + q_1q_3 \quad (5.70)$$

$$(q_1 + q_2)q_3 = q_1q_3 + q_2q_3. \quad (5.71)$$

## 5.10 Rotating vectors using quaternions

One excellent application for quaternions is rotating vectors, and readers requiring an introduction to this topic are directed to the author's book *Mathematics for Computer Graphics* [8].

It can be shown that a position vector  $\mathbf{p}$  can be rotated about an axis  $\hat{\mathbf{u}}$  by an angle  $\theta$  to  $\mathbf{p}'$  using the following operation:

$$\mathbf{p}' = q\mathbf{p}q^{-1} \quad (5.72)$$

where

$$\mathbf{p} = xi + yj + zk \quad (5.73)$$

$$p = 0 + ix + jy + kz \quad (5.74)$$

$$q = \cos(\theta/2) + \sin(\theta/2)\hat{\mathbf{u}} \quad (5.75)$$

$$q^{-1} = \cos(\theta/2) - \sin(\theta/2)\hat{\mathbf{u}} \quad (5.76)$$

and the axis of rotation is

$$\hat{\mathbf{u}} = [x_u \mathbf{i} + y_u \mathbf{j} + z_u \mathbf{k}] \quad (\|\hat{\mathbf{u}}\| = 1). \quad (5.77)$$

This is best demonstrated through an example.

Let the point to be rotated be

$$P(0, 1, 1). \quad (5.78)$$

Let the axis of rotation be

$$\hat{\mathbf{u}} = \mathbf{j}. \quad (5.79)$$

Let the angle of rotation be

$$\theta = 90^\circ. \quad (5.80)$$

Therefore,

$$p = 0 + i0 + j + k \quad (5.81)$$

$$q = \cos 45^\circ + \sin 45^\circ(i0 + j + k0)$$

$$q = \frac{\sqrt{2}}{2}(1 + i0 + j + k0) \quad (5.82)$$

$$q^{-1} = \cos 45^\circ - \sin 45^\circ(i0 + j + k0)$$

$$q^{-1} = \frac{\sqrt{2}}{2}(1 - i0 - j - k0). \quad (5.83)$$

The rotated point is given by

$$p' = qpq^{-1}$$

$$p' = \frac{\sqrt{2}}{2}(1 + i0 + j + k0)(0 + i0 + j + k) \frac{\sqrt{2}}{2}(1 - i0 - j - k0). \quad (5.84)$$

This is best expanded in two steps, and zero imaginary terms are included for clarity.

$qp$  followed by  $(qp)q^{-1}$ .

### Step 1

$$qp = \frac{\sqrt{2}}{2}(1 + i0 + j + k0)(0 + i0 + j + k)$$

$$qp = \frac{\sqrt{2}}{2}(-1 + i + j + k). \quad (5.85)$$

### Step 2

$$(qp)q^{-1} = \frac{\sqrt{2}}{2}(-1 + i + j + k) \frac{\sqrt{2}}{2}(1 - i0 - j - k0)$$

$$= \frac{1}{2}(-1 + 1 + j + i + j + k + i - k)$$

$$= \frac{1}{2}(0 + i2 + j2 + k0)$$

$$(qp)q^{-1} = 0 + i + j + k0. \quad (5.86)$$

The coordinates of the rotated point are stored in the pure part of the quaternion: (1, 1, 0).

## 5.11 Summary

Out of all the algebras we have so far considered, quaternion algebra paves the way to geometric algebra. In fact, as we will soon discover, GA shows that quaternions are a left-handed system and employ the concepts of GA. The good news is that if you understand quaternions, you will find it much easier to understand GA.



# 6 Geometric Conventions

## 6.1 Introduction

Algebra is a powerful numerical framework for solving real-world problems. But as mentioned in chapter 3 we must be careful when manipulating the quantity zero and taking square-roots of negative numbers. In this chapter we look at how geometric conventions give rise to negative areas and volumes which we must understand before proceeding with GA.

Readers already familiar with computer graphics will understand the importance of using a left-handed or right-handed axial system when designing computer programs. Failure to observe a consistent axial model can result in reflected images or reversed rotations, so let's begin by taking a look at spatial conventions.

## 6.2 Clockwise and anticlockwise

The number line used to visualize the distribution of the set of real numbers  $\mathbb{R}$  is conventionally organized as shown in Fig. 6.1a, with negative values to the left and positive numbers to the right.

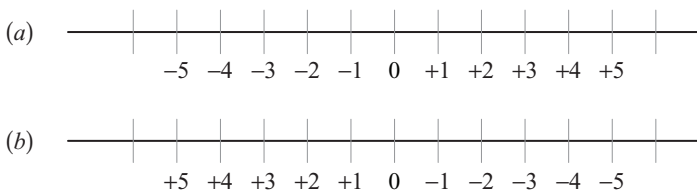


FIGURE 6.1.

Such a scheme is just a convention imposed upon us by previous civilizations. Even if the opposite convention had evolved, as shown in Figure 6.1b, it would not have affected the way we count or compute arithmetic operations. However, when we construct axial systems in  $\mathbb{R}^2$  four

models are available, as shown in Fig. 6.2. But if we look closely at these axes, (c) is (b) rotated 180°, and (d) is (a) rotated 180°, therefore, there are only two models: (a) and (b). So the important question is: Does it make any difference?

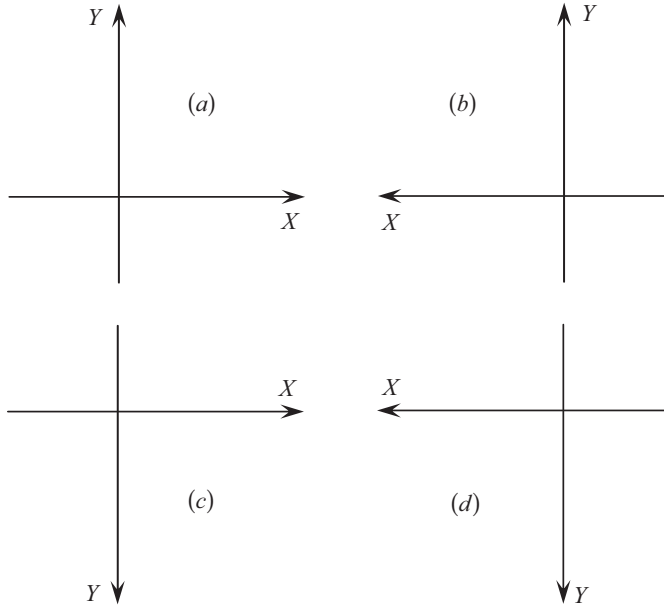


FIGURE 6.2.

The answer depends upon what we mean by ‘make any difference’. Obviously the slopes of graphs will look different, angles will be reversed, etc., but the rules of trigonometry will still hold, calculus will still work and areas will remain positive. But is the last statement true? Let’s find out.

The calculation of area is normally expressed as *width x height*, which results in an unsigned number. But say we compute the area of a triangle using the following Cartesian coordinate approach:

$$\text{area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2). \quad (6.1)$$

Figure 6.3 shows a triangle with its vertices organized in an anticlockwise sequence relative to a conventional axial system. Substituting the coordinates into Eq. (6.1) we obtain

$$\text{area} = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} = \frac{1}{2}(0 + 6 + 0 - 0 - 0 - 0) = 3 \quad (6.2)$$

and the result is +3 square units.

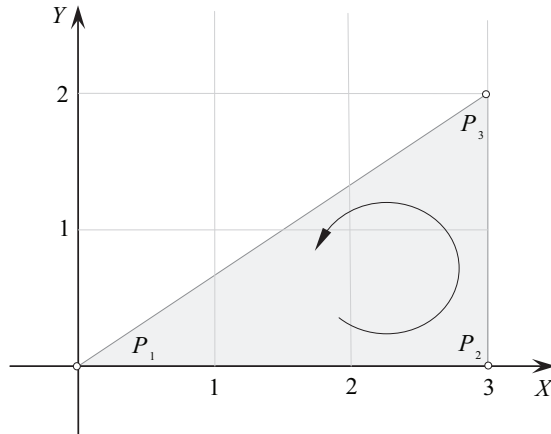


FIGURE 6.3.

If we transpose this triangle to an axial system where the positive x-axis is directed to the left we obtain the scenario shown in Fig. 6.4.

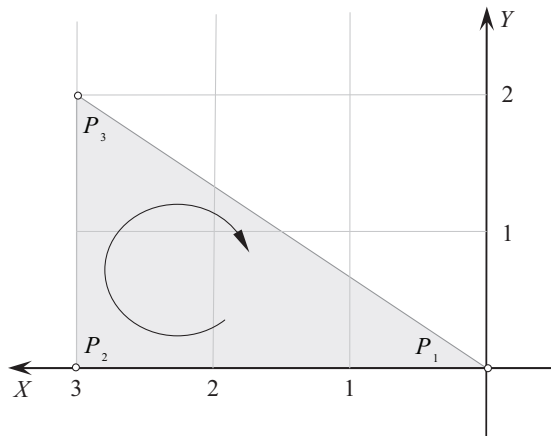


FIGURE 6.4.

Note that the triangle’s vertex sequence is no longer anticlockwise, but clockwise. The area is still +3 square units, but the notion of clockwise and anticlockwise has been reversed. This, then, is a problem if an algorithm or function is sensitive to the rotational sequence of vertices.

To demonstrate that Eq. (6.1) is sensitive to the rotational sequence of vertices, let’s reverse the coordinate sequence of the triangle and recompute its area:

$$\begin{aligned}
 \text{area} &= \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 3 & 2 & 1 \\ 3 & 0 & 1 \end{vmatrix} = \frac{1}{2}(0 + 0 + 0 - 0 - 0 - 6) \\
 \text{area} &= -3 \tag{6.3}
 \end{aligned}$$

this time the result is  $-3$  square units. So, simply reversing the vertices of a polygon switches the sign of its area.

So far, we have ignored one important convention — the fact that we are dealing with an ordered pair of coordinates. The French mathematician, René Descartes [1596–1650], is recognized as the inventor of Cartesian coordinates, and it is he who suggested the coordinate sequence:  $(x, y)$ . The actual letters are immaterial — what is important is that the horizontal coordinate is first and the vertical coordinate second. If Descartes had proposed to retain a horizontal  $x$ -axis and a vertical  $y$ -axis, and swap the ordered pair to  $(y, x)$ , this would have swapped the relationship between vertex sequence and sign of area. For example, if we swap the coordinates in Eq. (6.1) we obtain:

$$\begin{aligned} \text{area} &= \frac{1}{2} \begin{vmatrix} y_1 & x_1 & 1 \\ y_2 & x_2 & 1 \\ y_3 & x_3 & 1 \end{vmatrix} \\ \text{area} &= \frac{1}{2}(y_1x_2 + y_2x_3 + y_3x_1 - y_1x_3 - y_2x_1 - y_3x_2). \end{aligned} \quad (6.4)$$

Substituting the triangle's coordinates we get

$$\begin{aligned} \text{area} &= \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 3 & 1 \\ 2 & 3 & 1 \end{vmatrix} \\ \text{area} &= \frac{1}{2}(0 + 0 + 0 - 0 - 0 - 6) = -3 \end{aligned} \quad (6.5)$$

which flips the sign negative.

From this we observe that, starting with an ordered pair  $(x, y)$  and an axial system using horizontal  $(x)$  and vertical  $(y)$ , an anticlockwise vertex sequence results in a positive area, otherwise it is negative.

### 6.3 Left and right-handed axial systems

We also have two distinct axial systems in  $\mathbb{R}^3$  called left or right-handed systems. If humans had evolved with a single arm, and a single hand which had a central thumb in the middle of four fingers, we would have had to find an alternative name to label the two types of  $\mathbb{R}^3$  space. As it is, this anthropomorphic label is convenient.

Consider, then, a 2D system of axes as shown in Fig. 6.2a. It can be extended into a 3D system of axes by adding the  $z$ -axis in one of two directions, as shown in Fig. 6.5.

Figure 6.5a shows a right-handed system of axes, where using one's right-hand, the thumb aligns with the  $x$ -axis, the first-finger with the  $y$ -axis, and middle finger with the  $z$ -axis. Conversely, Fig. 6.5b shows a left-handed system of axes, where a similar alignment only works with a left hand. Either system works, but the right-hand system is widely used in computer graphics, which makes the sharing of computer algorithms that much easier.

Having seen that algebra supports the idea of positive and negative area, could it be that volume has a sign? Let's find out.

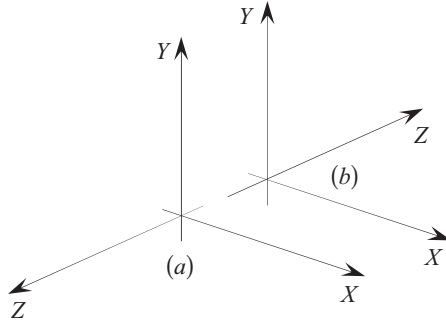


FIGURE 6.5.

The calculation of volume is normally expressed as *area x extension*, which results in an unsigned number. However, if we compute the volume of a parallelepiped using the following vector approach:

$$volume = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix} \tag{6.6}$$

it is possible to retrieve a signed value depending on the configuration of the vectors.

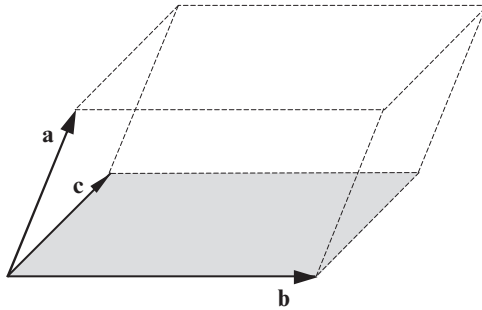


FIGURE 6.6.

The parallelepiped shown in Fig. 6.6 is associated with Eq. (6.6) and can be expressed in coordinates, rather than displacements. If we imagine that the vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are position vectors pointing to three points  $P_1(x_1, y_1, z_1)$ ,  $P_2(x_2, y_2, z_2)$  and  $P_3(x_3, y_3, z_3)$  respectively, we can write Eq. (6.6) as

$$volume = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$volume = x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1. \tag{6.7}$$



Substituting the coordinates of the box shown in Figure 6.7 into Eq. (6.7) we obtain

$$\begin{aligned}
 \text{volume} &= \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{vmatrix} \\
 \text{volume} &= 0 + 0 + 6 - 0 - 0 - 0 = 6
 \end{aligned} \tag{6.8}$$

and the result is +6 cubic units.

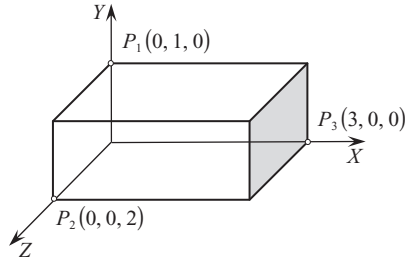


FIGURE 6.7.

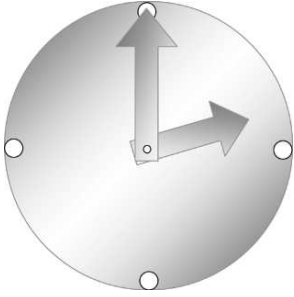
Note, that relative to the axial system, the vertices follow an anticlockwise path, which if reversed, switches the volume negative. This is effected by swapping  $P_2$  and  $P_3$ :

$$\begin{aligned}
 \text{volume} &= \begin{vmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix} \\
 \text{volume} &= 0 + 0 + 0 - 0 - 6 - 0 = -6.
 \end{aligned} \tag{6.9}$$

which flips the sign negative.

## 6.4 Summary

It is clear from the above examples that areas and volumes are signed quantities; what GA provides is a coherent algebraic framework that supports the manipulation of directed lines, areas and vectors, and any other higher-dimensional object.



# 7 Geometric Algebra

## 7.1 Introduction

In 1844 when Hamilton published his invention of quaternions, the German mathematician and schoolteacher, Hermann Gunther Grassmann [1809–1877], published the first edition of his geometric calculus *Lineale Ausdehnungslehre*, which offered an original algebra for describing geometric operations. The word ‘*Ausdehnungslehre*’ translates as ‘*theory of extension*’ and the principal algebraic product of the theory was the *exterior product*. The notion of *extension* is something that Euclid was aware of, in that the product of two lengths create an area, and the product of a length and an area create a volume. Grassmann discovered an algebra where his exterior product of vectors created areas, volumes and higher-dimensional objects.

Being *just* a schoolteacher and standing in the shadow of Hamilton, who was knighted, and a

Fellow of the American Society of Arts and Sciences,  
Fellow of the Society of Arts for Scotland,  
Fellow of the Royal Astronomical Society of London,  
Fellow of the Royal Northern Society of Antiquaries at Copenhagen,  
Honorary Member of the Institute of France,  
Honorary Member of the Imperial or Royal Academies of St. Petersburg,  
Berlin and Turin,  
Honorary Member of the Royal Societies of Edinburgh and Dublin,  
Honorary Member of the Cambridge Philosophical Society,  
Honorary Member of the New York Historical Society,  
Honorary Member of the Society of Natural Sciences at Lausanne,  
Honorary Member of other Scientific Societies in British and Foreign Countries,  
Andrews’ Professor of Astronomy in the University of Dublin, and  
Royal Astronomer of Ireland,

it is not surprising that few people bothered to buy or read Grassmann’s book!

Grassmann had not helped matters by writing a rather dense description of his geometric calculus. For in his book he presented new ideas on vector analysis, vector addition and subtraction, two vector products and vector differentiation, all interwoven with his philosophy on pure thought and existence. Not only that, it also applied to any number of dimensions [6].

In 1861 Grassmann published an updated version of his book with the title *Die Ausdehnungslehre: Vollständig und in strenger Form bearbeitet*, by which time he was Professor at the Stettin Gymnasium. But in spite of this academic promotion, Grassmann had to pay for the publishing costs, which covered a run of three-hundred books, and he died a few years later before mathematicians realized that he had been a genius of the first order.

Eventually, the English mathematician, William Kingdom Clifford [1845–1879], recognized the brilliance of Grassmann's ideas and formalized what today has become known as *geometric algebra*.

## 7.2 Foundations of geometric algebra

Basically, there are three ways authors approach an introduction to GA: The first group adopt an abstract algebraic approach where axioms give rise to an algebra — a Clifford algebra — which describe and resolve geometric problems in any number of dimensions. The second group starts with some simple algebraic axioms and show how GA flows naturally from these axioms. The third group take a vectorial approach and show how existing vector products lead to the principles of GA. Either approach is valid, but the author's personal preference is to support the second and third approaches, which are explored in this chapter, and we begin by reviewing some important ideas that should have emerged from the previous chapters. But one more point before we start. To distinguish vectors from scalars it is common practice to embolden vector names. Indeed, this convention was employed in the previous chapters. But in GA virtually everything is a vector of some sort and some authors have abandoned this convention and identify vectorial quantities using an italic font. Obviously, one has to be careful to distinguish scalars from vectorial quantities, which is accomplished by using letters from the Greek alphabet for scalars.

In the every-day algebra of real numbers we are familiar with its associative, distributive and commutative properties. In the algebra of complex numbers we make allowances for the fact that  $i^2 = -1$  and that multiplying a complex number by  $i$  effectively rotates it anticlockwise  $90^\circ$  on the complex plane. In vector algebra we discover that the vector product creates a third vector perpendicular to the plane containing the original vectors, and is antisymmetric. Well, it just so happens that GA is associative, distributive and involves an antisymmetric product, therefore we should not be surprised that it also has imaginary properties.

## 7.3 Introduction to geometric algebra

### 7.3.1 Length, area and volume

In the physical world of 3D space we measure the linear extension or something, i.e. its length; its planar coverage – its area; and its space filling capacity – its volume. This enables us to describe a room as being 3 meters high, a floor as being 16 square meters, and a room's volume being 48 cubic meters. It is difficult to think of a situation when in every-day parlance we would describe

something as having a negative length, area or volume, but in mathematics, such entities do exist, and GA provides a framework for their description.

Primarily, GA manipulates vectors, although scalar quantities are easily integrated into the equations, but, for the moment, we will concentrate on the role vectors play within the algebra.

A single vector, independent of its spatial dimension, has two qualities: orientation and magnitude. Its orientation is determined by the sign of its components, whilst its magnitude is represented by its length, which in turn is derived from its components. A vector's orientation is reversed, simply by switching the signs of its components.

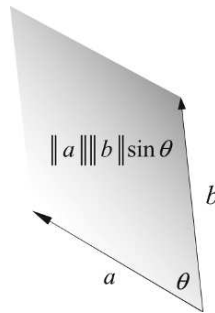


FIGURE 7.1.

The product of two vectors can be used to represent the area of a parallelogram as shown in Fig. 7.1, where the area is given by

$$area = \|a\| \|b\| \sin \theta. \quad (7.1)$$

Because  $\|a\|$  and  $\|b\|$  are scalars, their order is immaterial. Furthermore, we have assumed that the angle  $\theta$  is always positive, hence its sign is always positive, which is why area is normally regarded as a positive quantity.

Grassmann was aware that mathematics, especially determinants, supported positive *and* negative areas and volumes, and wanted to exploit this feature. His solution was to create a vector product that he called the *outer product* and written  $a \wedge b$ . The wedge symbol “ $\wedge$ ” is why the product is also known as the *wedge product*, and it is worth noting that this symbol is also used by French mathematicians for the vector (cross) product. The outer product is sensitive to the order of the vectors it manipulates, and permits us to distinguish between  $a \wedge b$  and  $b \wedge a$ . In fact, the algebra ensures that

$$a \wedge b = -b \wedge a. \quad (7.2)$$

Therefore, when using the outer product we must think carefully about their order, which is why in chapter 6 we discussed the order of axial systems. This idea is developed in Fig. 7.2 where we see the graphical difference between the two products.

Figure 7.2a shows that  $a \wedge b$  creates an area from vectors  $a$  and  $b$  forming an anticlockwise rotation, whereas Fig. 7.2b shows that  $b \wedge a$  creates an area from vectors  $b$  and  $a$  forming a clockwise rotation. The directed circle is included to remind us of the area's orientation.

From vector algebra we know that there are two important products: the scalar and the vector product. The scalar product creates a non-zero scalar value when the associated vectors are not perpendicular, and tells us something about the mutual alignment of the two vectors. Whereas,

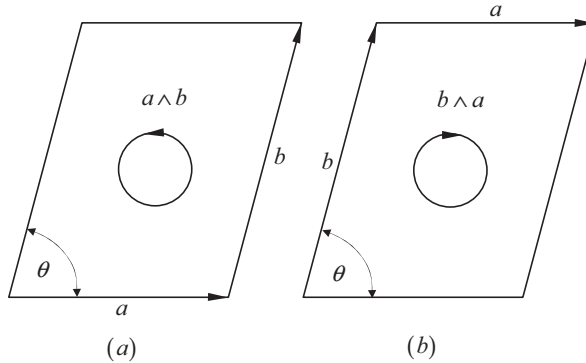


FIGURE 7.2.

the vector product creates a non-zero vector when the associated vectors are not parallel, and tells us something about the area of the parallelogram formed by the two vectors.

GA adopts these two products but changes the interpretation of the vector product. Hamilton interpreted the result  $a \times b$  as a third vector  $c$  perpendicular to the plane containing  $a$  and  $b$ . Although this interpretation works in three dimensions, it is ambiguous in higher dimensions. Grassmann interpreted the result of the vector product in terms of its capacity to compute a signed area, which is why he created the *outer product*.

## 7.4 The outer product

Now we already know that the magnitude of the vector product is given by

$$\|a \times b\| = \|a\| \|b\| \sin \theta \quad (7.3)$$

where  $\theta$  is the angle between the two vectors. The outer product preserves this value but abandons the concept of a perpendicular vector. Instead, the value  $\|a\| \|b\| \sin \theta$  is retained as the signed area formed by the two vectors.

Now although  $\|a \wedge b\| = \|a\| \|b\| \sin \theta$ , we must pose the question: What sort of object is  $a \wedge b$ ? Well, for a start, it is not a vector, nor is it a simple scalar. In fact, we have to invent a new name, which is always unsettling as it is difficult to relate it to things with which we are familiar. Where the cross product  $a \times b$  creates a vector, the outer product  $a \wedge b$  is called a *bivector*, which is a totally new concept to grasp.

A bivector describes the orientation of a plane in terms of two vectors, and its magnitude is the area of the parallelogram formed by the vectors. Reversing the vector sequence in the product flips the sign of the area. The outer product has the same components as the cross product, but instead of using the components to form a vector, they become the projective characteristics of a planar surface.

We are very familiar with the concept of a vector and accept that it has magnitude and orientation, where its components are expressed using orthogonal basis vectors. Reversing the direction of the vector reverses its components without changing its magnitude. Similarly, a bivector has

magnitude and orientation, where its components are expressed in terms of areas projected onto the bivectors formed by the orthogonal unit basis vectors. Reversing the direction of the bivector reverses its components without changing its magnitude. This is illustrated later in this chapter. For the moment, a bivector is just a name used to orient a planar area.

### 7.4.1 Some algebraic properties

Even with our sketchy knowledge of a bivector, it is possible to describe how the outer product responds to parallel vectors. For example

$$\|a \wedge a\| = \|a\| \|a\| \sin 0^\circ = 0. \quad (7.4)$$

Although the outer product is antisymmetric, it behaves just like the scalar product when multiplying a group of vectors:

$$\text{scalar: } a \cdot (b + c) = a \cdot b + a \cdot c \quad (7.5)$$

similarly

$$\text{outer: } a \wedge (b + c) = a \wedge b + a \wedge c. \quad (7.6)$$

### 7.4.2 Visualizing the outer product

The cross product is easy to visualize:  $a \times b = c$ , where  $c$  is orthogonal to the plane containing  $a$  and  $b$ . The relative direction of  $c$  is determined by the right-hand rule where using one's right hand, where the thumb aligns with  $a$ , the first finger with  $b$ , and the middle finger aligns with  $c$ . The magnitude of  $c$  equals  $\|a\| \|b\| \sin \theta$ , where  $\theta$  is the angle between  $a$  and  $b$ , and equals the area of the parallelogram formed by  $a$  and  $b$ . This relationship is shown in Fig. 7.3.

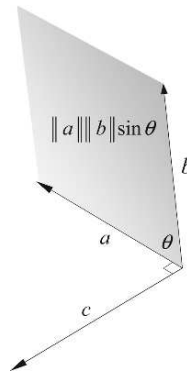


FIGURE 7.3.

Visualizing the outer product is slightly different. It is true that the magnitude  $\|a \wedge b\|$  is  $\|a\|\|b\| \sin \theta$  which represents the area of the parallelogram formed by  $a$  and  $b$ , but consider what happens if we form the product  $a' \wedge b$  where  $a' = a + \lambda b$ :

$$\begin{aligned} a' \wedge b &= (a + \lambda b) \wedge b \\ &= a \wedge b + \lambda b \wedge b \\ a' \wedge b &= a \wedge b. \end{aligned} \tag{7.7}$$

Two other vectors generate the same bivector! Figure 7.4 illustrates what is happening.

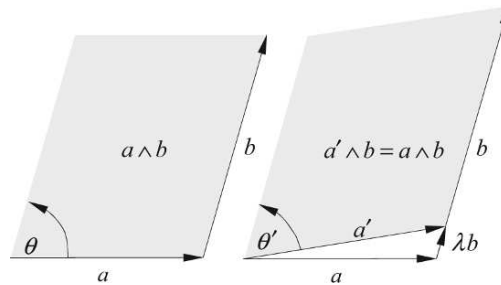


FIGURE 7.4.

The area created by  $a' \wedge b$  is identical to that created by  $a \wedge b$ , so there is no single parallelogram that represents  $a \wedge b$  — there are an infinite number! So why bother trying to represent  $a \wedge b$  as a parallelogram in the first place? Well, it was a starting point, but now that we have discovered this feature of the outer product, why not substitute another shape such as a circle instead of a parallelogram, and make the area of the circle equal to  $\|a\|\|b\| \sin \theta$ ? That was a rhetorical question, but a useful suggestion, and Fig. 7.5 shows what is implied.

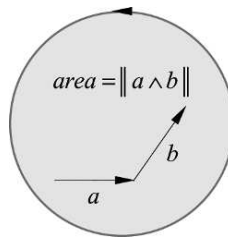


FIGURE 7.5.

### 7.4.3 Orthogonal bases

GA works in any number of dimensions, and anticipating the need to embrace a large number of dimensions we require a notation for the extended orthogonal axial systems. Conventionally,  $i$  and  $j$  represent the unit basis vectors for  $\mathbb{R}^2$ , and  $i$ ,  $j$  and  $k$  represent the unit basis vectors for  $\mathbb{R}^3$ .

If we continue with this notation the alphabet cannot support very high-dimensional spaces. An alternative convention is to use  $e_1, e_2, e_3, \dots, e_n$  to represent the orthogonal unit basis vectors.

Using this notation we define two vectors in  $\mathbb{R}^2$  as

$$a = a_1e_1 + a_2e_2 \quad (7.8)$$

$$b = b_1e_1 + b_2e_2. \quad (7.9)$$

We can now state the outer product as

$$a \wedge b = (a_1e_1 + a_2e_2) \wedge (b_1e_1 + b_2e_2) \quad (7.10)$$

which expands to

$$a \wedge b = a_1b_1(e_1 \wedge e_1) + a_1b_2(e_1 \wedge e_2) + a_2b_1(e_2 \wedge e_1) + a_2b_2(e_2 \wedge e_2). \quad (7.11)$$

Substituting the following observations

$$e_1 \wedge e_1 = e_2 \wedge e_2 = 0 \text{ and } e_2 \wedge e_1 = -e_1 \wedge e_2 \quad (7.12)$$

we obtain

$$a \wedge b = a_1b_2(e_1 \wedge e_2) - a_2b_1(e_1 \wedge e_2) \quad (7.13)$$

simplifying, we obtain

$$a \wedge b = (a_1b_2 - a_2b_1)(e_1 \wedge e_2). \quad (7.14)$$

The scalar term  $a_1b_2 - a_2b_1$  in Eq. (7.14) looks familiar — in fact, it is the magnitude of the imaginary term of Eq. (3.17), the value of which equals  $\|a\|\|b\|\sin\theta$ , which is the area of the parallelogram formed by  $a$  and  $b$ . So in this context, the outer product  $a \wedge b$  is a scalar area multiplying the unit bivector  $e_1 \wedge e_2$ , which just means that the area is associated with the plane defined by  $e_1 \wedge e_2$ . Figure 7.6 illustrates this relationship.

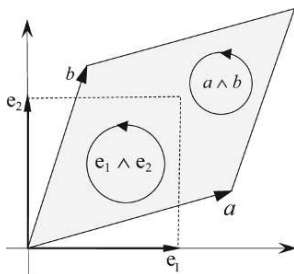


FIGURE 7.6.

Now let's compute  $b \wedge a$ :

$$b \wedge a = (b_1e_1 + b_2e_2) \wedge (a_1e_1 + a_2e_2)$$



which expands to

$$b \wedge a = a_1 b_1 (e_1 \wedge e_1) + a_2 b_1 (e_1 \wedge e_2) + a_1 b_2 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2). \quad (7.15)$$

Substituting the following observations

$$e_1 \wedge e_1 = e_2 \wedge e_2 = 0 \text{ and } e_2 \wedge e_1 = -e_1 \wedge e_2 \quad (7.16)$$

we obtain

$$b \wedge a = a_2 b_1 (e_1 \wedge e_2) - a_1 b_2 (e_1 \wedge e_2). \quad (7.17)$$

Simplifying, we obtain

$$b \wedge a = -(a_1 b_2 - a_2 b_1) (e_1 \wedge e_2) \quad (7.18)$$

which confirms that  $b \wedge a = -a \wedge b$ .

Now let's consider the outer product in  $\mathbb{R}^3$ :

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 \quad (7.19)$$

$$b = b_1 e_1 + b_2 e_2 + b_3 e_3. \quad (7.20)$$

The outer product is

$$a \wedge b = (a_1 e_1 + a_2 e_2 + a_3 e_3) \wedge (b_1 e_1 + b_2 e_2 + b_3 e_3) \quad (7.21)$$

which expands to

$$\begin{aligned} a \wedge b = & a_1 b_1 (e_1 \wedge e_1) + a_1 b_2 (e_1 \wedge e_2) + a_1 b_3 (e_1 \wedge e_3) + a_2 b_1 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2) \\ & + a_2 b_3 (e_2 \wedge e_3) + a_3 b_1 (e_3 \wedge e_1) + a_3 b_2 (e_3 \wedge e_2) + a_3 b_3 (e_3 \wedge e_3). \end{aligned} \quad (7.22)$$

Substituting

$$e_1 \wedge e_1 = e_2 \wedge e_2 = e_3 \wedge e_3 = 0 \quad (7.23)$$

and

$$e_2 \wedge e_1 = -e_1 \wedge e_2 \quad e_1 \wedge e_3 = -e_3 \wedge e_1 \quad e_3 \wedge e_2 = -e_2 \wedge e_3 \quad (7.24)$$

we obtain

$$\begin{aligned} a \wedge b = & a_1 b_2 (e_1 \wedge e_2) - a_1 b_3 (e_3 \wedge e_1) - a_2 b_1 (e_1 \wedge e_2) \\ & + a_2 b_3 (e_2 \wedge e_3) + a_3 b_1 (e_3 \wedge e_1) - a_3 b_2 (e_2 \wedge e_3). \end{aligned} \quad (7.25)$$

Simplifying, we obtain

$$a \wedge b = (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 + (a_2 b_3 - a_3 b_2) e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_3 \wedge e_1. \quad (7.26)$$

You may be wondering why the unit basis bivectors in Eq. (7.26) have been chosen in this way, especially  $e_3 \wedge e_1$ . This could easily be  $e_1 \wedge e_3$ . To understand why, refer to Fig. 7.7, which shows a right-handed axial system and where each orthogonal plane is defined by its associated unit basis bivectors.

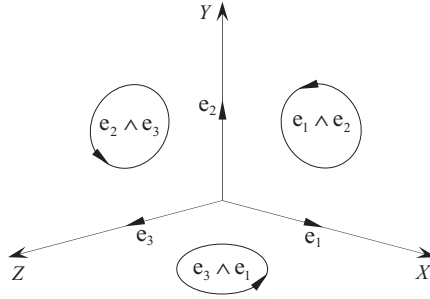


FIGURE 7.7.

Figure 7.7 also shows the orthogonal alignment of the Cartesian axes with the unit basis bivectors:

the  $x$ -axis is orthogonal to  $e_2 \wedge e_3$

the  $y$ -axis is orthogonal to  $e_3 \wedge e_1$

the  $z$ -axis is orthogonal to  $e_1 \wedge e_2$

and if Eq. (7.26) is rearranged in this sequence we obtain

$$a \wedge b = (a_2 b_3 - a_3 b_2)e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3)e_3 \wedge e_1 + (a_1 b_2 - a_2 b_1)e_1 \wedge e_2. \quad (7.27)$$

Now let's look at a definition of the cross product. We begin by declaring two vectors using the conventional orthogonal unit basis vectors  $i, j$  and  $k$ :

$$a = a_1 i + a_2 j + a_3 k \quad (7.28)$$

$$b = b_1 i + b_2 j + b_3 k. \quad (7.29)$$

The cross product is

$$a \times b = (a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k) \quad (7.30)$$

which expands to

$$a \times b = a_1 b_1 (i \times i) + a_1 b_2 (i \times j) + a_1 b_3 (i \times k) + a_2 b_1 (j \times i) + a_2 b_2 (j \times j) + a_2 b_3 (j \times k) + a_3 b_1 (k \times i) + a_3 b_2 (k \times j) + a_3 b_3 (k \times k). \quad (7.31)$$

The magnitude of the cross product is  $\|a\| \|b\| \sin \theta$ , which means that

$$i \times i = j \times j = k \times k = 0. \quad (7.32)$$

Therefore,

$$a \times b = a_1 b_2 (i \times j) + a_1 b_3 (i \times k) + a_2 b_1 (j \times i) + a_2 b_3 (j \times k) + a_3 b_1 (k \times i) + a_3 b_2 (k \times j). \quad (7.33)$$

Because the cross product is antisymmetric

$$j \times i = -i \times j \quad k \times j = -j \times k \quad i \times k = -k \times i. \quad (7.34)$$

Substituting these relationships:

$$\begin{aligned} a \times b &= a_1 b_2 (i \times j) - a_1 b_3 (k \times i) - a_2 b_1 (i \times j) \\ &\quad + a_2 b_3 (j \times k) + a_3 b_1 (k \times i) - a_3 b_2 (j \times k). \end{aligned} \quad (7.35)$$

Collecting up like terms:

$$a \times b = (a_2 b_3 - a_3 b_2) j \times k + (a_3 b_1 - a_1 b_3) k \times i + (a_1 b_2 - a_2 b_1) i \times j. \quad (7.36)$$

If we place Eqs. (7.27) and (7.36) together and substitute the  $e$  notation for  $i, j$  and  $k$ , we obtain

$$a \wedge b = (a_2 b_3 - a_3 b_2) e_2 \wedge e_3 + (a_3 b_1 - a_1 b_3) e_3 \wedge e_1 + (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 \quad (7.37)$$

$$a \times b = (a_2 b_3 - a_3 b_2) e_2 \times e_3 + (a_3 b_1 - a_1 b_3) e_3 \times e_1 + (a_1 b_2 - a_2 b_1) e_1 \times e_2. \quad (7.38)$$

In the cross product, the terms  $(a_2 b_3 - a_3 b_2)$ ,  $(a_3 b_1 - a_1 b_3)$  and  $(a_1 b_2 - a_2 b_1)$  are the components of an orthogonal vector, whereas in the outer product they become signed areas projected onto the planes defined by the unit bivectors  $e_2 \wedge e_3$ ,  $e_3 \wedge e_1$  and  $e_1 \wedge e_2$ . And in spite of there being such similarity between the two equations, it would be dangerous to conclude that  $a \wedge b \equiv a \times b$ .

What Hamilton had proposed was that

$$e_2 \times e_3 = e_1 \quad e_3 \times e_1 = e_2 \quad e_1 \times e_2 = e_3 \quad (7.39)$$

which is fine for  $\mathbb{R}^3$ , but is ambiguous for higher dimensions. So, in GA we substitute the outer product for the cross product and introduce the concept of a directed area, which holds for any number of dimensions.

Before we reveal the imaginary nature of the outer product in the next chapter, consider the scenario shown in Fig. 7.8. Two vectors  $a$  and  $b$  are shown forming a parallelogram created by their outer product  $a \wedge b$  with parallel projections of the parallelogram projected onto the three orthogonal planes. The projections will normally be parallelograms, but under some conditions they could collapse to a line. Whatever happens, at least one will be a parallelogram.

We define two vectors as

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3 \quad (7.40)$$

$$b = b_1 e_1 + b_2 e_2 + b_3 e_3. \quad (7.41)$$

Starting with the plane containing  $e_1$  and  $e_2$ , which is defined by  $e_1 \wedge e_2$ , the projections of  $a$  and  $b$  are  $a'''$  and  $b'''$ , respectively, where

$$a''' = a_1 e_1 + a_2 e_2 \quad (7.42)$$

$$b''' = b_1 e_1 + b_2 e_2. \quad (7.43)$$

Therefore,

$$\begin{aligned} a''' \wedge b''' &= (a_1 e_1 + a_2 e_2) \wedge (b_1 e_1 + b_2 e_2) \\ &= a_1 b_1 (e_1 \wedge e_1) + a_1 b_2 (e_1 \wedge e_2) + a_2 b_1 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2) \\ a''' \wedge b''' &= (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 \end{aligned} \quad (7.44)$$

which is the last term in Eq. (7.27).

Similarly, we can show that

$$a' \wedge b' = (a_2b_3 - a_3b_2)e_2 \wedge e_3 \tag{7.45}$$

$$a'' \wedge b'' = (a_3b_1 - a_1b_3)e_3 \wedge e_1. \tag{7.46}$$

Thus we see that instead of creating a new vector, the outer product projects the parallelogram onto the three orthogonal planes to create three new bivectors, whose area is positive or negative. The cross product, however, takes these areas and uses them to form a vector, which happens to be orthogonal to the original parallelogram.

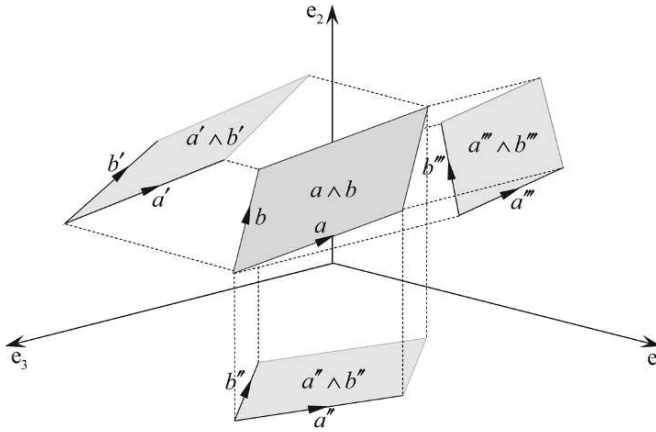


FIGURE 7.8.

To illustrate this concept, consider two vectors  $a$  and  $b$

$$a = a_1e_1 + a_2e_2 + a_3e_3 \tag{7.47}$$

$$b = b_1e_1 + b_2e_2 + b_3e_3 \tag{7.48}$$

where

$$\begin{aligned} a_1 = 1 & \quad a_2 = 0 & \quad a_3 = 1 \\ b_1 = 1 & \quad b_2 = 1 & \quad b_3 = 0 \end{aligned} \tag{7.49}$$

which makes

$$a = e_1 + e_3 \quad b = e_1 + e_2. \tag{7.50}$$

Using Eq. (7.26)

$$\begin{aligned} a \wedge b &= (a_1b_2 - a_2b_1)e_1 \wedge e_2 + (a_2b_3 - a_3b_2)e_2 \wedge e_3 + (a_3b_1 - a_1b_3)e_3 \wedge e_1 \\ a \wedge b &= (1)e_1 \wedge e_2 + (-1)e_2 \wedge e_3 + (1)e_3 \wedge e_1. \end{aligned} \tag{7.51}$$

The signed area on the plane  $e_1 \wedge e_2$  is +1 and is shown in Fig. 7.9. The projected area is shown crosshatched.

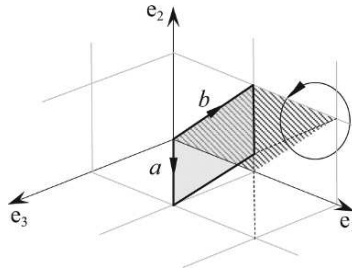


FIGURE 7.9.

Similarly, the signed area on the plane  $e_2 \wedge e_3$  is  $-1$  and is shown in Fig. 7.10. Note that the direction of the projected area opposes the direction of  $e_2 \wedge e_3$ .

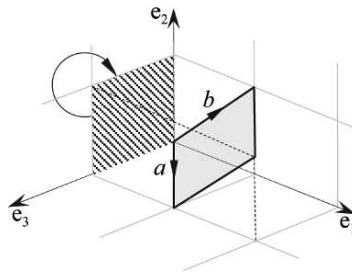


FIGURE 7.10.

And the signed area on the plane  $e_3 \wedge e_1$  is  $+1$ , and is shown in Fig. 7.11.

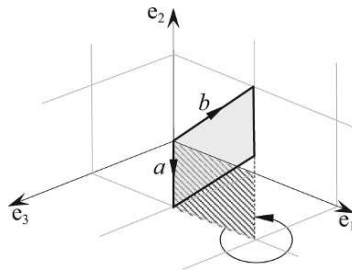


FIGURE 7.11.

Now let's compute the magnitude of the bivector  $a \wedge b$ .

To begin with, we need to know the angle between  $a$  and  $b$ , which is revealed using the dot product:

$$\theta = \cos^{-1} \left( \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\|a\| \|b\|} \right)$$

$$\theta = \cos^{-1} \left( \frac{1}{\sqrt{2}\sqrt{2}} \right) = 60^\circ. \quad (7.52)$$

Therefore,

$$\begin{aligned}\|a \wedge b\| &= \|a\| \|b\| \sin 60^\circ \\ \|a \wedge b\| &= \sqrt{2}\sqrt{2}\frac{\sqrt{3}}{2} = \sqrt{3}.\end{aligned}\tag{7.53}$$

The next question to pose is whether this value is related to the other three areas? Well the answer is “yes”, and for a very good reason:

$$\|a \wedge b\|^2 = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2\tag{7.54}$$

therefore,

$$\sqrt{3}^2 = (1)^2 + (-1)^2 + (1)^2 = 3.\tag{7.55}$$

Remember, that the cross product uses these coefficients as Cartesian components of the axial vector and satisfy the Pythagorean rule:

$$\|a\|^2 = a_1^2 + a_2^2 + a_3^2.\tag{7.56}$$

To prove that this holds, we need to show that Eq. (7.54) is correct.

Expanding the LHS of Eq. (7.54):

$$\begin{aligned}\|a \wedge b\|^2 &= \|a\|^2 \|b\|^2 \sin^2 \theta = \|a\|^2 \|b\|^2 (1 - \cos^2 \theta) \\ \|a \wedge b\|^2 &= \|a\|^2 \|b\|^2 - \|a\|^2 \|b\|^2 \cos^2 \theta.\end{aligned}\tag{7.57}$$

From the dot product

$$\cos^2 \theta = \frac{(a_1b_1 + a_2b_2 + a_3b_3)^2}{\|a\|^2 \|b\|^2}.\tag{7.58}$$

Therefore,

$$\begin{aligned}\|a \wedge b\|^2 &= \|a\|^2 \|b\|^2 - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ \|a \wedge b\|^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2\end{aligned}$$

and we obtain

$$\begin{aligned}\|a \wedge b\|^2 &= (a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2) + (a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2) \\ &\quad + (a_3^2b_1^2 - 2a_3a_1b_3b_1 + a_1^2b_3^2) \\ \|a \wedge b\|^2 &= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2.\end{aligned}\tag{7.59}$$

Therefore, Eq. (7.54) is correct.

Now, as

$$\|a \wedge b\| = \|a\| \|b\| \sin \theta\tag{7.60}$$

$$\|a \wedge b\|^2 = \|a\|^2 \|b\|^2 \sin^2 \theta\tag{7.61}$$

and

$$\|a\|\|b\| \sin^2 \theta = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 \quad (7.62)$$

therefore

$$\theta = \sin^{-1} \left( \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2}}{\|a\|\|b\|} \right). \quad (7.63)$$

Substituting the values for the above example:

$$\theta = \sin^{-1} \left( \frac{\sqrt{3}}{\sqrt{2}\sqrt{2}} \right) = 60^\circ. \quad (7.64)$$

The beauty of the outer product is that it works in any number of dimensions. For example, we can create two vectors in  $\mathbb{R}^4$  as follows:

$$a = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 \quad (7.65)$$

$$b = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 \quad (7.66)$$

and form their outer product:

$$a \wedge b = (a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \wedge (b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4). \quad (7.67)$$

This explodes into

$$\begin{aligned} a \wedge b = & a_1b_1(e_1 \wedge e_1) + a_1b_2(e_1 \wedge e_2) + a_1b_3(e_1 \wedge e_3) + a_1b_4(e_1 \wedge e_4) \\ & + a_2b_1(e_2 \wedge e_1) + a_2b_2(e_2 \wedge e_2) + a_2b_3(e_2 \wedge e_3) + a_2b_4(e_2 \wedge e_4) \\ & + a_3b_1(e_3 \wedge e_1) + a_3b_2(e_3 \wedge e_2) + a_3b_3(e_3 \wedge e_3) + a_3b_4(e_3 \wedge e_4) \\ & + a_4b_1(e_4 \wedge e_1) + a_4b_2(e_4 \wedge e_2) + a_4b_3(e_4 \wedge e_3) + a_4b_4(e_4 \wedge e_4) \end{aligned}$$

and collapses to

$$\begin{aligned} a \wedge b = & (a_1b_2 - a_2b_1)(e_1 \wedge e_2) + (a_2b_3 - a_3b_2)(e_2 \wedge e_3) + (a_3b_1 - a_1b_3)(e_3 \wedge e_1) \\ & + (a_1b_4 - a_4b_1)(e_1 \wedge e_4) + (a_2b_4 - a_4b_2)(e_2 \wedge e_4) + (a_3b_4 - a_4b_3)(e_3 \wedge e_4) \end{aligned} \quad (7.68)$$

which resolves the outer product into six bivectors.

These bivectors arise because there are six ways of making 2-tuples from four axes:

$${}_4C_2 = \frac{4!}{(4-2)!2!} = 6. \quad (7.69)$$

In five dimensions there are 10 bivectors.

$${}_5C_2 = \frac{5!}{(5-2)!2!} = 10. \quad (7.70)$$

As a final example, let's consider two vectors in  $\mathbb{R}^4$  and compute their outer product. The vectors are

$$a = e_1 + e_3 + e_4 \quad (7.71)$$

$$b = e_1 + e_2 + e_4. \quad (7.72)$$

Then

$$\|a\| = \sqrt{3} \quad \|b\| = \sqrt{3} \quad (7.73)$$

and the separating angle  $\theta$  is

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) \simeq 48.19^\circ. \quad (7.74)$$

Similarly,

$$\theta = \sin^{-1}\left(\frac{\sqrt{5}}{3}\right) \simeq 48.19^\circ. \quad (7.75)$$

Substituting the vectors into Eq. (7.68):

$$a \wedge b = (1)(e_1 \wedge e_2) + (-1)(e_2 \wedge e_3) + (1)(e_3 \wedge e_1) + (-1)(e_2 \wedge e_4) + (1)(e_3 \wedge e_4). \quad (7.76)$$

Therefore,  $\|a \wedge b\|$  is given by

$$\|a \wedge b\| = \|a\|\|b\| \sin \theta = \sqrt{3}\sqrt{3} \sin 48.19^\circ \simeq 2.2361. \quad (7.77)$$

Finally, let's show that the  $\mathbb{R}^4$  equivalent of Eq. (7.54) still holds:

$$\begin{aligned} \|a \wedge b\|^2 &= |a_1b_2 - a_2b_1|^2 + |a_2b_3 - a_3b_2|^2 + |a_3b_1 - a_1b_3|^2 \\ &\quad + |a_1b_4 - a_4b_1|^2 + |a_2b_4 - a_4b_2|^2 + |a_3b_4 - a_4b_3|^2 \\ 2.2361^2 &= (1)^2 + (-1)^2 + (1)^2 + (0)^2 + (-1)^2 + (1)^2 = 5. \end{aligned} \quad (7.78)$$

## 7.5 The outer product in action

Later, we investigate a number of scenarios where the outer product is used to solve problems in computer graphics, but at this point it is worth looking at three problems where it seems that we have been using the outer product without knowing.



### 7.5.1 Area of a triangle

There are many ways to find the area of a triangle, but the one proposed here uses a triangle's vertex coordinates, as shown in Fig. 7.12a. The triangle has vertices  $A, B, C$  defined in an anticlockwise order, and its area is given by

$$\text{area} = \frac{1}{2} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}. \quad (7.79)$$

Using the coordinates from Fig. 7.12a we have

$$\text{area} = \frac{1}{2} \begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix} \quad (7.80)$$

$$\text{area} = \frac{1}{2}(9 + 6 - 6 - 3) = +3 \quad (7.81)$$

which is correct. Note that reversing the triangle's vertex sequence creates a negative area:

$$\text{area} = \frac{1}{2} \begin{vmatrix} 0 & 2 & 1 \\ 3 & 3 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$\text{area} = \frac{1}{2}(3 + 6 - 6 - 9) = -3. \quad (7.82)$$

We can prove Eq. (7.79) algebraically, and if we create the right diagram, outer products come to our rescue. Figure 7.12b shows three position vectors  $a, b, c$  locating the vertices, which we use to form three outer products. The first product  $a \wedge b$  computes the area of the parallelogram  $OBCA$ , and  $\frac{1}{2}(a \wedge b)$  computes the area of the triangle  $\Delta OBA$ . The sequence of the vertices  $O, A, B$  create a clockwise outer product, which accounts for the negative signs in  $\Delta OBA$ .

The second product  $b \wedge c$  computes the area of the parallelogram  $OBEC$ , and  $\frac{1}{2}(b \wedge c)$  computes the area of the triangle  $\Delta OBC$ . The sequence of the vertices  $O, B, C$  create an anticlockwise outer product, which accounts for the positive signs in  $\Delta OBC$ .

The third product  $c \wedge a$  computes the area of the parallelogram  $OCCA$ , and  $\frac{1}{2}(c \wedge a)$  computes the area of the triangle  $\Delta OCA$ . The sequence of the vertices  $O, C, A$  create an anticlockwise outer product, which accounts for the positive signs in  $\Delta OCA$ .

The sum of the three outer products is

$$\frac{1}{2}(a \wedge b) + \frac{1}{2}(b \wedge c) + \frac{1}{2}(c \wedge a)$$

and creates three areas: two of the areas contribute toward the triangles  $\Delta ABC$  and  $\Delta OBA$ , whilst the third area cancels the area of triangle  $\Delta OBA$ , leaving behind the area of  $\Delta ABC$ .

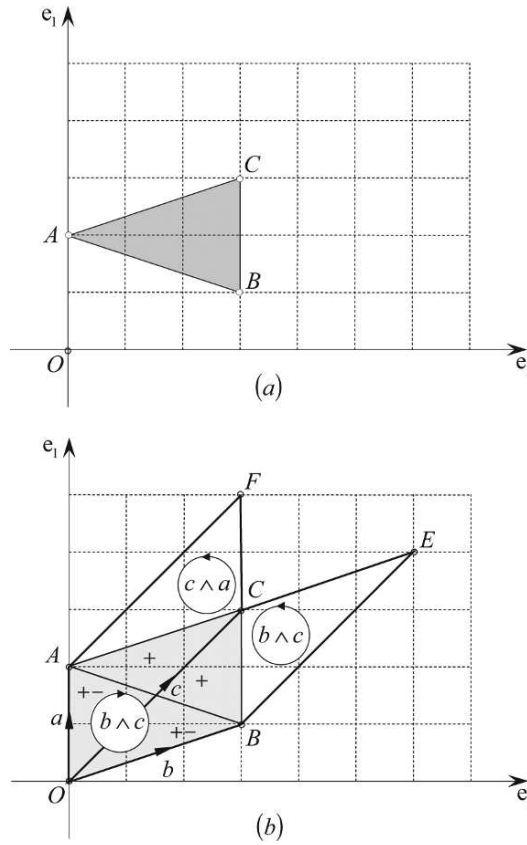


FIGURE 7.12.

The sum of the outer products become

$$area \Delta ABC = \frac{1}{2}[(a \wedge b) + (b \wedge c) + (c \wedge a)] \tag{7.83}$$

which expand to

$$area \Delta ABC = \frac{1}{2}(x_A y_B - y_A x_B + x_B y_C - y_B x_C + x_C y_A - y_C x_A)$$

and

$$area \Delta ABC = \frac{1}{2} \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}. \tag{7.84}$$

What is useful about summing these outer products is that it works for any irregular shape.

### 7.5.2 The sine rule

The traditional way of proving the sine rule is to take a triangle and drop a perpendicular from one of its vertices onto the opposite side to form two right-angled triangles, from which we define the sine ratio of two angles. Using Fig. 7.13 we can state that

$$\frac{H}{B} = \sin \alpha \quad \text{and} \quad \frac{H}{A} = \sin \beta \quad (7.85)$$

from which we can write

$$B \sin \alpha = A \sin \beta \quad (7.86)$$

or

$$\frac{A}{\sin \alpha} = \frac{B}{\sin \beta}. \quad (7.87)$$

Using another vertex and an associated perpendicular we can show that

$$\frac{A}{\sin \alpha} = \frac{B}{\sin \beta} = \frac{C}{\sin \chi}. \quad (7.88)$$

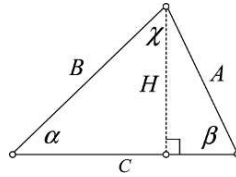


FIGURE 7.13.

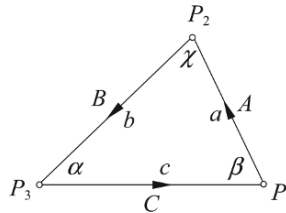


FIGURE 7.14.

The GA approach is to remember that the outer product includes a sine function and computes an area. Therefore, we develop Fig. 7.13 to include three vectors as shown in Fig. 7.14 where

$$A = \|a\| \quad B = \|b\| \quad C = \|c\| \quad (7.89)$$

From the figure we observe that

$$\text{area of } \Delta P_1P_2P_3 = \frac{1}{2} \|a \wedge -c\| = \frac{1}{2} AC \sin \beta \tag{7.90}$$

$$\text{area of } \Delta P_2P_3P_1 = \frac{1}{2} \|b \wedge -a\| = \frac{1}{2} BA \sin \chi \tag{7.91}$$

$$\text{area of } \Delta P_3P_1P_2 = \frac{1}{2} \|c \wedge -b\| = \frac{1}{2} CB \sin \alpha. \tag{7.92}$$

Therefore,

$$AC \sin \beta = CB \sin \alpha = BA \sin \chi \tag{7.93}$$

and

$$\frac{A}{\sin \alpha} = \frac{B}{\sin \beta} = \frac{C}{\sin \chi}. \tag{7.94}$$

### 7.5.3 Intersection of two lines

The traditional way of calculating the intersection point of two lines in a plane is to define two vectors as shown in Fig. 7.15, where

$$p = r + \lambda a \quad \lambda \in \mathbb{R} \tag{7.95}$$

$$p = s + \varepsilon b \quad \varepsilon \in \mathbb{R}. \tag{7.96}$$

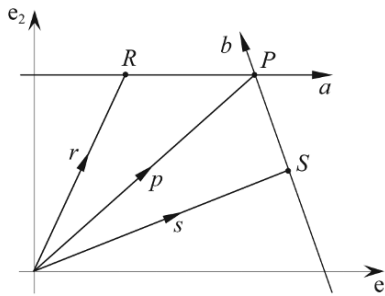


FIGURE 7.15.

Therefore,

$$r + \lambda a = s + \varepsilon b. \tag{7.97}$$

From Eq. (7.95) we can write

$$x_r + \lambda x_a = x_s + \varepsilon x_b \tag{7.98}$$

$$y_r + \lambda y_a = y_s + \varepsilon y_b. \tag{7.99}$$

To find  $\lambda$  we eliminate  $\varepsilon$  by multiplying Eq. (7.98) by  $y_b$  and Eq. (7.99) by  $x_b$ :

$$x_r y_b + \lambda x_a y_b = x_s y_b + \varepsilon x_b y_b \quad (7.100)$$

$$x_b y_r + \lambda x_b y_a = x_b y_s + \varepsilon x_b y_b. \quad (7.101)$$

Subtracting Eq. (7.101) from Eq. (7.100):

$$x_r y_b - x_b y_r + \lambda(x_a y_b - x_b y_a) = x_s y_b - x_b y_s \quad (7.102)$$

where

$$\lambda = \frac{x_b(y_r - y_s) - y_b(x_r - x_s)}{x_a y_b - x_b y_a}. \quad (7.103)$$

Let's test this with the following vectors

$$r = j \quad a = 2i - j \quad (7.104)$$

$$s = 2j \quad b = 2i - 2j \quad (7.105)$$

$$\lambda = \frac{2(1 - 2) + 2(0 - 0)}{-4 + 2} = \frac{-2}{-2} = 1 \quad (7.106)$$

therefore,

$$p = j + 2i - j = 2i \quad (7.107)$$

and the point of intersection is (2, 0).

Another approach is to reason that

$$p = \alpha a + \beta b \quad (7.108)$$

therefore, we can write

$$x_p = \alpha x_a + \beta x_b \quad (7.109)$$

$$y_p = \alpha y_a + \beta y_b. \quad (7.110)$$

To find  $\alpha$  we eliminate  $\beta$  by multiplying Eq. (7.109) by  $y_b$  and Eq. (7.110) by  $x_b$ :

$$x_p y_b = \alpha x_a y_b + \beta x_b y_b \quad (7.111)$$

$$x_b y_p = \alpha x_b y_a + \beta x_b y_b. \quad (7.112)$$

Subtracting Eq. (7.112) from Eq. (7.111) we obtain

$$x_p y_b - x_b y_p = \alpha x_a y_b - \alpha x_b y_a = \alpha(x_a y_b - x_b y_a) \quad (7.113)$$

where

$$\alpha = \frac{x_p y_b - x_b y_p}{x_a y_b - x_b y_a} = \frac{\begin{vmatrix} x_p & y_p \\ x_b & y_b \end{vmatrix}}{\begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}}. \quad (7.114)$$

To find  $\beta$  we eliminate  $\alpha$  by multiplying Eq. (7.109) by  $y_a$  and Eq. (7.110) by  $x_a$ :

$$x_p y_a = \alpha x_a y_a + \beta x_b y_a \quad (7.115)$$

$$x_a y_p = \alpha x_a y_a + \beta x_a y_b. \quad (7.116)$$

Subtracting Eq. (7.116) from Eq. (7.115) we obtain

$$x_p y_a - x_a y_p = \beta x_b y_a - \beta x_a y_b = \beta (x_b y_a - x_a y_b) \quad (7.117)$$

where

$$\beta = \frac{x_p y_a - x_a y_p}{x_b y_a - x_a y_b} = \frac{\begin{vmatrix} x_p & y_p \\ x_a & y_a \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}}. \quad (7.118)$$

Using Eq. (7.114) and Eq. (7.118) we can rewrite Eq. (7.108) as

$$p = \frac{\begin{vmatrix} x_p & y_p \\ x_b & y_b \end{vmatrix}}{\begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix}} a + \frac{\begin{vmatrix} x_p & y_p \\ x_a & y_a \end{vmatrix}}{\begin{vmatrix} x_b & y_b \\ x_a & y_a \end{vmatrix}} b. \quad (7.119)$$

The problem with Eq. (7.119) is that the determinants reference the coordinates of the point we are trying to discover. Nevertheless, let's continue and write Eq. (7.119) using outer products

$$p = \frac{p \wedge b}{a \wedge b} a + \frac{p \wedge a}{b \wedge a} b. \quad (7.120)$$

Figure 7.16a provides a graphical interpretation of part of Eq. (7.120) where the parallelogram formed by the outer product  $p \wedge a$  is identical to the outer product formed by  $r \wedge a$ . Which means that we can substitute  $r \wedge a$  for  $p \wedge a$  in Eq. (7.120):

$$p = \frac{p \wedge b}{a \wedge b} a + \frac{r \wedge a}{b \wedge a} b. \quad (7.121)$$

Similarly, in Fig. 7.16b the parallelogram formed by the outer product  $p \wedge b$  is identical to the outer product formed by  $s \wedge b$ . Which means that we can substitute  $s \wedge b$  for  $p \wedge b$  in Eq. (7.121):

$$p = \frac{s \wedge b}{a \wedge b} a + \frac{r \wedge a}{b \wedge a} b. \quad (7.122)$$

The positions of  $R$  and  $S$  are not very important, as they could be anywhere along the two vectors, even positioned as shown in Fig. 7.17:

In Fig. 7.17 the three parallelograms:  $OSTU$ ,  $OVWR$  and  $OVXU$  have areas:

$$\text{area } OSTU = s \wedge b \quad (7.123)$$

$$\text{area } OVWR = r \wedge a \quad (7.124)$$

$$\text{area } OVXU = a \wedge b. \quad (7.125)$$

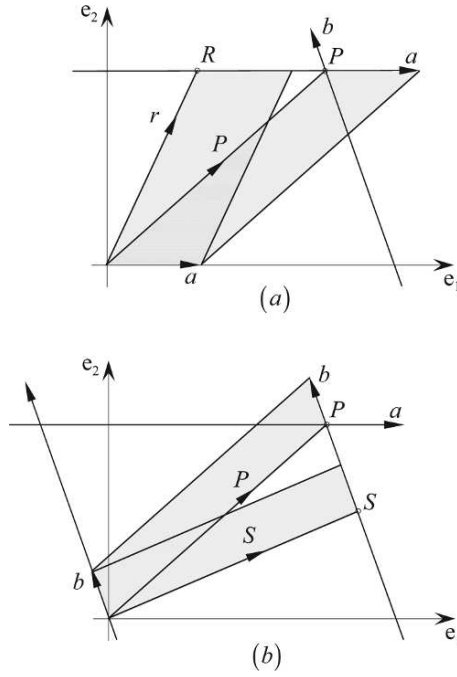


FIGURE 7.16.

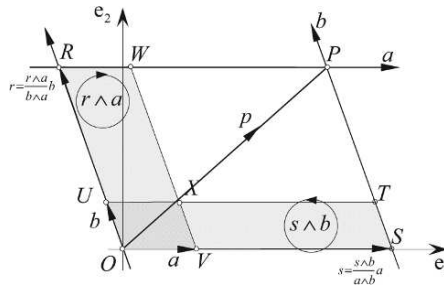


FIGURE 7.17.

Simply by relocating  $S$  and  $R$ , we have created a convenient visual symmetry where

$$s = \frac{s \wedge b}{a \wedge b} a \tag{7.126}$$

and

$$r = \frac{r \wedge a}{b \wedge a} b. \tag{7.127}$$

Note how  $s \wedge b$  and  $a \wedge b$  are in the same sense, whilst  $r \wedge a$  and  $b \wedge a$  are in the opposite sense. Observe, also, from Fig. (7.17) why

$$\frac{s}{a} = \frac{s \wedge b}{a \wedge b} \quad (7.128)$$

and

$$\frac{r}{b} = \frac{r \wedge a}{b \wedge a}. \quad (7.129)$$

It now becomes obvious that

$$p = s + r = \frac{s \wedge b}{a \wedge b} a + \frac{r \wedge a}{b \wedge a} b \quad (7.130)$$

where the solution to the problem is based upon the ratios of areas of parallelograms!

Let's test Eq. (7.130) using the same vectors above:

$$r = e_2 \quad a = 2e_1 - e_2 \quad (7.131)$$

$$s = 2e_2 \quad b = 2e_1 - 2e_2 \quad (7.132)$$

$$\begin{aligned} p &= \frac{(2e_2) \wedge (2e_1 - 2e_2)}{(2e_1 - e_2) \wedge (2e_1 - 2e_2)} (2e_1 - e_2) + \frac{e_2 \wedge (2e_1 - e_2)}{(2e_1 - 2e_2) \wedge (2e_1 - e_2)} (2e_1 - 2e_2) \\ p &= \frac{-4(e_1 \wedge e_2)}{-4(e_1 \wedge e_2) + 2(e_1 \wedge e_2)} (2e_1 - e_2) + \frac{-2(e_1 \wedge e_2)}{-2(e_1 \wedge e_2) + 4(e_1 \wedge e_2)} (2e_1 - 2e_2) \\ p &= 2(2e_1 - e_2) - (2e_1 - 2e_2) = 2e_1. \end{aligned} \quad (7.133)$$

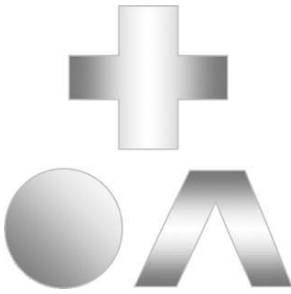
Therefore, the point of intersection is  $(2, 0)$ . Which is the same as the previous result.

We have spent some time exploring the above techniques, which in some cases are quite tedious. However, the conformal model, which is explored in chapter 11, simplifies the whole process.

## 7.6 Summary

It seems that the outer product is a very natural way of describing the orientation of two vectors, and has immediate applications in a variety of geometric problems. Let's now examine the properties of another product—the *geometric product*.





# 8 The Geometric Product

## 8.1 Introduction

Whenever we attempt to learn something completely new, all sorts of mental barriers are raised, especially if the subject matter appears foreign or irrational. This often happens when we learn a new language and wonder why its syntax differs from our own native language. Mathematics is a minefield for such experiences, and as we explore the world of GA, don't be surprised if you feel uncomfortable or bewildered by its structure and ideas.

If this is the first time you have studied GA this chapter will be both challenging and exciting. It will be challenging not because it is difficult, but because there is so much to remember as the algebra unfolds. For example, some elements of the algebra commute, whilst others anti-commute. Some functions are symmetric, whilst others are antisymmetric. Certain conditions arise with orthogonal elements, and others arise with parallel elements, and there is a sense of being overwhelmed by a world of axioms, rules and special conditions. It will be exciting as GA is extremely rich in new concepts that will draw you into its strange world of symbol manipulation that correspond with the world of space.

Basically, GA allows us manipulate scalars, vectors, areas and volumes using a simple and consistent notation. Combinations of such elements are called *multivectors*, which may be added, subtracted and multiplied. Adding or subtracting multivectors create no problems, as we simply add or subtract like elements. What is strange, however, are the products. In vector algebra there are only two products to consider: the inner product and the outer vector product. The inner product creates a scalar, whilst the vector product creates another vector normal to the original vectors. From these products arise all sorts of triple products such as:

$$(a \cdot b)c \quad a \cdot (b \times c) \quad (a \times b) \times c \quad a \times (b \times c) \quad (8.1)$$

which are easy to interpret and visualize. On the other hand, GA employs a new product called the *geometric product*, which operates upon multivectors containing scalars, vectors, areas and volumes. Visualizing these products can be difficult. For example, how should we visualize the product of two areas, or the product of a vector and a volume, or even the product of two volumes? These are new concepts and take some getting used to. What is even more strange is that

the algebra involves imaginary elements, which, unlike the reasonably familiar  $i$ , do not always commute with their neighbor. An unusual, but useful, feature of GA is that multivectors can be divided by vectors, which is something conventional vector algebra is unable to do.

The problem now is how best to reveal this algebra? The approach taken in this chapter is to split GA's features in two: the first part explores GA in 2D space followed by 3D space. But remember, the algebra can be applied to space of any number of dimensions. In the first part we look at vectors, bivectors (areas), pseudoscalars, multivectors and their products in an  $\mathbb{R}^2$  context. We also discover how these products give rise to rotations, much in the same way that complex numbers can be rotated. And because there is a close relationship between GA and complex numbers, we look at how it is possible to move between the two systems. In the second part we look at vectors, bivectors, trivectors (volumes), pseudoscalars, multivectors and their products in an  $\mathbb{R}^3$  context. We also discover how simple rotations arise from these products, and the close relationship between GA and quaternions.

This said, let's begin this journey with a description of Clifford's geometric product.

## 8.2 Clifford's definition of the geometric product

Clifford defined the geometric product of two vectors  $a$  and  $b$  as

$$ab = a \cdot b + a \wedge b \quad (8.2)$$

which is the sum of a scalar and a bivector. Now there is always a good reason why such definitions are made, and it is far from arbitrary. In order to develop this new product we start by defining the axioms associated with the algebra. These comprise an associative axiom, distributive axiom, and a definition of a modulus.

For the moment, let's put to one side what we have discovered about the outer product, and see whether its properties emerge from the following axioms.

### Associative axiom

$$a(bc) = (ab)c = abc \quad (8.3)$$

$$(\lambda a)b = \lambda(ab) = \lambda ab \quad [\lambda \in \mathbb{R}]. \quad (8.4)$$

### Distributive axiom

$$a(b + c) = ab + ac \quad (8.5)$$

and

$$(b + c)a = ba + ca. \quad (8.6)$$

### Modulus

$$a^2 = \pm \|a\|^2. \quad (8.7)$$

From these axioms we can derive the meaning of the product  $ab$ . Just in case the product is antisymmetric, we pay particular attention to the order of vectors.

We begin with two vectors  $a$  and  $b$  and represent their sum as

$$c = a + b. \tag{8.8}$$

Therefore,

$$c^2 = (a + b)^2 \tag{8.9}$$

and

$$c^2 = a^2 + b^2 + ab + ba. \tag{8.10}$$

To simplify this relationship we investigate how Eq. (8.10) behaves when vectors  $a$  and  $b$  are orthogonal, linearly dependent and linearly independent.

### 8.2.1 Orthogonal vectors

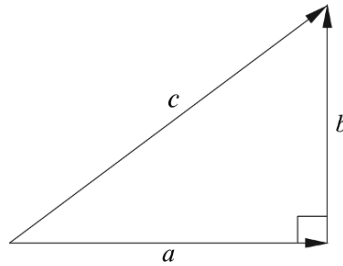


FIGURE 8.1.

With reference to Fig. 8.1, when

$$b \perp a$$

then

$$\|c\|^2 = \|a\|^2 + \|b\|^2. \tag{8.11}$$

Invoking the modulus axiom, we have

$$c^2 = a^2 + b^2 \tag{8.12}$$

which implies that in Eq. (8.10)

$$ab + ba = 0 \tag{8.13}$$

or

$$ab = -ba \tag{8.14}$$

which confirms that orthogonal vectors anticommute.

### 8.2.2 Linearly dependent vectors

With reference to Fig. 8.2, when

$$b \parallel a \text{ and } b = \lambda a \text{ where } [\lambda \in \mathbb{R}] \tag{8.15}$$

$$ab = a\lambda a = \lambda aa = ba \tag{8.16}$$

which confirms that linearly dependent vectors commute.

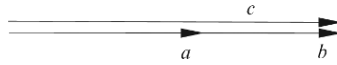


FIGURE 8.2.

Invoking the modulus axiom we have

$$\lambda a a = \lambda a^2 = \lambda \|a\|^2 \quad (8.17)$$

which is a scalar.

### 8.2.3 Linearly independent vectors

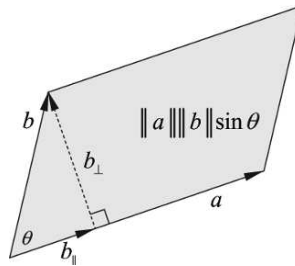


FIGURE 8.3.

With reference to Fig. 8.3

$$b = b_{\parallel} + b_{\perp}. \quad (8.18)$$

Therefore, we can write

$$ab = a(b_{\parallel} + b_{\perp}) \quad (8.19)$$

and

$$ab = ab_{\parallel} + ab_{\perp}. \quad (8.20)$$

Let's examine the RHS products of Eq. (8.20):

$ab_{\parallel}$ : As  $a$  and  $b_{\parallel}$  are linearly dependent,  $ab_{\parallel}$  is a scalar. Furthermore,

$$ab_{\parallel} = \|a\| \|b\| \cos \theta = a \cdot b \quad (8.21)$$

which is defined as the inner product, or the inner product, and is symmetric.

$ab_{\perp}$ : As  $a$  and  $b_{\perp}$  are orthogonal

$$ab_{\perp} = \|a\| \|b\| \sin \theta = a \wedge b \quad (8.22)$$

which is defined as the outer product and is antisymmetric; i.e.

$$a \wedge b = -b \wedge a. \quad (8.23)$$

The area of the parallelogram formed by  $a$  and  $b$  in Fig. 8.20 is

$$\|a\| \|b\| \sin \theta. \quad (8.24)$$

Therefore,

$$\|a \wedge b\| = \|a\| \|b\| \sin \theta \quad (8.25)$$

which enables us to write Eq. (8.20) as

$$ab = a \cdot b + a \wedge b. \quad (8.26)$$

The parallel and orthogonal components created by  $a \cdot b$  and  $a \wedge b$  describe everything about the vectors  $a$  and  $b$ , which is why Clifford combined them into his geometric product. Furthermore, because these product components are linearly independent, the modulus of  $ab$  is computed using the Pythagorean rule:

$$\begin{aligned} \|ab\|^2 &= \|a \cdot b\|^2 + \|a \wedge b\|^2 \\ \|ab\|^2 &= \|a\|^2 \|b\|^2 \cos^2 \theta + \|a\|^2 \|b\|^2 \sin^2 \theta \\ \|ab\|^2 &= \|a\|^2 \|b\|^2 (\cos^2 \theta + \sin^2 \theta) \end{aligned} \quad (8.27)$$

$$\|ab\| = \|a\| \|b\|. \quad (8.28)$$

Now we already know that  $a \cdot b$  is a pure scalar and  $a \wedge b$  is a directed area, which we suspect has an imaginary flavor. So it may seem strange adding two different mathematical objects together, but no stranger than a complex number. Nevertheless, we still require a name for this new object, which is a *multivector* and is described in section 8.5.

If we reverse the product to  $ba$  we have

$$ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b. \quad (8.29)$$

Note how the antisymmetry of the outer product introduces the negative sign.

Knowing that the geometric product is the sum of the inner and outer products, it is possible to define the inner and outer products in terms of the geometric product as follows.

Subtracting Eq. (8.29) from Eq. (8.26) we obtain

$$ab - ba = (a \cdot b + a \wedge b) - (a \cdot b - a \wedge b) = 2(a \wedge b) \quad (8.30)$$

therefore,

$$a \wedge b = \frac{1}{2}(ab - ba). \quad (8.31)$$

Similarly, adding Eq. (8.29) to Eq. (8.26) we obtain

$$ab + ba = 2a \cdot b \quad (8.32)$$

therefore,

$$a \cdot b = \frac{1}{2}(ab + ba). \quad (8.33)$$

These are important relationships and will be called upon frequently.

Now let's explore the geometric product further using the unit basis vectors for  $\mathbb{R}^2$ .

### 8.2.4 The product of identical basis vectors

Before we begin exploring this product, it is worth introducing a shorthand notation that simplifies our equations. Very often we have to write down a string of basis vectors such as  $e_1 e_2 e_1$  which can also be written as  $e_{121}$ , and saves space on the printed page. In general this is expressed as:

$$e_i e_j e_k \equiv e_{ijk}. \quad (8.34)$$

So let's start with the product  $e_1 e_1$ :

$$e_1 e_1 = e_1 \cdot e_1 + e_1 \wedge e_1. \quad (8.35)$$

Now we already know that  $e_1 \wedge e_1 = 0$  and  $e_1 \cdot e_1 = 1$ , which means that

$$e_1 e_1 = e_1^2 = 1. \quad (8.36)$$

Similarly,

$$e_2^2 = 1. \quad (8.37)$$

### 8.2.5 The product of orthogonal basis vectors

Next, the product  $e_1 e_2$ :

$$e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2. \quad (8.38)$$

Again, we know that  $e_1 \cdot e_2 = 0$ , which means that

$$e_1 e_2 = e_1 \wedge e_2. \quad (8.39)$$

So, whenever we find the unit bivector  $e_1 \wedge e_2$  we can substitute  $e_1 e_2$  or  $e_{12}$ .

Now let's compute the product  $e_2 e_1$ :

$$e_2 e_1 = e_2 \cdot e_1 + e_2 \wedge e_1 = e_2 \cdot e_1 - e_1 \wedge e_2. \quad (8.40)$$

But  $e_2 \cdot e_1 = 0$ , therefore,

$$e_2 e_1 = -e_1 \wedge e_2 = -e_{12}. \quad (8.41)$$

## 8.2.6 The imaginary properties of the outer product

The imaginary properties of the outer product are revealed by evaluating the product  $(e_1 \wedge e_2)^2$ :

$$(e_1 \wedge e_2)^2 = (e_1 \wedge e_2)(e_1 \wedge e_2) = e_1 e_2 e_1 e_2. \quad (8.42)$$

But as

$$e_2 e_1 = -e_1 e_2 \quad (8.43)$$

then

$$(e_1 \wedge e_2)^2 = -e_1 e_1 e_2 e_2 = -e_1^2 e_2^2. \quad (8.44)$$

But as

$$e_1^2 = e_2^2 = 1 \quad (8.45)$$

then

$$(e_1 \wedge e_2)^2 = -1. \quad (8.46)$$

So the unit bivector possess the same qualities as imaginary  $i$  in that it squares to  $-1$ .

Now this has all sorts of ramifications as it suggests that GA is related to complex numbers and possibly, quaternions, and could perform rotations in  $n$ -dimensions. At this point, the algebra explodes into many paths, which will have to be explored in turn.

## 8.3 The unit bivector pseudoscalar

GA uses the term *grade* to distinguish its algebraic elements. For example, a scalar is grade-0, a vector grade-1 and a bivector grade-2, etc. In each algebra, the highest grade element is called the *pseudoscalar* and its grade equals the dimension of the associated space, which in  $\mathbb{R}^2$  is the bivector  $e_1 \wedge e_2$  and is a two-dimensional element. Later on, we discover that the trivector in  $\mathbb{R}^3$  is also called a pseudoscalar.

Because the pseudoscalar has imaginary properties, some authors use the lowercase  $i$  to represent it, whilst others opt for the uppercase  $I$ . The reason for this is that  $i$  is normally associated with scalars, where there are no commuting problems. On the other hand, we will soon discover that the pseudoscalar anticommutes with vectors in  $\mathbb{R}^2$ , and it is safer to employ the symbol  $I$  so that its anticommuting properties do not get confused with those of  $i$ .

### 8.3.1 The rotational properties of the pseudoscalar

Now that we know that the unit bivector possesses imaginary properties, let's confirm that it rotates vectors in the same way we saw in section 7.2. We begin with the product  $e_1 I$ :

$$e_1 I = e_1 e_1 e_2 = e_1^2 e_2 = e_2. \quad (8.47)$$

Taking the result  $e_2$  and post-multiplying this by  $I$ :

$$e_2 I = e_2 e_1 e_2 = e_2 (-e_2 e_1) = -e_2^2 e_1 = -e_1. \quad (8.48)$$

Taking the result  $-e_1$  and post-multiplying this by  $I$ :

$$-e_1 I = -e_1 e_1 e_2 = -e_1^2 e_2 = -e_2. \quad (8.49)$$

Taking the result  $-e_2$  and multiplying this by  $I$ :

$$-e_2 I = -e_2 e_1 e_2 = -e_2 (-e_2 e_1) = e_2^2 e_1 = e_1 \quad (8.50)$$

which brings us back to the starting point. Similarly, when the product is reversed, the direction of rotation is reversed.

As a simple example of the algebra in action, consider post-multiplying a vector  $a$  by the pseudoscalar  $I$  where

$$a = a_1 e_1 + a_2 e_2. \quad (8.51)$$

Then

$$aI = a e_1 e_2 = (a_1 e_1 + a_2 e_2) e_1 e_2 = a_1 e_1^2 e_2 + a_2 e_2 e_1 e_2 \quad (8.52)$$

and

$$aI = a_1 e_2 - a_2 e_2^2 e_1 = -a_2 e_1 + a_1 e_2 \quad (8.53)$$

which has clearly rotated the vector  $90^\circ$  anticlockwise.

Pre-multiplying the vector  $a$  by  $I$  produces:

$$Ia = e_1 e_2 a = e_1 e_2 (a_1 e_1 + a_2 e_2) = a_1 e_1 e_2 e_1 + a_2 e_1 e_2^2 \quad (8.54)$$

and

$$Ia = -a_1 e_2 + a_2 e_1 = a_2 e_1 - a_1 e_2 \quad (8.55)$$

which has rotated the vector  $90^\circ$  clockwise.

Therefore,

$$aI = -Ia \quad (8.56)$$

and confirms that in  $\mathbb{R}^2$ , the pseudoscalar and vectors anticommute.

These rotations are illustrated in Fig. 8.4.

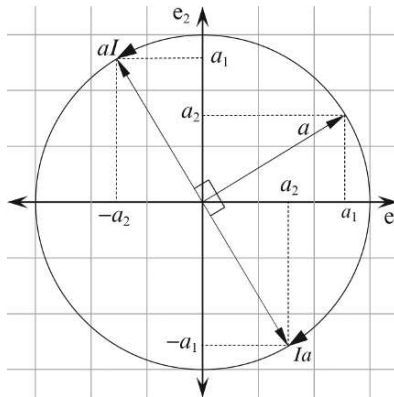


FIGURE 8.4.



### 8.4 Summary of the products

Table 8.1 summarizes the products we have encountered so far.

TABLE 8.1

Products in $\mathbb{R}^2$			
Type	Product	Absolute Value	Notes
inner	$e_1 \cdot e_1$	1	$e_2 \cdot e_2 = e_1 \cdot e_1$
outer	$e_1 \wedge e_1$	0	$e_2 \wedge e_2 = e_1 \wedge e_1$
geometric	$e_1^2$	1	$e_2^2 = e_1^2$ $e_1 I = -I e_1$
inner	$e_1 \cdot e_2$	0	$e_2 \cdot e_1 = e_1 \cdot e_2$
outer	$e_1 \wedge e_2$	1	$e_1 \wedge e_2 = -(e_2 \wedge e_1)$
geometric	$e_1 e_2$	1	$e_{12} = -e_{21}$ $e_{12} = I$ $I^2 = -1$
inner	$a \cdot a$	$\ a\ ^2$	
outer	$a \wedge a$	0	
geometric	$a^2$	$\ a\ ^2$	
inner	$a \cdot b$	$\ a\  \ b\  \cos \theta$ $a_1 b_1 + a_2 b_2$	$a \cdot b = \frac{1}{2}(ab + ba)$
outer	$a \wedge b$	$\ a\  \ b\  \sin \theta$ $a_1 b_2 - a_2 b_1$	$a \wedge b = \frac{1}{2}(ab - ba)$ $a \wedge b = (a_1 b_2 - a_2 b_1)e_1 \wedge e_2$
geometric	$ab$	$\ a\  \ b\ $	$ab = a \cdot b + a \wedge b$ $aI = -Ia$

### 8.5 Multivectors in $\mathbb{R}^2$

In Chapters 2, 3, 4 and 5 we reviewed four algebraic systems with their axioms and elements and saw that elementary algebra supports scalars; complex algebra supports complex numbers (a scalar and an imaginary); vector algebra supports vectors ( $n$ -tuples); and quaternion algebra supports quaternions (a scalar and a vector). Clifford required that geometric algebra should support an element containing scalars, vectors, bivectors and any other object that could be created using the geometric product, which seems to be an impossible task. But his deep understanding of algebra and geometry resulted in an object he called a *multivector* which can be added and multiplied together just like any other element. For example, a multivector in  $\mathbb{R}^2$  contains a scalar, vectors and a bivector, whereas in  $\mathbb{R}^3$  a multivector contains a scalar, vectors, bivectors and a *trivector*. Higher-dimensional spaces contain similar combinations of scalar and vector-based objects.

The multivector elements that exist in  $\mathbb{R}^2$  are scalars, vectors and bivectors, which are summarized in Table 8.2.

TABLE 8.2

Element	Symbol	Grade
1 scalar	$\lambda$	0
2 vectors	$\{e_1, e_2\}$	1
1 unit bivector	$e_1 \wedge e_2 = e_{12}$	2

A multivector is defined as a linear combination of the graded elements associated with the size of the linear space, which, in the case of  $\mathbb{R}^2$  are scalars, vectors and bivectors. Therefore a multivector  $A$  is defined as follows:

$$A = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_{12} \quad [\lambda_i \in \mathbb{R}] \quad (8.57)$$

Note that we have substituted the geometric product for the outer product, as this is much more convenient. Using arbitrary values, the following are possible multivectors:

$$A = 4 + 3e_1 + 4e_2 + 5e_{12} \quad (8.58)$$

$$B = 3 + 2e_1 + 3e_2 + 4e_{12} \quad (8.59)$$

which, allows us to write:

$$A + B = 7 + 5e_1 + 7e_2 + 9e_{12} \quad (8.60)$$

and

$$A - B = 1 + e_1 + e_2 + e_{12}. \quad (8.61)$$

But what about the product  $AB$ ? To answer this question, let's define  $B$  in general terms and form the product  $AB$ :

$$B = \beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12}. \quad (8.62)$$

Therefore,

$$AB = (\lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_{12})(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12}). \quad (8.63)$$

Expanding, we obtain

$$\begin{aligned} AB = & \lambda_0 \beta_0 + \lambda_0 \beta_1 e_1 + \lambda_0 \beta_2 e_2 + \lambda_0 \beta_3 e_{12} + \lambda_1 \beta_0 e_1 + \lambda_1 \beta_1 e_1^2 \\ & + \lambda_1 \beta_2 e_{12} + \lambda_1 \beta_3 e_{112} + \lambda_2 \beta_0 e_2 + \lambda_2 \beta_1 e_{21} + \lambda_2 \beta_2 e_2^2 \\ & + \lambda_2 \beta_3 e_{212} + \lambda_3 \beta_0 e_{12} + \lambda_3 \beta_1 e_{121} + \lambda_3 \beta_2 e_{122} + \lambda_3 \beta_3 e_{12}^2. \end{aligned} \quad (8.64)$$

Substituting

$$e_1^2 = e_2^2 = 1 \quad e_{21} = -e_{12} \quad e_{12}^2 = -1 \quad (8.65)$$

and collecting up like terms:

$$\begin{aligned} AB = & (\lambda_0 \beta_0 + \lambda_1 \beta_1 + \lambda_2 \beta_2 - \lambda_3 \beta_3) + (\lambda_0 \beta_1 + \lambda_1 \beta_0 + \lambda_3 \beta_2 - \lambda_2 \beta_3) e_1 \\ & + (\lambda_0 \beta_2 + \lambda_1 \beta_3 + \lambda_2 \beta_0 - \lambda_3 \beta_1) e_2 + (\lambda_0 \beta_3 + \lambda_1 \beta_2 + \lambda_3 \beta_0 - \lambda_2 \beta_1) e_{12}. \end{aligned} \quad (8.66)$$

Which confirms that the multivector product  $AB$  creates another multivector and consequently forms a closed algebra.

Using the above multivectors

$$AB = (4 + 3e_1 + 4e_2 + 5e_{12})(3 + 2e_1 + 3e_2 + 4e_{12}) \quad (8.67)$$

then

$$AB = 10 + 16e_1 + 26e_2 + 32e_{12}. \quad (8.68)$$

## 8.6 The relationship between bivectors, complex numbers and vectors

The geometric product reveals the relationship between bivectors and complex numbers, and is demonstrated by computing the product of two vectors in  $\mathbb{R}^2$ .

Given two vectors  $a$  and  $b$  where

$$a = a_1e_1 + a_2e_2 \quad (8.69)$$

$$b = b_1e_1 + b_2e_2 \quad (8.70)$$

then

$$\begin{aligned} ab &= (a_1e_1 + a_2e_2)(b_1e_1 + b_2e_2) \\ &= a_1b_1e_1^2 + a_1b_2e_{12} + a_2b_1e_{21} + a_2b_2e_2^2 \\ &= a_1b_1 + a_2b_2 + a_1b_2e_{12} - a_2b_1e_{12} \\ &= (a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)e_{12} \\ ab &= (a_1b_1 + a_2b_2) + (a_1b_2 - a_2b_1)I \end{aligned} \quad (8.71)$$

which is a complex number! Note that  $(a_1b_1 + a_2b_2)$  is a scalar whilst  $(a_1b_2 - a_2b_1)I$  is a bivector, which means that we can form the equivalent of a complex number  $Z$  by combining a scalar and a unit bivector as follows:

$$Z = a_1 + a_2e_{12} = a_1 + a_2I \quad (8.72)$$

where  $a_1$  is the real part, and  $a_2$  is the imaginary part.

Furthermore, we can convert a vector  $a$  into a complex number  $Z$  as follows.

Given a vector  $a$ :

$$a = a_1e_1 + a_2e_2 \quad (8.73)$$

then pre-multiplying  $a$  by  $e_1$  we obtain:

$$e_1a = e_1(a_1e_1 + a_2e_2) = a_1e_1^2 + a_2e_1e_2 = a_1 + a_2I. \quad (8.74)$$

Therefore,

$$e_1a = Z. \quad (8.75)$$

But what happens if we reverse  $e_1$  and  $a$ ?

$$ae_1 = (a_1e_1 + a_2e_2)e_1 = a_1e_1^2 + a_2e_2e_1 = a_1 - a_2I \quad (8.76)$$

which we recognize as the complex conjugate. Therefore,

$$ae_1 = Z^\dagger. \quad (8.77)$$

(The dagger symbol  $\dagger$  is sometimes used to represent the complex conjugate of a multivector).

## 8.7 Reversion

Reversing sequences of symbols happens to be a useful operation in GA. For instance, we may wish to reverse the sequence of three vectors  $abc$  to  $cba$ , or swap two bivectors  $AB$  to  $BA$ . Whatever the elements may be, the reversion operator performs this task and is used as follows:

$$(abc \cdots d)^\sim = (d \cdots cba). \quad (8.78)$$

The tilde superscript reminds us of this action, but other authors may employ the dagger symbol.

For any vectors  $a$  and  $b$

$$\begin{aligned} (ab)^\sim &= (a \cdot b + a \wedge b)^\sim \\ &= a \cdot b - b \wedge a \\ (ab)^\sim &= ba. \end{aligned} \quad (8.79)$$

Similarly, for any multivectors  $A$  and  $B$

$$(AB)^\sim = \tilde{B}\tilde{A}. \quad (8.80)$$

Unfortunately, some reversions involve a sign change, and are summarized in Table 8.3.

TABLE 8.3

Blade	$k$	Sign
scalar	0	+
vector	1	+
bivector	2	-
trivector	3	-
4-vector	4	+
5-vector	5	+
6-vector	6	-
7-vector	7	-
etc.		

This sign switching pattern is accommodated by the following formula:

$$\tilde{A}_k = (-1)^{\frac{k(k-1)}{2}} A_k. \quad (8.81)$$

## 8.8 Rotations in $\mathbb{R}^2$

In chapter 3 we saw that a complex number  $z$  is rotated through an angle  $\phi$  using

$$z' = ze^{i\phi} \quad (8.82)$$

where

$$e^{i\phi} = \cos \phi + i \sin \phi. \quad (8.83)$$

But as  $i^2 = I^2$

$$e^{I\phi} = \cos \phi + I \sin \phi. \quad (8.84)$$

Therefore,

$$z' = ze^{I\phi}. \quad (8.85)$$

But a multivector consisting of a scalar and a bivector is identical to a complex number, which means that we can write Eq. (8.85) as

$$Z' = Ze^{I\phi}. \quad (8.86)$$

So now let's see how a vector is rotated using a similar operation.

Pre-multiplying Eq. (8.75) by  $e_1$  we obtain

$$e_1 e_1 v = e_1 Z \quad (8.87)$$

and

$$v = e_1 Z. \quad (8.88)$$

Let's assume that there exists another vector  $v'$  with an associated multivector  $Z'$  such that

$$v' = e_1 Z'. \quad (8.89)$$

Substituting Eq. (8.86) we obtain

$$v' = e_1 Z e^{I\phi}. \quad (8.90)$$

Substituting Eq. (8.75) we obtain

$$v' = e_1 e_1 v e^{I\phi} = v e^{I\phi} \quad (8.91)$$

which rotates the vector  $v$  through an angle  $\phi$  to  $v'$ .

Let's illustrate Eq. (8.91) with an example.

Rotate  $v = e_1$  anticlockwise  $90^\circ$  in the plane  $e_{12}$ :

$$\begin{aligned} v' &= v e^{I\phi} = e_1 (\cos 90^\circ + e_{12} \sin 90^\circ) \\ v' &= e_1 e_{12} = e_2. \end{aligned} \quad (8.92)$$

Which is correct.

## 8.9 The vector-bivector product in $\mathbb{R}^2$

In section 8.3.1 we saw that a pseudoscalar rotates a vector  $90^\circ$  in the plane without scaling the vector. Now let's see what happens when we form the geometric product of a vector and a bivector. For example, given a vector  $a$  and a bivector  $B$  where

$$a = a_1 e_1 + a_2 e_2 \quad (8.93)$$

$$B = (b_1 e_1 + b_2 e_2) \wedge (c_1 e_1 + c_2 e_2) \quad (8.94)$$

then  $a'$  is the product  $aB$

$$\begin{aligned}
 a' &= aB \\
 &= (a_1e_1 + a_2e_2)((b_1e_1 + b_2e_2) \wedge (c_1e_1 + c_2e_2)) \\
 &= (a_1e_1 + a_2e_2)(b_1c_1e_1 \wedge e_1 + b_1c_2e_1 \wedge e_2 + b_2c_1e_2 \wedge e_1 + b_2c_2e_2 \wedge e_2) \\
 &= (a_1e_1 + a_2e_2)(b_1c_2 - b_2c_1)e_{12} \\
 &= a_1(b_1c_2 - b_2c_1)e_1^2e_2 + a_2(b_1c_2 - b_2c_1)e_{212} \\
 &= a_1(b_1c_2 - b_2c_1)e_2 - a_2(b_1c_2 - b_2c_1)e_1 \\
 a' &= -a_2(b_1c_2 - b_2c_1)e_1 + a_1(b_1c_2 - b_2c_1)e_2.
 \end{aligned} \tag{8.95}$$

But

$$\|B\| = b_1c_2 - b_2c_1. \tag{8.96}$$

Therefore,

$$a' = \|B\|(-a_2e_1 + a_1e_2). \tag{8.97}$$

It is clear from Eq. (8.97) that vector  $a$  has been rotated anticlockwise  $90^\circ$  and scaled by the magnitude of the bivector  $B$ . Reversing the product reverses the direction of rotation:

$$\begin{aligned}
 a' &= Ba \\
 &= ((b_1e_1 + b_2e_2) \wedge (c_1e_1 + c_2e_2))(a_1e_1 + a_2e_2) \\
 &= (b_1c_2 - b_2c_1)e_{12}(a_1e_1 + a_2e_2) \\
 &= a_1(b_1c_2 - b_2c_1)e_{121} + a_2(b_1c_2 - b_2c_1)e_{122} \\
 &= -a_1(b_1c_2 - b_2c_1)e_2 + a_2(b_1c_2 - b_2c_1)e_1 \\
 &= a_2(b_1c_2 - b_2c_1)e_1 - a_1(b_1c_2 - b_2c_1)e_2 \\
 a' &= \|B\|(a_2e_1 - a_1e_2).
 \end{aligned} \tag{8.98}$$

Equation (8.98) confirms that  $a$  has been rotated clockwise  $90^\circ$  and scaled by the magnitude of the bivector  $B$ .

## 8.10 Volumes and the trivector

By now you will have observed that geometric algebra is highly structured. We start with scalars, which in various tuples create vectors, which in turn create bivectors and ultimately lead to multivectors. The next element after the bivector is the *trivector* and is used to represent a directed volume. Starting with a bivector  $a \wedge b$ , which represents a directed area, we can imagine that this is moved along a third vector  $c$  to sweep out a parallelepiped as shown in Fig. 8.5 (a).

Remember that we are working with a right-handed axial system, and the bivector  $a \wedge b$  is anticlockwise as viewed from inside the volume and moves along the direction of vector  $c$  to create the trivector  $(a \wedge b) \wedge c$ . In Fig. 8.5 (b) the bivector  $b \wedge c$  is still anticlockwise as viewed from inside the volume and moves along the direction of vector  $a$  to create the trivector  $(b \wedge c) \wedge a$ .

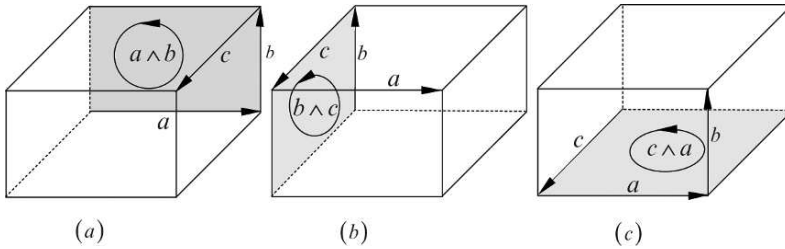


FIGURE 8.5.

Finally, in Fig. 8.5 (c), the bivector  $c \wedge a$  is still anticlockwise as viewed from inside the volume and moves along the direction of vector  $b$  to create the trivector  $(c \wedge a) \wedge b$ . It is obvious that the three volumes are identical, which allows us to state

$$(a \wedge b) \wedge c = (b \wedge c) \wedge a = (c \wedge a) \wedge b. \tag{8.99}$$

Although the volumes in Fig. 8.5 are rectangular parallelepipeds, the above reasoning still holds for general parallelepipeds. In fact, just as the parallelogram helped us visualize the area computing powers of the bivector, the parallelepiped is just a useful object to illustrate the volumetric computing powers of the trivector. However, any volume can be used to visualize a trivector.

When vectors  $a, b, c$  are described in terms of the unit basis vectors:

$$a = a_1e_1 + a_2e_2 + a_3e_3 \tag{8.100}$$

$$b = b_1e_1 + b_2e_2 + b_3e_3 \tag{8.101}$$

$$c = c_1e_1 + c_2e_2 + c_3e_3 \tag{8.102}$$

and multiplied together using  $a \wedge b \wedge c$ , it is obvious that this will give rise to terms such as:

$$e_1 \wedge e_1 \wedge e_1 = 0 \tag{8.103}$$

$$e_2 \wedge e_2 \wedge e_2 = 0 \tag{8.104}$$

$$e_3 \wedge e_3 \wedge e_3 = 0, \text{ etc.} \tag{8.105}$$

Furthermore, a variety of new terms arise involving triple outer products such as:

$$e_1 \wedge e_2 \wedge e_3 \tag{8.106}$$

$$e_1 \wedge e_2 \wedge e_1 \tag{8.107}$$

$$e_1 \wedge e_2 \wedge e_2, \text{ etc.} \tag{8.108}$$

The product  $e_1 \wedge e_2 \wedge e_3$  is interpreted as:

‘sweep the unit bivector  $e_1 \wedge e_2$  along the orthogonal vector  $e_3$  creating a volume represented by the trivector  $e_1 \wedge e_2 \wedge e_3$ .’

The product  $e_1 \wedge e_2 \wedge e_1$  is interpreted as:

‘sweep the unit bivector  $e_1 \wedge e_2$  along one of its vectors:  $e_1$ , which does not create a volume.’

The product  $e_1 \wedge e_2 \wedge e_2$  is interpreted as:

‘sweep the unit bivector  $e_1 \wedge e_2$  along one of its vectors:  $e_2$ , which also does not create a volume.’

Although these interpretations are correct, we require an algebraic explanation, which is provided as follows.

For completeness, let's expand the triple outer product:

$$\begin{aligned}
 a \wedge b \wedge c &= (a_1 e_1 + a_2 e_2 + a_3 e_3) \wedge (b_1 e_1 + b_2 e_2 + b_3 e_3) \wedge (c_1 e_1 + c_2 e_2 + c_3 e_3) \\
 &= \left( \begin{aligned} &a_1 b_1 e_1 \wedge e_1 + a_1 b_2 e_1 \wedge e_2 + a_1 b_3 e_1 \wedge e_3 + \\ &a_2 b_1 e_2 \wedge e_1 + a_2 b_2 e_2 \wedge e_2 + a_2 b_3 e_2 \wedge e_3 + \\ &a_3 b_1 e_3 \wedge e_1 + a_3 b_2 e_3 \wedge e_2 + a_3 b_3 e_3 \wedge e_3 \end{aligned} \right) \wedge (c_1 e_1 + c_2 e_2 + c_3 e_3) \\
 &= \left( \begin{aligned} &a_1 b_2 e_1 \wedge e_2 - a_1 b_3 e_3 \wedge e_1 - a_2 b_1 e_1 \wedge e_2 + \\ &a_2 b_3 e_2 \wedge e_3 + a_3 b_1 e_3 \wedge e_1 - a_3 b_2 e_2 \wedge e_3 \end{aligned} \right) \wedge (c_1 e_1 + c_2 e_2 + c_3 e_3) \\
 a \wedge b \wedge c &= \left( \begin{aligned} &(a_1 b_2 - a_2 b_1) e_1 \wedge e_2 + (a_2 b_3 - a_3 b_2) e_2 \wedge e_3 \\ &+ (a_3 b_1 - a_1 b_3) e_3 \wedge e_1 \end{aligned} \right) \wedge (c_1 e_1 + c_2 e_2 + c_3 e_3) \quad (8.109)
 \end{aligned}$$

At this point we can reject terms such as

$$e_1 \wedge e_2 \wedge e_1, \quad e_1 \wedge e_2 \wedge e_2, \quad e_2 \wedge e_3 \wedge e_2 \quad (8.110)$$

as they are zero volume elements, and means that we are left with the following trivector coefficients:

$$\begin{aligned}
 a \wedge b \wedge c &= (a_1 b_2 - a_2 b_1) c_3 e_{123} + (a_2 b_3 - a_3 b_2) c_1 e_{123} + (a_3 b_1 - a_1 b_3) c_2 e_{123} \\
 &= ((a_2 b_3 - a_3 b_2) c_1 + (a_3 b_1 - a_1 b_3) c_2 + (a_1 b_2 - a_2 b_1) c_3) e_{123}
 \end{aligned}$$

and

$$a \wedge b \wedge c = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} e_{123} \quad (8.111)$$

which we recognize as the volume of a parallelepiped. Thus a trivector computes a directed volume.

## 8.11 The unit trivector pseudoscalar

Earlier in this chapter we discovered that

$$(e_1 \wedge e_2)^2 = -1 \quad (8.112)$$

and the name pseudoscalar was given to this product. Now let's do the same for the trivector:

$$\begin{aligned}
 (e_1 \wedge e_2 \wedge e_3)^2 &= (e_1 e_2 e_3)^2 \\
 &= e_1 e_2 e_3 e_1 e_2 e_3 = e_1 e_2 e_1 e_3 e_3 e_2 \\
 (e_1 \wedge e_2 \wedge e_3)^2 &= e_1 e_2 e_1 e_2 = -1 \quad (8.113)
 \end{aligned}$$

which shows that the unit trivector also possesses imaginary properties.

With this new-found knowledge, let's compute the volume of a rectangular parallelepiped and a general parallelepiped.



We start by defining the edges of a box using the following vectors as shown in Fig. 8.6:

$$a = 2e_1 \quad b = 3e_2 \quad c = 4e_3. \tag{8.114}$$

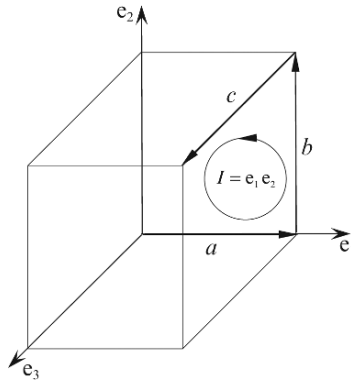


FIGURE 8.6.

Its volume  $V$  is defined as

$$\begin{aligned} V &= \|a \wedge b \wedge c\| \\ &= \|2e_1 \wedge 3e_2 \wedge 4e_3\| \\ &= \|24e_1 \wedge e_2 \wedge e_3\| \\ &= \|24e_{123}\| \\ V &= \|24I\|. \end{aligned} \tag{8.115}$$

Although the volume is represented as  $24I$ , we are only interested in its magnitude, which is 24. Hopefully, it is obvious that by reversing one of the vectors reverses the sign of the volume.

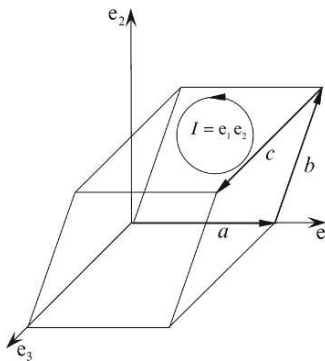


FIGURE 8.7.

For a second example, Fig. 8.7 illustrates a general parallelepiped where

$$a = 2e_1 \quad b = 0.5e_1 + 2e_2 \quad c = 3e_3. \tag{8.116}$$

Therefore, its volume  $V$  is given by

$$\begin{aligned}
 V &= \|a \wedge b \wedge c\| \\
 &= \|2e_1 \wedge (0.5e_1 + 2e_2) \wedge 3e_3\| \\
 &= \|4e_{12} \wedge 3e_3\| \\
 &= \|12e_{123}\| \\
 V &= 12.
 \end{aligned} \tag{8.117}$$

At this point it is worth summarizing a pseudoscalar's features. To begin with, the pseudoscalar squares to  $-1$ :

$$I^2 = -1 \tag{8.118}$$

which guarantees

$$\|I\|^2 = 1. \tag{8.119}$$

Secondly, the pseudoscalar defines orientation. For instance, the 2D unit bivector is defined by  $e_1 \wedge e_2$ , and if any other bivector has the same sign as  $e_1 \wedge e_2$  it shares the same orientation. Similarly, the 3D unit trivector is defined by  $e_1 \wedge e_2 \wedge e_3$ , and if any other trivector has the same sign as  $e_1 \wedge e_2 \wedge e_3$  it shares the same orientation. Convention dictates that  $e_1 \wedge e_2 \wedge e_3$  describes a right-handed system of axes.

## 8.12 The product of the unit basis vectors in $\mathbb{R}^3$

### 8.12.1 The product of identical basis vectors

The three unit basis vectors in  $\mathbb{R}^3$  are  $e_1, e_2$  and  $e_3$ , and although it is self-evident, for the sake of completeness, we will record that

$$e_1^2 = e_2^2 = e_3^2 = 1. \tag{8.120}$$

### 8.12.2 The product of orthogonal basis vectors

The third unit basis vector  $e_3$  gives rise to three orthogonal unit basis bivector combinations:

$$e_{12}, e_{23} \text{ and } e_{31}. \tag{8.121}$$

We already know that

$$e_{12} = e_1 \wedge e_2 \tag{8.122}$$

and it should come as no surprise that

$$e_{23} = e_2 \wedge e_3 \tag{8.123}$$

and

$$e_{31} = e_3 \wedge e_1. \tag{8.124}$$

### 8.12.3 The imaginary properties of the unit bivectors

In section 8.2.6 we discovered that

$$\mathbf{e}_{12}^2 = (\mathbf{e}_1 \wedge \mathbf{e}_2)^2 = -1 \quad (8.125)$$

and the same pattern is repeated for  $\mathbb{R}^3$ :

$$\mathbf{e}_{23}^2 = (\mathbf{e}_2 \wedge \mathbf{e}_3)^2 = -1 \quad (8.126)$$

and

$$\mathbf{e}_{31}^2 = (\mathbf{e}_3 \wedge \mathbf{e}_1)^2 = -1. \quad (8.127)$$

## 8.13 The vector-unit bivector product in $\mathbb{R}^3$

In section 8.3.1 we discovered that pre-multiplying a vector in  $\mathbb{R}^2$  by the pseudoscalar  $I = \mathbf{e}_{12}$  rotates the vector clockwise  $90^\circ$  and post-multiplying rotates the vector anticlockwise  $90^\circ$ . Let's see what happens when we multiply a vector in  $\mathbb{R}^3$  by a unit bivector. We begin by defining a vector and a unit bivector  $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3. \quad (8.128)$$

Therefore,

$$\begin{aligned} \mathbf{e}_{12}\mathbf{a} &= a_1\mathbf{e}_{12}\mathbf{e}_1 + a_2\mathbf{e}_{12}\mathbf{e}_2 + a_3\mathbf{e}_{12}\mathbf{e}_3 \\ &= -a_1\mathbf{e}_2 + a_2\mathbf{e}_1 + a_3\mathbf{e}_{123} \\ \mathbf{e}_{12}\mathbf{a} &= a_2\mathbf{e}_1 - a_1\mathbf{e}_2 + a_3\mathbf{e}_{123}. \end{aligned} \quad (8.129)$$

Equation (8.129) contains two elements:  
a vector

$$a_2\mathbf{e}_1 - a_1\mathbf{e}_2 \quad (8.130)$$

and a volume

$$a_3\mathbf{e}_{123}. \quad (8.131)$$

What has happened is this. The multiplier  $\mathbf{e}_{12}$  has:

1. Rotated the projection of  $\mathbf{a}$  on the bivector  $\mathbf{e}_1 \wedge \mathbf{e}_2$ , clockwise  $90^\circ$ . (Fig. 8.8a)
2. Formed a volume of  $a_3$  by sweeping  $\mathbf{e}_1 \wedge \mathbf{e}_2$  along  $\mathbf{e}_3$ . (Fig. 8.8b)

Reversing the product to  $\mathbf{a}\mathbf{e}_{12}$  produces

$$\begin{aligned} \mathbf{a}\mathbf{e}_{12} &= a_1\mathbf{e}_1\mathbf{e}_{12} + a_2\mathbf{e}_2\mathbf{e}_{12} + a_3\mathbf{e}_3\mathbf{e}_{12} \\ &= a_1\mathbf{e}_2 - a_2\mathbf{e}_1 + a_3\mathbf{e}_{123} \\ \mathbf{a}\mathbf{e}_{12} &= -a_2\mathbf{e}_1 + a_1\mathbf{e}_2 + a_3\mathbf{e}_{123}. \end{aligned} \quad (8.132)$$

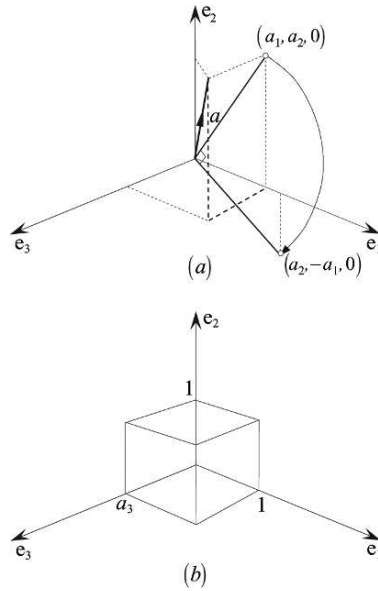


FIGURE 8.8.

Equation (8.13) confirms that the direction of rotation has been reversed to anticlockwise, whilst the sign of the volume remains unchanged.

Similar results are obtained with the products with  $e_{23}a$  and  $e_{31}a$ :

$$\begin{aligned} e_{23}a &= a_1e_{23}e_1 + a_2e_{23}e_2 + a_3e_{23}e_3 \\ &= a_1e_{123} - a_2e_3 + a_3e_2 \\ e_{23}a &= a_3e_2 - a_2e_3 + a_1e_{123} \text{ (Figs. 8.9(a) and (b))} \end{aligned}$$

and

$$ae_{23} = -a_3e_2 + a_2e_3 + a_1e_{123}. \tag{8.133}$$

and

$$\begin{aligned} e_{31}a &= a_1e_{31}e_1 + a_2e_{31}e_2 + a_3e_{31}e_3 \\ &= a_1e_3 + a_2e_{123} - a_3e_1 \\ e_{31}a &= a_1e_3 - a_3e_1 + a_2e_{123} \text{ (Figs. 8.10(a) and (b))} \end{aligned}$$

and

$$ae_{31} = -a_1e_3 + a_3e_1 + a_2e_{123}. \tag{8.134}$$

These are interesting patterns, so let's see what happens when the multiplying bivector is not a unit bivector.

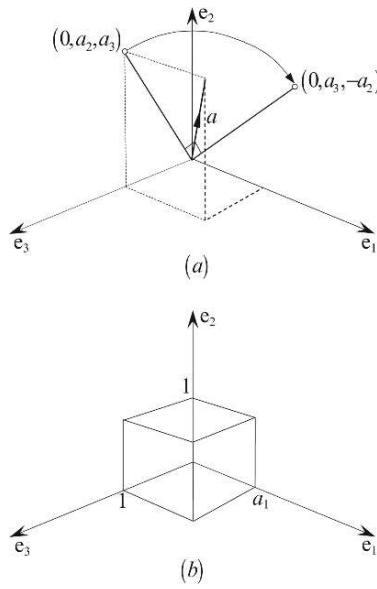


FIGURE 8.9.

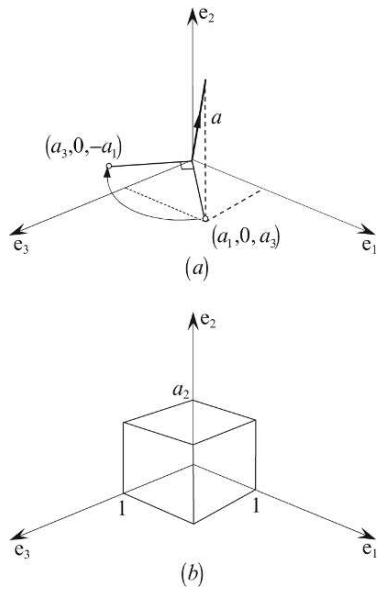


FIGURE 8.10.

### 8.14 The vector-bivector product in $\mathbb{R}^3$

Our bivector  $B$  is defined by the outer product of two vectors, whose precise values are not important, as any relevant combination will do. The vector  $a$  will have the form

$$a = a_1 e_1 + a_2 e_2 + a_3 e_3. \quad (8.135)$$

However, it is much more useful to express it in terms of two orthogonal components:

$$a = a_{\parallel} + a_{\perp} \quad (8.136)$$

where  $a_{\parallel}$  is parallel with  $B$  and  $a_{\perp}$  is perpendicular to  $B$ . This scenario is shown in Fig. 8.11.

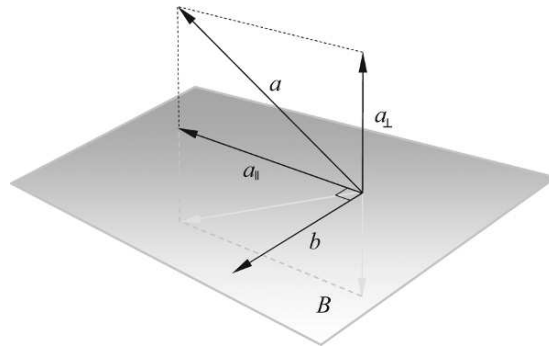


FIGURE 8.11.

Figure 8.11 shows an extra vector  $b$  which is orthogonal to  $a_{\parallel}$  and also lies in the plane  $B$ . The value of  $b$  is chosen such that

$$a_{\parallel} \wedge b = B. \quad (8.137)$$

Therefore, the geometric product  $a_{\parallel} b$  is

$$a_{\parallel} b = a_{\parallel} \cdot b + a_{\parallel} \wedge b = a_{\parallel} \wedge b = B \quad (8.138)$$

therefore,

$$B = a_{\parallel} b. \quad (8.139)$$

Now we explore the product of a vector and a bivector using  $a_{\parallel}$ ,  $a_{\perp}$  and  $B$ .

Starting with  $a_{\parallel} B$  we obtain

$$a_{\parallel} B = a_{\parallel} (a_{\parallel} b) = a_{\parallel}^2 b. \quad (8.140)$$

But

$$a_{\parallel}^2 = \|a_{\parallel}\|^2. \quad (8.141)$$

Therefore,

$$a_{\parallel} B = \|a_{\parallel}\|^2 b \quad (8.142)$$

which is a vector, and must lie in the plane  $B$ . In fact, Eq. (8.142) shows that when a vector and a bivector are coplanar, their product rotates the vector  $90^\circ$  and scales it.

Next, we investigate the product  $a_{\perp}B$ :

$$a_{\perp}B = a_{\perp}(a_{\parallel}b) = a_{\perp}a_{\parallel}b. \quad (8.143)$$

With reference to Fig. 8.11,  $a_{\perp}$ ,  $a_{\parallel}$  and  $b$  are three orthogonal vectors, and can be visualized as sweeping the bivector  $a_{\perp} \wedge a_{\parallel}$  along vector  $b$  creating a volume represented by the trivector  $a_{\perp}a_{\parallel}b$ .

Using Eqs. (8.142) and (8.143) we can express the product  $aB$  as

$$\begin{aligned} aB &= (a_{\parallel} + a_{\perp})B \\ &= a_{\parallel}B + a_{\perp}B \\ aB &= \|a_{\parallel}\|^2 b + a_{\perp}a_{\parallel}b \end{aligned} \quad (8.144)$$

which is the sum of a vector and a trivector. Although this will not always be the case, because vector  $a$  could be orthogonal to bivector  $B$ , which only creates a trivector, in general, we can predict that the product  $aB$  will contain two terms: a vector and a trivector.

We are now in a position to define the product  $aB$  in terms of the inner and outer products, where there is a temptation to assume that it obeys the same rule for the geometric product of two vectors:

$$aB = a \cdot B + a \wedge B, \quad (8.145)$$

which although is true, has to be proved.

We begin by declaring  $B$  as the outer product

$$B = b \wedge c. \quad (8.146)$$

Therefore,

$$aB = a(b \wedge c). \quad (8.147)$$

We now have to find a way of converting Eq. (8.147) into Eq. (8.145), which is achieved as follows:

Using the geometric product

$$b \wedge c = \frac{1}{2}(bc - cb) \quad (8.148)$$

then

$$a(b \wedge c) = a \frac{1}{2}(bc - cb) = \frac{1}{2}(abc - acb). \quad (8.149)$$

Similarly,

$$a \cdot b = \frac{1}{2}(ab + ba) \Rightarrow ab = 2a \cdot b - ba \quad (8.150)$$

and

$$a \cdot c = \frac{1}{2}(ac + ca) \Rightarrow ac = 2a \cdot c - ca. \quad (8.151)$$

Substituting Eqs. (8.150) and (8.151) in Eq. (8.149)

$$\begin{aligned}
 a(b \wedge c) &= \frac{1}{2}((2a \cdot b - ba)c - (2a \cdot c - ca)b) \\
 &= \frac{1}{2}(2(a \cdot b)c - bac - 2(a \cdot c)b + cab) \\
 &= (a \cdot b)c - (a \cdot c)b + \frac{1}{2}(cab - bac) \\
 &= (a \cdot b)c - (a \cdot c)b + \frac{1}{2}(abc + abc) \\
 a(b \wedge c) &= (a \cdot b)c - (a \cdot c)b + abc. \tag{8.152}
 \end{aligned}$$

Equation (8.152) shows that the product of a vector and a bivector ( $aB$ ) creates two components: a vector

$$(a \cdot b)c - (a \cdot c)b \tag{8.153}$$

and a trivector

$$abc. \tag{8.154}$$

Next, we will show that reversing the product ( $Ba$ ) creates a vector

$$-(a \cdot b)c + (a \cdot c)b \tag{8.155}$$

and a trivector

$$abc \tag{8.156}$$

where the vector is reversed, and the trivector remains unaltered.

What we want to do now is arrange that some combination of  $aB$  and  $Ba$  forms  $a \cdot B$  to create the vector component, and another combination forms  $a \wedge B$  to create the trivector component.

We can isolate each part using the following subterfuge:

Reversing the product  $a(b \wedge c)$  to  $(b \wedge c)a$

$$(b \wedge c)a = \frac{1}{2}(bc - cb)a = \frac{1}{2}(bca - cba). \tag{8.157}$$

Now

$$c \cdot a = \frac{1}{2}(ca + ac) \Rightarrow ca = 2a \cdot c - ac \tag{8.158}$$

and

$$b \cdot a = \frac{1}{2}(ba + ab) \Rightarrow ba = 2a \cdot b - ab. \tag{8.159}$$



Substituting Eqs. (8.158) and (8.159) in Eq. (8.157)

$$\begin{aligned}
 (b \wedge c)a &= \frac{1}{2}(b(2a \cdot c - ac) - c(2a \cdot b - ab)) \\
 &= \frac{1}{2}(2b(a \cdot c) - bac - 2c(a \cdot b) + cab) \\
 &= (a \cdot c)b - (a \cdot b)c + \frac{1}{2}(cab - bac) \\
 &= (a \cdot c)b - (a \cdot b)c + \frac{1}{2}(abc + abc) \\
 (b \wedge c)a &= (a \cdot c)b - (a \cdot b)c + abc.
 \end{aligned} \tag{8.160}$$

Subtracting Eq. (8.160) from Eq. (8.152)

$$a(b \wedge c) - (b \wedge c)a = 2(a \cdot b)c - 2(a \cdot c)b. \tag{8.161}$$

Therefore,

$$(a \cdot b)c - (a \cdot c)b = \frac{1}{2}(aB - Ba). \tag{8.162}$$

As  $\frac{1}{2}(aB - Ba)$  creates a vector, which is a lower grade object compared to a bivector, it is defined using the dot symbol as:

$$a \cdot B = \frac{1}{2}(aB - Ba) = (a \cdot b)c - (a \cdot c)b. \tag{8.163}$$

Adding Eq. (8.160) and (8.152) together, we obtain

$$a(b \wedge c) + (b \wedge c)a = 2abc. \tag{8.164}$$

Therefore,

$$\frac{1}{2}(aB + Ba) = abc. \tag{8.165}$$

As  $\frac{1}{2}(aB + Ba)$  creates a higher grade object compared to a bivector, it is defined using the outer symbol as:

$$a \wedge B = \frac{1}{2}(aB + Ba) = abc. \tag{8.166}$$

Combining Eqs. (8.163) and (8.166) we define the geometric product  $aB$  as

$$aB = a \cdot B + a \wedge B \tag{8.167}$$

$$aB = (a \cdot b)c - (a \cdot c)b + abc \tag{8.168}$$

where

$$B = b \wedge c. \tag{8.169}$$

Now let's derive formulae for the reverse product  $Ba$ .

Subtracting Eq. (8.168) from Eq. (8.160) we have

$$(b \wedge c)a - a(b \wedge c) = 2(a \cdot c)b - 2(a \cdot b)c \quad (8.170)$$

therefore,

$$\frac{1}{2}(Ba - aB) = (a \cdot c)b - (a \cdot b)c. \quad (8.171)$$

As this is a grade lowering operation, it is defined as an inner product:

$$B \cdot a = \frac{1}{2}(Ba - aB) = (a \cdot c)b - (a \cdot b)c. \quad (8.172)$$

Adding Eq. (8.168) to Eq. (8.160) we have

$$(b \wedge c)a + a(b \wedge c) = 2abc \quad (8.173)$$

therefore,

$$\frac{1}{2}(Ba + aB) = abc. \quad (8.174)$$

As this is a grade raising operation, it is defined as an outer product:

$$B \wedge a = \frac{1}{2}(Ba + aB) = abc. \quad (8.175)$$

Therefore,

$$B \cdot a + B \wedge a = \frac{1}{2}(Ba - aB) + \frac{1}{2}(Ba + aB) = Ba \quad (8.176)$$

and

$$Ba = B \cdot a + B \wedge a. \quad (8.177)$$

Let's bring these results to life with two simple examples.

We start with three vectors

$$a = 2e_1 + e_2 - e_3 \quad (8.178)$$

$$b = e_1 - e_2 + e_3 \quad (8.179)$$

$$c = 2e_1 + 2e_2 + e_3. \quad (8.180)$$

We now construct a bivector  $B$ :

$$\begin{aligned} B &= b \wedge c = (e_1 - e_2 + e_3) \wedge (2e_1 + 2e_2 + e_3) \\ &= 2e_{12} - e_{31} + 2e_{12} - e_{23} + 2e_{31} - 2e_{23} \\ B &= 4e_{12} - 3e_{23} + e_{31}. \end{aligned} \quad (8.181)$$

Therefore,

$$\begin{aligned} aB &= (2e_1 + e_2 - 2e_3)(4e_{12} - 3e_{23} + e_{31}) \\ &= 8e_2 - 6e_{123} - 2e_3 - 4e_1 - 3e_3 + e_{123} - 8e_{123} - 6e_2 - 2e_1 \\ aB &= -6e_1 + 2e_2 - 5e_3 - 13e_{123}. \end{aligned} \quad (8.182)$$

Similarly,

$$\begin{aligned}
 Ba &= (4e_{12} - 3e_{23} + e_{31})(2e_1 + e_2 - 2e_3) \\
 &= -8e_2 + 4e_1 - 8e_{123} - 6e_{123} + 3e_3 + 6e_2 + 2e_3 + e_{123} + 2e_1 \\
 Ba &= 6e_1 - 2e_2 + 5e_3 - 13e_{123}.
 \end{aligned} \tag{8.183}$$

Now let's calculate the inner and outer products.

The inner product:

$$\begin{aligned}
 a \cdot B &= \frac{1}{2}(aB - Ba) \\
 &= \frac{1}{2}(-6e_1 + 2e_2 - 5e_3 - 13e_{123} - 6e_1 + 2e_2 - 5e_3 + 13e_{123}) \\
 &= \frac{1}{2}(-12e_1 + 4e_2 - 10e_3) \\
 a \cdot B &= -6e_1 + 2e_2 - 5e_3.
 \end{aligned} \tag{8.184}$$

The outer product:

$$\begin{aligned}
 a \wedge B &= \frac{1}{2}(aB + Ba) \\
 &= \frac{1}{2}(-6e_1 + 2e_2 - 5e_3 - 13e_{123} + 6e_1 - 2e_2 + 5e_3 - 13e_{123}) \\
 a \wedge B &= -13e_{123}.
 \end{aligned} \tag{8.185}$$

Thus

$$\begin{aligned}
 aB &= a \cdot B + a \wedge B \\
 aB &= -6e_1 + 2e_2 - 5e_3 - 13e_{123}.
 \end{aligned} \tag{8.186}$$

It is clear from these examples how the inner and outer products identify the two parts of the geometric product.

In vector algebra the inner product is also known as the scalar product or dot product. Conversely, in geometric algebra we can create products between vectors, bivectors, trivectors, etc, and any combination such as a vector and a bivector, or a bivector and trivector. Because these objects have different grades, we require a new interpretation of the dot symbol, which also embraces the original definition. Thus the dot product in Eq. (8.167) means the “*lowest grade part of the product.*” Similarly, the outer product in Eq. (8.167) means the “*highest grade part of the product.*”

If you believe that you have seen the RHS of Eq. (8.163) before, you may recall from vector algebra that

$$(b \times c) \times a = (a \cdot b)c - (a \cdot c)b \tag{8.187}$$

and

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c. \tag{8.188}$$

Later in this chapter we show how GA can be used to derive these relationships.

### 8.15 Unit bivector-bivector products in $\mathbb{R}^3$

In  $\mathbb{R}^3$  we have to consider the possible products that exist between  $e_{12}$ ,  $e_{23}$  and  $e_{31}$ . We already know that

$$e_{12}^2 = e_{23}^2 = e_{31}^2 = -1 \quad (8.189)$$

and it is easy to show that

$$e_{12}e_{23} = e_{13} = -e_{31} \quad (8.190)$$

$$e_{23}e_{31} = e_{21} = -e_{12} \quad (8.191)$$

$$e_{31}e_{12} = e_{32} = -e_{23} \quad (8.192)$$

$$e_{12}e_{31} = e_{23} \quad (8.193)$$

$$e_{23}e_{12} = e_{31} \quad (8.194)$$

$$e_{31}e_{23} = e_{12} \quad (8.195)$$

Thus we see that unit bivectors anticommute.

These results are summarized in Table 8.4

TABLE 8.4

GP	$e_{12}$	$e_{23}$	$e_{31}$
$e_{12}$	-1	$-e_{31}$	$e_{23}$
$e_{23}$	$e_{31}$	-1	$-e_{12}$
$e_{31}$	$-e_{23}$	$e_{12}$	-1

Accordingly, when we encounter an expression such as  $\alpha e_{12}\beta e_{23}$  we can rewrite it as

$$\alpha\beta e_{12}e_{23} = -\alpha\beta e_{31} \quad (8.196)$$

### 8.16 Unit vector-trivector product in $\mathbb{R}^3$

To begin with, let's consider the products  $e_1e_{123}$ ,  $e_2e_{123}$  and  $e_3e_{123}$ :

$$e_1e_{123} = e_{23} \quad (8.197)$$

$$e_2e_{123} = e_{31} \quad (8.198)$$

$$e_3e_{123} = e_{12} \quad (8.199)$$

Similarly,

$$e_{123}e_1 = e_{23} \quad (8.200)$$

$$e_{123}e_2 = e_{31} \quad (8.201)$$

$$e_{123}e_3 = e_{12} \quad (8.202)$$

Thus we see that vectors and trivectors commute.

Let's illustrate this with a simple example  $a5e_{123}$  where

$$a = 2e_1 + 3e_2 + 4e_3. \quad (8.203)$$

Therefore

$$\begin{aligned} a5e_{123} &= (2e_1 + 3e_2 + 4e_3)5e_{123} \\ a5e_{123} &= 20e_{12} + 10e_{23} + 15e_{31}. \end{aligned} \quad (8.204)$$

The volumetric element has been reduced to three bivector terms.

### 8.17 Unit bivector-trivector product in $\mathbb{R}^3$

To begin, let's consider the products  $e_{12}e_{123}$ ,  $e_{23}e_{123}$  and  $e_{31}e_{123}$ :

$$e_{12}e_{123} = -e_3 \quad (8.205)$$

$$e_{23}e_{123} = -e_1 \quad (8.206)$$

$$e_{31}e_{123} = -e_2. \quad (8.207)$$

Similarly,

$$e_{123}e_{12} = -e_3 \quad (8.208)$$

$$e_{123}e_{23} = -e_1 \quad (8.209)$$

$$e_{123}e_{31} = -e_2. \quad (8.210)$$

Thus we see that bivectors and trivectors commute.

Again, let's illustrate this product with an example  $B5e_{123}$  where

$$B = 2e_{12} + 3e_{23} + 4e_{31}. \quad (8.211)$$

Therefore,

$$\begin{aligned} B5e_{123} &= (2e_{12} + 3e_{23} + 4e_{31})5e_{123} \\ B5e_{123} &= -15e_1 - 20e_2 - 10e_3. \end{aligned} \quad (8.212)$$

The volumetric element has been reduced to three vector terms.

### 8.18 Unit trivector-trivector product in $\mathbb{R}^3$

We have already discovered in section 8.10 that the square of the pseudoscalar equals  $-1$ .

### 8.19 Higher products in $\mathbb{R}^3$

Having considered the product of two trivectors in  $\mathbb{R}^3$ , it is worth exploring the concept of expanded outer products. For example, in  $\mathbb{R}^3$ , what is meant by

$$a \wedge b \wedge c \wedge d? \quad (8.213)$$

We can resolve this question by reasoning that if  $a, b, c$  are not coplanar, then  $d$  must be a linear combination of  $a, b, c$ :

$$d = \lambda_a a + \lambda_b b + \lambda_c c \quad [\lambda_a, \lambda_b, \lambda_c \in \mathbb{R}] \quad (8.214)$$

therefore,

$$a \wedge b \wedge c \wedge d = a \wedge b \wedge c \wedge (\lambda_a a + \lambda_b b + \lambda_c c)$$

and

$$a \wedge b \wedge c \wedge d = \lambda_a a \wedge b \wedge c \wedge a + \lambda_b a \wedge b \wedge c \wedge b + \lambda_c a \wedge b \wedge c \wedge c. \quad (8.215)$$

Recall that

$$a \wedge b = -b \wedge a \quad (8.216)$$

therefore,

$$a \wedge b \wedge c \wedge a = b \wedge a \wedge a \wedge c. \quad (8.217)$$

But

$$a \wedge a = 0 \quad (8.218)$$

therefore,

$$a \wedge b \wedge c \wedge a = 0. \quad (8.219)$$

Similarly,

$$a \wedge b \wedge c \wedge b = a \wedge b \wedge c \wedge c = 0 \quad (8.220)$$

therefore,

$$a \wedge b \wedge c \wedge d = 0. \quad (8.221)$$

### 8.20 Blades

Now that we covered bivectors and trivectors and tentatively explored higher dimensions, it appears that each space possesses a unique element created by the outer product. Starting with vectors, the outer product produces bivectors, trivectors, and even quadvectors, and there is no reason why higher  $n$ -vector elements cannot exist. Such a pattern was recognized by Hestenes who proposed the name *blade* for these elements [14]. Thus a blade is any multivector that can be formed as the outer product of a set of vectors. However, this definition has been widened by some authors to embrace scalars as 0-blades, vectors as 1-blades, bivectors as 2-blades, etc.

## 8.21 Duality transformation

An interesting relationship exists between blades and the pseudoscalar that is referred to as the *duality transformation*. Consider the following products involving the pseudoscalar  $I = e_1 \wedge e_2$  and 2D basis vectors:

$$Ie_1 = e_1e_2e_1 = -e_2 \quad (8.222)$$

$$Ie_2 = e_1e_2e_2 = e_1 \quad (8.223)$$

and

$$e_1I = e_1e_1e_2 = e_2 \quad (8.224)$$

$$e_2I = e_2e_1e_2 = -e_1. \quad (8.225)$$

Note that they anticommute. Now consider the following products involving the pseudoscalar  $I = e_1 \wedge e_2 \wedge e_3$  and the 3D basis vectors:

$$Ie_1 = e_1e_2e_3e_1 = e_2e_3 \quad (8.226)$$

$$Ie_2 = e_1e_2e_3e_2 = e_3e_1 \quad (8.227)$$

$$Ie_3 = e_1e_2e_3e_3 = e_1e_2 \quad (8.228)$$

and

$$e_1I = e_1e_1e_2e_3 = e_2e_3 \quad (8.229)$$

$$e_2I = e_2e_1e_2e_3 = e_3e_1 \quad (8.230)$$

$$e_3I = e_3e_1e_2e_3 = e_1e_2. \quad (8.231)$$

Note that they commute, and further analysis shows that spaces with an odd number of dimensions, the pseudoscalar commutes with all vectors and multivectors, whereas spaces with an even number of dimensions they anticommute. This relationship is summarized by the relationship:

$$I_n A_r = (-1)^{r(n-1)} A_r I_n. \quad (8.232)$$

Doran and Lasenby [15] show how this relationship can be used to relate the inner and outer products:

$$\begin{aligned} a \cdot (A_r I) &= \frac{1}{2}(aA_r I - (-1)^{n-r} A_r I a) \\ &= \frac{1}{2}(aA_r I - (-1)^{n-r} (-1)^{n-1} A_r a I) \\ &= \frac{1}{2}(aA_r + (-1)^r A_r a) I \\ a \cdot (A_r I) &= a \wedge A_r I. \end{aligned} \quad (8.233)$$

The product  $aI$  is a grade lowering operation as a volumetric element is reduced into bivector elements, and consequently is denoted using the dot product:

$$aI = a \cdot I. \tag{8.234}$$

Figure 8.12 shows the duality relationship between bivectors and vectors in 3D space.

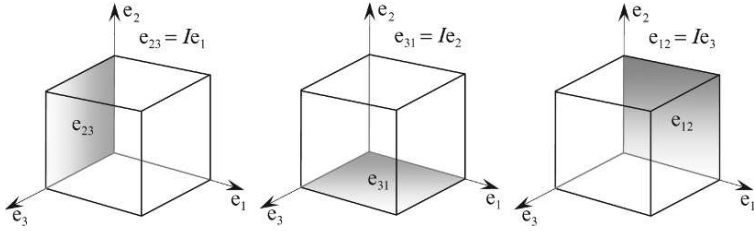


FIGURE 8.12.

A convenient notation used to represent the dual of  $A$  is  $A^*$ .

## 8.22 Summary of products in $\mathbb{R}^3$

We are now in a position to summarize the above products in tabular form as shown in Table 8.5.

TABLE 8.5

Inner product

Vectors commute	$a \cdot b = b \cdot a$
Vectors and bivectors anticommute	$a \cdot B = -B \cdot a$
	$a \cdot B = \frac{1}{2}(aB - Ba)$
	$a \cdot B = (a \cdot b)c - (a \cdot c)b$
	$B \cdot a = \frac{1}{2}(Ba - aB)$
	$B \cdot a = (a \cdot c)b - (a \cdot b)c$

Outer product

Vectors anticommute	$a \wedge b = -b \wedge a$
Vectors and bivectors commute	$a \wedge B = B \wedge a$
	$a \wedge B = \frac{1}{2}(aB + Ba)$
	$a \wedge B = abc$
	$B \wedge a = \frac{1}{2}(Ba + aB)$
	$B \wedge a = abc$



TABLE 8.5 (continued)

Geometric product	
Orthogonal vectors anticommute	$e_{12} = -e_{21}$
Orthogonal bivectors anticommute	$e_{12}e_{23} = -e_{23}e_{12}$
Bivectors square to $-1$	$e_{12}^2 = e_{23}^2 = e_{31}^2 = -1$
Definition	$ab = a \cdot b + a \wedge b$
Vectors and bivectors anticommute	$aB = -Ba$
	$aB = a \cdot B + a \wedge B$
	$aB = (a \cdot b)c - (a \cdot c)b + abc$
	$Ba = B \cdot a + B \wedge a$
	$Ba = (a \cdot c)b - (a \cdot b)c + abc$
Trivector commutes with all multivectors in the space	$aT = Ta \quad BT = TB$
The pseudoscalar	$e_{123} = I$
Vectors and the pseudoscalar commute	$aI = Ia$
	$aI = a \cdot I$
Duality transformation	$e_{23} = Ie_1$
	$e_{31} = Ie_2$
	$e_{12} = Ie_3$
The trivector squares to $-1$	$I^2 = -1$

Where  $a$  and  $b$  are vectors,  $B$  is a bivector, and  $T$  is a trivector.

Table 8.6 summarizes the commutative rules that exist between vectors, bivectors and trivectors when using the inner, outer and geometric products. The fact that every product is resolved in terms of the table's indices means that the product of two multivectors forms a closed algebra.

TABLE 8.6

GP	$\lambda$	$e_1$	$e_2$	$e_3$	$e_{12}$	$e_{23}$	$e_{31}$	$e_{123}$
$\lambda$	$\lambda^2$	$\lambda e_1$	$\lambda e_2$	$\lambda e_3$	$\lambda e_{12}$	$\lambda e_{23}$	$\lambda e_{31}$	$\lambda e_{123}$
$e_1$	$\lambda e_1$	1	$e_{12}$	$-e_{31}$	$e_2$	$e_{123}$	$-e_3$	$e_{23}$
$e_2$	$\lambda e_2$	$-e_{12}$	1	$e_{23}$	$-e_1$	$e_3$	$e_{123}$	$e_{31}$
$e_3$	$\lambda e_3$	$e_{31}$	$-e_{23}$	1	$e_{123}$	$-e_2$	$e_1$	$e_{12}$
$e_{12}$	$\lambda e_{12}$	$-e_2$	$e_1$	$e_{123}$	$-1$	$-e_{31}$	$e_{23}$	$-e_3$
$e_{23}$	$\lambda e_{23}$	$e_{123}$	$-e_3$	$e_2$	$e_{31}$	$-1$	$-e_{12}$	$-e_1$
$e_{31}$	$\lambda e_{31}$	$e_3$	$e_{123}$	$-e_1$	$-e_{23}$	$e_{12}$	$-1$	$-e_2$
$e_{123}$	$\lambda e_{123}$	$e_{23}$	$e_{31}$	$e_{12}$	$-e_3$	$-e_1$	$-e_2$	$-1$

### 8.23 Multivectors in $\mathbb{R}^3$

In section 8.5 we defined a multivector in  $\mathbb{R}^2$  as a linear combination of scalars, vectors and bivectors. We now extend this definition to include trivectors. Table 8.7 summarizes the elements and confirms that we have 1 scalar, 3 vectors, 3 bivectors and 1 trivector.

TABLE 8.7

Element	Symbol	Grade
1 scalar	$\lambda$	0
3 vectors	$\{e_1, e_2, e_3\}$	1
3 bivectors	$e_1 \wedge e_2 = e_{12}$ $e_2 \wedge e_3 = e_{23}$ $e_3 \wedge e_1 = e_{31}$	2
1 trivector	$e_{123}$	3

For completeness, let's form the product of two multivectors to demonstrate that we have a closed algebra.

We begin by defining two multivectors  $A$  and  $B$ :

$$A = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_{12} + \lambda_4 e_{23} + \lambda_5 e_{31} + \lambda_6 e_{123} \quad [\lambda_i \in \mathbb{R}] \quad (8.235)$$

$$B = \beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123} \quad [\beta_i \in \mathbb{R}] \quad (8.236)$$

Therefore,

$$\begin{aligned} AB = & \lambda_0(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_1 e_1(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_2 e_2(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_3 e_{12}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_4 e_{23}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_5 e_{31}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \\ & + \lambda_6 e_{123}(\beta_0 + \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_{12} + \beta_4 e_{23} + \beta_5 e_{31} + \beta_6 e_{123}) \end{aligned} \quad (8.237)$$

expanding

$$\begin{aligned} AB = & \lambda_0 \beta_0 + \lambda_0 \beta_1 e_1 + \lambda_0 \beta_2 e_2 + \lambda_0 \beta_3 e_{12} + \lambda_0 \beta_4 e_{23} + \lambda_0 \beta_5 e_{31} + \lambda_0 \beta_6 e_{123} + \lambda_1 \beta_0 e_1 + \lambda_1 \beta_1 \\ & + \lambda_1 \beta_2 e_{12} + \lambda_1 \beta_3 e_2 + \lambda_1 \beta_4 e_{123} - \lambda_1 \beta_5 e_3 + \lambda_1 \beta_6 e_{23} + \lambda_2 \beta_0 e_2 - \lambda_2 \beta_1 e_{12} + \lambda_2 \beta_2 \\ & - \lambda_2 \beta_3 e_1 + \lambda_2 \beta_4 e_3 + \lambda_2 \beta_5 e_{123} + \lambda_2 \beta_6 e_{31} + \lambda_3 \beta_0 e_{12} - \lambda_3 \beta_1 e_2 + \lambda_3 \beta_2 e_1 - \lambda_3 \beta_3 \\ & - \lambda_3 \beta_4 e_{31} + \lambda_3 \beta_5 e_{23} - \lambda_3 \beta_6 e_3 + \lambda_4 \beta_0 e_{23} + \lambda_4 \beta_1 e_{123} - \lambda_4 \beta_2 e_3 + \lambda_4 \beta_3 e_{31} - \lambda_4 \beta_4 \\ & - \lambda_4 \beta_5 e_{12} - \lambda_4 \beta_6 e_1 + \lambda_5 \beta_0 e_{31} + \lambda_5 \beta_1 e_3 + \lambda_5 \beta_2 e_{123} - \lambda_5 \beta_3 e_{23} + \lambda_5 \beta_4 e_{12} - \lambda_5 \beta_5 \\ & - \lambda_5 \beta_6 e_2 + \lambda_6 \beta_0 e_{123} + \lambda_6 \beta_1 e_{23} + \lambda_6 \beta_2 e_{31} - \lambda_6 \beta_3 e_3 - \lambda_6 \beta_4 e_1 - \lambda_6 \beta_5 e_2 - \lambda_6 \beta_6 \end{aligned} \quad (8.238)$$

simplifying and collecting up like terms

$$\begin{aligned} AB = & \lambda_0 \beta_0 + \lambda_1 \beta_1 + \lambda_2 \beta_2 - \lambda_3 \beta_3 - \lambda_4 \beta_4 - \lambda_5 \beta_5 - \lambda_6 \beta_6 \\ & + (\lambda_0 \beta_1 + \lambda_1 \beta_0 - \lambda_2 \beta_3 + \lambda_3 \beta_2 - \lambda_4 \beta_6 - \lambda_6 \beta_4) e_1 \\ & + (\lambda_0 \beta_2 + \lambda_1 \beta_3 + \lambda_2 \beta_0 - \lambda_3 \beta_1 - \lambda_5 \beta_6 - \lambda_6 \beta_5) e_2 \end{aligned}$$

$$\begin{aligned}
 &+ (-\lambda_1\beta_5 + \lambda_2\beta_4 - \lambda_3\beta_6 - \lambda_4\beta_2 + \lambda_5\beta_1 - \lambda_6\beta_3)e_3 \\
 &+ (\lambda_0\beta_3 + \lambda_1\beta_2 - \lambda_2\beta_1 + \lambda_3\beta_0 - \lambda_4\beta_5 + \lambda_5\beta_4)e_{12} \\
 &+ (\lambda_0\beta_4 + \lambda_1\beta_6 + \lambda_3\beta_5 + \lambda_4\beta_0 - \lambda_5\beta_3 + \lambda_6\beta_1)e_{23} \\
 &+ (\lambda_0\beta_5 + \lambda_2\beta_6 - \lambda_3\beta_4 + \lambda_4\beta_3 + \lambda_5\beta_0 + \lambda_6\beta_2)e_{31} \\
 &+ (\lambda_0\beta_6 + \lambda_1\beta_4 + \lambda_2\beta_5 + \lambda_4\beta_1 + \lambda_5\beta_2 + \lambda_6\beta_0)e_{123}
 \end{aligned} \tag{8.239}$$

which is another multivector and forms a closed algebra.

A multivector that contains terms of only a single grade is said to be *homogeneous*.

You may have noticed an obvious pattern associated with the number of elements in each multivector. Table 8.8 summarizes the number of elements for  $\mathbb{R}^2$ ,  $\mathbb{R}^3$  and  $\mathbb{R}^4$  where it is obvious that Pascal’s numbers are in control.

TABLE 8.8

	Scalar	Vector	Bivector	Trivector	Quadvector
$\mathbb{R}^2$	1	2	1		
$\mathbb{R}^3$	1	3	3	1	
$\mathbb{R}^4$	1	4	6	4	1

As a multivector contains elements with a variety of grades, it is useful to isolate each grade using the following notation:  $\langle A \rangle_n$ , where  $n$  is the required grade. For example, given

$$A = 3 + 2e_1 + e_2 - 3e_3 + 5e_1 \wedge 2e_2 + 7e_{123} \tag{8.240}$$

then

$$\langle A \rangle_0 = 3 \tag{8.241}$$

$$\langle A \rangle_1 = 2e_1 + e_2 - 3e_3 \tag{8.242}$$

$$\langle A \rangle_2 = 5e_1 \wedge 2e_2 \tag{8.243}$$

$$\langle A \rangle_3 = 7e_{123}. \tag{8.244}$$

In the case of the geometric product

$$ab = a \cdot b + a \wedge b \tag{8.245}$$

$$\langle ab \rangle_0 = a \cdot b \tag{8.246}$$

$$\langle ab \rangle_2 = a \wedge b. \tag{8.247}$$

## 8.24 Relationship between vector algebra and geometric algebra

We are now in a position to compare vector algebra with geometric algebra, especially with the way vectors relate to complex numbers, and how rotations in the plane are effected.

Table 8.9 summarizes the two algebras beginning with a vector, the mapping from a vector into a complex number, and the technique for rotating.

TABLE 8.9

Vector Algebra		Geometric Algebra	
vector	$v = a_1i + a_2j$	vector	$v = a_1e_1 + a_2e_2$
map	$a = a_1 \quad b = a_2$	map	$Z = e_1v$
complex number	$z = a + bi$	multivector	$Z = a_1 + a_2I$
rotor	$z' = ze^{i\phi}$	rotor	$Z' = Ze^{I\phi}$ $v' = ve^{I\phi}$
90° rotor	$v' = -a_2i + a_1j$	90° rotor	$v' = vI$

**8.25 Relationship between the outer product and the cross product**

In chapter 7 we discovered the close similarity between the outer product and the cross product and saw that the bivector coefficients for the outer product are identical to the coefficients for the axial vector resulting from the cross product. We are now in a position to discover the algebraic relationship between the two products.

Starting with two vectors  $a$  and  $b$ , their cross and outer products are

$$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = (a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3 \tag{8.248}$$

and

$$a \wedge b = (a_2b_3 - a_3b_2)e_{23} + (a_3b_1 - a_1b_3)e_{31} + (a_1b_2 - a_2b_1)e_{12}. \tag{8.249}$$

Let's see what happens when we pre-multiply Eq. (8.249) by the pseudoscalar  $e_{123}$ :

$$e_{123}(a \wedge b) = (a_2b_3 - a_3b_2)e_{123}e_{23} + (a_3b_1 - a_1b_3)e_{123}e_{31} + (a_1b_2 - a_2b_1)e_{123}e_{12}$$

$$e_{123}(a \wedge b) = -(a_2b_3 - a_3b_2)e_1 - (a_3b_1 - a_1b_3)e_2 - (a_1b_2 - a_2b_1)e_3. \tag{8.250}$$

This has almost created the cross product in Eq. (8.248) apart from the inverted sign. So let's multiply Eq. (8.250) by  $-1$ :

$$-e_{123}(a \wedge b) = (a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3 \tag{8.251}$$

which creates the cross product in Eq. (8.248). Thus we can state that

$$a \times b = -e_{123}(a \wedge b) \text{ or } -I(a \wedge b). \tag{8.252}$$

Therefore, given a bivector  $B$ , the vector  $v$  orthogonal to the planar surface is given by

$$v = -IB \tag{8.253}$$

which is determined algebraically.

For example, Fig. 8.13 shows two vectors  $a$  and  $b$  where

$$a = -e_2 + e_3 \tag{8.254}$$

$$b = e_1 - e_2. \tag{8.255}$$

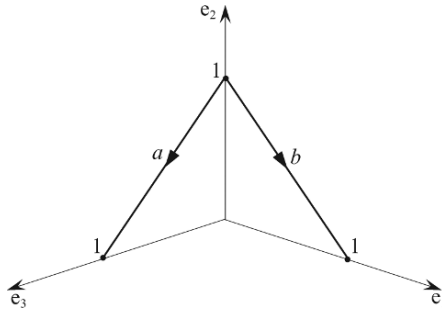


FIGURE 8.13.

The cross product is given by

$$\begin{aligned} a \times b = c &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} \\ &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \end{aligned} \quad (8.256)$$

which is expected from the symmetry of the vectors.

Now let's compute  $c$  using Eq. (8.253):

$$\begin{aligned} B &= a \wedge b \\ &= (-\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 - \mathbf{e}_2) \\ &= -\mathbf{e}_2 \wedge \mathbf{e}_1 + \mathbf{e}_2 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_1 - \mathbf{e}_3 \wedge \mathbf{e}_2 \\ B &= \mathbf{e}_{12} + \mathbf{e}_{31} + \mathbf{e}_{23}. \end{aligned} \quad (8.257)$$

Therefore,

$$\begin{aligned} c &= -IB \\ &= -\mathbf{e}_{123}(\mathbf{e}_{12} + \mathbf{e}_{31} + \mathbf{e}_{23}) \\ &= \mathbf{e}_3 + \mathbf{e}_2 + \mathbf{e}_1 \\ c &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \end{aligned} \quad (8.258)$$

which is identical to the previous result.

Now that we have a mechanism to move between GA and the cross product, we can prove various identities in vector analysis using GA. For example, let's expand the vector triple product  $(a \times b) \times c$ .

Starting with  $a \times b$ :

$$\begin{aligned}
 a \times b &= -I(a \wedge b) \\
 &= -e_{123}(a_1e_1 + a_2e_2 + a_3e_3) \wedge (b_1e_1 + b_2e_2 + b_3e_3) \\
 &= -e_{123}((a_2b_3 - a_3b_2)e_{23} + (a_3b_1 - a_1b_3)e_{31} + (a_1b_2 - a_2b_1)e_{12}) \\
 a \times b &= (a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3
 \end{aligned} \tag{8.259}$$

substitute  $\lambda_i$  for each coefficient

$$a \times b = \lambda_1e_1 + \lambda_2e_2 + \lambda_3e_3. \tag{8.260}$$

Expand  $(a \times b) \times c$ :

$$\begin{aligned}
 (a \times b) \times c &= -I(\lambda_1e_1 + \lambda_2e_2 + \lambda_3e_3) \wedge (c_1e_1 + c_2e_2 + c_3e_3) \\
 &= -e_{123}((\lambda_2c_3 - \lambda_3c_2)e_{23} + (\lambda_3c_1 - \lambda_1c_3)e_{31} + (\lambda_1c_2 - \lambda_2c_1)e_{12}) \\
 (a \times b) \times c &= (\lambda_2c_3 - \lambda_3c_2)e_1 + (\lambda_3c_1 - \lambda_1c_3)e_2 + (\lambda_1c_2 - \lambda_2c_1)e_3.
 \end{aligned}$$

Re-substitute for each  $\lambda_i$

$$\begin{aligned}
 (a \times b) \times c &= ((a_3b_1 - a_1b_3)c_3 - (a_1b_2 - a_2b_1)c_2)e_1 \\
 &\quad + ((a_1b_2 - a_2b_1)c_1 - (a_2b_3 - a_3b_2)c_3)e_2 \\
 &\quad + ((a_2b_3 - a_3b_2)c_2 - (a_3b_1 - a_1b_3)c_1)e_3.
 \end{aligned} \tag{8.261}$$

Rearrange the order

$$\begin{aligned}
 (a \times b) \times c &= (a_2c_2 + a_3c_3)b_1e_1 + (a_1c_1 + a_3c_3)b_2e_2 + (a_1c_1 + a_2c_2)b_3e_3 \\
 &\quad - ((b_2c_2 + b_3c_3)a_1e_1 + (b_1c_1 + b_3c_3)a_2e_2 + (b_1c_1 + b_2c_2)a_3e_3)
 \end{aligned} \tag{8.262}$$

Now we add the following zero term to complete the inner products:

$$(a_1c_1)b_1e_1 + (a_2c_2)b_2e_2 + (a_3c_3)b_3e_3 - (b_1c_1)a_1e_1 - (b_2c_2)a_2e_2 - (b_3c_3)a_3e_3 = 0 \tag{8.263}$$

$$\begin{aligned}
 (a \times b) \times c &= (a_1c_1 + a_2c_2 + a_3c_3)b_1e_1 + (a_1c_1 + a_2c_2 + a_3c_3)b_2e_2 + (a_1c_1 + a_2c_2 + a_3c_3)b_3e_3 \\
 &\quad - ((b_1c_1 + b_2c_2 + b_3c_3)a_1e_1 + (b_1c_1 + b_2c_2 + b_3c_3)a_2e_2 + (b_1c_1 + b_2c_2 + b_3c_3)a_3e_3)
 \end{aligned} \tag{8.264}$$

therefore,

$$(a \times b) \times c = (a \cdot c)b - (b \cdot c)a. \tag{8.265}$$

## 8.26 Relationship between geometric algebra and quaternions

In chapter 6 we reviewed the ideas behind quaternions and saw that a quaternion is defined as the sum of a scalar and a vector, where Hamilton's imaginaries  $i, j$  and  $k$  obey the product rules shown in Table 8.10.

TABLE 8.10

	<i>i</i>	<i>j</i>	<i>k</i>
<i>i</i>	-1	<i>k</i>	- <i>j</i>
<i>j</i>	- <i>k</i>	-1	<i>i</i>
<i>k</i>	<i>j</i>	- <i>i</i>	-1

and  $ijk = -1$ .

If we let

$$B_1 = e_2 \wedge e_3 \tag{8.266}$$

$$B_2 = e_3 \wedge e_1 \tag{8.267}$$

$$B_3 = e_1 \wedge e_2 \tag{8.268}$$

the bivector products obey the rules shown in Table 8.11.

TABLE 8.11

	$B_1$	$B_2$	$B_3$
$B_1$	-1	$-B_3$	$B_2$
$B_2$	$B_3$	-1	$-B_1$
$B_3$	$-B_2$	$B_1$	-1

and

$$B_1 B_2 B_3 = +1. \tag{8.269}$$

The subtle difference between Table 8.10 and Table 8.11 is that, apart from the diagonal, the signs are reversed, which suggests that there is a difference in the handedness of the axial systems. To confirm this, Fig. 8.14 shows a left-handed set of bivectors, which obey the rules shown in Table 8.12.

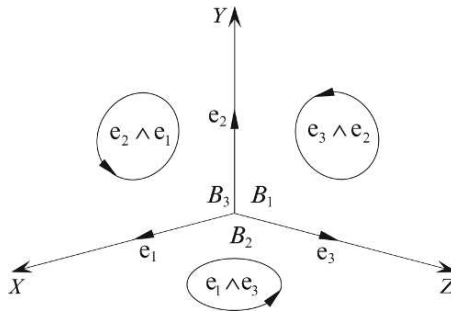


FIGURE 8.14.

TABLE 8.12

	$B_1$	$B_2$	$B_3$
$B_1$	-1	$B_3$	$-B_2$
$B_2$	$-B_3$	-1	$B_1$
$B_3$	$B_2$	$-B_1$	-1

We can see from Tables 8.10 and 8.12 the intimate relationship between Hamilton's imaginaries and a left-handed set of bivectors, which is elegantly described by Chris Doran and Anthony Lasenby in their book *Geometric Algebra for Physicists* [14]. What is strange, is that even the greatest mathematicians can misinterpret their discoveries, and what is so ironic is that Grassmann's algebra embraced vectors, bivectors and quaternions and would have changed the path of mathematics had it been adopted at the time.

## 8.27 Inverse of a vector

Associative algebras such as the algebra of real numbers and complex numbers permit division. For example, if

$$\alpha\beta = \delta \quad (8.270)$$

then

$$\alpha = \delta\beta^{-1}. \quad (8.271)$$

Similarly, given these complex numbers

$$(a + ib)(c + id) = e + if \quad (8.272)$$

then we can state that

$$(a + ib) = (e + if)(c + id)^{-1}. \quad (8.273)$$

And as geometric algebra is associative, we can divide by vectors. For example, given that a multivector  $B = ab$ , then we can multiply throughout by  $b$  and state that

$$Bb = (ab)b = ab^2 \quad (8.274)$$

which means that

$$B \frac{b}{b^2} = a \quad (8.275)$$

or

$$Bb^{-1} = a \quad (8.276)$$

where

$$b^{-1} = \frac{b}{b^2} = \frac{b}{\|b\|^2}. \quad (8.277)$$

We can illustrate this with an example.



Two vectors  $a$  and  $b$  are given by

$$a = 3e_1 + 4e_2 \quad (8.278)$$

$$b = e_1 + e_2 \quad (8.279)$$

and a multivector  $B$  is given by

$$\begin{aligned} B &= ab \\ &= (3e_1 + 4e_2)(e_1 + e_2) \\ &= 3 + 3e_{12} - 4e_{12} + 4 \\ B &= 7 - e_{12}. \end{aligned} \quad (8.280)$$

Now let's compute  $b^{-1}$

$$\begin{aligned} b^{-1} &= \frac{b}{b^2} \\ &= \frac{e_1 + e_2}{\|\sqrt{e_1^2 + e_2^2}\|^2} \\ b^{-1} &= \frac{1}{2}(e_1 + e_2). \end{aligned} \quad (8.281)$$

Therefore, we can recover  $a$  from  $B$  as follows:

$$\begin{aligned} a &= Bb^{-1} \\ &= \frac{1}{2}(7 - e_{12})(e_1 + e_2) \\ &= \frac{1}{2}(7e_1 + 7e_2 - e_{12}e_1 - e_{12}e_2) \\ &= \frac{1}{2}(7e_1 + 7e_2 + e_2 - e_1) \\ a &= 3e_1 + 4e_2 \end{aligned} \quad (8.282)$$

which is correct.

Next, we consider a multivector  $C = abc$ , where

$$a = 3e_1 + 4e_2 \quad (8.283)$$

$$b = e_1 + e_2 \quad (8.284)$$

$$c = e_3. \quad (8.285)$$

Therefore,

$$\begin{aligned} C &= (3e_1 + 4e_2)(e_1 + e_2)e_3 \\ &= (3e_1^2 + 3e_{12} - 4e_{12} + 4e_2^2)e_3 \\ &= (7 - e_{12})e_3 \\ C &= 7e_3 - e_{123}. \end{aligned} \quad (8.286)$$

Now, if we are given  $c$ , we can find the bivector term as follows:

$$\begin{aligned}
 ab &= Cc^{-1} \\
 &= \frac{Cc}{\|a\|^2} \\
 &= (7e_3 - e_{123})e_3 \\
 ab &= 7 - e_{12}
 \end{aligned} \tag{8.287}$$

which is correct.

### 8.28 The meet operation

For all sorts of reasons we are always interested in the intersections of lines, planes, spheres, cylinders, etc., and GA's *meet* operation provides a way of calculating such intersections. For example, the meet of  $A$  and  $B$  is written  $A \vee B$ , and without proof, is defined as

$$A \vee B = A^* \cdot B. \tag{8.288}$$

To illustrate how this operation works we first examine the intersections of the three basis 2-blades, followed by the intersection of two arbitrary blades.

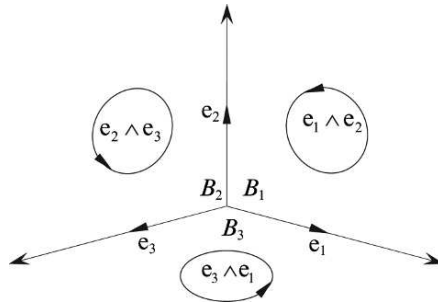


FIGURE 8.15.

Figure 8.15 shows the three basis 2-blades  $B_1, B_2, B_3$ , and it is obvious that

$$B_1 \vee B_2 = e_2 \quad B_2 \vee B_3 = e_3 \quad B_3 \vee B_1 = e_1. \tag{8.289}$$

Now let's demonstrate how the meet operation confirms this result.

$$B_1 = e_1 \wedge e_2 \quad B_2 = e_2 \wedge e_3 \quad B_3 = e_3 \wedge e_1. \tag{8.290}$$

Therefore,

$$\begin{aligned}
 B_1 \vee B_2 &= B_1^* \cdot B_2 \\
 &= (e_{123}e_{12}) \cdot e_{23} \\
 B_1 \vee B_2 &= -e_3 \cdot e_{23}.
 \end{aligned} \tag{8.291}$$

Using the identity  $a \cdot B = \frac{1}{2}(aB - Ba)$

$$\begin{aligned}
 B_1 \vee B_2 &= \frac{1}{2}(-e_{323} + e_{233}) \\
 B_1 \vee B_2 &= e_2.
 \end{aligned}
 \tag{8.292}$$

Similarly,

$$\begin{aligned}
 B_2 \vee B_3 &= B_2^* \cdot B_3 \\
 &= (e_{123}e_{23}) \cdot e_{31} \\
 &= -e_1 \cdot e_{31} \\
 &= \frac{1}{2}(-e_{131} + e_{311}) \\
 B_2 \vee B_3 &= e_3
 \end{aligned}
 \tag{8.293}$$

and

$$\begin{aligned}
 B_3 \vee B_1 &= B_3^* \cdot B_1 \\
 &= (e_{123}e_{31}) \cdot e_{12} \\
 &= -e_2 \cdot e_{12} \\
 &= \frac{1}{2}(-e_{212} + e_{122}) \\
 B_3 \vee B_1 &= e_1.
 \end{aligned}
 \tag{8.294}$$

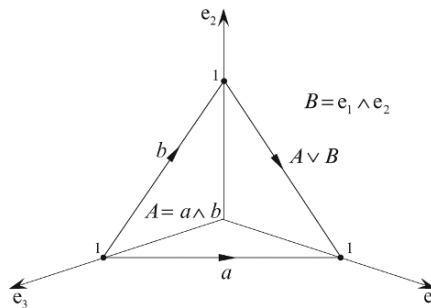


FIGURE 8.16.

The next example is shown in Fig. 8.16 where one of the planes is away from the origin. The meet of the two blades  $A$  and  $B$  is a line passing through the two points  $(1, 0, 0)$  and  $(0, 1, 0)$ , whose direction vector is given by  $\pm(e_1 - e_2)$ . Let's compute the product  $A^* \cdot B$  to confirm this prediction.

Given

$$a = e_1 - e_3 \tag{8.295}$$

$$b = e_2 - e_3 \tag{8.296}$$

$$\begin{aligned}
 A &= a \wedge b \\
 &= (e_1 - e_3) \wedge (e_2 - e_3) \\
 A &= e_{12} - e_{13} - e_{32}
 \end{aligned} \tag{8.297}$$

$$B = e_{12} \tag{8.298}$$

then

$$\begin{aligned}
 A \vee B &= A^* \cdot B \\
 &= e_{123}(e_{12} - e_{13} - e_{32}) \cdot e_{12} \\
 A \vee B &= (-e_3 - e_2 - e_1) \cdot e_{12}.
 \end{aligned} \tag{8.299}$$

Expand using  $a \cdot B = \langle aB \rangle_1$

$$\begin{aligned}
 A \vee B &= \langle (-e_3 - e_2 - e_1)e_{12} \rangle_1 \\
 A \vee B &= e_1 - e_2
 \end{aligned} \tag{8.300}$$

which is correct.

We explore other applications of the meet operation in the following chapters.

## 8.29 Summary

This chapter has covered a large number of topics, which, if understood completely, can be summarized as follows:

Geometric algebra provides a coordinate free, algebraic framework for describing geometry in any number of dimensions. At the heart of the algebra is an associative, geometric product which has real and imaginary parts and is defined as the sum of the inner and outer products. It is also invertible. The inner product is the familiar inner product  $a \cdot b$  whereas the outer product is defined as the outer product  $a \wedge b$ . Thus the geometric product of two vectors is defined as

$$ab = a \cdot b + a \wedge b. \tag{8.301}$$

The outer product defines a directed area, which, unlike the cross product, exists in space of any number of dimensions. However, like the cross product, it is antisymmetric:

$$a \wedge b = -b \wedge a. \tag{8.302}$$

The outer product creates a new entity called a bivector, which is a directed area defined by a pair of vectors. In  $\mathbb{R}^2$  there is only one unit bivector:  $e_1 \wedge e_2 = e_{12}$ , whereas in  $\mathbb{R}^3$  there are three:  $e_1 \wedge e_2$ ,  $e_2 \wedge e_3$  and  $e_3 \wedge e_1$ . Thus, the outer product of two vectors in  $\mathbb{R}^2$  is represented as

$$a \wedge b = \lambda_1(e_1 \wedge e_2) \quad \{\lambda_1 \in \mathbb{R}\} \tag{8.303}$$

and in  $\mathbb{R}^3$  it is represented as

$$a \wedge b = \lambda_1(e_1 \wedge e_2) + \lambda_2(e_2 \wedge e_3) + \lambda_3(e_3 \wedge e_1) \quad \{\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}. \tag{8.304}$$

In  $\mathbb{R}^3$  the outer product of three vectors  $a \wedge b \wedge c$  (i.e. a trivector) represents a directed volume.

It is possible to linearly combine scalars, vectors, bivectors and trivectors to create multivectors that form a closed algebra. These elements possess a natural hierarchy which is described in terms of their grade where scalars are grade 0, vectors are grade 1, bivectors are grade 2, and trivectors are grade 3. The inner product has grade reducing qualities as it transforms two vectors into a scalar, whereas the outer product has grade raising qualities as it transforms two vectors into a bivector.

The inner and outer products can be defined in terms of the geometric product using

$$a \cdot b = \frac{1}{2}(ab + ba) \quad (8.305)$$

and

$$a \wedge b = \frac{1}{2}(ab - ba). \quad (8.306)$$

The axioms defining the algebra are

$$a(bc) = (ab)c \quad (8.307)$$

$$a(b + c) = ab + ac \quad (8.308)$$

$$(b + c)a = ba + ca \quad (8.309)$$

$$\lambda a = a\lambda \quad (8.310)$$

$$a^2 = \pm \|a\|^2. \quad (8.311)$$

An unusual feature of geometric algebra is that the highest graded element for any space (bivector for  $\mathbb{R}^2$ , trivector for  $\mathbb{R}^3$ ) squares to  $-1$ , which introduces imaginary features to multivectors. These elements are called pseudoscalars.

Multivectors can be added, subtracted, multiplied together and even divided by a vector. When adding or subtracting multivectors, like elements are combined individually. However, the product of two multivectors is computed using the rules summarized in Table 8.5.

The number of elements belonging to a multivector is determined by the number of combinations of  $n$  elements selected  $p$  at a time  ${}_n C_p$ . For example, in  $\mathbb{R}^2$  we have 1 scalar, 2 unit basis vectors and 1 unit bivector. Whereas in  $\mathbb{R}^3$ , we have 1 scalar, three unit basis vectors, three unit bivectors and 1 unit trivector. In  $\mathbb{R}^4$  we have 1 scalar, 4 unit basis vectors, 6 unit bivectors, 4 unit trivectors and 1 unit quadvector.

In  $\mathbb{R}^2$  the product of a unit bivector (pseudoscalar)  $I$  and a vector rotate the vector  $90^\circ$ . For example

$$e_1 I = e_2 \quad (8.312)$$

whereas

$$I e_1 = -e_2. \quad (8.313)$$

In  $\mathbb{R}^3$  premultiplying a vector by a bivector performs two operations:

- first, it rotates the projection of the vector on the bivector clockwise  $90^\circ$
- second, it creates a volume by sweeping the bivector along the perpendicular component of the vector.

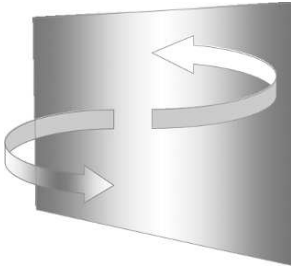
In  $\mathbb{R}^3$  premultiplying a vector by a trivector creates a multivector consisting of bivector terms. In the case of the unit basis vectors we have

$$\mathbf{e}_{123}\mathbf{e}_1 = \mathbf{e}_{23} \quad (8.314)$$

$$\mathbf{e}_{123}\mathbf{e}_2 = \mathbf{e}_{31} \quad (8.315)$$

$$\mathbf{e}_{123}\mathbf{e}_3 = \mathbf{e}_{12}. \quad (8.316)$$

Apart from the rotations described above, GA contains some powerful 3D rotation features that are described in the following chapter.



# 9 Reflections and Rotations

## 9.1 Introduction

Rotating objects and virtual cameras are central to computer animation and computer games and are traditionally effected using matrix transforms representing Euler angle rotations. For example, to rotate a 2D point about the origin we use

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \quad (9.1)$$

To rotate a 3D point about the origin we use one transform for each axis: to rotate about the  $x$ -axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\text{pitch}) & -\sin(\text{pitch}) \\ 0 & \sin(\text{pitch}) & \cos(\text{pitch}) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (9.2)$$

to rotate about the  $y$ -axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\text{yaw}) & 0 & \sin(\text{yaw}) \\ 0 & 1 & 0 \\ -\sin(\text{yaw}) & 0 & \cos(\text{yaw}) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (9.3)$$

and to rotate about the  $z$ -axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(\text{roll}) & -\sin(\text{roll}) & 0 \\ \sin(\text{roll}) & \cos(\text{roll}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \quad (9.4)$$

These rotations are not very intuitive to use, especially when we need to rotate points about an arbitrary axis, for which the following transform is used:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a^2K + \cos \theta & abK - c \sin \theta & acK + b \sin \theta \\ abK + c \sin \theta & b^2K + \cos \theta & bcK - a \sin \theta \\ acK - b \sin \theta & bcK + a \sin \theta & c^2K + \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (9.5)$$

where

$$K = 1 - \cos \theta \quad (9.6)$$

and the axis of rotation is defined by

$$\hat{v} = ai + bj + ck. \quad (9.7)$$

In recent years, Hamilton's quaternions have been embraced by the computer animation and games communities where a point  $P$  is rotated to  $P'$ , through an angle  $\theta$ , about an axis  $\hat{v}$  using the pure quaternion  $q$ . The process involves the following steps:

1. Convert the point  $P(x, y, z)$  to a pure quaternion  $p$ :

$$p = [0 + xi + yj + zk]. \quad (9.8)$$

2. Define the axis of rotation as a unit vector  $\hat{v}$ :

$$\hat{v} = [x_v i + y_v j + z_v k]. \quad (9.9)$$

3. Define the transforming quaternion  $q$ :

$$q = [\cos(\theta/2) + \sin(\theta/2)\hat{v}]. \quad (9.10)$$

4. Define the inverse of the transforming quaternion  $q^{-1}$ :

$$q^{-1} = [\cos(\theta/2) - \sin(\theta/2)\hat{v}]. \quad (9.11)$$

5. Compute  $p'$ :

$$p' = qpq^{-1}. \quad (9.12)$$

6. Unpack  $(x', y', z')$  from  $p'$ :

$$p' = [0 + x'i + y'j + z'k]. \quad (9.13)$$

Given a quaternion  $[w + xi + yj + zk]$ , its equivalent matrix is

$$\begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{bmatrix} \quad (9.14)$$

where

$$w^2 + x^2 + y^2 + z^2 = 1. \quad (9.15)$$

We saw in the previous chapter that there is a strong relationship between quaternions and GA, and in this chapter this relationship is further strengthened when we examine how GA implements reflections and rotations.



## 9.2 Reflections

### 9.2.1 Reflecting vectors

Solving problems using vector algebra is not always straight forward, for much depends upon the nature of the diagram used to annotate the relevant vectors. For instance, say we are given a mirror whose orientation is defined by an orthogonal unit vector  $\hat{n}$ , and the task is to find the reflection of the vector  $a$  in the mirror. Figure 9.1 shows the advantages of placing the vector so that its tail touches the mirror. It then becomes obvious that the vector's head is a distance  $a \cdot \hat{n}$  in front of the mirror, which means that the head of the reflected vector  $a'$  is an equal distance behind the mirror. Thus, we can write the following vector equation for  $a'$  as

$$a' = a - (2a \cdot \hat{n})\hat{n} \tag{9.16}$$

which is rather succinct.

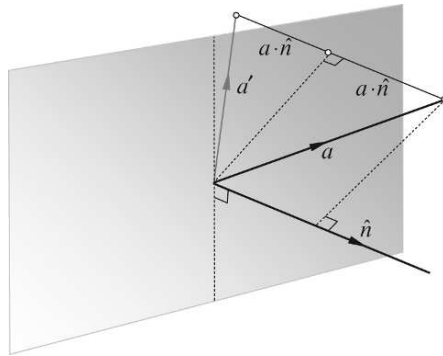


FIGURE 9.1.

Now let's solve the problem using GA.

Figure 9.2 shows the same diagram, but annotated differently.

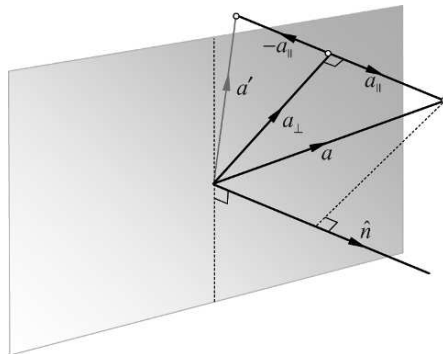


FIGURE 9.2.

As the mirror's surface normal is defined as a unit vector, then  $\hat{n}^2 = 1$ , which permits us to write

$$a = \hat{n}^2 a = \hat{n}(\hat{n}a) \quad (9.17)$$

and substituting the geometric product we have

$$a = \hat{n}(\hat{n} \cdot a + \hat{n} \wedge a). \quad (9.18)$$

From Fig. 9.2 it is obvious that

$$a = a_{\perp} + a_{\parallel} \quad (9.19)$$

and

$$a_{\parallel} = (\hat{n} \cdot a)\hat{n}. \quad (9.20)$$

Therefore,

$$a = \hat{n}(\hat{n} \wedge a) + a_{\parallel}$$

which means that

$$a_{\perp} = \hat{n}(\hat{n} \wedge a). \quad (9.21)$$

From Fig. 9.2 it is also obvious that

$$a' = a_{\perp} - a_{\parallel}. \quad (9.22)$$

Substituting Eqs. (9.20) and (9.21) in Eq. (9.22) we have

$$a' = \hat{n}(\hat{n} \wedge a) - (\hat{n} \cdot a)\hat{n}.$$

Reordering the products we get

$$a' = -(\hat{n} \cdot a)\hat{n} + \hat{n}(\hat{n} \wedge a). \quad (9.23)$$

Recalling from the previous chapter that vectors and bivectors anticommute, i.e.  $aB = -Ba$ , therefore,

$$\hat{n}(\hat{n} \wedge a) = -(\hat{n} \wedge a)\hat{n} \quad (9.24)$$

which means that we can write Eq. (9.23) as

$$a' = -(\hat{n} \cdot a)\hat{n} - (\hat{n} \wedge a)\hat{n} \quad (9.25)$$

which simplifies to

$$a' = -((\hat{n} \cdot a) + (\hat{n} \wedge a))\hat{n}. \quad (9.26)$$

The reason behind the above strategy was to create the geometric product  $\hat{n}a$  within Eq. (9.26), which now simplifies to

$$a' = -\hat{n}a\hat{n}. \quad (9.27)$$

This sandwiching effect is reminiscent of the structure for using quaternions to rotate vectors, and will become even more obvious when we consider rotations. For the moment, let's test Eq. (9.27) with an example.

Figure 9.3 shows a vector  $a$  with reflection  $a'$  in the plane defined by  $e_{12}$ , with surface normal  $\hat{n} = e_3$ .

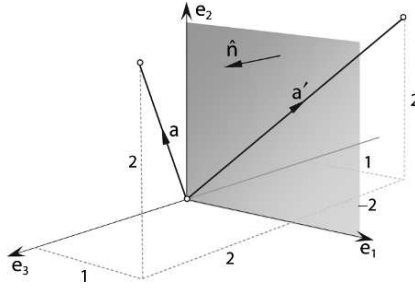


FIGURE 9.3.

If, for example,

$$a = e_1 + 2e_2 + 2e_3 \quad (9.28)$$

then, from Fig. 9.3 it is obvious that its reflection is

$$a' = e_1 + 2e_2 - 2e_3. \quad (9.29)$$

We can confirm this result using Eq. (9.27):

$$\begin{aligned} a' &= -\hat{n}a\hat{n} \\ &= -e_3(e_1 + 2e_2 + 2e_3)e_3 \\ &= -(e_{313} + 2e_{323} + 2e_{333}) \\ a' &= e_1 + 2e_2 - 2e_3. \end{aligned} \quad (9.30)$$

It is important to note that this reflection formula assumes that the line and plane intersect at the origin. Now let's investigate what happens when a bivector is reflected.

## 9.2.2 Reflecting bivectors

As a bivector is formed from a pair of vectors, its reflection must be formed from the reflections of its vectors, as shown in Fig. 9.4.

If  $B = a \wedge b$  then its reflection is  $B' = a' \wedge b'$  where  $a'$  and  $b'$  are the reflections of the original vectors. Algebraically, we proceed as follows using Eq. (9.27):

$$a' = -\hat{n}a\hat{n} \quad (9.31)$$

and

$$b' = -\hat{n}b\hat{n}. \quad (9.32)$$

Therefore,

$$\begin{aligned} B' &= (-\hat{n}a\hat{n}) \wedge (-\hat{n}b\hat{n}) \\ B' &= (\hat{n}a\hat{n}) \wedge (\hat{n}b\hat{n}). \end{aligned} \quad (9.33)$$

But we know that

$$B = a \wedge b = \frac{1}{2}(ab - ba) \quad (9.34)$$

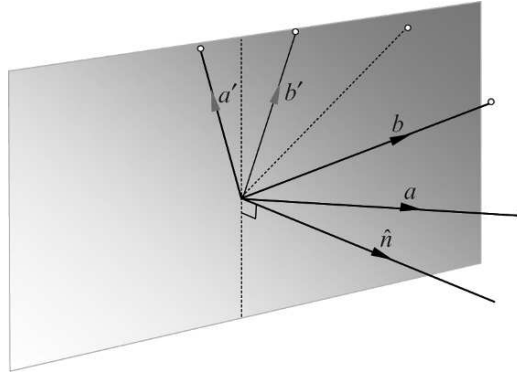


FIGURE 9.4.

therefore,

$$\begin{aligned}
 B' &= \frac{1}{2}(\hat{n}a\hat{n}b\hat{n} - \hat{n}b\hat{n}a\hat{n}) \\
 &= \frac{1}{2}(\hat{n}ab\hat{n} - \hat{n}ba\hat{n}) \\
 &= \frac{1}{2}\hat{n}(ab - ba)\hat{n} \\
 B' &= \hat{n}B\hat{n}
 \end{aligned}
 \tag{9.35}$$

which, apart from the minus sign, is identical to the equation for reflecting a vector.

Again, it's worth testing the action of Eq. (9.35) with an example.

Figure 9.5 shows a bivector  $B = a \wedge b$  reflected in the plane defined by  $\hat{n} = e_3$ .

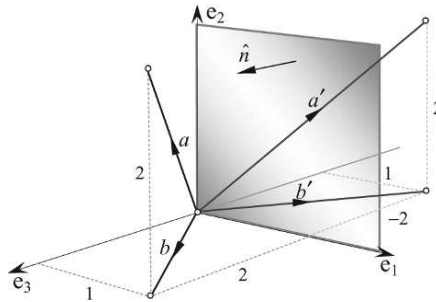


FIGURE 9.5.

If

$$a = e_1 + 2e_2 + 2e_3 \tag{9.36}$$

and

$$b = e_1 + 2e_3 \tag{9.37}$$

then

$$B = (e_1 + 2e_2 + 2e_3) \wedge (e_1 + 2e_3). \tag{9.38}$$

To save time evaluating outer products, the following *aide-mémoire* reminds us how to calculate the coefficients of the bivector terms.

	$e_1$	$e_2$	$e_3$
$m$	$m_1$	$m_2$	$m_3$
$n$	$n_1$	$n_2$	$n_3$
$m \wedge n$	$x = m_2n_3 - m_3n_2$	$y = m_3n_1 - m_1n_3$	$z = m_1n_2 - m_2n_1$
	$e_{23}$	$e_{31}$	$e_{12}$

therefore,

	$e_1$	$e_2$	$e_3$
$a$	1	2	2
$b$	1	0	2
$B$	4	0	-2
	$e_{23}$	$e_{31}$	$e_{12}$

and

$$B = -2e_{12} + 4e_{23}. \tag{9.39}$$

Now let's calculate the reflections of  $a$  and  $b$ .

From Eq. (9.30)

$$a' = e_1 + 2e_2 - 2e_3. \tag{9.40}$$

From Eq. (9.27)

$$\begin{aligned} b' &= -\hat{n}b\hat{n} \\ &= -e_3(e_1 + 2e_3)e_3 \\ b' &= e_1 - 2e_3. \end{aligned} \tag{9.41}$$

Therefore,

$$\begin{aligned} B' &= a' \wedge b' \\ &= (e_1 + 2e_2 - 2e_3) \wedge (e_1 - 2e_3) \end{aligned}$$

	$e_1$	$e_2$	$e_3$
$a'$	1	2	-2
$b'$	1	0	-2
$B'$	-4	0	-2
	$e_{23}$	$e_{31}$	$e_{12}$

and

$$B' = -2e_{12} - 4e_{23}. \tag{9.42}$$

Comparing Eq. (9.39) and Eq. (9.42) we see that the sign of the unit basis bivector  $e_{23}$  coefficient has flipped.

Alternatively, we can calculate  $B'$  using Eq. (9.35):

$$\begin{aligned}
 B' &= \hat{n}\hat{n} \\
 &= e_3(-2e_1 \wedge e_2 + 4e_2 \wedge e_3)e_3 \\
 &= e_3(-2e_{12} + 4e_{23})e_3 \\
 &= -2e_{3123} + 4e_{3233} \\
 B' &= -2e_{12} - 4e_{23}.
 \end{aligned}
 \tag{9.43}$$

It should be obvious from Fig. 9.5 why the coefficients of  $e_{12}$  and  $e_{23}$  are negative, and why the coefficient of  $e_{31}$  is zero.

### 9.2.3 Reflecting trivectors

Finally, let's examine how trivectors behave when reflected in a mirror. Experience confirms that when we hold up our right hand in front of a mirror, we see a reflection identical to an image of our left hand – and vice versa. Therefore, a set of right-handed orthogonal axes should appear reflected as a left-handed set, as shown in Fig. 9.6. We can confirm this algebraically as follows:

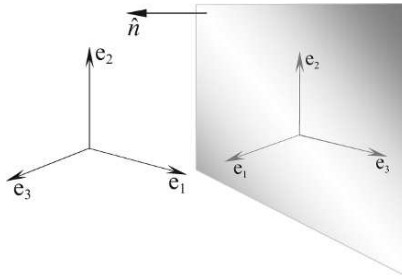


FIGURE 9.6.

Starting with the unit trivector, which is also a pseudoscalar:

$$I = e_1 \wedge e_2 \wedge e_3 \tag{9.44}$$

its reflection consists of three reflected unit vectors:

$$-\hat{n}e_1\hat{n} \quad -\hat{n}e_2\hat{n} \quad -\hat{n}e_3\hat{n} \tag{9.45}$$

which form the reflected unit trivector

$$(-\hat{n}e_1\hat{n}) \wedge (-\hat{n}e_2\hat{n}) \wedge (-\hat{n}e_3\hat{n}). \tag{9.46}$$

Expanding the first two terms of Eq. (9.46) using  $a \wedge b = \frac{1}{2}(ab - ba)$  we have:

$$\begin{aligned} (-\hat{n}e_1\hat{n}) \wedge (-\hat{n}e_2\hat{n}) &= \frac{1}{2}((-\hat{n}e_1\hat{n})(-\hat{n}e_2\hat{n}) - (-\hat{n}e_2\hat{n})(-\hat{n}e_1\hat{n})) \\ &= \frac{1}{2}(\hat{n}e_{12}\hat{n} - \hat{n}e_{21}\hat{n}) \\ &= \frac{1}{2}(\hat{n}e_{12}\hat{n} + \hat{n}e_{12}\hat{n}) \end{aligned}$$

therefore,

$$(-\hat{n}e_1\hat{n}) \wedge (-\hat{n}e_2\hat{n}) = \hat{n}e_{12}\hat{n}. \quad (9.47)$$

Expanding the rest of Eq. (9.46) we have

$$(\hat{n}e_{12}\hat{n}) \wedge (-\hat{n}e_3\hat{n}) \quad (9.48)$$

and using  $B \wedge a = \frac{1}{2}(Ba + aB)$  we have:

$$\begin{aligned} \hat{n}e_{12}\hat{n} \wedge (-\hat{n}e_3\hat{n}) &= \frac{1}{2}((\hat{n}e_{12}\hat{n}) \wedge (-\hat{n}e_3\hat{n}) + (-\hat{n}e_3\hat{n}) \wedge (\hat{n}e_{12}\hat{n})) \\ &= \frac{1}{2}((\hat{n}e_{12}\hat{n})(-\hat{n}e_3\hat{n}) + (-\hat{n}e_3\hat{n})(\hat{n}e_{12}\hat{n})) \\ &= \frac{1}{2}(-\hat{n}e_{123}\hat{n} - \hat{n}e_{312}\hat{n}) \\ &= -\hat{n}e_{123}\hat{n} \end{aligned}$$

therefore,

$$(-\hat{n}e_1\hat{n}) \wedge (-\hat{n}e_2\hat{n}) \wedge (-\hat{n}e_3\hat{n}) = -\hat{n}I\hat{n} \quad (9.49)$$

where  $I$  is the pseudoscalar, which commutes with vectors. Therefore,

$$(-\hat{n}e_1\hat{n}) \wedge (-\hat{n}e_2\hat{n}) \wedge (-\hat{n}e_3\hat{n}) = -\hat{n}\hat{n}I = -I. \quad (9.50)$$

Equation (9.50) confirms that the sign of the trivector's reflection has switched from positive to negative, as predicted.

It is possible to show that the reflection of a general trivector behaves in exactly the same way.

## 9.3 Rotations

### 9.3.1 Rotating by double reflecting

The reason why we started with reflections is that they provide a way to rotate vectors. To illustrate this, consider Fig. 9.7(a) showing a mirror  $m$  and a vector  $a$  forming an angle  $\alpha$  with the mirror. By the laws of reflection,  $a$ 's reflection is  $b$  forming an equal angle  $\alpha$  on the other side of the mirror. Now consider Fig. 9.7(b) which shows two superimposed mirrors  $m$  and  $n$ , where  $a$ 's reflection in  $m$  is  $b$ , and  $b$ 's reflection in  $n$  is  $c$ , which must coincide with  $a$ . We can reason that

as the separating angle between the mirrors is  $0^\circ$ , the separating angle between  $a$  and its double reflection  $c$  is also  $0^\circ$ .

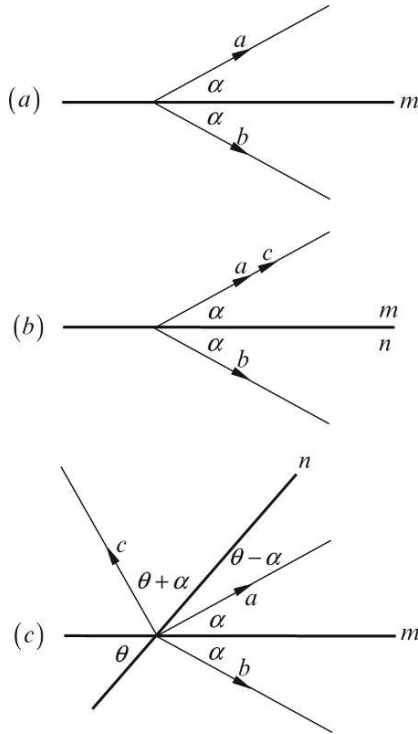


FIGURE 9.7.

Now consider Fig. 9.7(c) where the two mirrors  $m$  and  $n$  are separated by an angle  $\theta$ . Vector  $a$ 's reflection is still  $b$ , whilst  $b$ 's reflection in  $n$  has rotated anticlockwise to  $c$ . By inspection, the angle of rotation between  $b$  and  $n$  is  $\theta + \alpha$ , which places  $c$  at an angle  $\theta + \alpha$  on the opposite side of  $n$ . The interesting result about this double mirror arrangement is that the angle between  $a$  and  $c$  is  $2\theta$ , exactly double the angle between the mirrors. Now let's make a subtle substitution by representing the mirror  $m$  by a perpendicular unit vector  $\hat{m}$ , and mirror  $n$  by a perpendicular unit vector  $\hat{n}$ . This in no way changes the geometry, but allows us to describe the double reflection using GA. We will also define the plane supporting the mirrors by the outer product  $\hat{m} \wedge \hat{n}$ , as this represents the order of the mirrors.

Vector  $a$ 's reflection  $b$  is given by

$$b = -\hat{n}a\hat{m} \tag{9.51}$$

which, in turn, is reflected in  $n$  to create  $c$ :

$$\begin{aligned} c &= -\hat{n}b\hat{n} \\ &= -\hat{n}(-\hat{m}a\hat{m})\hat{n} \\ c &= \hat{n}\hat{m}a\hat{m}\hat{n}. \end{aligned} \tag{9.52}$$



Substituting  $R = \hat{n}\hat{m}$  in Eq. (9.52) we have

$$c = Ra\tilde{R} \tag{9.53}$$

where  $\tilde{R} = \hat{m}\hat{n}$ , the reverse product of  $R$ .

Although this description is based on an imaginary 2D scenario, it works in any number of dimensions, however, we are particularly interested in  $\mathbb{R}^3$ .

Figure 9.8 shows two mirrors  $m$  and  $n$  represented by their normal vectors  $\hat{m}$  and  $\hat{n}$ , separated by an angle  $\theta$ . Vector  $a$ 's reflection is still  $b$ , and  $b$ 's reflection in  $n$  is still  $c$ , and effectively,  $a$  has been rotated  $2\theta$  to  $c$ .

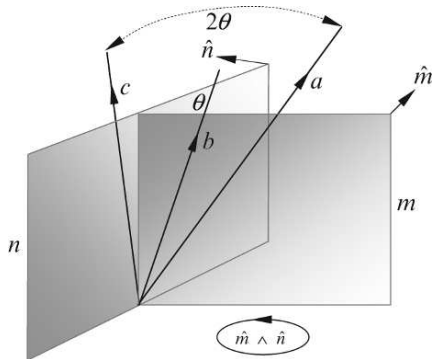


FIGURE 9.8.

To illustrate this double reflection, consider the two mirrors shown in Fig. 9.9 with normal vectors

$$\hat{m} = -e_3 \tag{9.54}$$

$$\hat{n} = -e_1. \tag{9.55}$$

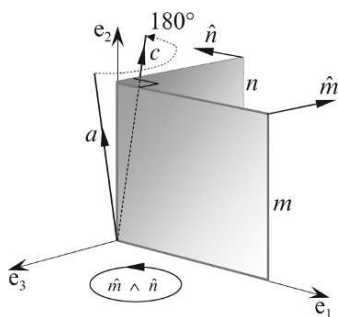


FIGURE 9.9.

If the vector  $a$  is

$$a = e_1 + e_2 + e_3 \tag{9.56}$$

then

$$\begin{aligned}
 R &= \hat{n}\hat{m} \\
 R &= (-e_1)(-e_3) = e_{13}
 \end{aligned}
 \tag{9.57}$$

and

$$\begin{aligned}
 \tilde{R} &= \hat{m}\hat{n} \\
 \tilde{R} &= (-e_3)(-e_1) = e_{31}.
 \end{aligned}
 \tag{9.58}$$

Therefore,

$$\begin{aligned}
 c &= Ra\tilde{R} \\
 &= e_{13}(e_1 + e_2 + e_3)e_{31} \\
 &= (-e_3 + e_{132} + e_1)e_{31} \\
 &= -e_1 + e_2 - e_3 \\
 c &= -e_1 + e_2 - e_3
 \end{aligned}
 \tag{9.59}$$

which is as expected.

We have seen from the previous examples that the mirrors and the angle of rotation are controlled by the bivector associated with the plane perpendicular to the mirrors, so let's drop the idea of mirrors and reflections and adopt the idea of rotating vectors using a bivector.

Figure 9.10 shows two vectors  $m$  and  $n$  forming the bivector  $m \wedge n$  directed anticlockwise. As the internal angle of the bivector is  $60^\circ$ , vector  $a$  will be rotated  $120^\circ$  anticlockwise, which we can predict will be  $e_1 + e_3$ . Now let's construct the geometric products to perform the rotation.

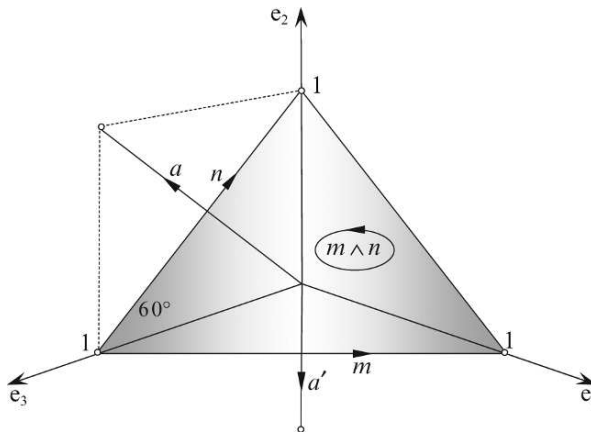


FIGURE 9.10.

First we define the unit vectors  $\hat{m}$  and  $\hat{n}$ :

$$\hat{m} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_3) \quad (9.60)$$

$$\hat{n} = \frac{1}{\sqrt{2}}(\mathbf{e}_2 - \mathbf{e}_3) \quad (9.61)$$

and

$$\mathbf{a} = \mathbf{e}_2 + \mathbf{e}_3. \quad (9.62)$$

Therefore,

$$\begin{aligned} R &= \hat{n}\hat{m} \\ &= \frac{1}{2}(\mathbf{e}_2 - \mathbf{e}_3)(\mathbf{e}_1 - \mathbf{e}_3) \\ R &= \frac{1}{2}(\mathbf{e}_{21} - \mathbf{e}_{23} - \mathbf{e}_{31} + 1) \end{aligned} \quad (9.63)$$

and

$$\begin{aligned} \hat{n}\hat{m}\mathbf{a} &= \frac{1}{2}(\mathbf{e}_{21} - \mathbf{e}_{23} - \mathbf{e}_{31} + 1)(\mathbf{e}_2 + \mathbf{e}_3) \\ &= \frac{1}{2}(\mathbf{e}_{212} + \mathbf{e}_{213} - \mathbf{e}_{232} - \mathbf{e}_{233} - \mathbf{e}_{312} - \mathbf{e}_{313} + \mathbf{e}_2 + \mathbf{e}_3) \\ &= \frac{1}{2}(-\mathbf{e}_1 - \mathbf{e}_{123} + \mathbf{e}_3 - \mathbf{e}_2 - \mathbf{e}_{123} + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\ \hat{n}\hat{m}\mathbf{a} &= \mathbf{e}_3 - \mathbf{e}_{123}. \end{aligned} \quad (9.64)$$

Now we compute the reverse product

$$\begin{aligned} \tilde{R} &= \hat{m}\hat{n} \\ &= \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_3)(\mathbf{e}_2 - \mathbf{e}_3) \\ \tilde{R} &= \frac{1}{2}(\mathbf{e}_{12} - \mathbf{e}_{13} - \mathbf{e}_{32} + 1) \end{aligned} \quad (9.65)$$

and

$$\begin{aligned} Ra\tilde{R} &= \frac{1}{2}(\mathbf{e}_3 - \mathbf{e}_{123})(\mathbf{e}_{12} - \mathbf{e}_{13} - \mathbf{e}_{32} + 1) \\ &= \frac{1}{2}(\mathbf{e}_{312} - \mathbf{e}_{313} - \mathbf{e}_{332} + \mathbf{e}_3 - \mathbf{e}_{12312} + \mathbf{e}_{12313} + \mathbf{e}_{12332} - \mathbf{e}_{123}) \\ Ra\tilde{R} &= \mathbf{e}_1 + \mathbf{e}_3 \end{aligned} \quad (9.66)$$

which confirms our prediction.

### 9.3.2 Rotors

Much of mathematics is about patterns, especially in formulae. One such pattern is about to emerge, and whoever discovered it deserves some sort of recognition. In chapter 3 we saw that a complex number is rotated in the complex plane by multiplying it by  $e^{i\theta}$ , which is equivalent to  $\cos \theta + i \sin \theta$ . We are about to discover that a multivector can also be rotated in a plane defined by a unit bivector, which plays a similar role to the imaginary  $i$ .

As sandwiching a multivector between  $R$  and  $\tilde{R}$  results in a rotation,  $R$  is called a *rotor*, much like  $e^{i\theta}$ . What is strange, though, is that the bivector defining the plane is  $\hat{m} \wedge \hat{n}$ , whilst the rotor sequence is  $R = \hat{n}\hat{m}$ . The vectors are switched, and we will have to watch out for this.

We start the process as follows:

$$R = \hat{n}\hat{m} \quad (9.67)$$

which, using the geometric product, expands to

$$R = \hat{n} \cdot \hat{m} + \hat{n} \wedge \hat{m}. \quad (9.68)$$

But

$$\hat{n} \cdot \hat{m} = \|\hat{n}\| \|\hat{m}\| \cos \theta = \cos \theta \quad (9.69)$$

therefore,

$$R = \cos \theta + \hat{n} \wedge \hat{m}. \quad (9.70)$$

This is where we begin looking for a pattern. We already know that

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (9.71)$$

so could it be that  $R$  has a similar structure? To find the answer to this question consider the following expansion  $(\hat{m} \wedge \hat{n})^2$  using

$$\hat{m} \wedge \hat{n} = \hat{m}\hat{n} - \hat{m} \cdot \hat{n}$$

and

$$\hat{m} \wedge \hat{n} = \hat{m} \cdot \hat{n} - \hat{n}\hat{m}. \quad (9.72)$$

Therefore,

$$\begin{aligned} (\hat{m} \wedge \hat{n})^2 &= (\hat{m}\hat{n} - \hat{m} \cdot \hat{n})(\hat{m} \cdot \hat{n} - \hat{n}\hat{m}) \\ &= \hat{m}\hat{n}(\hat{m} \cdot \hat{n}) - \hat{m}\hat{n}^2\hat{m} - (\hat{m} \cdot \hat{n})^2 + \hat{n}\hat{m}(\hat{m} \cdot \hat{n}) \\ &= \hat{m} \cdot \hat{n}(\hat{m}\hat{n} + \hat{n}\hat{m}) - \hat{m}\hat{n}^2\hat{m} - (\hat{m} \cdot \hat{n})^2 \\ (\hat{m} \wedge \hat{n})^2 &= -\hat{m}^2\hat{n}^2 - (\hat{m} \cdot \hat{n})^2. \end{aligned} \quad (9.73)$$

But as

$$(\hat{m} \cdot \hat{n})^2 = \|\hat{m}\|^2 \|\hat{n}\|^2 \cos^2 \theta \quad (9.74)$$

and

$$\hat{m}^2\hat{n}^2 = \|\hat{m}\|^2 \|\hat{n}\|^2 \quad (9.75)$$

then

$$\begin{aligned}
 (\hat{m} \wedge \hat{n})^2 &= -\|\hat{m}\|^2\|\hat{n}\|^2 - \|\hat{m}\|^2\|\hat{n}\|^2 \cos^2 \theta \\
 &= -1 - \cos^2 \theta \\
 (\hat{m} \wedge \hat{n})^2 &= -\sin^2 \theta.
 \end{aligned} \tag{9.76}$$

Note the imaginary feature of this result, which can be interpreted as follows:

$$\hat{m} \wedge \hat{n} = \hat{B} \sin \theta \tag{9.77}$$

where  $\hat{B}$  is the unit bivector in the  $\hat{m} \wedge \hat{n}$  plane where  $\hat{B}^2 = -1$ .

Similarly,

$$\hat{n} \wedge \hat{m} = -\hat{B} \sin \theta \tag{9.78}$$

which can be substituted in Eq. (9.70)

$$R = \cos \theta - \hat{B} \sin \theta \tag{9.79}$$

which has a similar structure to Eq. (9.71) apart from a negative imaginary component.

We can convert Eq. (9.79) to its exponential form as follows:

$$R = \exp(-\hat{B}\theta). \tag{9.80}$$

Remembering that the double reflection technique doubles the angle of rotation, we must compensate for this by halving the original angle:

$$R = \exp(-\hat{B}\theta/2). \tag{9.81}$$

Similarly,

$$\tilde{R} = \exp(\hat{B}\theta/2) \tag{9.82}$$

which enables us to write the result

$$c = e^{-\hat{B}\theta/2} a e^{\hat{B}\theta/2}. \tag{9.83}$$

More generally, the vector  $a$  is rotated through an angle  $\theta$  in the plane defined by the unit bivector  $\hat{B}$  using

$$a' = e^{-\hat{B}\theta/2} a e^{\hat{B}\theta/2}. \tag{9.84}$$

So now we have two ways of visualizing a rotor: either as a bivector or as an exponential, which is readily represented as

$$\exp(-\hat{B}\theta/2) = \cos(\theta/2) - \hat{B} \sin(\theta/2) \tag{9.85}$$

Therefore, we can rewrite Eq. (9.84) as

$$a' = (\cos(\theta/2) - \hat{B} \sin(\theta/2))a(\cos(\theta/2) + \hat{B} \sin(\theta/2)). \tag{9.86}$$

Let's test Eq. (9.86) with an example.

Figure 9.11 shows two vectors  $m$  and  $n$  forming a bivector  $m \wedge n$ . The angle of rotation is  $120^\circ$ , which means that the vector  $a = e_2 + e_3$  will be rotated to  $a' = e_1 + e_3$  as shown in Fig. 9.12.

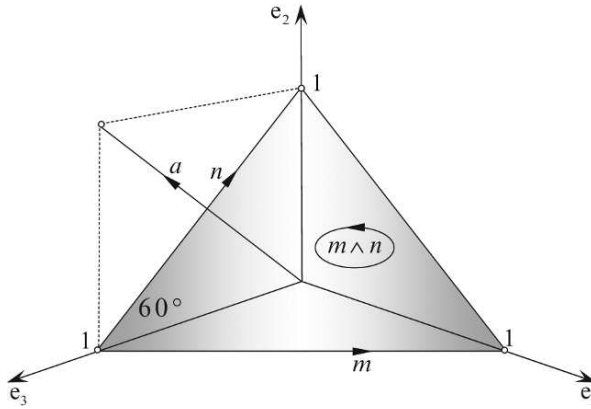


FIGURE 9.11.

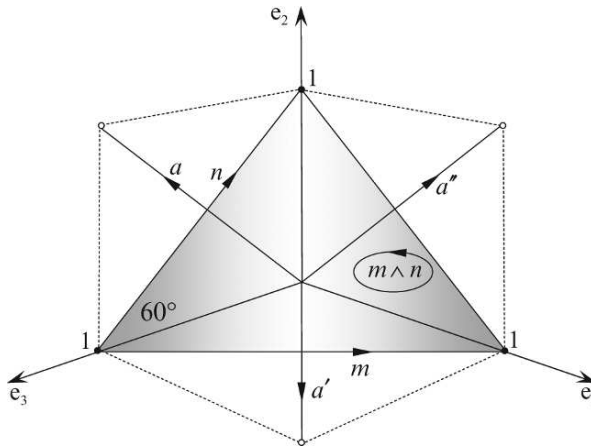


FIGURE 9.12.

Using Eq. (9.86) we have

$$\begin{aligned}
 a' &= (\cos 60^\circ - \hat{B} \sin 60^\circ)(e_2 + e_3)(\cos 60^\circ + \hat{B} \sin 60^\circ) \\
 &= \left(\frac{1}{2} - \hat{B}\sqrt{3}/2\right)(e_2 + e_3)\left(\frac{1}{2} + \hat{B}\sqrt{3}/2\right) \\
 a' &= \frac{1}{4}(1 - \hat{B}\sqrt{3})(e_2 + e_3)(1 + \hat{B}\sqrt{3}).
 \end{aligned}
 \tag{9.87}$$

Given that

$$m = e_1 - e_3 \tag{9.88}$$

and

$$n = e_2 - e_3 \tag{9.89}$$

we evaluate the outer product using our *aide-mémoire*.

	$e_1$	$e_2$	$e_3$
$m$	1	0	-1
$n$	0	1	-1
$m \wedge n$	$x = 1$	$y = 1$	$z = 1$
	$e_{23}$	$e_{31}$	$e_{12}$

where

$$m \wedge n = e_{23} + e_{31} + e_{12}. \quad (9.90)$$

But we require a unit bivector, which makes

$$\hat{B} = \frac{1}{\sqrt{3}}(e_{23} + e_{31} + e_{12}). \quad (9.91)$$

Therefore,

$$\begin{aligned} a' &= \frac{1}{4}(1 - e_{23} - e_{31} - e_{12})(e_2 + e_3)(1 + e_{23} + e_{31} + e_{12}) \\ &= \frac{1}{4}(e_2 + e_3 - e_{232} - e_{233} - e_{312} - e_{313} - e_{122} - e_{123})(1 + e_{23} + e_{31} + e_{12}) \\ &= \frac{1}{2}(e_3 - e_{123})(1 + e_{23} + e_{31} + e_{12}) \\ &= \frac{1}{2}(e_3 + e_{323} + e_{331} + e_{312} - e_{123} - e_{12323} - e_{12331} - e_{12312}) \\ a' &= e_1 + e_3 \end{aligned} \quad (9.92)$$

which is what we predicted.

### 9.3.3 Rotor matrix

Another way of implementing a rotor is using a matrix, which is created as follows. We begin with the bivector defining the plane  $m \wedge n$ , about which the rotation is effected, where

$$m = m_1 e_1 + m_2 e_2 + m_3 e_3 \quad (9.93)$$

and

$$n = n_1 e_1 + n_2 e_2 + n_3 e_3. \quad (9.94)$$

Notice in the following how the bivectors are associated with their perpendicular axes.

Therefore,

$$\begin{aligned} R &= mn \\ R &= w + x e_{23} + y e_{31} + z e_{12} \end{aligned} \quad (9.95)$$

and

$$\begin{aligned}\tilde{R} &= nm \\ \tilde{R} &= w - xe_{23} - ye_{31} - ze_{12}\end{aligned}\quad (9.96)$$

where

$$w^2 + x^2 + y^2 + z^2 = 1 \quad (9.97)$$

and  $x, y$  and  $z$  are computed using the outer product *aide-mémoire*.

We derive the matrix  $[[R]]$  representing  $Ra\tilde{R}$  by expanding the individual elements for  $Re_1\tilde{R}$ ,  $Re_2\tilde{R}$  and  $Re_3\tilde{R}$ :

$$\begin{aligned}Re_1\tilde{R} &= (w + xe_{23} + ye_{31} + ze_{12})e_1(w - xe_{23} - ye_{31} - ze_{12}) \\ &= (we_1 + xe_{123} + ye_3 - ze_2)(w - xe_{23} - ye_{31} - ze_{12}) \\ Re_1\tilde{R} &= (w^2 + x^2 - y^2 - z^2)e_1 + 2(-wz + xy)e_2 + 2(wy + xz)e_3.\end{aligned}\quad (9.98)$$

The first term is simplified by substituting

$$w^2 + x^2 = 1 - y^2 - z^2 \quad (9.99)$$

$$Re_1\tilde{R} = (1 - 2(y^2 + z^2))e_1 + 2(xy - wz)e_2 + 2(xz + wy)e_3. \quad (9.100)$$

Next is  $Re_2\tilde{R}$

$$\begin{aligned}Re_2\tilde{R} &= (w + xe_{23} + ye_{31} + ze_{12})e_2(w - xe_{23} - ye_{31} - ze_{12}) \\ &= (we_2 - xe_3 + ye_{123} + ze_1)(w - xe_{23} - ye_{31} - ze_{12}) \\ Re_2\tilde{R} &= 2(xy + wz)e_1 + (w^2 - x^2 + y^2 - z^2)e_2 + 2(yz - wx)e_3.\end{aligned}\quad (9.101)$$

Substituting

$$w^2 + y^2 = 1 - x^2 - z^2 \quad (9.102)$$

$$Re_2\tilde{R} = 2(xy + wz)e_1 + (1 - 2(x^2 + z^2))e_2 + 2(yz - wx)e_3. \quad (9.103)$$

Finally  $Re_3\tilde{R}$

$$\begin{aligned}Re_3\tilde{R} &= (w + xe_{23} + ye_{31} + ze_{12})e_3(w - xe_{23} - ye_{31} - ze_{12}) \\ &= (we_3 + xe_2 - ye_1 + ze_{123})(w - xe_{23} - ye_{31} - ze_{12}) \\ Re_3\tilde{R} &= 2(xz - wy)e_1 + 2(yz + wx)e_2 + (w^2 - x^2 - y^2 + z^2)e_3.\end{aligned}\quad (9.104)$$

Substituting

$$w^2 + z^2 = 1 - x^2 - y^2 \quad (9.105)$$

$$Re_3\tilde{R} = 2(xz - wy)e_1 + 2(yz + wx)e_2 + (1 - 2(x^2 + y^2))e_3. \quad (9.106)$$



Therefore, the final matrix is

$$\llbracket R \rrbracket = \begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{bmatrix} \quad (9.107)$$

which is also used in its transposed form. Notice that it is identical to the matrix representing a quaternion. (Eq. (9.52)).

Let's illustrate this matrix using the previous example.

Figure 9.11 shows the bivector  $m \wedge n$  which will be used to rotate the vector  $a$  through an angle  $120^\circ$ .

Given the following vectors:

$$m = e_1 - e_3 \quad (9.108)$$

$$n = e_2 - e_3 \quad (9.109)$$

$$a = e_2 + e_3 \quad (9.110)$$

then

$$m \cdot n = (e_1 - e_3) \cdot (e_2 - e_3) = 1 \quad (9.111)$$

and

$$m \wedge n = e_{23} + e_{31} + e_{12}. \quad (9.112)$$

Therefore,

$$\begin{aligned} R &= mn \\ &= m \cdot n + m \wedge n \\ R &= 1 + e_{23} + e_{31} + e_{12}. \end{aligned} \quad (9.113)$$

But this has to be normalized, which makes the scaling factor  $\frac{1}{2}$  and in matrix form using Eq. (9.107) becomes

$$\llbracket R \rrbracket = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \quad (9.114)$$

Multiplying vector  $a$  by this matrix we have

$$a' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (9.115)$$

which shows that  $a'$  is now pointing to  $(1, 0, 1)$ .

If  $a'$  is subjected to the same rotation we obtain

$$a'' = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (9.116)$$

which shows that  $a''$  is now pointing to  $(1, 1, 0)$ .

If  $a''$  is subjected to the same rotation we should return to the original vector:

$$a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{9.117}$$

which does reassuringly, brings us back to the original vector  $a$ . These rotations are shown in Fig. 9.12.

This seems too good to be true! So let's test it with another example. This time, let's reverse the bivector as shown in Fig. 9.13, where the bivector creates a clockwise rotation of  $45^\circ$ .

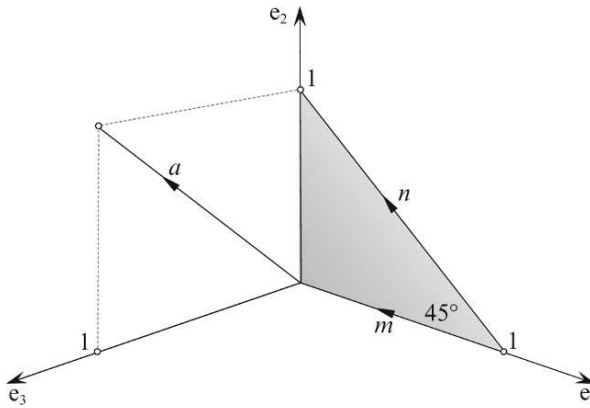


FIGURE 9.13.

Given the following vectors:

$$m = -e_1 \tag{9.118}$$

$$n = -e_1 + e_2 \tag{9.119}$$

$$a = e_2 + e_3. \tag{9.120}$$

	$e_1$	$e_2$	$e_3$
$m$	-1	0	0
$n$	-1	1	0
$m \wedge n$	$x = 0$	$y = 0$	$z = -1$
	$e_{23}$	$e_{31}$	$e_{12}$

Therefore,

$$m \cdot n = (-e_1) \cdot (-e_1 + e_2) = 1 \tag{9.121}$$

and

$$m \wedge n = -e_{12}. \tag{9.122}$$

Therefore,

$$\begin{aligned} R &= mn = m \cdot n + m \wedge n \\ R &= 1 - e_{12}. \end{aligned} \tag{9.123}$$

But this has to be normalized, which makes the scaling factor  $1/\sqrt{2}$  and in matrix form using Eq. (9.107) becomes

$$\llbracket R \rrbracket = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.124}$$

Multiplying vector  $a$  by this matrix we have

$$a' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{9.125}$$

which shows that  $a'$  is now pointing to  $(1, 0, 1)$ .

If  $a'$  is subjected to the same rotation we obtain

$$a'' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \tag{9.126}$$

which shows that  $a''$  is now pointing to  $(0, -1, 1)$ .

If  $a''$  is subjected to the same rotation we obtain

$$a''' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \tag{9.127}$$

Finally, If  $a'''$  is subjected to the same rotation we obtain

$$a = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{9.128}$$

which again reassuringly, brings us back to the original vector  $a$ . These rotations are shown in Fig. 9.14.

So once again, we see how close GA is to the algebra discovered by Hamilton. And even though Grassmann had discovered many of the ideas outlined above, he was unable to persuade mathematicians of the day to adopt his algebra, and it was left to Clifford to unify both men's work. Furthermore, it has taken several decades for GA to be applied seriously to science and physics, and only during the past decade has GA found application within computer graphics.

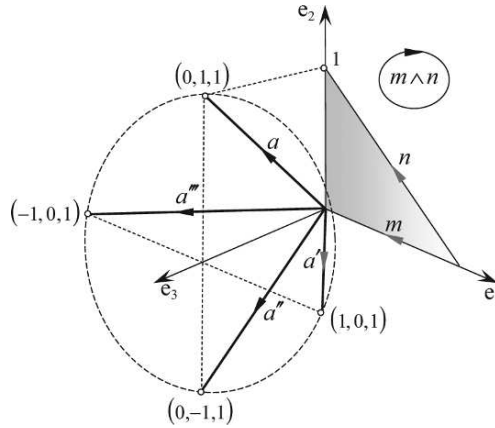


FIGURE 9.14.

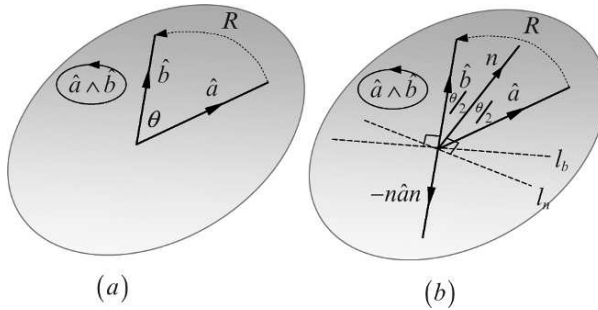


FIGURE 9.15.

### 9.3.4 Building rotors

In the previous section we discovered how to rotate a vector using a bivector. In this section we investigate how to derive the rotor that rotates one vector into another. Figure 9.15(a) illustrates the problem, where we see two vectors  $a$  and  $b$  in the plane defined by  $a \wedge b$ , and the objective is to find a rotor  $R$  that rotates  $a$  into  $b$ .

To do this, we bisect the angle  $\theta$  between the two vectors and create a mid-vector  $n$  using

$$n = \frac{\hat{a} + \hat{b}}{\|\hat{a} + \hat{b}\|}. \tag{9.129}$$

Perpendicular to vector  $n$  is a reflector  $l_n$  which is used to create a reflection of  $\hat{a}$ :  $-n\hat{a}n$ , as shown in Fig. 9.15(b), which must equal  $-\hat{b}$ . But rather than use

$$\hat{b} = n\hat{a}n \tag{9.130}$$

we create a reflection about the line  $l_b$  perpendicular to  $\hat{b}$ :

$$\begin{aligned}\hat{b} &= -\hat{b}(-n\hat{a}n)\hat{b} \\ \hat{b} &= \hat{b}n\hat{a}n\hat{b}\end{aligned}\tag{9.131}$$

which enables us to define  $\hat{b}n$  as the rotor:

$$R = \hat{b}n\tag{9.132}$$

and

$$\hat{b} = R\hat{a}\tilde{R}.\tag{9.133}$$

We now have a geometric product which expands to

$$\begin{aligned}R &= \hat{b}n \\ &= \hat{b} \left( \frac{\hat{a} + \hat{b}}{\|\hat{a} + \hat{b}\|} \right) \\ R &= \frac{1 + \hat{b}\hat{a}}{\|\hat{a} + \hat{b}\|}.\end{aligned}\tag{9.134}$$

We can simplify the denominator to avoid unnecessary arithmetic by the following subterfuge. Figure 9.16 shows part of the geometry associated with vectors  $\hat{a}$  and  $\hat{b}$ , where we see that  $d = \cos(\theta/2)$ , which means that using the half-angle identity

$$\cos(\theta/2) = \sqrt{\frac{1 + \cos \theta}{2}}\tag{9.135}$$

we have

$$\begin{aligned}\|\hat{a} + \hat{b}\| &= 2d \\ &= 2 \cos(\theta/2) \\ \|\hat{a} + \hat{b}\| &= \sqrt{2(1 + \cos \theta)}.\end{aligned}\tag{9.136}$$

This permits us to substitute

$$\cos \theta = \hat{a} \cdot \hat{b}\tag{9.137}$$

and

$$R = \frac{1 + \hat{b}\hat{a}}{\sqrt{2(1 + \hat{b} \cdot \hat{a})}}\tag{9.138}$$

which has the effect of rotating  $\hat{a}$  to  $\hat{b}$ .

In chapter 3 we showed that a complex number is rotated through an angle  $\theta$  in the complex plane using

$$z' = ze^{i\theta}.\tag{9.139}$$

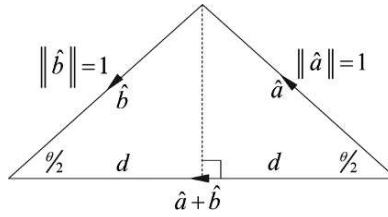


FIGURE 9.16.

Could it be that in the above scenario that  $\hat{a}$  can be rotated into  $\hat{b}$  using a similar formula? In fact, the answer is “yes”, and we can prove it as follows. Using Eq. (9.138) we have

$$\begin{aligned} R\hat{a} &= \frac{(1 + \hat{b}\hat{a})\hat{a}}{\sqrt{2(1 + \hat{b} \cdot \hat{a})}} \\ R\hat{a} &= \frac{\hat{a} + \hat{b}}{\sqrt{2(1 + \hat{b} \cdot \hat{a})}} \end{aligned} \quad (9.140)$$

and

$$\tilde{R} = \frac{1 + \hat{a}\hat{b}}{\sqrt{2(1 + \hat{b} \cdot \hat{a})}} \quad (9.141)$$

therefore

$$\begin{aligned} \hat{a}\tilde{R} &= \frac{\hat{a}(1 + \hat{a}\hat{b})}{\sqrt{2(1 + \hat{b} \cdot \hat{a})}} \\ \hat{a}\tilde{R} &= \frac{\hat{a} + \hat{b}}{\sqrt{2(1 + \hat{b} \cdot \hat{a})}} \end{aligned} \quad (9.142)$$

and

$$R\hat{a} = \hat{a}\tilde{R}. \quad (9.143)$$

Equation (9.143) confirms that pre-multiplying a vector by a rotor is equivalent to post-multiplying it by the rotor’s inverse, which leads to

$$\begin{aligned} R^2\hat{a} &= \hat{a}\tilde{R}^2 \\ &= R\hat{a}\tilde{R} \\ R^2\hat{a} &= \hat{b}. \end{aligned} \quad (9.144)$$

But we showed above in Eq. (9.81) that

$$R = \exp(-\hat{B}\theta/2) \quad (9.145)$$

where  $\hat{B}$  is the unit bivector representing the plane of rotation. Therefore, applying the rules of exponentiation to Eq. (9.144) we have

$$\begin{aligned} R^2 &= (e^{-\hat{B}\theta/2})^2 \\ &= e^{-\hat{B}\theta} \\ R^2 &= \exp(-\hat{B}\theta). \end{aligned} \tag{9.146}$$

From Eq. (9.144) we have

$$\begin{aligned} \hat{b} &= R^2 \hat{a} \\ &= \exp(-\hat{B}\theta) \hat{a} \\ &= \hat{a} e^{\hat{B}\theta} \\ \hat{b} &= \hat{a}(\cos \theta + \hat{B} \sin \theta). \end{aligned} \tag{9.147}$$

Let's illustrate this process with an example.

Figure 9.17 shows two vectors

$$\hat{a} = e_1 \quad \hat{b} = e_2 \tag{9.148}$$

which belong to the plane defined by  $\hat{B} = e_{12}$ . The separating angle is  $\pi/2$  radians. Using Eq. (9.147) we have

$$\begin{aligned} \hat{b} &= e_1 e^{e_{12}\pi/2} \\ &= e_1(\cos(\pi/2) + e_{12} \sin(\pi/2)) \\ \hat{b} &= e_1 e_{12} = e_2 \end{aligned} \tag{9.149}$$

which is correct.

Now let's try another combination, as shown in Figure 9.18 using vectors

$$\hat{a} = e_1 \quad \hat{b} = -e_1. \tag{9.150}$$

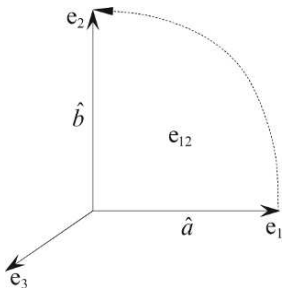


FIGURE 9.17.

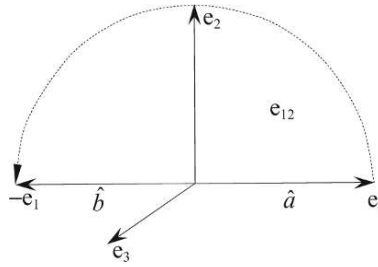


FIGURE 9.18.

Using Eq. (9.147) we have

$$\begin{aligned} \hat{b} &= e_1 e^{e_{12}\pi} \\ &= e_1 (\cos \pi + e_{12} \sin \pi) \\ \hat{b} &= -e_1 \end{aligned} \tag{9.151}$$

which is correct.

### 9.3.5 Interpolating rotors

Interpolating scalars is a trivial exercise and is readily implemented using the linear interpolant

$$s = s_1(1 - \lambda) + s_2\lambda \quad 0 \leq \lambda \leq 1. \tag{9.152}$$

And there is no reason why we cannot use the same equation for interpolating two vectors:

$$v = v_1(1 - \lambda) + v_2\lambda \quad 0 \leq \lambda \leq 1 \tag{9.153}$$

apart from the fact that the magnitude of the interpolated vector is not preserved, and could collapse to zero under some conditions. To overcome this problem a *slerp* (spherical linear interpolant) [8] is used

$$v = \frac{\sin((1 - \lambda)\theta)}{\sin \theta} v_1 + \frac{\sin(\lambda\theta)}{\sin \theta} v_2 \quad 0 \leq \lambda \leq 1 \tag{9.154}$$

where  $\theta$  is the angle between two vectors or quaternions, which preserves the integrity of their magnitude during the interpolation.

Fortunately, this *slerp* can also be used to interpolate between two rotors as follows:

$$R = \frac{\sin((1 - \lambda)\theta/2)}{\sin(\theta/2)} R_1 + \frac{\sin(\lambda\theta/2)}{\sin(\theta/2)} R_2 \quad 0 \leq \lambda \leq 1 \tag{9.155}$$

where  $\theta$  is the angle of rotation. An example will quickly reveal the action of Eq. (9.155).

Figure 9.19 shows a vector  $a = e_1$  and a plane of rotation defined by the bivector  $e_{12}$ . We will now design an interpolant that will interpolate between two rotors using the scalar  $\lambda$ , where  $0 \leq \lambda \leq 1$ .



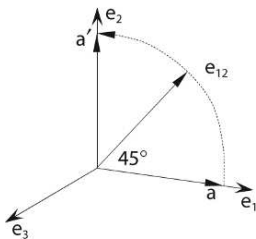


FIGURE 9.19.

We begin by defining the two rotors  $R_1$  and  $R_2$ , where  $R_1$  is the rotor locating  $a$  and  $R_2$  rotates  $a$  to  $a'$ . Using Eq. (9.85)

$$R = \cos(\theta/2) - \hat{B} \sin(\theta/2) \quad (9.156)$$

then

$$\begin{aligned} R_1 &= \cos 0^\circ - e_{12} \sin 0^\circ \\ R_1 &= 1 \end{aligned} \quad (9.157)$$

and

$$\begin{aligned} R_2 &= \cos 45^\circ - e_{12} \sin 45^\circ \\ R_2 &= \sqrt{2}/2(1 - e_{12}). \end{aligned} \quad (9.158)$$

Therefore,  $R_1$  and  $R_2$  can be substituted in Eq. (9.155) to produce

$$R = \frac{\sin((1-\lambda)45^\circ)}{\sin(45^\circ)} + \frac{\sin(\lambda 45^\circ)}{\sin(45^\circ)} \frac{\sqrt{2}(1 - e_{12})}{2}. \quad (9.159)$$

We can see from Eq. (9.159) that when  $\lambda = 0$ ,  $R_0 = 1$ , and when  $\lambda = 1$ ,

$$R_1 = \sqrt{2}/2(1 - e_{12}). \quad (9.160)$$

Using  $R_0$  to rotate vector  $a$  we have

$$a' = R_0 a \tilde{R}_0 = a \quad (9.161)$$

which is expected.

Using  $R_1$  to rotate  $a$  we have

$$\begin{aligned}
 a' &= R_1 a \tilde{R}_1 \\
 &= \sqrt{2}/2(1 - e_{12})a\sqrt{2}/2(1 + e_{12}) \\
 &= \frac{1}{2}(1 - e_{12})e_1(1 + e_{12}) \\
 a' &= \frac{1}{2}(e_1 + e_2)(1 + e_{12})
 \end{aligned}
 \tag{9.162}$$

and

$$\begin{aligned}
 a' &= \frac{1}{2}(e_1 + e_2 + e_2 - e_1) \\
 a' &= e_2
 \end{aligned}
 \tag{9.163}$$

which is also correct.

Now let's compute a half-way rotor when  $\lambda = \frac{1}{2}$ .

$$\begin{aligned}
 R_{\frac{1}{2}} &= \frac{\sin(45^\circ/2)}{\sin(45^\circ)} + \frac{\sin(45^\circ/2)}{\sin(45^\circ)} \frac{\sqrt{2}(1 - e_{12})}{2} \\
 &= \frac{\sin(45^\circ/2)}{\sin(45^\circ)} \left( 1 + \frac{\sqrt{2}(1 - e_{12})}{2} \right) \\
 R_{\frac{1}{2}} &\simeq 0.9238 - 0.3827e_{12}.
 \end{aligned}
 \tag{9.164}$$

Using  $R_{\frac{1}{2}}$  to rotate  $a$  we have

$$a' \simeq (0.9239 - 0.3827e_{12})e_1(0.9239 + 0.3827e_{12})$$

TABLE 9.1

Action	Algebra
Reflecting a vector	$a' = -\hat{n}a\hat{n}$
Reflecting a bivector	$B' = \hat{n}B\hat{n}$
Reflecting a trivector	$T' = -\hat{n}T\hat{n}$
Rotating a vector	$a' = Ra\tilde{R}$
Rotor	$R = \exp(-\hat{B}\theta/2) = \cos(\theta/2) - \hat{B} \sin(\theta/2)$
Rotor matrix	$  \llbracket R \rrbracket = \begin{bmatrix} 1 - 2(y^2 + z^2) & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & 1 - 2(x^2 + z^2) & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & 1 - 2(x^2 + y^2) \end{bmatrix}  $ <p>where <math>w^2 + z^2 = 1 - x^2 - y^2</math>.</p>
Interpolating between two rotors	$R = \frac{\sin((1 - \lambda)\theta/2)}{\sin(\theta/2)}R_1 + \frac{\sin(\lambda\theta/2)}{\sin(\theta/2)}R_2 \quad 0 \leq \lambda \leq 1$

and

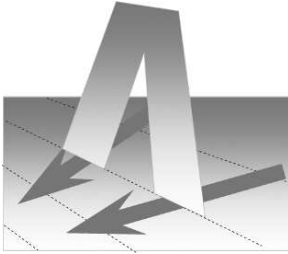
$$a' \simeq 0.7071e_1 + 0.7071e_2 \quad (9.165)$$

which shows that  $a$  has been rotated  $45^\circ$  anticlockwise.

Hopefully, the reader is convinced that the interpolant works for all other values of  $\lambda$ !

## 9.4 Summary

Reflections and rotations are one of GA's strengths and it is interesting to discover a notation that does not require an explicit matrix transform, even though one is lurking just beneath the surface. Finally, one must be extremely careful to ensure that the correct sign is used for the different blades. Table 9.1 summarizes most of the important formulae associated with reflections and rotations.



# 10 Geometric Algebra and Geometry

## 10.1 Introduction

In this chapter we explore how GA can be used to resolve simple geometric problems in computer graphics. This is not to persuade you to adopt these methods, rather demonstrate that GA can be used alongside classic vectorial techniques. The classic vectorial approach often depends upon the scalar and cross products and normal vectors to lines and planes. The GA approach is to avoid such vectors and work with the inner, outer and geometric products, bivectors and trivectors. To begin, we investigate how the outer product can be used to test whether a point is inside a triangle.

## 10.2 Point inside a triangle

### 10.2.1 Point inside a 2D triangle

Various methods exist to determine whether a point is inside the boundary of a 2D triangle, but let's examine one that employs the outer product.

Figure 10.1 shows a triangle  $\Delta P_1P_2P_3$  with a fourth point  $P_0$  that may or may not be inside the triangle's boundary. Note that the vectors  $v_{12}, v_{23}, v_{31}$  form an anticlockwise path through the points  $P_1, P_2, P_3$ .

The vectors are:

$$\begin{aligned} v_{12} &= [6 \quad 1]^T & v_{23} &= [-3 \quad 2]^T & v_{31} &= [-3 \quad -3]^T \\ v_{10} &= [3 \quad 2]^T & v_{20} &= [-3 \quad 1]^T & v_{30} &= [0 \quad -1]^T. \end{aligned} \quad (10.1)$$

We now form the outer products

$$v_{12} \wedge v_{10} \quad v_{23} \wedge v_{20} \quad v_{31} \wedge v_{30} \quad (10.2)$$

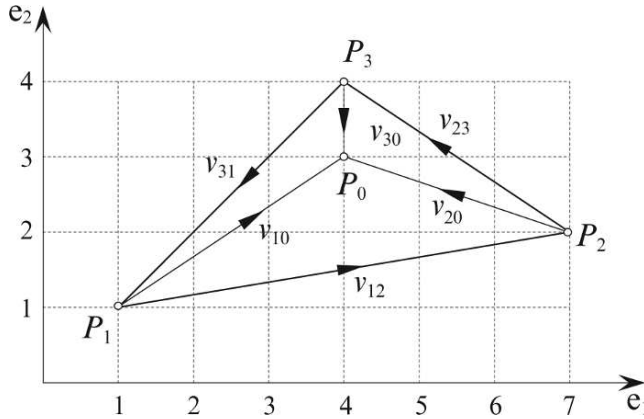


FIGURE 10.1.

and compare their signs. If they are all positive, i.e. they align with the sign of  $e_1 \wedge e_2$ ,  $P_0$  is inside the triangle's boundary. Let's compute these products for the triangle in Fig. 10.1.

$$v_{12} \wedge v_{10} = \begin{vmatrix} 6 & 1 \\ 3 & 2 \end{vmatrix} = 9 \quad (10.3)$$

$$v_{23} \wedge v_{20} = \begin{vmatrix} -3 & 2 \\ -3 & 1 \end{vmatrix} = 3 \quad (10.4)$$

$$v_{31} \wedge v_{30} = \begin{vmatrix} -3 & -3 \\ 0 & -1 \end{vmatrix} = 3 \quad (10.5)$$

and as they are all positive, the point  $P_0$  is inside the triangle's boundary.

Basically, what we are doing is computing the area of the parallelogram formed by the pairs of vectors.

Figure 10.2 shows the point  $P_0$  positioned on the boundary of the triangle, which produces the following vectors and outer products:

$$v_{10} = [2 \ 2]^T \quad v_{20} = [-4 \ 1]^T \quad v_{30} = [-1 \ -1]^T. \quad (10.6)$$

$$v_{12} \wedge v_{10} = \begin{vmatrix} 6 & 1 \\ 2 & 2 \end{vmatrix} = 10 \quad (10.7)$$

$$v_{23} \wedge v_{20} = \begin{vmatrix} -3 & 2 \\ -4 & 1 \end{vmatrix} = 5 \quad (10.8)$$

$$v_{31} \wedge v_{30} = \begin{vmatrix} -3 & -3 \\ -1 & -1 \end{vmatrix} = 0. \quad (10.9)$$

The third outer product is zero and confirms that  $P_0$  is on the boundary.

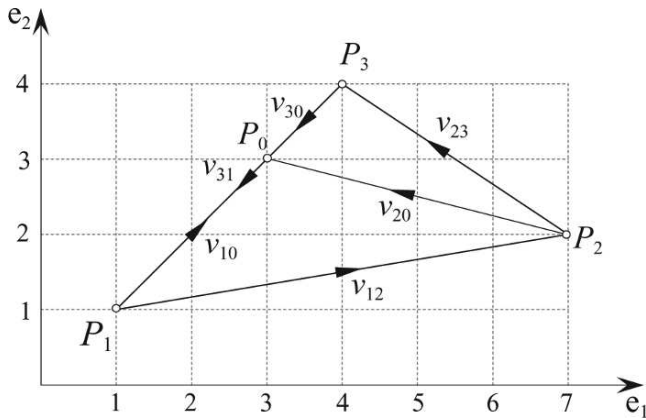


FIGURE 10.2.

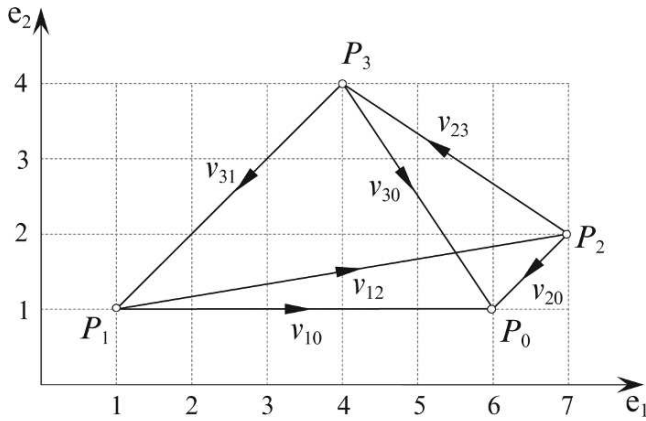


FIGURE 10.3.

Finally, Fig. 10.3 shows the point  $P_0$  outside the triangle, which produces the following vectors and outer products:

$$v_{10} = [5 \ 0]^T \quad v_{20} = [-1 \ -1]^T \quad v_{30} = [2 \ -3]^T. \tag{10.10}$$

$$v_{12} \wedge v_{10} = \begin{vmatrix} 6 & 1 \\ 5 & 0 \end{vmatrix} = -5 \tag{10.11}$$

$$v_{23} \wedge v_{20} = \begin{vmatrix} -3 & 2 \\ -1 & -1 \end{vmatrix} = 5 \tag{10.12}$$

$$v_{31} \wedge v_{30} = \begin{vmatrix} -3 & -3 \\ 2 & -3 \end{vmatrix} = 15. \tag{10.13}$$

The first outer product is negative and confirms that the point is outside the triangle.

Note that the sum of the products remains constant at 15, which is twice the triangle's area.

### 10.2.2 Point inside a 3D triangle

To prove that a point is inside a 3D triangle we use a similar technique to that used for a 2D triangle. But before we start, we must make sure that the point resides on the triangle's plane, which is accomplished by computing the trivector associated with the points.

Figure 10.4 shows a triangle  $\Delta P_1P_2P_3$  with a fourth point  $P_0$ . First, we take two vectors  $a, b$  from the triangle and associate them with another vector  $c$  to the point  $P_0$ .

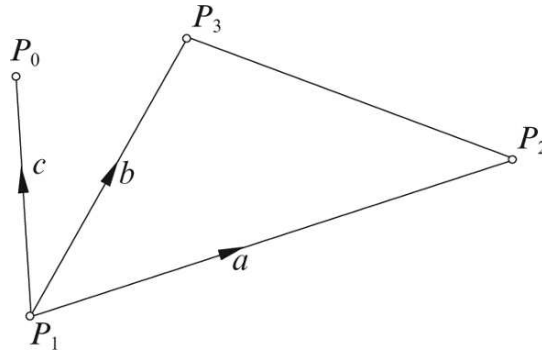


FIGURE 10.4.

The outer product  $a \wedge b$  computes the area of the parallelogram associated with the vectors  $a, b$ , which when wedged with  $c$  computes the swept volume:

$$volume = a \wedge b \wedge c \tag{10.14}$$

$$volume = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} e_{123}. \tag{10.15}$$

A zero volume implies that the point is coplanar with the bivector. If this test is passed we can proceed to the second step.

In the 2D case there was only one bivector plane to consider, however, in the 3D case we have 3 basis bivectors:  $e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1$ . To illustrate this, consider the scenario shown in Fig. 10.5.

Computing the three outer products as before:

$$v_{12} \wedge v_{10} \quad v_{23} \wedge v_{20} \quad v_{31} \wedge v_{30} \tag{10.16}$$

each one will contain three basis bivectors  $e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1$ . And as we are only interested in their signs we can write them as follows:

$$v_{12} \wedge v_{10} = \alpha_{12}e_{12} + \alpha_{23}e_{23} + \alpha_{31}e_{31} \tag{10.17}$$

$$v_{23} \wedge v_{20} = \beta_{12}e_{12} + \beta_{23}e_{23} + \beta_{31}e_{31} \tag{10.18}$$

$$v_{31} \wedge v_{30} = \chi_{12}e_{12} + \chi_{23}e_{23} + \chi_{31}e_{31}. \tag{10.19}$$

Let's illustrate this with three examples.

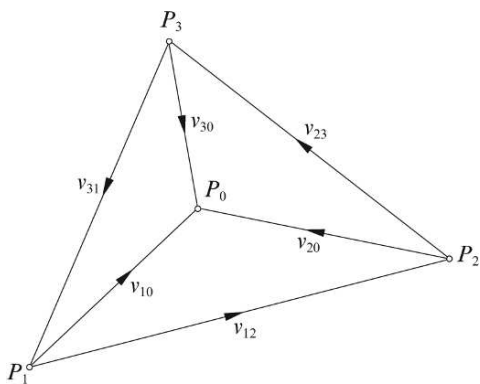


FIGURE 10.5.

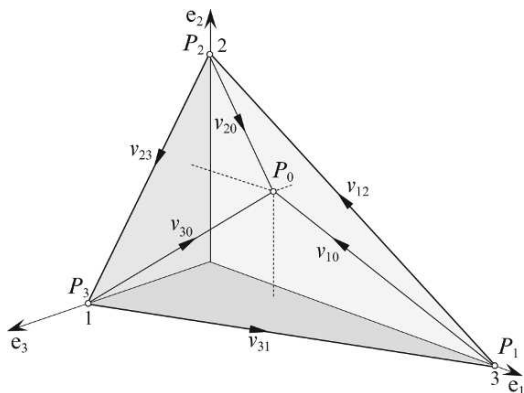


FIGURE 10.6.

Figure 10.6 shows a triangle leaning against the three axes. The points and vectors are as follows:

$$\begin{aligned}
 P_1(3, 0, 0) & & P_2(0, 2, 0) & & P_3(0, 0, 1) \\
 v_{12} = [-3 \ 2 \ 0]^T & & v_{23} = [0 \ -2 \ 1]^T & & v_{31} = [3 \ 0 \ -1]^T.
 \end{aligned}
 \tag{10.20}$$

Although we do not require the plane equation, it is useful to confirm that  $P_0(1, 1, \frac{1}{6})$  is on the plane:

$$2x + 3y + 6z - 6 = 0. \tag{10.21}$$

Next we compute the vectors  $v_{10}, v_{20}, v_{30}$ :

$$v_{10} = \left[ -2 \ 1 \ \frac{1}{6} \right]^T \quad v_{20} = \left[ 1 \ -1 \ \frac{1}{6} \right]^T \quad v_{30} = \left[ 1 \ 1 \ -\frac{5}{6} \right]^T. \tag{10.22}$$



Now we can compute the bivector coefficients:

$$\begin{aligned} v_{12} \wedge v_{10} &= (-3\mathbf{e}_1 + 2\mathbf{e}_2) \wedge \left( -2\mathbf{e}_1 + \mathbf{e}_2 + \frac{1}{6}\mathbf{e}_3 \right) \\ v_{12} \wedge v_{10} &= \mathbf{e}_{12} + \frac{1}{3}\mathbf{e}_{23} + \frac{1}{2}\mathbf{e}_{31} \end{aligned} \quad (10.23)$$

$$\begin{aligned} v_{23} \wedge v_{20} &= (-2\mathbf{e}_2 + \mathbf{e}_3) \wedge \left( \mathbf{e}_1 - \mathbf{e}_2 + \frac{1}{6}\mathbf{e}_3 \right) \\ v_{23} \wedge v_{20} &= 2\mathbf{e}_{12} + \frac{2}{3}\mathbf{e}_{23} + \mathbf{e}_{31} \end{aligned} \quad (10.24)$$

$$\begin{aligned} v_{31} \wedge v_{30} &= (3\mathbf{e}_1 - \mathbf{e}_3) \wedge \left( \mathbf{e}_1 + \mathbf{e}_2 - \frac{5}{6}\mathbf{e}_3 \right) \\ v_{31} \wedge v_{30} &= 3\mathbf{e}_{12} + \mathbf{e}_{23} + \frac{3}{2}\mathbf{e}_{31}. \end{aligned} \quad (10.25)$$

Equations (10.23), (10.24) and (10.25) identify the relevant coefficients and Table 10.1 shows them in tabular form for clarity. From the table we can see that every column has consistent signs, which confirm that the point is inside the triangle.

TABLE 10.1

bivector	$\mathbf{e}_{12}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$
$v_{12} \wedge v_{10}$	1	$\frac{1}{3}$	$\frac{1}{2}$
$v_{23} \wedge v_{20}$	2	$\frac{2}{3}$	1
$v_{31} \wedge v_{30}$	3	1	$\frac{3}{2}$

Now let's repeat the exercise for the scenario shown in Fig. 10.7. This time the test point is  $P_0(1, 1, 0)$  which is on the border and generates the vectors:

$$v_{10} = [-2 \ 1 \ 0]^T \quad v_{20} = [1 \ -1 \ 0]^T \quad v_{30} = [1 \ 1 \ -1]^T. \quad (10.26)$$

and the following outer products:

$$\begin{aligned} v_{12} \wedge v_{10} &= (-3\mathbf{e}_1 + 2\mathbf{e}_2) \wedge (-2\mathbf{e}_1 + \mathbf{e}_2) \\ v_{12} \wedge v_{10} &= \mathbf{e}_{12} \end{aligned} \quad (10.27)$$

$$\begin{aligned} v_{23} \wedge v_{20} &= (-2\mathbf{e}_2 + \mathbf{e}_3) \wedge (\mathbf{e}_1 - \mathbf{e}_2) \\ v_{23} \wedge v_{20} &= 2\mathbf{e}_{12} + \mathbf{e}_{23} + \mathbf{e}_{31} \end{aligned} \quad (10.28)$$

$$\begin{aligned} v_{31} \wedge v_{30} &= (3\mathbf{e}_1 - \mathbf{e}_3) \wedge (\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3) \\ v_{31} \wedge v_{30} &= 3\mathbf{e}_{12} + \mathbf{e}_{23} + 2\mathbf{e}_{31}. \end{aligned} \quad (10.29)$$

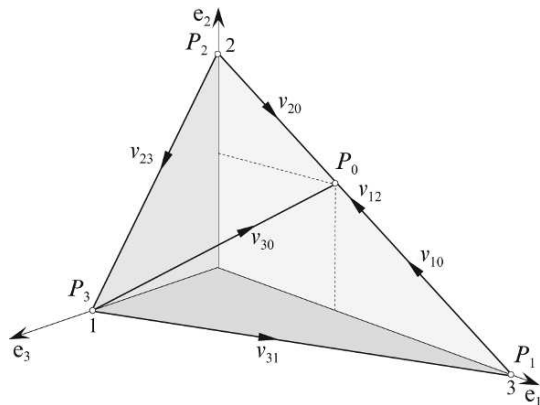


FIGURE 10.7.

The coefficients are shown in Table 10.2

TABLE 10.2

bivector	$e_{12}$	$e_{23}$	$e_{31}$
$v_{12} \wedge v_{10}$	1	0	0
$v_{23} \wedge v_{20}$	2	1	1
$v_{31} \wedge v_{30}$	3	1	2

The two zeros in the  $v_{12} \wedge v_{10}$  row confirm that  $P_0$  is on the  $v_{12}$  edge.

Finally, let's position  $P_0$  such that it is still on the plane, but outside the triangle's boundary. Figure 10.8 shows the scenario with  $P_0(2, 2, -\frac{2}{3})$  which is outside the triangle and generates the vectors:

$$v_{10} = \left[-1 \quad 2 \quad -\frac{2}{3}\right]^T \quad v_{20} = \left[2 \quad 0 \quad -\frac{2}{3}\right]^T \quad v_{30} = \left[2 \quad 2 \quad -\frac{5}{3}\right]^T. \tag{10.30}$$

and the following outer products:

$$\begin{aligned} v_{12} \wedge v_{10} &= (-3e_1 + 2e_2) \wedge \left(-e_1 + 2e_2 - \frac{2}{3}e_3\right) \\ v_{12} \wedge v_{10} &= -4e_{12} - \frac{4}{3}e_{23} - 2e_{31} \end{aligned} \tag{10.31}$$

$$\begin{aligned} v_{23} \wedge v_{20} &= (-2e_2 + e_3) \wedge \left(2e_1 - \frac{2}{3}e_3\right) \\ v_{23} \wedge v_{20} &= 4e_{12} + \frac{4}{3}e_{23} + 2e_{31} \end{aligned} \tag{10.32}$$

$$\begin{aligned} v_{31} \wedge v_{30} &= (3e_1 - e_3) \wedge \left(2e_1 + 2e_2 - \frac{5}{3}e_3\right) \\ v_{31} \wedge v_{30} &= 6e_{12} + 3e_{31} + 2e_{23} \end{aligned} \tag{10.33}$$

Once more, the bivector coefficients are shown in Table 10.3 for clarity.

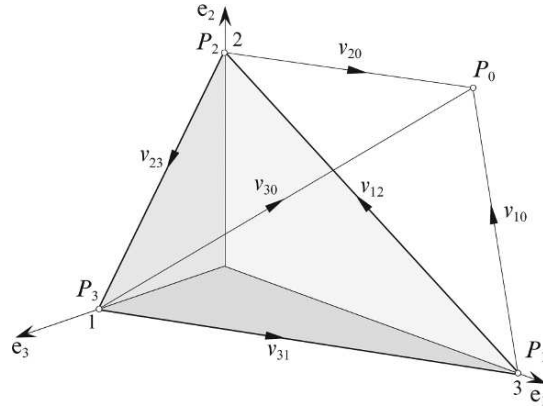


FIGURE 10.8.

TABLE 10.3

bivector	$e_{12}$	$e_{23}$	$e_{31}$
$v_{12} \wedge v_{10}$	-4	$-\frac{4}{3}$	-2
$v_{23} \wedge v_{20}$	4	$\frac{4}{3}$	2
$v_{31} \wedge v_{30}$	6	3	2

The negative signs in the  $v_{12} \wedge v_{10}$  row confirm that  $P_0$  is outside the triangle's boundary.

Note that the sum of the coefficients in each table is constant at  $11 = (6 + 2 + 3)$ , which confirms the consistency of our calculations. The area of the triangle is related to these values as follows:

$$area = \sqrt{6^2 + 2^2 + 3^2} = 7. \tag{10.34}$$

### 10.3 The relationship between bivectors and direction cosines

Direction cosines [8] are the cosines of angles between a vector and the basis unit vectors as shown in Fig. 10.9.

They are particularly useful for transforming the coordinates of a point in one axial system to another. For instance, a point  $(x, y, z)$  in  $[e_1 e_2 e_3]$  has coordinates  $(x', y', z')$  in  $[e'_1 e'_2 e'_3]$  defined by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{10.35}$$

where

- $r_{11} \ r_{12} \ r_{13}$  are the direction cosines for  $e'_1$
- $r_{21} \ r_{22} \ r_{23}$  are the direction cosines for  $e'_2$
- $r_{31} \ r_{32} \ r_{33}$  are the direction cosines for  $e'_3$ .

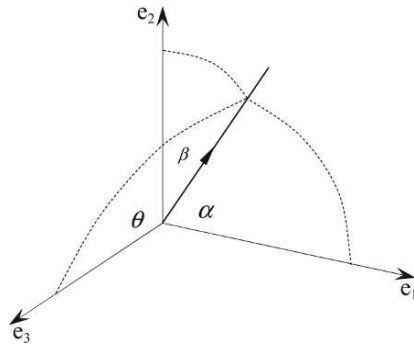


FIGURE 10.9.

As an example, consider the two axial systems shown in Fig. 10.10.

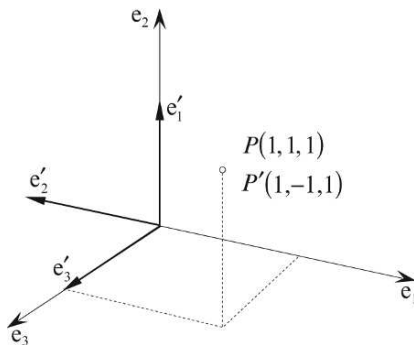


FIGURE 10.10.

The direction cosines are as shown using the following transformation:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{10.36}$$

and if we substitute the point (1, 1, 1) we obtain (1, -1, 1):

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{10.37}$$

Their relationship with bivectors is revealed using the dual operation.

With reference to Fig. 10.11 we associate three vectors with the rotated axes as follows:

$$v'_1 = e_2 \quad v'_2 = -e_1 \quad v'_3 = e_3 \tag{10.38}$$

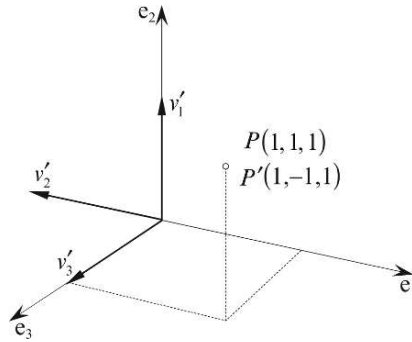


FIGURE 10.11.

and form the following outer products:

$$v'_1 \wedge v'_2 = e_2 \wedge (-e_1) = e_{12} \tag{10.39}$$

$$v'_2 \wedge v'_3 = (-e_1) \wedge e_3 = e_{31} \tag{10.40}$$

$$v'_3 \wedge v'_1 = e_3 \wedge e_2 = -e_{23}. \tag{10.41}$$

Each outer product contains only one basis bivector, the other basis bivectors are zero, which is due to the example chosen. But if we rewrite the products and include the zero coefficients we obtain:

$$v'_1 \wedge v'_2 = 1e_{12} + 0e_{23} + 0e_{31} \tag{10.42}$$

$$v'_2 \wedge v'_3 = 0e_{12} + 0e_{23} + 1e_{31} \tag{10.43}$$

$$v'_3 \wedge v'_1 = 0e_{12} - 1e_{23} + 0e_{31}. \tag{10.44}$$

For clarity, Table 10.4 shows the bivector coefficients:

TABLE 10.4

bivector	$e_{12}$	$e_{23}$	$e_{31}$
$v'_1 \wedge v'_2$	1	0	0
$v'_2 \wedge v'_3$	0	0	1
$v'_3 \wedge v'_1$	0	-1	0

We have nine coefficients which must have some relationship with the direction cosine transformation, but they appear to be in the wrong order! However, if we take the dual of all the elements and put them in the conventional order as shown in Table 10.5, this reveals the direction cosine transformation.

TABLE 10.5

bivector	$e_1 = (e_{23})^*$	$e_2 = (e_{31})^*$	$e_3 = (e_{12})^*$
$v'_1 = (v'_2 \wedge v'_3)^*$	0	1	0
$v'_2 = (v'_3 \wedge v'_1)^*$	-1	0	0
$v'_3 = (v'_1 \wedge v'_2)^*$	0	0	1

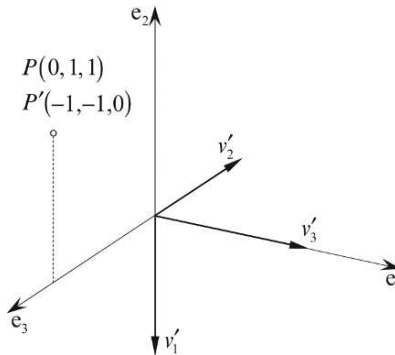


FIGURE 10.12.

To demonstrate this once more, consider the scenario shown in Fig.10.12. The vectors representing the new axial system are:

$$v'_1 = -e_2 \quad v'_2 = -e_3 \quad v'_3 = e_1 \tag{10.45}$$

and their outer products are

$$v'_1 \wedge v'_2 = 0e_{12} + 1e_{23} + 0e_{31} \tag{10.46}$$

$$v'_2 \wedge v'_3 = 0e_{12} + 0e_{23} - 1e_{31} \tag{10.47}$$

$$v'_3 \wedge v'_1 = -1e_{12} + 0e_{23} + 0e_{31}. \tag{10.48}$$

Table 10.6 shows the individual coefficients, which when subjected to the dual operation reveals the following direction cosine transform:

$$\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{10.49}$$

TABLE 10.6

bivector	$e_{12}$	$e_{23}$	$e_{31}$
$v'_1 \wedge v'_2$	0	1	0
$v'_2 \wedge v'_3$	0	0	-1
$v'_3 \wedge v'_1$	-1	0	0

There is no need to invoke the dual operation as the transform comes direct from the outer products:

$$v'_1 \wedge v'_2 = r_{33}e_{12} + r_{31}e_{23} + r_{32}e_{31} \quad (10.50)$$

$$v'_2 \wedge v'_3 = r_{13}e_{12} + r_{11}e_{23} + r_{12}e_{31} \quad (10.51)$$

$$v'_3 \wedge v'_1 = r_{23}e_{12} + r_{21}e_{23} + r_{22}e_{31}. \quad (10.52)$$

## 10.4 Lines and planes

### 10.4.1 Relative orientation of a point and a line

The standard parametric equation for a straight line is

$$p = t + \lambda v \quad (10.53)$$

where

$p$  points to any point  $P$  on the line

$t$  points to a known point  $T$  on the line

$v$  is the line's direction vector, and

$\lambda \in \mathbb{R}$ .

Figure 10.13 shows this scenario.

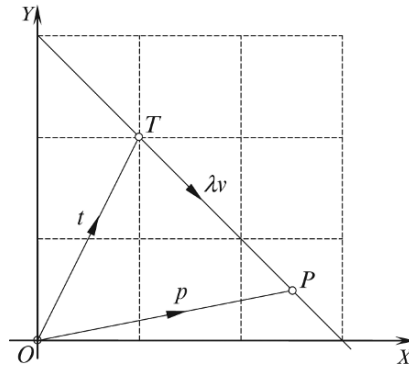


FIGURE 10.13.

A GA approach using the outer product allows us to determine whether points are on or off the line. Fig. 10.14 shows two known points  $T$  and  $P$  and the associated direction vector  $v$  where

$$v = p - t. \quad (10.54)$$

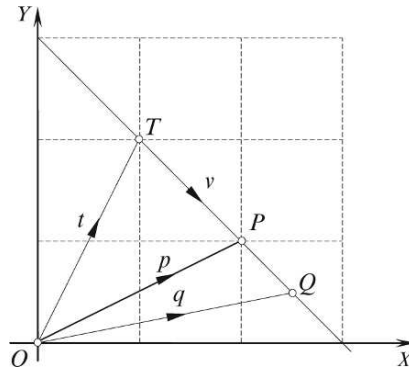


FIGURE 10.14.

Multiplying Eq. (10.54) by  $v$  using the outer product, we obtain

$$v \wedge v = v \wedge (p - t) \tag{10.55}$$

and as  $v \wedge v = 0$ , then

$$v \wedge (p - t) = 0. \tag{10.56}$$

This conclusion could have easily been realized without the algebra.

Equation (10.56) can now be used to determine whether another point  $Q$  is on the line:

$$v \wedge (q - t) = 0 \tag{10.57}$$

Using the following points:

$$T(-1, 1) \quad P(1, 2) \quad Q(3, 3) \tag{10.58}$$

and the associated vectors:

$$t = -e_1 + e_2 \tag{10.59}$$

$$p = e_1 + 2e_2 \tag{10.60}$$

$$v = p - t = 2e_1 + e_2 \tag{10.61}$$

$$q = 3e_1 + 3e_2 \tag{10.62}$$

then

$$\begin{aligned} v \wedge (q - t) &= (2e_1 + e_2) \wedge ((3e_1 + 3e_2) - (-e_1 + e_2)) \\ &= (2e_1 + e_2) \wedge (4e_1 + 2e_2) \\ &= 4e_{12} + 4e_{21} = 4e_{12} - 4e_{12} \\ v \wedge (q - t) &= 0. \end{aligned} \tag{10.63}$$

The zero result confirms that  $Q$  is on the line, as shown in Fig. 10.15. Although this is an example in  $\mathbb{R}^2$ , it obviously applies to  $\mathbb{R}^3$ .



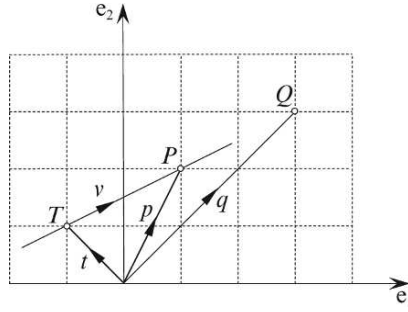


FIGURE 10.15.

The following examples, however, apply only to  $\mathbb{R}^2$ , and concern the orientation of a point relative to a line. For example, let's change the location of  $Q$  to  $(3, 2)$  as shown in Fig. 10.16.

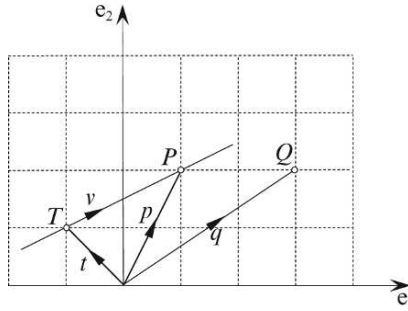


FIGURE 10.16.

Equation (10.56) can be used to determine the orientation of  $Q$  relative to the line, i.e., whether  $Q$  is to the right or left, relative to the line's direction. Substituting the new value of  $q$  in Eq. (10.57) we obtain

$$\begin{aligned}
 v \wedge (q - t) &= (2e_1 + e_2) \wedge ((3e_1 + 2e_2) - (-e_1 + e_2)) \\
 &= (2e_1 + e_2) \wedge (4e_1 + e_2) \\
 v \wedge (q - t) &= 2e_{12} - 4e_{12} = -2e_{12}.
 \end{aligned}
 \tag{10.64}$$

The negative result confirms that the orientation of the blade  $v \wedge (q - t)$  opposes the reference blade  $e_1 \wedge e_2$ , and is to the right of the line moving from  $T$  to  $P$ , whilst the value of 2 represents twice the area of the triangle  $\Delta TPQ$ .

For completeness, let's move  $Q$  to  $(2, 3)$  which is to the left of the line:

$$\begin{aligned}
 v \wedge (q - t) &= (2e_1 + e_2) \wedge ((2e_1 + 3e_2) - (-e_1 + e_2)) \\
 &= (2e_1 + e_2) \wedge (3e_1 + 2e_2) \\
 &= 4e_{12} - 3e_{12} \\
 v \wedge (q - t) &= e_{12}.
 \end{aligned}
 \tag{10.65}$$

The positive result confirms that the orientation of the blade  $v \wedge (q - t)$  matches the reference blade  $e_1 \wedge e_2$ , and is to the left of the line moving from  $T$  to  $P$ , whilst the value 1 represents twice the area of the triangle  $\Delta TPQ$ .

### 10.4.2 Relative orientation of a point and a plane

Having solved the relative orientation of a point and a line, we now discover that a similar solution resolves the relationship between a point and a plane. Figure 10.17 shows the scenario where a plane defined by the blade  $A$  intersects the points  $T$  and  $P$ , which allows us to write

$$v = p - t. \tag{10.66}$$

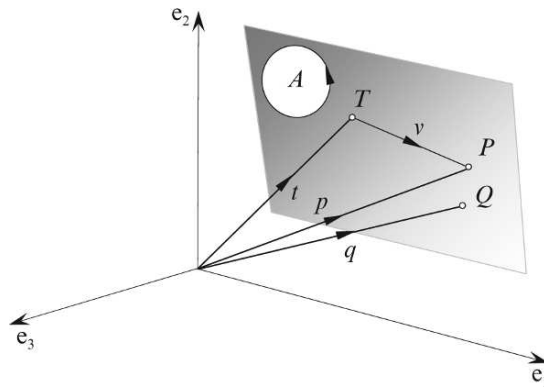


FIGURE 10.17.

Multiplying Eq. (10.66) by  $A$  using the outer product, we obtain

$$A \wedge v = A \wedge (p - t) \tag{10.67}$$

and because  $v$  is parallel to the bivector,  $A \wedge v = 0$ , and

$$A \wedge (p - t) = 0. \tag{10.68}$$

Equation (10.68) can also be used to determine whether another point  $Q$  is on the plane:

$$A \wedge (q - t) = 0. \tag{10.69}$$

Let's test Eq. (10.69) with the scenario shown in Fig. 10.18.

Given

$$a = e_1 - e_3 \tag{10.70}$$

$$b = e_2 - e_3 \tag{10.71}$$

$$t = e_3 \tag{10.72}$$

$$q = \frac{1}{2}e_1 + \frac{1}{2}e_2 \tag{10.73}$$

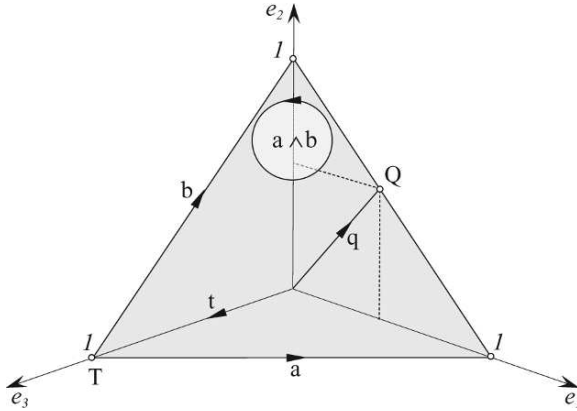


FIGURE 10.18.

then the bivector  $A$  is

$$\begin{aligned}
 A &= a \wedge b \\
 &= (e_1 - e_3) \wedge (e_2 - e_3) \\
 &= e_{12} - e_{13} - e_{32} \\
 A &= e_{12} + e_{23} + e_{31}.
 \end{aligned} \tag{10.74}$$

Therefore,

$$\begin{aligned}
 A \wedge (q - t) &= (e_{12} + e_{23} + e_{31}) \wedge \left( \left( \frac{1}{2}e_1 + \frac{1}{2}e_2 \right) - e_3 \right) \\
 &= -e_{123} + \frac{1}{2}e_{231} + \frac{1}{2}e_{312} \\
 &= -e_{123} + \frac{1}{2}e_{123} + \frac{1}{2}e_{123} \\
 A \wedge (q - t) &= 0.
 \end{aligned} \tag{10.75}$$

The zero result confirms that  $Q$  is on the blade  $A$ .

Now let's move  $Q$  to  $(1, 1, 0)$  which is clearly above the plane:

$$\begin{aligned}
 A \wedge (q - t) &= (e_{12} + e_{23} + e_{31}) \wedge ((e_1 + e_2) - e_3) \\
 &= -e_{123} + e_{231} + e_{312} \\
 &= -e_{123} + e_{123} + e_{123} \\
 A \wedge (q - t) &= e_{123}.
 \end{aligned} \tag{10.76}$$

The positive result confirms that  $Q$  is above the blade.

Finally, let's move  $Q$  to  $(0, 0, 0)$  which is clearly below the plane:

$$\begin{aligned} A \wedge (q - t) &= (e_{12} + e_{23} + e_{31}) \wedge (-e_3) \\ A \wedge (q - t) &= -e_{123}. \end{aligned} \quad (10.77)$$

The negative result confirms that  $Q$  is below the blade. Thus we have a simple mechanism for determining the orientation of a point relative to a plane.

### 10.4.3 Shortest distance from a point to a plane

Deriving the equation for the shortest distance from a point to a plane using vector analysis is determined by the definition of the plane. For example, using the plane equation

$$ax + by + cz = \delta \quad (10.78)$$

where the unit surface normal is

$$\hat{n} = ai + bj + ck$$

$\delta$  is the perpendicular distance from the origin to the plane

$p$  is the position vector for the given point  $P$ , and

$q$  is the position vector for the nearest point  $Q$  on the plane,

then  $q$  is given by

$$q = p - ((\hat{n} \cdot p) - \delta)\hat{n}, \quad (10.79)$$

and the distance between  $P$  and  $Q$  is  $\|p - q\|$ .

However, if we define the plane using a unit normal vector  $\hat{n}$ , and a point  $T$  on the plane with position vector  $t$ , then  $q$  is given by

$$q = p - \hat{n} \cdot (p - t)\hat{n}. \quad (10.80)$$

Before proceeding with a GA solution, let's demonstrate that Eqs. (10.79) and (10.80) give identical results. For example, Fig. 10.19 shows a plane, whose equation is

$$x + y + z = 1 \quad (10.81)$$

therefore

$$\hat{n} = \frac{\sqrt{3}}{3}(i + j + k) \quad (10.82)$$

and

$$\delta = \frac{\sqrt{3}}{3}. \quad (10.83)$$

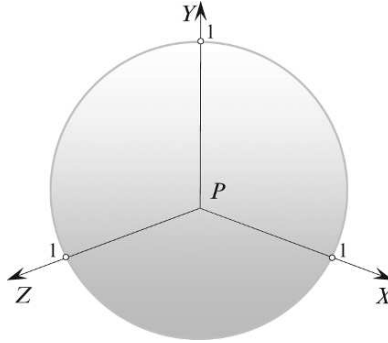


FIGURE 10.19.

If we let the point  $P$  be  $(1, 1, 1)$ , then

$$\begin{aligned}
 q &= p - ((\hat{n} \cdot p) - \delta)\hat{n} \\
 &= (i + j + k) - \left( \frac{\sqrt{3}}{3}(i + j + k) \cdot (i + j + k) - \frac{\sqrt{3}}{3} \right) \frac{\sqrt{3}}{3}(i + j + k) \\
 &= (i + j + k) - \left( \sqrt{3} - \frac{\sqrt{3}}{3} \right) \frac{\sqrt{3}}{3}(i + j + k) \\
 q &= \frac{1}{3}(i + j + k). \tag{10.84}
 \end{aligned}$$

The distance between  $P$  and  $Q$  is

$$\begin{aligned}
 \|p - q\| &= \left\| (i + j + k) - \frac{1}{3}(i + j + k) \right\| \\
 \|p - q\| &= \frac{2}{3}\sqrt{3}. \tag{10.85}
 \end{aligned}$$

which is correct.

Using Eq. (10.80), the plane's unit normal vector is

$$\hat{n} = \frac{\sqrt{3}}{3}(i + j + k) \tag{10.86}$$

with the point  $T$  on the plane equal to  $(1, 0, 0)$ . Therefore,

$$\begin{aligned}
 q &= p - \hat{n} \cdot (p - t)\hat{n} \\
 &= (i + j + k) - \left( \frac{\sqrt{3}}{3}(i + j + k) \cdot (i + j + k - i) \right) \frac{\sqrt{3}}{3}(i + j + k)
 \end{aligned}$$

$$\begin{aligned}
 &= (i + j + k) - \frac{2}{3}(i + j + k) \\
 q &= \frac{1}{3}(i + j + k) \tag{10.87}
 \end{aligned}$$

which gives the same distance as above.

Now let's investigate a GA approach using the outer product.

Figure 10.20 shows a blade  $A$  containing two points  $T$  and  $Q$  with their respective position vectors  $t$  and  $q$ .  $Q$  is the point on the blade nearest to the point  $P$  which makes the vector  $d = q - p$  perpendicular to  $A$ .

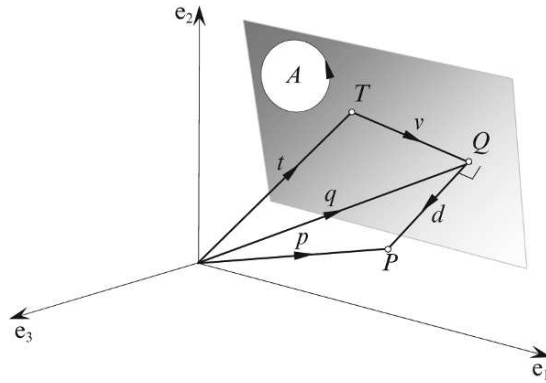


FIGURE 10.20.

From Fig. 10.20, we see that

$$d = p - t - v \tag{10.88}$$

and

$$A \wedge v = 0 \tag{10.89}$$

and that  $\|d\|$  is the distance of  $P$  to the blade. So, the objective is to secure a value for  $d$ , which is achieved using the geometric product  $Ad$ :

$$Ad = A \cdot d + A \wedge d. \tag{10.90}$$

But as  $d$  is perpendicular to  $A$ ,  $A \cdot d = 0$ , therefore,

$$Ad = A \wedge d. \tag{10.91}$$

To reveal  $d$ , we pre-multiply both sides by  $A^{-1}$

$$A^{-1}Ad = A^{-1}(A \wedge d) \tag{10.92}$$

therefore,

$$d = A^{-1}(A \wedge d). \tag{10.93}$$

Substituting Eq. (10.88) in Eq. (10.93) we obtain

$$d = A^{-1}(A \wedge (p - t - v)). \tag{10.94}$$

Expanding

$$d = A^{-1}(A \wedge p - A \wedge t - A \wedge v) \tag{10.95}$$

and

$$d = A^{-1}(A \wedge (p - t)). \tag{10.96}$$

Let's test Eq. (10.96) with the previous example, as shown in Fig. 10.21.

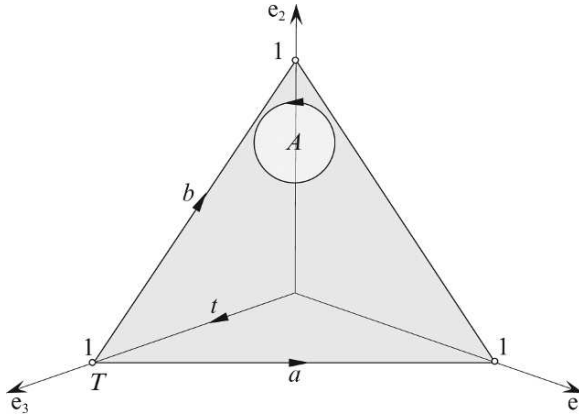


FIGURE 10.21.

Given the following vectors

$$a = e_1 - e_3 \tag{10.97}$$

$$b = e_2 - e_3 \tag{10.98}$$

$$t = e_3 \tag{10.99}$$

then

$$\begin{aligned} A &= a \wedge b \\ &= (e_1 - e_3) \wedge (e_2 - e_3) \\ A &= e_{12} - e_{13} - e_{32}. \end{aligned} \tag{10.100}$$

The inverse of  $A$  is determined as follows:

$$\begin{aligned} A^{-1} &= -\frac{A}{\|A\|^2} \\ &= -\frac{e_{12} - e_{13} - e_{32}}{3} \\ A^{-1} &= \frac{1}{3}(e_{13} + e_{32} - e_{12}). \end{aligned} \tag{10.101}$$

Let the point  $T$  be  $(0, 0, 1)$  and  $P$  be  $(1, 1, 1)$ , therefore,

$$t = e_3 \quad (10.102)$$

and

$$p = e_1 + e_2 + e_3. \quad (10.103)$$

Now we can compute  $d$ :

$$\begin{aligned} d &= A^{-1}(A \wedge (p - t)) \\ &= \frac{1}{3}(e_{13} + e_{32} - e_{12})((e_{12} - e_{13} - e_{32}) \wedge (e_1 + e_2 + e_3 - e_3)) \\ &= \frac{1}{3}(e_{13} + e_{32} - e_{12})(-e_{132} - e_{321}) \\ &= -\frac{2}{3}(e_{13} + e_{32} - e_{12})e_{132} \\ &= -\frac{2}{3}(e_{13132} + e_{32132} - e_{12132}) \\ &= -\frac{2}{3}(-e_2 - e_1 - e_3) \\ d &= \frac{2}{3}(e_1 + e_2 + e_3). \end{aligned} \quad (10.104)$$

Therefore, the distance from  $P$  to the blade is  $\|d\| = \frac{2}{3}\sqrt{3}$ , which is correct.

The point  $Q$  is given by

$$\begin{aligned} q &= p - d \\ q &= (e_1 + e_2 + e_3) - \frac{2}{3}(e_1 + e_2 + e_3) \end{aligned} \quad (10.105)$$

and

$$q = \frac{1}{3}(e_1 + e_2 + e_3) \quad (10.106)$$

which makes  $Q(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ .

#### 10.4.4 A line intersecting a plane

A classic vectorial approach to solve the intersection between a line and a plane is to define the line parametrically as follows:

$$p = t + \lambda v \quad (10.107)$$

where

$p$  is the position vector for any point  $P$  on the line

$t$  is the position vector for a point  $T$  on the line

$v$  is the line's direction vector, and

$\lambda \in \mathbb{R}$ .



The plane is defined using

$$ax + by + cz = \delta \quad (10.108)$$

where the normal vector is

$$n = ai + bj + ck, \text{ and} \quad (10.109)$$

$\delta$  is the perpendicular distance from the origin to the plane

when  $\|n\| = 1$ .

The position vector  $p$  for the intersection point  $P$  is given by

$$p = t + \left( \frac{\delta - n \cdot t}{n \cdot v} \right) v. \quad (10.110)$$

For example, consider the scenario shown in Fig. 10.22.

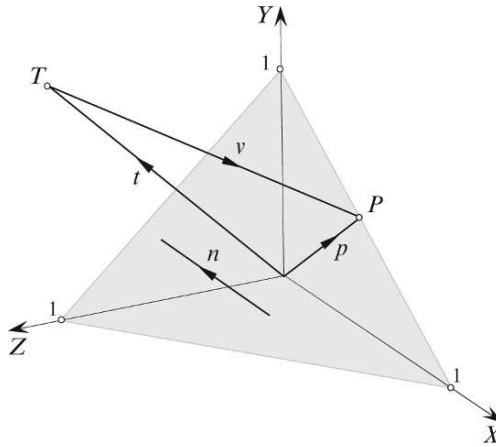


FIGURE 10.22.

The plane equation is

$$x + y + z = 1 \quad (10.111)$$

therefore,

$$n = i + j + k \quad (10.112)$$

and

$$\delta = 1. \quad (10.113)$$

The point  $T$  is  $(0, 1, 1)$  and  $t = j + k$ .

The line's direction vector is

$$v = \frac{1}{2}i - \frac{1}{2}j - k. \quad (10.114)$$

Therefore,

$$\begin{aligned}
 p &= t + \left( \frac{\delta - n \cdot t}{n \cdot v} \right) v \\
 &= (j+k) + \left( \frac{1 - (i+j+k) \cdot (j+k)}{(i+j+k) \cdot (\frac{1}{2}i - \frac{1}{2}j - k)} \right) \left( \frac{1}{2}i - \frac{1}{2}j - k \right) \\
 &= (j+k) + \left( \frac{-1}{-1} \right) \left( \frac{1}{2}i - \frac{1}{2}j - k \right) \\
 p &= \frac{1}{2}i + \frac{1}{2}j
 \end{aligned} \tag{10.115}$$

which makes  $Q(\frac{1}{2}, \frac{1}{2}, 0)$ .

Now let's explore a GA approach to this problem. Figure 10.23 shows a blade  $A$  intersected by a line represented by the parametric equation

$$p = t + \lambda v. \tag{10.116}$$

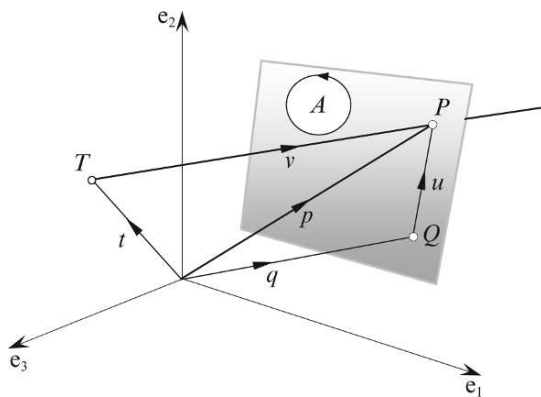


FIGURE 10.23.

The 2-blade is represented by  $A$  and a known point  $Q$  on the blade. From Fig. 10.23 we can see that

$$u = p - q \tag{10.117}$$

and

$$A \wedge u = 0 \tag{10.118}$$

therefore,

$$A \wedge (p - q) = 0. \tag{10.119}$$

Substituting Eq. (10.116) in Eq. (10.119) we have

$$\begin{aligned}
 A \wedge (t + \lambda v - q) &= 0 \\
 A \wedge (t - q) + A \wedge \lambda v &= 0
 \end{aligned} \tag{10.120}$$

therefore,

$$\lambda = \frac{A \wedge (q - t)}{A \wedge v} \quad (10.121)$$

and

$$p = t + \left( \frac{A \wedge (q - t)}{A \wedge v} \right) v. \quad (10.122)$$

Let's test Eq. (10.122) using the example above as shown in Fig. 10.24.

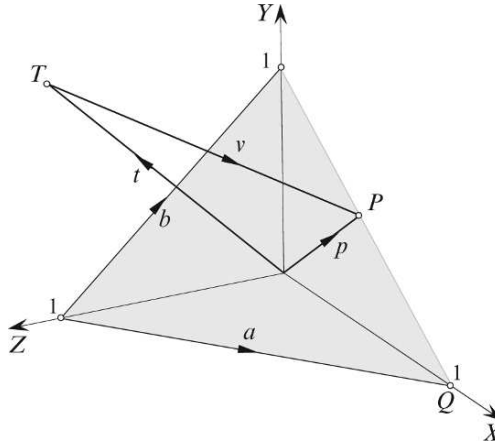


FIGURE 10.24.

The vectors are as follows:

$$q = e_1 \quad (10.123)$$

$$a = e_1 - e_3 \quad (10.124)$$

$$b = e_2 - e_3 \quad (10.125)$$

then

$$\begin{aligned} A &= a \wedge b \\ &= (e_1 - e_3) \wedge (e_2 - e_3) \\ A &= e_{12} - e_{13} - e_{32}. \end{aligned} \quad (10.126)$$

The point  $T$  is  $(0, 1, 1)$  and  $t = e_2 + e_3$ .

The point  $Q$  is  $(1, 0, 0)$  and  $q = e_1$ .

The line's direction vector is

$$v = \frac{1}{2}e_1 - \frac{1}{2}e_2 - e_3. \quad (10.127)$$

Therefore,

$$\begin{aligned}
 p &= t + \left( \frac{A \wedge (q - t)}{A \wedge v} \right) v \\
 &= (e_2 + e_3) + \left( \frac{(e_{12} - e_{13} - e_{32}) \wedge (e_1 - e_2 - e_3)}{(e_{12} - e_{13} - e_{32}) \wedge (\frac{1}{2}e_1 - \frac{1}{2}e_2 - e_3)} \right) \left( \frac{1}{2}e_1 - \frac{1}{2}e_2 - e_3 \right) \\
 &= (e_2 + e_3) + \left( \frac{-e_{123} + e_{132} - e_{321}}{-e_{123} + \frac{1}{2}e_{132} - \frac{1}{2}e_{321}} \right) \left( \frac{1}{2}e_1 - \frac{1}{2}e_2 - e_3 \right) \\
 &= (e_2 + e_3) + \left( \frac{1}{2}e_1 - \frac{1}{2}e_2 - e_3 \right) \\
 p &= \frac{1}{2}e_1 + \frac{1}{2}e_2
 \end{aligned} \tag{10.128}$$

which makes  $Q(\frac{1}{2}, \frac{1}{2}, 0)$ .

### 10.5 Perspective projection

Calculating the perspective projection of an object is relatively easy and normally only requires the use of similar triangles. For example, Fig. 10.25 shows a typical scenario where the observer is positioned at the origin gazing along the z-axis. The image plane is located orthogonal to the gaze direction and a distance  $d$  from the origin.

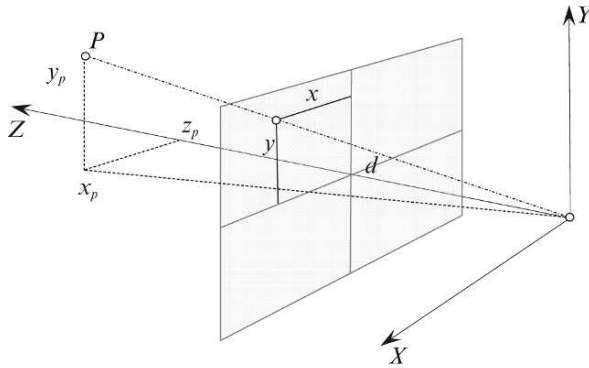


FIGURE 10.25.

We can now make the following observations:

$$\frac{x_p}{x} = \frac{z_p}{d} \tag{10.129}$$

and

$$\frac{y_p}{y} = \frac{z_p}{d} \tag{10.130}$$

which leads to:

$$x = d \frac{x_p}{z_p} \quad \text{and} \quad y = d \frac{y_p}{z_p}. \quad (10.131)$$

Figure 10.26 shows how we would approach the problem using vectors.

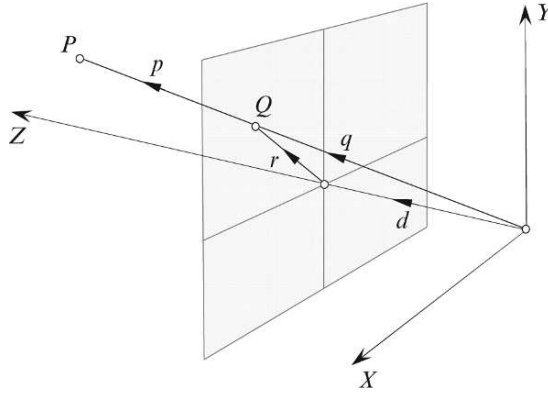


FIGURE 10.26.

With reference to Fig. 10.26 we can state that

$$q = d + r \quad (10.132)$$

and

$$q = \lambda p \quad (10.133)$$

therefore,

$$\lambda p = d + r. \quad (10.134)$$

Multiplying Eq. (10.134) by  $d$  using the inner product we obtain:

$$\begin{aligned} \lambda d \cdot p &= d \cdot (d + r) \\ \lambda d \cdot p &= d \cdot d + d \cdot r. \end{aligned} \quad (10.135)$$

But as  $d \cdot r = 0$

$$\lambda d \cdot p = d \cdot d \quad (10.136)$$

and

$$\lambda = \frac{d \cdot d}{d \cdot p} \quad (10.137)$$

therefore,

$$q = \left( \frac{d \cdot d}{d \cdot p} \right) p. \quad (10.138)$$

Now let's see how we could employ the outer product.

As we are dealing with a projection onto a plane, we can treat the plane as a 2-blade and represent it as

$$A = e_1 \wedge e_2 = e_{12}. \quad (10.139)$$

Therefore,

$$A \wedge r = A \wedge (q - d) = 0 \quad (10.140)$$

$$A \wedge q = A \wedge d = \lambda A \wedge p \quad (10.141)$$

$$\lambda = \frac{A \wedge d}{A \wedge p} \quad (10.142)$$

and

$$q = \left( \frac{A \wedge d}{A \wedge p} \right) p. \quad (10.143)$$

Now let's test Eqs. (10.138) and (10.143) with an example.

Given a point  $P(10, 10, 20)$  and  $d = 10$  we have, using Eq. (10.138)

$$\begin{aligned} q &= \left( \frac{d \cdot d}{d \cdot p} \right) p \\ &= \frac{10\mathbf{k} \cdot 10\mathbf{k}}{10\mathbf{k} \cdot (10\mathbf{i} + 10\mathbf{j} + 20\mathbf{k})} (10\mathbf{i} + 10\mathbf{j} + 20\mathbf{k}) \\ &= \frac{100}{200} (10\mathbf{i} + 10\mathbf{j} + 20\mathbf{k}) \\ q &= 5\mathbf{i} + 5\mathbf{j} + 10\mathbf{k} \end{aligned} \quad (10.144)$$

giving

$$Q(5, 5, 10).$$

Or using Eq. (10.143)

$$\begin{aligned} q &= \left( \frac{A \wedge d}{A \wedge p} \right) p \\ &= \left( \frac{e_{12} \wedge 10e_3}{e_{12} \wedge (10e_1 + 10e_2 + 20e_3)} \right) (10e_1 + 10e_2 + 20e_3) \\ &= \frac{10e_{123}}{20e_{123}} (10e_1 + 10e_2 + 20e_3) \\ q &= 5e_1 + 5e_2 + 10e_3 \end{aligned} \quad (10.145)$$

giving

$$Q(5, 5, 10).$$

Now let's explore another approach that introduces homogeneous coordinates.

Figure 10.27 shows the point  $P$ 's position vector  $p$  intersecting an image plane orthogonal to the gaze vector  $n$ . The point of intersection  $Q$  is determined as follows:

$$n + q = \lambda p. \quad (10.146)$$

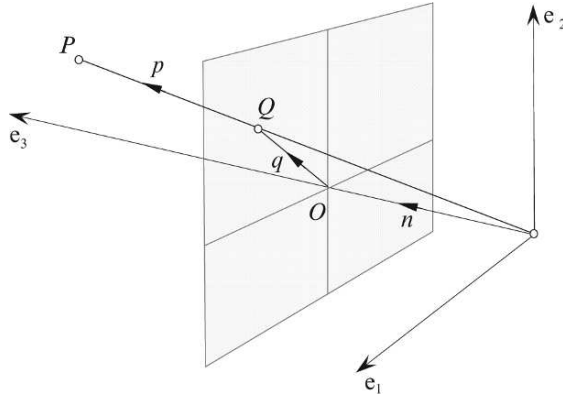


FIGURE 10.27.

Multiplying Eq. (10.146) by  $n$  using the inner product we obtain

$$n \cdot n + q \cdot n = \lambda p \cdot n \quad (10.147)$$

but as  $q \cdot n = 0$

$$\lambda = \frac{n^2}{p \cdot n} \quad (10.148)$$

therefore,

$$\begin{aligned} q &= \frac{n^2 p}{p \cdot n} - n \\ q &= \frac{n^2 p - (p \cdot n)n}{p \cdot n}. \end{aligned} \quad (10.149)$$

If we make  $n^2 = 1$ , then

$$\lambda = \frac{1}{p \cdot n} \quad (10.150)$$

and

$$q = \frac{p - (p \cdot n)n}{p \cdot n}. \quad (10.151)$$

Using the geometric product

$$p \cdot n = pn - p \wedge n \quad (10.152)$$

therefore,

$$\begin{aligned} q &= \frac{p - (pn - p \wedge n)n}{p \cdot n} \\ &= \frac{p - pnn + (p \wedge n)n}{p \cdot n} \end{aligned}$$

and

$$q = \left( \frac{p \wedge n}{p \cdot n} \right) n. \tag{10.153}$$

Equation (10.153) contains a 2-blade  $B$ , where

$$B = \frac{p \wedge n}{p \cdot n} \tag{10.154}$$

and a vector  $n$ . The bivector represents the 2-D line  $OQ$  on the image plane, and if we set the vector  $n = e_3$ , we establish an orthogonal axial system.

Now, say for example

$$p = p_1 e_1 + p_2 e_2 + p_3 e_3 \tag{10.155}$$

then

$$\begin{aligned} B &= \frac{(p_1 e_1 + p_2 e_2 + p_3 e_3) \wedge e_3}{(p_1 e_1 + p_2 e_2 + p_3 e_3) \cdot e_3} \\ &= \frac{p_1 e_{13} + p_2 e_{23}}{p_3} \\ B &= \frac{p_1}{p_3} e_{13} + \frac{p_2}{p_3} e_{23}. \end{aligned} \tag{10.156}$$

Substituting Eq. (10.156) in Eq. (10.153) we have

$$\begin{aligned} q &= \left( \frac{p_1}{p_3} e_{13} + \frac{p_2}{p_3} e_{23} \right) e_3 \\ q &= \frac{p_1}{p_3} e_1 + \frac{p_2}{p_3} e_2 \end{aligned} \tag{10.157}$$

which confirms the previous results.

Rather innocently, we have stumbled across the idea of homogeneous coordinates, which we expand upon in the next section. But before we do, let's explore how the outer product can be used for back-face removal.

## 10.6 Back-face removal

To minimize processing time, back-facing polygons are sometimes removed prior to rendering. A traditional technique for achieving this is to compute the angle between a polygon's normal vector  $n$  and the vector back to the observer  $v$ , as shown in Fig. 10.28.

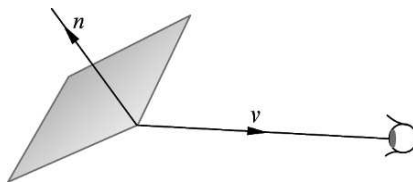


FIGURE 10.28.



In fact, by computing the dot product  $n \cdot v$ , we can discover if the angle between  $n$  and  $v$  equals or exceeds  $90^\circ$ . Thus if  $n \cdot v \leq 0$  the polygon is invisible to the observer.

Another way is to choose two edges as vectors and project them onto the image plane. The sign of the projected outer product determines whether the convex polygon is forward- or back-facing. For example, Fig. 10.29 shows a convex polygon whose vertices are anticlockwise when forward facing, therefore, by definition, the outer product  $v_1 \wedge v_2$  is positive. However, as the polygon is facing away from the observer, the outer product  $v'_1 \wedge v'_2$  is negative, and the polygon can be ignored. This technique was used in the author's PICASO computer animation system in 1970.

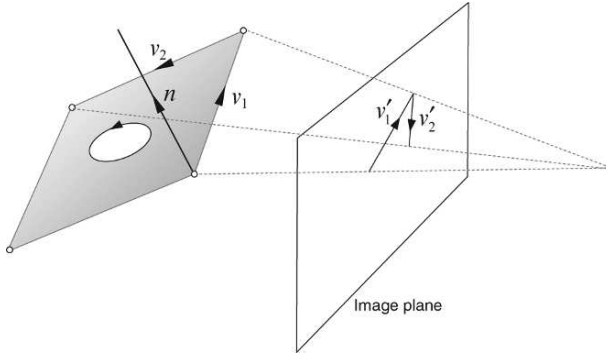


FIGURE 10.29.

## 10.7 Homogeneous coordinates

### 10.7.1 Introduction

Homogeneous space is a useful subterfuge employed in the design of CGI algorithms to provide an inclusive mathematical environment where scaling, shearing, reflection, rotation and translating transforms can be integrated. Normally, a point in  $\mathbb{R}^2$  is represented by its coordinates  $(x, y)$ . However, if we embed this space in one with an extra dimension, there are an infinite range of values for this third coordinate. This is readily overcome by representing the point as  $(wx, wy, w)$ , then we can set  $w$  to any non-zero value. In fact,  $w = 1$  is a rather useful value.

Figure 10.30 shows such a scenario where  $P(x, y)$  is to be translated to  $Q(x + \Delta_x, y + \Delta_y)$ . The transformation to achieve this is

$$Q = \begin{bmatrix} 1 & 0 & \Delta_x \\ 0 & 1 & \Delta_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}. \quad (10.158)$$

The point  $P$  can also be rotated and scaled relative to  $(0, 0, 1)$  using the following transformations:

$$R = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (10.159)$$

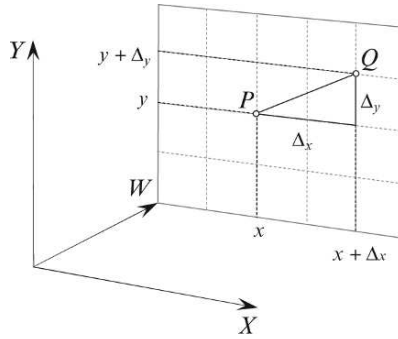


FIGURE 10.30.

$$S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10.160}$$

There are two important consequences for adopting this notation: the first is that the transforms can be multiplied together; the second is that it applies to any dimensional space. Consequently, points in  $\mathbb{R}^3$  are extended to  $(x, y, z, 1)$ , which, unfortunately, is impossible to illustrate accurately.

Homogeneous space offers similar advantages to GA, where the convention is to employ  $e$  or  $e_0$  to represent the homogeneous coordinate. The latter notation is adopted in this book. Let's begin by embedding the Euclidean plane in a three-dimensional space, as shown in Fig. 10.31.

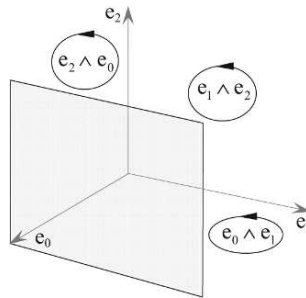


FIGURE 10.31.

The extra dimension creates three orthogonal bivectors:  $e_1 \wedge e_2, e_2 \wedge e_0$  and  $e_0 \wedge e_1$ , which means that

$$e_1 \cdot e_2 = e_2 \cdot e_0 = e_0 \cdot e_1 = 0 \tag{10.161}$$

and

$$e_1 \cdot e_1 = \|e_1\| = e_2 \cdot e_2 = \|e_2\| = e_0 \cdot e_0 = \|e_0\| = 1. \tag{10.162}$$

### 10.7.2 Representing 2D lines in 3D homogeneous space

Recall that to represent a line passing through the points  $T$  and  $P$  with a direction vector  $v$  (Fig. 10.13) we use the equation

$$p = t + \lambda v. \tag{10.163}$$

Although this is reasonably easy to use, GA provides an alternative approach, which gives lines and their manipulation a new lease of life.

Figure 10.32 shows how two homogeneous points  $P$  and  $Q$  are used to define a line using their outer product  $p_h \wedge q_h$ . This is sufficient to define the line, which is represented by the intersection of the bivector  $p_h \wedge q_h$  and the Euclidean plane, which is orthogonal to the  $e_0$  axis. At first, this seems too good to be true, so let's illustrate why this is so.

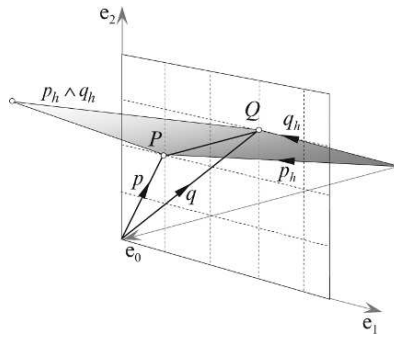


FIGURE 10.32.

We define  $P$  and  $Q$  as follows:

$$P = (x_p, y_p, 1) \text{ and } Q = ((x_p + \Delta_x), (y_p + \Delta_y), 1) \tag{10.164}$$

where  $Q$  is displaced from  $P$  by  $(\Delta_x, \Delta_y)$ .

The homogeneous position vectors  $p_h$  and  $q_h$  are

$$p_h = x_p e_1 + y_p e_2 + e_0 \tag{10.165}$$

$$q_h = (x_p + \Delta_x) e_1 + (y_p + \Delta_y) e_2 + e_0 \tag{10.166}$$

therefore,

$$\begin{aligned} p_h \wedge q_h &= p_h \wedge ((x_p + \Delta_x) e_1 + (y_p + \Delta_y) e_2 + e_0) \\ &= p_h \wedge (p_h + \Delta_x e_1 + \Delta_y e_2 + e_0) \\ &= p_h \wedge (\Delta_x e_1 + \Delta_y e_2 + e_0) \\ &= (x_p e_1 + y_p e_2 + e_0) \wedge (\Delta_x e_1 + \Delta_y e_2 + e_0) \\ p_h \wedge q_h &= (x_p \Delta_y - y_p \Delta_x) e_{12} + \Delta_x e_{01} + \Delta_y e_{02}. \end{aligned} \tag{10.167}$$

There are three coefficients for the three basis bivectors  $e_{12}$ ,  $e_{01}$  and  $e_{02}$ :

$$(x_p \Delta_y - y_p \Delta_x), \quad \Delta_x \text{ and } \Delta_y. \tag{10.168}$$

Encoded within these coefficients is the line's equation, and as a consequence, the perpendicular vector from the origin to the line. First, let's show how the line equation is encoded.

Equation (10.167) seems to possess a structure that arises with line equations, and is worth investigating further.

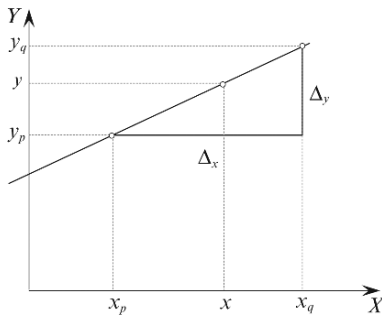


FIGURE 10.33.

Figure 10.33 shows a line passing through two points  $P(x_p, y_p)$  and  $Q(x_q, y_q)$ . From the figure we can see that

$$\frac{y - y_p}{x - x_p} = \frac{\Delta_y}{\Delta_x} \tag{10.169}$$

and

$$(y - y_p) \Delta_x = (x - x_p) \Delta_y \tag{10.170}$$

$$y \Delta_x - y_p \Delta_x = x \Delta_y - x_p \Delta_y \tag{10.171}$$

$$x \Delta_y - y \Delta_x - (x_p \Delta_y - y_p \Delta_x) = 0. \tag{10.172}$$

Equation (10.172) looks extremely similar to Eq. (10.167) and we can increase the similarity by taking the dual of Eq. (10.167):

$$\begin{aligned} (p_h \wedge q_h)^* &= ((x_p \Delta_y - y_p \Delta_x) e_{12} + \Delta_x e_{01} + \Delta_y e_{02}) e_{021} \\ &= (x_p \Delta_y - y_p \Delta_x) e_{12021} + \Delta_x e_{01021} + \Delta_y e_{02021} \\ &= (x_p \Delta_y - y_p \Delta_x) e_0 + \Delta_x e_2 - \Delta_y e_1. \end{aligned}$$

Rearranging the terms, we have

$$(p_h \wedge q_h)^* = e_1 \Delta_y - e_2 \Delta_x - e_0 (x_p \Delta_y - y_p \Delta_x). \tag{10.173}$$

This implies that when we take the dual of the bivector  $p_h \wedge q_h$ , the three coefficients of the homogeneous orthogonal axes have a one-to-one correspondence with the parameters of the line equation passing through the original points  $P$  and  $Q$ . For example, Fig. 10.32 shows that

$$p = e_1 + 2e_2 \tag{10.174}$$

$$q = 3e_1 + 3e_2 \tag{10.175}$$

which makes

$$\Delta_x = 2 \text{ and } \Delta_y = 1.$$

Therefore, using Eq. (10.167) we have

$$\begin{aligned} p_h \wedge q_h &= (x_p \Delta_y - y_p \Delta_x)e_{12} + \Delta_x e_{01} - \Delta_y e_{20} \\ &= -3e_{12} + 2e_{01} - e_{20} \\ p_h \wedge q_h &= -e_{20} + 2e_{01} - 3e_{12}. \end{aligned} \tag{10.176}$$

Whereas, using the dual of the bivector in Eq. (10.173) we have

$$(p_h \wedge q_h)^* = e_1 - 2e_2 + 3e_0. \tag{10.177}$$

Which means that the line equation for  $PQ$  is

$$x - 2y + 3 = 0 \tag{10.178}$$

which is correct!

Just to convince you, consider the scenario shown in Fig. 10.34 where

$$p = e_1 + 3e_2 \tag{10.179}$$

$$q = 3e_1 + 2e_2 \tag{10.180}$$

which makes

$$\Delta_x = 2 \text{ and } \Delta_y = -1.$$

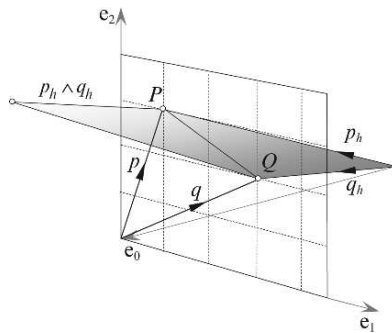


FIGURE 10.34.

Therefore, using Eq. (10.167) we have

$$\begin{aligned} p_h \wedge q_h &= (x_p \Delta_y - y_p \Delta_x) e_{12} + \Delta_x e_{01} - \Delta_y e_{20} \\ &= -7e_{12} + 2e_{01} + e_{20} \\ p_h \wedge q_h &= e_{20} + 2e_{01} - 7e_{12}. \end{aligned} \quad (10.181)$$

Whereas, using the dual of the bivector in Eq. (10.173) we have

$$(p_h \wedge q_h)^* = -e_1 - 2e_2 + 7e_0. \quad (10.182)$$

Which means that the line equation for  $PQ$  is

$$-x - 2y + 7 = 0 \quad (10.183)$$

which is also correct!

Now let's demonstrate that  $p \wedge q$  also encodes the vector perpendicular to the line from the origin. For example, consider the scenario shown in Fig. 10.35 where we have two points  $P$  and  $Q$  with position vectors  $p$  and  $q$  respectively. The line's direction vector is  $v = q - p$ .

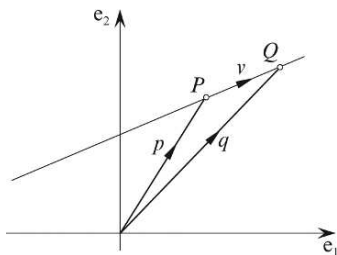


FIGURE 10.35.

Now say we translate points  $P$  and  $Q$  along the line by  $\lambda v$ , then their new position vectors become

$$p + \lambda v \text{ and } q + \lambda v \quad (10.184)$$

and their outer product is

$$\begin{aligned} (p + \lambda v) \wedge (q + \lambda v) &= p \wedge q + \lambda p \wedge v + \lambda v \wedge q \\ &= p \wedge q + \lambda v \wedge q - \lambda v \wedge p \\ &= p \wedge q + \lambda v \wedge (q - p) \\ &= p \wedge q + \lambda v \wedge v \\ (p + \lambda v) \wedge (q + \lambda v) &= p \wedge q. \end{aligned} \quad (10.185)$$

Which proves a very important feature of the outer product: the outer product of two points, returns the same result, no matter how the two points are moved along the line in tandem. Consequently, we can identify two points  $D$  and  $E$  on the line such that  $v = e - d$ , as shown in Fig. 10.36.

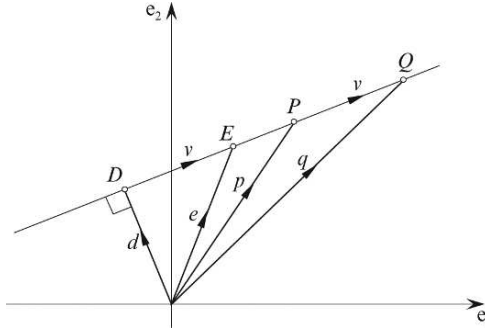


FIGURE 10.36.

Therefore, using the geometric product

$$dv = d \cdot v + d \wedge v \quad (10.186)$$

$$dv = d \wedge v = p \wedge v \quad (10.187)$$

and

$$\begin{aligned} d &= \frac{p \wedge v}{v} \\ &= \frac{p \wedge (q - p)}{q - p} \\ d &= \frac{p \wedge q}{q - p}. \end{aligned} \quad (10.188)$$

Let's apply Eq. (10.188) to the previous example where

$$p = e_1 + 3e_2 \quad (10.189)$$

$$q = 3e_1 + 2e_2 \quad (10.190)$$

as shown in Fig. 10.34.

$$\begin{aligned} d &= \frac{p \wedge q}{q - p} \\ &= \frac{(e_1 + 3e_2) \wedge (3e_1 + 2e_2)}{(3e_1 + 2e_2) - (e_1 + 3e_2)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{-7\mathbf{e}_{12}}{2\mathbf{e}_1 - \mathbf{e}_2} \\
 &= -7\mathbf{e}_{12}\left(\frac{2\mathbf{e}_1 - \mathbf{e}_2}{5}\right) \\
 d &= \frac{7}{5}\mathbf{e}_1 - \frac{14}{5}\mathbf{e}_2
 \end{aligned} \tag{10.191}$$

and

$$\begin{aligned}
 \delta &= \|d\| \\
 &= \left\| \frac{7}{5}\mathbf{e}_1 - \frac{14}{5}\mathbf{e}_2 \right\| \\
 \delta &= \frac{7}{\sqrt{5}}.
 \end{aligned} \tag{10.192}$$

This can be confirmed from the line equation

$$-x - 2y + 7 = 0 \tag{10.193}$$

where, given

$$ax + by + c = 0 \tag{10.194}$$

$$\delta = \frac{|c|}{\sqrt{a^2 + b^2}} = \frac{7}{\sqrt{5}}. \tag{10.195}$$

Another advantage of this homogeneous representation is that a line can be rotated simply by rotating its bivector. For example, to rotate  $PQ$   $180^\circ$  to  $P'Q'$  we sandwich the bivector between  $\mathbf{e}_{12}$  and  $\tilde{\mathbf{e}}_{12}$  as follows:

$$P'Q' = \mathbf{e}_{12}(p_h \wedge q_h)\mathbf{e}_{21}. \tag{10.196}$$

Using the bivector illustrated in Fig. 10.32

$$p_h \wedge q_h = -3\mathbf{e}_{12} + 2\mathbf{e}_{01} - \mathbf{e}_{20} \tag{10.197}$$

$$\begin{aligned}
 p'_h \wedge q'_h &= \mathbf{e}_{12}(-3\mathbf{e}_{12} + 2\mathbf{e}_{01} - \mathbf{e}_{20})\mathbf{e}_{21} \\
 &= -3\mathbf{e}_{121221} + 2\mathbf{e}_{120121} - \mathbf{e}_{122021}
 \end{aligned}$$

$$p'_h \wedge q'_h = -3\mathbf{e}_{12} - 2\mathbf{e}_{01} + \mathbf{e}_{20}. \tag{10.198}$$

The projection of  $p'_h \wedge q'_h$  on the Euclidean plane is the same as  $p_h \wedge q_h$ , whilst the signs of the other coefficients are reversed, which is expected. And just in case you are not convinced that this is so, let's rotate  $PQ$   $180^\circ$  to  $P'Q'$  and compute its bivector.

We have

$$P'(-1, -2, 1) \text{ and } Q'(-3, -3, 1), \tag{10.199}$$

therefore,

$$p'_h = -\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_0 \tag{10.200}$$



and

$$q'_h = -3e_1 - 3e_2 + e_0 \quad (10.201)$$

therefore,

$$\begin{aligned} p'_h \wedge q'_h &= (-e_1 - 2e_2 + e_0) \wedge (-3e_1 - 3e_2 + e_0) \\ p'_h \wedge q'_h &= -3e_{12} - 2e_{01} + e_{20}. \end{aligned} \quad (10.202)$$

Which is reassuring!

To complete this example, let's find the equation of this rotated line:

$$\begin{aligned} (p'_h \wedge q'_h)^* &= (-3e_{12} - 2e_{01} + e_{20})e_{021} \\ &= -3e_{12021} - 2e_{01021} + e_{20021} \\ (p'_h \wedge q'_h)^* &= e_1 - 2e_2 - 3e_0 \end{aligned} \quad (10.203)$$

which means that the equation for the line  $P'Q'$  is

$$x - 2y - 3 = 0 \quad (10.204)$$

which is correct.

### 10.7.3 Intersection of two lines in $\mathbb{R}^2$

Computing the intersection point of two lines in  $\mathbb{R}^2$  is relatively easy. For example, if the lines are defined by the line equations

$$a_1x + b_1y + c_1 = 0 \quad (10.205)$$

$$a_2x + b_2y + c_2 = 0 \quad (10.206)$$

using Cramer's rule, the intersection point is

$$x_i = \frac{c_2b_1 - c_1b_2}{a_1b_2 - a_2b_1} \text{ and } y_i = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}. \quad (10.207)$$

The lines are parallel if

$$a_1b_2 - a_2b_1 = 0. \quad (10.208)$$

For example, the two lines in Fig. 10.37 have equations

$$PQ: \quad -x + y - 1 = 0 \quad (10.209)$$

$$RS: \quad x + y - 7 = 0 \quad (10.210)$$

which makes their intersection point

$$x_i = \frac{-7 + 1}{-2} = 3 \text{ and } y_i = \frac{-1 - 7}{-2} = 4 \quad (10.211)$$

and is correct.

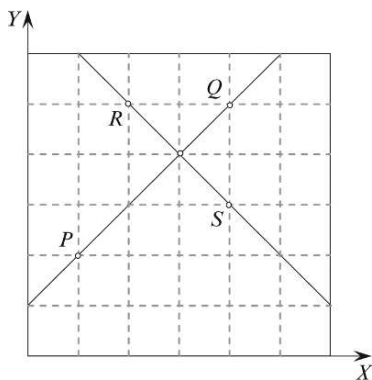


FIGURE 10.37.

The GA homogeneous approach provides an alternative solution using a common notation and without invoking Cramer’s rule! But before we look at this, we need to remind ourselves of some notation.

In chapter 8 we discovered that

$$aB = a \cdot B + a \wedge B \tag{10.212}$$

where  $a \cdot B$  represents the lowest grade part of the product and  $a \wedge B$  is the highest grade part of the product. More generally, given a vector  $a$  and a multivector  $A_r$ , whose highest grade element is  $r$ , we can write:

$$aA_r = \langle aA_r \rangle_{r-1} + \langle aA_r \rangle_{r+1}. \tag{10.213}$$

This means that the geometric product creates two parts: the inner product, consisting of those elements with a grade lower than  $r$ , and the outer product, consisting of those elements with a grade higher than  $r$ :

$$a \cdot A_r = \langle aA_r \rangle_{r-1} \tag{10.214}$$

$$a \wedge A_r = \langle aA_r \rangle_{r+1}. \tag{10.215}$$

We use this relationship in the next example.

Figure 10.38 shows the homogeneous model where the four points are identified by their position vectors:

$$p_h = e_1 + 2e_2 + e_0 \tag{10.216}$$

$$q_h = 4e_1 + 5e_2 + e_0 \tag{10.217}$$

$$r_h = 2e_1 + 5e_2 + e_0 \tag{10.218}$$

$$s_h = 4e_1 + 3e_2 + e_0. \tag{10.219}$$

The intersection point is given by:

$$(p_h \wedge q_h) \vee (r_h \wedge s_h) = (p_h \wedge q_h)^* \cdot (r_h \wedge s_h) \tag{10.220}$$

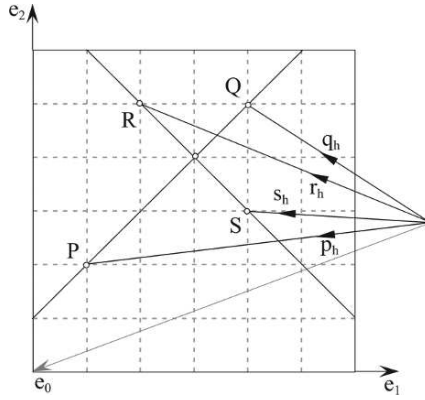


FIGURE 10.38.

First, let's calculate the two bivectors:

$$\begin{aligned}
 p_h \wedge q_h &= (e_1 + 2e_2 + e_0) \wedge (4e_1 + 5e_2 + e_0) \\
 p_h \wedge q_h &= -3e_{12} - 3e_{20} + 3e_{01}
 \end{aligned}
 \tag{10.221}$$

$$\begin{aligned}
 r_h \wedge s_h &= (2e_1 + 5e_2 + e_0) \wedge (4e_1 + 3e_2 + e_0) \\
 r_h \wedge s_h &= -14e_{12} + 2e_{20} + 2e_{01}
 \end{aligned}
 \tag{10.222}$$

Next we compute the dual  $(p_h \wedge q_h)^*$ :

$$\begin{aligned}
 (p_h \wedge q_h)^* &= e_{012}(-3e_{12} - 3e_{20} + 3e_{01}) \\
 (p_h \wedge q_h)^* &= 3e_0 + 3e_1 - 3e_2.
 \end{aligned}
 \tag{10.223}$$

As the inner product  $(p_h \wedge q_h)^* \cdot (r_h \wedge s_h)$  involves a vector and a bivector we invoke Eq. (10.220), which only requires a cross-product to be computed:

	$e_0$	$e_1$	$e_2$
$(p_h \wedge q_h)^*$	3	3	-3
$r_h \wedge s_h$	-14	2	2
	12	36	48

$$(p_h \wedge q_h)^* \cdot (r_h \wedge s_h) = 12e_0 + 36e_1 + 48e_2.
 \tag{10.224}$$

The  $12e_0$  term is the homogeneous scaling factor, which means that we must divide the result by 12:

$$3e_1 + 4e_2 + e_0
 \tag{10.225}$$

and implies that the point of intersection is (3, 4). What could be simpler?

### 10.7.4 Representing 3D lines in 4D homogeneous space

Having seen how to embed the Euclidean plane in a 3D homogeneous space, we now consider embedding Euclidean space in a 4D homogeneous space. Although it is impossible to illustrate this concept diagrammatically, the mathematical solution is trivial, as it requires only extending 3D coordinates  $(x, y, z)$  with an extra coordinate, which, for convenience, is chosen as 1. However, as in the 3D case, if after any calculations we arrive at a point with coordinates  $(wx, wy, wz, w)$ , we divide throughout by  $w$  to return to the original space.

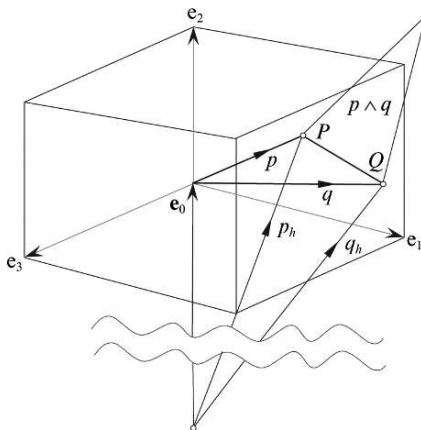


FIGURE 10.39.

Figure 10.39 attempts to depict Euclidean space and the fourth dimension  $e_0$  which is orthogonal to the three basis vectors  $e_1, e_2$  and  $e_3$ , and consequently, every vector within this space. Two points  $P$  and  $Q$  will then have homogeneous position vectors  $p_h$  and  $q_h$  respectively, together with their local position vectors  $p$  and  $q$ . We can now create a bivector  $p_h \wedge q_h$  which resolves the 4D parallelogram into six orthogonal bivector projections. Why six, you may ask? Well, the extra dimension  $e_0$  can be combined orthogonally with  $e_1, e_2$  and  $e_3$ , making the following basis bivectors:

$$e_{12}, e_{23}, e_{31}, e_{10}, e_{20}, e_{30}. \tag{10.226}$$

Although the first three bivectors follow a right-handed system, the remaining three bivectors have an arbitrary axial sequence because 4D space is not handed in the accepted sense. To illustrate the process, let's compute the outer product of two position vectors for the points  $P(2, 1, 2)$  and  $Q(4, 3, 4)$ . In homogeneous space they become  $P(2, 1, 2, 1)$  and  $Q(4, 3, 4, 1)$  with position vectors

$$p_h = 2e_1 + e_2 + 2e_3 + e_0 \tag{10.227}$$

$$q_h = 4e_1 + 3e_2 + 4e_3 + e_0. \tag{10.228}$$

The outer product  $p_h \wedge q_h$  is

$$\begin{aligned} p_h \wedge q_h &= (2e_1 + e_2 + 2e_3 + e_0) \wedge (4e_1 + 3e_2 + 4e_3 + e_0) \\ &= 6e_{12} + 8e_{13} + 2e_{10} + 4e_{21} + 4e_{23} + e_{20} + 8e_{31} + 6e_{32} + 2e_{30} + 4e_{01} + 3e_{02} + 4e_{03} \\ p_h \wedge q_h &= 2e_{12} - 2e_{23} - 2e_{10} - 2e_{20} - 2e_{30}. \end{aligned} \quad (10.229)$$

As it happens, it appears that there are only five coefficients, but that is because one of the six is zero.

### 10.7.5 Representing lines and planes in 4D homogeneous space

Perhaps the simplest way of representing a plane in homogeneous space is by taking three points on the plane and computing their outer product. For example, given the points  $P$ ,  $Q$  and  $R$ , the outer product of their homogeneous position vectors  $p_h$ ,  $q_h$  and  $r_h$  respectively, is  $p_h \wedge q_h \wedge r_h$ , and describes the plane completely. For example, Figure 10.40 shows a triangle with coordinates  $P(0, 0, 1)$ ,  $Q(1, 0, 0)$  and  $R(0, 1, 0)$ , and homogeneous position vectors:

$$p_h = e_3 + e_0 \quad (10.230)$$

$$q_h = e_1 + e_0 \quad (10.231)$$

$$r_h = e_2 + e_0. \quad (10.232)$$

Therefore, their outer product  $\pi$  is

$$\begin{aligned} \pi &= p_h \wedge q_h \wedge r_h \\ &= (e_3 + e_0) \wedge (e_1 + e_0) \wedge (e_2 + e_0) \\ &= (e_{31} + e_{30} + e_{01}) \wedge (e_2 + e_0) \\ &= e_{312} + e_{310} + e_{302} + e_{012} \\ \pi &= e_{123} + e_{310} - e_{320} + e_{120}. \end{aligned} \quad (10.233)$$

Anticipating that we will require the dual of  $\pi$ , let's compute it now:

$$\begin{aligned} \pi^* &= e_{0123}A = e_{0123}(e_{123} + e_{310} - e_{320} + e_{120}) \\ &= e_{0123123} + e_{0123310} - e_{0123320} + e_{0123120} \\ &= -e_0 + e_2 + e_1 + e_3 \\ \pi^* &= e_1 + e_2 + e_3 - e_0. \end{aligned} \quad (10.234)$$

Similarly, the line  $SQ$  is completely described by the outer product of its homogeneous vectors. As shown in Fig. 10.40 this product is

$$l = s_h \wedge q_h \quad (10.235)$$

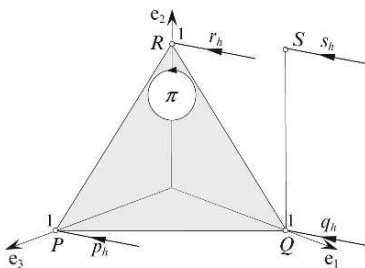


FIGURE 10.40.

and is defined by

$$\begin{aligned}
 s_h \wedge q_h &= (e_1 + e_2 + e_0) \wedge (e_1 + e_0) \\
 &= e_{10} + e_{21} + e_{20} + e_{01} \\
 s_h \wedge q_h &= e_{21} + e_{20}.
 \end{aligned}
 \tag{10.236}$$

So, let's exploit these definitions and compute the intersection between the line and the plane using the meet operator, which is given by

$$\pi^* \cdot l = (e_1 + e_2 + e_3 - e_0) \cdot (e_{21} + e_{20}).
 \tag{10.237}$$

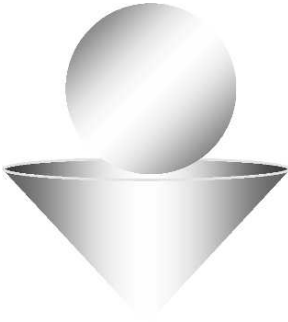
Once more, we have the inner product of a vector and bivector, which is evaluated by extracting the vector portion of the geometric product (Eq. (10.214)).

$$\begin{aligned}
 \pi^* \cdot l &= \langle \pi^* l \rangle_1 \\
 &= \langle (e_1 + e_2 + e_3 - e_0)(e_{21} + e_{20}) \rangle_1 \\
 &= \langle e_{121} + e_{120} + e_{221} + e_{220} + e_{321} + e_{320} - e_{021} - e_{020} \rangle_1 \\
 &= \langle -e_2 + e_{120} + e_1 + e_0 + e_{321} + e_{320} + e_{120} + e_2 \rangle_1 \\
 \pi^* \cdot l &= e_1 + e_0.
 \end{aligned}
 \tag{10.238}$$

As the coefficient of  $e_0$  is unity, there is no homogeneous scaling, and the intersection point is 1 on the  $e_1$  axis, which is correct.

## 10.8 Summary

Hopefully, the variety of examples in this chapter have demonstrated the potential of GA as a problem-solving tool. Obviously, much work needs to be done to identify elegant solutions to these problems and there is plenty of scope for authors wishing to explore new ways of solving geometric problems.



# 11 Conformal Geometry

## 11.1 Introduction

In this chapter we examine briefly the *conformal model*, which is an alternative space to Euclidean space for solving 3D geometric problems. Various arguments have been proposed for adopting this model such as, conformal space:

- is homogeneous and consequently removes the origin as being a special point
- supports points and lines at infinity
- provides a single geometric mechanism for representing lines, circles, planes and spheres, which introduces a rare quality of elegance to problem solving
- enjoys all the normal features of Euclidean space.

A model of space that possesses such a range of positive features sounds too good to be true! But remember, in mathematics there is no such thing as a free lunch. So what price must we pay for this model?

### 11.1.1 Spatial dimension

Although the humble Euclidean space  $\mathbb{R}^3$  requires a unique point called the origin, the space is easy to visualize and design meaningful drawings of 3D geometric scenes. Whereas conformal space requires five dimensions, which precludes any true form of mental visualization and graphic illustration. Furthermore, one of the dimensions has, what is called a *negative signature*, which transforms the space into a *Minkowski space*.

### 11.1.2 Algebraic underpinning

The toolkit employed by CGI programmers is a ‘rag bag’ of formulae, transforms, vectors and 2D/3D geometric theorems, which are relatively easy to learn, even without strong mathematical skills.

Whereas the conformal model draws on concepts such as hyperplanes, hyperspheres, bijective mappings, projective splits, which are all on top of the new notation introduced by GA.

### 11.1.3 Mathematical language and notation

To illustrate the abstractness of the language and notation, here are two extracts from recent publications:

“– if they intersect, the intersection  $(s_1 \wedge s_2)^\sim$  is an  $(n - 2)$ -plane. When both  $\tilde{s}_1$  and  $\tilde{s}_2$  are hyperspaces, the intersection corresponds to the  $(n - 2)$ -space  $(s_1 \wedge s_2)I_n$  in  $\mathbb{R}^n$ , where  $I_n$  is the unit pseudoscalar of  $\mathbb{R}^n$ ; otherwise the intersection is in the hyperspace  $(e_0 \cdot (s_1 \wedge s_2))^\sim$  and has the same normal and distance from the origin as the hyperplane  $(P_{s_1 \wedge s_2}(e_0))^\sim$ .  
– if they are parallel, the distance between them is  $|e_0 \cdot P_{s_2}^\perp(s_1)|/|s_1|$ .”

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$$\text{“} \log(T_t R_{I_0} S_\gamma) = \frac{\gamma}{1 - e^\gamma} ((t \wedge I)/I) \infty/2 + T_{(1 - e^\gamma R^2)^{-1}(t)I} I[-I\phi/2 + \gamma o \wedge \infty/2] \text{”}$$

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### 11.1.4 Protection

The conformal model is covered by the following US Patent:

#### US Patent 6,853,964

##### System for encoding and manipulating models of objects

United States Patent 6853964

David Hestenes, Hongbo Li, Alyn Rockwood

Abstract:

A method models an object composed of one or more components. Data are input to a memory of a computer system for each component of the object. The data include Cartesian coordinates expressed in Euclidean space of a plurality of points  $x$  of each component. Each component point  $x$  is encoded as a vector  $x$  in a general homogeneous space by  $x = (x + 1/2x \cdot \text{sup.} 2 e + e \cdot \text{sub.} *)E = xE - 1/2x \cdot \text{sup.} 2 e + e \cdot \text{sub.} *$ , where  $e$  and  $e \cdot \text{sub.} *$  are basis null vectors of a Minkowski space  $E$ . General homogeneous operators are associated with each data point to generate a model of the object. The general homogeneous operators are applied to each encoded point of the associated component for each component to manipulate the model of the object.

Licences to employ this system for commercial purposes are available through the patent holders, although there are no restrictions for academic research and educational use.

All of the above may be a small price to pay for the benefits associated with the conformal model, and I will do my best to explain the model in this chapter.



## 11.2 Stereographic projection

Computer graphics uses projective geometry to create an environment where scaling, shearing, rotation and translation transforms are combined seamlessly. In spite of the obvious advantages of projective space, there are some disadvantages. For example, the ratio of lengths along a line are not preserved, neither is the concept of *betweenness*, as illustrated in Fig. 11.1.

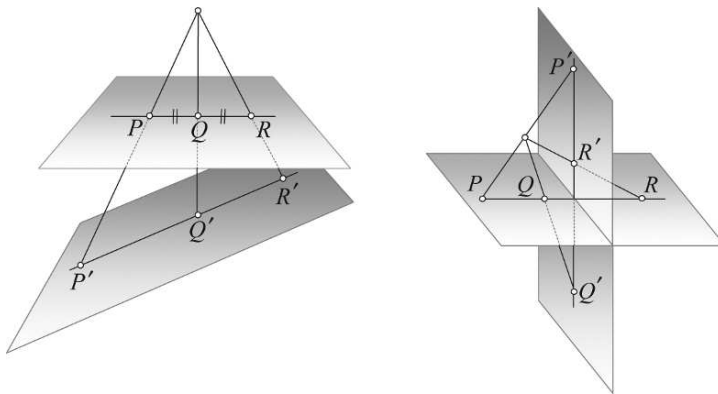


FIGURE 11.1.

However, the cross-ratio of four points is preserved under all projective transformations. For example, given four collinear points  $A, B, C, D$  on some plane, then their cross ratio is given by

$$(ABCD) = \frac{AC/CB}{AD/DB}. \tag{11.1}$$

Figure 11.2 shows two sets of points  $A, B, C, D$  and  $A', B', C', D'$  whose cross ratios are identical:

$$\frac{AC/CB}{AD/DB} = \frac{A'C'/C'B'}{A'D'/D'B'}. \tag{11.2}$$

If, for example

$$AB = 2 \quad BC = 4 \quad CD = 2 \tag{11.3}$$

then

$$(ABCD) = \frac{6/4}{8/6} = 36/32 = 9/8 \tag{11.4}$$

which also applies for  $A', B', C', D'$ .

The cross ratio is used in CGI to extract 3D information from 2D images and it is important that this ratio is preserved under all sorts of projections.

In the 1820s, the German mathematician August Ferdinand Möbius [1790–1868] discovered a geometry that preserves ratios of lengths along lines, and maps lines to lines and parallel lines to parallel lines. Möbius called this *affine geometry* because projected figures retain a close *affinity* to the original figures [16]. Möbius is also known for his transformations which have the property of

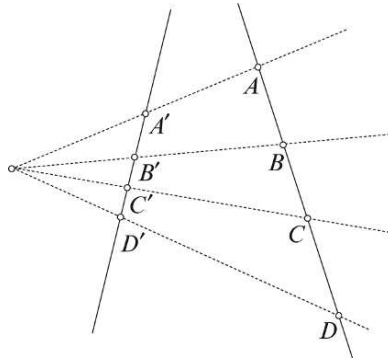


FIGURE 11.2.

preserving the magnitude and orientation of angles and map generalized circles into generalized circles.

The reason we are interested in projective geometry at this point is because the conformal model employs the *stereographic projection* to convert points in Euclidean space to conformal space, and vice versa. We can derive the mapping functions using a variety of techniques such as 3D geometry, vector analysis or geometric algebra, but perhaps the most useful is vector analysis as this can be generalized to higher dimensions.

Figure 11.3 shows a unit radius hemisphere positioned on the complex plane such that the origin of the complex plane coincides with the sphere's centre. The line connecting the North Pole  $N$  to a point  $X$  on the plane must intersect the sphere at  $P$ , which provides the mechanism for relating points on the plane to corresponding points on the sphere. This configuration is called the *Riemann sphere*, after the German mathematician Georg Riemann [1826–1866].

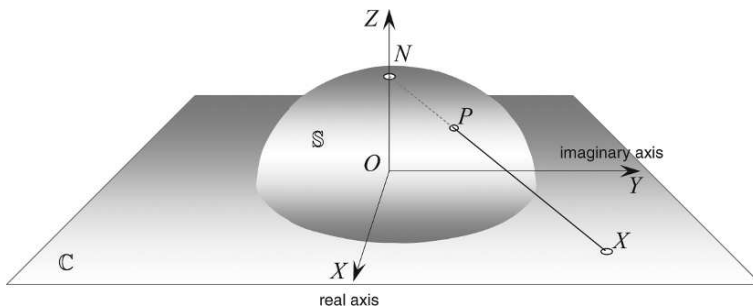


FIGURE 11.3.

Figure 11.4 shows the unit sphere and three points  $A, P, X$  with their position vectors  $a, p, x$  respectively. From the figure it is obvious that

$$p = a + \lambda(x - a) \quad \lambda \in \mathbb{R}. \tag{11.5}$$

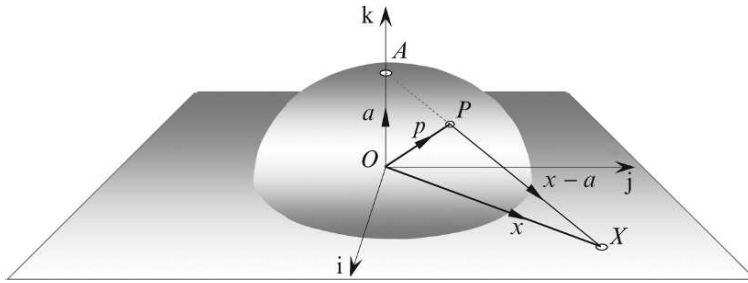


FIGURE 11.4.

We also know that

$$a \cdot x = 0 \quad a \cdot a = p \cdot p = 1. \tag{11.6}$$

Squaring both sides using the dot product we have

$$p \cdot p = (a + \lambda(x - a)) \cdot (a + \lambda(x - a)) \tag{11.7}$$

$$= a \cdot a + 2\lambda a \cdot (x - a) + \lambda^2(x - a) \cdot (x - a)$$

$$1 = 1 - 2\lambda + \lambda^2(1 + x \cdot x)$$

$$\lambda = \frac{2}{1 + x \cdot x}. \tag{11.8}$$

Substituting  $\lambda$  in Eq. (11.4) we have

$$\begin{aligned} p &= a + \left( \frac{2}{1 + x \cdot x} \right) (x - a) \\ &= \frac{a(1 + x \cdot x) + 2(x - a)}{1 + x \cdot x} \\ p &= \frac{(x \cdot x)a + 2x - a}{1 + x \cdot x}. \end{aligned} \tag{11.9}$$

Writing Eq. (11.9) using the geometric product, we have

$$p = \frac{x^2 a + 2x - a}{1 + x^2}. \tag{11.10}$$

The stereographic projection ensures that circles on the sphere map onto generalized circles on the plane as shown in Fig. 11.5.

Furthermore, if the circle on the sphere passes through the North Pole it creates a circle of infinite radius and effectively creates a straight line as shown in Fig. 11.6. The North Pole is mapped onto a circle of infinite radius concentric with the sphere, and is the point on the sphere corresponding to a point at infinity on the line. It can also be shown that the angle between two circles on the sphere equals the angle between the lines on the plane, which is the conformal property that is exploited in GA.

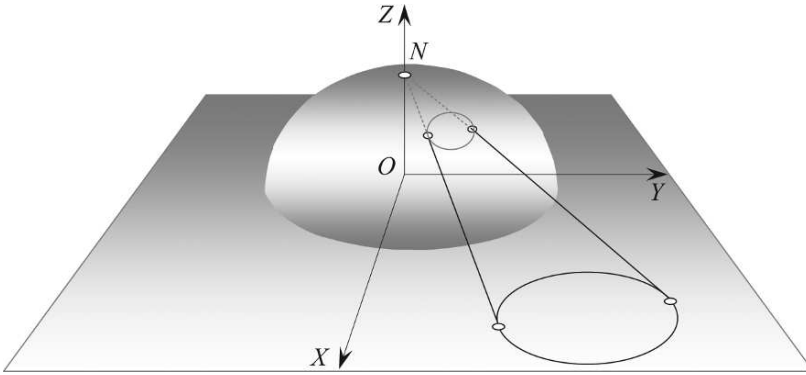


FIGURE 11.5.

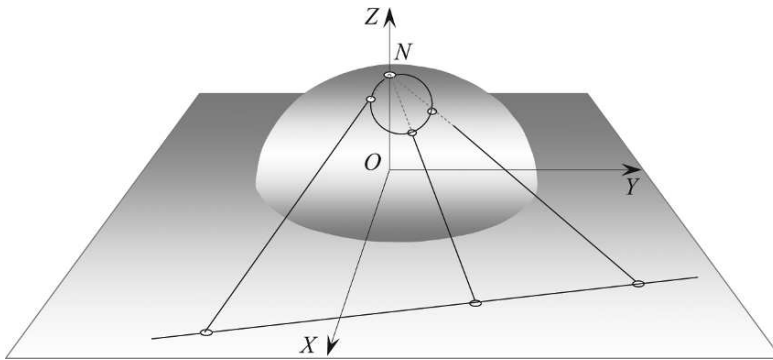


FIGURE 11.6.

### 11.3 Signatures and null vectors

Vectors in Euclidean space generally have a magnitude and direction, and satisfy the dot product  $n \cdot n = \|n\|^2$ . This arises because 3D Euclidean space employs basis vectors whose dot products  $e_1 \cdot e_1 = e_2 \cdot e_2 = e_3 \cdot e_3 = 1$ . But there is no absolute reason why such a rule should always prevail. In fact, the Russian mathematician Hermann Minkowski [1864–1909] laid the mathematical foundations for Einstein's Special and General Theories of Relativity by abandoning this rule. Minkowski's four-dimensional *spacetime* has a set of orthogonal basis vectors  $\{e_1, e_2, e_3, e_4\}$  where  $e_1 \cdot e_1 = e_2 \cdot e_2 = e_3 \cdot e_3 = 1$  but  $e_4 \cdot e_4 = -1$ . The positive and negative values define the *signature* of the space and denoted  $\mathbb{R}^{3,1}$  which is also known as *Minkowski space*. In general, the signature of a space is normally denoted  $\mathbb{R}^{p,q}$  where  $p$  is the number of positive dimensions and  $q$  the number of negative dimensions. One of the results of having a space with mixed signatures is the possibility of *null vectors*. And although null vectors possess all the normal attributes associated with vectors, because the space in which they are embedded possesses mixed signatures, the absolute value of a squared vector is zero.

In conformal geometry points in space of signature  $\mathbb{R}^{p,q}$  are represented as null vectors in a space of signature  $\mathbb{R}^{p+1,q+1}$ , which means that points in Euclidean space  $\mathbb{R}^{3,0}$  are represented by

null vectors in a space with signature  $\mathbb{R}^{4,1}$ . Thus we are working in a five-dimensional space which permits the modeling of lines, circles, spheres and planes. However, it just so happens that lines are circles with an infinite radius, and planes are spheres with an infinite radius.

In terms of Eq. (11.8), if we introduce an extra vector  $\bar{e}$  which has a negative signature:

$$\begin{aligned}
 p' &= \frac{x^2 a + 2x - a}{1 + x^2} + \bar{e} \\
 p' &= \frac{(x^2 - 1)a + 2x + (1 + x^2)\bar{e}}{1 + x^2}
 \end{aligned}
 \tag{11.11}$$

we can show that this transforms the vector into a null vector by squaring the numerator using the dot product as follows:

$$\begin{aligned}
 p' \cdot p' &= (x^2 - 1)^2 a \cdot a + 2(x^2 - 1)a \cdot x + (x^2 - 1)(1 + x^2)a \cdot \bar{e} + \\
 & 2(x^2 - 1)x \cdot a + 4x \cdot x + 2(1 + x^2)x \cdot \bar{e} + \\
 & (1 + x^2)(x^2 - 1)\bar{e} \cdot a + 2(1 + x^2)\bar{e} \cdot x + (1 + x^2)^2 \bar{e}^2.
 \end{aligned}
 \tag{11.12}$$

However

$$a \cdot a = 1 \quad \bar{e} \cdot \bar{e} = -1 \quad a \cdot x = a \cdot \bar{e} = x \cdot \bar{e} = 0
 \tag{11.13}$$

therefore,

$$\begin{aligned}
 p' \cdot p' &= (x^2 - 1)^2 + 4x^2 - (1 + x^2)^2 \\
 &= x^4 - 2x^2 + 1 + 4x^2 - 1 - 2x^2 - x^4 \\
 p' \cdot p' &= 0.
 \end{aligned}
 \tag{11.14}$$

Having proved that the vector is null also shows that it is homogeneous, which means that it can be scaled, especially by its denominator, which produces the final conformal vector equation

$$X = (x^2 - 1)a + 2x + (1 + x^2)\bar{e},
 \tag{11.15}$$

well almost!

The above reasoning was based upon Fig. 11.4 which is concerned with the mapping between a plane and a hemisphere. But even if we had started with the mapping between a semicircle and a line, a similar equation would have resulted. In fact, the nature of the analysis permits us to generalize the result to higher dimensions.

Let's now rewrite Eq. (11.15) in a form that reflects the notation employed in most GA publications. We start by expanding the terms as follows:

$$\begin{aligned}
 X &= x^2 a - a + 2x + \bar{e} + x^2 \bar{e} \\
 X &= x^2(a + \bar{e}) + 2x - (a - \bar{e}).
 \end{aligned}
 \tag{11.16}$$

Next we substitute  $e = a$  where  $e$  has a positive signature, then

$$n = e + \bar{e} \text{ and } \bar{n} = e - \bar{e}
 \tag{11.17}$$

and

$$X = 2x + x^2n - \bar{n} \tag{11.18}$$

which is the conformal representation of the Euclidean vector  $x$ .

For example, if

$$x = e_1 + 2e_2 + 3e_3 \tag{11.19}$$

then

$$\begin{aligned} X &= 2e_1 + 4e_2 + 6e_3 + (1 + 4 + 9)n - \bar{n} \\ X &= 2e_1 + 4e_2 + 6e_3 + 14n - \bar{n}. \end{aligned} \tag{11.20}$$

A possible source of confusion arises in the different notation employed by mathematicians. For example, Table 11.1 summarizes the notation from three publications. However, there is nothing to stop us scaling Eq. (11.18) by  $\frac{1}{2}$ :

$$X = \frac{1}{2}(2x + x^2n - \bar{n}) \tag{11.21}$$

which almost brings the equations into line. Fortunately, the inconsistency in the last term makes no difference to the equation’s validity. Dorst, Fontijne & Mann [18] (page 360) do note that “standards have not quite been established yet” which is why such conflicts have arisen. For this book, I will employ Eq. (11.21) as it avoids a range of scaling factors when the equation is manipulated.

TABLE 11.1

Authors	Publication	Infinity Vector	Origin Vector	Map
Li, Hestenes & Rockwood	[17] p33	$e$	$e_0$	$x = \mathbf{x} + \frac{1}{2}\mathbf{x}^2e + e_0$
Dorst, Fontijne & Mann	[18] p360	$\infty$	$o$	$p = \mathbf{p} + \frac{1}{2}\mathbf{p}^2\infty + o$
Doran & Lasenby	[15] p353	$n$	$\bar{n}$	$X = 2x + x^2n - \bar{n}$

As mentioned at the start of this chapter, visualizing 5D space is impossible. But wouldn’t it be nice if we could have some inkling of what is happening? Well, most authors who have written about this model suggest that the diagram shown in Fig. 11.7 is a useful mental model. The three dimensions of Euclidean space are compressed into one dimension depicted by  $\mathbb{R}^3$ , which permits the diagrammatic representation of the other two dimensions  $e$  and  $\bar{e}$ , together with  $n$  and  $\bar{n}$ . The cone represents the set of null vectors defined by Eq. (11.21). Intersections between the cone and other geometric elements reveal points, lines, circles and spheres. In the author’s opinion, at this stage, very little extra understanding is gained by struggling to relate conformal algebra with a 5D visual interpretation.

Before continuing, let’s investigate the following products:

$$\begin{aligned} n \cdot n &= (e + \bar{e}) \cdot (e + \bar{e}) \\ &= e \cdot e + 2e \cdot \bar{e} + \bar{e} \cdot \bar{e} \\ n \cdot n &= 1 + 0 - 1 = 0 \end{aligned} \tag{11.22}$$

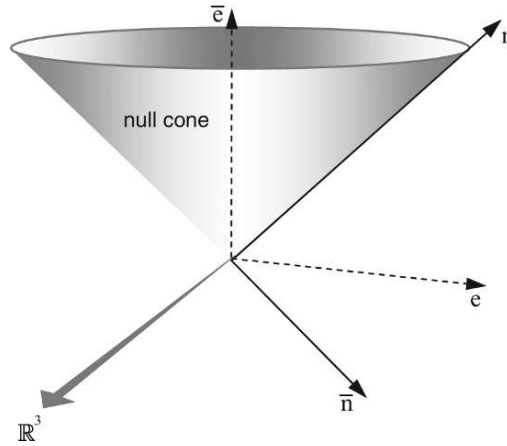


FIGURE 11.7.

$$\begin{aligned} \bar{n} \cdot \bar{n} &= (e - \bar{e}) \cdot (e - \bar{e}) \\ &= e \cdot e - 2e \cdot \bar{e} + \bar{e} \cdot \bar{e} \\ \bar{n} \cdot \bar{n} &= 1 - 0 - 1 = 0 \end{aligned} \tag{11.23}$$

$$n \cdot x_i = \bar{n} \cdot x_i = 0 \tag{11.24}$$

$$\begin{aligned} n \cdot \bar{n} &= (e + \bar{e}) \cdot (e - \bar{e}) \\ &= e \cdot e - \bar{e} \cdot \bar{e} \\ n \cdot \bar{n} &= 1 + 1 = 2 \end{aligned} \tag{11.25}$$

$$\begin{aligned} n \wedge \bar{n} &= (e + \bar{e}) \wedge (e - \bar{e}) \\ &= -2e \wedge \bar{e} \\ n \wedge \bar{n} &= -2e\bar{e}. \end{aligned} \tag{11.26}$$

All the products are shown in Table 11.2.

TABLE 11.2

GP	e	e <sub>i</sub>	ē	·	e	e <sub>i</sub>	ē	∧	e	e <sub>i</sub>	ē
e	1	-e <sub>i</sub> e	ēe	e	1	0	0	e	0	-e <sub>i</sub> ∧ e	e ∧ ē
e <sub>i</sub>	e <sub>i</sub> e	1	e <sub>i</sub> ē	e <sub>i</sub>	0	1	0	e <sub>i</sub>	e <sub>i</sub> ∧ e	0	e <sub>i</sub> ∧ ē
ē	-ēe	-e <sub>i</sub> ē	-1	ē	0	0	-1	ē	-e ∧ ē	-e <sub>i</sub> ∧ ē	0
GP	n	e <sub>i</sub>	n̄	·	n	e <sub>i</sub>	n̄	∧	n	e <sub>i</sub>	n̄
n	0	-e <sub>i</sub> n	2 - 2ēe	n	0	0	2	n	0	-e <sub>i</sub> ∧ n	-2ēe
e <sub>i</sub>	e <sub>i</sub> n	1	e <sub>i</sub> n̄	e <sub>i</sub>	0	1	0	e <sub>i</sub>	e <sub>i</sub> ∧ n	0	e <sub>i</sub> ∧ n̄
n̄	2 + 2ēe	-e <sub>i</sub> n̄	0	n̄	2	0	0	n̄	2ēe	-e <sub>i</sub> ∧ n̄	0

Here are some other products that will arise later on in this chapter:

$$(\bar{n}\bar{n} + \bar{n}n) = (e + \bar{e})(e - \bar{e}) + (e - \bar{e})(e + \bar{e}) = 4 \quad (11.27)$$

$$(\bar{n}n + n\bar{n}) = (e - \bar{e})(e + \bar{e}) + (e + \bar{e})(e - \bar{e}) = 4 \quad (11.28)$$

$$n\bar{n}n = (e + \bar{e})(e - \bar{e})(e + \bar{e}) = 4n \quad (11.29)$$

$$\bar{n}n\bar{n} = (e - \bar{e})(e + \bar{e})(e - \bar{e}) = 4n. \quad (11.30)$$

We must now try to discover the geometric meaning of  $n$  and  $\bar{n}$ . Starting with  $\bar{n}$  and using Eq. (11.21), if we let  $x$  point to the origin in  $\mathbb{R}^3$ , then

$$X = -\frac{1}{2}\bar{n} \quad (11.31)$$

which implies that  $\bar{n}$  also points to the origin.

Next  $n$ , which emerges using the following subterfuge: First we take the inner product  $X \cdot \bar{n}$ :

$$X \cdot \bar{n} = \frac{1}{2}(2x \cdot \bar{n} + x^2(\bar{n} \cdot n) - \bar{n} \cdot \bar{n}) = x^2 \quad (11.32)$$

therefore,

$$\begin{aligned} 2\frac{X}{X \cdot \bar{n}} &= 2 \left( \frac{\frac{1}{2}(2x + x^2n - \bar{n})}{x^2} \right) \\ 2\frac{X}{X \cdot \bar{n}} &= n + \frac{2x - \bar{n}}{x^2}. \end{aligned} \quad (11.33)$$

We can see from Eq. (11.33) that as  $x^2 \rightarrow \infty$ , the fraction  $\frac{2x - \bar{n}}{x^2} \rightarrow 0$ , which means that  $n$  points to infinity, which is required for the stereographic projection.

Equation (11.21) has something new to reveal, especially when we use it to take the inner product of two null vectors. For example, given two null vectors

$$X_1 = \frac{1}{2}(2x_1 + x_1^2n - \bar{n}) \quad (11.34)$$

$$X_2 = \frac{1}{2}(2x_2 + x_2^2n - \bar{n}) \quad (11.35)$$

then

$$\begin{aligned} X_1 \cdot X_2 &= \frac{1}{4}(2x_1 + x_1^2n - \bar{n}) \cdot (2x_2 + x_2^2n - \bar{n}) \\ X_1 \cdot X_2 &= \frac{1}{4}(4x_1 \cdot x_2 + 2x_2^2n \cdot x_1 - 2x_1 \cdot \bar{n} \\ &\quad + 2x_1^2x_2 \cdot n + x_1^2x_2^2n \cdot n - x_1^2n \cdot \bar{n} \\ &\quad - 2\bar{n} \cdot x_2 - x_2^2\bar{n} \cdot n + \bar{n} \cdot \bar{n}) \end{aligned} \quad (11.36)$$



but

$$\mathbf{n} \cdot \mathbf{n} = \bar{\mathbf{n}} \cdot \bar{\mathbf{n}} = \mathbf{n} \cdot x_i = \bar{\mathbf{n}} \cdot x_i = 0, \quad \mathbf{n} \cdot \bar{\mathbf{n}} = 2 \quad (11.37)$$

therefore,

$$\begin{aligned} X_1 \cdot X_2 &= x_1 \cdot x_2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \\ X_1 \cdot X_2 &= -\frac{1}{2}(x_1 - x_2)^2. \end{aligned} \quad (11.38)$$

The  $(x_1 - x_2)$  term is a vector between the two points and obviously encodes the Euclidean distance between them. This is rather fortunate, and confirms that angles *and* distances are invariant in conformal space.

Equation (11.21) possesses a convenient quality in that

$$-X \cdot \mathbf{n} = -\frac{1}{2}(2x + x^2\mathbf{n} - \bar{\mathbf{n}}) \cdot \mathbf{n} = 1. \quad (11.39)$$

Thus when working with arbitrary null vectors we can normalize them by dividing by  $-X \cdot \mathbf{n}$ . For example, say we started with Eq. (11.21) and after various calculations arrived at the null vector

$$X = 24\mathbf{e} + 40\bar{\mathbf{e}} + 32\mathbf{e}_1 + 32\mathbf{e}_2, \quad (11.40)$$

the terms  $32\mathbf{e}_1 + 32\mathbf{e}_2$  only refer to original Euclidean coordinates if  $X$  is normalized, by dividing by  $-X \cdot \mathbf{n}$ :

$$\begin{aligned} -X \cdot \mathbf{n} &= -(24\mathbf{e} + 40\bar{\mathbf{e}} + 32\mathbf{e}_1 + 32\mathbf{e}_2) \cdot (\mathbf{e} + \bar{\mathbf{e}}) \\ &= -24 + 40 \\ -X \cdot \mathbf{n} &= 16. \end{aligned} \quad (11.41)$$

Therefore, the normalized form of  $X$  is

$$\begin{aligned} \hat{X} &= \frac{1}{16}(24\mathbf{e} + 40\bar{\mathbf{e}} + 32\mathbf{e}_1 + 32\mathbf{e}_2) \\ \hat{X} &= \frac{3}{2}\mathbf{e} + \frac{5}{2}\bar{\mathbf{e}} + 2\mathbf{e}_1 + 2\mathbf{e}_2 \end{aligned} \quad (11.42)$$

which implies that  $2\mathbf{e}_1 + 2\mathbf{e}_2$  refers to a true point in Euclidean space.

The normalization value depends upon the equation used to represent the null vector. For instance, if we employed

$$X = 2x + x^2\mathbf{n} - \bar{\mathbf{n}} \quad (11.43)$$

then

$$-X \cdot \mathbf{n} = (2x + x^2\mathbf{n} - \bar{\mathbf{n}}) \cdot \mathbf{n} = 2. \quad (11.44)$$

## 11.4 The basis blades for the conformal model

Knowing that the conformal model involves five dimensions, the basis for any multivector includes 32 elements: 1 scalar, 5 vectors, 10 bivectors, 10 trivectors, 5 quadvectors and 1 pseudoscalar. These are listed in Table 11.3.

TABLE 11.3

Grade	Name	Blades
0	1 scalar	1
1	5 vectors	$e_1, e_2, e_3, \bar{n}, n$
2	10 bivectors	$e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge \bar{n}, e_2 \wedge \bar{n},$ $e_3 \wedge \bar{n}, e_1 \wedge n, e_2 \wedge n, e_3 \wedge n, \bar{n} \wedge n$
3	10 trivectors	$e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge \bar{n}, e_1 \wedge e_2 \wedge n, e_1 \wedge e_3 \wedge \bar{n},$ $e_1 \wedge e_3 \wedge n, e_1 \wedge \bar{n} \wedge n, e_2 \wedge e_3 \wedge \bar{n}, e_2 \wedge e_3 \wedge n,$ $e_2 \wedge \bar{n} \wedge n, e_3 \wedge \bar{n} \wedge n$
4	5 quadvectors	$e_1 \wedge e_2 \wedge e_3 \wedge \bar{n}, e_1 \wedge e_2 \wedge e_3 \wedge n, e_1 \wedge e_2 \wedge \bar{n} \wedge n,$ $e_1 \wedge e_3 \wedge \bar{n} \wedge n, e_2 \wedge e_3 \wedge \bar{n} \wedge n$
5	1 pseudoscalar	$e_1 \wedge e_2 \wedge e_3 \wedge \bar{n} \wedge n$

## 11.5 Representing geometric objects

The standard geometric primitives used in GA comprise a point, point pair, line, circle, plane and sphere. GA provides two ways to describe them: one using the null cone inner product and the null cone outer product. For instance, a circle can be defined by the intersection of two spheres, and also by the outer product of three distinct points, and if we move one of the points to infinity, we form a line. Let's examine these primitives starting with points.

### 11.5.1 Points

Using Eq. (11.21):

$$X = \frac{1}{2}(2x + x^2 n - \bar{n}) \quad (11.45)$$

where

$$-X \cdot n = 1. \quad (11.46)$$

If we set  $x$  to the origin  $(0, 0, 0)$  then  $x^2 = 0$  and

$$X = -\frac{1}{2}\bar{n} \quad (11.47)$$

which, by definition, is the conformal origin. Similarly, if we set  $x$  to the point  $(0, 2, 0)$  we have

$$X = \frac{1}{2}(4e_2 + 4n - \bar{n})$$

$$X = 2e_2 + 2n - \frac{1}{2}\bar{n}. \quad (11.48)$$

Furthermore, if we take the inner product  $X \cdot X$ , we should get a zero answer, as the distance between two identical points is zero:

$$\begin{aligned}
 X \cdot X &= \left(2e_2 + 2n - \frac{1}{2}\bar{n}\right) \cdot \left(2e_2 + 2n - \frac{1}{2}\bar{n}\right) \\
 &= 4e_2 \cdot e_2 + 4e_2 \cdot n - e_2 \cdot \bar{n} + 4n \cdot e_2 \\
 &\quad + 4n \cdot n - n \cdot \bar{n} - \bar{n} \cdot e_2 - \bar{n} \cdot n + \frac{1}{4}\bar{n} \cdot \bar{n} \\
 X \cdot X &= 4 - 2 - 2 = 0
 \end{aligned} \tag{11.49}$$

which confirms our prediction and enables us to define a point as

$$P = \frac{1}{2}(2x + x^2n - \bar{n}). \tag{11.50}$$

### 11.5.2 Point pair

We now know that a single point in Euclidean space is represented by a null vector in conformal space, and we also know that

$$X_1 \cdot X_1 = 0 \tag{11.51}$$

and

$$X_1 \cdot X_2 = -\frac{1}{2}(x_1 - x_2)^2 \tag{11.52}$$

where the inner product is a measure of distance between two points. It turns out that the outer product of two points encodes the geometry associated with a pair of points. For example, consider two Euclidean points  $x_1, x_2$  represented as null vectors

$$X_1 = \frac{1}{2}(2x_1 + x_1^2n - \bar{n}) \tag{11.53}$$

$$X_2 = \frac{1}{2}(2x_2 + x_2^2n - \bar{n}) \tag{11.54}$$

then

$$\begin{aligned}
 X_1 \wedge X_2 &= \frac{1}{4}(2x_1 + x_1^2n - \bar{n}) \wedge (2x_2 + x_2^2n - \bar{n}) \\
 &= x_1 \wedge x_2 + \frac{1}{2}x_2^2x_1 \wedge n - \frac{1}{2}x_1 \wedge \bar{n} + \frac{1}{2}x_1^2n \wedge x_2 \\
 &\quad - \frac{1}{4}x_1^2n \wedge \bar{n} - \frac{1}{2}\bar{n} \wedge x_2 - \frac{1}{4}x_2^2\bar{n} \wedge n \\
 X_1 \wedge X_2 &= x_1 \wedge x_2 + \frac{1}{2}(x_2^2x_1 - x_1^2x_2) \wedge n \\
 &\quad + \frac{1}{2}(x_2 - x_1) \wedge \bar{n} + \frac{1}{4}(x_2^2 - x_1^2)n \wedge \bar{n}.
 \end{aligned} \tag{11.55}$$

It is obvious that Eq. (11.55) directly stores the outer product of the original two points, together with seven other bivectors formed by the Euclidean basis vectors with  $n$  and  $\bar{n}$ .

### 11.5.3 Lines

A line in the conformal model is defined as

$$L = P_1 \wedge P_2 \wedge n \quad (11.56)$$

where  $P_1$  and  $P_2$  are two points on the line, and  $n$  is the point at infinity. Let's see what happens when we substitute two points

$$P_1 = \frac{1}{2}(2x_1 + x_1^2 n - \bar{n}) \quad (11.57)$$

$$P_2 = \frac{1}{2}(2x_2 + x_2^2 n - \bar{n}). \quad (11.58)$$

But as  $P_1$  and  $P_2$  are to be wedged with  $n$ , the  $n$  terms can be dropped, as they create zero elements:

$$\begin{aligned} L &= \frac{1}{4}(2x_1 - \bar{n}) \wedge (2x_2 - \bar{n}) \wedge n \\ &= \frac{1}{4}(4x_1 \wedge x_2 - 2x_1 \wedge \bar{n} + 2x_2 \wedge \bar{n}) \wedge n \\ &= x_1 \wedge x_2 \wedge n + \frac{1}{2}x_1 \wedge n \wedge \bar{n} - \frac{1}{2}x_2 \wedge n \wedge \bar{n} \\ L &= x_1 \wedge x_2 \wedge n - \frac{1}{2}(x_2 - x_1) \wedge n \wedge \bar{n}. \end{aligned} \quad (11.59)$$

The first trivector term encodes the bivector  $x_1 \wedge x_2$  which is a measure of the area formed by the two vectors and the origin. When  $x_2 = \lambda x_1$ , i.e. the line connecting  $P_1$  and  $P_2$  passes through the origin, the trivector collapses to zero. The second trivector term encodes the direction vector representing the line. For example, to represent a line parallel with the  $x$  axis we can let  $P_1 = \bar{n}$  and  $P_2$  be the point  $(1, 0, 0)$ . Thus using Eq. (11.59) we have

$$\begin{aligned} L &= \bar{n} \wedge e_1 \wedge n - \frac{1}{2}(e_1 - \bar{n}) \wedge n \wedge \bar{n} \\ &= \bar{n} \wedge e_1 \wedge n - \frac{1}{2}e_1 \wedge n \wedge \bar{n} \\ L &= \frac{1}{2}e_1 \wedge n \wedge \bar{n} \end{aligned} \quad (11.60)$$

We can rewrite Eq. (11.60) in terms of  $e$  and  $\bar{e}$  as follows:

$$L = -e_1 e \bar{e}. \quad (11.61)$$

Thus we can see that the original Euclidean line is recognizable within this conformal description. Before moving on, let's try two other points  $x_1 = (1, 1, 0)$   $x_2 = (1, 2, 0)$ :

$$\begin{aligned} L &= (e_1 + e_2) \wedge (e_1 + 2e_2) \wedge n - \frac{1}{2}e_2 \wedge n \wedge \bar{n} \\ &= e_1 \wedge e_2 \wedge n - \frac{1}{2}e_2 \wedge n \wedge \bar{n} \\ L &= e_1 e_2 \wedge n + e_2 e \bar{e}. \end{aligned} \quad (11.62)$$

Once more, it is possible to recognize the original line's characteristics.

Now let's see what happens if we introduce a third point  $X$ , which must also have a null vector representation i.e.  $X^2 = 0$  then  $L \wedge X = 0$ . We can demonstrate this using the last example with  $x = (1, 3, 0)$ , which is on the same line:

$$L \wedge X = \left( e_1 \wedge e_2 \wedge n - \frac{1}{2} e_2 \wedge n \wedge \bar{n} \right) \wedge \frac{1}{2} (2e_1 + 6e_2 + 10n - \bar{n})$$

$$L \wedge X = -\frac{1}{2} e_1 \wedge e_2 \wedge n \wedge \bar{n} + \frac{1}{2} e_1 \wedge e_2 \wedge n \wedge \bar{n} = 0. \quad (11.63)$$

Finally, if we chose a point that is not on the line, e.g. the origin, then

$$L \wedge X = \left( e_1 \wedge e_2 \wedge n - \frac{1}{2} e_2 \wedge n \wedge \bar{n} \right) \wedge \frac{1}{2} (-\bar{n})$$

$$L \wedge X = e_1 e_2 e \bar{e} \quad (11.64)$$

which is confirmation.

### 11.5.4 Circles

One of the reasons for using the conformal model is to exploit the point at infinity, which has the effect of straightening circles into lines, and spheres into planes. Knowing that a line is represented by

$$L = P_1 \wedge P_2 \wedge n \quad (11.65)$$

a circle then becomes

$$L = P_1 \wedge P_2 \wedge P_3 \quad (11.66)$$

where  $P_1, P_2, P_3$  are three points on the circle.

Let's investigate this equation where

$$x_1 = e_1 \quad P_1 = \frac{1}{2} (2e_1 + n - \bar{n}) \quad (11.67)$$

$$x_2 = e_2 \quad P_2 = \frac{1}{2} (2e_2 + n - \bar{n}) \quad (11.68)$$

$$x_3 = -e_1 \quad P_3 = \frac{1}{2} (-2e_1 + n - \bar{n}) \quad (11.69)$$

as shown in Fig. 11.8.

Then the circle  $C$  is given by

$$C = \frac{1}{2} (2e_1 + n - \bar{n}) \wedge \frac{1}{2} (2e_2 + n - \bar{n}) \wedge \frac{1}{2} (-2e_1 + n - \bar{n})$$

$$= (e_1 + \bar{e}) \wedge (e_2 + \bar{e}) \wedge (-e_1 + \bar{e})$$

$$= (e_1 \wedge e_2 + e_1 \wedge \bar{e} - e_2 \wedge \bar{e}) \wedge (-e_1 + \bar{e})$$

$$= e_1 \wedge e_2 \wedge \bar{e} + e_2 \wedge \bar{e} \wedge e_1$$

$$C = 2e_1 e_2 \bar{e}. \quad (11.70)$$

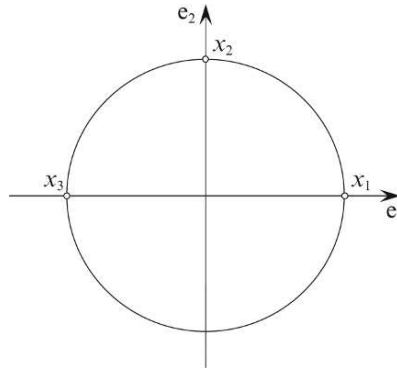


FIGURE 11.8.

The homogeneous multiplier 2 has no geometric significance and can be ignored. Therefore, the circle is given by

$$C = e_1 e_2 \bar{e} \tag{11.71}$$

The radius of the circle is given by

$$\rho^2 = \frac{-C^2}{(C \wedge n)^2} \tag{11.72}$$

which in the above example is

$$\begin{aligned} \rho^2 &= \frac{-e_1 e_2 \bar{e} e_1 e_2 \bar{e}}{(e_1 e_2 \bar{e} \wedge n)^2} \\ &= \frac{-e_1 e_2 \bar{e} e_1 e_2 \bar{e}}{e_1 e_2 \bar{e} e e_1 e_2 \bar{e} e} \\ \rho^2 &= \frac{-e_1 e_2 \bar{e} e_1 e_2 \bar{e}}{e_1 e_2 \bar{e} e e_1 e_2 \bar{e} e^2} \end{aligned} \tag{11.73}$$

remember that we can cancel vectors:

$$\rho^2 = \frac{-1}{e^2} = 1 \tag{11.74}$$

which returns a unit radius.

The circle's centre is given by

$$CnC \tag{11.75}$$

which in the above example is

$$\begin{aligned} CnC &= (e_1 e_2 \bar{e})n(e_1 e_2 \bar{e}) \\ &= (e_1 e_2 \bar{e})n(e_1 e_2 \bar{e}) \\ CnC &= -\bar{e}n\bar{e}. \end{aligned} \tag{11.76}$$

But this is a reflection condition, and with reference to Fig. 11.7 shows that  $-\bar{e}n\bar{e}$  is equivalent to reflecting  $n$  in  $\bar{e}$ , which makes

$$CnC = -\bar{n} \tag{11.77}$$

which is the null vector for the origin, and the circle's centre.

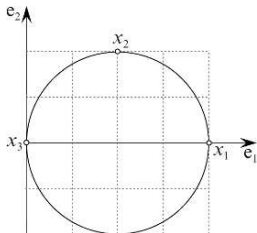


FIGURE 11.9.

Just to show how the algebra behaves in a more normal setting, consider the situation shown in Fig. 11.9 where the radius is 2 and the centre is  $2e_1$ .

$$x_1 = 4e_1 \quad P_1 = \frac{1}{2}(8e_1 + 16n - \bar{n}) \tag{11.78}$$

$$x_2 = 2e_1 + 2e_2 \quad P_2 = \frac{1}{2}(4e_1 + 4e_2 + 8n - \bar{n}) \tag{11.79}$$

$$x_3 = 0 \quad P_3 = -\frac{1}{2}\bar{n}. \tag{11.80}$$

Therefore,

$$\begin{aligned} C &= P_1 \wedge P_2 \wedge P_3 \\ &= \frac{1}{2}(8e_1 + 16n - \bar{n}) \wedge \frac{1}{2}(4e_1 + 4e_2 + 8n - \bar{n}) \wedge \left(-\frac{1}{2}\bar{n}\right) \\ C &= -4e_1 \wedge e_2 \wedge \bar{n} + 8e_2 \wedge n \wedge \bar{n}. \end{aligned} \tag{11.81}$$

Substituting

$$n = e + \bar{e} \quad \bar{n} = e - \bar{e} \tag{11.82}$$

we obtain

$$C = 4(-e_1e_2e + e_1e_2\bar{e} - 4e_2e\bar{e}). \tag{11.83}$$

It follows

$$C \wedge n = -4(2e_1e_2e\bar{e}). \tag{11.84}$$

Therefore

$$\begin{aligned} \rho^2 &= -\frac{4^2(-e_1e_2e + e_1e_2\bar{e} - 4e_2e\bar{e})(-e_1e_2e + e_1e_2\bar{e} - 4e_2e\bar{e})}{4^2(2e_1e_2e\bar{e})(2e_1e_2e\bar{e})} \\ \rho^2 &= -\frac{16}{-4} = 4 \end{aligned} \tag{11.85}$$

and the radius  $\rho = 2$ .

The circle's centre is given by

$$\begin{aligned}
 \text{CnC} &= 4^2(-e_1e_2e + e_1e_2\bar{e} - 4e_2e\bar{e})n(-e_1e_2e + e_1e_2\bar{e} - 4e_2e\bar{e}) \\
 &= 4^2(-e_1e_2e + e_1e_2\bar{e} - 4e_2e\bar{e})(e + \bar{e})(-e_1e_2e + e_1e_2\bar{e} - 4e_2e\bar{e}) \\
 &= 4^2(12e + 20\bar{e} + 16e_1) \\
 \text{CnC} &= 64(3e + 5\bar{e} + 4e_1). \tag{11.86}
 \end{aligned}$$

Before extracting the  $e_1$  coordinate we must normalize the null vector by dividing  $\text{CnC}$  by  $-(\text{CnC}) \cdot n$ :

$$\begin{aligned}
 \text{CnC} &= \frac{3e + 5\bar{e} + 4e_1}{-(3e + 5\bar{e} + 4e_1) \cdot (e + \bar{e})} \\
 &= \frac{3e + 5\bar{e} + 4e_1}{-3 + 5} \\
 \text{CnC} &= \frac{3}{2}e + \frac{5}{2}\bar{e} + 2e_1 \tag{11.87}
 \end{aligned}$$

which confirms that the centre is  $2e_1$ .

### 11.5.5 Planes

A plane in the conformal model is defined as

$$\pi = P_1 \wedge P_2 \wedge P_3 \wedge n \tag{11.88}$$

where  $P_1, P_2, P_3$  are three points on the plane, and  $n$  is the point at infinity. This time we see a quadvector encodes a plane. We can represent the plane  $e_1 \wedge e_2$  by letting  $P_1$  be the point  $(1, 0, 0)$ ,  $P_2$  be the point  $(0, 1, 0)$  and  $P_3$  is the origin. We can drop the  $n$  terms in  $P_1$  and  $P_2$  as they are being wedged with  $n$  and create zero terms.

$$\begin{aligned}
 \pi &= \frac{1}{2}(2e_1 - \bar{n}) \wedge \frac{1}{2}(2e_2 - \bar{n}) \wedge -\frac{1}{2}\bar{n} \wedge n \\
 &= \left(e_1 - \frac{1}{2}\bar{n}\right) \wedge \left(e_2 - \frac{1}{2}\bar{n}\right) \wedge -\frac{1}{2}\bar{n} \wedge n \\
 &= \left(e_1e_2 - \frac{1}{2}e_1 \wedge \bar{n} + \frac{1}{2}e_2 \wedge \bar{n}\right) \wedge -\frac{1}{2}\bar{n} \wedge n \\
 &= -\frac{1}{2}e_1e_2 \wedge \bar{n} \wedge n \\
 \pi &= e_1e_2\bar{e} \tag{11.89}
 \end{aligned}$$

which is a quadvector formed by the bivector  $e_1 \wedge e_2$  the infinity and origin vectors.



If we introduce a fourth point  $X$  on the plane, which also has a null vector representation, i.e.  $X^2 = 0$ , then  $\pi \wedge X = 0$ . For example, the point  $(1, 1, 0)$  is on the plane and

$$X = \frac{1}{2}(2e_1 + 2e_2 + 2n - \bar{n}) \tag{11.90}$$

and

$$\begin{aligned} \pi \wedge X &= (e_1 e_2 \bar{e} e) \wedge \frac{1}{2}(2e_1 + 2e_2 + 2n - \bar{n}) \\ \pi \wedge X &= (e_1 e_2 \bar{e} e) \wedge \frac{1}{2}(2e_1 + 2e_2 + e + 3\bar{e}) = 0. \end{aligned} \tag{11.91}$$

There is no need to demonstrate that a point off the bivector creates a non-zero answer.

### 11.5.6 Spheres

A sphere is defined as

$$S = P_1 \wedge P_2 \wedge P_3 \wedge P_4 \tag{11.92}$$

and using the points for a unit radius sphere centered at the origin, as shown in Fig. 11.10 we have:

$$x_1 = e_1 \quad P_1 = \frac{1}{2}(2e_1 + n - \bar{n}) = e_1 + \bar{e} \tag{11.93}$$

$$x_2 = e_2 \quad P_2 = \frac{1}{2}(2e_2 + n - \bar{n}) = e_2 + \bar{e} \tag{11.94}$$

$$x_3 = e_3 \quad P_3 = \frac{1}{2}(2e_3 + n - \bar{n}) = e_3 + \bar{e} \tag{11.95}$$

$$x_4 = -e_1 \quad P_4 = \frac{1}{2}(-2e_1 + n - \bar{n}) = -e_1 + \bar{e} \tag{11.96}$$

$$\begin{aligned} S &= (e_1 + \bar{e}) \wedge (e_2 + \bar{e}) \wedge (e_3 + \bar{e}) \wedge (-e_1 + \bar{e}) \\ &= (e_1 \wedge e_2 + e_1 \wedge \bar{e} - e_2 \wedge \bar{e}) \wedge (-e_3 \wedge e_1 + e_3 \wedge \bar{e} + e_1 \wedge \bar{e}) \end{aligned}$$

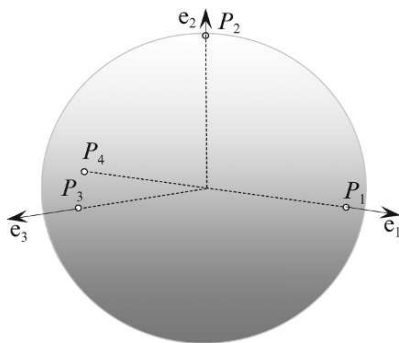


FIGURE 11.10.

$$\begin{aligned}
&= \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \bar{\mathbf{e}} \\
S &= \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \bar{\mathbf{e}}.
\end{aligned} \tag{11.97}$$

The radius of the sphere is given by

$$\rho^2 = \frac{-S^2}{(S \wedge \mathbf{n})^2} \tag{11.98}$$

which in the above example is

$$\begin{aligned}
\rho^2 &= -\frac{\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \bar{\mathbf{e}} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \bar{\mathbf{e}}}{(\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \bar{\mathbf{e}} \wedge (\mathbf{e} + \bar{\mathbf{e}}))^2} \\
\rho^2 &= -\frac{1}{-1} = 1
\end{aligned} \tag{11.99}$$

and is correct.

The sphere's centre is given by

$$\begin{aligned}
\varepsilon &= S \mathbf{n} S = (\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \bar{\mathbf{e}}) \mathbf{n} (\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \bar{\mathbf{e}}) \\
&= \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \bar{\mathbf{e}} \mathbf{n} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \bar{\mathbf{e}} \\
\varepsilon &= -\bar{\mathbf{e}} \mathbf{n} \bar{\mathbf{e}}.
\end{aligned} \tag{11.100}$$

But this is the same reflection condition found with the circle and is equivalent to reflecting  $\mathbf{n}$  in  $\bar{\mathbf{e}}$  and makes

$$\varepsilon = -\bar{\mathbf{n}} \tag{11.101}$$

which is the null vector for the origin and the sphere's centre.

## 11.6 Conformal transformations

The inspiration for this section is Doran and Lasenby [15] where readers will find a rigorous description of conformal transformations. In the following section I will provide readers with an outline of the concepts with some relevant examples.

The transforms we normally encounter in Euclidean space comprise translation, rotation and scaling, and are normally implemented using matrices. Such transforms are available in conformal space but are implemented using rotors. To clarify the following description, Euclidean vector space is referred to as  $\mathcal{V}(p, q)$ .

### 11.6.1 Translations

In chapter 9 we discovered that a rotor has the form

$$R = e^{-\hat{\mathbf{B}}\theta/2} \tag{11.102}$$

where  $\hat{B}$  is a unit bivector and  $\theta$  is the angle of rotation. In the conformal model, a rotor takes the form

$$R = e^{na/2} \tag{11.103}$$

where  $n$  is the null infinity vector and  $a$  is a vector in  $\mathcal{V}(p, q)$ . These vectors are orthogonal, therefore  $a \cdot n = 0$  and the product  $na$  represents a bivector, but it is null:

$$(na)^2 = nana = -anna = 0. \tag{11.104}$$

Equation (11.103) can be expanded using the Taylor series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \dots \tag{11.105}$$

however, when  $x = na/2$ , something nice happens – the series terminates after the second term, simply because the null vector creates a zero value. Therefore,

$$T_a = e^{na/2} = 1 + \frac{na}{2}. \tag{11.106}$$

We have already discovered that rotors work like quaternions, where a vector is sandwiched between two quaternions, one of which is the reverse of the other. Fortunately, in conformal space the same model prevails. For example, the rotor  $T_a$  is used as follows:

$$T_a x \tilde{T}_a. \tag{11.107}$$

In fact, let's see how the above expands:

$$\begin{aligned} T_a x \tilde{T}_a &= \left(1 + \frac{na}{2}\right) x \left(1 - \frac{na}{2}\right) \\ &= x + \frac{1}{2}nax - \frac{1}{2}xna - \frac{1}{4}naxna \\ T_a x \tilde{T}_a &= x + \frac{1}{2}nax + \frac{1}{2}nxa. \end{aligned} \tag{11.108}$$

But

$$a \cdot x = \frac{1}{2}(ax + xa) \tag{11.109}$$

therefore,

$$T_a x \tilde{T}_a = x + n(a \cdot x). \tag{11.110}$$

It is also worth exploring what this rotor does to the null vectors  $n$  and  $\bar{n}$ :

$$\begin{aligned} T_a n \tilde{T}_a &= \left(1 + \frac{na}{2}\right) n \left(1 - \frac{na}{2}\right) \\ &= \left(n + \frac{nan}{2}\right) \left(1 - \frac{na}{2}\right) \\ &= n + \frac{nan}{2} - \frac{nna}{2} - \frac{nanna}{4} \\ T_a n \tilde{T}_a &= n - \frac{n^2 a}{2} - \frac{n^2 a}{2} - \frac{nan^2 a}{4} = n. \end{aligned} \tag{11.111}$$

Similarly,

$$\begin{aligned}
 T_a \bar{n} \tilde{T}_a &= \left(1 + \frac{na}{2}\right) \bar{n} \left(1 - \frac{na}{2}\right) \\
 &= \left(\bar{n} + \frac{na\bar{n}}{2}\right) \left(1 - \frac{na}{2}\right) \\
 &= \bar{n} - \frac{\bar{n}na}{2} + \frac{na\bar{n}}{2} - \frac{na\bar{n}na}{4} \\
 &= \bar{n} - \frac{(\bar{n}n + n\bar{n})a}{2} + \frac{n\bar{n}na^2}{4} \\
 T_a \bar{n} \tilde{T}_a &= \bar{n} - 2a - a^2 n.
 \end{aligned} \tag{11.112}$$

To summarize:

$$T_a x \tilde{T}_a = x + n(a \cdot x) \tag{11.113}$$

$$T_a n \tilde{T}_a = n \tag{11.114}$$

$$T_a \bar{n} \tilde{T}_a = \bar{n} - 2a - a^2 n. \tag{11.115}$$

The reason why we have computed these products is that they cover all the terms found in the conformal null vector:

$$F(x) = 2x + x^2 n - \bar{n} \tag{11.116}$$

which means that we can combine the above results to evaluate  $T_a F(x) \tilde{T}_a$ :

$$\begin{aligned}
 T_a F(x) \tilde{T}_a &= T_a(2x) \tilde{T}_a + T_a(x^2 n) \tilde{T}_a - T_a \bar{n} \tilde{T}_a \\
 &= 2(x + n(a \cdot x)) + x^2 n - \bar{n} + 2a + a^2 n \\
 &= 2x + 2n(a \cdot x) + x^2 n - \bar{n} + 2a + a^2 n \\
 T_a F(x) \tilde{T}_a &= 2(x + a) + (x + a)^2 n - \bar{n}.
 \end{aligned} \tag{11.117}$$

Equation (11.117) looks familiar. In fact it has a structure identical to Eq. (11.116) apart from  $x + a$  replaces the original  $x$ . The geometric interpretation is that  $x$  has been translated by  $a$ ! Let's illustrate exactly how this operates.

Consider an object located at  $(1, 0, 0)$  which is to be translated by  $2e_2$ . This can be undertaken within conformal space using the following steps:

1st Convert the point to be translated to a null vector:

$$(1, 0, 0) \rightarrow \frac{1}{2}(2e_1 + n - \bar{n}). \tag{11.118}$$

2nd Create the rotor

$$\left(1 + \frac{2ne_2}{2}\right) \cdots \left(1 - \frac{2ne_2}{2}\right) = (1 + ne_2) \cdots (1 - ne_2). \tag{11.119}$$

3rd Combine the above elements

$$\begin{aligned}
 T_a F(x) \tilde{T}_a &= \frac{1}{2} (1 + ne_2)(2e_1 + n - \bar{n})(1 - ne_2) \\
 &= \frac{1}{2} (2e_1 + n - \bar{n} + 2ne_2e_1 - ne_2\bar{n})(1 - ne_2) \\
 &= \frac{1}{2} \begin{pmatrix} 2e_1 + n - \bar{n} + 2ne_2e_1 - ne_2\bar{n} - 2e_1ne_2 + \\ \bar{n}ne_2 - 2ne_2e_1ne_2 - ne_2\bar{n}ne_2 \end{pmatrix} \\
 &= \frac{1}{2} (2e_1 + n - \bar{n} + e_2(n\bar{n} + \bar{n}n) + e_2n\bar{n}ne_2) \\
 &= \frac{1}{2} (2e_1 + n - \bar{n} + 4e_2 + e_24ne_2) \\
 T_a F(x) \tilde{T}_a &= e_1 + 2e_2 + \frac{5}{2}n - \frac{1}{2}\bar{n}, \tag{11.120}
 \end{aligned}$$

which is correct.

Now the above may seem a rather convoluted way of adding  $2e_2$  to  $e_1$  to produce  $e_1 + 2e_2$ , but this example does not reflect what conformal transformations about. So let's examine how rotations are implemented.

## 11.6.2 Rotations

We have seen that rotations in GA take the form

$$x' = Rx\tilde{R} \tag{11.121}$$

where the vector  $x$  is rotated about the origin. The rotor  $R$  takes the form

$$R = \cos(\theta/2) - \hat{B} \sin(\theta/2) \tag{11.122}$$

where  $\hat{B}$  is a unit bivector and  $\theta$  is the angle of rotation. For example, to rotate the point  $3e_1$  about the origin  $180^\circ$  in the plane  $e_3e_1$ , the rotor is

$$\begin{aligned}
 R &= \cos 90^\circ - e_3e_1 \sin 90^\circ \\
 R &= e_1e_3. \tag{11.123}
 \end{aligned}$$

The point  $3e_1$  is rotated as follows:

$$\begin{aligned}
 x' &= e_1e_3(3e_1)e_3e_1 \\
 &= 3e_1e_3e_1e_3e_1 \\
 x' &= -3e_1. \tag{11.124}
 \end{aligned}$$

But say we wanted to rotate the point about  $2e_1$  in the same plane? Well, if we were using matrices, we would translate the point  $-2e_1$ , perform the rotation about the origin, and translate back  $2e_1$ .

Now the extremely useful feature of the conformal model is that the operations of translation and rotation are integrated in a rather elegant way, as we will now discover.

To begin with, conformal rotations use the same technique of sandwiching a conformal vector using

$$x' = RF(x)\tilde{R} \quad (11.125)$$

where  $F(x)$  represents the null vector. For instance, using the above example we proceed as follows:

$$\begin{aligned} x' &= e_1 e_3 F(3e_1) e_3 e_1 \\ &= e_1 e_3 \frac{1}{2} (6e_1 + 9n - \bar{n}) e_3 e_1 \\ &= \frac{1}{2} (6e_1 e_3 e_1 + 9e_1 e_3 n - e_1 e_3 \bar{n}) e_3 e_1 \\ &= \frac{1}{2} (-6e_3 e_3 e_1 + 9e_1 e_3 n e_3 e_1 - e_1 e_3 \bar{n} e_3 e_1) \\ &= \frac{1}{2} (-6e_1 - 9n - \bar{n}) \\ x' &= -3e_1 - 5e - 4\bar{e} \end{aligned} \quad (11.126)$$

and we see that the point  $3e_1$  has indeed been rotated to  $-3e_1$ . Now let's see what needs to be done if we offset the axis of rotation.

We begin by translating the conformal null vector back to the origin by the required distance:

$$T_{-a}F(x)\tilde{T}_{-a}. \quad (11.127)$$

Next, we rotate it about the origin:

$$R(T_{-a}F(x)\tilde{T}_{-a})\tilde{R} \quad (11.128)$$

Then, we translate it back to where it came from:

$$F(x') = T_a(R(T_{-a}F(x)\tilde{T}_{-a})\tilde{R})\tilde{T}_a. \quad (11.129)$$

The parentheses are included to clarify the process, but they are not important in this associative expression:

$$F(x') = T_a R T_{-a} F(x) \tilde{T}_{-a} \tilde{R} \tilde{T}_a. \quad (11.130)$$

However, we can introduce some new parentheses to reveal an interesting pattern:

$$F(x') = (T_a R T_{-a}) F(x) (\tilde{T}_{-a} \tilde{R} \tilde{T}_a) \quad (11.131)$$

A useful equality is that  $T_{-a} \equiv \tilde{T}_a$  therefore we have:

$$F(x') = (T_a R \tilde{T}_a) F(x) (\tilde{T}_{-a} \tilde{R} \tilde{T}_{-a}) \quad (11.132)$$

or if we let

$$R = T_a R \tilde{T}_a \quad \text{and} \quad \tilde{R} = \tilde{T}_{-a} \tilde{R} \tilde{T}_{-a} \quad (11.133)$$

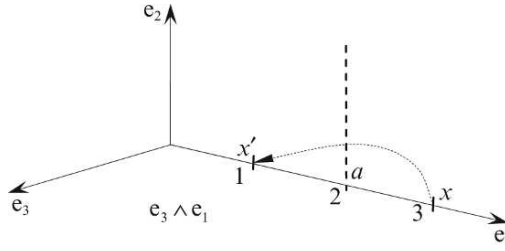


FIGURE 11.11.

then

$$F(x') = RF(x)\tilde{R}. \tag{11.134}$$

The important discovery here is that the rotor  $R$  incorporates a rotation and a translation, which we will illustrate with an example.

Consider the scenario shown in Fig. 11.11 where the point  $3e_1$  is rotated  $180^\circ$  about the point  $2e_2$  in the plane  $e_3 \wedge e_1$ . The rotors are defined as follows:

$$\begin{aligned} a &= 2e_1 \quad x = 3e_1 & (11.135) \\ T_a &= 1 + \frac{1}{2}na = 1 + ne_1 \quad R_a = e_1e_3 \quad \tilde{T}_a = 1 + \frac{1}{2}an = 1 + e_1n. \end{aligned}$$

Therefore,

$$\begin{aligned} R &= T_a R_a \tilde{T}_a = (1 + ne_1)e_1e_3(1 + e_1n) \\ &= e_1e_3 + ne_3 + e_1e_3e_1n \\ R &= e_1e_3 - 2e_3n \end{aligned} \tag{11.136}$$

and

$$\tilde{R} = e_3e_1 + 2e_3n. \tag{11.137}$$

Now we compute

$$\begin{aligned} RF(3e_1)\tilde{R} &= (e_1e_3 - 2e_3n)\frac{1}{2}(6e_1 + 9n - \bar{n})(e_3e_1 + 2e_3n) \\ &= \frac{1}{2}(-6e_3 + 9e_1e_3n - e_1e_3\bar{n} - 12e_1ne_3 + 2e_3n\bar{n})(e_3e_1 + 2e_3n) \\ &= \frac{1}{2}(-6e_3 - 3e_1e_3n - e_1e_3\bar{n} + 2e_3n\bar{n})(e_3e_1 + 2e_3n) \\ &= \frac{1}{2} \begin{pmatrix} -6e_1 - 3e_1e_3ne_3e_1 - e_1e_3\bar{n}e_3e_1 + 2e_3n\bar{n}e_3e_1 - \\ 12n - 2e_1e_3\bar{n}e_3n + 4e_3n\bar{n}e_3n \end{pmatrix} \\ &= \frac{1}{2}(-6e_1 - 15n - \bar{n} + 2n\bar{n}e_1 + 2e_1\bar{n}n + 4n\bar{n}n) \\ &= \frac{1}{2}(2e_1 + n - \bar{n}) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}(2e_1 + 2\bar{e}) \\
RF(x)\tilde{R} &= e_1 + \bar{e} \tag{11.138}
\end{aligned}$$

therefore, the rotated point is  $e_1$ , which is correct.

### 11.6.3 Dilations

The term *dilation* refers to the stretching of vectors and *contraction* refers to the shrinking of vectors. However, we can accommodate both operations by treating stretching as dilations greater than 1, and shrinking as dilations less than 1. So what we want to find now is a rotor that performs this operation. Doran and Lasenby [15] offer the following argument:

We start with the premise that a dilation relative to the origin is effected by

$$x' = e^{-\alpha}x, \quad \text{where } \alpha \in \mathbb{R}. \tag{11.139}$$

As  $e^{-\alpha}$  is a scalar, it cannot alter the angle of  $x$ , therefore it is a conformal operation. The null vector representing  $x'$  is given by

$$\begin{aligned}
F(x') &= 2e^{-\alpha}x + (e^{-\alpha})^2x^2n - \bar{n} \\
F(x') &= e^{-\alpha}(2x + e^{-\alpha}x^2n - e^{\alpha}\bar{n}). \tag{11.140}
\end{aligned}$$

Therefore, we require a map to ensure

$$n \mapsto e^{-\alpha}n \quad \text{and} \quad \bar{n} \mapsto e^{\alpha}\bar{n}. \tag{11.141}$$

Fortunately such a map exists and takes the form of a rotor  $D_{\alpha}$  using

$$D_{\alpha}F(x)\tilde{D}_{\alpha}. \tag{11.142}$$

The rotor  $D_{\alpha}$  is defined as

$$D_{\alpha} = e^{\alpha N/2} = \cosh(\alpha/2) + \sinh(\alpha/2)N \tag{11.143}$$

where, with reference to Table 11.2,  $N$  is defined as

$$N = e\bar{e} = \frac{1}{2}\bar{n} \wedge n. \tag{11.144}$$

Before showing how the rotor works, we require the following products in the proof:

$$nN = (e + \bar{e})e\bar{e} = n. \tag{11.145}$$

$$Nn = e\bar{e}(e + \bar{e}) = -n \tag{11.146}$$

$$\bar{n}N = (e - \bar{e})e\bar{e} = -\bar{n} \tag{11.147}$$

$$N\bar{n} = e\bar{e}(e - \bar{e}) = \bar{n} \tag{11.148}$$

$$N^2 = (e\bar{e})^2 = 1. \tag{11.149}$$



Notice that  $n$  and  $N$ , and  $\bar{n}$  and  $N$  anticommute, which permits us to write

$$D_\alpha n \tilde{D}_\alpha = D_\alpha^2 n \quad \text{and} \quad D_\alpha \bar{n} \tilde{D}_\alpha = D_\alpha^2 \bar{n}. \quad (11.150)$$

We can now prove the action of the rotor as follows:

$$\begin{aligned} D_\alpha n \tilde{D}_\alpha &= D_\alpha^2 n = (\cosh(\alpha/2) + \sinh(\alpha/2)N)^2 n \\ &= (\cosh^2(\alpha/2) + \sinh^2(\alpha/2))n - 2 \cosh(\alpha/2) \sinh(\alpha/2)n \\ &= (\cosh \alpha - \sinh \alpha)n \\ D_\alpha n \tilde{D}_\alpha &= e^{-\alpha} n. \end{aligned} \quad (11.151)$$

Similarly,

$$\begin{aligned} D_\alpha \bar{n} \tilde{D}_\alpha &= D_\alpha^2 \bar{n} = (\cosh(\alpha/2) + \sinh(\alpha/2)N)^2 \bar{n} \\ &= (\cosh^2(\alpha/2) + \sinh^2(\alpha/2))\bar{n} + 2 \cosh(\alpha/2) \sinh(\alpha/2)\bar{n} \\ &= (\cosh \alpha + \sinh \alpha)\bar{n} \\ D_\alpha \bar{n} \tilde{D}_\alpha &= e^\alpha \bar{n} \end{aligned} \quad (11.152)$$

which confirms that Eq. (11.143) is the desired rotor.

Consequently, we can write

$$F(e^{-\alpha}x) = e^{-\alpha} D_\alpha F(x) \tilde{D}_\alpha. \quad (11.153)$$

For example, if we wanted to dilate the vector by a factor of 2, we compute  $e^\alpha = \frac{1}{2}$  which makes  $\alpha = \ln(0.5) = -0.693$ .

From this

$$D_\alpha = \cosh(-0.693/2) + \sinh(-0.693/2)N \quad (11.154)$$

and

$$\tilde{D}_\alpha = \cosh(-0.693/2) - \sinh(-0.693/2)N \quad (11.155)$$

this ensures that the following expression holds.

$$F(2x) = e^{0.693} D_\alpha F(x) \tilde{D}_\alpha. \quad (11.156)$$

A dilation about an arbitrary point is achieved by including the translation rotor  $T_a$ :

$$D'_\alpha = T_a D_\alpha \tilde{T}_a = e^{\alpha N'/2}, \quad (11.157)$$

where

$$N' = T_a N \tilde{T}_a. \quad (11.158)$$

But

$$\begin{aligned} \bar{n} \wedge n &= (e - \bar{e}) \wedge (e + \bar{e}) \\ &= 2e\bar{e} \\ \bar{n} \wedge n &= 2N \end{aligned} \quad (11.159)$$

therefore

$$N = \frac{1}{2} \bar{\mathbf{n}} \wedge \mathbf{n} \quad (11.160)$$

and

$$\begin{aligned} N' &= \frac{1}{2} T_a \bar{\mathbf{n}} \wedge \mathbf{n} \tilde{T}_a \\ N' &= \frac{1}{2} T_a \bar{\mathbf{n}} \tilde{T}_a \wedge T_n \tilde{T}_a. \end{aligned} \quad (11.161)$$

Substituting Eqs. (11.114) and (11.115) in Eq. (11.161) we have

$$N' = -\frac{1}{2} F(a) \wedge \mathbf{n} = -\frac{1}{2} A \wedge \mathbf{n} \quad (11.162)$$

where  $A = F(a)$ . Therefore, the rotor  $D'_\alpha$  is given by

$$\begin{aligned} D'_\alpha &= e^{\alpha N'/2} \\ &= e^{-\alpha A \wedge \mathbf{n}/4} \\ D'_\alpha &= \exp(-\alpha A \wedge \mathbf{n}/4). \end{aligned} \quad (11.163)$$

Even this can be tidied up to

$$D'_\alpha = \exp\left(\frac{\alpha}{2} \frac{A \wedge \mathbf{n}}{A \cdot \mathbf{n}}\right). \quad (11.164)$$

## 11.6.4 Reflections

In chapter 9 we discovered that GA handles reflections in a rather elegant fashion:

$$a' = -\hat{\mathbf{n}} a \hat{\mathbf{n}} \quad (11.165)$$

where  $a$  is the vector to be reflected and  $\hat{\mathbf{n}}$  is the unit normal vector of the reflector. However, there is one proviso with this formula – the vector and plane must intersect the origin. The conformal model overcomes this restriction as we will discover, but first, let's examine planes and lines that intersect the origin.

Figure 11.12 shows three points

$$(0, 0, 0) \quad (-e_3, e_2, 0) \quad (e_3, e_2, 0) \quad (11.166)$$

on a plane defined by

$$\pi = P_1 \wedge P_2 \wedge P_3 \wedge \mathbf{n} \quad (11.167)$$

where the three null vectors are

$$P_1 = -\frac{1}{2} \bar{\mathbf{n}} \quad (11.168)$$

$$P_2 = \frac{1}{2} (2e_2 - 2e_3 + 2\mathbf{n} - \bar{\mathbf{n}}) = e_2 - e_3 + \mathbf{n} - \frac{1}{2} \bar{\mathbf{n}} \quad (11.169)$$

$$P_3 = \frac{1}{2} (2e_2 + 2e_3 + 2\mathbf{n} - \bar{\mathbf{n}}) = e_2 + e_3 + \mathbf{n} - \frac{1}{2} \bar{\mathbf{n}}. \quad (11.170)$$

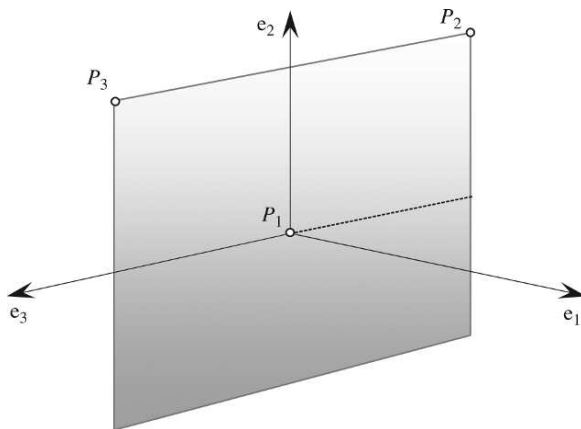


FIGURE 11.12.

The plane is

$$\begin{aligned}
 \pi &= -\frac{1}{2}\bar{n} \wedge \left( e_2 - e_3 + n - \frac{1}{2}\bar{n} \right) \wedge \left( e_2 + e_3 + n - \frac{1}{2}\bar{n} \right) \wedge n \\
 &= \left( -\frac{1}{2}\bar{n} \wedge e_2 + \frac{1}{2}\bar{n} \wedge e_3 - \frac{1}{2}\bar{n} \wedge n \right) \wedge \left( e_2 \wedge n + e_3 \wedge n - \frac{1}{2}\bar{n} \wedge n \right) \\
 &= -\frac{1}{2}\bar{n} \wedge e_2 \wedge e_3 \wedge n + \frac{1}{2}\bar{n} \wedge e_3 \wedge e_2 \wedge n \\
 \pi &= e_2 \wedge e_3 \wedge n \wedge \bar{n}. \tag{11.171}
 \end{aligned}$$

But as  $n \wedge \bar{n} = -2e\bar{e}$ , then

$$\pi = -e_2 e_3 e\bar{e}. \tag{11.172}$$

The plane equation contains two parts:  $-e_2 e_3$  and  $e\bar{e}$ . This construction is typical of a plane intersecting the origin, for as soon as the plane is displaced away from the origin, new terms appear. Now let's consider lines intersecting the origin.

Figure 11.13 shows two points

$$(0, 0, 0) \quad (e_1, e_2, 0) \tag{11.173}$$

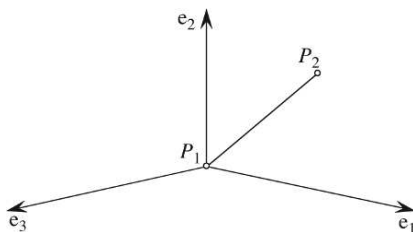


FIGURE 11.13.

on a line intersecting the origin, defined by

$$L = P_1 \wedge P_2 \wedge n \quad (11.174)$$

where the two null vectors are

$$P_1 = -\frac{1}{2}\bar{n} \quad (11.175)$$

$$P_2 = \frac{1}{2}(2e_1 + 2e_2 + 2n - \bar{n}) = e_1 + e_2 + n - \frac{1}{2}\bar{n}. \quad (11.176)$$

The line is

$$\begin{aligned} L &= -\bar{n} \wedge \left( e_1 + e_2 + n - \frac{1}{2}\bar{n} \right) \wedge n \\ &= -\bar{n} \wedge e_1 \wedge n - \bar{n} \wedge e_2 \wedge n \\ L &= -(e_1 + e_2) \wedge n \wedge \bar{n}. \end{aligned} \quad (11.177)$$

But as  $n \wedge \bar{n} = -2e\bar{e}$ , then

$$L = 2(e_1 + e_2)e\bar{e}. \quad (11.178)$$

The line equation contains two parts:  $2(e_1 + e_2)$  and  $e\bar{e}$ . This construction is also typical of a line intersecting the origin, for as soon as the line is displaced away from the origin, new terms appear.

In the conformal model the reflection of a line in a plane is achieved using

$$L' = \pi L \pi. \quad (11.179)$$

We can readily illustrate this using the above plane and line formulae:

$$\begin{aligned} L' &= \pi L \pi = (-e_2 e_3 e\bar{e})(2(e_1 + e_2)e\bar{e})(-e_2 e_3 e\bar{e}). \\ &= 2e_2 e_3 e\bar{e}(e_1 + e_2)e\bar{e}e_2 e_3 e\bar{e} \\ &= 2e_2 e_3 e\bar{e}(e_1 + e_2)e_2 e_3 \\ &= 2(e_2 e_3 e\bar{e}e_1 e_2 e_3 + e_2 e_3 e\bar{e}e_2 e_2 e_3) \\ &= 2(-e_1 e\bar{e} + e_2 e\bar{e}) \\ L' &= 2(-e_1 + e_2)e\bar{e}. \end{aligned} \quad (11.180)$$

The line has been reflected in the plane and clearly intersects the origin. Now here is the surprise: This reflection formula also works for lines and planes that do not intersect the origin! To illustrate this, consider the scenario shown in Fig. 11.14.

The figure shows a line  $L$  defined by the points  $P_4, P_5$ , whilst the plane  $\pi$  is defined by points  $P_1, P_2, P_3$ . In fact, we have already calculated the plane's conformal equation:

$$\pi = -e_2 e_3 e\bar{e}. \quad (11.181)$$

The line's points are

$$(0, e_2, 0) \quad (e_1, 2e_2, 0) \quad (11.182)$$

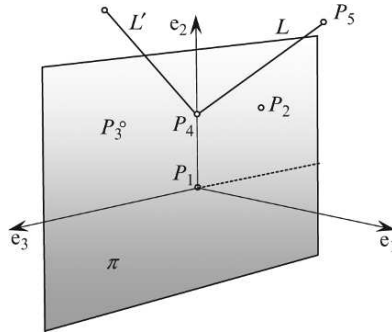


FIGURE 11.14.

where the two null vectors are

$$P_1 = \frac{1}{2}(2e_2 + n - \bar{n}) \tag{11.183}$$

$$P_2 = \frac{1}{2}(2e_1 + 4e_2 + 5n - \bar{n}). \tag{11.184}$$

However, as  $P_1$  and  $P_2$  are to be wedged with  $n$ , the  $n$  terms can be dropped as they create zero terms:

$$P_1 = e_2 - \frac{1}{2}\bar{n} \tag{11.185}$$

$$P_2 = e_1 + 2e_2 - \frac{1}{2}\bar{n}. \tag{11.186}$$

The conformal line is

$$\begin{aligned} L &= \left( e_2 - \frac{1}{2}\bar{n} \right) \wedge \left( e_1 + 2e_2 - \frac{1}{2}\bar{n} \right) \wedge n \\ &= e_2e_1 \wedge n - \frac{1}{2}e_2 \wedge \bar{n} \wedge n - \frac{1}{2}\bar{n} \wedge e_1 \wedge n - \bar{n} \wedge e_2 \wedge n \\ &= e_2e_1 \wedge n - \frac{1}{2}(e_1 + e_2) \wedge n \wedge \bar{n} \\ L &= e_2e_1(e + \bar{e}) + (e_1 + e_2)e\bar{e}. \end{aligned} \tag{11.187}$$

$$\pi = e_2e_3e\bar{e}. \tag{11.188}$$

The reflected line is  $L'$  and calculated as follows:

$$L' = \pi L \pi$$

$$L' = (e_2e_3e\bar{e})(e_2e_1(e + \bar{e}) + (e_1 + e_2)e\bar{e})(e_2e_3e\bar{e})$$

which reduces to

$$L' = e_1e_2(e + \bar{e}) + (-e_1 + e_2)e\bar{e} \tag{11.189}$$

where we see the  $e_1$  component reversed, creating the reflection.

### 11.6.5 Intersections

The following formulae are given without proof:

The intersection of two lines in a plane, where the lines are encoded by the trivectors  $L_1$  and  $L_2$  in the geometric space  $G(3, 1)$  is given by:

$$B = (L_1^* \wedge L_2^*)^* = I(L_1 \times L_2) \quad (11.190)$$

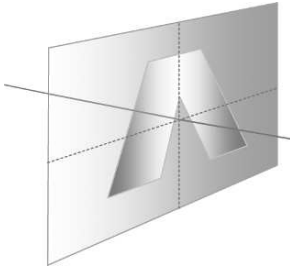
where  $I$  is the conformal pseudoscalar.

The intersection of a line and a surface, where the line is encoded by a 3-blade, and a plane or sphere by a 4-blade, is given by

$$B = (P^* \wedge L^*)^* = (IP) \cdot L = I(P \times L). \quad (11.191)$$

## 11.7 Summary

The objective of this chapter was not to write a rigorous, axiomatic description of the conformal model, rather a rudimentary introduction to the ideas and concepts employed supported by some examples. Hopefully, it will make life easier if you wish to delve deeper into this strange geometric world. I cannot deny that this has been a challenging chapter to write. In fact, if it had not been for Chris Doran and Anthony Lasenby's book [15] it would have never been written!



# 12 Applications of Geometric Algebra

## 12.1 Introduction

In the previous chapters we have seen how GA is easily used to solve various geometric problems in 2D and 3D, and obviously there are no restrictions for the algebra – after all, it is just an algebra for vectorial quantities. It is difficult to think of an area of CGI where vectors are not used – and sometimes when GA has been used, there are minor considerations of efficiency in execution time, although there is general agreement that it outperforms all other approaches for analytic elegance, and it is just a question of time before GA will be applied to all aspects of CGI.

In this chapter we explore a few more examples of how GA is providing new approaches to familiar problems.

## 12.2 3D Linear transformations

Ron Goldman has revealed how a family of linear transformations can be derived using a general vector equation [19]. The technique assumes the existence of two non-zero vectors  $a$  and  $b$ , subject to the conditions expressed in Eqs. (12.1), (12.2) and (12.3):

$$a \cdot b \neq 0 \tag{12.1}$$

$$a \cdot x = \lambda \quad \lambda \in \mathbb{R} \tag{12.2}$$

$$b \times x = c \tag{12.3}$$

where the unknown vector  $x$  is given by

$$x = \frac{\lambda b + c \times a}{a \cdot b}. \tag{12.4}$$

An example provided by Goldman computes the reflection of a line in an arbitrary plane. Figure 12.1 shows a plane  $\pi$  with its unit normal vector  $a$ , the vector to be reflected  $v$  and its reflection  $v'$  in the plane.

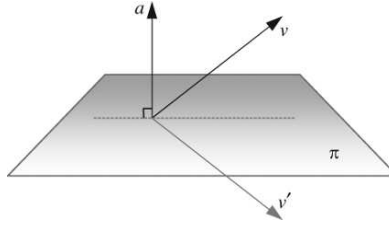


FIGURE 12.1.

Using Goldman's vector names, we make the following substitutions:

$$x = v' \quad (12.5)$$

$$b = a \quad (12.6)$$

$$a = b = a \quad (12.7)$$

which enables us to rewrite Eqs. (12.1), (12.2) and (12.3) as:

$$a \cdot a \neq 0 \quad (12.8)$$

$$a \cdot v' = \lambda \quad \lambda \in \mathbb{R} \quad (12.9)$$

$$a \times v' = c. \quad (12.10)$$

From Fig. 12.1 we can reason that

$$a \cdot v' = -a \cdot v \quad (12.11)$$

and

$$a \times v = a \times v' \quad (12.12)$$

which allows us to define  $\lambda$  and  $a \times v'$  in terms of the known vectors:

$$\lambda = -a \cdot v \quad (12.13)$$

$$c = a \times v' = a \times v. \quad (12.14)$$

We are now in a position to evaluate Eq. (12.4):

$$\frac{\lambda b + c \times a}{a \cdot b} = \frac{(-a \cdot v)a + (a \times v) \times a}{a \cdot a} \quad (12.15)$$

$$v' = (-a \cdot v)a + (a \times v) \times a. \quad (12.16)$$

Using the triple product identity [12] allows us to simplify our partial solution

$$(a \times b) \times c = (a \cdot c)b - (b \cdot c)a \quad (12.17)$$

$$v' = (-a \cdot v)a + (a \cdot a)v - (v \cdot a)a \quad (12.18)$$

$$v' = v - 2(a \cdot v)a, \quad (12.19)$$

which is the well-known formula for computing the reflection of a vector [20].



Hua Zhang *et al.* have developed Goldman's idea further, using GA [21]. To begin with, the plane is defined by the bivector  $B$ :

$$B = a \wedge x \quad (12.20)$$

where  $x$  is the unknown vector and

$$a \cdot x = \lambda. \quad (12.21)$$

Therefore, using the geometric product we have

$$ax = a \cdot x + a \wedge x = \lambda + B \quad (12.22)$$

and

$$x = \frac{\lambda + B}{a}. \quad (12.23)$$

Zhang *et al.* show that there is a correspondence between Eq. (12.4) and Eq. (12.23) and identify the relationship as follows:

To begin, Eq. (12.4) is rearranged to include the definition of  $c$ :

$$\frac{\lambda b + c \times a}{a \cdot b} = \frac{\lambda b + (b \times x) \times a}{a \cdot b}, \quad (12.24)$$

and the triple product identity substituted

$$= \frac{\lambda b + (b \cdot a)x - (a \cdot x)b}{a \cdot b}. \quad (12.25)$$

Next, Eq. (12.23) is rearranged to introduce a geometric product

$$\frac{\lambda + B}{a} = \frac{a(\lambda + B)}{aa} = \frac{\lambda a + aB}{a^2} \quad (12.26)$$

and expanded becomes:

$$= \frac{\lambda a + a \cdot B + a \wedge B}{\|a\|^2} \quad (12.27)$$

then the bivector is substituted:

$$= \frac{\lambda a + a \cdot (a \wedge x) + a \wedge (a \wedge x)}{\|a\|^2} \quad (12.28)$$

$$\frac{\lambda + B}{a} = \frac{\lambda a + (a \cdot a)x - (a \cdot x)a}{\|a\|^2}. \quad (12.29)$$

Comparing Eq. (12.25) with (12.29) we observe that they are identical if  $a = b$ , which, with this proviso, means that Eq. (12.23) can be used to resolve certain linear transformations. Let's demonstrate how Eq. (12.23) is used in practice to solve a scale and refraction transform.

### 12.2.1 Scale transform

Figure 12.2 shows a unit vector  $a$  which is to be scaled by a factor  $\varepsilon$ . Vectors  $v$  and  $v'$  reside in the plane  $\pi$  containing  $a$ , such that  $v$  identifies the head of  $a$ , and  $v'$  identifies the head of  $\varepsilon a$ .

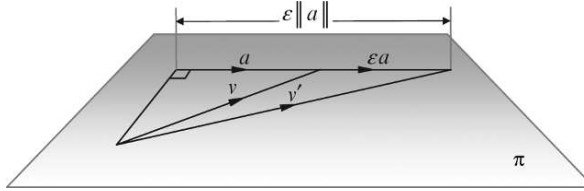


FIGURE 12.2.

As we know vectors  $a$  and  $v$ , we define the bivector as

$$B = a \wedge v = a \wedge v'. \quad (12.30)$$

The objective now is to discover the value of  $v'$ .

We equate

$$x = v' \quad (12.31)$$

and

$$\lambda = a \cdot v' = \varepsilon(a \cdot v). \quad (12.32)$$

Substituting  $v'$ ,  $\lambda$  and  $B$  in Eq. (12.23) produces

$$v' = a^{-1}(\lambda + B) = a^{-1}(\varepsilon(a \cdot v) + a \wedge v). \quad (12.33)$$

Multiplying throughout by  $a/a$  gives:

$$v' = \varepsilon(a \cdot v)a + a(a \wedge v). \quad (12.34)$$

Expanding the geometric product:

$$v' = \varepsilon(a \cdot v)a + a \cdot (a \wedge v) + a \wedge (a \wedge v). \quad (12.35)$$

Applying the identity:

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b \quad (12.36)$$

$$v' = \varepsilon(a \cdot v)a + (a \cdot a)v - (a \cdot v)a \quad (12.37)$$

we obtain

$$v' = v + (\varepsilon - 1)(a \cdot v)a. \quad (12.38)$$

Let's apply Eq. (12.38) to the example shown in Fig. 12.3.

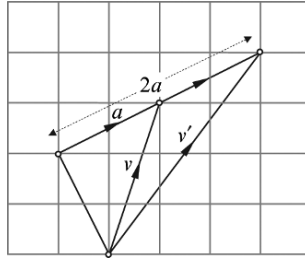


FIGURE 12.3.

From the figure we see that

$$\varepsilon = 2 \quad (12.39)$$

$$a = \frac{1}{\sqrt{5}}(2e_1 + e_2) \quad (12.40)$$

$$v = e_1 + 3e_2 \quad (12.41)$$

$$a \cdot v = \frac{1}{\sqrt{5}}(2e_1 + e_2) \cdot (e_1 + 3e_2)$$

$$a \cdot v = \frac{5}{\sqrt{5}} \quad (12.42)$$

$$\begin{aligned} v' &= v + (\varepsilon - 1)(a \cdot v)a \\ &= e_1 + 3e_2 + \frac{5}{\sqrt{5}} \frac{1}{\sqrt{5}}(2e_1 + e_2) \\ &= e_1 + 3e_2 + 2e_1 + e_2 \\ v' &= 3e_1 + 4e_2 \end{aligned} \quad (12.43)$$

which is correct.

## 12.2.2 Refraction transform

Refraction is the bending of light rays when they pass from one medium to another. Changes in the propagation speed alter the direction of the ray's associated wavefront and is measured by the refractive indices associated with the two media. In this example  ${}_{air}\mu_{m1}$  is the absolute refractive index for the medium  $m1$  above the plane, and  ${}_{air}\mu_{m2}$  is the absolute refractive index for the medium  $m2$  below the plane. Figure 12.4 shows the incident ray  $v$  and the refracted ray  $v_R$  with their respective angles of incidence and refraction  $\alpha$  and  $\beta$ . All vectors are unit vectors.

The illustrated scenario is described by Snell's law:

$$\frac{\sin \alpha}{\sin \beta} = \frac{{}_{air}\mu_{m2}}{{}_{air}\mu_{m1}} = \frac{1}{\mu}. \quad (12.44)$$

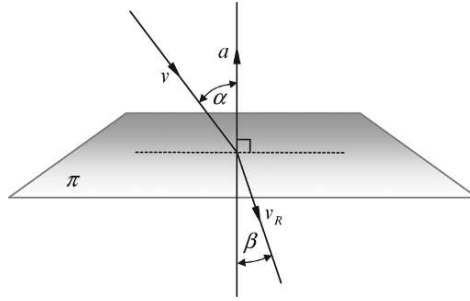


FIGURE 12.4.

Squaring Eq. (12.44) and rearranging the terms, we have

$$\mu^2 \sin^2 \alpha = \sin^2 \beta. \quad (12.45)$$

Substituting the following identities in Eq. (12.45) we have

$$\sin^2 \beta = 1 - \cos^2 \beta \quad (12.46)$$

and

$$\sin^2 \alpha = 1 - \cos^2 \alpha \quad (12.47)$$

$$\cos^2 \beta = 1 - \mu^2(1 - \cos^2 \alpha) \quad (12.48)$$

and

$$\cos \beta = \sqrt{1 - \mu^2(1 - \cos^2 \alpha)} \quad (12.49)$$

but

$$a \cdot v = \cos \alpha \quad (12.50)$$

$$\cos \beta = \sqrt{1 - \mu^2(1 - (a \cdot v)^2)}. \quad (12.51)$$

Introducing the outer product, we note that

$$a \wedge v = \|a\| \|v\| \sin \alpha \quad (12.52)$$

and

$$a \wedge v_R = \|a\| \|v_R\| \sin \beta \quad (12.53)$$

therefore

$$a \wedge v_R = \frac{\sin \beta}{\sin \alpha} (a \wedge v). \quad (12.54)$$

We are now in a position to invoke Eq. (12.5):

$$x = a^{-1}(\lambda + B) \quad (12.55)$$

where

$$x = v_R \quad (12.56)$$

$$\lambda = -\cos \beta \quad (12.57)$$

$$B = \frac{\sin \beta}{\sin \alpha} (a \wedge v) = \mu (a \wedge v) \quad (12.58)$$

$$v_R = a^{-1} (-\cos \beta + \mu (a \wedge v)). \quad (12.59)$$

Multiplying throughout by  $a/a$  we have

$$v_R = -a \cos \beta + \mu a (a \wedge v) \quad (12.60)$$

and expanding the geometric product we obtain

$$v_R = -a \cos \beta + \mu (a \cdot (a \wedge v) + a \wedge (a \wedge v)). \quad (12.61)$$

But

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b \quad (12.62)$$

therefore,

$$\begin{aligned} v_R &= -a \cos \beta + \mu ((a \cdot a)v - (a \cdot v)a) \\ &= -a \cos \beta + \mu (v - (a \cdot v)a) \\ v_R &= (-\cos \beta - \mu (a \cdot v))a + \mu v. \end{aligned} \quad (12.63)$$

Substituting Eq. (12.45) in (12.63) we have

$$v_R = \left( -\sqrt{1 - \mu^2(1 - (a \cdot v)^2)} - \mu(a \cdot v) \right) a + \mu v \quad (12.64)$$

which is the desired result.

Let's test Eq. (12.64) using the refraction caused when light moves from water to glass, where  ${}_{\text{air}}\mu_{\text{water}} = 1.33$  and  ${}_{\text{air}}\mu_{\text{glass}} = 1.5$

Given the following vectors and  $\mu$ :

$$a = e_2 \quad (12.65)$$

$$v = \frac{1}{\sqrt{2}}(e_1 - e_2) \quad (12.66)$$

$$\mu = 1.5/1.33 = 1.13 \quad (12.67)$$

$$v_R = \left( -\sqrt{1 - 1.13^2 \left( 1 - \left( \frac{1}{\sqrt{2}} \right)^2 \right)} - 1.13 \left( \frac{1}{\sqrt{2}} \right) \right) e_2 + 1.13 \left( \frac{1}{\sqrt{2}}(e_1 - e_2) \right) \quad (12.68)$$

$$\begin{aligned} &= \left( -\sqrt{1 - 1.13^2 \times 0.5} - 1.13 \left( \frac{1}{\sqrt{2}} \right) \right) e_2 + 1.13 \left( \frac{1}{\sqrt{2}}(e_1 - e_2) \right) \\ &= (-0.6 - 0.8)e_2 + 0.8e_1 - 0.8e_2 \end{aligned}$$

$$v_R = 0.8e_1 - 2.2e_2. \quad (12.69)$$

Figure 12.5 shows the incident and refracted vectors.

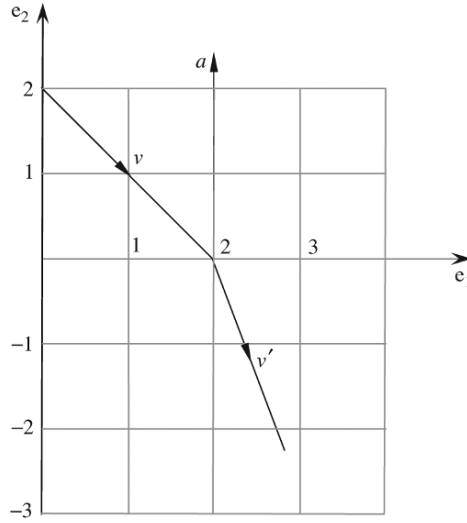


FIGURE 12.5.

### 12.3 Rigid-body pose control

Richard Wareham and Joan Lasenby have reported on their work on rigid-body pose and position interpolation using GA [22]. Their algorithm employs rotors within the conformal model that are interpolated resulting in smooth interpolation of displacements and rotations. Their approach lends itself to piece-wise linear interpolation and quadratic interpolation. The first part of their paper proves that “any displacement of Euclidean geometry can be mapped smoothly onto a linear sub-space of the bivectors”, where the three bivectors  $e_1 \wedge e_2$ ,  $e_2 \wedge e_3$ ,  $e_3 \wedge e_1$  representing the orthogonal planes identify a set of target poses  $\{P_1, P_2, \dots, P_n\}$ . Associated with these poses is a set of rotors which transform an origin pose to the target poses  $\{R_1, R_2, \dots, R_n\}$  which are mapped onto a set of bivectors  $\{l(R_1), l(R_2), \dots, l(R_n)\}$ . Whatever interpolation scheme is chosen, an interpolated bivector  $B'_\lambda$  is revealed from which is computed a pose using  $\exp(B'_\lambda)$ .

When using piece-wise interpolation the following steps are used:

1. Compute the rotor  $\Delta R_n$  that moves from the current rotor  $R_n$  to the next rotor  $R_{n+1}$  in the interpolation sequence:

$$R_{n+1} = (\Delta R_n)R_n \tag{12.70}$$

$$\Delta R_n = R_{n+1}\tilde{R}_n. \tag{12.71}$$

2. Find the bivector  $\Delta B_n$  that generates

$$\Delta R_n = \exp(\Delta B_n). \tag{12.72}$$

3. Form a rotor interpolating between  $R_n$  and  $R_{n+1}$ :

$$R_{n,\lambda} = \exp(\lambda \Delta B_n)R_n \tag{12.73}$$

where

$$\lambda \in [0, 1] \text{ and } R_{n,0} = R_n \text{ and } R_{n,1} = R_{n+1}. \quad (12.74)$$

A potential weakness with this form of piece-wise interpolation is that the interpolated poses only takes into consideration two key poses, which can cause abrupt changes in direction. An alternative interpolation strategy is to use quadratic interpolation where a quadratic is fitted through three interpolation poses  $\{B_1, B_2, B_3\}$  with the interpolating parameter  $\lambda \in [-1, +1]$ :

$$B'_\lambda = \left( \frac{B_3 + B_1}{2} - B_2 \right) \lambda^2 + \frac{B_3 - B_1}{2} \lambda + B_2 \quad (12.75)$$

giving

$$B'_{-1} = B_1, \quad B'_0 = B_2 \quad \text{and} \quad B'_{+1} = B_3. \quad (12.76)$$

This time the interpolated poses still pass through the key poses but enjoy a smooth transition as they anticipate the next pose in the sequence.

TABLE 12.1

Algebra	Line representation	Line-plane intersection
3D linear algebra	$\mathbf{p} = \mathbf{q} + \lambda \mathbf{u}$ $\mathbf{q}$ : vector from origin to point on the line. $\mathbf{u}$ : direction of the line	$\mathbf{q}_i = \mathbf{q}_l - \frac{((\mathbf{q}_l \cdot \mathbf{n}) - \delta) \mathbf{u}}{\mathbf{u} \cdot \mathbf{n}}$ $\mathbf{q}_l$ : point on the line $\mathbf{u}$ : direction of the line $\mathbf{n}$ : normal vector of plane $\delta$ : distance of plane to origin
3D geometric algebra	$\mathbf{p} = \mathbf{q} + \lambda \mathbf{u}$ $\mathbf{q}$ : vector from origin to point on the line. $\mathbf{u}$ : direction of the line	$\mathbf{q}_i = \mathbf{q}_l - \frac{((\mathbf{q}_l \wedge \mathbf{p})^* - \delta) \mathbf{u}}{(\mathbf{u} \wedge \mathbf{B})^*}$ $\mathbf{q}_l$ : point on the line $\mathbf{u}$ : direction of the line $\mathbf{B}$ : Bivector of the plane $\delta$ : distance of plane to origin
4D linear algebra	$\mathbf{l} = (\vec{q}_1 - \vec{q}_2 : \vec{q}_1 \times \vec{q}_2)$ $\mathbf{q}_1 = (\vec{q}_1 : 1), \mathbf{q}_2 = (\vec{q}_2 : 1)$ : two points	$\mathbf{l} = (\vec{u} : \vec{v})$ : line $\mathbf{p} = [\vec{n} : \delta]$ $\mathbf{q}_i = \left( \frac{\vec{v} \times \vec{n} + \delta \vec{u}}{\vec{u} \cdot \vec{n}} : 1 \right)$
4D geometric algebra	$\mathbf{l} = \mathbf{q}_1 \wedge \mathbf{q}_2$ $\mathbf{q}_1, \mathbf{q}_2$ : two points	$\mathbf{q}_i = \mathbf{p}^* \cdot \mathbf{l}$ $\mathbf{l}, \mathbf{p}$ : line, plane
Conformal model	$\mathbf{l} = \mathbf{q}_1 \wedge \mathbf{q}_2 \wedge \mathbf{n}$ $\mathbf{q}_1, \mathbf{q}_2$ : two points $\mathbf{n}$ : point at infinity	$\mathbf{q} \wedge \mathbf{n} = \mathbf{p}^* \cdot \mathbf{l}$ $\mathbf{l}, \mathbf{p}$ : line, plane $\mathbf{n}$ : point at infinity plus one flat point decomposition, if required.

## 12.4 Ray tracing

Ray tracing provides a perfect environment for evaluating new algorithms, as it involves a range of geometric primitives, ray casting, reflections and refraction calculations. Daniel Fontijne and Leo Dorst have reported on their implementation of a GA software package [23].

Their evaluation compares five ways of implementing a ray tracer: 3D linear algebra, 3D geometric algebra, 4D linear algebra, 4D geometric algebra and the conformal model. Table 12.1 lists the geometric representations used for lines and line-plane intersection.

Needless to say, the 3D linear algebra algorithm reported the fastest rendering time and the conformal model the longest. However, these are early days for GA and execution times are only part of the efficiency equation. It is extremely difficult to measure an algorithm's elegance and the ease with which it can be used, but they are extremely important considerations.

## 12.5 Summary

In this chapter and chapters 9 and 10 we have explored a wide range of applications for GA. The solutions may have appeared verbose, but this is because I deliberately worked through the algebra to show how it behaves at each stage. A normal linear algebra approach could have also appeared tedious if every step of a matrix operation had been shown. Therefore, it is wrong to make any judgments about the efficiency of GA by comparing the relative size of the two approaches.

The purpose of the examples was to show the algebraic reasoning that is adopted when solving different types of problem. For example, we have seen how useful it is to divide an expression by a single or multiple vectors. Similarly, multiplying an expression by a vector using the inner or outer product can be a useful way of removing unwanted elements. What is important is to acquire a toolkit of problem solving strategies that can be applied whenever a problem arises.

Traditional linear algebra underpins most of the algorithms employed in computer graphics. The algebra works, it is fast, accurate, stable and easy to use. Consequently, any beneficial features that GA possesses must be strong enough to convince a community that is entrenched in doing things in a particular way. If it could be shown that GA algorithms reduced execution time, reduced memory requirement and were easy to design, it would be like pushing at an open door. Unfortunately, at the moment, none of these are true, which is not surprising. GA is still relatively young, and time is needed to assemble a coherent body of work that will convince the CGI community that they can benefit by moving across to this alternative approach to problem solving.



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0001111110011110000
0011111100011110000
0101111000001111000
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```

# 13 Programming Tools for Geometric Algebra

## 13.1 Introduction

Geometric algebra has only recently been embraced by the computer graphics community with major centers of research at Cambridge University (UK), University of Amsterdam (Netherlands) and MPI Saarbruecken (Germany). Research at Cambridge University focuses on the areas of physics, computer vision, motion analysis and computer graphics [24], [25], [26], [27], [28], [29]. Research at the University of Amsterdam is in computer science and computer graphics [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], whilst pose estimation and marker-less motion capture is investigated at MPI Saarbruecken [40], [41], [42].

As mentioned in the Preface, the Internet is a rich source of material for anyone wanting to learn about GA – most of the information is extremely good although some websites are unreadable. Nevertheless, what is important is the level of interest being taken in an algebraic system; this is extremely healthy, and will help resolve the notation and future use of the algebra. This process is much more democratic than what happened in the late 19<sup>th</sup> century when the nature of vector analysis was decided by a few influential people.

In this brief chapter I draw the reader's attention to some of the software tools available to programmers wanting to embrace GA.

## 13.2 Programming implications

Integrating 3D/4D linear algebra within a programming environment is a trivial task. Points, lines, planes, spheres, planar-bounded volumes, vectors and transforms all have simple descriptions, and it is relatively easy to establish a library of algorithms that permit the efficient manipulation of these geometric and algebraic elements. Furthermore, as execution speed is always an important parameter in measuring the success of new code, especially in real-time applications, programmers have no choice but to “hack” their code to release the very last ounce of speed. Consequently, the apparent relative simplicity associated with the high-level mathematical description crumbles away as one approaches the binary reality of executable code.

One of the reasons why geometric algebra is being positively promoted within computer graphics is the coherence it brings when designing algorithms. For example, reflections are expressed as

$$a' = -\hat{n}a\hat{n} \quad (13.1)$$

and rotations as

$$a' = Ra\tilde{R}. \quad (13.2)$$

One could not ask for anything simpler. However, this high-level simplicity is only skin deep, for beneath this symbolic surface resides a world of vectors, bivectors, trivectors, quadvectors, pseudoscalars, inner, outer, geometric products, meets, joins, duals, reverses, versors, rotors and blades, all hungry for memory and processing time. So one immediate issue is how best to represent all of these elements – on the one hand we must minimize memory storage and maintain readability, and the on the other hand maximize execution speed.

### 13.3 Programming tools

For programmers interested in designing GA algorithms, a wide variety of programming tools is available and comprise everything from symbolic algebraic systems to interactive environments. Systems like Maple [43], Mathematica [44] and MatLab [45] provide environments where algebraic expressions are evaluated and readily converted into images. There are also websites such as [46] where Maple tutorials are available to teach Clifford algebra. GABLE is a MatLab GA Tutorial although it is no longer supported. The authors have moved on to C++, however the system can still be downloaded [47].

A very important piece of software is GAIGEN (GA Implementation Generator) developed by Fontijne, Bouma and Dorst [37]. Gaigen 2 is a C++ implementation of GA and GAVIEWER is an interactive program for teaching GA. Both systems can be downloaded from:

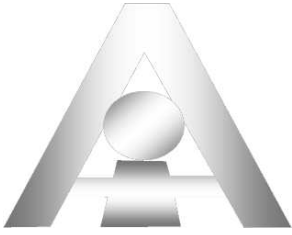
<http://www.geometricalgebra.net>.

C. Perwass has written a software program called CLUCALC [48] which can visualize the geometric interpretation of many algebraic entities. This is available from:

<http://www.clucalc.info>.

### 13.4 Summary

Grassmann and Clifford had no access to computers and computer graphic software and possibly their interpretation of GA was algebraic rather than visual. Perhaps an algebraic approach is the most effective way of maintaining the mathematical rigor required by GA, but this is a problem for those of us who think in pictures rather than in symbols. Those readers possessing strong mathematical skills will be able to pick up the rest of GA without any graphical aids whatsoever, and I am certain that the above computer graphic systems will be extremely useful in bringing linear subspaces, conformal space meets, joins and intersections alive for many readers.



# 14 Conclusion

The aim of this book was to offer a gentle introduction to geometric algebra through the eyes of a consumer of mathematics, rather than through the eyes of a mathematician. To begin with, I do not possess the eyes of a mathematician, therefore I could not have written a book that would have met the rigorous standards required by the mathematical community. However, having taught mathematics to hundreds of students who graduated and became programmers within the computer animation and computer games sectors, I know that there are busy people out there who need to use mathematics to solve their everyday problems. This is the community with whom I want to communicate.

If you have managed to read and understand the previous dozen chapters you will have appreciated the importance of GA. There is no doubt that GA unifies various branches of mathematics and brings an exciting degree of clarity to physics, which will keep academics and researchers busy for many decades to come. However, before GA is universally embraced, mathematicians must agree on a notation that is adopted universally. A major frustration that accompanied the research for this book was the wide range and conflicting notation adopted by different authors. This must be resolved fast. Then the task of drawing up a coherent axiomatic description of GA must begin.

GA has changed my life and will probably change yours, and in time, could change the entire CG sector. I do hope that in the future, more mathematicians will work with CG experts to create a GA toolset that will transform the way we currently solve geometric problems.

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