Advanced analysis of local fractional calculus applied to Rice theory in fractal fracture mechanics

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Abstract

In this chapter, the recent results for the analysis of local fractional calculus are considered for the first time. The local fractional derivative (LFD) and local fractional integral (LFI) in the fractional real and complex sets, the series and transforms involving the Mattig-Leffer function defined on Cantor sets are introduced and reviewed. The unique of the solutions of the local fractional differential and integral equations and local fractional inequalities are considered in detail. The local fractional vector calculus is applied to describe the Rice theory in fractal fracture mechanics.

Keywords:

Local fractional calculus, local fractional derivative, local fractional integral, local fractional vector calculus, local fractional partial differential equation, local fractional integral transform, local fractional integral equation, local fractional inequality, Rice theory, fractal fracture mechanics, fractals.

1 Introduction

Fractional calculus (FC) have successfully been utilized to describe the fractal problems in engineering practices [1, 2, 3, 4]. The important examples are the fractal Fokker– Planck equations [5] and fractal description of stress and strain in elasticity [6, 7, 8]. There are several alternative approaches for handling the complex and fractal behaviors in nature [9, 10, 11, 12].

The theory of the local fractional calculus (LFC) is a mathematical tool for handling the non-differentiable problems under the consideration of the complex and fractal behaviors of the real world problems [13, 14, 15, 16, 17, 18, 19]. The local fractional derivative (LFD) and local fractional integral (LFI) were used to present the approaches for describing the fractal phenomena in mathematical physics (see [20, 21, 22, 23]). For the details of the applications of the LFC, we see as follows: the LFC to model the shallow water surfaces [24, 25], LC-electric circuit [26, 27, 28], local fractional partial differential equations (PDEs) [29, 30, 31, 32], local fractional ordinary differential equations (ODEs) [33, 34] and so on. The special inequalities via LFI, such as the Ostrowski type [35], Steffensen type [36], and Pompeiu type [37] inequalities for the LFIs and other (see [38, 39, 40, 41, 42, 43, 44, 45, 46]) were considered.

The local fractional integral transforms via LFC were proposed in [9, 10, 47] and developed in [12, 16]. The local fractional Fourier type integral transform was investigated in [48, 49, 50, 51]. The local fractional Laplace type integral transform was investigated in [51, 52, 53, 54, **?**, 56, 57, 58]. They were applied to find the non-differentiable solutions for the local fractional PDEs (see[12, 59]). From the functional analysis point of view, the unique of the solutions of the local fractional ODE and local fractional integral equations were considered in [9, 10] for the first time. The existence and unique of the solutions of some local fractional abstract differential equations were presented in [60]. The existence and uniqueness of solutions for local fractional differential equations and its applications were reported in [9, 10, 59, 61]. The local fractional vector calculus and applications in the fractal heat conduction problems were presented in [2, 11].

The brief aim of the chapter is to investigate the properties of the LFC, the series and transforms involving the Mattig-Leffer function defined on Cantor sets, analysis of the local fractional differential and integral equations, local fractional inequalities and local fractional vector calculus, and to present the applications of the extended version of the Rice theory in fractal fracture mechanics.

The structure of the chapter is as follows. In Section 2, the theory of the LFD and LFI in the fractional real and complex sets is presented. In Section 3, the analysis of the local fractional differential and integral equations is derived. In Section 4, the local fractional inequalities are discussed in detail. In Section 5, the series and transforms involving the Mattig-Leffer function defined on Cantor sets are reported. In Section 6, the local fractional vector calculus and its application in fractal fracture mechanics are considered in detail. Finally, the conclusions are given in Section 7.

2 The LFD and LFI in the fractional real and complex sets

In this section, we introduce the LFC of the real and complex variables and consider α as the fractal dimension in the chapter.

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be sets of the natural numbers, real numbers and complex numbers.

Let $\mathbb{N}^\alpha$, $\mathbb{R}^\alpha$ and $\mathbb{C}^\alpha$ be the fractional sets of the natural numbers, real numbers and complex numbers [9, 10, 11, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46].

Definition 1 *The complex number defined on the fractal set $\mathbb{C}^\alpha$, is given as [9, 10, 11, 12, 13]*

$$z^\alpha = x^\alpha + i^\alpha y^\alpha, \quad x, y \in \mathrm{R}, \tag{1}$$

and its conjugate by

$$\overline{z^\alpha} = x^\alpha - i^\alpha y^\alpha, \quad \overline{z^\alpha} \in \mathbb{C}^\alpha, x, y \in \mathbb{R}, \tag{2}$$

with its fractional modulus is defined as [9, 10, 11, 12, 13]

$$\left|\overline{z^\alpha}\right| = \left|z^\alpha\right| = \sqrt{\overline{z^\alpha} \cdot z^\alpha} = \sqrt{x^{2\alpha} + y^{2\alpha}}. \tag{3}$$

The complex number defined on the fractal set $\mathbb{C}^\alpha$ is represented in the form:

$$z^\alpha = Re\left(z^\alpha\right) + i^\alpha Im\left(z^\alpha\right) = x^\alpha = x^\alpha + i^\alpha y^\alpha,$$

where $Re\left(z^\alpha\right) = x^\alpha$ is the purely real part and $Im\left(z^\alpha\right) = y^\alpha$ is the purely imaginary part, which can be expressed as [9, 10, 11, 12, 13]

$$z^\alpha = x^\alpha + i^\alpha y^\alpha = \sqrt{x^{2\alpha} + y^{2\alpha}}\left(\cos_\alpha\left(x^\alpha\right) + i^\alpha \sin_\alpha\left(x^\alpha\right)\right),$$

with

$$\cos_\alpha\left(x^\alpha\right) = \frac{x^\alpha}{\sqrt{x^{2\alpha} + y^{2\alpha}}},$$

$$\sin_\alpha\left(x^\alpha\right) = \frac{y^\alpha}{\sqrt{x^{2\alpha} + y^{2\alpha}}},$$

where

$$\cos_\alpha\left(z^\alpha\right) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2\alpha k}}{\Gamma\left(1 + 2\alpha k\right)}, \tag{4}$$

$$\sin_\alpha\left(z^\alpha\right) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{(2k+1)\alpha}}{\Gamma\left[1 + \alpha\left(2k+1\right)\right]}. \tag{5}$$

Definition 2 *The complex Mittag-Leffler function on the fractal set $\mathbb{C}^\alpha$ is defined as [9, 10, 11, 12, 13]*

$$E_\alpha\left(z^\alpha\right) := \sum_{k=0}^{\infty} \frac{z^{\alpha k}}{\Gamma\left(1 + k\alpha\right)}, \tag{6}$$

where $z^{\alpha} \in \mathbb{C}^{\alpha}$, which leads to the formulation in the form: [9, 10, 11, 12, 13]

$$\begin{aligned}
& z^{\alpha} \\
& = x^{\alpha} + i^{\alpha} y^{\alpha} \\
& = \sqrt{x^{2\alpha} + y^{2\alpha}} \left(\cos_{\alpha} \left(x^{\alpha} \right) + i^{\alpha} \sin_{\alpha} \left(x^{\alpha} \right) \right) \\
& = \sqrt{x^{2\alpha} + y^{2\alpha}} E_{\alpha} \left(i^{\alpha} z^{\alpha} \right),
\end{aligned}$$

where

$$E_{\alpha} \left(i^{\alpha} z^{\alpha} \right) := \cos_{\alpha} \left(z^{\alpha} \right) + i^{\alpha} \sin_{\alpha} \left(z^{\alpha} \right). \tag{7}$$

2.1 The LFD and LFI in the fractional real set

Definition 3 *A function $f(x)$ is said to be local fractional continuous at $x = x_0$ if for each $\varepsilon > 0$ there exists for $\delta > 0$ such that [9, 10, 11, 12, 13, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]*

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha}, \tag{8}$$

whenever $0 < |x - x_0| < \delta$.

It is to say that

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{9}$$

If $f(x)$ is local fractional continuous in the domain $I = (a, b)$, then we write it as [22, 23, 24, 25, 26, 27, 28, 29, 30]

$$f(x) \in H_{\alpha}(a, b). \tag{10}$$

Definition 4 *Let$f(x) \in H_{\alpha}(a, b)$. The LFD of the function $f(x)$ of order αat $x = x_0$, denoted as $f^{(\alpha)}(x_0)$ or $\frac{d^{\alpha} f(x)}{dx^{\alpha}} |_{x=x_0}$, is defined as [9, 10, 11, 12, 13]*

$$D^{(\alpha)} f(x) = f^{(\alpha)}(x_0) = \frac{d^{\alpha} f(x)}{dx^{\alpha}} |_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha} \left(f(x) - f(x_0) \right)}{\left(x - x_0 \right)^{\alpha}}, \tag{11}$$

where $\Delta^{\alpha} \left(f(x) - f(x_0) \right) \cong \Gamma(1 + \alpha) \Delta \left(f(x) - f(x_0) \right)$.

Let $f(x), g(x) \in H_{\alpha}(a, b)$.

The properties of the LFD are presented as follows [9, 10, 11, 12, 13]:

(1)

$$\frac{d^{\alpha}}{dx^{\alpha}} \left(f(x) \pm g(x) \right) = \frac{d^{\alpha} f(x)}{dx^{\alpha}} \pm \frac{d^{\alpha} g(x)}{dx^{\alpha}};$$

(2)

$$\frac{d^{\alpha} \left(f(x) g(x) \right)}{dx^{\alpha}} = g(x) \frac{d^{\alpha} f(x)}{dx^{\alpha}} + f(x) \frac{d^{\alpha} g(x)}{dx^{\alpha}};$$

(3)

$$\frac{d^\alpha}{dx^\alpha}\left(\frac{f(x)}{g(x)}\right)=\frac{1}{g(x)^2}\left(g(x)\frac{d^\alpha f(x)}{dx^\alpha}+f(x)\frac{d^\alpha g(x)}{dx^\alpha}\right),$$

where $g(x)\neq 0$;

(4)

$$\frac{d^\alpha(hf(x))}{dx^\alpha}=h\frac{d^\alpha f(x)}{dx^\alpha},$$

where h is a constant;

(5)

If $y(x)=(f\circ u)(x)$, where $u(x)=g(x)$, then we have

$$\frac{d^\alpha y(x)}{dx^\alpha}=f^{(\alpha)}(g(x))\left(g^{(1)}(x)\right)^\alpha.$$

The LFDs of the elementary functions defined on fractal sets are given as follows [9, 10, 11, 12, 13]:

(1)

$$\frac{d^\alpha}{dx^\alpha}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}=\frac{x^{(k-1)\alpha}}{\Gamma(1+(k-1)\alpha)};$$

(2)

$$\frac{d^\alpha E_\alpha(x^\alpha)}{dx^\alpha}=E_\alpha(x^\alpha);$$

(3)

$$\frac{d^\alpha E_\alpha(kx^\alpha)}{dx^\alpha}=kE_\alpha(kx^\alpha),$$

where k is a constant.

(4)

$$\frac{d^\alpha\sin_\alpha(x^\alpha)}{dx^\alpha}=\cos_\alpha(x^\alpha);$$

(5)

$$\frac{d^\alpha\cos_\alpha(x^\alpha)}{dx^\alpha}=-\sin_\alpha(x^\alpha).$$

Theorem 1 *(The mean value theorem for the LFD)*

If $f(x)\in H_\alpha[a,b]$, then there exists a point $x_0\in(a,b)$ such that [9, 10, 11, 12, 13]

$$f(b)-f(a)=f^{(\alpha)}(x_0)\frac{(b-a)^\alpha}{\Gamma(1+\alpha)}.$$

Definition 5 *Let $f(x)\in H_\alpha[a,b]$. The LFI of the function $f(x)$ of order α $(0<\alpha\leq 1)$ is defined as [9, 10, 11, 12, 13]*

$$
{}_aI_b^{(\alpha)}f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta x_k\to 0}\sum_{k=0}^{N-1}f(x_k)(\Delta x_k)^\alpha,
$$

where $\Delta x_k = x_{k+1}-x_k$ *with* $x_0=a<x_1<\cdots<x_{N-1}<x_N=b$.

Let $f(x), g(x) \in H_\alpha(a,b)$. The properties of the LFI are presented as follows [9, 10, 11, 12, 13]:

(1)

$$
{}_aI_b^{(\alpha)}(f(x)\pm g(x))={}_aI_b^{(\alpha)}f(x)\pm {}_aI_b^{(\alpha)}g(x);
$$

(2)

$$
{}_aI_b^{(\alpha)}(hf(x))=h\,{}_aI_b^{(\alpha)}f(x),
$$

where h is a constant.

The LFIs of the elementary functions defined on fractal sets are given as follows [9, 10, 11, 12, 13]:

(1)

$$
\frac{1}{\Gamma(1+\alpha)}\int_a^b E_\alpha(x^\alpha)(dx)^\alpha=E_\alpha(b^\alpha)-E_\alpha(a^\alpha);
$$

(2)

$$
\frac{1}{\Gamma(1+\alpha)}\int_a^b \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}(dx)^\alpha=\frac{a^{(k+1)\alpha}}{\Gamma(1+(k+1)\alpha)}-\frac{b^{(k+1)\alpha}}{\Gamma(1+(k+1)\alpha)};
$$

(3)

$$
\frac{1}{\Gamma(1+\alpha)}\int_a^b \sin_\alpha(x^\alpha)(dx)^\alpha=\cos_\alpha(a^\alpha)-\cos_\alpha(b^\alpha);
$$

(4)

$$
\frac{1}{\Gamma(1+\alpha)}\int_a^b \cos_\alpha(x^\alpha)(dx)^\alpha=\sin_\alpha(b^\alpha)-\sin_\alpha(a^\alpha)
$$

Theorem 2 *(The mean value theorem for the LFI)*

If $f(x)\in H_\alpha[a,b]$, *then there exists a point* $\xi\in(a,b)$ *such that [9, 10, 11, 12, 13]*

$$
{}_aI_b^{(\alpha)}f(x)=f(\xi)\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}.
$$

Theorem 3 *If $f(x) \in H_\alpha[a,b]$, then there exists a point $\xi \in (a,b)$ such that [9, 10, 11, 12, 13]*

$$f(b) - f(a) = \frac{f^{(\alpha)}(\xi)(b-a)^\alpha}{\Gamma(1+\alpha)}.$$

Theorem 4 *Suppose that $f(x) \in H_\alpha[a,b]$, then there is a function [9, 10, 11, 12, 13]*

$$\Pi(x) = {}_aI_x^{(\alpha)} f(x),$$

such that it has the LFD,

$$\frac{d^\alpha \Pi(x)}{dx^\alpha} = f(x), a \le x \le b.$$

Theorem 5 *(The LFI is anti-differentiation)*

If $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have [9, 10, 11, 12, 13]

$${}_aI_b^{(\alpha)} f(x) = g(b) - g(a).$$

Theorem 6 *Theorem 6 (The LFI by parts)*

If $f^{(\alpha)}(x), g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have [9, 10, 11, 12, 13]

$${}_aI_b^{(\alpha)} f(t) g^{(\alpha)}(t) = [f(t) g(t)]_a^b - {}_aI_b^{(\alpha)} f^{(\alpha)}(t) g(t).$$

Theorem 7 *(The local fractional Taylor' theorem)*

Suppose that $f^{((k+1)\alpha)}(x) \in C_\alpha(a,b)$, $fork = 0, 1, ..., n$, then we have [9, 10, 11, 12, 13]

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k\alpha)}(x_0)}{\Gamma(1+k\alpha)} (x-x_0)^{k\alpha} + \frac{f^{((n+1)\alpha)}(\xi)}{\Gamma(1+(n+1)\alpha)} (x-x_0)^{(n+1)\alpha}$$

with $a < x_0 < \xi < x < b$ and $\forall x \in (a,b)$, where $f^{((k+1)\alpha)}(x) = \overbrace{D_x^{(\alpha)} ... D_x^{(\alpha)}}^{k+1^{t}imes} f(x)$.

2.2 The LFD and LFI in the fractional complex set

Let the complex function $f(z)$ be defined in a neighborhood of a point z_0.

Definition 6 *The LFD of $f(z)$ at the point z_0, denoted by ${}_{z_0}D_z^\alpha f(z)$, $\frac{d^\alpha}{dz^\alpha} f(z)|_{z=z_0}$ or $f^{(\alpha)}(z_0)$, is defined as [9, 10, 11]:*

$${}_{z_0}D_z^\alpha f(z) =: \lim_{z \to z_0} \frac{\Delta^\alpha f(z)}{(z-z_0)^\alpha},\ 0 < \alpha \le 1 \tag{12}$$

where $\Delta^\alpha f(z) = \Gamma(1+\alpha)[f(z) - f(z_0)]$.

If this limit exists, then the function $f(z)$ is said to be local fractional analytic at z_0.

If this limit exists for all $z_0$in a region $\aleph^\alpha \in \mathbb{C}^\alpha$, then the function $f(z)$ is said to be local fractional analytic in a region $\aleph^\alpha \in \mathbb{C}^\alpha$.

Let $f(z)$ and $g(z)$ be local fractional analytic functions. Then there is as follows [9, 10, 11]:

(1)

$$\frac{d^\alpha (f(z) \pm g(z))}{dz^\alpha} = \frac{d^\alpha f(z)}{dz^\alpha} \pm \frac{d^\alpha g(z)}{dz^\alpha};$$

(2)

$$\frac{d^\alpha (f(z) g(z))}{dz^\alpha} = g(z) \frac{d^\alpha f(z)}{dz^\alpha} + f(z) \frac{d^\alpha g(z)}{dz^\alpha};$$

(3)

$$\frac{d^\alpha}{dz^\alpha} \left(\frac{f(z)}{g(z)} \right) = \frac{1}{g(z)^2} \left(g(z) \frac{d^\alpha f(z)}{dz^\alpha} + f(z) \frac{d^\alpha g(z)}{dz^\alpha} \right)$$

where $g(z) \neq 0$;

(4)

$$\frac{d^\alpha (h f(z))}{dz^\alpha} = h \frac{d^\alpha f(z)}{dz^\alpha}$$

where h is a constant.

Definition 7 *Let $f(z)$ be defined, single-valued and local fractional continuous in a region $\aleph^\alpha \in \mathrm{C}^\alpha$. The LFI of the complex function $f(z)$ along the contour C in $\aleph^\alpha \in \mathbb{C}^\alpha$ from point z_p to point z_q is defined as [9, 10, 11]*

$$I_C^\alpha f(z) = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta z \to 0} \sum_{i=0}^{n-1} f(z_i) (\Delta z)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int\limits_C f(z) (dz)^\alpha, \quad (13)$$

where $(\Delta z_i)^\alpha = z_i^\alpha - z_{i-1}^\alpha$, $z_0 = z_p$, $z_n = z_q$ and $i \in \mathbb{N}_0$.

Theorems for the LFC of the complex variables are presented as follows:

Theorem 8 *If the contour C have the end points z_p and z_q with the orientation z_p to z_q, then we have [9, 10, 11]*

$$\frac{1}{\Gamma(1+\alpha)} \int\limits_C f(z) (dz)^\alpha = F(z_q) - F(z_p) \quad (14)$$

where the functionf (z) has the primitive $F(z)$ on the contour C.

Theorem 9 *Let the function $f(z)$ be a primitive on C, where C is a simple closed contour. Then we have [9, 10, 11]*

$$\frac{1}{\Gamma(1+\alpha)}\oint_C f(z)(dz)^{\alpha}=0 \tag{15}$$

Theorem 10 *If $f(z)$ is local fractional analytic on C_1 , C_2 and between them, and the contours C_1 and C_2 have same end points, then we have [9, 10, 11]*

$$\frac{1}{\Gamma(1+\alpha)}\int_{C_1} f(z)(dz)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_{C_2} f(z)(dz)^{\alpha} \tag{16}$$

Theorem 11 *If the closed contours C_1 and C_2 are such that C_2 lies inside C_1, then we have [9, 10, 11]*

$$\frac{1}{\Gamma(1+\alpha)}\int_{C_1} f(z)(dz)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_{C_2} f(z)(dz)^{\alpha}, \tag{17}$$

where $f(z)$ is local fractional analytic on C_1 , C_2 and between them.

Theorem 12 *If $f(z)$ is local fractional analytic within and on a simple closed contour C and z_0 is any point interior to the contour C, then we have [9, 10, 11]*

$$\frac{1}{(2\pi)^{\alpha} i^{\alpha}}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{f(z)}{(z-z_0)^{\alpha}}(dz)^{\alpha}=f(z_0) \tag{18}$$

Theorem 13 *If $f(z)$ is local fractional analytic within and on a simple closed contour C and z_0 is any point interior to the contour C, then we have [9, 10, 11]*

$$\frac{1}{(2\pi)^{\alpha} i^{\alpha}}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{f(z)}{(z-z_0)^{(n+1)\alpha}}(dz)^{\alpha}=f^{(n\alpha)}(z_0). \tag{19}$$

Theorem 14 *If $f(z)$ is local fractional analytic within and on a simple closed contour C and z_0 is any point interior to the contour C, then we have [9, 10, 11]*

$$\frac{1}{(2\pi)^{\alpha}}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{(dz)^{\alpha}}{(z-z_0)^{\alpha}}=i^{\alpha} \tag{20}$$

Theorem 15 *If $f(z)$ is local fractional analytic within and on a simple closed contour C and z_0 is any point interior to the contour C, then we have [9, 10, 11]*

$$\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{(dz)^{\alpha}}{(z-z_0)^{n\alpha}}=0 \tag{21}$$

where $n>1$

Definition 8 *Let* $f(z) = \varphi(z) / (z - z_0)^{n\alpha}$ *and* $\varphi(z) \neq 0$*, where* $\varphi(z)$ *is local fractional analytic everywhere in a region including* $z = z_0$*. There are as follows [9, 10, 11]:*

(1) If n *is a positive integer, then* $f(z)$ *has an isolated singularity at* $z = z_0$*, the point is called as a pole of order* n*, where* n *is a positive integer.*

(2) If $n = 1$*, the pole is often called a simple pole;*

(3) if $n = 2$*, it is called as a double pole.*

Theorem 16 *If* $f(z)$ *has a pole of order* n *at* $z = z_0$ *but is local fractional analytic at every other point inside and on a contour* C *with the center at the point* z_0*, then* $(z - z_0)^{n\alpha} f(z)$ *is local fractional analytic at all points inside and on the contour* C *and has a local fractional Laurent type series about* $z = z_0$ *so that* $f(z) = \sum\limits_{i=-\infty}^{\infty} a_k (z - z_0)^{k\alpha},\ 0 < \alpha \leq 1$ *where [9, 10, 11]*

$$a_k = \frac{1}{(2\pi)^{\alpha}} \cdot \frac{1}{i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint\limits_{C} \frac{f(z)}{(z - z_0)^{(k+1)\alpha}} (dz)^{\alpha} \tag{22}$$

for the contour $C : |z - z_0|^{\alpha} \leq R^{\alpha}$*.*

Theorem 17 *If* $f(z)$ *is local fractional analytic within and on the boundary* C *of a region* $\aleph^{\alpha} \in \mathbb{C}^{\alpha}$ *except at a number of poles a within* $\Re$*, then we have [9, 10, 11]*

$$\frac{1}{(2\pi)^{\alpha} i^{\alpha} \Gamma(1+\alpha)} \oint\limits_{C} f(z) (dz)^{\alpha} = \operatorname*{Res}_{z=z_0} f(z) = a_{-1}. \tag{23}$$

where $\operatorname*{Res}_{z=z_0} f(z) = a_{-1}$ *is the residue of the function* $f(z)$*.*

3 Analysis of the local fractional differential and integral equations

Here, we introduce the local fractional continuity, convergence, and completeness in a generalized metric space.

Definition 9 *A metric space on a fractal set* E *is a map* $\rho_{\alpha} : E \times E \to \mathbb{R}^{\alpha}$ *such that for all* $x^{\alpha}, y^{\alpha}, z^{\alpha} \in E$*.*

The following rules hold [9, 10, 11, 12]:

(1)$\rho_{\alpha}(x^{\alpha}, y^{\alpha}) \geq 0$ *with the equality* $\rho_{\alpha}(x^{\alpha}, y^{\alpha}) = 0$ *if* $x^{\alpha} = y^{\alpha}$*;*

(2) $\rho_{\alpha}(x^{\alpha}, y^{\alpha}) = \rho_{\alpha}(y^{\alpha}, x^{\alpha})$*;*

(3) $\rho_{\alpha}(x^{\alpha}, z^{\alpha}) \leq \rho_{\alpha}(x^{\alpha}, y^{\alpha}) + \rho_{\alpha}(y^{\alpha}, z^{\alpha})$*.*

The pair (E, ρ_{α}) *is a generalized metric space in the fractal space with the fractal dimension* α*.*

Let E is a generalized metric space and $a^{\alpha}, b^{\alpha}, c^{\alpha} \in E$. Then we have

$$|\rho_{\alpha}(a^{\alpha}, b^{\alpha}) - \rho_{\alpha}(b^{\alpha}, c^{\alpha})| \leq \rho_{\alpha}(a^{\alpha}, c^{\alpha}). \tag{24}$$

Definition 10 *Suppose that X, Y are generalized metric spaces and f is a mapping of X into Y. If for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho_\alpha (f(a), f(x)) < \varepsilon^\alpha$ whenever $x^\alpha \in X$ and $\rho(a, x) < \delta$, then f is called local fractional continuous at the point $a^\alpha \in X$, which is noted as follows [9, 10, 11]:*

$$\lim_{x \to a} f(x) = f(a). \tag{25}$$

Definition 11 *Let X be a generalized metric space.*

A sequence $\{x_n^\alpha\}_{n=1}^\infty$ in a generalized metric space X is called a Cauchy sequence if for each $\varepsilon > 0$ there exists a positive integer N such that [9, 10, 11]

$$\rho_\alpha (x_m^\alpha, x_n^\alpha) < \varepsilon^\alpha \tag{26}$$

whenever $m, n \geq N$. This is equivalent to the requirement that

$$\lim_{m,n \to \infty} \rho_\alpha (x_m^\alpha, x_n^\alpha) = 0. \tag{27}$$

Definition 12 *Let X be a generalized metric space. If each Cauchy sequence in the space X converges in X, the generalized metric space X is complete [9, 10, 11].*

We notice that $\mathbb{R}_n^\alpha$ and $\mathbb{C}_n^\alpha$ are complete.

Definition 13 *Let (X, ρ_α) be a generalized metric space and $T : X \to X$.*

If there exists a number $\beta \in (0, 1)$ such that [9, 10, 11]

$$\rho_\alpha (T(x^\alpha), T(y^\alpha)) \leq \beta^\alpha \rho_\alpha (x^\alpha, y^\alpha) \tag{28}$$

for all $x^\alpha, y^\alpha \in X$.

We say that T is a contraction mapping on the generalized metric space X.

Definition 14 *(see [9, 10, 11])*

Let (X, ρ_α) be a generalized metric space.

If $x^\alpha \in X$ and $Tx^\alpha = x^\alpha$, then we say that x^α is a fixed point of T.

Theorem 18 *(see [9, 10, 11])*

Let X be a generalized metric space. A convergent sequence in the fractal space X may have more than one limit in X.

Theorem 19 *(Contraction Mapping Theorem) (see [9, 10, 11])*

A contraction mapping T defined on the complete generalized metric space (X, ρ_α) has a unique fixed point.

Theorem 20 *(Generalized Contraction Mapping Theorem) (see [9, 10, 11])*

Let $T : X \to X$ be a map on the complete metric space (X, ρ_α). Then, for some $m \geq 1$, T^m is a contraction, and

$$\rho_\alpha (T^m(x^\alpha), T^m(x^\alpha)) \leq \beta^\alpha \rho_\alpha (x^\alpha, y^\alpha) \tag{29}$$

for all $x^\alpha, y^\alpha \in X$.

3.1 The unique of the solutions of the local fractional differential equations

In this subsection, we discuss the unique of the solutions of the local fractional differential equations.

Theorem 21 *Suppose that $x_0 \in [a,b]$ and $y_0 \in \mathbb{R}^{\alpha}$, $F : [a,b] \times \mathbb{R}_1^{\alpha} \to \mathbb{R}_1^{\alpha}$ is local fractional continuous. For all $x \in [a,b]$, there is a continuous condition given as (see [9, 10, 11, 61])*

$$\left| F\left(x, y_1\right) - F\left(x, y_2\right) \right| \le k^{\alpha} \left| y_1 - y_2 \right|^{\alpha}. \tag{30}$$

where $1 > k > 0$ and $1 \ge \alpha > 0$.

Then local fractional differential equation

$$\frac{d^{\alpha} y}{dx^{\alpha}} = F\left(x, y\right) \tag{31}$$

subject to the initial condition $y_0 = y\left(x_0\right)$ has a unique solution in the space $C_{\alpha}\left[a,b\right]$.

Proof 1 *We consider the map $T : C_{\alpha}\left[a,b\right] \to C_{\alpha}\left[a,b\right]$ defined as*

$$Tf\left(x\right) = y_0 + \frac{1}{\Gamma\left(1+\alpha\right)} \int_{x_0}^{x} F\left(t, f\left(t\right)\right) \left(dt\right)^{\alpha}.$$

We claim that for all n,

$$\left| T^n f_1\left(x\right) - T^n f_2\left(x\right) \right| \le k^{n\alpha} \frac{\left| x - x_0 \right|^{n\alpha}}{\Gamma\left(1+n\alpha\right)} \rho_{\alpha}\left(f_1, f_2\right).$$

The proof is by the induction on n.

The case $n = 0$ is trivial (and $n = 1$ is already done).

The induction step is as follows:

$$\begin{aligned}
&\left| T^{n+1} f_1\left(x\right) - T^{n+1} f_2\left(x\right) \right| \\
&= \left| \tfrac{1}{\Gamma(1+\alpha)} \int_{x_0}^{x} F\left(t, T^n f_1\left(x\right)\right) - F\left(t, T^n f_2\left(x\right)\right) \left(dt\right)^{\alpha} \right| \\
&\le \left| \tfrac{1}{\Gamma(1+\alpha)} \int_{x_0}^{x} k^{\alpha} \left| F\left(t, T^n f_1\left(x\right)\right) - F\left(t, T^n f_2\left(x\right)\right) \right| \left(dt\right)^{\alpha} \right| \\
&\le \left| \tfrac{1}{\Gamma(1+\alpha)} \int_{x_0}^{x} \tfrac{k^{(n+1)\alpha} |x-x_0|^{n\alpha}}{\Gamma(1+n\alpha)} \rho_{\alpha}\left(f_1, f_2\right) \left(dt\right)^{\alpha} \right| \\
&\le \left| \tfrac{1}{\Gamma(1+\alpha)} \int_{x_0}^{x} k^{(n+1)\alpha} \tfrac{|x-x_0|^{n\alpha}}{\Gamma(1+n\alpha)} \rho_{\alpha}\left(f_1, f_2\right) \left(dt\right)^{\alpha} \right| \\
&\le k^{(n+1)\alpha} \tfrac{|x-x_0|^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \rho_{\alpha}\left(f_1, f_2\right) \\
&\le k^{(n+1)\alpha} \tfrac{|b-a|^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} \rho_{\alpha}\left(f_1, f_2\right).
\end{aligned}$$

We have

$$k^{(n+1)\alpha} \frac{\left| b - a \right|^{(n+1)\alpha}}{\Gamma\left(1 + \left(n+1\right)\alpha\right)} \rho_{\alpha}\left(f_1, f_2\right) \to 0$$

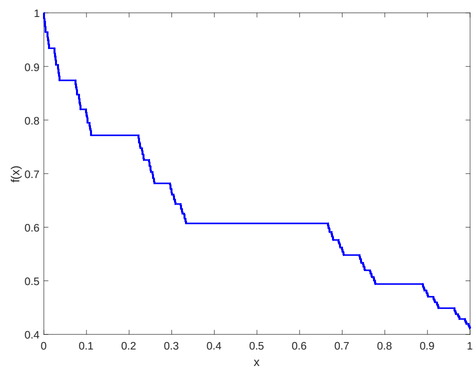

Figure 1: The plot of the solution of the local fractional differential equation when $\alpha = \ln 2/\ln 3$.

as $n \to 0$.

If n *is sufficiently large, we have*

$$0 < k^{(n+1)\alpha} \frac{|b-a|^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)} < 1$$

such that T^n *is a contraction on the space* $C_\alpha[a,b]$.

Hence, T *has a unique fixed point in the space* $C_\alpha[a,b]$*, which gives a unique solution to the local fractional differential equation.*

Example 1 *The local fractional differential equation*

$$\frac{d^\alpha f(x)}{dx^\alpha} + f(x) = 0,$$

has the unique solution given as $f(x) = E_\alpha(-x^\alpha)$ *and its graph is shown in Figure 1.*

3.2 The unique of the solutions of the local fractional integral equations

In this subsection, we discuss the unique of the solutions of the local fractional integral equations.

Theorem 22 *Let* $C_\alpha[a,b] = \{x(t) : x(t)$ *be local fractional continuous on the interval* $[a,b]$. *The metric on the space* $C_\alpha[a,b]$ *is defined as (see [9, 10, 11])*

$$\rho_\alpha(x,y) = \{\max|x(t)-y(t)| : t \in [a,b], x, y \in C_\alpha[a,b]\}. \tag{32}$$

Let us consider that the local fractional integral equation

$$f\left(x\right)=\frac{\lambda^{\alpha}}{\Gamma\left(1+\alpha\right)}\int_{a}^{x}F\left(x,y\right)f\left(y\right)\left(dy\right)^{\alpha}+\varphi\left(x\right), \tag{33}$$

has a unique solution in $C_{\alpha}\left[a,b\right]$*, where* $\lambda^{\alpha}\in\mathbb{R}^{\alpha}$*,* $\varphi\in C_{\alpha}\left[a,b\right]$ *and* $F\left(x,y\right)\in C_{\alpha}\left[a,b\right]\times C_{\alpha}\left[a,b\right]$*.*

Proof 2 *We define* $T:C_{\alpha}\left[a,b\right]\rightarrow C_{\alpha}\left[a,b\right]$ *by*

$$Tf\left(x\right)=\frac{\lambda^{\alpha}}{\Gamma\left(1+\alpha\right)}\int_{a}^{x}F\left(x,y\right)f\left(y\right)\left(dy\right)^{\alpha}+\varphi\left(x\right).$$

Let $f_1,\ f_2\in C_{\alpha}\left[a,b\right]$*. Then*

$$\begin{aligned}
&\rho_{\alpha}\left(Tf_1,Tf_2\right)\\
&=\max_{x\in[a,b]}\left|Tf_1-Tf_2\right|\\
&=\max_{x\in[a,b]}\left|\tfrac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_a^x F\left(x,y\right)\left(f_1\left(y\right)-f_2\left(y\right)\right)\left(dy\right)^{\alpha}\right|\\
&\leq\tfrac{\left|\lambda\right|^{\alpha}M}{\Gamma(1+\alpha)}\left[\max_{x\in[a,b]}\left|f_1\left(x\right)-f_2\left(x\right)\right|\right]\left|\int_a^x\left(dy\right)^{\alpha}\right|\\
&\leq\tfrac{\left|\lambda\right|^{\alpha}M\rho_{\alpha}(f_1,f_2)}{\Gamma(1+\alpha)}\left|\int_a^x\left(dy\right)^{\alpha}\right|\\
&=\tfrac{\left|\lambda\right|^{\alpha}M\rho_{\alpha}(f_1,f_2)}{\Gamma(1+\alpha)}\left|x-a\right|^{\alpha}\\
&\leq\tfrac{\left|\lambda\right|^{\alpha}M\left|b-a\right|^{\alpha}}{\Gamma(1+\alpha)}\rho_{\alpha}\left(f_1,f_2\right)
\end{aligned}$$

where $M=\max\leq\left\{\left|F\left(x,y\right)\right|:x,y\in\left[a,b\right]\right\}$*.*

We claim that for all n,

$$\rho_{\alpha}\left(T^nf_1,T^nf_2\right)\leq\frac{\left|\lambda\right|^{n\alpha}M^n\left|x-a\right|^{n\alpha}}{\Gamma\left(1+n\alpha\right)}\rho_{\alpha}\left(f_1,f_2\right)\leq\frac{\left|\lambda\right|^{n\alpha}M^n\left|b-a\right|^{n\alpha}}{\Gamma\left(1+n\alpha\right)}\rho_{\alpha}\left(f_1,f_2\right).$$

The induction step is as follows:

$$\begin{aligned}
&\rho_{\alpha}\left(T^{n+1}f_1,T^{n+1}f_2\right)=\max_{x\in[a,b]}\left|T^{n+1}f_1-T^{n+1}f_2\right|\\
&=\max_{x\in[a,b]}\left|\tfrac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_a^x F\left(x,y\right)\left(T^nf_1\left(y\right)-T^nf_2\left(y\right)\right)\left(dy\right)^{\alpha}\right|\\
&\leq\tfrac{\left|\lambda\right|^{(n+1)\alpha}M^{n+1}}{\Gamma(1+n\alpha)}\left[\max_{x\in[a,b]}\left|f_1\left(x\right)-f_2\left(x\right)\right|\right]\left|\tfrac{1}{\Gamma(1+\alpha)}\int_a^x\left(x-a\right)^{n\alpha}\left(dy\right)^{\alpha}\right|\\
&\leq\tfrac{\left|\lambda\right|^{(n+1)\alpha}M^{n+1}\rho_{\alpha}(f_1,f_2)}{\Gamma(1+(n+1)\alpha)}\left|x-a\right|^{(n+1)\alpha}\\
&\leq\tfrac{\left|\lambda\right|^{(n+1)\alpha}M^{n+1}\left|b-a\right|^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}\rho_{\alpha}\left(f_1,f_2\right)
\end{aligned}$$

For each $\lambda^{\alpha}\in\mathbb{R}^{\alpha}$*, there exists* $N\in\mathbb{N}$ *such that*

$$0<\frac{\left|\lambda\right|^{n\alpha}M^n\left|b-a\right|^{n\alpha}}{\Gamma\left(1+n\alpha\right)}\rho_{\alpha}\left(f_1,f_2\right)<1,$$

where $n>N$*.*

It is to say that T^n *is a contraction mapping and has a unique fixed point* f*.*

Thus, f *provides the unique local fractional continuous solution to the local fractional integral equation.*

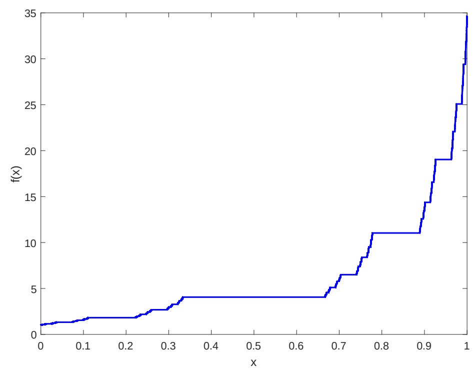

Figure 2: The plot of the solution of the local fractional integral equation when $\lambda = 2$ and $\alpha = \ln 2/\ln 3$.

Example 2 *The local fractional integral equation*

$$f(x) - \frac{\lambda}{\Gamma(1+\alpha)} \int_0^x f(x)\,(dx)^\alpha = 1,$$

has the unique solution given as $f(x) = E_\alpha(\lambda x^\alpha)$ *and its graph is shown in Figure 2.*

4. Local fractional inequalities

In this chapter, we present the inequalities within local fractional integral, such as the Hölder type, Cauchy–Schwarz type and Minkowski type inequalities.

Let E be a fractal set.

The Hölder type, Cauchy–Schwarz type and Minkowski type inequalities in the fractal finite series are presented as follows:

Theorem 23 *(Generalized Hölder type inequality) (see [9, 10, 11])*
Let $|x_i^\alpha| > 0$, $|y_i^\alpha| > 0$, $p > 0$, $q > 0$, $i \in \mathbb{N}$ *and* $1/p + 1/q = 1$. *Then we have*

$$\sum_{i=1}^{n} |x_i^\alpha|\,|y_i^\alpha| \le \left(\sum_{i=1}^{n} |x_i^\alpha|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i^\alpha|^q\right)^{\frac{1}{q}}, \tag{34}$$

where $p > 1$, $q > 1$ *and* $0 < \alpha \le 1$.

Theorem 24 *(Generalized Cauchy–Schwarz type inequality) (see [9, 10, 11])*
Let $|x_i^\alpha| > 0$, $|y_i^\alpha| > 0$ *and* $i \in \mathbb{N}$. *Then we have*

$$\sum_{i=1}^{n} |x_i^\alpha|\,|y_i^\alpha| \le \left(\sum_{i=1}^{n} |x_i^\alpha|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} |y_i^\alpha|^2\right)^{\frac{1}{2}}. \tag{35}$$

Theorem 25 *(Generalized Minkowski type inequality) (see [9, 10, 11])*

$$\left(\sum_{i=1}^{n}|x_i^\alpha - y_i^\alpha|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n}|x_i^\alpha|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|y_i^\alpha|^p\right)^{\frac{1}{p}}, \tag{36}$$

where $p > 1$ *and* $0 < \alpha \leq 1$.

For the linear space of bounded infinite sequences, denoted as $E = l_{p,\alpha}$, the generalized normed linear space on E is defined by (see [9, 10, 11]):

$$\|x^\alpha\|_{p,\alpha} =: \left(\sum_{i=1}^{\infty}|x_i^\alpha|^p\right)^{\frac{1}{p}} < \infty, \tag{37}$$

where $1 \leq p < \infty$.

Theorem 26 *(The infinite version of the generalized Minkowski type inequality)*

The infinite version of generalized Minkowski type inequality can be write as [9, 10, 11]:

$$\left(\sum_{i=1}^{\infty}|x_i^\alpha - y_i^\alpha|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty}|x_i^\alpha|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty}|y_i^\alpha|^p\right)^{\frac{1}{p}},$$

where $\infty > p \geq 1$ *and* $0 < \alpha \leq 1$.

Let $E = L_{p,\alpha}[a,b]$. Then the normed space with the $p-$norm is given as ([9, 10, 11]):

$$\|f\|_{p,\alpha} =: \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(t)|^p (dt)^\alpha\right)^{\frac{1}{p}} < \infty, \tag{38}$$

where $0 < \alpha \leq 1$ and $\infty > p \geq 1$.

The following rules hold ([9, 10, 11]):

1. If $\|f\|_{1,\alpha} = 0$, then $f(x) = 0$;
2. $\|ag\|_{1,\alpha} = |a|^\alpha \|f\|_{p,\alpha}$;
3. $\|f+g\|_{1,\alpha} \leq \|f\|_{1,\alpha} + \|g\|_{1,\alpha}$.

Theorem 5 (The integral form of the generalized Hölder type inequality)

Let $f, g \in L_{p,\alpha}[\mathbb{R}], 1 \leq p < \infty$. Then we have (see [9, 10, 11])

$$\|fg\|_{1,\alpha} \leq \|f\|_{p,\alpha}\|g\|_{q,\alpha}, \tag{39}$$

where $p \geq 1,\ q \geq 1$ and $1/q + 1/p = 1$.

Theorem 6 (The integral form of the generalized Minkowski type inequality)

Let $f, g \in L_{p,\alpha}[\mathbb{R}],\ 1 \le p < \infty$. Then we have (see [9, 10, 11])

$$\|f+g\|_{p,\alpha} \le \|f\|_{p,\alpha} + \|g\|_{p,\alpha}. \tag{40}$$

For more details of the Hölder type, Cauchy–Schwarz type and Minkowski type inequalities defined on the fractal domain, see [9, 10, 11].

5. The series and transforms involving the Mattig-Leffer function defined on Cantor sets

In this section, we consider the concepts and theorems of the series and transforms involving the Mattig-Leffer function defined on Cantor sets.

5.1 The Fourier type series via Mattig-Leffer function defined on Cantor sets

In this subsection, we introduce the concepts and theorems of the series involving the Mattig-Leffer function defined on Cantor sets.

Definition 15 *Let $f(x)$ be 2π-periodic. For $n \in \mathbb{Z}$, the complex Mittag-Leffler form of the local fractional Fourier type series of $f(x)$ involving the Mattig-Leffer function defined on Cantor sets is defined as (see [9, 10, 11, 13])*

$$f(x) \sim \sum_{k=-\infty}^{\infty} C_n E_\alpha\left(i^\alpha (nx)^\alpha\right), \tag{41}$$

where the Fourier coefficients are represented as (see [9, 10, 11, 13]):

$$C_n = \frac{1}{(2\pi)^\alpha} \int_{-\pi}^{\pi} f(x) E_\alpha\left(-i^\alpha (nx)^\alpha\right)(dx)^\alpha. \tag{42}$$

Theorem 27 *Suppose that $f(x)$ is 2π-periodic, bounded and local fractional integral on $[-\pi,\pi]$. Then, the local fractional series of the function $f(x)$ involving the Mattig-Leffer function defined on Cantor sets converges to $f(x)$ at $x \in [-\pi,\pi]$, and (see [9, 10, 11, 13])*

$$\frac{f(x+0)+f(x-0)}{2} = \sum_{k=-\infty}^{\infty} C_n E_\alpha\left(i^\alpha (nx)^\alpha\right), \tag{43}$$

where the Fourier type coefficients are expressed by

$$C_n = \frac{1}{(2l)^\alpha} \int_{-l}^{l} f(x) E_\alpha\left(\pi^\alpha i^\alpha (nx)^\alpha\right)(dx)^\alpha. \tag{44}$$

Definition 16 *Let $f(x)$ be $2l$-periodic. For $n \in \mathbb{Z}$, the complex generalized Mittag-Leffler form of the local fractional Fourier type series of the function $f(x)$ involving the Mattig-Leffer function defined on Cantor sets is defined as (see [9, 10, 11, 13])*

$$f(x) \sim \sum_{k=-\infty}^{\infty} C_n E_\alpha \left(\frac{\pi^\alpha i^\alpha (nx)^\alpha}{l^\alpha} \right), \tag{45}$$

where the Fourier type coefficients are given as

$$C_n = \frac{1}{(2l)^\alpha} \int_{-l}^{l} f(x) E_\alpha \left(\frac{-\pi^\alpha i^\alpha (nx)^\alpha}{l^\alpha} \right) (dx)^\alpha. \tag{46}$$

Theorem 28 *Suppose that $f(x)$ is $2l$-periodic, bounded and local fractional integral on $[-l,l]$. Then, the local fractional series of the function $f(x)$ involving the Mattig-Leffer function defined on Cantor sets converges to $f(x)$ at $x \in [-l,l]$, and (see [9, 10, 11, 13])*

$$\frac{f(x+0)+f(x-0)}{2} = \sum_{k=-\infty}^{\infty} C_n E_\alpha \left(\frac{\pi^\alpha i^\alpha (nx)^\alpha}{l^\alpha} \right), \tag{47}$$

where the Fourier type coefficients are represented as

$$C_n = \frac{1}{(2l)^\alpha} \int_{-l}^{l} f(x) E_\alpha \left(\frac{-\pi^\alpha i^\alpha (nx)^\alpha}{l^\alpha} \right) (dx)^\alpha. \tag{48}$$

5.2 The Fourier type transform via Mattig-Leffer function defined on Cantor sets

In this subsection, we introduce the concepts and theorems of the Fourier type transform involving the Mattig-Leffer function defined on Cantor sets.

Definition 17 *The local fractional Fourier type transform of the function $f(x)$ involving the Mattig-Leffer function defined on Cantor sets is defined as (see [9, 10, 11, 13])*

$$F_\alpha \{f(x)\} = f_\omega^{F,\alpha}(\omega) := \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} E_\alpha(-i^\alpha \omega^\alpha x^\alpha) f(x) (dx)^\alpha,$$

where the latter converges.

The sufficient condition for convergence is given as (see [9, 10, 11, 13])

$$\left| \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(x) E_\alpha(-i^\alpha \omega^\alpha x^\alpha) (dx)^\alpha \right| \le \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} |f(x)| (dx)^\alpha = \|f\|_{1,\alpha} < \infty,$$

which can be written as $f \in L_{1,\alpha}[\mathbb{R}]$.

If $f \in L_{1,\alpha}[\mathbb{R}]$, then local fractional Fourier type transform of the function $f(x)$ exists.

The inverse local fractional Fourier type transform involving the Mattig-Leffer function defined on Cantor sets is defined as (see [9, 10, 11, 13])

$$f(x) = F_\alpha^{-1}\left(f_\omega^{F,\alpha}(\omega)\right) := \frac{1}{(2\pi)^\alpha}\int_{-\infty}^{\infty} E_\alpha\left(i^\alpha\omega^\alpha x^\alpha\right) f_\omega^{F,\alpha}(\omega)(d\omega)^\alpha,$$

where the latter converges.

Definition 18 *The local fractional convolution of the functions $f_1(x)$ and $f_2(x)$ is defined as (see [9, 10, 11, 13])*

$$f_1(x) * f_2(x) = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} f_1(t) f_2(x-t)(dt)^\alpha.$$

There are the equalities as follows (see [9, 10, 11, 13]):

$$f_1(x) * f_2(x) = f_2(x) * f_1(x),$$

$$f_1(x) * (f_2(x) + f_3(x)) = f_1(x) * f_2(x) + f_1(x) * f_3(x).$$

The theorems for the local fractional Fourier type transform are presented as follows (see [9, 10, 11, 13]):

Let $f, f_1, f_2 \in L_{1,\alpha}[\mathbb{R}]$, $F_\alpha\{f(x)\} = f_\omega^{F,\alpha}(\omega)$, $F_\alpha\{f_1(x)\} = f_{\omega,1}^{F,\alpha}(\omega)$ and $F_\alpha\{f_2(x)\} = f_{\omega,2}^{F,\alpha}(\omega)$. Then, we have the following:

(1) $F_\alpha\{f_1(x) + f_2(x)\} = F_\alpha\{f_1(x)\} + F_\alpha\{f_2(x)\}$;

(2) $F_\alpha\{f_1(x) * f_2(x)\} = f_{\omega,1}^{F,\alpha}(\omega) f_{\omega,2}^{F,\alpha}(\omega)$;

(3) $F_\alpha\left\{f^{(\alpha)}(x)\right\} = i^\alpha\omega^\alpha F_\alpha\{f(x)\}$, where $\lim\limits_{|x|\to\infty} f(x) = 0$;

(4) $F_\alpha\left\{{}_{-\infty}I_x^{(\alpha)} f(x)\right\} = F_\alpha\{f(x)\}/(i^\alpha\omega^\alpha)$, where $\lim\limits_{x\to\infty} {}_{-\infty}I_x^{(\alpha)} f(x) \to 0$.

6. Local fractional vector calculus with an application in fractal fracture mechanics

In this chapter, we introduce the theory of the local fractional vector calculus and present an application to the Rice theory in the fractal fracture mechanics.

6.1 Local fractional vector calculus

In this subsection, we introduce the basic theory and theorems of the local fractional vector calculus.

Definition 19 *For $1 > \alpha > 0$, the local fractional line integral of the function $\mathbf{u}(x_P, y_P, z_P)$ along a fractal line l^α is defined as (see [2, 12])*

$$\int\limits_{l^{(\alpha)}} \mathbf{u}(x_P, y_P, z_P)\cdot d\mathbf{l}^{(\alpha)} = \lim_{N\to\infty}\sum_{P=1}^{N}\mathbf{u}(x_P, y_P, z_P)\cdot\Delta\mathbf{l}_P^{(\alpha)} \tag{49}$$

where the elements of line $\Delta\mathbf{l}_P^{(\alpha)}$ is required that all $|\Delta l_P^\alpha| \to 0$as $N\to\infty$ and $\beta = 2\alpha$.

Definition 20 *For $\gamma = \frac{3}{2}\beta = 3\alpha, 1 > \alpha > 0$, the local fractional surface integral of $u(r_P)$ is defined as (see [2, 12]):*

$$\iint u(r_P)\, d\mathbf{S}^{(\beta)} = \lim_{N\to\infty} \sum_{P=1}^{N} u(r_P)\, \mathbf{n}_P \Delta S_P^{(\beta)} \tag{50}$$

where $d\mathbf{S}^{(\beta)}$ is N elements of area with a unit normal local fractional vector n_P, $\Delta S_P^{(\beta)} \to 0$ as $N \to \infty$.

Definition 21 *For $\gamma = \frac{3}{2}\beta = 3\alpha, 1 > \alpha > 0$, the local fractional volume integral of the function $\mathbf{u}(r_P)$ is defined as (see [2, 12]):*

$$\iiint \mathbf{u}(r_P)\, dV^{(\gamma)} = \lim_{N\to\infty} \sum_{P=1}^{N} \mathbf{u}(r_P)\, \Delta V_P^{(\gamma)}, \tag{51}$$

where $\Delta V_P^{(\gamma)}$ is the elements of volume $\Delta V_P^{(\gamma)} \to 0$ as $N \to \infty$.

Basic operators of the local fractional vector integrals are as follows (see [2, 12]):

$$\int_{l^{(\alpha)}} (\mathbf{u}_1 + \mathbf{u}_2) \cdot d\mathbf{l}^{(\alpha)} = \int_{l^{(\alpha)}} \mathbf{u}_1 \cdot d\mathbf{l}^{(\alpha)} + \int_{l^{(\alpha)}} \mathbf{u}_2 \cdot d\mathbf{l}^{(\alpha)},$$

$$\int_{l^{(\alpha)}} \mathbf{u} \cdot d\mathbf{l}^{(\alpha)} = \int_{l_1^{(\alpha)}} \mathbf{u} \cdot d\mathbf{l}^{(\alpha)} + \int_{l_2^{(\alpha)}} \mathbf{u} \cdot d\mathbf{l}^{(\alpha)},$$

$$\iint_{S^{(\beta)}} (\mathbf{u}_1 + \mathbf{u}_2) \cdot d\mathbf{S}^{(\beta)} = \iint_{S^{(\beta)}} \mathbf{u}_1 \cdot d\mathbf{S}^{(\beta)} + \iint_{S^{(\beta)}} \mathbf{u}_2 \cdot d\mathbf{S}^{(\beta)},$$

$$\iint_{S^{(\beta)}} \mathbf{u} \cdot d\mathbf{S}^{(\beta)} = \iint_{S_1^{(\beta)}} \mathbf{u} \cdot d\mathbf{S}^{(\beta)} + \iint_{S_2^{(\beta)}} \mathbf{u} \cdot d\mathbf{S}^{(\beta)},$$

$$\iiint_{V^{(\gamma)}} (\mathbf{u}_1 + \mathbf{u}_2) \cdot dV^{(\gamma)} = \iiint_{V^{(\gamma)}} \mathbf{u}_1 \cdot dV^{(\gamma)} + \iiint_{V^{(\gamma)}} \mathbf{u}_2 \cdot dV^{(\gamma)},$$

$$\iiint_{V^{(\gamma)}} (\mathbf{u}_1 + \mathbf{u}_2) \cdot dV^{(\gamma)} = \iiint_{V_1^{(\gamma)}} \mathbf{u} \cdot dV^{(\gamma)} + \iiint_{V_2^{(\gamma)}} \mathbf{u} \cdot dV^{(\gamma)},$$

where $\mathbf{l}^{(\alpha)} = \mathbf{l}_1^{(\alpha)} + \mathbf{l}_2^{(\alpha)}$, $\mathbf{S}^{(\beta)} = \mathbf{S}_1^{(\beta)} + \mathbf{S}_2^{(\beta)}$ and $V^{(\gamma)} = V_1^{(\gamma)} + V_2^{(\gamma)}$.

Definition 22 *For $\gamma = \frac{3}{2}\beta = 3\alpha, 1 > \alpha > 0$, the local fractional gradient of the scale function φ is defined as (see [2, 12])*

$$\nabla^\alpha \varphi = \lim_{dV^{(\gamma)}\to 0} \left(\frac{1}{dV^{(\gamma)}} \oiint_{S^{(\beta)}} \varphi d\mathbf{S}^{(\beta)} \right) = \frac{\partial^\alpha \varphi}{\partial x_1^\alpha} e_1^\alpha + \frac{\partial^\alpha \varphi}{\partial x_2^\alpha} e_2^\alpha + \frac{\partial^\alpha \varphi}{\partial x_3^\alpha} e_3^\alpha, \tag{52}$$

where $V^{(\gamma)}$ is a small fractal volume enclosing P, $S^{(\beta)}$ is its bounding fractal surface, and ∇^α is a local fractional Hamilton operator.

Definition 23 *For $\gamma = \frac{3}{2}\beta = 3\alpha, 1 > \alpha > 0$, the local fractional divergence of the vector function $\mathbf{u}$ is defined by (see [2, 12])*

$$\nabla^\alpha \bullet \mathbf{u} = \lim_{dV^{(\gamma)} \to 0} \left(\frac{1}{dV^{(\gamma)}} \oiint\limits_{S^{(2\alpha)}} \mathbf{u} \bullet d\mathbf{S}^{(\beta)} \right) = \frac{\partial^\alpha u_1}{\partial x_1^\alpha} + \frac{\partial^\alpha u_2}{\partial x_2^\alpha} + \frac{\partial^\alpha u_3}{\partial x_3^\alpha}, \tag{53}$$

where $\mathbf{u} = u_1 e_1^\alpha + u_2 e_2^\alpha + u_3 e_3^\alpha$.

Definition 24 *For $\gamma = \frac{3}{2}\beta = 3\alpha, 1 > \alpha > 0$, the local fractional curl of the vector function $\mathbf{u}$ is defined by (see [2, 12]):*

$$\begin{aligned} \nabla^\alpha \times \mathbf{u} &= \lim_{dS^{(\beta)} \to 0} \left(\tfrac{1}{dS^{(\beta)}} \oint\limits_{l^{(\alpha)}} \mathbf{u} \cdot d\mathbf{l}^{(\alpha)} \right) \mathbf{n}_P \\ &= \left(\tfrac{\partial^\alpha u_3}{\partial x_2^\alpha} - \tfrac{\partial^\alpha u_2}{\partial x_3^\alpha} \right) e_1^\alpha + \left(\tfrac{\partial^\alpha u_1}{\partial x_3^\alpha} - \tfrac{\partial^\alpha u_3}{\partial x_1^\alpha} \right) e_2^\alpha + \left(\tfrac{\partial^\alpha u_2}{\partial x_1^\alpha} - \tfrac{\partial^\alpha u_1}{\partial x_2^\alpha} \right) e_3^\alpha, \end{aligned} \tag{54}$$

where $\mathbf{u} = u_1 e_1^\alpha + u_2 e_2^\alpha + u_3 e_3^\alpha$.

Theorem 29 *(Local fractional Gauss theorem)*

For $\gamma = \frac{3}{2}\beta = 3\alpha, 1 > \alpha > 0$, the local fractional Gauss theorem of the fractal vector field states that (see [2, 12])

$$\iiint\limits_{V^{(\gamma)}} \nabla^\alpha \cdot \mathbf{u} dV^{(\gamma)} = \oiint\limits_{S^{(\beta)}} \mathbf{u} \cdot d\mathbf{S}^{(\beta)}. \tag{55}$$

Theorem 30 *(Local fractional Stokes' theorem)*

For $\beta = 2\alpha, 1 > \alpha > 0$, the local fractional Stokes' theorem of the fractal field states that (see [2, 12])

$$\oint\limits_{l^{(\alpha)}} \mathbf{u} \cdot d\mathbf{l}^\alpha = \iint\limits_{S^{(\beta)}} (\nabla^\alpha \times \mathbf{u}) \cdot d\mathbf{S}^{(\beta)}.$$

For more details of the local fractional vector calculus, see [2, 12].

6.2 An application to Rice theory in fractal mechanics

Let us consider the work of the traction in fractal boundary, the elastic energy in fractal medium and the fractal losing energy be

$$W_1 = \iint\limits_{S^{(\beta)}} \mathbf{p} \cdot \mathbf{u} d\mathbf{S}^{(\beta)}, \ W_2 = -\iiint\limits_{V^{(\gamma)}} w dV^{(\gamma)}$$

and

$$W_3 = \int\limits_{l^{(\alpha)}} D \cdot d\mathbf{l}^{(\alpha)},$$

respectively, where $\mathbf{p}$ is the traction in the fractal boundary, $\mathbf{u}$ is the fractal displacement, w is the fractal elastic energy density, and D is the fractal losing energy in unit fractal line.

The energy in fractal medium can be written as

$$W=\int\limits_{l^{(\alpha)}} p_i u_i d\mathbf{l}^{(\alpha)} - \iint\limits_{S^{(\beta)}} w d\mathbf{S}^{(\beta)}, \tag{56}$$

where p_i and u_i are components of both traction in the fractal boundary and the fractal displacement.

Consider the fractal losing energy and finding the LFD, we give

$$\frac{\partial^\alpha W}{\partial t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha}\int\limits_{l^{(\alpha)}} p_i u_i d\mathbf{l}^{(\alpha)} - \frac{\partial^\alpha}{\partial t^\alpha}\iint\limits_{S^{(\beta)}} w d\mathbf{S}^{(\beta)} - \frac{\partial^\alpha D}{\partial t^\alpha}. \tag{57}$$

With the use of

$$\frac{\partial^\alpha}{\partial t^\alpha}\int\limits_{l^{(\alpha)}} p_i u_i d\mathbf{l}^{(\alpha)} = \int\limits_{l^{(\alpha)}} p_i \frac{\partial^\alpha u_i}{\partial t^\alpha} d\mathbf{l}^{(\alpha)} = \int\limits_{l^{(\alpha)}} p_i \frac{\partial^\alpha u_i}{\partial a^\alpha}\left(\frac{\partial a}{\partial t}\right)^\alpha d\mathbf{l}^{(\alpha)},$$

$$\frac{\partial^\alpha}{\partial t^\alpha}\iint\limits_{S^{(\beta)}} w d\mathbf{S}^{(\beta)} = \iint\limits_{S^{(\beta)}} \frac{\partial^\alpha w}{\partial t^\alpha} d\mathbf{S}^{(\beta)} = \iint\limits_{S^{(\beta)}} \frac{\partial^\alpha w}{\partial a^\alpha}\left(\frac{\partial a}{\partial t}\right)^\alpha d\mathbf{S}^{(\beta)},$$

$$\frac{\partial^\alpha D}{\partial t^\alpha} = \frac{\partial^\alpha D}{\partial a^\alpha}\left(\frac{\partial a}{\partial t}\right)^\alpha,$$

where a is the length of crack, we obtain from Eq.(57) that

$$\begin{aligned} &\tfrac{\partial^\alpha W}{\partial t^\alpha} \\ &= \int\limits_{l^{(\alpha)}} p_i \tfrac{\partial^\alpha u_i}{\partial a^\alpha}\left(\tfrac{\partial a}{\partial t}\right)^\alpha d\mathbf{l}^{(\alpha)} - \iint\limits_{S^{(\beta)}} \tfrac{\partial^\alpha w}{\partial a^\alpha}\left(\tfrac{\partial a}{\partial t}\right)^\alpha d\mathbf{S}^{(\beta)} - \tfrac{\partial^\alpha D}{\partial a^\alpha}\left(\tfrac{\partial a}{\partial t}\right)^\alpha \\ &= \left(\tfrac{\partial a}{\partial t}\right)^\alpha\left(\int\limits_{l^{(\alpha)}} p_i \tfrac{\partial^\alpha u_i}{\partial a^\alpha} d\mathbf{l}^{(\alpha)} - \iint\limits_{S^{(\beta)}} \tfrac{\partial^\alpha w}{\partial a^\alpha} d\mathbf{S}^{(\beta)} - \tfrac{\partial^\alpha D}{\partial a^\alpha}\right). \end{aligned} \tag{58}$$

When $\partial^\alpha W/\partial t^\alpha = 0$, we have from Eq.(58) that

$$\int\limits_{l^{(\alpha)}} p_i \frac{\partial^\alpha u_i}{\partial a^\alpha} d\mathbf{l}^{(\alpha)} - \iint\limits_{S^{(\beta)}} \frac{\partial^\alpha w}{\partial a^\alpha} d\mathbf{S}^{(\beta)} - \frac{\partial^\alpha D}{\partial a^\alpha} = 0. \tag{59}$$

The J-integral in fractal medium is defined as

$$J_\alpha = \frac{\partial^\alpha D}{\partial a^\alpha}.$$

From Eq.(59), we obtain that

$$J_\alpha = \int\limits_{l^{(\alpha)}} p_i \frac{\partial^\alpha u_i}{\partial a^\alpha} d\mathbf{l}^{(\alpha)} - \iint\limits_{S^{(\beta)}} \frac{\partial^\alpha w}{\partial a^\alpha} d\mathbf{S}^{(\beta)}. \tag{60}$$

As an extended version of the Rice's theory, we give that

$$\frac{\partial^\alpha W}{\partial t^\alpha} \geq 0. \tag{61}$$

From Eq.(61), there are two cases:

Case 1. When the crack tip is super-static, there is $\partial^\alpha W/\partial t^\alpha > 0$;

Case 2. When the crack tip is sub-static, there is $\partial^\alpha W/\partial t^\alpha = 0$.

When the crack length has is greater and the horizontal coordinate value is smaller, there is relationship of both increment of crack length and increment of horizontal coordinate value given as

$$(dx)^\alpha = -(da)^\alpha \tag{62}$$

which leads to

$$J_\alpha = \int\limits_{l^{(\alpha)}} p_i \frac{\partial^\alpha u_i}{\partial a^\alpha} d\mathbf{l}^{(\alpha)} - \iint\limits_{S^{(\beta)}} \frac{\partial^\alpha w}{\partial a^\alpha} d\mathbf{S}^{(\beta)} = \int\limits_{l^{(\alpha)}} w\,(dy)^\alpha\, d\mathbf{l}^{(\alpha)} - \int\limits_{l^{(\alpha)}} p_i \frac{\partial^\alpha u_i}{\partial a^\alpha} d\mathbf{l}^{(\alpha)}. \tag{63}$$

By using the traction on the fractal boundary given as

$$\mathbf{P} = \mathbf{N} \cdot \sigma, \tag{64}$$

we have

$$(dx)^\alpha = (N_1) \cdot d\mathbf{l}^{(\alpha)}, (dy)^\alpha = N_2 d\mathbf{l}^{(\alpha)}, \tag{65}$$

where

$$N_1 = \frac{(dx)^\alpha}{\sqrt{(dx)^\alpha + (dy)^\alpha}},\ N_2 = -\frac{(dy)^\alpha}{\sqrt{(dx)^\alpha + (dy)^\alpha}}. \tag{66}$$

Suppose that $w = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\,(\varepsilon_{ij})^\alpha$, where $\sigma_{ij} = \partial^\alpha w/\partial\,(\varepsilon_{ij})^\alpha$ and $\varepsilon_{ij} = \partial^\alpha u_i/\partial x_j^\alpha$, we have

$$\int\limits_{l^{(\alpha)}} w\,(dy)^\alpha\, d\mathbf{l}^{(\alpha)} = \iint\limits_{S^{(\beta)}} \frac{\partial^\alpha w}{\partial x^\alpha} d\mathbf{S}^{(\beta)} = \iint\limits_{S^{(\beta)}} \sigma_{ij} \frac{\partial^\alpha \varepsilon_{ij}}{\partial x^\alpha} d\mathbf{S}^{(\beta)} = \oint\limits_{l^{(\alpha)}} \sigma_{ij} N_j \frac{\partial^\alpha u_i}{\partial x^\alpha} d\mathbf{l}^{(\alpha)}, \tag{67}$$

which leads to

$$J_\alpha = \oint\limits_{l^{(\alpha)}} (\sigma_{ij} N_j - p_i) \frac{\partial^\alpha u_i}{\partial x^\alpha} d\mathbf{l}^{(\alpha)} = 0, \tag{68}$$

where $l^{(\alpha)}$ is the closed circle.

The result states the crack tip is always super-static or sub-static in the real materials and the two cases always take place in the real crack progression in the differential fractal dimension of the material surface (see [12]).

7. Conclusion

In the present work, we introduce the analysis of the LFC for the first time. The concepts and properties of the LFD and LFI in the fractional real and complex sets, the series and transforms involving the Mattig-Leffer function defined on Cantor sets are investigated in detail. The unique of the solutions of the local fractional differential and integral equations and local fractional inequalities were also discussed. The local fractional vector calculus were used to describe the extended version of the Rice theory in fractal fracture mechanics with aid of the LFC operator. The results are accurate and efficient for handling a family of the fractal problems by using the local fractional differential and integral equations from the functional analysis point of view.

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