School Mathematics

Volume 6

Real World Applications

Includes worked examples, and test questions with answers.

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Preface

The volumes in the School Mathematics series contain brief review notes, selected questions with solutions, and test questions with answers. We hope you find the material here useful.

Most questions have been selected, with some modification, from the books *Integrated Mathematics for Explorers by* Adeline Ng and R. Parwani, and *Real World Mathematics* by W.K. Ng and R. Parwani. The solutions are edited from the corresponding *Solutions Manuals* by C.L. Ching and Sun Jie, and Y.L. Len and M.H. Thong.

Singapore, Jan 2016.

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Integrated Mathematics for Explorers
by Adeline Ng and R. Parwani
This book is for mathematics explorers in school, college and beyond. Includes worked examples, exercises, challenges, investigations, puzzles, and unsolved mathematical problems.

Real World Mathematics by Wei Khim Ng and Rajesh R. Parwani
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Chapter 1

Worked Examples

Review Notes and Formulae
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1. When two resistors R_1 and R_2 are connected in parallel in a electrical circuit, their combined resistance R is determined by $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$. All resistances are positive.

   (a) Prove that $R \leq R_1$ and $R \leq R_2$.

   (b) Express R as a rational function of R_1 and R_2.

– –

Solutions:

(a) Since

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}\ , \tag{1.1}$$

and all variables are positive, so $\dfrac{1}{R} \geq \dfrac{1}{R_1}$ and $\dfrac{1}{R} \geq \dfrac{1}{R_2}$. Therefore, $R \leq R_1$ and $R \leq R_2$.

(b) Re-arranging the terms: $\dfrac{1}{R} = \dfrac{R_2 + R_1}{R_1 R_2}$. Hence, $R = \dfrac{R_1 R_2}{(R_1 + R_2)}$.

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2. You are instructed to construct an open top box from a square, $a \times a$, piece of cardboard: First cut out squares of dimensions x by x at each corner of the cardboard, then fold the cardboard to form a box of dimensions x by $(a - 2x)$ by $(a - 2x)$.

   (a) Find the values of x which make the volume of the box a minimum and explain the physical situation those values correspond to.

   (b) Hence, or otherwise, explain why for some value of x the volume of the box would be a maximum.

   (c) Determine the maximum volume using calculus.

– –

Solutions:

(a) The volume of the box is $V(x) = x(a-2x)^2$. As physically $V \geq 0$, and $(a - 2x)^2 \geq 0$, so we see by inspection that the minimum volume corresponds to the situation $x = 0$ (no height) or $x = a/2$ (no base area). Both minima correspond to zero volume.

(b) As $V(x) \geq 0$ is a smooth function, it must have a maximum in the physical range $0 < x < a/2$ since it reaches its minimum value (zero) at the end points.

(c) The turning points satisfy the equation

$$\frac{dV}{dx} = a^2 - 8ax + 12x^2 = 0,$$

which gives us the values $x = a/2$ and $x = a/6$. To determine the nature of the these extrema (minima or maxima), we need to check the value of $d^2V/dx^2 = -8a + 24x$. For $x = a/2$, we have $d^2V/dx^2 > 0$, so this is a minimum point (as we already knew from part (a)). For the other extremum at $x = a/6$, we have $d^2V/dx^2 = -4a$, so $x = a/6$ is at least a local maximum. Since at the boundaries $x = 0$ and $x = a/2$, the smooth function

$V(x)$ does not exceed its value at the local maximum, so the local maximum is actually a global maximum.
The maximum volume is given by $V(x = a/6) = 2a^3/27$.

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3. The pH value of a solution indicates its hydrogen ion concentration, denoted by $[H^+]$ in moles per litre, through the formula

$$pH = -\log_{10}[H^+] \ .$$

Neutral solutions have a pH of 7, acidic solutions have pH values below 7, while base solutions have a pH above 7.

(a) Determine the hydrogen ion concentration of pure water, a neutral solution.

(b) Do acidic solutions have a larger, or smaller, hydrogen ion concentration than neutral solutions?

– –

Solutions:

(a) Neutral solution means pH=7, so $7 = -\log_{10}[H^+]$, or $[H^+] = 10^{-7}$ moles per litre.

(b) We have $[H^+] = 10^{-(\mathrm{pH})}$. Since acidic solutions have a pH less than 7, so they will have a higher hydrogen ion concentration than a neutral solution.

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4. The curve formed by a freely hanging cable supported only at its ends is called a catenary. Its equation is $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$. If the lowest point of the cable reaches 100 m below its support level, and if $a = 10$ m, determine the coordinates, in metres, of the support point.

– –

Solution:
For convenience, we first define the hyperbolic cosine function $\cosh(x) \equiv \left(\frac{e^x + e^{-x}}{2}\right)$. Then, we have $y = a\cosh(x/a)$. Similarly, we define $z \equiv a\sinh(x/a)$, where $\sinh(x) \equiv \left(\frac{e^x - e^{-x}}{2}\right)$ is the hyperbolic sine function. It can be easily verified that $\frac{d\cosh(x)}{dx} = \sinh(x)$. The minimum point of y is at where $\frac{dy}{dx} = 0$, which is satisfied only when $x = 0$. Therefore, we have $y = a = 10\,\text{m}$ at the minimum. The support level is 100 m above the minimum, that is, at $y = 110\,\text{m}$.

The x-coordinates of the two support points are obtained by solving $110 = 10\cosh(x/10)$. Explicitly, $x = \pm 10\cosh^{-1}(11) = \pm 30.89$, where $\cosh^{-1}()$ is the inverse hyperbolic cosine function. The $\pm$ sign comes from the fact that y is unchanged upon the substitution $x \to -x$.

Ans: $x = \pm 30.89,\ y = 110$.

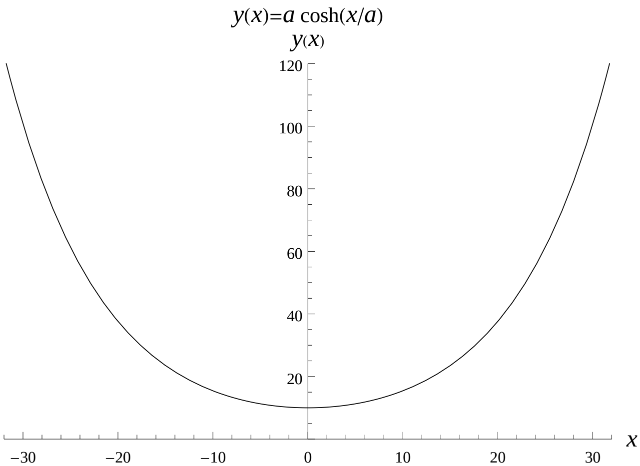

Figure 1.1: Plot of $y(x)$, with $a = 10$.

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5. A car can reach a maximum speed of 80 km/hr. Its fuel consumption (in some units) when moving at v km/hr is $C = (v/20)^3$ units per hour. The car must be driven from one town to another, a distance of 300 km, in less than 6 hours, but the total fuel consumption must not exceed 150 units. Assuming, for simplicity, that the car travels at constant speed, determine the range of values for v which achieve the objective.

– –

Solution:
We have $v = d/t$ where $d = 300$ is the distance travelled and t the time in hours. So $t < 6 \Rightarrow 300/v < 6$, that is $v > 300/6 = 50$ km/h. The fuel consumption rate is $C = (v/20)^3$. The total fuel consumption after t hours is $F = C \times t = Cd/v$. We require $F < 150$, which implies $300v^2/8000 < 150$, or $v < 63.25$.

Ans: $50 < v < 63.25$.

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6. Eona determines the height of a tree using a metre rule placed vertically on the ground: She finds that the shadow of the metre rule is 1.3m long when the shadow of her tree is 5.5m long. How tall is Eona's tree? [Ans: 4.23 m]

– –

Solution:
As shown in the figure, the angle is the same. That is, we have $\tan(\theta) = 1/1.3 = x/5.5$, so $x = 5.5/1.3 = 4.23\,\text{m}$.

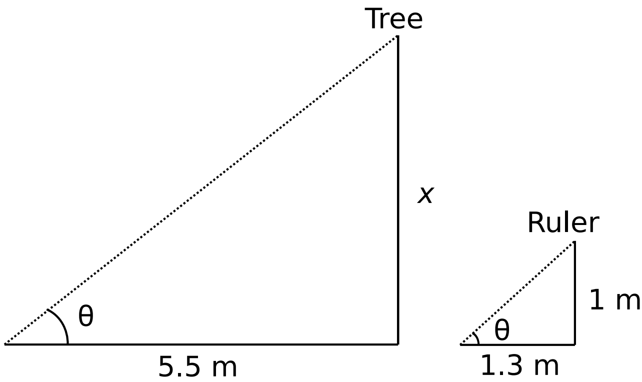

Figure 1.2: Similar Triangles.

7. A guitar string is tied at its two ends located at $x = 0$ and $x = L$. When plucked, it creates "standing waves" whose vertical displacement is given by $y(x) = A \sin \frac{2\pi x}{\lambda}$ where A is the maximum amplitude and λ is the wavelength of the wave.

   (a) Using the condition $y(L) = 0$ (since the tied end does not move), find the possible values for λ in terms of a positive integer n.

   (b) For $n = 2$, locate the points of the standing wave with zero amplitude (called "nodes") and the points with maximum amplitude (called "anti-nodes").

\- -

Solutions:

(a) $y(L) = A \sin \frac{2\pi L}{\lambda} = 0 \Rightarrow 2\pi L/\lambda = n\pi$, that is $\lambda = 2L/n$ with n a positive integer.

(b) For $n = 2$, $y(x) = A \sin \frac{2\pi x}{L}$. The nodes occur at values of $x \leq L$ that satisfy $y = 0$, that is $2\pi x/L = m\pi$ with m a positive integer. So the nodes are at $x = 0,\ L/2,\ L$. Similarly, the anti-nodes occur when $|y| = A$, that is at $x = L/4,\ 3L/4$.

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8. An object with initial speed v is brought to stop in a distance d by a constant opposing force. Theory predicts that $d \propto v^2$. If the opposing force is kept constant while v is varied, suggest a plot involving d and v that would produce a straight line.

_ _

Solutions:
We have $d = kv^2$, where k is a constant. For a straight line plot involving d and v, we can either

(a) Plot d against v^2, such that the plot will have a gradient of k, or

(b) Plot $\log d$ against $\log v$. In this case, $\log d = \log k + 2\log v$, so the gradient will be 2.

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9. The von Bertalanffy model for tumour growth is given by the equation

$$\frac{dV}{dt} = aV^p - bV^q\,,$$

where V is the size of the tumour, t the time and a, b, p, q are positive constants.

(a) The two terms on the right hand side of the equation represent growth and degradation. Identify those respective terms.

(b) Show that for $q > p$ there is a maximum size to the tumour.

_ _

Solutions:

(a) We are given $\dfrac{dV}{dt} = aV^p - bV^q$ where a, b, p, q are positive constants. V represents the size of the tumour, and is therefore positive. Thus, both the terms aV^p and bV^q will be non-negative. $\dfrac{dV}{dt}$ is the rate of change in the tumour size, and hence the positive term aV^p represents growth, while the negative term $-bV^q$ is for degradation.

(b) We want to show that for $q > p$, there is always a maximum size to the tumour. We first re-write $\dfrac{dV}{dt} = aV^p\left[1 - \left(\dfrac{b}{a}\right)V^{q-p}\right]$.
Local extrema occur when the derivative vanishes, that is when $V = 0$ or $V \equiv V_c = \left(\dfrac{a}{b}\right)^{1/(q-p)}$.
We first note that as $V \to 0$, $dV/dt \approx aV^p$ as it will be dominated by the smaller power of V. Since $V \geq 0$, this implies that near $V = 0$, dV/dt is positive, and V increases as it deviates from zero. Thus $V = 0$ is a minimum point.
Next, we re-write the derivative again as $\dfrac{dV}{dt} = aV^p\left[1 - \left(\dfrac{V}{V_c}\right)^{q-p}\right]$.
From this we see that the term in square-brackets is positive for $V < V_c$ and negative for $V > V_c$. That is, dV/dt is positive for $0 < V < V_c$ and negative for $V > V_c$.

We conclude that for $V > 0$, the tumour increases in size up to a maximum of V_c.

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10. A thin-skinned rubber sphere is inflated by pumping it with water (which is essentially incompressible). Water is pumped in at the constant rate of 3 $\text{cm}^3\ \text{s}^{-1}$. If the volume of the sphere is 200 cm^3 when $t = 10$, determine

(a) The radius of the sphere at $t = 1$.

(b) The rate of change of the radius at $t = 1$.

– –

Solutions:

(a) We are given $\dfrac{dV}{dt} = 3$. So $V = \int dV = \int \frac{dV}{dt}\, dt = \int 3dt = 3t + C$ where C is the integration constant. Since $V = 200$ at $t = 10$, so $C = 170$ and $V = 3t + 170$. Hence $V(t = 1) = 173$, and using $V = 4\pi r^3/3$ gives us $r = 3.457$ cm.

(b) From the expression $V = 4\pi r^3/3$, we get $dV/dr = 4\pi r^2$. We also have from the chain rule,

$$\frac{dV}{dt} = \left(\frac{dV}{dr}\right)\left(\frac{dr}{dt}\right) = 4\pi r^2 \left(\frac{dr}{dt}\right)$$

Therefore,

$$\frac{dr}{dt} = \frac{3}{4\pi r^2} = 0.02 \text{ cm s}^{-1}$$

where we used $r = 3.457$ from the first part.

Chapter 2

Test Yourself

1. In Worked example (1), if the value of R_1 is fixed while R_2 is a variable resistor, determine the range of values for R.

2. In Worked example (2), use a plot to estimate the maximum volume of the box when $a = 2$. Compare with the result using calculus.

3. In Worked example (3), if a solution has a hydrogen ion concentration of 5×10^{-7} moles/litre, determine its pH value. Is the solution acidic or alkaline (basic)?

4. In Worked example (6), what was the elevation of the Sun when Eona did her measurement?

5. In Worked example (10), what is the rate of change of the surface area at $t = 1$.

6. Challenge: Show that near its lowest point, the catenary mentioned in worked example (4) can be approximated by a parabola.

Chapter 3

Answers to Test

1. $0 \leq R \leq R_1$.

2. $V_{max} = 16/27$ at $x = 1/3$. The plot of V against x is shown in Fig. (3.1).

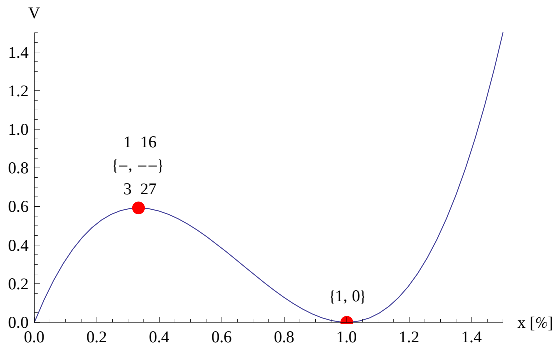

Figure 3.1: The maximum point is at $x = 1/3$, with $V = 16/27$. Note: The physical region is $0 \leq x \leq 1$.

3. 6.3, acidic.

4. $37.6°$.

5. 1.7375 cm s^{-2}.

6. The bottom of the catenary is at $x = 0$. Near its bottom, we can thus expand y as $y = \dfrac{(1 + x + x^2/2 \cdots) + (1 - x + x^2/2 + \cdots)}{2a} \approx \dfrac{2 + x^2}{2a}$, which is a parabola.

Did You Know?

A freely hanging cable takes the shape of a catenary, see worked example (4).
Near its lowest point, a catenary can be approximated by a parabola (see test question (6)).

However, if the cable supports a bridge, and the weight of the cable is negligible compared to the weight it supports, then the shape of the cable is closer to a parabola than a catenary.