General Methods For Solving Ordinary Differential Equations 1

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Abstract

The method of this paper is my original creation. A new method for solving linear differential equations is proposed in this paper. The important conclusion of this paper is that arbitrary order linear ordinary differential equations with variable coefficients can be solved by the method of recursion and reduction of order under very loose conditions.

Keywords: general method, ordinary differential equation(ODE), linear ordinary differential equations with variable coefficients, reduction of order, recurrence.

1 Introduction

The method proposed in this paper is different from the traditional known methods. It is my original creation.

This paper is based on the new mathematical expansion of a given function that I proposed in [1] (Daiyuan Zhang, 2014). The new mathematical expansion is composed of the derivatives of the given function. Because of this, the new mathematical expansion proposed in [1] (Daiyuan Zhang, 2014) can be used to solve differential equations.

In this paper, linear ordinary differential equations with high order variable coefficients are discussed, and its conclusion is of course applicable to linear ordinary differential equations with constant coefficients. In the case of no confusion, they are all called linear ordinary differential equations. Using the formula in the literature [1] (Daiyuan Zhang, 2014), the linear ordinary differential equation can be solved by reducing the order. In particular, under fairly loose conditions, a given linear ordinary differential equation with order k (k is a positive integer) can be transformed into a linear ordinary differential equation with order $k-1$. By using the recursive method, the $(k-1)$th order linear ordinary differential equation can be reduced to the $(k-2)$th order linear ordinary differential equation. Therefore, we can get the first order linear ordinary differential equation and 0th order linear ordinary differential equation, so that we can get the solution to the linear ordinary differential equation directly.

In order not to make space too long, I only give some conclusions and examples in this paper, The theoretical proof will be given in my follow-up papers.

2 General method for solving high order linear ordinary differential equations

This section discusses the general method for solving high order linear ordinary differential equations. The method can be applied to linear ordinary differential equations with variable coefficients, and can also be applied to linear ordinary differential equations with constant coefficients.

Any kth$(k=1,2,\cdots)$ order linear differential equation can be written in the following general form:

$$y^{(k)}=p_{k1}y^{(k-1)}+p_{k2}y^{(k-2)}+\cdots+p_{k(k-1)}y'+p_{kk}y+p_{k(k+1)} \tag{2.1}$$

where p_{k1}, p_{k2}, $\cdots$, $p_{k(k+1)}$ are all known functions of x.

The problem now is to seek the solution of variable coefficient linear differential equation (2.1) under the initial condition of function value type $x=x_1$, $y=y\left(x_1\right)$, $x=x_2$, $y=y\left(x_2\right)$, $\cdots$, $x=x_k$, $y=y\left(x_k\right)$, that is, to solve the following variable coefficient linear differential equation:

$$\begin{cases} y^{(k)}=p_{k1}y^{(k-1)}+p_{k2}y^{(k-2)}+\cdots+p_{k(k-1)}y'+p_{kk}y+p_{k(k+1)} \\ x=x_1,\ y=y\left(x_1\right) \\ x=x_2,\ y=y\left(x_2\right) \\ \cdots\cdots \\ x=x_k,\ y=y\left(x_k\right) \end{cases} \tag{2.2}$$

For the kth order linear differential equation (2.1), we can get the reduced $(k-1)$th order linear differential equation after careful deduction:

$$\begin{cases} y^{(k-1)} = p_{(k-1)1}y^{(k-2)} + p_{(k-1)2}y^{(k-3)} + \cdots + p_{(k-1)(k-2)}y' + p_{(k-1)(k-1)}y + p_{(k-1)k} \\ x = x_1,\ y = y(x_1) \\ x = x_2,\ y = y(x_2) \\ \cdots\cdots \\ x = x_{k-1},\ y = y(x_{k-1}) \end{cases} \tag{2.3}$$

The initial condition $x = x_k$, $y = y(x_k)$ has been included in the equation (2.3), where

$$\begin{cases} p_{(k-1)1} = -\dfrac{Q_{(k-1)2}}{Q_{(k-1)1}} \\ p_{(k-1)2} = -\dfrac{Q_{(k-1)3}}{Q_{(k-1)1}} \\ \cdots\cdots \\ p_{(k-1)(k-2)} = -\dfrac{Q_{(k-1)(k-1)}}{Q_{(k-1)1}} \\ p_{(k-1)(k-1)} = -\dfrac{Q_{(k-1)k}}{Q_{(k-1)1}} \\ p_{(k-1)k} = \dfrac{Q_{(k-1)(k+1)}}{Q_{(k-1)1}} \end{cases} \tag{2.4}$$

where $Q_{(k-1)1} \neq 0$.

$$Q_{(k-1)1} = \sum_{n=k}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n1} + (-1)^{k-1} \frac{(x-a)^{k-1}}{(k-1)!} \tag{2.5}$$

$$Q_{(k-1)2} = \sum_{n=k}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n2} + (-1)^{k-2} \frac{(x-a)^{k-2}}{(k-2)!} \tag{2.6}$$

$$\cdots\cdots$$

$$Q_{(k-1)(k-1)} = \sum_{n=k}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n(k-1)} + (-1)^1 \frac{(x-a)}{1!} \tag{2.7}$$

$$Q_{(k-1)k} = \sum_{n=k}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{nk} + (-1)^0 \frac{(x-a)^0}{0!} \tag{2.8}$$

$$Q_{(k-1)(k+1)} = \sum_{n=k}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n(k+1)} - y(a) \tag{2.9}$$

$$\begin{cases} p_{n1} = p'_{(n-1)1} + p_{(n-1)1}p_{k1} + p_{(n-1)2} \\ p_{n2} = p'_{(n-1)2} + p_{(n-1)1}p_{k2} + p_{(n-1)3} \\ \cdots \\ p_{n(k-1)} = p'_{(n-1)(k-1)} + p_{(n-1)1}p_{k(k-1)} + p_{(n-1)k} \\ p_{nk} = p'_{(n-1)k} + p_{(n-1)1}p_{kk} \\ p_{n(k+1)} = p'_{(n-1)(k+1)} + p_{(n-1)1}p_{k(k+1)} \end{cases} \tag{2.10}$$

where $n = k+1,\ k+2,\ \cdots$. For kth order linear differential equations, p_{k1}, p_{k2}, $\cdots$, $p_{k(k+1)}$ are all known quantities.

With the same method, the$(k-1)$th order linear differential equation (2.3) can be reduced to the $(k-2)$th order linear differential equation in the following:

$$\begin{cases} y^{(k-2)} = p_{(k-2)1}y^{(k-3)} + p_{(k-2)2}y^{(k-4)} + \cdots + p_{(k-2)(k-3)}y' + p_{(k-2)(k-2)}y + p_{(k-2)(k-1)} \\ x = x_1,\ y = y\left(x_1\right) \\ x = x_2,\ y = y\left(x_2\right) \\ \cdots\cdots \\ x = x_{k-2},\ y = y\left(x_{k-2}\right) \end{cases} \tag{2.11}$$

The initial conditions $x = x_k$, $y = y\left(x_k\right)$, $x = x_k,\ y = y\left(x_k\right)$, $x = x_{k-1},\ y = y\left(x_{k-1}\right)$ have been included in the equation (2.11), where

$$\begin{cases} p_{(k-2)1} = -\dfrac{Q_{(k-2)2}}{Q_{(k-2)1}} \\ p_{(k-2)2} = -\dfrac{Q_{(k-2)3}}{Q_{(k-2)1}} \\ \cdots\cdots \\ p_{(k-2)(k-3)} = -\dfrac{Q_{(k-2)(k-2)}}{Q_{(k-2)1}} \\ p_{(k-2)(k-2)} = -\dfrac{Q_{(k-2)(k-1)}}{Q_{(k-2)1}} \\ p_{(k-2)(k-1)} = \dfrac{Q_{(k-2)k}}{Q_{(k-2)1}} \end{cases} \tag{2.12}$$

where $Q_{(k-2)1} \neq 0$, and

$$Q_{(k-2)1} = \sum_{n=k-1}^{\infty} (-1)^n \frac{\left(x - x_{k-1}\right)^n}{n!} p_{n1} + (-1)^{k-2} \frac{\left(x - x_{k-1}\right)^{k-2}}{(k-2)!} \tag{2.13}$$

$$Q_{(k-2)2} = \sum_{n=k-1}^{\infty} (-1)^n \frac{(x - x_{k-1})^n}{n!} p_{n2} + (-1)^{k-3} \frac{(x - x_{k-1})^{k-3}}{(k-3)!} \tag{2.14}$$

$$\cdots\cdots$$

$$Q_{(k-2)(k-2)} = \sum_{n=k-1}^{\infty} (-1)^n \frac{(x - x_{k-1})^n}{n!} p_{n(k-2)} + (-1)^1 \frac{(x - x_{k-1})}{1!} \tag{2.15}$$

$$Q_{(k-2)(k-1)} = \sum_{n=k-1}^{\infty} (-1)^n \frac{(x - x_{k-1})^n}{n!} p_{n(k-1)} + (-1)^0 \frac{(x - x_{k-1})^0}{0!} \tag{2.16}$$

$$Q_{(k-2)k} = \sum_{n=k-1}^{\infty} (-1)^n \frac{(x - x_{k-1})^n}{n!} p_{nk} - y(x_{k-1}) \tag{2.17}$$

$$\begin{cases} p_{n1} = p'_{(n-1)1} + p_{(n-1)1} p_{(k-1)1} + p_{(n-1)2} \\ p_{n2} = p'_{(n-1)2} + p_{(n-1)1} p_{(k-1)2} + p_{(n-1)3} \\ \cdots \\ p_{n(k-2)} = p'_{(n-1)(k-2)} + p_{(n-1)1} p_{(k-1)(k-2)} + p_{(n-1)(k-1)} \\ p_{n(k-1)} = p'_{(n-1)(k-1)} + p_{(n-1)1} p_{(k-1)(k-1)} \\ p_{nk} = p'_{(n-1)k} + p_{(n-1)1} p_{(k-1)k} \end{cases} \tag{2.18}$$

where $n = k,\ k+1,\ \cdots$. For $(k-1)$th order linear differential equations, $p_{(k-1)1},\ p_{(k-1)2},\ \cdots,\ p_{(k-1)k}$ are all known quantities, which is calculated by formula (2.4).

After a series of reduction of order in this way, we can get a first order linear ordinary differential equation. As the first order linear differential equation obtained, the solution to equation (2.1) can be found by doing a order reduction once more.

The above is the core of this paper, they are based on the following theorem.

Theorem 2.1. *Suppose that on a closed interval I, $x_1 \in I$, $x_2 \in I$, $\cdots$, $x_k \in I$, and x_1, x_2,$\cdots$, x_k are different from one another, p_{k1}, p_{k2}, $\cdots$, $p_{k(k+1)}$ are functions of x, assume that the solution $y = y(x)$ of equation (2.2) and p_{k1}, p_{k2}, $\cdots$, $p_{k(k+1)}$ have derivatives of arbitrary order on the interval I, and*

$$\lim_{n\to\infty} \frac{1}{n!} \int_a^x (x_i - t)^n y^{(n+1)}(t)\, dt = 0 \tag{2.19}$$

where $i = 1, 2, \cdots, k$. Then the order k of linear ordinary differential equation (2.2) can be reduced by (2.4)–(2.10), and becomes the linear ordinary differential equation (2.3) with the order of $k-1$.

I will give a proof of theorem(2.1) in one of my subsequent papers.

Theorem(2.1) points out that any kth order linear ordinary differential equation (2.2) satisfying the conditions of (2.1) can be solved by reduction of order using recursive method.

3 Second order linear ordinary differential equations

Since the second order linear ordinary differential equations are widely applied in physics and engineering technology, the following focuses on the second order linear ordinary differential equations.

In many documents, the second order linear ordinary differential equations are discussed in detail, such as [2] (Mircea V. Soare, Petre P. Teodorescu and Ileana Toma, 2007) [3](Shair Ahmad, Antonio Ambrosetti, 2014), [4] (Mircea V. Soare, Petre P. Teodorescu and Ileana, 2010), etc. However, the method proposed in this paper is original and novel, which is totally different from the known methods.

First, consider the following second order linear ordinary differential equation:

$$\begin{cases} y'' = p_{21}y' + p_{22}y + p_{23} \\ x = a, \ \ y = y\left(a\right) \\ x = b, \ \ y = y\left(b\right) \end{cases} \tag{3.1}$$

where $a \neq b$.

In the recursive formula (2.10), let $k = 2$, we have

$$\begin{cases} p_{n1} = p'_{(n-1)1} + p_{(n-1)1}p_{21} + p_{(n-1)2} \\ p_{n2} = p'_{(n-1)2} + p_{(n-1)1}p_{22} \\ p_{n3} = p'_{(n-1)3} + p_{(n-1)1}p_{23} \end{cases} \tag{3.2}$$

In order to specify the method proposed in this paper, we get a few expressions based on recurrence formula (3.2) in the following:

$$p_{31} = p'_{21} + p_{21}^2 + p_{22} \tag{3.3}$$

$$p_{32} = p'_{22} + p_{21}p_{22} \tag{3.4}$$

$$p_{33} = p'_{23} + p_{21}p_{23} \tag{3.5}$$

$$\begin{aligned} p_{41} &= p'_{31} + p_{31}p_{21} + p_{32} \\ &= \left(p'_{21} + p_{21}^2 + p_{22}\right)' + \left(p'_{21} + p_{21}^2 + p_{22}\right) p_{21} + p'_{22} + p_{21}p_{22} \\ &= p''_{21} + 2p_{21}p'_{21} + p'_{22} + p_{21}p'_{21} + p_{21}^3 + p_{21}p_{22} + p'_{22} + p_{21}p_{22} \\ &= p''_{21} + 3p_{21}p'_{21} + 2p'_{22} + p_{21}^3 + 2p_{21}p_{22} \end{aligned} \tag{3.6}$$

$$\begin{aligned} p_{42} &= p'_{32} + p_{31}p_{22} \\ &= (p'_{22} + p_{21}p_{22})' + \left(p'_{21} + p_{21}^2 + p_{22}\right) p_{22} \\ &= p''_{22} + p'_{21}p_{22} + p_{21}p'_{22} + p'_{21}p_{22} + p_{21}^2p_{22} + p_{22}^2 \\ &= p''_{22} + 2p'_{21}p_{22} + p'_{22}p_{21} + p_{21}^2p_{22} + p_{22}^2 \end{aligned} \tag{3.7}$$

$$\begin{aligned} p_{43} &= p'_{33} + p_{31}p_{23} \\ &= (p'_{23} + p_{21}p_{23})' + \left(p'_{21} + p_{21}^2 + p_{22}\right) p_{23} \\ &= p''_{23} + p'_{21}p_{23} + p_{21}p'_{23} + p'_{21}p_{23} + p_{21}^2p_{23} + p_{22}p_{23} \\ &= p''_{23} + 2p'_{21}p_{23} + p'_{23}p_{21} + p_{21}^2p_{23} + p_{22}p_{23} \end{aligned} \tag{3.8}$$

Let $k = 2$, from (2.5)–(2.9), we have

$$\begin{aligned} Q_{11} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n1} - (x-a) \\ &= -(x-a) + \frac{(x-a)^2}{2!} p_{21} - \frac{(x-a)^3}{3!} p_{31} + \frac{(x-a)^4}{4!} p_{41} - \cdots \\ &= -(x-a) + \frac{(x-a)^2}{2!} p_{21} - \frac{(x-a)^3}{3!} \left(p'_{21} + p_{21}^2 + p_{22}\right) \\ &+ \frac{(x-a)^4}{4!} \left(p''_{21} + 3p_{21}p'_{21} + 2p'_{22} + p_{21}^3 + 2p_{21}p_{22}\right) - \cdots \end{aligned} \tag{3.9}$$

$$\begin{aligned} Q_{12} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n2} + 1 \\ &= 1 + \frac{(x-a)^2}{2!} p_{22} - \frac{(x-a)^3}{3!} p_{32} + \frac{(x-a)^4}{4!} p_{42} - \cdots \\ &= 1 + \frac{(x-a)^2}{2!} p_{22} - \frac{(x-a)^3}{3!} (p'_{22} + p_{21}p_{22}) \\ &+ \frac{(x-a)^4}{4!} \left(p''_{22} + 2p'_{21}p_{22} + p'_{22}p_{21} + p_{21}^2p_{22} + p_{22}^2\right) - \cdots \end{aligned} \tag{3.10}$$

$$\begin{aligned} Q_{13} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n3} - y(a) \\ &= -y(a) + \frac{(x-a)^2}{2!} p_{23} - \frac{(x-a)^3}{3!} p_{33} + \frac{(x-a)^4}{4!} p_{43} - \cdots \\ &= -y(a) + \frac{(x-a)^2}{2!} p_{23} - \frac{(x-a)^3}{3!} \left(p'_{23} + p_{21}p_{23}\right) \\ &+ \frac{(x-a)^4}{4!} \left(p''_{23} + 2p'_{21}p_{23} + p'_{23}p_{21} + p_{21}^2 p_{23} + p_{22}p_{23}\right) - \cdots \end{aligned} \tag{3.11}$$

Therefore, the second order linear ordinary differential equation can be reduced to the first order linear ordinary differential equation in the following:

$$\begin{cases} y' = p_{11}y + p_{12} \\ x = b, \ y = y(b) \end{cases} \tag{3.12}$$

where

$$p_{11} = -\frac{Q_{12}}{Q_{11}} \tag{3.13}$$

$$p_{12} = -\frac{Q_{13}}{Q_{11}} \tag{3.14}$$

For the first order linear ordinary differential equations obtained in (3.12), we can find the solution to equation (3.1) by doing another order reduction in the following:

$$y = -\frac{Q_{02}}{Q_{01}} \tag{3.15}$$

where (see (2.5)–(2.9))

$$Q_{01} = \sum_{n=1}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n1} + 1 \tag{3.16}$$

$$Q_{02} = \sum_{n=1}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n2} - y(a) \tag{3.17}$$

In recursive formula (2.10), let $k = 1$, we get the following results:

$$\begin{cases} p_{n1} = p'_{(n-1)1} + p_{(n-1)1}p_{11} \\ p_{n2} = p'_{(n-1)2} + p_{(n-1)1}p_{12} \end{cases}, \ n = 2, 3, \cdots \tag{3.18}$$

Using recursive formula (3.18), we get the following items:

$$\begin{aligned} p_{21} &= p'_{11} + p_{11}^2 \\ &= \left(-\frac{Q_{12}}{Q_{11}}\right)' + \left(-\frac{Q_{12}}{Q_{11}}\right)^2 \\ &= \frac{Q'_{11}Q_{12} - Q'_{12}Q_{11} + Q_{12}^2}{Q_{11}^2} \end{aligned} \tag{3.19}$$

$$\begin{aligned} p_{22} &= p'_{12} + p_{11}p_{12} \\ &= \left(-\frac{Q_{13}}{Q_{11}}\right)' + \left(-\frac{Q_{12}}{Q_{11}}\right)\left(-\frac{Q_{13}}{Q_{11}}\right) \\ &= \frac{Q'_{11}Q_{13} - Q'_{13}Q_{11} + Q_{12}Q_{13}}{Q_{11}^2} \end{aligned} \tag{3.20}$$

$$\begin{aligned} p_{31} &= p'_{21} + p_{21}p_{11} \\ &= \left(p'_{11} + p_{11}^2\right)' + \left(p'_{11} + p_{11}^2\right)p_{11} \\ &= p''_{11} + 3p'_{11}p_{11} + p_{11}^3 \\ &= \left(-\frac{Q_{12}}{Q_{11}}\right)'' + 3\left(-\frac{Q_{12}}{Q_{11}}\right)'\left(-\frac{Q_{12}}{Q_{11}}\right) + \left(-\frac{Q_{12}}{Q_{11}}\right)^3 \\ &= \left(\frac{Q'_{11}Q_{12} - Q'_{12}Q_{11}}{Q_{11}^2}\right)' + 3\left(\frac{Q'_{11}Q_{12} - Q'_{12}Q_{11}}{Q_{11}^2}\right)\left(-\frac{Q_{12}}{Q_{11}}\right) + \left(-\frac{Q_{12}}{Q_{11}}\right)^3 \\ &= \frac{Q''_{11}Q_{12}Q_{11} - Q''_{12}Q_{11}^2 - 2(Q'_{11})^2Q_{12}}{Q_{11}^3} \\ &+ \frac{2Q'_{11}Q'_{12}Q_{11} - 3Q'_{11}Q_{12}^2 + 3Q'_{12}Q_{11}Q_{12} - Q_{12}^3}{Q_{11}^3} \end{aligned} \tag{3.21}$$

$$\begin{aligned} p_{32} &= p'_{22} + p_{21}p_{12} \\ &= \left(\frac{Q'_{11}Q_{13} - Q'_{13}Q_{11} + Q_{12}Q_{13}}{Q_{11}^2}\right)' + \frac{Q'_{11}Q_{12} - Q'_{12}Q_{11} + Q_{12}^2}{Q_{11}^2}\left(-\frac{Q_{13}}{Q_{11}}\right) \\ &= \frac{Q''_{11}Q_{11}Q_{13} - Q''_{13}Q_{11}^2 + 2Q'_{12}Q_{11}Q_{13} + Q'_{13}Q_{11}Q_{12}}{Q_{11}^3} \\ &+ \frac{-2(Q'_{11})^2Q_{13} + 2Q'_{11}Q'_{13}Q_{11} - 3Q'_{11}Q_{12}Q_{13} - Q_{12}^2Q_{13}}{Q_{11}^3} \end{aligned} \tag{3.22}$$

By substituting (3.19)–(3.22) into formulas (3.16) and (3.17), we have

$$
\begin{aligned}
Q_{01} &= \sum_{n=1}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n1} + 1 \\
&= 1 - (x-a)\, p_{11} + \frac{(x-a)^2}{2!} p_{21} - \frac{(x-a)^3}{3!} p_{31} + \frac{(x-a)^4}{4!} p_{41} - \cdots \\
&= 1 + (x-a)\, \frac{Q_{12}}{Q_{11}} + \frac{(x-a)^2}{2!} \left(\frac{Q'_{11}Q_{12} - Q'_{12}Q_{11} + Q_{12}^2}{Q_{11}^2} \right) \\
&- \frac{(x-a)^3}{3!} \left(\begin{array}{c} \frac{Q''_{11}Q_{12}Q_{11} - Q''_{12}Q_{11}^2 - 2(Q'_{11})^2 Q_{12}}{Q_{11}^3} + \\ \frac{2Q'_{11}Q'_{12}Q_{11} - 3Q'_{11}Q_{12}^2 + 3Q'_{12}Q_{11}Q_{12} - Q_{12}^3}{Q_{11}^3} \end{array} \right) + \cdots
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
Q_{02} &= \sum_{n=1}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n2} - y(a) \\
&= -y(a) - (x-a)\, p_{12} + \frac{(x-a)^2}{2!} p_{22} - \frac{(x-a)^3}{3!} p_{32} + \cdots \\
&= -y(a) + (x-a)\, \frac{Q_{13}}{Q_{11}} + \frac{(x-a)^2}{2!} \left(\frac{Q'_{11}Q_{13} - Q'_{13}Q_{11} + Q_{12}Q_{13}}{Q_{11}^2} \right) \\
&- \frac{(x-a)^3}{3!} \left(\begin{array}{c} \frac{Q''_{11}Q_{11}Q_{13} - Q''_{13}Q_{11}^2 + 2Q'_{12}Q_{11}Q_{13} + Q'_{13}Q_{11}Q_{12}}{Q_{11}^3} \\ + \frac{-2(Q'_{11})^2 Q_{13} + 2Q'_{11}Q'_{13}Q_{11} - 3Q'_{11}Q_{12}Q_{13} - Q_{12}^2 Q_{13}}{Q_{11}^3} \end{array} \right) + \cdots
\end{aligned}
\tag{3.24}
$$

Then we obtain the solution to the second order linear ordinary differential equation (3.1) in the following:

$$
y = -\frac{Q_{02}}{Q_{01}} \tag{3.25}
$$

where Q_{01} and Q_{02} are calculated by formulas (3.23) and (3.24) respectively, while Q_{11}, Q_{12} and Q_{13} are calculated by formulas (3.9), (3.10) and (3.11) respectively.

It can be seen that Q_{01} and Q_{02} are functions of Q_{11}, Q_{12} and Q_{13}, while Q_{11}, Q_{12} and Q_{13} are functions of x, p_{21}, p_{22} and p_{23}. Therefore, the solution to the second order linear ordinary differential equation (3.1) is the function of x, p_{21}, p_{22} and p_{23}, i.e.

$$
y = f(x, p_{21}, p_{22}, p_{23}) \tag{3.26}
$$

An important feature of the method proposed in this paper is to get the solution for a given equation by a series of recursion calculations from the known quantity of p_{21}, p_{22} and p_{23} (the given coefficients of the linear ordinary differential equation). These calculations that include p_{21}, p_{22} and p_{23} are the add, subtract, multiply and divide and differential calculations, but does not include the integral operation.

Now, a special case is considered below, which is the following second order linear ordinary differential equation with two initial conditions.

$$\begin{cases} y'' = p_{22}y \\ x = a,\ y = y(a) \\ x = b,\ y = y(b) \end{cases} \tag{3.27}$$

where $p_{21} = p_{23} = 0$, according to recursive formula (3.2), we have

$$\begin{cases} p_{n1} = p'_{(n-1)1} + p_{(n-1)1}p_{21} + p_{(n-1)2} = p'_{(n-1)1} + p_{(n-1)2} \\ p_{n2} = p'_{(n-1)2} + p_{(n-1)1}p_{22} \\ p_{n3} = p'_{(n-1)3} + p_{(n-1)1}p_{23} = p'_{(n-1)3} \end{cases},\ n = 3, 4, \cdots \tag{3.28}$$

Therefore,

$$p_{23} = p_{33} = p_{43} = \cdots = 0 \tag{3.29}$$

$$p_{31} = p'_{21} + p_{22} = p_{22} \tag{3.30}$$

$$p_{32} = p'_{22} + p_{21}p_{22} = p'_{22} \tag{3.31}$$

$$p_{41} = p'_{31} + p_{32} = 2p'_{22} \tag{3.32}$$

$$p_{42} = p'_{32} + p_{31}p_{22} = p''_{22} + p_{22}^2 \tag{3.33}$$

$$p_{51} = p'_{41} + p_{42} = (2p'_{22})' + p''_{22} + p_{22}^2 = 3p''_{22} + p_{22}^2 \tag{3.34}$$

$$\begin{aligned} p_{52} &= p'_{42} + p_{41}p_{22} = (p''_{22} + p_{22}p_{22})' + (2p'_{22})\,p_{22} \\ &= p'''_{22} + 4p'_{22}p_{22} \end{aligned} \tag{3.35}$$

$$\begin{aligned} p_{61} &= p'_{51} + p_{52} = \left(3p''_{22} + p_{22}^2\right)' + p'''_{22} + 4p'_{22}p_{22} \\ &= 4p'''_{22} + 6p'_{22}p_{22} \end{aligned} \tag{3.36}$$

$$\begin{aligned} p_{62} &= p'_{52} + p_{51}p_{22} = (p'''_{22} + 4p'_{22}p_{22})' + \left(3p''_{22} + p_{22}^2\right) p_{22} \\ &= p_{22}^{(4)} + 7p''_{22}p_{22} + 4(p'_{22})^2 + p_{22}^3 \end{aligned} \tag{3.37}$$

Let $k = 2$, by substituting into (2.5)–(2.9), we have

$$\begin{aligned} Q_{11} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n1} - (x-a) \\ &= -(x-a) + \frac{(x-a)^2}{2!} p_{21} - \frac{(x-a)^3}{3!} p_{31} + \frac{(x-a)^4}{4!} p_{41} \\ &- \frac{(x-a)^5}{5!} p_{51} + \frac{(x-a)^6}{6!} p_{61} - \cdots \\ &= -(x-a) + \frac{(x-a)^2}{2!} p_{21} - \frac{(x-a)^3}{3!} p_{22} + \frac{2(x-a)^4}{4!} p'_{22} \\ &- \frac{(x-a)^5}{5!} \left(3p''_{22} + p_{22}^2\right) + \frac{(x-a)^6}{6!} (4p'''_{22} + 6p'_{22}p_{22}) - \cdots \end{aligned} \tag{3.38}$$

$$\begin{aligned} Q_{12} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n2} + 1 \\ &= 1 + \frac{(x-a)^2}{2!} p_{22} - \frac{(x-a)^3}{3!} p_{32} + \frac{(x-a)^4}{4!} p_{42} \\ &+ \frac{(x-a)^5}{5!} p_{52} + \frac{(x-a)^6}{6!} p_{62} - \cdots \\ &= 1 + \frac{(x-a)^2}{2!} p_{22} - \frac{(x-a)^3}{3!} p'_{22} + \frac{(x-a)^4}{4!} \left(p''_{22} + p_{22}^2\right) \\ &+ \frac{(x-a)^5}{5!} (p'''_{22} + 4p'_{22}p_{22}) + \frac{(x-a)^6}{6!} \left(p_{22}^{(4)} + 7p''_{22}p_{22} + 4(p'_{22})^2 + p_{22}^3\right) - \cdots \end{aligned} \tag{3.39}$$

$$\begin{aligned} Q_{13} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n3} - y(a) \\ &= -y(a) \end{aligned} \tag{3.40}$$

Therefore, the second order linear ordinary differential equation is reduced to the first order linear ordinary differential equation in the following:

$$\begin{cases} y' = p_{11}y + p_{12} \\ x = b, \ y = y(b) \end{cases} \tag{3.41}$$

where,

$$p_{11} = -\frac{Q_{12}}{Q_{11}} = -\frac{\begin{pmatrix} 1 + \frac{(x-a)^2}{2!} p_{22} - \frac{(x-a)^3}{3!} p_{32} + \frac{(x-a)^4}{4!} p_{42} \\ + \frac{(x-a)^5}{5!} (p'''_{22} + 4p'_{22} p_{22}) + \frac{(x-a)^6}{6!} \left(p_{22}^{(4)} + 7p''_{22} p_{22} + 4(p'_{22})^2 + p_{22}^3 \right) - \cdots \end{pmatrix}}{\begin{pmatrix} -(x-a) + \frac{(x-a)^2}{2!} p_{21} - \frac{(x-a)^3}{3!} p_{22} + \frac{2(x-a)^4}{4!} p'_{22} \\ - \frac{(x-a)^5}{5!} \left(3p''_{22} + p_{22}^2 \right) + \frac{(x-a)^6}{6!} (4p'''_{22} + 6p'_{22} p_{22}) - \cdots \end{pmatrix}} \tag{3.42}$$

$$p_{12} = -\frac{Q_{13}}{Q_{11}} = \frac{y(a)}{\begin{matrix} -(x-a) + \frac{(x-a)^2}{2!} p_{21} - \frac{(x-a)^3}{3!} p_{22} + \frac{2(x-a)^4}{4!} p'_{22} \\ - \frac{(x-a)^5}{5!} \left(3p''_{22} + p_{22}^2 \right) + \frac{(x-a)^6}{6!} (4p'''_{22} + 6p'_{22} p_{22}) - \cdots \end{matrix}} \tag{3.43}$$

Using the formulas (3.38), (3.39) and (3.40), the above formulas (3.42) and (3.43) are substitued into the recursive formula (3.18), then, the following items can be obtained according to the formulas (3.19)–(3.22).

$$\begin{aligned} p_{21} &= p'_{11} + p_{11}^2 \\ &= \frac{Q'_{11} Q_{12} - Q'_{12} Q_{11} + Q_{12}^2}{Q_{11}^2} \end{aligned} \tag{3.44}$$

$$\begin{aligned} p_{22} &= p'_{12} + p_{11} p_{12} \\ &= \frac{Q'_{11} Q_{13} - Q'_{13} Q_{11} + Q_{12} Q_{13}}{Q_{11}^2} \\ &= \frac{Q'_{11} Q_{13} + Q_{12} Q_{13}}{Q_{11}^2} \end{aligned} \tag{3.45}$$

$$\begin{aligned} p_{31} &= p'_{21} + p_{21} p_{11} \\ &= \frac{Q''_{11} Q_{12} Q_{11} - Q''_{12} Q_{11}^2 - 2(Q'_{11})^2 Q_{12}}{Q_{11}^3} \\ &+ \frac{2Q'_{11} Q'_{12} Q_{11} - 3Q'_{11} Q_{12}^2 + 3Q'_{12} Q_{11} Q_{12} - Q_{12}^3}{Q_{11}^3} \end{aligned} \tag{3.46}$$

$$
\begin{aligned}
p_{32} &= {p'}_{22} + p_{21}p_{12} \\
&= \frac{{Q''}_{11}Q_{11}Q_{13} - {Q''}_{13}Q_{11}^2 + 2{Q'}_{12}Q_{11}Q_{13} + {Q'}_{13}Q_{11}Q_{12}}{Q_{11}^3} \\
&+ \frac{-2({Q'}_{11})^2Q_{13} + 2{Q'}_{11}{Q'}_{13}Q_{11} - 3{Q'}_{11}Q_{12}Q_{13} - Q_{12}^2Q_{13}}{Q_{11}^3} \\
&= \frac{{Q''}_{11}Q_{11}Q_{13} + 2{Q'}_{12}Q_{11}Q_{13} - 2({Q'}_{11})^2Q_{13} - 3{Q'}_{11}Q_{12}Q_{13} - Q_{12}^2Q_{13}}{Q_{11}^3}
\end{aligned}
\tag{3.47}
$$

Among them, Q_{11}, Q_{12} and Q_{13} are calculated by formulas (3.38), (3.39) and (3.40) respectively.

From formulas (3.16) and (3.17), we have

$$
\begin{aligned}
Q_{01} &= \sum_{n=1}^{\infty}(-1)^n\frac{(x-b)^n}{n!}p_{n1} + 1 \\
&= 1 - (x-b)\,p_{11} + \frac{(x-b)^2}{2!}p_{21} - \frac{(x-b)^3}{3!}p_{31} + \cdots \\
&= 1 + (x-b)\,\frac{Q_{12}}{Q_{11}} + \frac{(x-b)^2}{2!}\left(\frac{{Q'}_{11}Q_{12} - {Q'}_{12}Q_{11} + Q_{12}^2}{Q_{11}^2}\right) \\
&- \frac{(x-b)^3}{3}\left(\begin{array}{c}\dfrac{{Q''}_{11}Q_{12}Q_{11} - {Q''}_{12}Q_{11}^2 - 2({Q'}_{11})^2Q_{12}}{Q_{11}^3} + \\ \dfrac{2{Q'}_{11}{Q'}_{12}Q_{11} - 3{Q'}_{11}Q_{12}^2 + 3{Q'}_{12}Q_{11}Q_{12} - Q_{12}^3}{Q_{11}^3}\end{array}\right) + \cdots
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
Q_{02} &= \sum_{n=1}^{\infty}(-1)^n\frac{(x-b)^n}{n!}p_{n2} - y\,(b) \\
&= -y\,(b) - (x-b)\,p_{12} + \frac{(x-b)^2}{2!}p_{22} - \frac{(x-b)^3}{3!}p_{32} + \cdots \\
&= -y\,(b) + (x-b)\,\frac{Q_{13}}{Q_{11}} + \frac{(x-a)^2}{2!}\left(\frac{{Q'}_{11}Q_{13} + Q_{12}Q_{13}}{Q_{11}^2}\right) \\
&- \frac{(x-b)^3}{3!}\left(\frac{{Q''}_{11}Q_{11}Q_{13} + 2{Q'}_{12}Q_{11}Q_{13} - 2({Q'}_{11})^2Q_{13} - 3{Q'}_{11}Q_{12}Q_{13} - Q_{12}^2Q_{13}}{Q_{11}^3}\right) + \cdots
\end{aligned}
\tag{3.49}
$$

So we get the solution of linear ordinary differential equation (3.27) in the following:

$$y = -\frac{Q_{02}}{Q_{01}} \tag{3.50}$$

Q_{01} and Q_{02} are calculated by formulas (3.48) and (3.49) respectively, while Q_{11}, Q_{12} and Q_{13} are calculated by formulas (3.38), (3.39) and (3.40) respectively.

4 Examples

Two calculation examples are given in this section. Example 4.1 is a simple one for sovling second order linear ordinary differential equation with constant coefficients of known solutions. This simple example is chosen to verify the correctness of the method proposed in this paper.

Example 4.2 is to solve second order linear ordinary differential equation with variable coefficients. The process and steps of the solution are the same as that of Example 4.1.

Example 4.1. *Solving the following equation.*

$$\begin{cases} y'' = y \\ x = a, \ y = y(a) \\ x = b, \ y = y(b) \end{cases} \tag{4.1}$$

Solutionn: Obviously, we have

$$\begin{cases} p_{21} = 0 \\ p_{22} = 1 \\ p_{23} = 0 \end{cases} \tag{4.2}$$

Using formulas (3.29)–(3.40), *we get*

$$p_{31} = p_{22} = 1 \tag{4.3}$$

$$p_{32} = {p'}_{22} = 0 \tag{4.4}$$

$$p_{33} = 0 \tag{4.5}$$

$$p_{41} = 2{p'}_{22} = 0 \tag{4.6}$$

$$p_{42} = {p''}_{22} + p_{22}^2 = 1 \tag{4.7}$$

$$p_{43} = 0 \tag{4.8}$$

$$p_{51} = 3p''{}_{22} + p_{22}^2 = 1 \tag{4.9}$$

$$p_{52} = p'''{}_{22} + 4p'{}_{22}p_{22} = 0 \tag{4.10}$$

$$p_{53} = 0 \tag{4.11}$$

$$p_{61} = 4p'''{}_{22} + 6p'{}_{22}p_{22} = 0 \tag{4.12}$$

$$p_{62} = p_{22}^{(4)} + 7p''{}_{22}p_{22} + 4(p'{}_{22})^2 + p_{22}^3 = 1 \tag{4.13}$$

$$p_{63} = 0 \tag{4.14}$$

$$\cdots\cdots$$

$$\begin{aligned} Q_{11} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n1} - (x-a) \\ &= -\sum_{n=0}^{\infty} \frac{(x-a)^{2n+1}}{(2n+1)!} \\ &= -\left(\frac{x-a}{1} + \frac{(x-a)^3}{3!} + \frac{(x-a)^5}{5!} + \cdots \right) \\ &= -\sinh(x-a) \end{aligned} \tag{4.15}$$

$$\begin{aligned} Q_{12} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n2} + 1 \\ &= \sum_{n=0}^{\infty} \frac{(x-a)^{2n}}{(2n)!} \\ &= 1 + \frac{(x-a)^2}{2!} + \frac{(x-a)^4}{4!} + \cdots \\ &= \cosh(x-a) \end{aligned} \tag{4.16}$$

$$\begin{aligned} Q_{13} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n3} - y(a) \\ &= -y(a) \end{aligned} \tag{4.17}$$

After order reduction, we get the first order linear ordinary differential equation in the following:

$$\begin{cases} y' = p_{11}y + p_{12} \\ x = b, \ y = y(b) \end{cases} \tag{4.18}$$

where,

$$p_{11} = -\frac{Q_{12}(x)}{Q_{11}(x)} = \frac{\cosh(x-a)}{\sinh(x-a)} = \coth(x-a) \tag{4.19}$$

$$p_{12} = -\frac{Q_{13}(x)}{Q_{11}(x)} = -\frac{y(a)}{\sinh(x-a)} \tag{4.20}$$

$$\begin{aligned} Q_{01} &= \sum_{n=1}^{\infty} (-1)^n \frac{(x-b)^n}{n!} p_{n1} + 1 \\ &= 1 - (x-b)\,p_{11} + \frac{(x-b)^2}{2!} p_{21} - \frac{(x-b)^3}{3!} p_{31} + \frac{(x-b)^4}{4!} p_{41} - \cdots \\ &= 1 - (x-b)\coth(x-a) + \frac{(x-b)^2}{2!} - \frac{(x-b)^3}{3!}\coth(x-a) + \frac{(x-b)^4}{4!} - \cdots \\ &= 1 + \frac{(x-b)^2}{2!} + \frac{(x-b)^4}{4!} + \cdots - \coth(x-a)\left((x-b) + \frac{(x-b)^3}{3!} + \cdots\right) \\ &= \cosh(x-b) - \coth(x-a)\sinh(x-b) \end{aligned} \tag{4.21}$$

$$\begin{aligned} Q_{02} &= \sum_{n=1}^{\infty} (-1)^n \frac{(x-b)^n}{n!} p_{n2} - y(b) \\ &= -y(b) - (x-b)\,p_{12} + \frac{(x-b)^2}{2!} p_{22} - \frac{(x-b)^3}{3!} p_{32} + \cdots \\ &= -y(b) - (x-b)\left(-\frac{y(a)}{\sinh(x-a)}\right) - \frac{(x-b)^3}{3!}\left(-\frac{y(a)}{\sinh(x-a)}\right) + \cdots \\ &= -y(b) + \frac{y(a)}{\sinh(x-a)}\left((x-b) + \frac{(x-b)^3}{3!} + \frac{(x-b)^5}{5!} + \cdots\right) \\ &= -y(b) + \frac{y(a)}{\sinh(x-a)}\sinh(x-b) \end{aligned} \tag{4.22}$$

Therefore,

$$\begin{aligned} y &= -\frac{Q_{02}}{Q_{01}} = -\frac{-y(b) + \frac{y(a)}{\sinh(x-a)}\sinh(x-b)}{\cosh(x-b) - \coth(x-a)\sinh(x-b)} \\ &= -\frac{-y(b)\left(e^{x-a} - e^{-(x-a)}\right) + y(a)\left(e^{x-b} - e^{-(x-b)}\right)}{e^{b-a} - e^{-(b-a)}} \\ &= \frac{y(a)e^{-b} - y(b)e^{-a}}{e^{a-b} - e^{b-a}}e^{x} + \frac{y(b)e^{a} - y(a)e^{b}}{e^{a-b} - e^{b-a}}e^{-x} \end{aligned} \tag{4.23}$$

If using traditional method, the general solution is as follows:

$$y(x) = C_1 e^{x} + C_2 e^{-x} \tag{4.24}$$

Substituting the initial conditions $x = a$, $y = y(a)$ *and* $x = b$, $y = y(b)$ *into* (4.24), *we get*

$$y(x) = \frac{y(a)e^{-b} - y(b)e^{-a}}{e^{a-b} - e^{b-a}}e^{x} + \frac{y(b)e^{a} - y(a)e^{b}}{e^{a-b} - e^{b-a}}e^{-x} \tag{4.25}$$

The above formula is the same as (4.23), *that is, the same solution is obtained by the traditional method, which shows the correctness of the method proposed in this paper.*

Although this method is more complex than the traditional method for simple problems, this method is a general method, and the procedure of solving process has consistency, not only for linear ordinary differential equations with constant coefficients, but also for linear ordinary differential equations with variable coefficients.

This example illustrates the method proposed in this paper in detail. It can be seen that this method only needs algebraic computation (addition, subtraction, multiplication and division) and derivative calculation, and does not need to solve the characteristic equation, and does not need integral calculation.

Example 4.2. *Solving the solution of the following differential equation with variable coefficients.*

$$\begin{cases} y'' - xy = 0 \\ x = a,\ y = y(a) \\ x = b,\ y = y(b) \end{cases} \tag{4.26}$$

Solution: Since

$$\begin{cases} p_{21} = 0 \\ p_{22} = x \\ p_{23} = 0 \end{cases} \tag{4.27}$$

From (3.29)–(3.37), *we have*

$$p_{31} = {p'}_{21} + p_{22} = p_{22} = x \tag{4.28}$$

$$p_{32} = {p'}_{22} = 1 \tag{4.29}$$

$$p_{33} = 0 \tag{4.30}$$

$$p_{41} = 2{p'}_{22} = 2 \tag{4.31}$$

$$p_{42} = {p''}_{22} + p_{22}^2 = x^2 \tag{4.32}$$

$$p_{43} = 0 \tag{4.33}$$

$$p_{51} = 3{p''}_{22} + p_{22}^2 = x^2 \tag{4.34}$$

$$p_{52} = {p'''}_{22} + 4{p'}_{22}p_{22} = 4x \tag{4.35}$$

$$p_{53} = 0 \tag{4.36}$$

$$p_{61} = 4{p'''}_{22} + 6{p'}_{22}p_{22} = 6x \tag{4.37}$$

$$p_{62} = p_{22}^{(4)} + 7{p''}_{22}p_{22} + 4({p'}_{22})^2 + p_{22}^3 = x^3 + 4 \tag{4.38}$$

$$p_{63} = 0 \tag{4.39}$$

From formulas (3.9)–(3.11)*, we can write the first six items in the following:*

$$\begin{aligned} Q_{11} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n1} - (x-a) \\ &= -(x-a) - \frac{x(x-a)^3}{3!} + \frac{2(x-a)^4}{4!} \\ &- \frac{x^2(x-a)^5}{5!} + \frac{6x(x-a)^6}{6!} + \cdots \end{aligned} \tag{4.40}$$

$$\begin{aligned} Q_{12} &= \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n2} + 1 \\ &= 1 + \frac{x(x-a)^2}{2!} - \frac{(x-a)^3}{3!} + \frac{x^2(x-a)^4}{4!} \\ &- \frac{4x(x-a)^5}{5!} + \frac{(x^3+4)(x-a)^6}{6!} + \cdots \end{aligned} \tag{4.41}$$

$$Q_{13} = \sum_{n=2}^{\infty} (-1)^n \frac{(x-a)^n}{n!} p_{n3} - y(a) = -y(a) \tag{4.42}$$

After descending order, we have

$$\begin{cases} y' = p_{11} y + p_{12} \\ x = b, \ y = y(b) \end{cases} \tag{4.43}$$

where,

$$\begin{aligned} p_{11} &= -\frac{Q_{12}}{Q_{11}} \\ &= -\frac{1 + \frac{x(x-a)^2}{2!} - \frac{(x-a)^3}{3!} + \frac{x^2(x-a)^4}{4!} - \frac{4x(x-a)^5}{5!} + \frac{\left(x^3+4\right)(x-a)^6}{6!} + \cdots}{-(x-a) - \frac{x(x-a)^3}{3!} + \frac{2(x-a)^4}{4!} - \frac{x^2(x-a)^5}{5!} + \frac{6x(x-a)^6}{6!} + \cdots} \end{aligned} \tag{4.44}$$

$$\begin{aligned} p_{12} &= -\frac{Q_{13}}{Q_{11}} \\ &= \frac{y(a)}{-(x-a) - \frac{x(x-a)^3}{3!} + \frac{2(x-a)^4}{4!} - \frac{x^2(x-a)^5}{5!} + \frac{6x(x-a)^6}{6!} + \cdots} \end{aligned} \tag{4.45}$$

$$\begin{aligned} Q_{01} &= \sum_{n=1}^{\infty} (-1)^n \frac{(x-b)^n}{n!} p_{n1} + 1 \\ &= 1 - (x-b) p_{11} + \frac{(x-b)^2}{2!} p_{21} - \frac{(x-b)^3}{3!} p_{31} + \frac{(x-b)^4}{4!} p_{41} - \cdots \end{aligned} \tag{4.46}$$

$$\begin{aligned} Q_{02} &= \sum_{n=1}^{\infty} (-1)^n \frac{(x-b)^n}{n!} p_{n2} - y(b) \\ &= -y(b) - (x-b) p_{12} + \frac{(x-b)^2}{2!} p_{22} - \frac{(x-b)^3}{3!} p_{32} + \cdots \end{aligned} \tag{4.47}$$

So the solution to equation (4.26) *is solved in the following:*

$$y = -\frac{Q_{02}}{Q_{01}} = \frac{\sum\limits_{n=1}^{\infty} (-1)^n \frac{(x-b)^n}{n!} p_{n2} - y(b)}{\sum\limits_{n=1}^{\infty} (-1)^n \frac{(x-b)^n}{n!} p_{n1} + 1} \tag{4.48}$$

Where p_{n1} *and* p_{n2} *are obtained by recursive formula* (3.18).

5 Conclusions

The method proposed in this paper is a general method, and the process steps are consistent. The core process is recursion and reduction of order. It can not only solve the linear ordinary differential equations with constant coefficient, but also solve the variable coefficient ordinary differential equations.

In this paper, we only need algebraic computation (addition , subtraction, multiplication and division) and derivative calculation, which do not need to solve the characteristic equation and do not need integral calculation.

In this paper, the initial conditions are automatically satisfied.

The correctness of the method is verified by a simple example.

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