Lokenath Debnath

Nonlinear Partial Differential Equations

for Scientists and Engineers

Third Edition

Birkhäuser

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TO MY MOTHER

with love, gratitude, and admiration

True Laws of Nature cannot be linear.

Albert Einstein

… the progress of physics will to a large extent depend on the progress of nonlinear mathematics, of methods to solve nonlinear equations … and therefore we can learn by comparing different nonlinear problems.

Werner Heisenberg

Our present analytical methods seem unsuitable for the solution of the important problems arising in connection with nonlinear partial differential equations and, in fact, with virtually all types of nonlinear problems in pure mathematics. The truth of this statement is particularly striking in the field of fluid dynamics.…

John Von Neumann

However varied may be the imagination of man, nature is a thousand times richer, … Each of the theories of physics … presents (partial differential) equations under a new aspect … without these theories, we should not know partial differential equations.

Henri Poincaré

Since a general solution must be judged impossible from want of analysis, we must be content with the knowledge of some special cases, and that all the more, since the development of various cases seems to be the only way to bringing us at last to a more perfect knowledge.

Leonard Euler

… as Sir Cyril Hinshelwood has observed … fluid dynamicists were divided into hydraulic engineers who observed things that could not be explained and mathematicians who explained things that could not be observed.

James Lighthill

Preface to the Third Edition

A teacher can never truly teach unless he is still learning himself. A lamp can never light another lamp unless it continues to burn its own flame. The teacher who has come to the end of his subject, who has no living traffic with his knowledge but merely repeats his lessons to his students, can only load their minds; he cannot quicken them.

Rabindranath Tagore
An Indian Poet
1913 Nobel Prize Winner for Literature

The previous two editions of my book were very well received and used as a senior undergraduate or graduate-level text and research reference in the United States and abroad for many years. We received many comments and suggestions from many students, faculty, and researchers around the world. These comments and criticisms have been very helpful, beneficial, and encouraging. This third edition is the result of the input.

Another reason for adding this third edition to the literature is the fact that there have been major discoveries of new ideas, results and methods for the solutions of nonlinear partial differential equations in the second half of the twentieth century. It is becoming even more desirable for mathematicians, scientists, and engineers to pursue study and research on these topics. So what has changed, and will continue to change, is the nature of the topics that are of interest in mathematics, applied mathematics, physics, and engineering, the evolution of books such as this one is a history of these shifting concerns.

This new and revised edition preserves the basic content and style of the second edition published in 2005. As with the previous editions, this book has been revised primarily as a comprehensive text for senior undergraduates or beginning graduate students and a research reference for professionals in mathematics, engineering, and other applied sciences. The main goal of the book is to develop required analytical skills on the part of the reader, rather than to focus on the importance of more abstract formulation, with full mathematical rigor. Indeed, our major emphasis is to provide an accessible working knowledge of the analytical and numerical methods with proofs required in mathematics, applied mathematics, physics, and engineering.

In general, changes have been made to modernize the contents and to improve the exposition and clarity of the previous edition, to include additional topics, comments, and observations, to add many examples of applications and exercises, and in some cases to entirely rewrite and reorganize many sections. There is plenty of material in the book for a year-long course or seminar. Some of the material need not be covered in a course work and can be left for the readers to study on their own in order to prepare them for further study and research. This edition contains a collection of over 1000 worked examples and exercises with answers and hints to selected exercises. Some of the major changes and additions include the following:

1. Many sections of almost all chapters have been revised and expanded to modernize the contents. We have also taken advantage of this new edition to correct typographical errors and to include several new figures for a clear understanding of physical explanations.
2. Several nonlinear models including the Camassa–Holm (CH) equation, the Degasperis–Procesi (DP) equation, and the Toda lattice equation (TLE) have been presented with their physical significance in Chapter 2. Included are also new sections on the small-amplitude gravity-capillary waves on water of finite and infinite depth, the energy equation and energy flux.
3. A new section on the Lorenz nonlinear system, the Lorenz attractor, and deterministic chaos has been added in Chapter 6.
4. Included is a new section on the Camassa–Holm equation, the Degasperis–Procesi equation, and the Euler–Poincaré (EP) equation in Chapter 9 to describe the wave breaking (singular) phenomena. A new section on the derivation of the KdV equation for the gravity-capillary wave, the gravity-capillary solitary wave solutions, and the two-dimensional periodic flow in an inviscid, incompressible fluid with constant vorticity has been added. Special attention is given to both analytical and computational solutions of these problems with physical significance.
5. A new example describing nonlinear quasi-harmonic waves and modulational instability has been added in Chapter 10.
6. The nonlinear lattices and the Toda lattice equation have been treated in some detail at the end of Chapter 11.
7. All tables of Fourier transforms, Fourier sine and cosine transforms, Laplace transforms, Hankel transforms, and finite Hankel transforms have been revised and expanded so that they become more useful for the study of partial differential equations.
8. In order to make the book self-contained, two new appendices on some special functions and their basic properties, Fourier series, generalized functions, Fourier and Laplace transforms have been added. Special attention has been given to algebraic and analytical properties of the Fourier and Laplace convolutions with applications.

9. The whole section on Answers and Hints to Selected Exercises has been revised and expanded to provide additional help to students.
10. The entire bibliography has been revised and expanded to include new and current research papers and books so that it can stimulate new interest in future study and research.
11. The Index has been revised and reorganized to make it more useful for the reader.

Some of the highlights in this edition include the following:

- The book offers a detailed and clear explanation of every concept and method that is introduced, accompanied by carefully selected worked examples, with special emphasis given to those topics in which students experience difficulty.
- A wide variety of modern examples of applications has been selected from areas of partial differential equations, quantum mechanics, fluid dynamics, solid mechanics, calculus of variations, linear and nonlinear wave propagation, telecommunication, soliton dynamics, and nonlinear stability analysis.
- The book is organized with sufficient flexibility to enable instructors to select chapters appropriate for courses of differing lengths, emphases, and levels of difficulty as chapters are significantly independent of each other.
- A wide spectrum of exercises has been carefully chosen and included at the end of each chapter so the reader may further develop both rigorous skills in the theory and applications of partial differential equations and a deeper insight into the subject.
- Many new research papers and standard books have been added to the bibliography to stimulate new interest in future study and research. The Index of the book has also been completely revised in order to include a wide variety of topics.
- The book provides information that puts the reader at the forefront of current research.

With the improvements and many challenging worked out problems and exercises, we hope this edition will continue to be a useful textbook for students as well as a research reference for professionals in mathematics, applied mathematics, physics, and engineering.

It is my pleasure to express our grateful thanks to many friends, colleagues, and students around the world who offered their suggestions and help at various stages of the preparation of the book. Special thanks to Mrs. Veronica Chavarria for drawing some figures, typing the manuscript with constant changes and revisions. In spite of the best efforts of everyone involved, some typographical errors doubtless remain. Finally, we wish to express our special thanks to Mr. Tom Grasso, Senior Editor, and the staff of Birkhäuser, Boston, for their help and cooperation. I also wish to thank Mr. Donatas Akmanavičius and his staff for their meticulous job in preparing the final revised manuscript for printing the third edition. I am indebted to my wife, Sadhana, for her understanding and tolerance while the third edition was being written.

Edinburgh, Texas Lokenath Debnath

Preface to the Second Edition

This is a revised and expanded version of the first edition, published in 1997. The first edition was well received and used as a graduate level text and research reference in the United States and abroad for the last several years. I have received many criticisms and suggestions from graduate students and faculty members around the world. Their suggestions for improvement have been very helpful, beneficial, and encouraging. Most of the changes have been made in response to that input. However, an attempt has been made to preserve the character of the first edition. I believe that this new edition will remain a major source of linear and nonlinear partial differential equations and provide a useful working knowledge of the most important analytical methods of the solutions of the equations. Finding and interpreting the solutions of these equations is a central part of the modern applied mathematics, and a thorough understanding of partial differential equations is essential for mathematicians, scientists, and engineers. The main emphasis of the book is on the development of the required analytical skills on the part of the reader, rather than the importance of more abstract formulation with full mathematical rigor. However, because the study of partial differential equations is a subject at the forefront of current research, I have made an effort to include many new ideas, remarkable observations, and new evolution equations as further research topics for the ambitious reader to pursue.

I have taken advantage of this new edition to add some recent exciting developments of the subject, to update the bibliography and correct typographical errors, to include many new topics, examples, exercises, comments, and observations, and, in some cases, to entirely rewrite whole sections. The most significant difference from the first edition is the inclusion of many new sections, such as those on Sturm–Liouville (SL) systems and some major general results including eigenvalues, eigenfunctions, and completeness of SL system, energy integrals and higher dimensional wave and diffusion equations in different coordinate systems, solutions of fractional partial differential equations with new examples of applications, the Euler–Lagrange variational principle and the Hamilton variational principle with important examples of applications, and the Hamilton–Jacobi equation and its applications. Included also are the Euler equation and the continuity equation, which provide the fundamental basis of the study of modern theories of water waves, Stokes' analysis of nonlinear finite amplitude water waves, Whitham's equation, peaking and breaking of water waves, and conservation laws of the Whitham equation. This edition also contains some recent unexpected results and discoveries including a new class of strongly dispersive nonlinear evolution equations and compactons, new intrinsic localized modes in anharmonic crystals, and the derivation of the Korteweg–de Vries (KdV) equation, Kadomtsev–Petviashvili (KP) equation, Boussinesq equation, axisymmetric KdV equation, and Johnson concentric equation derived from the asymptotic expansion of the nonlinear water wave equations. As an example of an application of compactons, the solution of nonlinear vibration of an anharmonic mass–spring system is presented. Included are the existence of peakon (singular) solutions of a new strongly nonlinear model in shallow water described by Camassa and Holm equation, and the Harry Dym equation, which arises as a generalization of the class of isospectral flows of the Schrödinger operator. Furthermore, asymptotic expansions and the method of multiple scales, formal derivations of the nonlinear Schrödinger equation, and the Davey–Stewartson nonlinear evolution equations with several conservation laws have been added to this edition. Several short tables of the Fourier, Laplace, and Hankel transforms are provided in Chapter 13 for additional help to the reader.

A systematic mathematical treatment of linear and nonlinear partial differential equations is presented in the most straightforward manner, with worked examples and simple cases carefully explained with physical significance. Many and varied useful aspects, relevant proofs and calculations, and additional examples are provided in the numerous exercises at the end of each chapter. This edition contains over 600 worked examples and exercises with answers and hints to selected exercises, accompanied by original reference sources which include research papers and other texts. There is plenty of material in the book for a year-long course. Some of the material need not be covered in a course work and can be left for the readers to study on their own in order to prepare them for further study and research.

It is my pleasure to express my grateful thanks to the many friends and colleagues around the world who offered their suggestions and generous help at various stages of the preparation of this book. I offer my special thanks to Dr. Andras Balogh for drawing all figures, to Dr. Dambaru Bhatta for proofreading the whole book, and to Ms. Veronica Martinez for typing the manuscript with constant changes and revisions. In spite of the best efforts of everyone involved, some typographical errors doubtless remain. Finally, I wish to express my special thanks to Mr. Tom Grasso and the staff of Birkhäuser, Boston, for their help and cooperation. I am deeply indebted to my wife, Sadhana, for her understanding and tolerance while the second edition was being written.

University of Texas–Pan America Lokenath Debnath

Preface to the First Edition

Overview

Historically, partial differential equations originated from the study of surfaces in geometry and for solving a wide variety of problems in mechanics. During the second half of the nineteenth century, a large number of mathematicians became actively involved in the investigation of numerous problems presented by partial differential equations. The primary reason for this research was that partial differential equations both express many fundamental laws of nature and frequently arise in the mathematical analysis of diverse problems in science and engineering. The next phase of the development of linear partial differential equations is characterized by the efforts to develop the general theory and various methods of solutions of these linear equations. In fact, partial differential equations have been found to be essential to develop the theory of surfaces on the one hand and to the solution of physical problems on the other. These two areas of mathematics can be seen as linked by the bridge of the calculus of variations. With the discovery of the basic concepts and properties of distributions, the modern theory of the linear partial differential equations is now well established. The subject plays a central role in modern mathematics, especially in physics, geometry, and analysis.

Although the origin of nonlinear partial differential equations is very old, they have undergone remarkable new developments during the last half of the twentieth century. One of the main impulses for developing nonlinear partial differential equations has been the study of nonlinear wave propagation problems. These problems arise in different areas of applied mathematics, physics, and engineering, including fluid dynamics, nonlinear optics, solid mechanics, plasma physics, quantum field theory, and condensed-matter physics. Nonlinear wave equations in particular have provided several examples of *new* solutions that are remarkably different from those obtained for linear wave problems. The best known examples of these are the corresponding shock waves, water waves, solitons and solitary waves. One of the remarkable properties of solitons is a localized wave form that is retained after interaction with other solitons, confirming solitons' 'particle-like' behavior. Indeed, the theory of nonlinear waves and solitons has experienced a revolution over the past three decades. During this revolution, many remarkable and unexpected phenomena have also been observed in physical, chemical, and biological systems. Other major achievements of twentieth-century applied mathematics include the discovery of soliton interactions, the Inverse Scattering Transform (IST) method for finding the explicit exact solution for several canonical partial differential equations, and asymptotic perturbation analysis for the investigation of nonlinear evolution equations.

One of the major goals of the present book is to provide an accessible working knowledge of some of the current analytical methods required in modern mathematics, physics, and engineering. So the writing of the book was greatly influenced by the emphasis which Lord Rayleigh and Richard Feynman expressed as follows:

> *In the mathematical investigation I have usually employed such methods as present themselves naturally to a physicist. The pure mathematician will complain, and (it must be confessed) sometimes with justice, of deficient rigor. But to this question there are two sides. For, however important it may be to maintain a uniformly high standard in pure mathematics, the physicist may occasionally do well to rest content with arguments which are fairly satisfactory and conclusive from his point of view. To his mind, exercised in a different order of ideas, the more severe procedure of the pure mathematician may appear not more but less demonstrative. And further, in many cases of difficulty to insist upon highest standard would mean the exclusion of the subject altogether in view of the space that would be required.*
>
> *Lord Rayleigh*

> *. . . However, the emphasis should be somewhat more on how to do the mathematics quickly and easily, and what formulas are true, rather than the mathematicians' interest in methods of rigorous proof.*
>
> *Richard P. Feynman*

Audience and Organization

This book provides an introduction to nonlinear partial differential equations and to the basic methods that have been found useful for finding the solutions of these equations. While teaching a course on partial differential equations, the author has had difficulty choosing textbooks to accompany the lectures on some modern topic in nonlinear partial differential equations. The book was developed as a result of many years of experience teaching partial differential equations at the senior undergraduate and/or graduate levels. Parts of this book have also been used to accompany lecturers on special topics in nonlinear partial differential equations at Indian universities during my recent visit on a Senior Fulbright Fellowship. Based on my experience, I believe that nonlinear partial differential equations are best approached through a sound knowledge of linear partial differential equations. In order to make the book self-contained, the first chapter deals with linear partial differential equations and their methods of solution with examples of applications. There is plenty of material in this book for a two-semester graduate level course for mathematics, science, and engineering students. Many new examples of applications to problems in fluid dynamics, plasma physics, nonlinear optics, gas dynamics, analytical dynamics, and acoustics are included. Special emphasis is given to physical, chemical, biological, and engineering problems involving nonlinear wave phenomena. It is *not* essential for the reader to have a thorough knowledge of the physical aspect of these topics, but limited knowledge of at least some of them would be helpful. Besides, the book is intended to serve as a reference work for those seriously interested in advanced study and research in the subject, whether for its own sake or for its applications to other fields of applied mathematics, mathematical physics, and engineering science.

Another reason for adding this book to the literature is the fact that studies are continually being added to the theory, methods of solutions, and applications of nonlinear partial differential equations. It is becoming even more desirable for applied mathematicians, physicists, and engineering scientists to pursue study and research on these and related topics. Yet it is increasingly difficult to do so, because major articles appear in journals and research monographs of widely different natures. Some of these occur in papers scattered widely through the vast literature, and their connections are not readily apparent. This difficulty might be alleviated if a single book on nonlinear partial differential equations contained a coherent account of the recent developments, especially if written to be accessible to both graduate and postgraduate students. The field is growing fast. It is my hope that the book will first interest, then prepare readers to undertake research projects on nonlinear wave phenomena, reaction-diffusion phenomena, soliton dynamics, nonlinear instability and other nonlinear real-world problems, by providing that background of fundamental ideas, results, and methods essential to understanding the specialized literature of this vast area. The book is aimed at the reader interested in a broad perspective on the subject, the wide variety of phenomena encompassed by it and a working knowledge of the most important methods for solving the nonlinear equations. Those interested in more rigorous treatment of the mathematical theory of the subjects covered may consult some outstanding advanced books and treatises, listed in the Bibliography. Many ideas, principles, results, methods, examples of applications, and exercises presented in the book are either motivated by, or borrowed from works cited in the Bibliography. The author wishes to express his gratitude to the authors of these works.

The first chapter provides an introduction to linear partial differential equations and to the methods of solutions of these equations, and to the basic properties of these solutions, that gives the reader a clear understanding of the subject and its varied examples of applications.

Chapter 2 deals with nonlinear model equations and variational principles and the Euler–Lagrange equations. Included are variational principles for the nonlinear Klein–Gordon equation and for the nonlinear water waves.

The third and fourth chapters are devoted to the first-order quasi-linear and nonlinear equations and to the method of characteristics for solving them. Examples of applications of these equations to analytical dynamics and nonlinear optics are included.

Chapters 5 and 6 deal with conservation laws and shock waves, and kinematic waves and specific real-world nonlinear problems. The concept of weak or discontinuous solutions is introduced in Section 5.4. Several sections of Chapter 6 discuss the properties of solutions of several real-world nonlinear models that include traffic flow, flood waves, chromatographic models, sediment transport in rivers, glacier flow, and roll waves.

Chapter 7 is devoted to nonlinear dispersive waves, Whitham's equations, and Whitham's averaged variational principle. This is followed by the Whitham instability analysis and its applications to nonlinear water waves.

In Chapter 8, we study the nonlinear diffusion-reaction phenomena, and Burgers' and Fisher's equations with physical applications. Special attention is given to traveling wave solutions and their stability analysis, similarity methods and similarity solutions of diffusion equations.

Chapter 9 develops the theory of solitons and the Inverse Scattering Transform. Many recent results on the basic properties of the Korteweg–de Vries (KdV) and Boussinesq equations are discussed in some detail. Included are Bäcklund transformations, the nonlinear superposition principle, the Lax formulation and its KdV hierarchy.

The nonlinear Schrödinger equation and solitary waves are the main focus of Chapter 10. Special attention is paid to examples of applications to fluid dynamics, plasma physics, and nonlinear optics.

Chapter 11 is concerned with the theory of nonlinear Klein–Gordon and sine-Gordon equations with applications. The soliton and anti-soliton solutions of the sine-Gordon equation are described. The inverse scattering method, the similarity method and the method of separation of variables for the sine-Gordon equation are developed with examples.

The final chapter deals with nonlinear evolution equations and asymptotic methods. Several asymptotic perturbation methods and the method of multiple scales are developed for the solutions of quasilinear dissipative systems, weakly and strongly dispersive systems.

Salient Features

The book contains 450 worked examples, examples of applications, and exercises which include some selected from many standard treatises as well as from recent research papers. It is hoped that they will serve as helpful self-tests for understanding of the theory and mastery of the nonlinear partial differential equations. These examples and examples of applications were chosen from the areas of partial differential equations, geometry, vibration and wave propagation, heat conduction in solids, electric circuits, dynamical systems, fluid mechanics, plasma physics, quantum mechanics, nonlinear optics, physical chemistry, mathematical modeling, population dynamics, and mathematical biology. This varied number of examples and exercises should provide something of interest for everyone. The exercises truly complement the text and range from the elementary to the challenging.

This book is designed as a new source for modern topics dealing with nonlinear phenomena and their applications for future development of this important and useful subject. Its main features are listed below:

1. A systematic mathematical treatment of some nonlinear partial differential equations, the methods of the solutions of these equations, and the basic properties of these solutions is presented, that gives the reader a clear understanding of the subject and its varied applications.
2. A detailed and clear explanation of every concept and method which is introduced, accompanied by carefully selected worked examples, is included with special emphasis being given to those topics in which students experience difficulty.
3. The book presents a wide variety of modern examples of applications carefully selected from areas of fluid dynamics, plasma physics, nonlinear optics, soliton dynamics, analytical dynamics, gas dynamics, and acoustics to provide motivation, and to illustrate the wide variety of real-world nonlinear problems.
4. Most of the recent developments in the subject since the early 1960s appear here in book form for the first time.
5. Included also is a broad coverage of the essential standard material on nonlinear partial differential equations and their applications that is *not* readily found in any texts or reference books.
6. A striking balance between the mathematical and physical aspects of the subject is maintained.
7. The book is organized with sufficient flexibility so as to enable instructors to select chapters according to length, emphasis and level of different courses.
8. A wide spectrum of exercises has been carefully chosen and included at the end of each chapter so the reader may further develop both manipulative skills in the applications of nonlinear equations and a deeper insight into this modern subject.
9. The book provides information that puts the reader at the forefront of current research. An updated Bibliography is included to stimulate new interest in future study and research.
10. Answers and hints to selected exercises with original source are provided at the end of the book for additional help to students.

Acknowledgements

In preparing the book, the author has been encouraged by and has benefited from the helpful comments and criticism of a number of faculty, post-doctoral and doctoral students of several universities in the United States, Canada, and India. The author expresses his grateful thanks to these individuals for their interest in the book. My special thanks to Jackie Callahan and Ronee Trantham who typed the manuscript with many diagrams and cheerfully put up with constant changes and revisions. In spite of the best efforts of everyone involved, some typographical errors doubtless remain. I do hope that these are both few and obvious, and will cause minimum confusion. Finally, the author wishes to express his special thanks to Mr. Wayne Yuhasz, Executive Editor, and the staff of Birkhäuser for their help and cooperation. I am deeply indebted to my wife, Sadhana, for her understanding and tolerance while the book was being written.

University of Central Florida Lokenath Debnath

Contents

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Linear Partial Differential Equations

However varied may be the imagination of man, nature is still a thousand times richer, ... Each of the theories of physics ... presents (partial differential) equations under a new aspect ... without these theories, we should not know partial differential equations.

Henri Poincaré

Since a general solution must be judged impossible from want of analysis, we must be content with the knowledge of some special cases, and that all the more, since the development of various cases seems to be the only way to bringing us at last to a more perfect knowledge.

Leonard Euler

1.1 Introduction

Partial differential equations arise frequently in the formulation of fundamental laws of nature and in the mathematical analysis of a wide variety of problems in applied mathematics, mathematical physics, and engineering science. This subject plays a central role in modern mathematical sciences, especially in physics, geometry, and analysis. Many problems of physical interest are described by partial differential equations with appropriate initial and/or boundary conditions. These problems are usually formulated as initial-value problems, boundary-value problems, or initial boundary-value problems. In order to prepare the reader for study and research in nonlinear partial differential equations, a broad coverage of the essential standard material on linear partial differential equations and their applications is required.

This chapter provides a review of basic concepts, principles, model equations, and their methods of solutions. This is followed by a systematic mathematical treatment of the theory and methods of solutions of second-order linear partial differential

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equations that gives the reader a clear understanding of the subject and its varied applications. Linear partial differential equations of the second order can be classified as one of the three types: hyperbolic, parabolic, and elliptic, and reduced to an appropriate canonical or normal form. The classification and method of reduction are described in Section 1.5. Special emphasis is given to various methods of solution of the initial-value and/or boundary-value problems associated with the three types of linear equations, each of which shows an entirely different behavior in properties and construction of solutions. Section 1.6 deals with the solutions of linear partial differential equations using the method of separation of variables combined with the superposition principle. A brief discussion of Fourier, Laplace, and Hankel transforms is included in Sections 1.7–1.10. These integral transforms are then applied to solve a large variety of initial and boundary problems described by partial differential equations. The transform solution combined with the convolution theorem provides an elegant representation of the solution for initial-value and boundary-value problems. Section 1.11 is devoted to Green's functions for solving a wide variety of inhomogeneous partial differential equations of most common interest. This method can be made considerably easier by using generalized functions combined with appropriate integral transforms. The Sturm–Liouville systems and their general properties are discussed in Section 1.12. Section 1.13 deals with energy integrals, the law of conservation of energy, uniqueness theorems, and higher dimensional wave and diffusion equations. The final section contains some recent examples of fractional order diffusion–wave equations and their solutions.

1.2 Basic Concepts and Definitions

A partial differential equation for a function $u(x, y, \ldots)$ is a relationship between u and its partial derivatives $u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots$, and can be written as

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0, \tag{1.2.1}$$

where F is some function, $x, y, \ldots$ are independent variables and $u(x, y, \ldots)$ is called a *dependent variable*.

The *order* of a partial differential equation is defined in analogy with an ordinary differential equation as the highest-order of a derivative appearing in (1.2.1). The most general *first-order* partial differential equation can be written

$$F(x, y, u, u_x, u_y) = 0. \tag{1.2.2}$$

Similarly, the most general *second-order* partial differential equation in two independent variables x, y has the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{1.2.3}$$

and so on for higher-order equations.

For example,

$$x\, u_x + y\, u_y = 0, \tag{1.2.4}$$

$$x\,u_x + y\,u_y = x^2 + y^2, \tag{1.2.5}$$
$$u\,u_x + u_t = u, \tag{1.2.6}$$
$$u_x^2 + u_y^2 = 1 \tag{1.2.7}$$

are first-order equations, and

$$u_{xx} + 2u_{xy} + u_{yy} = 0, \tag{1.2.8}$$
$$u_{xx} + u_{yy} = 0, \tag{1.2.9}$$
$$u_{tt} - c^2 u_{xx} = f(x,t) \tag{1.2.10}$$

are second-order equations. Finally,

$$u_t + uu_x + u_{xxx} = 0, \tag{1.2.11}$$
$$u_{tt} + u_{xxxx} = 0 \tag{1.2.12}$$

are examples of the third-order and fourth-order equations, respectively.

A partial differential equation is called *linear* if it is linear in the unknown function and all its derivatives with coefficients depend only on the independent variables. It is called *quasi-linear*, if it is linear in the highest-order derivative of the unknown function. For example, (1.2.4), (1.2.5), (1.2.8)–(1.2.10) and (1.2.12) are linear equations, whereas (1.2.6) and (1.2.11) are quasi-linear equations.

It is possible to write a partial differential equation in the operator form

$$L_x u(\mathbf{x}) = f(\mathbf{x}), \tag{1.2.13}$$

where L_x is an operator. The operator L_x is called a *linear operator* if it satisfies the property

$$L_x(au + bv) = a\,L_x u + b\,L_x v \tag{1.2.14}$$

for any two functions u and v and for any two constants a and b.

Equation (1.2.13) is called *linear* if L_x is a linear operator. Equation (1.2.13) is called an *inhomogeneous* (or *nonhomogeneous*) linear equation. If $f(\mathbf{x}) \equiv 0$, (1.2.13) is called a *homogeneous* equation. Equations (1.2.4), (1.2.8), (1.2.9), and (1.2.12) are linear homogeneous equations, whereas (1.2.5) and (1.2.10) are linear inhomogeneous equations.

An equation which is not linear is called a *nonlinear equation*. If L_x is not linear, then (1.2.13) is called a *nonlinear* equation. Equations (1.2.6), (1.2.7), and (1.2.11) are examples of nonlinear equations.

A *classical solution* (or simply a *solution*) of (1.2.1) is an ordinary function $u = u(x, y, \ldots)$ defined on some domain D which is continuously differentiable such that all its partial derivatives involved in the equation exist and satisfy (1.2.1) identically.

However, this notion of classical solution can be extended by relaxing the requirement that u is continuously *differentiable* over D. The solution $u = u(x, y, \ldots)$ is called a weak (or *generalized*) *solution* of (1.2.1) if u or its partial derivatives are discontinuous at some or all points in D.

To introduce the idea of a *general solution* of a partial differential equation, we solve a simple equation for $u = u(x, y)$ of the form

$$u_{xy} = 0. \tag{1.2.15}$$

Integrating this equation with respect to x (keeping y fixed), we obtain

$$u_y = h(y),$$

where $h(y)$ is an arbitrary function of y. We then integrate it with respect to y to find

$$u(x, y) = \int h(y)\, dy + f(x),$$

where $f(x)$ is an arbitrary function. Or equivalently,

$$u(x, y) = f(x) + g(y), \tag{1.2.16}$$

where $f(x)$ and $g(y)$ are arbitrary functions. The solution (1.2.16) is called the *general solution* of the second-order equation (1.2.15).

Usually, the general solution of a partial differential equation is an expression that involves arbitrary functions. This is a striking contrast to the general solution of an ordinary differential equation which involves arbitrary constants. Further, a simple equation (1.2.15) has infinitely many solutions. This can be illustrated by considering the problem of construction of partial differential equations from given arbitrary functions. For example, if

$$u(x, t) = f(x - ct) + g(x + ct), \tag{1.2.17}$$

where f and g are arbitrary functions of $(x - ct)$ and $(x + ct)$, respectively, then

$$\begin{aligned} u_{xx} &= f''(x - ct) + g''(x + ct), \\ u_{tt} &= c^2 f''(x - ct) + c^2 g''(x + ct) = c^2 u_{xx}, \end{aligned}$$

where primes denote differentiation with respect to the appropriate argument. Thus, we obtain the second-order linear equation, called the *wave equation*,

$$u_{tt} - c^2 u_{xx} = 0. \tag{1.2.18}$$

Thus, the function $u(x, t)$ defined by (1.2.17) satisfies (1.2.18) irrespective of the functional forms of $f(x-ct)$ and $g(x+ct)$, provided f and g are at least twice differentiable functions. Thus, the general solution of equation (1.2.18) is given by (1.2.17) which contains arbitrary functions.

In the case of only two independent variables x, y, the solution $u(x, y)$ of the equation (1.2.1) is visualized *geometrically as a surface*, called an *integral surface* in the (x, y, u) space.

1.3 The Linear Superposition Principle

The general solution of a linear homogeneous ordinary differential equation of order n is a linear combination of n linearly independent solutions with n arbitrary constants. In other words, if $u_1(x), u_2(x), \ldots, u_n(x)$ are n linearly independent solutions of an nth order, linear, homogeneous, ordinary differential equation of the form

$$Lu(x) = 0, \tag{1.3.1}$$

then, for any arbitrary constants $c_1, c_2, \ldots, c_n$,

$$u(x) = \sum_{k=1}^{n} c_k u_k(x) \tag{1.3.2}$$

represents the most general solution of (1.3.1). This is called the *linear superposition principle* for ordinary differential equations. We note that the general solution of (1.3.1) depends on *exactly n arbitrary constants*.

In the case of linear homogeneous partial differential equations of the form

$$L_x u(\mathbf{x}) = 0, \tag{1.3.3}$$

the general solution depends on arbitrary functions rather than arbitrary constants. So there are infinitely many solutions of (1.3.3). If we represent this infinite set of solutions of (1.3.3) by $u_1(\mathbf{x}), u_2(\mathbf{x}), \ldots, u_n(\mathbf{x}), \ldots$, then the infinite linear combinations

$$u(\mathbf{x}) = \sum_{n=1}^{\infty} c_n u_n(\mathbf{x}), \tag{1.3.4}$$

where c_n are arbitrary constants, in general, may *not* be again a solution of (1.3.3) because the infinite series may not be convergent. So, for the case of partial differential equations, the superposition principle may not be true, in general. However, if there are only a *finite* number of solutions $u_1(\mathbf{x}), u_2(\mathbf{x}), \ldots, u_n(\mathbf{x})$ of the partial differential equation (1.3.3), then

$$u(\mathbf{x}) = \sum_{n=1}^{n} c_n u_n(\mathbf{x}) \tag{1.3.5}$$

again is a solution of (1.3.3) as can be verified by direct substitution. As with linear homogeneous ordinary differential equations, the principle of superposition applies to linear homogeneous partial differential equations and $u(\mathbf{x})$ represents a solution of (1.3.3), provided that the infinite series (1.3.4) is convergent and the operator L_x can be applied to the series term by term.

In order to generate such an infinite set of solutions $u_n(\mathbf{x})$, the method of separation of variables is usually used. This method, combined with the superposition of solutions, is usually known as *Fourier's method*, which will be described in a subsequent section.

Another type of infinite linear combination is used to find the solution of a given partial differential equation. This is concerned with a family of solutions $u(\mathbf{x}, k)$ depending on a continuous real parameter k and a function $c(k)$ such that

$$\int_a^b c(k)u(\mathbf{x}, k)\, dk \quad \text{or} \quad \int_{-\infty}^{\infty} c(k)u(\mathbf{x}, k)\, dk \tag{1.3.6}$$

is convergent. Then, under certain conditions, this integral is again a solution. This may also be regarded as the *linear superposition principle*.

In almost all cases, the general solution of a partial differential equation is of little use since it has to satisfy other supplementary conditions, usually called *initial* or *boundary* conditions. As indicated earlier, the general solution of a linear partial differential equation contains arbitrary functions. This means that there are infinitely many solutions and only by specifying the initial and/or boundary conditions can we determine a specific solution of interest.

Usually, both initial and boundary conditions arise from the physics of the problem. In the case of partial differential equations in which one of the independent variables is the time t, an initial condition(s) specifies the physical state of the dependent variable $u(\mathbf{x}, t)$ at a particular time $t = t_0$ or $t = 0$. Often $u(\mathbf{x}, 0)$ and/or $u_t(\mathbf{x}, 0)$ are specified to determine the function $u(\mathbf{x}, t)$ at later times. Such conditions are called the *Cauchy* (*or initial*) *conditions*. It can be shown that these conditions are necessary and sufficient for the existence of a unique solution. The problem of finding the solution of the initial-value problem with prescribed Cauchy data on the line $t = 0$ is called the *Cauchy problem* or the *initial-value problem*.

In each physical problem, the governing equation is to be solved within a given domain D of space with prescribed values of the dependent variable $u(\mathbf{x}, t)$ given on the boundary ∂D of D. Often, the boundary need not enclose a finite volume—in which case, part of the boundary is at infinity. For problems with a boundary at infinity, boundedness conditions on the behavior of the solution at infinity must be specified. This kind of problem is typically known as a *boundary-value problem,* and it is one of the most fundamental problems in applied mathematics and mathematical physics.

There are three important types of boundary conditions which arise frequently in formulating physical problems. These are

(a) *Dirichlet conditions*, where the solution u is prescribed at each point of a boundary ∂D of a domain D. The problem of finding the solution of a given equation $L_x u(\mathbf{x}) = 0$ inside D with prescribed values of u on ∂D is called the *Dirichlet boundary-value problem*;

(b) *Neumann conditions*, where values of normal derivative $\frac{\partial u}{\partial n}$ of the solution on the boundary ∂D are specified. In this case, the problem is called the *Neumann boundary-value problem*;

(c) *Robin conditions*, where $(\frac{\partial u}{\partial n} + au)$ is specified on ∂D. The corresponding problem is called the *Robin boundary-value problem*.

A problem described by a partial differential equation in a given domain with a set of initial and/or boundary conditions (or other supplementary conditions) is said to be *well-posed* (or *properly posed*) provided the following criteria are satisfied:

(i) *Existence*: There exists at least one solution of the problem.
(ii) *Uniqueness*: There is at most one solution.
(iii) *Stability*: The solution must be stable in the sense that it depends continuously on the data. In other words, a small change in the given data must produce a small change in the solution.

The stability criterion is essential for physical problems. A mathematical problem is usually considered physically realistic if a small change in given data produces correspondingly a small change in the solution.

According to the Cauchy–Kowalewski theorem, the solution of an analytic Cauchy problem for partial differential equations exists and is unique. However, a Cauchy problem for Laplace's equation is *not* always *well-posed*. A famous example of a *non-well-posed* (or *ill-posed*) problem was first given by Hadamard. *Hadamard's example* deals with Cauchy's initial-value problem for the Laplace equation

$$\nabla^2 u \equiv u_{xx} + u_{yy} = 0, \quad 0 < y < \infty, \ x \in R \tag{1.3.7}$$

with the Cauchy data

$$u(x,0) = 0 \quad \text{and} \quad u_y(x,0) = \left(\frac{1}{n}\right) \sin nx, \tag{1.3.8}$$

where n is an integer representing the wavenumber. These data tend to zero uniformly as $n \to \infty$.

It can easily be verified that the unique solution of this problem is given by

$$u(x,y) = \left(\frac{1}{n^2}\right) \sinh ny \sin nx. \tag{1.3.9}$$

As $n \to \infty$, this solution does not tend to the solution $u = 0$. In fact, solution (1.3.9) represents oscillations in x with unbounded amplitude $n^{-2} \sinh ny$ which tends to infinity as $n \to \infty$. In other words, although the data change by an arbitrarily small amount, the change in the solution is infinitely large. So the problem is certainly *not* well-posed, that is, the solution does not depend continuously on the initial data. Even if the wavenumber n is a fixed, finite quantity, the solution is clearly unstable in the sense that $u(x,y) \to \infty$ as $y \to \infty$ for any fixed x, such that $\operatorname{sinc} nx \neq 0$.

On the other hand, the Cauchy problem (see Example 1.5.3) for the simplest hyperbolic equation (1.5.29) with the initial data (1.5.35ab) is a *well posed problem*. As to the domain of dependence for the solution, $u(x,t)$ depends *only* on those values of $f(\xi)$ and $g(\xi)$ for which $x - ct \le \xi \le x + ct$. Similarly, the Cauchy problems for parabolic equations are generally *well posed*.

We conclude this section with a general remark. The existence, uniqueness, and stability of solutions are the basic requirements for a complete description of a physical problem with appropriate initial and boundary conditions. However, there are many situations in applied mathematics which deal with ill-posed problems. In recent years, considerable progress has been made on the theory of ill-posed problems, but the discussion of such problems is beyond the scope of this book.

1.4 Some Important Classical Linear Model Equations

We start with a special type of second-order linear partial differential equation for the following reasons. First, second-order equations arise more frequently in a wide variety of applications. Second, their mathematical treatment is simpler and easier to understand than that of first-order equations, in general. Usually, in almost all physical phenomena, the dependent variable $u = u(x, y, z, t)$ is a function of the three space variables and the time variable t. Included here are only examples of equations of most common interest.

Example 1.4.1. The wave equation is

$$u_{tt} - c^2 \nabla^2 u = 0, \tag{1.4.1}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{1.4.2}$$

and c is a constant. This equation describes the propagation of a wave (or disturbance), and it arises in a wide variety of physical problems. Some of these problems include a vibrating string, vibrating membrane, longitudinal vibrations of an elastic rod or beam, shallow water waves, acoustic problems for the velocity potential for a fluid flow through which sound can be transmitted, transmission of electric signals along a cable, and both electric and magnetic fields in the absence of charge and dielectric.

Example 1.4.2. The heat or diffusion equation is

$$u_t - \kappa \nabla^2 u = 0, \tag{1.4.3}$$

where κ is the constant of diffusivity. This equation describes the diffusion of thermal energy in a homogeneous medium. It can be used to model the flow of a quantity, such as heat, or a concentration of particles. It is also used as a model equation for growth and diffusion, in general, and growth of a solid tumor, in particular. The diffusion equation describes the unsteady boundary-layer flow in the Stokes and Rayleigh problems and also the diffusion of vorticity from a vortex sheet.

Example 1.4.3. The Laplace equation is

$$\nabla^2 u = 0. \tag{1.4.4}$$

This equation is used to describe electrostatic potential in the absence of charges, gravitational potential in the absence of mass, equilibrium displacement of an elastic membrane, velocity potential for an incompressible fluid flow, temperature in a steady-state heat conduction problem, and many other physical phenomena.

Example 1.4.4. The Poisson equation is

$$\nabla^2 u = f(x, y, z), \tag{1.4.5}$$

where $f(x, y, z)$ is a given function describing a source or sink. This is an inhomogeneous Laplace equation, and hence, the Poisson equation is used to study all phenomena described by the Laplace equation in the presence of external sources or sinks.

Example 1.4.5. The Helmholtz equation is

$$\nabla^2 u + \lambda u = 0, \tag{1.4.6}$$

where λ is a constant. This is a time-independent wave equation (1.4.1) with λ as a separation constant. In particular, its solution in acoustics represents an acoustic radiation potential.

Example 1.4.6. The telegraph equation is given in a general form as

$$u_{tt} - c^2 u_{xx} + a u_t + b u = 0, \tag{1.4.7}$$

where a, b, and c are constants. This equation arises in the study of propagation of electrical signals in a cable of a transmission line. Both current I and voltage V satisfy an equation of the form (1.4.7). This equation also arises in the propagation of pressure waves in the study of pulsatile blood flow in arteries and in one-dimensional random motion of bugs along a hedge.

Example 1.4.7. The Klein–Gordon (or KG) equation is

$$\Box\psi + \left(\frac{mc^2}{\hbar}\right)^2 \psi = 0, \tag{1.4.8}$$

where

$$\Box \equiv \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 \tag{1.4.9}$$

is the d'Alembertian operator, $h(= 2\pi\hbar)$ is the Planck constant, and m is a constant mass of the particle. Klein (1927) and Gordon (1926) derived a relativistic equation for a charged particle in an electromagnetic field. It is of conservative dispersive type and played an important role in our understanding of the elementary particles. This equation is also used to describe dispersive wave phenomena, in general.

Example 1.4.8. The time-independent Schrödinger equation in quantum mechanics is

$$\left(\frac{\hbar^2}{2m}\right)\nabla^2\psi + (E - V)\psi = 0, \tag{1.4.10}$$

where $h(= 2\pi\hbar)$ is the Planck constant, m is the mass of the particle whose wave function is $\psi(x, y, z, t)$, E is a constant, and V is the potential energy. If $V = 0$, (1.4.10) reduces to the Helmholtz equation.

Example 1.4.9. The linear Korteweg–de Vries (or KdV) equation is

$$u_t + \alpha u_x + \beta u_{xxx} = 0, \tag{1.4.11}$$

where α and β are constants. This describes the propagation of linear, long, water waves and of plasma waves in a dispersive medium.

Example 1.4.10. The linear Boussinesq equation is

$$u_{tt} - \alpha^2 \nabla^2 u - \beta^2 \nabla^2 u_{tt} = 0, \tag{1.4.12}$$

where α and β are constants. This equation arises in elasticity for longitudinal waves in bars, long water waves, and plasma waves.

Example 1.4.11. The biharmonic wave equation is

$$u_{tt} + c^2 \nabla^4 u = 0, \tag{1.4.13}$$

where c is a constant. In elasticity, the displacement of a thin elastic plate by small vibrations satisfies this equation. When u is independent of time t, (1.4.13) reduces to what is called the *biharmonic equation,* namely

$$\nabla^4 u = 0. \tag{1.4.14}$$

This describes the equilibrium equation for the distribution of stresses in an elastic medium satisfied by Airy's stress function $u(x, y, z)$. In fluid dynamics, this equation is satisfied by the stream function $\psi(x, y, z)$ in a viscous fluid flow.

Example 1.4.12. The electromagnetic wave equations for the electric field E and the polarization P are

$$\mathcal{E}_0 \left(E_{tt} - c_0^2 E_{xx} \right) + P_{tt} = 0, \tag{1.4.15}$$

$$\left(P_{tt} + \omega_0^2 P \right) - \mathcal{E}_0 \omega_p^2 E = 0, \tag{1.4.16}$$

where $\mathcal{E}_0$ is the permittivity (or dielectric constant) of free space, ω_0 is the natural frequency of the oscillator, c_0 is the speed of light in a vacuum, and ω_p is the plasma frequency.

1.5 Second-Order Linear Equations and Method of Characteristics

The general second-order linear partial differential equation in two independent variables x, y is given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{1.5.1}$$

where A, B, C, D, E, F, and G are given functions of x and y or constants.

The classification of second-order equations is based upon the possibility of reducing equation (1.5.1) by a coordinate transformation to a canonical or standard form at a point. We consider the transformation from x, y to ξ, η defined by

$$\xi = \phi(x, y), \qquad \eta = \psi(x, y), \tag{1.5.2ab}$$

where ϕ and ψ are twice continuously differentiable and the Jacobian $J(x, y) = \phi_x \psi_y - \psi_x \phi_y$ is nonzero in a domain of interest so that x, y can be determined uniquely from the system (1.5.2ab). Then, by the chain rule,

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x, \qquad u_y = u_\xi \xi_y + u_\eta \eta_y, \\
(u_x)_x &= (u_x)_\xi \xi_x + (u_x)_\eta \eta_x \\
&= (u_\xi \xi_x + u_\eta \eta_x)_\xi \xi_x + (u_\xi \xi_x + u_\eta \eta_x)_\eta \eta_x, \\
u_{xx} &= u_{\xi\xi} \xi_x^2 + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}, \\
u_{yy} &= u_{\xi\xi} \xi_y^2 + 2u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}, \\
u_{xy} &= u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + u_{\eta\eta} \eta_x \eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}.
\end{aligned}$$

Substituting these results into equation (1.5.1) gives

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*, \tag{1.5.3}$$

where

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x \xi_y + C\xi_y^2, \\
B^* &= 2A\xi_x \eta_x + B(\xi_x \eta_y + \xi_y \eta_x) + 2C\xi_y \eta_y, \\
C^* &= A\eta_x^2 + B\eta_x \eta_y + C\eta_y^2, \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y, \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y, \\
F^* &= F, \quad \text{and} \quad G^* = G.
\end{aligned}$$

Now, the problem is to determine ξ and η so that equation (1.5.3) takes the simplest possible form. We choose ξ and η such that $A^* = C^* = 0$ and $B^* \neq 0$. Or, more explicitly,

$$A^* = A\xi_x^2 + B\xi_x \xi_y + C\xi_y^2 = 0, \tag{1.5.4}$$

$$C^* = A\eta_x^2 + B\eta_x \eta_y + C\eta_y^2 = 0. \tag{1.5.5}$$

These two equations can be combined into a single quadratic equation for $\zeta = \xi$ or η

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0. \tag{1.5.6}$$

We consider the level curves $\xi = \phi(x, y) = \text{const.} = C_1$ and $\eta = \psi(x, y) = \text{const.} = C_2$. On these curves

$$d\xi = \xi_x\,dx + \xi_y\,dy = 0, \qquad d\eta = \eta_x\,dx + \eta_y\,dy = 0, \tag{1.5.7ab}$$

that is, the slopes of these curves are given by

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y}, \qquad \frac{dy}{dx} = -\frac{\eta_x}{\eta_y}. \tag{1.5.8ab}$$

Thus, the slopes of both level curves are the roots of the same quadratic equation which is obtained from (1.5.6) as

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0, \tag{1.5.9}$$

and the roots of this equation are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right). \tag{1.5.10ab}$$

These equations are known as the *characteristic equations* for (1.5.1), and their solutions are called the *characteristic curves*, or simply the *characteristics* of equation (1.5.1). The solution of the two ordinary differential equations (1.5.10ab) defines two distinct families of characteristics $\phi(x,y) = C_1$ and $\psi(x,y) = C_2$. There are three possible cases to consider.

Case I. $B^2 - 4AC > 0$.

Equations for which $B^2 - 4AC > 0$ are called *hyperbolic*. Integrating (1.5.10ab) gives two real and distinct families of characteristics $\phi(x,y) = C_1$ and $\psi(x,y) = C_2$, where C_1 and C_2 are constants of integration. Since $A^* = C^* = 0$, and $B^* \neq 0$, and dividing by B^*, equation (1.5.3) reduces to the form

$$u_{\xi\eta} = -\frac{1}{B^*}\left(D^* u_\xi + E^* u_\eta + F^* u - G^*\right) = H_1(\text{say}). \tag{1.5.11}$$

This is called the *first canonical form of the hyperbolic equation*.

If the new independent variables

$$\alpha = \xi + \eta, \qquad \beta = \xi - \eta \tag{1.5.12ab}$$

are introduced, then

$$u_\xi = u_\alpha \alpha_\xi + u_\beta \beta_\xi = u_\alpha + u_\beta, \qquad u_\eta = u_\alpha \alpha_\eta + u_\beta \beta_\eta = u_\alpha - u_\beta,$$
$$(u_\eta)_\xi = (u_\eta)_\alpha \alpha_\xi + (u_\eta)_\beta \beta_\xi = (u_\alpha - u_\beta)_\alpha \cdot 1 + (u_\alpha - u_\beta)_\beta \cdot 1$$
$$= u_{\alpha\alpha} - u_{\beta\beta}.$$

Consequently, equation (1.5.11) becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = H_2(\alpha, \beta, u, u_\alpha, u_\beta). \tag{1.5.13}$$

This is called the *second canonical form of the hyperbolic equation*.

It is important to point out that characteristics play a fundamental role in the theory of hyperbolic equations.

Case II. $B^2 - 4AC = 0$.

There is only one family of real characteristics whose slope, due to (1.5.10ab), is given by

$$\frac{dy}{dx} = \frac{B}{2A}. \tag{1.5.14}$$

Integrating this equation gives $\xi = \phi(x, y) = \text{const.}$ (or $\eta = \psi(x, y) = \text{const.}$).

Since $B^2 = 4AC$ and $A^* = 0$, we obtain

$$0 = A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = \left(\sqrt{A}\xi_x + \sqrt{C}\xi_y\right)^2.$$

It then follows that

$$\begin{aligned} B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ &= 2\left(\sqrt{A}\xi_x + \sqrt{C}\xi_y\right)\left(\sqrt{A}\eta_x + \sqrt{C}\eta_y\right) = 0 \end{aligned}$$

for an arbitrary value of η which is independent of ξ. For example, if $\eta = y$, the Jacobian is nonzero in the domain of parabolicity.

Dividing (1.5.3) by $C^* \neq 0$ yields

$$u_{\eta\eta} = H_3(\xi, \eta, u, u_\xi, u_\eta). \tag{1.5.15}$$

This is known as the *canonical form of the parabolic equation*.

On the other hand, if we choose $\eta = \psi(x, y) = \text{const.}$ as the integral of (1.5.14), equation (1.5.3) assumes the form

$$u_{\xi\xi} = H_3^*(\xi, \eta, u, u_\xi, u_\eta). \tag{1.5.16}$$

Equations for which $B^2 - 4AC = 0$ are called *parabolic*.

Case III. $B^2 - 4AC < 0$.

Equations for which $B^2 - 4AC < 0$ are called *elliptic*. In this case, equations (1.5.10ab) have no real solutions. So there are two families of complex characteristics. Since the roots ξ, η of (1.5.10ab) are complex conjugates of each other, we introduce the new real variables as

$$\alpha = \frac{1}{2}(\xi + \eta), \qquad \beta = \frac{1}{2i}(\xi - \eta), \tag{1.5.17ab}$$

so that $\xi = \alpha + i\beta$ and $\eta = \alpha - i\beta$.

We use (1.5.17ab) to transform (1.5.3) into the form

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} = H_4(\alpha, \beta, u, u_\alpha, u_\beta), \tag{1.5.18}$$

where the coefficients of this equation assume the same form as the coefficients of (1.5.3). It can easily be verified that $A^* = 0$ and $C^* = 0$ take the form

$$A^{**} - C^{**} \pm iB^{**} = 0,$$

which are satisfied if and only if

$$A^{**} = C^{**} \quad \text{and} \quad B^{**} = 0.$$

Thus, dividing by A^{**}, equation (1.5.18) reduces to the form

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{H_4}{A^{**}} = H_5(\alpha, \beta, u, u_\alpha, u_\beta). \tag{1.5.19}$$

This is called the *canonical form of the elliptic equation*.

In summary, we state that equation (1.5.1) is called *hyperbolic*, *parabolic*, or *elliptic* at a point (x_0, y_0) accordingly as

$$B^2(x_0, y_0) - 4A(x_0, y_0)C(x_0, y_0) >=< 0. \tag{1.5.20}$$

If it is true at all points in a given domain, then the equation is said to be *hyperbolic*, *parabolic*, or *elliptic* in that domain. Finally, it has been shown above that, for the case of two independent variables, a transformation can always be found to transform the given equation to the canonical form. However, in the case of several independent variables, in general, it is *not* possible to find such a transformation.

These three types of partial differential equations arise in many areas of mathematical and physical sciences. Usually, boundary-value problems are associated with elliptic equations, whereas the initial-value problems arise in connection with hyperbolic and parabolic equations.

Example 1.5.1. Show that

(a) the wave equation $u_{tt} - c^2 u_{xx} = 0$ is hyperbolic,
(b) the diffusion equation $u_t - \kappa u_{xx} = 0$ is parabolic,
(c) the Laplace equation $u_{xx} + u_{yy} = 0$ is elliptic,
(d) the Tricomi equation $u_{xx} + x u_{yy} = 0$ is elliptic for $x > 0$, parabolic for $x = 0$, and hyperbolic for $x < 0$.

For case (a), $A = -c^2$, $B = 0$, and $C = 1$. Hence, $B^2 - 4AC = c^2 > 0$ for all x and t. So, the wave equation is hyperbolic everywhere. Similarly, the reader can show (b) and (c). Finally, for case (d), $A = 1$, $B = 0$, $C = x$, hence, $B^2 - 4AC = -4x < 0, = 0$, or > 0 accordingly as $x > 0$, $x = 0$, or $x < 0$, and the result follows.

Example 1.5.2. Find the characteristic equations and characteristics and then reduce the equation

$$x u_{xx} + u_{yy} = x^2 \tag{1.5.21}$$

to canonical form.

In this problem, $A = x$, $B = 0$, $C = 1$, $B^2 - 4AC = -4x$. Thus, the equation is hyperbolic if $x < 0$, parabolic if $x = 0$, and elliptic if $x > 0$.

The characteristics equations are

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \pm \frac{1}{\sqrt{-x}}. \tag{1.5.22ab}$$

Hence,

$$y = \pm 2\sqrt{-x} = \text{const.} = \pm 2\sqrt{-x} + c,$$

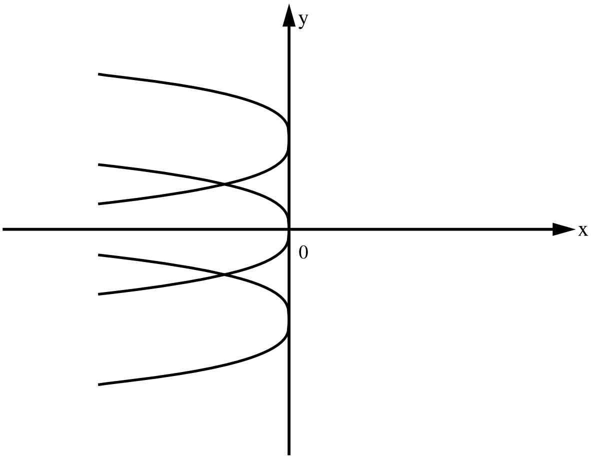

Fig. 1.1 Characteristics are parabolas for $x < 0$.

or equivalently,

$$\xi = y + 2\sqrt{-x} = \text{const.}, \qquad \eta = y - 2\sqrt{-x} = \text{const.} \tag{1.5.23ab}$$

These represent two branches of the parabolas $(y-c)^2 = -4x$ where c is a constant. The former equation ($\xi = \text{const.}$) gives a branch with positive slopes, whereas the latter equation ($\eta = \text{const.}$) represents a branch with negative slopes as shown in Figure 1.1. Both branches are tangent to the y-axis which is the single characteristic in the parabolic region. Indeed, the y-axis is the envelope of the characteristics for the hyperbolic region $x < 0$.

For $x < 0$, we use the transformations

$$\xi = y + 2\sqrt{-x}, \qquad \eta = y - 2\sqrt{-x} \tag{1.5.24ab}$$

to reduce (1.5.21) to the canonical form.

We find

$$\xi_x = -\frac{1}{\sqrt{-x}}, \qquad \xi_y = 1, \qquad \xi_{xx} = -\frac{1}{2}\frac{1}{(-x)^{3/2}}, \qquad \xi_{yy} = 0,$$

$$\eta_x = +\frac{1}{\sqrt{-x}}, \qquad \eta_y = 1, \qquad \eta_{xx} = \frac{1}{2}\frac{1}{(-x)^{3/2}}, \qquad \eta_{yy} = 0,$$

$$(\xi - \eta) = 4\sqrt{-x}, \quad \text{and} \quad (\xi - \eta)^4 = (16x)^2.$$

Consequently, the equation

$$xu_{xx} + u_{yy} = x^2$$

reduces to the form

$$x\left(u_{\xi\xi}\xi_x^2+2u_{\xi\eta}\xi_x\eta_x+u_{\eta\eta}\eta_x^2+u_\xi\xi_{xx}+u_\eta\eta_{xx}\right)$$
$$+\left(u_{\xi\xi}\xi_y^2+2u_{\xi\eta}\xi_y\eta_y+u_{\eta\eta}\eta_y^2+u_\xi\xi_{yy}+u_\eta\eta_{yy}\right)=x^2.$$

Or equivalently,

$$x\left[u_{\xi\xi}\left(-\frac{1}{x}\right)+2u_{\xi\eta}\left(\frac{1}{x}\right)-u_{\eta\eta}\left(\frac{1}{x}\right)-\frac{1}{2}\frac{1}{(-x)^{3/2}}u_\xi+\frac{1}{2}\frac{1}{(-x)^{3/2}}u_\eta\right]$$
$$+\left[u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}\right]=x^2.$$

This leads to the transformed equation which is

$$4u_{\xi\eta}+\frac{1}{2}\frac{1}{\sqrt{-x}}(u_\xi-u_\eta)=\frac{1}{(16)^2}(\xi-\eta)^4,$$

or equivalently,

$$u_{\xi\eta}=\frac{1}{4}\cdot\frac{1}{(16)^2}(\xi-\eta)^4-\frac{1}{2}\frac{1}{(\xi-\eta)}(u_\xi-u_\eta). \tag{1.5.25}$$

This is the first canonical form.

For $x>0$, we use the transformations

$$\xi=y+2i\sqrt{x},\qquad \eta=y-2i\sqrt{x}$$

so that

$$\alpha=\frac{1}{2}(\xi+\eta)=y,\qquad \beta=\frac{1}{2i}(\xi-\eta)=2\sqrt{x}. \tag{1.5.26ab}$$

Clearly,

$$\alpha_x=0,\qquad \alpha_y=1,\qquad \alpha_{xx}=0,\qquad \alpha_{yy}=0,\qquad \alpha_{xy}=0,$$
$$\beta_x=\frac{1}{\sqrt{x}},\qquad \beta_y=0,\qquad \beta_{xx}=-\frac{1}{2}\frac{1}{x^{3/2}},\qquad \beta_{yy}=0.$$

So, equation (1.5.21) reduces to the canonical form

$$x\left(u_{\alpha\alpha}\alpha_x^2+2u_{\alpha\beta}\alpha_x\beta_x+u_{\beta\beta}\beta_x^2+u_\alpha\alpha_{xx}+u_\beta\beta_{xx}\right)$$
$$+\left(u_{\alpha\alpha}\alpha_y^2+2u_{\alpha\beta}\alpha_y\beta_y+u_{\beta\beta}\beta_y^2+u_\alpha\alpha_{yy}+u_\beta\beta_{yy}\right)=\left(\frac{\beta}{2}\right)^4,$$

or

$$u_{\alpha\alpha}+u_{\beta\beta}-\frac{1}{2}\frac{1}{\sqrt{x}}u_\beta=\left(\frac{\beta}{2}\right)^4$$
$$u_{\alpha\alpha}+u_{\beta\beta}=\frac{1}{\beta}u_\beta+\left(\frac{\beta}{2}\right)^4. \tag{1.5.27}$$

This is the desired canonical form of the elliptic equation.

Finally, for the parabolic case $(x=0)$, equation (1.5.21) reduces to the canonical form

$$u_{yy} = 0. \tag{1.5.28}$$

In this case, the characteristic determined from $\frac{dx}{dy} = 0$ is $x = 0$. That is, the y-axis is the characteristic curve, and it represents a curve across which a transition from hyperbolic to elliptic form takes place.

Example 1.5.3 (The Cauchy Problem for the Wave Equation). The one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{1.5.29}$$

is a special case of (1.5.1) with $A = -c^2$, $B = 0$, and $C = 1$. Hence, $B^2 - 4AC = 4c^2 > 0$, and therefore, the equation is hyperbolic, as mentioned before. According to (1.5.10ab), the equations of characteristics are

$$\frac{dt}{dx} = \pm\frac{1}{c}, \tag{1.5.30}$$

or equivalently,

$$\xi = x - ct = \text{const.}, \qquad \eta = x + ct = \text{const.} \tag{1.5.31ab}$$

This shows that the characteristics are straight lines in the (x,t)-plane. The former represents a family of lines with positive slopes, and the latter a family of lines with negative slopes in the (x,t)-plane. In terms of new coordinates ξ and η, we obtain

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}, \qquad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$

so that the wave equation (1.5.29) becomes

$$-4c^2 u_{\xi\eta} = 0. \tag{1.5.32}$$

Since $c \neq 0$, $u_{\xi\eta} = 0$ which can be integrated twice to obtain the solution

$$u(\xi,\eta) = \phi(\xi) + \psi(\eta), \tag{1.5.33}$$

where ϕ and ψ are arbitrary functions. Thus, in terms of the original variables, we obtain

$$u(x,t) = \phi(x - ct) + \psi(x + ct). \tag{1.5.34}$$

This represents the general solution provided ϕ and ψ are arbitrary but twice differentiable functions. The first term $\phi(x - ct)$ represents a wave (or disturbance) traveling to the right with constant speed c. Similarly, $\psi(x + ct)$ represents a wave moving to the left with constant speed c. Thus, the general solution $u(x,t)$ is a linear superposition of two such waves.

The typical *initial-value problem* for the wave equation (1.5.29) is the *Cauchy problem* of an infinite string with initial data

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \tag{1.5.35ab}$$

where $f(x)$ and $g(x)$ are given functions representing the initial displacement and initial velocity, respectively. The conditions (1.5.35ab) imply that

$$\phi(x) + \psi(x) = f(x), \tag{1.5.36}$$
$$-c\phi'(x) + c\psi'(x) = g(x), \tag{1.5.37}$$

where the prime denotes the derivative with respect to the argument. Integrating equation (1.5.37) gives

$$-c\phi(x) + c\psi(x) = \int_{x_0}^{x} g(\tau)\,d\tau, \tag{1.5.38}$$

where x_0 is an arbitrary constant. Equations (1.5.36) and (1.5.38) now yield

$$\phi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\tau)\,d\tau,$$
$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\tau)\,d\tau.$$

Obviously, (1.5.34) gives the so-called *d'Alembert solution* of the Cauchy problem as

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau. \tag{1.5.39}$$

It can be verified by direct substitution that $u(x,t)$ satisfies equation (1.5.29) provided f is twice differentiable and g is differentiable. Further, the d'Alembert solution (1.5.39) can be used to show that this problem is *well posed*. The solution (1.5.39) consists of terms involving $f(x \pm ct)$ and the term involving the integral of g. Both terms combined together suggest that the value of the solution at position x and time t depends only on the initial values of $f(x)$ at points $x \pm ct$ and the value of the integral of g between these points. The interval $(x-ct, x+ct)$ is called the *domain of dependence* of (x,t). The terms involving $f(x \pm ct)$ in (1.5.39) show that waves are propagated along the characteristics with constant velocity c.

In particular, if $g(x) = 0$, the solution is represented by the first two terms in (1.5.39), that is,

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big]. \tag{1.5.40}$$

Physically, this solution shows that the initial data is split into two equal waves, similar in shape to the initial displacement, but of *half* the amplitude.

These waves propagate in the opposite direction with the same constant speed c as shown in Figure 1.2.

To investigate the physical significance of the d'Alembert solution, it is convenient to rewrite the solution in the form

$$\begin{aligned} u(x,t) &= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_{0}^{x-ct} g(\tau)\,d\tau + \frac{1}{2}f(x+ct) \\ &\quad + \frac{1}{2c}\int_{0}^{x+ct} g(\tau)\,d\tau \qquad (1.5.41) \\ &= \Phi(x-ct) + \Psi(x+ct), \qquad (1.5.42) \end{aligned}$$

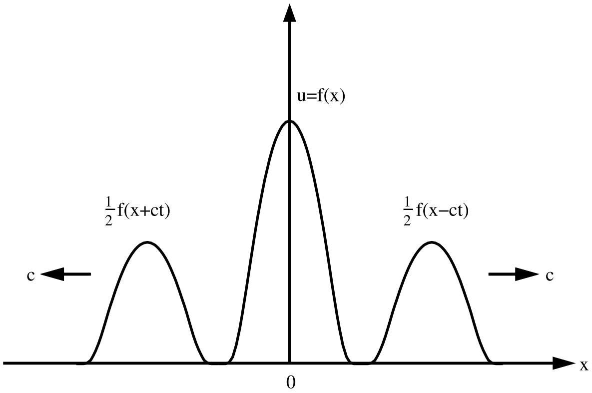

Fig. 1.2 Splitting of initial data into equal waves.

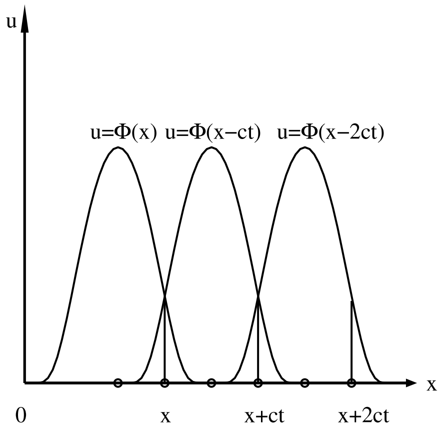

Fig. 1.3 Graphical representation of solution

where

$$\Phi(\xi) = \frac{1}{2}f(\xi) - \frac{1}{2c}\int_0^{\xi} g(\tau)\,d\tau, \tag{1.5.43a}$$

$$\Psi(\eta) = \frac{1}{2}f(\eta) + \frac{1}{2c}\int_0^{\eta} g(\tau)\,d\tau. \tag{1.5.43b}$$

Physically, $\Phi(x - ct)$ represents a progressive wave propagating in the positive x-direction with constant speed c without change of shape as shown in Figure 1.3. Similarly, $\Psi(x + ct)$ also represents a progressive wave traveling in the negative x-direction with the same speed without change of shape.

A more general form of the wave equation is

$$u_{tt} - a^2(x)u_{xx} = 0, \tag{1.5.44}$$

where a is a function of x only. The characteristic coordinates are now given by

$$\xi = t - \int^x \frac{d\tau}{a(\tau)}, \qquad \eta = t + \int^x \frac{d\tau}{a(\tau)}. \tag{1.5.45ab}$$

Thus,

$$\begin{aligned}
u_x &= -\frac{1}{a}u_\xi + \frac{1}{a}u_\eta, \qquad u_t = u_\xi + u_\eta, \\
u_{xx} &= \frac{1}{a^2}(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) - (u_\eta - u_\xi)\frac{a'(x)}{a^2}, \\
u_{tt} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.
\end{aligned}$$

Consequently, equation (1.5.44) reduces to

$$4u_{\xi\eta} + a'(x)(u_\eta - u_\xi) = 0. \tag{1.5.46}$$

In order to express a' in terms of ξ and η, we observe that

$$\eta - \xi = 2\int^x \frac{d\tau}{a(\tau)}, \tag{1.5.47}$$

so that x is a function of $(\eta - \xi)$. Thus, $a'(x)$ will be some function of $(\eta - \xi)$.

In particular, if $a(x) = Ax^n$, where A is a constant, so that $a'(x) = nAx^{n-1}$, and when $n \neq 1$, result (1.5.47) gives

$$\eta - \xi = -\frac{2}{A}\frac{1}{(n-1)}\frac{1}{x^{n-1}} \tag{1.5.48}$$

so that

$$a'(x) = -\frac{2n}{(n-1)} \cdot \frac{1}{\eta - \xi}.$$

Thus, equation (1.5.46) reduces to the form

$$4u_{\xi\eta} - \frac{2n}{(n-1)}\frac{1}{(\eta - \xi)}(u_\eta - u_\xi) = 0.$$

Finally, we find that

$$u_{\xi\eta} = \frac{n}{2(n-1)}\frac{1}{(\eta - \xi)}(u_\eta - u_\xi). \tag{1.5.49}$$

When $n = 1$, $a(x) = Ax$, and $a'(x) = A$, substituting $\xi = \frac{\alpha}{A}$ and $\eta = \frac{\beta}{A}$ can be used to reduce equation (1.5.46) to

$$u_{\alpha\beta} = \frac{1}{4}(u_\alpha - u_\beta). \tag{1.5.50}$$

Equation (1.5.49) is called the *Euler–Darboux equation* which has the hyperbolic form

$$u_{xy} = \frac{m}{x - y}(u_x - u_y), \tag{1.5.51}$$

where m is a positive integer.

We next note that

$$\frac{\partial^2}{\partial x \partial y}\big[(x-y)u\big] = \frac{\partial}{\partial x}\left[(x-y)\frac{\partial u}{\partial y} - u\right] = (x-y)u_{xy} + (u_y - u_x). \quad (1.5.52)$$

When $m = 1$, equation (1.5.51) becomes

$$(x-y)u_{xy} = u_x - u_y$$

so that (1.5.52) reduces to

$$\frac{\partial^2}{\partial x \partial y}\big[(x-y)u\big] = 0. \quad (1.5.53)$$

This shows that the solution of (1.5.53) is $(x-y)u = \phi(x) + \psi(y)$. Hence, the solution of (1.5.51) with $m = 1$ is

$$u(x,y) = \frac{\phi(x) + \psi(y)}{x-y}, \quad (1.5.54)$$

where ϕ and ψ are arbitrary functions.

We multiply (1.5.51) by $(x-y)$, and apply the derivative $\frac{\partial^2}{\partial x \partial y}$, so that the result is, due to (1.5.52),

$$(x-y)\frac{\partial^2}{\partial x \partial y}(u_{xy}) + \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)(u_{xy}) = m\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u_{xy}.$$

Or

$$(x-y)\frac{\partial^2}{\partial x \partial y}(u_{xy}) = (m+1)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(u_{xy}). \quad (1.5.55)$$

Hence, if u is a solution of (1.5.51), then u_{xy} is a solution of (1.5.51) with m replaced by $m+1$. When $m = 1$, the solution is given by (1.5.54), and hence, the solution of (1.5.51) takes the form

$$u(x,y) = \frac{\partial^{2m-2}}{\partial x^{m-1} \partial y^{m-1}}\left[\frac{\phi(x) + \psi(y)}{x-y}\right], \quad (1.5.56)$$

where ϕ and ψ are arbitrary functions.

1.6 The Method of Separation of Variables

This method combined with the principle of superposition is widely used to solve initial boundary-value problems involving linear partial differential equations. Usually, the dependent variable $u(x,y)$ is expressed in the separable form as $u(x,y) = X(x)Y(y)$ where X and Y are functions of x and y, respectively. In many cases, the partial differential equation reduces to two ordinary differential equations for X and Y. A similar treatment can be applied to equations in three or more independent variables. However, the question of separability of a partial differential equation into two or more ordinary differential equations is by no means a trivial one. In spite of this question, the method is widely used in finding solutions of a large class of initial boundary-value problems. This method of solution is known as the *Fourier method* (or *the method of eigenfunction expansion*). Thus, the procedure outlined above leads to the important ideas of eigenvalues, eigenfunctions, and orthogonality, all of which are very general and powerful for dealing with linear problems. The following examples illustrate the general nature of this method.

Example 1.6.1 (Transverse Vibration of a String). We consider the one-dimensional linear wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \ell,\ t > 0, \tag{1.6.1}$$

where $c^2 = T^*/\rho$, T^* is a constant tension, and ρ is the constant line density of the string. The initial and boundary conditions are

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le \ell, \tag{1.6.2ab}$$
$$u(0,t) = 0 = u(\ell,t), \quad t > 0, \tag{1.6.3ab}$$

where f and g are the initial displacement and initial velocity, respectively.

According to the method of separation of variables, we assume a separable solution of the form

$$u(x,t) = X(x)T(t) \neq 0, \tag{1.6.4}$$

where X is a function of x only, and T is a function of t only.

Substituting this solution into equation (1.6.1) yields

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2}. \tag{1.6.5}$$

Since the left side of this equation is a function of x only and the right-hand side is a function of t only, it follows that (1.6.5) can be true only if both sides are equal to the same constant value. We then write

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = \lambda, \tag{1.6.6}$$

where λ is an arbitrary separation constant. Thus, this leads to the pair of ordinary differential equations

$$\frac{d^2X}{dx^2} = \lambda X, \tag{1.6.7a}$$
$$\frac{d^2T}{dt^2} = \lambda c^2 T. \tag{1.6.7b}$$

We solve this pair of equations by using the boundary conditions which are obtained from (1.6.3ab) as

$$u(0,t) = X(0)T(t) = 0 \quad \text{for } t > 0, \tag{1.6.8}$$
$$u(\ell,t) = X(\ell)T(t) = 0 \quad \text{for } t > 0. \tag{1.6.9}$$

Hence, we take $T(t) \neq 0$ to obtain

$$X(0) = 0 = X(\ell). \tag{1.6.10ab}$$

There are three possible cases: (i) $\lambda > 0$, (ii) $\lambda = 0$, (iii) $\lambda < 0$.

For case (i), $\lambda = \alpha^2 > 0$. The solution of (1.6.7a) is

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}, \tag{1.6.11}$$

which together with (1.6.10ab) leads to $A = B = 0$. This leads to a trivial solution $u(x,t) = 0$.

For case (ii), $\lambda = 0$. In this case, the solution of (1.6.7a) is

$$X(x) = Ax + B. \tag{1.6.12}$$

Then. we use (1.6.10ab) to obtain $A = B = 0$. This also leads to the trivial solution $u(x,t) = 0$.

For case (iii), $\lambda < 0$, and hence, we write $\lambda = -\alpha^2$ so that the solution of equation (1.6.7a) gives

$$X = A\cos\alpha x + B\sin\alpha x, \tag{1.6.13}$$

whence, using (1.6.10ab), we derive the nontrivial solution

$$X(x) = B\sin\alpha x, \tag{1.6.14}$$

where B is an arbitrary nonzero constant. Clearly, since $B \neq 0$ and $X(\ell) = 0$,

$$\sin\alpha\ell = 0, \tag{1.6.15}$$

which gives the solution for the parameter α

$$\alpha = \alpha_n = \left(\frac{n\pi}{\ell}\right), \quad n = 1, 2, 3, \ldots. \tag{1.6.16}$$

Note that $n = 0$ $(\alpha = 0)$ leads to the trivial solution $u(x,t) = 0$, and hence, the case $n = 0$ is to be excluded.

Evidently, we see from (1.6.16) that there exists an infinite set of discrete values of α for which the problem has a nontrivial solution. These values α_n are called the *eigenvalues*, and the corresponding solutions are

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{1.6.17}$$

We next solve (1.6.7a) with $\lambda = -\alpha_n^2$ to find the solution for $T_n(t)$ as

$$T_n(t) = C_n\cos(\alpha_n ct) + D_n\sin(\alpha_n ct), \tag{1.6.18}$$

where C_n and D_n are constants of integration. Combining (1.6.17) with (1.6.18) yields the solution from (1.6.4) as

$$u_n(x,t) = \left[a_n \cos\left(\frac{n\pi ct}{\ell}\right) + b_n \sin\left(\frac{n\pi ct}{\ell}\right)\right] \sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.19}$$

where $a_n = C_n B_n$, $b_n = B_n D_n$ are new arbitrary constants and $n = 1, 2, 3, \ldots$. These solutions $u_n(x,t)$, corresponding to eigenvalues $\alpha_n = (\frac{n\pi}{\ell})$, are called the *eigenfunctions*. Finally, since the problem is linear, the most general solution is obtained by the principle of superposition in the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi ct}{\ell} + b_n \sin\frac{n\pi ct}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.20}$$

provided the series converges and it is twice continuously differentiable with respect to x and t. The arbitrary coefficients a_n and b_n are determined from the initial conditions (1.6.2ab) which give

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.21}$$

$$u_t(x,0) = g(x) = \left(\frac{\pi c}{\ell}\right) \sum_{n=1}^{\infty} n b_n \sin\left(\frac{n\pi x}{\ell}\right). \tag{1.6.22}$$

Either by a Fourier series method (see Appendix B) or by direct multiplication of (1.6.21) and (1.6.22) by $\sin(\frac{m\pi x}{\ell})$ and integrating from 0 to ℓ, we can find a_n and b_n as

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \qquad b_n = \frac{2}{n\pi c} \int_0^{\ell} g(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \tag{1.6.23ab}$$

in which we have used the result

$$\int_0^{\ell} \sin\left(\frac{m\pi x}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) dx = \frac{\ell}{2}\delta_{mn}, \tag{1.6.24}$$

where δ_{mn} are *Kronecker* deltas. Thus, (1.6.20) represents the solution where a_n and b_n are given by (1.6.23ab). Hence, the problem is completely solved.

We examine the physical significance of the solution (1.6.19) in the context of the free vibration of a string of length ℓ. The eigenfunctions

$$u_n(x,t) = (a_n \cos\omega_n t + b_n \sin\omega_n t) \sin\left(\frac{n\pi x}{\ell}\right), \quad \left(\omega_n = \frac{n\pi c}{\ell}\right), \tag{1.6.25}$$

are called the nth *normal modes* of vibration or the nth harmonic, and ω_n represents the discrete spectrum of *circular (or radian) frequency* or $\nu_n = \frac{\omega_n}{2\pi} = \frac{nc}{2\ell}$, which are called the *angular frequencies*. The first harmonic $(n = 1)$ is called the *fundamental*

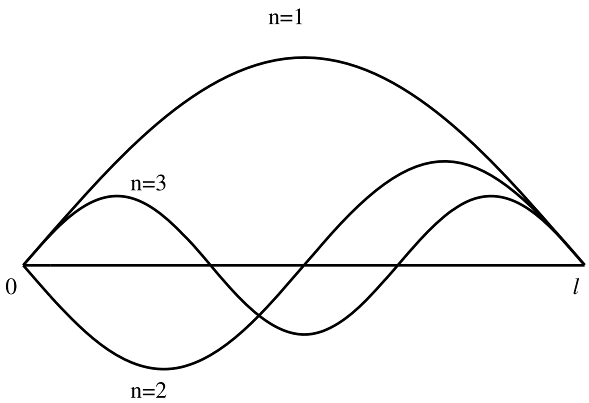

Fig. 1.4 Several modes of vibration in a string.

harmonic and all other harmonics $(n > 1)$ are called *overtones*. The frequency of the fundamental mode is given by

$$\omega_1 = \frac{\pi c}{\ell}, \qquad \nu_1 = \frac{1}{2\ell}\sqrt{\frac{T^*}{\rho}}. \tag{1.6.26ab}$$

Result (1.6.26ab) is considered the fundamental law (or *Mersenne law*) of a stringed musical instrument. The angular frequency of the fundamental mode of transverse vibration of a string varies as the square root of the tension, inversely as the length, and inversely as the square root of the density. The period of the fundamental mode is $T_1 = \frac{2\pi}{\omega_1} = \frac{2\ell}{c}$, which is called the *fundamental period*. Finally, the solution (1.6.20) describes the motion of a plucked string as a superposition of all normal modes of vibration with frequencies which are all integral multiples $(\omega_n = n\omega_1$ or $\nu_n = n\nu_1)$ of the fundamental frequency. This is the main reason for the fact that stringed instruments produce more sweet musical sounds (or tones) than drum instruments.

In order to describe waves produced in the plucked string with zero initial velocity $(u_t(x,0) = 0)$, we write the solution (1.6.25) in the form

$$u_n(x,t) = a_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell}\right), \quad n = 1,2,3,\ldots. \tag{1.6.27}$$

These solutions are called *standing waves* with amplitude $a_n \sin(\frac{n\pi x}{\ell})$, which vanishes at

$$x = 0, \frac{\ell}{n}, \frac{2\ell}{n}, \ldots, \ell.$$

These are called the *nodes* of the *n*th harmonic. The string displays n loops separated by the nodes as shown in Figure 1.4.

It follows from elementary trigonometry that (1.6.27) takes the form

$$u_n(x,t) = \frac{1}{2}a_n\left[\sin\frac{n\pi}{\ell}(x - ct) + \sin\frac{n\pi}{\ell}(x + ct)\right]. \tag{1.6.28}$$

This shows that a standing wave is expressed as a sum of two progressive waves of equal amplitude traveling in opposite directions. This result is in agreement with the d'Alembert solution.

Finally, we can rewrite the solution (1.6.19) of the *n*th normal modes in the form

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell} - \varepsilon_n\right), \tag{1.6.29}$$

where $c_n = (a_n^2 + b_n^2)^{\frac{1}{2}}$ and $\tan\varepsilon_n = (\frac{b_n}{a_n})$.

This solution represents transverse vibrations of the string at any point x and at any time t with amplitude $c_n \sin(\frac{n\pi x}{\ell})$ and circular frequency $\omega_n = \frac{n\pi c}{\ell}$. This form of the solution enables us to calculate the kinetic and potential energies of the transverse vibrations. The total kinetic energy (K.E.) is obtained by integrating with respect to x from 0 to ℓ, that is,

$$K_n = K.E. = \int_0^\ell \frac{1}{2}\rho\left(\frac{\partial u_n}{\partial t}\right)^2 dx, \tag{1.6.30}$$

where ρ is the line density of the string. Similarly, the total potential energy (P.E.) is given by

$$V_n = P.E. = \frac{1}{2}T^*\int_0^\ell \left(\frac{\partial u_n}{\partial x}\right)^2 dx. \tag{1.6.31}$$

Substituting (1.6.29) in (1.6.30) and (1.6.31) gives

$$\begin{aligned} K_n &= \frac{1}{2}\rho\left(\frac{n\pi c}{\ell}c_n\right)^2 \sin^2\left(\frac{n\pi ct}{\ell} - \varepsilon_n\right)\int_0^\ell \sin^2\left(\frac{n\pi x}{\ell}\right) dx \\ &= \frac{\rho c^2\pi^2}{4\ell}(nc_n)^2 \sin^2\left(\frac{n\pi ct}{\ell} - \varepsilon_n\right) = \frac{1}{4}\rho\ell\omega_n^2 c_n^2 \sin^2(\omega_n t - \varepsilon_n), \end{aligned} \tag{1.6.32}$$

where $\omega_n = \frac{n\pi c}{\ell}$.

Similarly,

$$\begin{aligned} V_n &= \frac{1}{2}T^*\left(\frac{n\pi c_n}{\ell}\right)^2 \cos^2\left(\frac{n\pi ct}{\ell} - \varepsilon_n\right)\int_0^\ell \cos^2\left(\frac{n\pi x}{\ell}\right) dx \\ &= \frac{\pi^2 T^*}{4\ell}(nc_n)^2 \cos^2\left(\frac{n\pi ct}{\ell} - \varepsilon_n\right) = \frac{1}{4}\rho\ell\omega_n^2 c_n^2 \cos^2(\omega_n t - \varepsilon_n). \end{aligned} \tag{1.6.33}$$

Thus, the total energy of the *n*th normal modes of vibrations is given by

$$E_n = K_n + V_n = \frac{1}{4}\rho\ell(\omega_n c_n)^2 = \text{const.} \tag{1.6.34}$$

For a given string oscillating in a normal mode, the total energy is proportional to the square of the circular frequency and to the square of the amplitude.

Finally, the total energy of the system is given by

$$E = \sum_{n=1}^{\infty} E_n = \frac{1}{4}\rho\ell \sum_{n=1}^{\infty} \omega_n^2 c_n^2, \tag{1.6.35}$$

which is constant because $E_n = \text{const.}$

Example 1.6.2 (One-Dimensional Diffusion Equation). The temperature distribution $u(x,t)$ in a homogeneous rod of length ℓ satisfies the diffusion equation

$$u_t = \kappa u_{xx}, \qquad 0 < x < \ell,\ t > 0, \tag{1.6.36}$$

with the boundary and initial conditions

$$u(0,t) = 0 = u(\ell,t), \qquad t \geq 0, \tag{1.6.37ab}$$

$$u(x,0) = f(x), \qquad 0 \leq x \leq \ell, \tag{1.6.38}$$

where κ is a diffusivity constant.

We assume a separable solution of (1.6.36) in the form

$$u(x,t) = X(x)T(t) \neq 0. \tag{1.6.39}$$

Substituting (1.6.39) in (1.6.36) gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\kappa T}\frac{dT}{dt}. \tag{1.6.40}$$

Since the left-hand side depends only on x and the right-hand side is a function of time t only, result (1.6.40) can be true only if both sides are equal to the same constant λ. Thus, we obtain two ordinary differential equations

$$\frac{d^2X}{dx^2} - \lambda X = 0, \qquad \frac{dT}{dt} - \lambda\kappa T = 0. \tag{1.6.41ab}$$

For $\lambda \geq 0$, the only solution of the form (1.6.39) consistent with the given boundary conditions is $u(x,t) \equiv 0$. Hence, for negative $\lambda = -\alpha^2$,

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0, \qquad \frac{dT}{dt} + \kappa\alpha^2 T = 0, \tag{1.6.42ab}$$

which admit solutions as

$$X(x) = A\cos\alpha x + B\sin\alpha x \tag{1.6.43}$$

and

$$T(t) = C\exp\left(-\kappa\alpha^2 t\right), \tag{1.6.44}$$

where A, B, and C are constants of integration.

The boundary conditions for $X(x)$ are

$$X(0) = 0 = X(\ell), \tag{1.6.45}$$

which are used to find A and B in solution (1.6.43). It turns out that $A = 0$ and $B \neq 0$. Hence,

$$\sin \alpha\ell = 0, \tag{1.6.46}$$

which gives the eigenvalues

$$\alpha = \alpha_n = \frac{n\pi}{\ell}, \quad n = 1, 2, 3, \ldots. \tag{1.6.47}$$

The value $n = 0$ is excluded because it leads to a trivial solution. Thus, the eigenfunctions are given by

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.48}$$

where B_n are nonzero constants.

With $\alpha = \alpha_n = \frac{n\pi}{\ell}$, we combine (1.6.44) with (1.6.48) to obtain the solution for $u_n(x,t)$ as

$$u_n(x,t) = a_n \exp\left[-\left(\frac{n\pi}{\ell}\right)^2 \kappa t\right] \sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.49}$$

where $a_n = B_n C_n$ is a new constant. Thus, (1.6.47) and (1.6.49) constitute an infinite set of eigenvalues and eigenfunctions. Thus, the most general solution is obtained by the principle of superposition in the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left[-\left(\frac{n\pi}{\ell}\right)^2 \kappa t\right] \sin\left(\frac{n\pi x}{\ell}\right). \tag{1.6.50}$$

Now, the initial condition implies that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right), \tag{1.6.51}$$

which determines a_n, in view of (1.6.24), as

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx. \tag{1.6.52}$$

Thus, the final form of the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{\ell} \int_0^{\ell} f(x') \sin\left(\frac{n\pi x'}{\ell}\right) dx'\right] \exp\left[-\left(\frac{n\pi}{\ell}\right)^2 \kappa t\right] \sin\left(\frac{n\pi x}{\ell}\right). \tag{1.6.53}$$

It follows from the series solution (1.6.53) that the series satisfies the given boundary and initial conditions. It also satisfies equation (1.6.36) because the series is convergent for all $x(0 \leq x \leq \ell)$ and $t \geq 0$ and can be differentiated term by term. Physically, the temperature distribution decays exponentially with time t. This shows a striking contrast to the wave equation, whose solution oscillates in time t. The time scale of decay for the nth mode is $T_d \sim \frac{1}{\kappa}(\frac{\ell}{n\pi})^2$ which is directly proportional to ℓ^2 and inversely proportional to the thermal diffusivity.

The method of separation of variables is applicable to the wave equation and the diffusion equation, and also to problems involving Laplace's equation and other equations in two or three dimensions with a wide variety of initial and boundary conditions. We consider the following examples.

Example 1.6.3 (Two-Dimensional Diffusion Equation). We consider

$$u_t = \kappa(u_{xx} + u_{yy}), \quad 0 < x < a,\ 0 < y < b,\ t > 0, \tag{1.6.54}$$

$$u(x, y, t) = f(x, y) \quad \text{at } t = 0, \tag{1.6.55}$$

$$u(x, y, t) = 0 \quad \text{on } \partial D, \tag{1.6.56}$$

where ∂D is the boundary of the rectangle defined by $0 \leq x \leq a$, $0 \leq y \leq b$.

The method here is precisely the same as in the previous examples except that we seek a solution of (1.6.54) in the form

$$u(x, y, z) = S(x, y)T(t) \neq 0, \tag{1.6.57}$$

so that S and T satisfy the equations

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \lambda S = 0, \tag{1.6.58}$$

$$\frac{\partial T}{\partial t} + \kappa\lambda T = 0. \tag{1.6.59}$$

For $\lambda \leq 0$, the separable solution (1.6.57) with the given boundary data leads only to a trivial solution $u(x, y, t) \equiv 0$. Hence, for positive λ, we solve (1.6.58), (1.6.59) subject to the given boundary and initial conditions. Equation (1.6.58) is an elliptic equation, and here we seek a solution $S(x, y)$ which satisfies the boundary conditions

$$S(0, y) = 0 = S(a, y) \quad \text{for } 0 \leq y \leq b, \tag{1.6.60}$$

$$S(x, 0) = 0 = S(x, b) \quad \text{for } 0 \leq x \leq a. \tag{1.6.61}$$

We also seek a separable solution of (1.6.58) in the form

$$S(x, y) = X(x)Y(y) \neq 0 \tag{1.6.62}$$

and find that $X(x)$ and $Y(y)$ satisfy the equation

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0,$$

that is,

$$\frac{X''}{X} = -\mu = -\left(\frac{Y''}{Y} + \lambda\right). \tag{1.6.63}$$

Or

$$X'' + \mu X = 0, \tag{1.6.64a}$$
$$Y'' + (\lambda - \mu)Y = 0. \tag{1.6.64b}$$

These equations have to be solved with the boundary conditions

$$X(0) = 0 = X(a), \tag{1.6.65a}$$
$$Y(0) = 0 = Y(b). \tag{1.6.65b}$$

The eigenvalue problem (1.6.64a), (1.6.64b) with (1.6.65a), (1.6.65b) gives the eigenvalues

$$\mu_m = \left(\frac{m\pi}{a}\right)^2, \tag{1.6.66}$$

and the corresponding eigenfunctions

$$X_m(x) = A_m \sin\left(\frac{m\pi x}{a}\right), \tag{1.6.67}$$

when $m = 1, 2, 3, \ldots$. Thus, equation (1.6.64b) becomes

$$Y'' + (\lambda - \mu_m)Y = 0, \tag{1.6.68}$$

which has to be solved with (1.6.65b). This is another eigenvalue problem similar to that already considered and leads to the eigenvalues

$$\lambda_n - \mu_m = \left(\frac{n\pi}{b}\right)^2 \tag{1.6.69}$$

and the corresponding eigenfunctions

$$Y_n(y) = B_n \sin\left(\frac{n\pi y}{b}\right), \tag{1.6.70}$$

where $n = 1, 2, 3, \ldots$. In other words, the solution of equation (1.6.58) becomes

$$S_{mn}(x, y) = X_m(x)Y_n(y) = A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \tag{1.6.71}$$

where $A_{mn} = A_m B_n$ are constants together with the eigenvalues

$$\lambda_{mn} = \mu_m + \left(\frac{n\pi}{b}\right)^2 = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^2, \tag{1.6.72}$$

where $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$.

With λ_{mn} as eigenvalues, we solve (1.6.59) to obtain

$$T_{mn}(t) = B_{mn}\exp(-\lambda_{mn}\kappa t), \tag{1.6.73}$$

where B_{mn} are integrating constants.

Finally, the solution (1.6.57) can be expressed as a double series

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\exp(-\lambda_{mn}\kappa t), \tag{1.6.74}$$

where a_{mn} are constants to be determined from the initial condition so that

$$f(x,y) = u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right). \tag{1.6.75}$$

To find constants a_{mn}, we multiply (1.6.75) by $\sin(\frac{r\pi x}{a})$ and integrate the result with respect to x from 0 to a with fixed y, so that

$$\frac{a}{2}\sum_{n=1}^{\infty} a_{mn}\sin\left(\frac{n\pi y}{b}\right) = \int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)dx. \tag{1.6.76}$$

The right-hand side is a function of y and is set equal to $g(y)$, so that

$$\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)dx = g(y). \tag{1.6.77}$$

Then, the coefficients a_{mn} (m fixed) in (1.6.76) are found by multiplying it by $\sin(\frac{n\pi y}{b})$ and integrating with respect to y from 0 to b, so that

$$\left(\frac{ab}{4}\right)a_{mn} = \int_0^b g(y)\sin\left(\frac{n\pi y}{b}\right)dy, \tag{1.6.78}$$

whence

$$a_{mn} = \left(\frac{4}{ab}\right)\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy. \tag{1.6.79}$$

Thus, the solution of the problem is given by (1.6.74) where a_{mn} is represented by (1.6.79). The method of construction of the solution shows that the initial and boundary conditions are satisfied by the solution. Moreover, the uniform convergence of the double series justifies differentiation of the series, and this, in turn, permits us to verify the solution by direct substitution in the original diffusion equation (1.6.54).

Example 1.6.4 (Dirichlet's Problem for a Circle). We consider the Laplace equation in cylindrical polar coordinates (r,θ,z) as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < a,\ 0 \le \theta < 2\pi, \tag{1.6.80}$$

with the boundary condition

$$u(a, \theta) = f(\theta) \quad \text{for all } \theta. \tag{1.6.81}$$

According to the method of separation of variables, we seek a solution in the form

$$u(r, \theta) = R(r)\Theta(\theta) \neq 0. \tag{1.6.82}$$

Substituting this solution in equation (1.6.80) gives

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

Hence,

$$r^2 R'' + rR' - \lambda R = 0 \quad \text{and} \tag{1.6.83a}$$

$$\Theta'' + \lambda\Theta = 0. \tag{1.6.83b}$$

For $\Theta(\theta)$, we naturally require periodic boundary conditions

$$\Theta(\theta + 2\pi) = \Theta(\theta) \quad \text{for } -\infty < z < \infty. \tag{1.6.84}$$

Due to the periodicity condition, for $\lambda < 0$, the solution (1.6.82) leads to a trivial solution. So, there are two cases: (i) $\lambda = 0$ and (ii) $\lambda > 0$.

For case (i), we have the solution

$$u(r, \theta) = (A + B \log r)(C\theta + D). \tag{1.6.85}$$

Since $\log r$ is singular at $r = 0$, hence, $B = 0$. For u to be periodic with period 2π, $C = 0$. Hence, the solution u must be constant for $\lambda = 0$.

For $\lambda > 0$, the solution of equation (1.6.83b) is

$$\Theta(\theta) = A \cos \sqrt{\lambda}\theta + B \sin \sqrt{\lambda}\theta. \tag{1.6.86}$$

Since $\Theta(\theta)$ is periodic with period 2π, $\sqrt{\lambda}$ must be an integer n so that $\lambda = n^2$, $n = 1, 2, 3, \ldots$. Thus, solution (1.6.86) becomes

$$\Theta(\theta) = A \cos n\theta + B \sin n\theta. \tag{1.6.87}$$

The equation (1.6.83a) is the Euler equation with $\lambda = n^2$. It gives solutions of the form $R(r) = r^\alpha \neq 0$ so that (1.6.83a) gives

$$\left[\alpha(\alpha - 1) + \alpha - n^2\right] r^\alpha = 0,$$

whence $\alpha = \pm n$. Thus, the solution for $R(r)$ is given by

$$R(r) = Cr^n + Dr^{-n}. \tag{1.6.88}$$

Since $R(r) \to \infty$ as $r \to 0$ because of the term r^{-n}, we get $D = 0$. Thus, the solution (1.6.82) reduces to

$$u(r,\theta) = Cr^n(A\cos n\theta + B\sin n\theta). \tag{1.6.89}$$

By the superposition principle, the solution of the Laplace equation within a circular region including the origin $r = 0$ is

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta), \tag{1.6.90}$$

where a_0, a_n, and b_n are constants to be determined from the boundary conditions, and the first term $\frac{1}{2}a_0$ represents the constant solution for $\lambda = 0$ $(n = 0)$.

Finally, using the boundary condition (1.6.81), we derive

$$f(\theta) = u(a,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a^n(a_n\cos n\theta + b_n\sin n\theta). \tag{1.6.91}$$

This is exactly the Fourier series representation for $f(\theta)$, and hence, the coefficients are given by

$$a_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\phi)\cos n\phi\,d\phi, \quad n = 0,1,2,3,\ldots,$$
$$b_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\phi)\sin n\phi\,d\phi, \quad n = 1,2,3,\ldots.$$

Substituting the values for a_n and b_n into (1.6.91) yields the solution

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi \\ &\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \int_0^{2\pi}(\cos n\theta\cos n\phi + \sin n\theta\sin n\phi)f(\phi)\,d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right\}d\phi, \end{aligned} \tag{1.6.92}$$

where the term inside the set of braces in the above integral can be summed by writing it as a geometric series, that is,

$$\begin{aligned} &1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \exp\{in(\theta-\phi)\} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \exp\{-in(\theta-\phi)\} \\ &= 1 + \frac{r\exp\{i(\theta-\phi)\}}{a - r\exp\{i(\theta-\phi)\}} + \frac{r\exp\{-i(\theta-\phi)\}}{a - r\exp\{-i(\theta-\phi)\}} \\ &= \frac{(a^2-r^2)}{a^2 - 2ar\cos(\theta-\phi) + r^2}. \end{aligned}$$

Thus, the final form of the solution is

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2) f(\phi)\, d\phi}{a^2 - 2ar\cos(\theta - \phi) + r^2}. \tag{1.6.93}$$

This formula is known as *Poisson's integral formula* representing the solution of the Laplace equation within the circle of radius a in terms of values prescribed on the circle. It has several important consequences. First, we set $r = 0$ and $\theta = 0$ in (1.6.93) to obtain

$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi. \tag{1.6.94}$$

This states that the value of u at the center is equal to the mean value of u on the boundary of the circle. This is called the *mean value property*.

We rewrite (1.6.93) in the form

$$u(r,\theta) = \int_0^{2\pi} P(r, \theta - \phi) f(\phi)\, d\phi, \tag{1.6.95}$$

where $P(r, \theta - \phi)$ is called the *Poisson kernel* given by

$$P(r, \theta - \phi) = \frac{1}{2\pi} \cdot \frac{(a^2 - r^2)}{a^2 - 2ar\cos(\theta - \phi) + r^2}, \tag{1.6.96}$$

which is zero for $r = a$ but $\theta \neq \phi$. Further,

$$f(\theta) = \lim_{r \to a-} u(r,\theta) = \int_0^{2\pi} \Big[\lim_{r \to a-} P(r, \theta - \phi)\Big] f(\phi)\, d\phi,$$

which implies that

$$\lim_{r \to a-} P(r, \theta - \phi) = \delta(\theta - \phi), \tag{1.6.97}$$

where $\delta(x)$ is the Dirac delta function.

1.7 Fourier Transforms and Initial Boundary-Value Problems

The Fourier transform of $u(x,t)$ with respect to $x \in R$ is denoted by $\mathcal{F}\{u(x,t)\} = U(k,t)$ and is defined by the integral

$$\mathcal{F}\big\{u(x,t)\big\} = U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,t)\, dx, \tag{1.7.1}$$

where k is real and is called the *transform variable*. The *inverse Fourier transform*, denoted by $\mathcal{F}^{-1}\{U(k,t)\} = u(x,t)$, is defined by

$$\mathcal{F}^{-1}\big\{U(k,t)\big\} = u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} U(k,t)\, dk. \tag{1.7.2}$$

The basic properties of the Fourier transforms including the convolution are discussed in Appendix B.

Example 1.7.1.

$$\text{(a)}\quad \mathcal{F}\{\exp(-ax^2)\} = \frac{1}{\sqrt{2a}}\exp\left(-\frac{k^2}{4a}\right), \quad a>0, \tag{1.7.3}$$

$$\text{(b)}\quad \mathcal{F}\{\exp(-a|x|)\} = \sqrt{\frac{2}{\pi}}\cdot\frac{a}{(k^2+a^2)}, \quad a>0. \tag{1.7.4}$$

If $u(x,t)\to 0$, as $|x|\to\infty$, then

$$\mathcal{F}\left\{\frac{\partial u}{\partial x}\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial u}{\partial x}\,dx,$$

which, by integrating by parts, is

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}}\left[e^{-ikx}u(x,t)\right]_{-\infty}^{\infty} + \frac{ik}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}u(x,t)\,dx \\ &= ik\mathcal{F}\{u(x,t)\} = ikU(k,t). \end{aligned} \tag{1.7.5}$$

Similarly, if $u(x,t)$ is continuously n times differentiable and $\frac{\partial^k u}{\partial x^k}\to 0$, as $|x|\to\infty$ for $k=1,2,3,\ldots,(n-1)$, then

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n\mathcal{F}\{u(x,t)\} = (ik)^n U(k,t). \tag{1.7.6}$$

It also follows from the definition (1.7.1) that

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{dU}{dt}, \qquad \mathcal{F}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \frac{d^2U}{dt^2},\ldots, \qquad \mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^nU}{dt^n}. \tag{1.7.7}$$

The definition of the Fourier transform (1.7.1) shows that a sufficient condition for $u(x,t)$ to have a Fourier transform is that $u(x,t)$ is absolutely integrable on $-\infty<x<\infty$. This existence condition is too strong for many practical applications. Many simple functions, such as a constant function, $\sin\omega x$, and $x^n H(x)$, do not have Fourier transforms, even though they occur frequently in applications. The above definition of the Fourier transform has been extended for a more general class of functions to include the above and other functions. We simply state the fact that there is a sense, useful in practical applications, in which the above stated functions and many others do have Fourier transforms. The following are examples of such functions and their Fourier transforms:

$$\mathcal{F}\{H(a-|x|)\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right), \tag{1.7.8}$$

where $H(x)$ is the Heaviside unit step function,

$$\mathcal{F}\{\delta(x-a)\} = \frac{1}{\sqrt{2\pi}}\exp(-iak), \tag{1.7.9}$$

where $\delta(x-a)$ is the Dirac delta function, and

$$\mathcal{F}\{H(x-a)\} = \sqrt{\frac{\pi}{2}}\left[\frac{1}{i\pi k} + \delta(k)\right]\exp(-iak). \quad (1.7.10)$$

Theorem 1.7.1 (Convolution Theorem). *If $\mathcal{F}\{f(x)\} = F(k)$ and $\mathcal{F}\{g(x)\} = G(k)$, then*

$$\mathcal{F}\{f(x) * g(x)\} = F(k)G(k). \quad (1.7.11)$$

Or equivalently,

$$\mathcal{F}^{-1}\{F(k)G(k)\} = f(x) * g(x), \quad (1.7.12)$$

where $f(x) * g(x)$ is called the convolution of two integrable functions $f(x)$ and $g(x)$ and is defined by

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi. \quad (1.7.13)$$

Hence, the result (1.7.12) can also be written as

$$\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi = \int_{-\infty}^{\infty} e^{ikx}F(k)G(k)\,dk. \quad (1.7.14)$$

It can easily be verified that the convolution has the following simple properties (*see Appendix B*)*:*

$$f(x) * g(x) = g(x) * f(x) \quad \textit{(commutative)}, \quad (1.7.15)$$
$$f(x) * \sqrt{2\pi}\delta(x) = f(x) = \sqrt{2\pi}\delta(x) * f(x) \quad \textit{(identity)}. \quad (1.7.16)$$

The Fourier transforms are very useful in solving a wide variety of initial boundary-value problems governed by linear partial differential equations. The following examples of applications illustrate the method of Fourier transforms.

Example 1.7.2 (The Cauchy Problem for the Linear Wave Equation). Obtain the d'Alembert solution of the initial-value problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0, \quad (1.7.17)$$

with the initial data

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad -\infty < x < \infty. \quad (1.7.18ab)$$

Application of the Fourier transform $\mathcal{F}\{u(x,t)\} = U(k,t)$ to this system gives

$$\frac{d^2U}{dt^2} + c^2k^2U = 0,$$
$$U(k,0) = F(k), \qquad \left(\frac{dU}{dt}\right)_{t=0} = G(k).$$

The solution of the transformed system is

$$U(k,t) = Ae^{ickt} + Be^{-ickt},$$

where A and B are constants of integration to be determined from the transformed data $A + B = F(k)$ and $A - B = \frac{1}{ikc}G(k)$. Solving for A and B, we obtain the solution

$$U(k,t) = \frac{1}{2}F(k)\left(e^{ickt} + e^{-ickt}\right) + \frac{G(k)}{2ick}\left(e^{ickt} - e^{-ickt}\right). \tag{1.7.19}$$

Thus, the inverse Fourier transform of (1.7.19) yields the solution

$$\begin{aligned} u(x,t) = {} & \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\left\{e^{ik(x+ct)} + e^{ik(x-ct)}\right\}dk\right] \\ & + \frac{1}{2c}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{G(k)}{ik}\left\{e^{ik(x+ct)} - e^{ik(x-ct)}\right\}dk\right]. \end{aligned} \tag{1.7.20}$$

We use the following results:

$$f(x) = \mathcal{F}^{-1}\{F(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}F(k)\,dk,$$
$$g(x) = \mathcal{F}^{-1}\{G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}G(k)\,dk,$$

to obtain the solution in the form

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(k)\,dk \int_{x-ct}^{x+ct} e^{ik\xi}\,d\xi \\ &= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} d\xi \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik\xi}G(k)\,dk\right] \\ &= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi. \end{aligned} \tag{1.7.21}$$

This is the well-known *d'Alembert solution* of the wave equation.

In particular, if $f(x) = \exp(-x^2)$ and $g(x) = 0$, then the d'Alembert solution (1.7.21) reduces to

$$u(x,t) = \frac{1}{2}\left[\exp\left\{-(x-ct)^2\right\} + \exp\left\{-(x+ct)^2\right\}\right]. \tag{1.7.22}$$

This shows that the initial wave profile breaks up into two identical traveling waves of half the amplitude moving in opposite directions with speed c.

On the other hand, if $f(x) = 0$ and $g(x) = \delta(x)$, the d'Alembert solution (1.7.21) becomes

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} \delta(\xi)\,d\xi = \frac{1}{2c}\int_{x-ct}^{x+ct} H'(\xi)\,d\xi \\ &= \frac{1}{2c}\big[H(x+ct) - H(x-ct)\big] \\ &= \frac{1}{2c}\begin{cases} 1 & \text{if } |x| < ct, \\ 0 & \text{if } |x| > ct > 0 \end{cases} \qquad (1.7.23) \\ &= \frac{1}{2c} H\big(c^2t^2 - x^2\big). \qquad (1.7.24) \end{aligned}$$

Example 1.7.3 (The Linearized Korteweg–de Vries Equation). The linearized Korteweg–de Vries (KdV) equation for the free surface elevation $\eta(x,t)$ in inviscid water of constant depth h is

$$\eta_t + c\eta_x + \frac{ch^2}{6}\eta_{xxx} = 0, \quad -\infty < x < \infty,\ t > 0, \tag{1.7.25}$$

where $c = \sqrt{gh}$ is the shallow water speed.

We solve equation (1.7.25) with the initial condition

$$\eta(x,0) = f(x), \quad -\infty < x < \infty. \tag{1.7.26}$$

Application of the Fourier transform $\mathcal{F}\{\eta(x,t)\} = E(k,t)$ to the KdV system gives the solution for $E(k,t)$ in the form

$$E(k,t) = F(k)\exp\left[ikct\left(\frac{k^2h^2}{6} - 1\right)\right].$$

The inverse transform gives

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\exp\left[ik\left\{(x-ct) + \left(\frac{cth^2}{6}\right)k^2\right\}\right]dk. \tag{1.7.27}$$

In particular, if $f(x) = \delta(x)$, then (1.7.27) reduces to the Airy integral

$$\eta(x,t) = \frac{1}{\pi}\int_0^{\infty} \cos\left[k(x-ct) + \left(\frac{cth^2}{6}\right)k^3\right]dk, \tag{1.7.28}$$

which, in terms of the Airy function,

$$= \left(\frac{cth^2}{2}\right)^{-\frac{1}{3}} Ai\left[\left(\frac{cth^2}{2}\right)^{-\frac{1}{3}}(x-ct)\right], \tag{1.7.29}$$

where the Airy function $Ai(az)$ is defined by

$$Ai(az) = \frac{1}{\pi a}\int_0^{\infty} \cos\left(kz + \frac{k^3}{3a^3}\right)dk. \tag{1.7.30}$$

Example 1.7.4 (Dirichlet's Problem in the Half Plane). We consider the solution of the Laplace equation in the half plane

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty,\ y \geq 0, \tag{1.7.31}$$

with the boundary conditions

$$u(x,0) = f(x), \quad -\infty < x < \infty, \tag{1.7.32}$$
$$u(x,y) \to 0 \quad \text{as } |x| \to \infty,\ y \to \infty. \tag{1.7.33}$$

We apply the Fourier transform with respect to x to the system (1.7.31)–(1.7.33) to obtain

$$\frac{d^2U}{dy^2} - k^2U = 0, \tag{1.7.34}$$

$$U(k,0) = F(k), \quad \text{and} \quad U(k,y) \to 0 \quad \text{as } y \to \infty. \tag{1.7.35ab}$$

Thus, the solution of this transformed system is

$$U(k,y) = F(k)\exp\bigl(-|k|y\bigr). \tag{1.7.36}$$

Application of the Convolution Theorem 1.7.1 gives the solution

$$u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi, \tag{1.7.37}$$

where

$$g(x) = \mathcal{F}^{-1}\bigl\{e^{-|k|y}\bigr\} = \sqrt{\frac{2}{\pi}}\frac{y}{(x^2+y^2)}. \tag{1.7.38}$$

Consequently, the solution (1.7.37) becomes

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{(x-\xi)^2+y^2}, \quad y > 0. \tag{1.7.39}$$

This is the well-known *Poisson integral formula* in the half plane. It is noted that

$$\begin{aligned}\lim_{y\to 0+} u(x,y) &= \int_{-\infty}^{\infty} f(\xi)\left[\lim_{y\to 0+}\frac{y}{\pi}\cdot\frac{1}{(x-\xi)^2+y^2}\right]d\xi \\ &= \int_{-\infty}^{\infty} f(\xi)\delta(x-\xi)\,d\xi,\end{aligned} \tag{1.7.40}$$

where Cauchy's definition of the delta function is used, that is,

$$\delta(x-\xi) = \lim_{y\to 0+}\frac{y}{\pi}\cdot\frac{1}{(x-\xi)^2+y^2}. \tag{1.7.41}$$

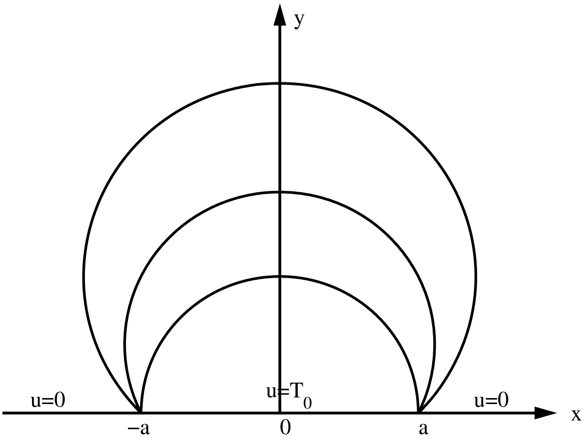

Fig. 1.5 Isothermal curves representing a family of circular arcs.

This may be recognized as a solution of the Laplace equation for a dipole source at $(x, y) = (\xi, 0)$.

In particular, when

$$f(x) = T_0 H\big(a - |x|\big), \tag{1.7.42}$$

the solution (1.7.39) reduces to

$$\begin{aligned} u(x, y) &= \frac{yT_0}{\pi} \int_{-a}^{a} \frac{d\xi}{(x-\xi)^2 + y^2} \\ &= \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{x+a}{y}\right) - \tan^{-1}\left(\frac{x-a}{y}\right)\right] \\ &= \frac{T_0}{\pi}\tan^{-1}\left(\frac{2ay}{x^2+y^2-a^2}\right). \end{aligned} \tag{1.7.43}$$

The curves in the upper half plane, for which the steady-state temperature is constant, are known as *isothermal curves*. In this case, these curves represent a family of circular arcs

$$x^2 + y^2 - \alpha y = a^2 \tag{1.7.44}$$

with centers on the y-axis and fixed end points on the x-axis at $x = \pm a$, as shown in Figure 1.5.

Another special case deals with

$$f(x) = \delta(x). \tag{1.7.45}$$

This solution for this case follows from (1.7.39) and is given by

$$u(x, t) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\delta(\xi)\, d\xi}{(x-\xi)^2 + y^2} = \frac{y}{\pi}\frac{1}{(x^2+y^2)}. \tag{1.7.46}$$

Further, we can readily reduce the solution of the *Neumann problem* in the half plane from the solution of the Dirichlet problem.

Example 1.7.5 (Neumann's Problem in the Half Plane). Find a solution of the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty,\ y > 0, \tag{1.7.47}$$

with the Neumann boundary condition

$$u_y(x,0) = f(x), \quad -\infty < x < \infty. \tag{1.7.48}$$

Condition (1.7.48) specifies the normal derivative on the boundary, and, physically, it describes the fluid flow or heat flux at the boundary.

We define a new function $v(x,y) = u_y(x,y)$ so that

$$u(x,y) = \int^y v(x,\eta)\,d\eta, \tag{1.7.49}$$

where an arbitrary constant can be added to the right-hand side. Clearly, the function v satisfies the Laplace equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} = \frac{\partial}{\partial y}(u_{xx} + u_{yy}) = 0,$$

with the boundary condition

$$v(x,0) = u_y(x,0) = f(x) \quad \text{for } -\infty < x < \infty.$$

Thus, $v(x,y)$ satisfies the Laplace equation with the Dirichlet condition on the boundary. Obviously, the solution is given by (1.7.39), that is,

$$v(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{(x-\xi)^2 + y^2}. \tag{1.7.50}$$

Then, the solution $u(x,y)$ can be obtained from (1.7.49) in the form

$$\begin{aligned} u(x,y) &= \int^y v(x,\eta)\,d\eta = \frac{1}{\pi}\int^y \eta\,d\eta \int_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{(x-\xi)^2 + \eta^2} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi \int^y \frac{\eta\,d\eta}{(x-\xi)^2 + \eta^2}, \quad y > 0 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\log\left[(x-\xi)^2 + y^2\right] d\xi, \end{aligned} \tag{1.7.51}$$

where an arbitrary constant can be added to this solution. In other words, the solution of any Neumann problem is uniquely determined up to an arbitrary constant.

Example 1.7.6 (The Cauchy Problem for the Diffusion Equation). We consider the initial-value problem for a one-dimensional diffusion equation with no sources or sinks:

$$u_t = \kappa u_{xx}, \quad -\infty < x < \infty,\ t > 0, \tag{1.7.52}$$

where κ is a diffusivity constant with the initial condition

$$u(x,0) = f(x), \quad -\infty < x < \infty. \tag{1.7.53}$$

We solve this problem using the Fourier transform in the space variable x defined by (1.7.1). Application of this transform to (1.7.52), (1.7.53) gives

$$U_t = -\kappa k^2 U, \quad t > 0, \tag{1.7.54}$$

$$U(k,0) = F(k). \tag{1.7.55}$$

The solution of the transformed system is

$$U(k,t) = F(k)\exp\bigl(-\kappa k^2 t\bigr). \tag{1.7.56}$$

The inverse Fourier transform gives the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\exp\bigl[\bigl(ikx - \kappa k^2 t\bigr)\bigr]\,dk,$$

which is, by the Convolution Theorem 1.7.1,

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi, \tag{1.7.57}$$

where

$$g(x) = \mathcal{F}^{-1}\bigl\{e^{-\kappa k^2 t}\bigr\} = \frac{1}{\sqrt{2\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right), \quad \text{by (1.7.3)}.$$

Thus, solution (1.7.57) becomes

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} f(\xi)\exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]d\xi. \tag{1.7.58}$$

The integrand involved in the solution consists of the initial value $f(x)$ and the *Green's function* (or the *fundamental solution*) $G(x-\xi,t)$ of the diffusion equation for the infinite interval:

$$G(x-\xi,t) = \frac{1}{\sqrt{4\pi\kappa t}}\exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]. \tag{1.7.59}$$

So, in terms of $G(x-\xi,t)$, solution (1.7.58) can be written as

$$u(x,t)=\int_{-\infty}^{\infty} f(\xi)G(x-\xi,t)\,d\xi, \tag{1.7.60}$$

so that, in the limit as $t \to 0+$, this formally becomes

$$u(x,0)=f(x)=\int_{-\infty}^{\infty} f(\xi)\lim_{t\to 0+} G(x-\xi,t)\,d\xi.$$

The limit of $G(x-\xi,t)$, as $t\to 0+$, represents the Dirac delta function

$$\delta(x-\xi)=\lim_{t\to 0+}\frac{1}{\sqrt{4\pi\kappa t}}\exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]. \tag{1.7.61}$$

It is important to point out that the integrand in (1.7.60) consists of the initial temperature distribution $f(x)$ and the Green's function $G(x-\xi,t)$ which represents the temperature response along the rod at time t due to an initial unit impulse of heat at $x=\xi$. The physical meaning of the solution (1.7.60) is that the initial temperature distribution $f(x)$ is decomposed into a spectrum of impulses of magnitude $f(\xi)$ at each point $x=\xi$ to form the resulting temperature $f(\xi)G(x-\xi,t)$. According to the *linear superposition principle* (1.3.6), the resulting temperature is integrated to find solution (1.7.60).

We make the change of variable

$$\frac{\xi-x}{2\sqrt{\kappa t}}=\zeta, \qquad d\zeta=\frac{d\xi}{2\sqrt{\kappa t}}$$

to express solution (1.7.58) in the form

$$u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f\left(x+2\sqrt{\kappa t}\zeta\right)\exp\left(-\zeta^2\right)d\zeta. \tag{1.7.62}$$

The integral solution (1.7.62) or (1.7.58) is called the *Poisson integral representation* of the temperature distribution. This integral is convergent for all time $t>0$, and the integrals obtained from (1.7.62) by differentiation under the integral sign with respect to x and t are uniformly convergent in the neighborhood of the point (x,t). Hence, the solution $u(x,t)$ and its derivatives of all orders exist for $t>0$.

Finally, we consider two special cases:

(a) $f(x)=\delta(x)$ and
(b) $f(x)=T_0H(x)$, where T_0 is a constant.

For case (a), the solution (1.7.58) reduces to

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty}\delta(\xi)\exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]d\xi \\ &= \frac{1}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right). \end{aligned} \tag{1.7.63}$$

This is usually called *Green's function* or the *fundamental solution* of the diffusion equation and is shown in Figure 1.6.

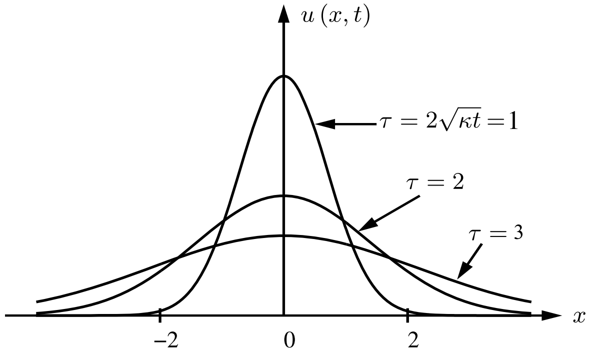

Fig. 1.6 The temperature distribution $u(x,t)$ due to a point source for different values of $\tau = 2\sqrt{\kappa t}$.

At any time t, the solution $u(x,t)$ is Gaussian. The peak height of $u(x,t)$ decreases inversely with $\sqrt{\kappa t}$, whereas the width of the solution $(x \sim \sqrt{\kappa t})$ increases with $\sqrt{\kappa t}$. In fact, the initially sharply peaked profile is gradually smoothed out as $t \to \infty$ under the action of diffusion. These are remarkable features for diffusion phenomena.

For case (b), the initial data is discontinuous. In this case, the solution is

$$u(x,t) = \frac{T_0}{2\sqrt{\pi \kappa t}} \int_0^\infty \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right] d\xi. \tag{1.7.64}$$

Introducing the change of variable $\eta = \frac{\xi - x}{2\sqrt{\kappa t}}$, we can express solution (1.7.64) in the form

$$\begin{aligned} u(x,t) &= \frac{T_0}{\sqrt{\pi}} \int_{\frac{-x}{2\sqrt{\kappa t}}}^\infty e^{-\eta^2}\, d\eta = \frac{T_0}{2} \operatorname{erfc}\left(-\frac{x}{2\sqrt{\kappa t}}\right) \\ &= \frac{T_0}{2}\left[1 + \operatorname{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]. \end{aligned} \tag{1.7.65}$$

This shows that, at $t = 0$, the solution coincides with the initial data $u(x,0) = T_0$. The graph of $\frac{1}{T_0} u(x,t)$ against x is shown in Figure 1.7. As t increases, the discontinuity is gradually smoothed out, whereas the width of the transition zone increases as $\sqrt{\kappa t}$.

Example 1.7.7 (The Schrödinger Equation in Quantum Mechanics). The time-dependent Schrödinger equation for a particle of mass m is

$$i\hbar \psi_t = \left[V(\mathbf{x}) - \frac{\hbar^2}{2m}\nabla^2\right]\psi = H\psi, \tag{1.7.66}$$

where $h(= 2\pi\hbar)$ is the Planck constant, $\psi(\mathbf{x},t)$ is the wave function, $V(\mathbf{x})$ is the potential, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three-dimensional Laplacian, and H is called the *Hamiltonian*.

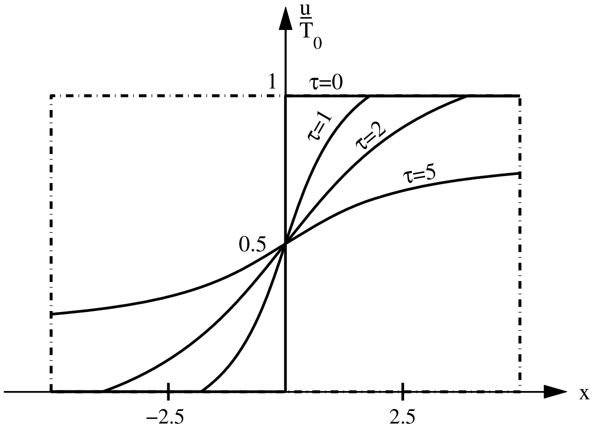

Fig. 1.7 The temperature distribution due to discontinuous initial data for different values of $\tau = 2\sqrt{\kappa t} = 0, 1, 2, 5$.

If $V(\mathbf{x}) = \text{const.} = V$, we seek a plane wave solution of the form

$$\psi(\mathbf{x}, t) = A \exp\left[i(\kappa \cdot \mathbf{x} - \omega t)\right], \tag{1.7.67}$$

where A is a constant amplitude, $\kappa = (k, l, m)$ is the wavenumber vector, and ω is the frequency.

Substituting this solution into (1.7.66), we conclude that this solution is possible provided the following relation is satisfied:

$$i\hbar(-i\omega) = V - \frac{\hbar^2}{2m}(i\kappa)^2, \qquad \kappa^2 = k^2 + l^2 + m^2,$$

or

$$\hbar\omega = V + \frac{\hbar^2 \kappa^2}{2m}. \tag{1.7.68}$$

This is called the *dispersion relation* for the de Broglie wave and shows that the sum of the potential energy V and the kinetic energy $\frac{(\hbar k)^2}{2m}$ equals the total energy $\hbar\omega$. Further, the kinetic energy

$$K.E. = \frac{1}{2m}(\hbar\kappa)^2 = \frac{p^2}{2m}, \tag{1.7.69}$$

where $p = \hbar\kappa$ is the momentum of the particle. In the one-dimensional case, the group velocity is

$$C_g = \frac{\partial \omega}{\partial k} = \frac{\hbar\kappa}{m} = \frac{p}{m} = \frac{mv}{m} = v. \tag{1.7.70}$$

This shows that the group velocity is equal to the classical particle velocity v.

We now use the Fourier transform method to solve the one-dimensional Schrödinger equation for a free particle ($V \equiv 0$), that is,

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx}, \quad -\infty < x < \infty,\ t > 0, \tag{1.7.71}$$

$$\psi(x,0) = \psi_0(x), \quad -\infty < x < \infty, \tag{1.7.72}$$

$$\psi(x,t) \to 0 \quad \text{as } |x| \to \infty,\ t > 0. \tag{1.7.73}$$

Application of the Fourier transform to (1.7.71)–(1.7.73) gives

$$\Psi_t = -\frac{i\hbar k^2}{2m}\Psi, \tag{1.7.74}$$

$$\Psi(k,0) = \Psi_0(k). \tag{1.7.75}$$

The solution of this transformed system is

$$\Psi(k,t) = \Psi_0(k)\exp\left(-i\alpha k^2 t\right), \quad \alpha = \frac{\hbar}{2m}. \tag{1.7.76}$$

This solution is similar to (1.7.56) with $\kappa = i\alpha$ so that the inverse Fourier transform gives the formal solution for the wave function $\psi(x,t)$ similar to (1.7.58) in the form

$$\psi(x,t) = \frac{1}{\sqrt{4\pi i\alpha t}}\int_{-\infty}^{\infty}\psi_0(\xi)\exp\left[-\frac{(x-\xi)^2}{4i\alpha t}\right]d\xi \tag{1.7.77}$$

$$= \frac{(1-i)}{\sqrt{8\pi\alpha t}}\int_{-\infty}^{\infty}\psi_0(\xi,0)\exp\left[\frac{i(x-\xi)^2}{4\alpha t}\right]d\xi. \tag{1.7.78}$$

This is the integral solution of the one-dimensional Schrödinger equation.

1.8 Multiple Fourier Transforms and Partial Differential Equations

Definition 1.8.1. Under the assumptions on $f(\mathbf{x})$ similar to those made for the one-dimensional case, the *multiple Fourier transform* of $f(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the n-dimensional vector, is defined by

$$\mathcal{F}\{f(\mathbf{x})\} = F(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{x})\}f(\mathbf{x})\,d\mathbf{x}, \tag{1.8.1}$$

where $\boldsymbol{\kappa} = (k_1, k_2, \ldots, k_n)$ is the n-dimensional transform vector and $\boldsymbol{\kappa}\cdot\mathbf{x} = (k_1x_1 + k_2x_2 + \cdots + k_nx_n)$.

The *inverse Fourier transform* is similarly defined by

$$\mathcal{F}^{-1}\{F(\boldsymbol{\kappa})\} = f(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\{i(\boldsymbol{\kappa}\cdot\mathbf{x})\}F(\boldsymbol{\kappa})\,d\boldsymbol{\kappa}. \tag{1.8.2}$$

In particular, the *double Fourier transform* is defined by

$$\mathcal{F}\{f(x,y)\} = F(k,\ell) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{r})\} f(x,y)\,dx\,dy, \qquad (1.8.3)$$

where $\mathbf{r} = (x, y)$ and $\boldsymbol{\kappa} = (k, \ell)$.

The *inverse double Fourier transform* is given by

$$\mathcal{F}\{F(k,\ell)\} = f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r})\} F(k,\ell)\,dk\,d\ell. \qquad (1.8.4)$$

Similarly, the *three-dimensional Fourier transform and its inverse* are defined by the integrals

$$\mathcal{F}\{f(x,y,z)\} = F(k,\ell,m) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-i(\boldsymbol{\kappa}\cdot\mathbf{r})\} f(x,y,z)\,dx\,dy\,dz, \qquad (1.8.5)$$

$$\mathcal{F}^{-1}\{F(k,\ell,m)\} = f(x,y,z) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r})\} F(k,\ell,m)\,dk\,d\ell\,dm. \qquad (1.8.6)$$

The operational properties of these multiple Fourier transforms are similar to those of the one-dimensional case. In particular, results (1.7.5) and (1.7.6) relating the Fourier transforms of derivatives to the Fourier transforms of given functions are also valid for the higher dimensional case. In higher dimensions, they are applied to the transforms of partial derivatives of $f(\mathbf{x})$ under the assumptions that $f(x_1, x_2, \ldots, x_n)$ and its partial derivatives vanish at infinity.

We illustrate the *multiple Fourier transform* method by the following examples of applications.

Example 1.8.1 (The Dirichlet Problem for the Three-Dimensional Laplace Equation in the Half-Space). The boundary-value problem for $u(x, y, z)$ satisfies the following equation and boundary conditions:

$$\nabla^2 u \equiv u_{xx} + u_{yy} + u_{zz} = 0, \quad -\infty < x, y < \infty,\ z > 0, \qquad (1.8.7)$$

$$u(x,y,0) = f(x,y), \quad -\infty < x, y < \infty, \qquad (1.8.8)$$

$$u(x,y,z) \to 0, \quad \text{as } r = \sqrt{x^2+y^2+z^2} \to \infty. \qquad (1.8.9)$$

We apply the double Fourier transform defined by (1.8.3) to the system (1.8.7)–(1.8.9) which reduces to

$$\frac{d^2U}{dz^2} - \kappa^2 U = 0 \quad \text{for } z > 0,$$

$$U(k,\ell,0) = F(k,\ell).$$

Thus, the solution of this transformed problem is

$$U(k,\ell,z) = F(k,\ell)\exp\bigl(-|\kappa|z\bigr) = F(k,\ell)G(k,\ell), \tag{1.8.10}$$

where $\boldsymbol{\kappa} = (k,\ell)$ and $G(k,\ell) = \exp(-|\kappa|z)$, so that

$$g(x,y) = \mathcal{F}^{-1}\bigl\{\exp\bigl(-|\kappa|z\bigr)\bigr\} = \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}. \tag{1.8.11}$$

Applying the Fourier Convolution Theorem to (1.8.10) gives the formal solution

$$\begin{aligned} u(x,y,z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)g(x-\xi,y-\eta,z)\,d\xi\,d\eta \\ &= \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(\xi,\eta)\,d\xi\,d\eta}{[(x-\xi)^2+(y-\eta)^2+z^2]^{\frac{3}{2}}}. \end{aligned} \tag{1.8.12}$$

Example 1.8.2 (The Two-Dimensional Diffusion Equation). We solve the two-dimensional diffusion equation

$$u_t = K\nabla^2 u, \quad -\infty < x, y < \infty,\ t > 0, \tag{1.8.13}$$

with the initial conditions

$$u(x,y,0) = f(x,y), \quad -\infty < x,y < \infty, \tag{1.8.14}$$

$$u(x,y,0) \to 0, \quad \text{as } r = \sqrt{x^2+y^2} \to \infty, \tag{1.8.15}$$

where K is the diffusivity constant.

The double Fourier transform of $u(x,y,t)$, defined by (1.8.3), is used to reduce the system (1.8.13)–(1.8.15) into the form

$$\begin{aligned} \frac{dU}{dt} &= -\kappa^2 KU, \quad t > 0, \\ U(k,\ell,0) &= F(k,\ell). \end{aligned}$$

The solution of this system is

$$U(k,\ell,t) = F(k,\ell)\exp\bigl(-tK\kappa^2\bigr) = F(k,\ell)G(k,\ell), \tag{1.8.16}$$

where

$$G(k,\ell) = \exp\bigl(-K\kappa^2 t\bigr),$$

so that

$$g(x,y) = \mathcal{F}^{-1}\bigl\{\exp\bigl(-K\kappa^2 t\bigr)\bigr\} = \frac{1}{2Kt}\exp\left[-\frac{x^2+y^2}{4Kt}\right]. \tag{1.8.17}$$

Finally, applying the Convolution Theorem to (1.8.16) gives the formal solution

$$u(x,y,t) = \frac{1}{4\pi Kt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\exp\left[-\frac{(x-\xi)^2+(y-\eta)^2}{4Kt}\right]d\xi\,d\eta, \tag{1.8.18}$$

or equivalently,

$$u(x,y,t)=\frac{1}{4\pi Kt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\big(\mathbf{r}'\big)\exp\bigg\{-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4Kt}\bigg\}\,d\mathbf{r}', \tag{1.8.19}$$

where $\mathbf{r}'=(\xi,\eta)$.

We make the change of variable $(\mathbf{r}-\mathbf{r}')=\sqrt{4Kt}\mathbf{R}$ to reduce (1.8.19) into the form

$$u(x,y,t)=\frac{1}{\pi\sqrt{4Kt}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\big(\mathbf{r}+\sqrt{4Kt}\mathbf{R}\big)\exp\big(-R^2\big)\,d\mathbf{R}. \tag{1.8.20}$$

Similarly, the formal solution of the initial-value problem for the three-dimensional diffusion equation

$$u_t=K(u_{xx}+u_{yy}+u_{zz}), \quad -\infty<x,y,z<\infty,\ t>0, \tag{1.8.21}$$

$$u(x,y,z,0)=f(x,y,z), \quad -\infty<x,y,z<\infty, \tag{1.8.22}$$

is given by

$$u(x,y,z,t)=\frac{1}{(4\pi Kt)^{\frac{3}{2}}}\iiint_{-\infty}^{\infty} f(\xi,\eta,\zeta)\exp\bigg(-\frac{r^2}{4Kt}\bigg)\,d\xi\,d\eta\,d\zeta, \tag{1.8.23}$$

where

$$r^2=(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2.$$

Or equivalently,

$$u(x,y,z,t)=\frac{1}{(4\pi Kt)^{\frac{3}{2}}}\iiint_{-\infty}^{\infty} f\big(\mathbf{r}'\big)\exp\bigg\{-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4Kt}\bigg\}\,d\xi\,d\eta\,d\zeta, \tag{1.8.24}$$

where $\mathbf{r}=(x,y,z)$ and $\mathbf{r}'=(\xi,\eta,\zeta)$.

Making the change of variable $\mathbf{r}'-\mathbf{r}=\sqrt{4tK}\mathbf{R}$, solution (1.8.24) reduces to the form

$$u(x,y,z,t)=\frac{1}{\pi^{\frac{3}{2}}4Kt}\iiint_{-\infty}^{\infty} f\big(\mathbf{r}+\sqrt{4Kt}\mathbf{R}\big)\exp\big(-R^2\big)\,d\mathbf{R}. \tag{1.8.25}$$

This is known as the *Fourier integral solution.*

Example 1.8.3 (The Cauchy Problem for the Two-Dimensional Wave Equation). The initial-value problem for the wave equation in two dimensions is governed by

$$u_{tt}=c^2(u_{xx}+u_{yy}), \quad -\infty<x,y<\infty,\ t>0, \tag{1.8.26}$$

with the initial data

$$u(x,y,0)=0, \quad u_t(x,y,0)=f(x,y), \quad -\infty<x,y<\infty, \tag{1.8.27ab}$$

where c is a constant. We assume that $u(x,y,t)$ and its first partial derivatives vanish at infinity.

We apply the two-dimensional Fourier transform defined by (1.8.3) to the system (1.8.26), (1.8.27ab), which becomes

$$\frac{d^2U}{dt^2} + c^2\kappa^2 U = 0, \quad \kappa^2 = k^2 + \ell^2,$$
$$U(k,\ell,0) = 0, \qquad \left(\frac{dU}{dt}\right)_{t=0} = F(k,\ell).$$

The solution of this transformed system is

$$U(k,\ell,t) = F(k,\ell)\,\frac{\sin(c\kappa t)}{c\kappa}. \tag{1.8.28}$$

The inverse Fourier transform gives the formal solution

$$u(x,y,t) = \frac{1}{2\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(i\boldsymbol{\kappa}\cdot\mathbf{r})\frac{\sin(c\kappa t)}{\kappa}F(\boldsymbol{\kappa})\,d\boldsymbol{\kappa} \tag{1.8.29}$$

$$= \frac{1}{4i\pi c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F(\boldsymbol{\kappa})}{\kappa}\left[\exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}+ct\right)\right\} - \exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}-ct\right)\right\}\right]d\boldsymbol{\kappa}. \tag{1.8.30}$$

The form of this solution reveals some interesting features of the wave equation. The exponential terms $\exp\{i\kappa(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa} \pm ct)\}$ involved in the integral solution (1.8.30) represent plane wave solutions of the wave equation (1.8.26). Thus, the solutions remain constant on the planes $\boldsymbol{\kappa}\cdot\mathbf{r} =$ const. that move parallel to themselves with velocity c. Evidently, solution (1.8.30) represents a superposition of the plane wave solutions traveling in all possible directions.

Similarly, the solution of the Cauchy problem for the three-dimensional wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}), \quad -\infty < x,y,z < \infty,\ t > 0, \tag{1.8.31}$$

$$u(x,y,z,0) = 0, \quad u_t(x,y,z,0) = f(x,y,z), \quad -\infty < x,y,z < \infty, \tag{1.8.32ab}$$

is given by

$$u(\mathbf{r},t) = \frac{1}{2ic(2\pi)^{\frac{3}{2}}}\iiint_{-\infty}^{\infty}\frac{F(\boldsymbol{\kappa})}{\kappa}\left[\exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}+ct\right)\right\} - \exp\left\{i\kappa\left(\frac{\boldsymbol{\kappa}\cdot\mathbf{r}}{\kappa}-ct\right)\right\}\right]d\boldsymbol{\kappa}, \tag{1.8.33}$$

where $\mathbf{r} = (x,y,z)$ and $\boldsymbol{\kappa} = (k,\ell,m)$.

For any given $\boldsymbol{\kappa}$, the terms $\exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r} \pm \kappa ct)\}$ represent the *plane traveling wave solution* of the wave equation (1.8.31), since they remain constant on the planes

$\boldsymbol{\kappa}\cdot\mathbf{r} = \pm c\kappa t$, which move perpendicular to themselves in the direction $\pm\boldsymbol{\kappa}$ at constant speed c. Since the integral solution (1.8.33) is the weighted sum of these plane waves at different wavenumbers $\boldsymbol{\kappa}$, the solution represents a *continuous spectrum* of plane waves propagating in *all directions*. It is somewhat easier to interpret solutions like (1.8.33) as $t \to \infty$ with (r/t) fixed. This can be done using the method of stationary phase approximation.

In particular, when $f(x,y,z) = \delta(x)\delta(y)\delta(z)$, so that $F(\boldsymbol{\kappa}) = (2\pi)^{-\frac{3}{2}}$, solution (1.8.33) becomes

$$u(\mathbf{r},t) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{\sin c\kappa t}{c\kappa} \exp\{i(\boldsymbol{\kappa}\cdot\mathbf{r})\}\, d\boldsymbol{\kappa}. \tag{1.8.34}$$

In terms of the spherical polar coordinates (κ,θ,ϕ) where the polar axis (the z-axis) is taken along the $\mathbf{r}$ direction with $\boldsymbol{\kappa}\cdot\mathbf{r} = \kappa r\cos\theta$, we write (1.8.34) in the form

$$\begin{aligned} u(r,t) &= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} \exp(i\kappa r\cos\theta) \frac{\sin c\kappa t}{c\kappa} \cdot \kappa^2 \sin\theta\, d\kappa \\ &= \frac{1}{2\pi^2 cr} \int_0^{\infty} \sin(c\kappa t)\sin(\kappa r)\, d\kappa \\ &= \frac{1}{8\pi^2 cr} \operatorname{Re} \int_{-\infty}^{\infty} \left[e^{i\kappa(ct-r)} - e^{i\kappa(ct+r)}\right] d\kappa, \end{aligned}$$

or

$$u(r,t) = \frac{1}{4\pi cr}\left[\delta(ct-r) - \delta(ct+r)\right]. \tag{1.8.35}$$

For $t > 0$, $ct + r > 0$, so that $\delta(ct+r) = 0$ and, hence, the solution is

$$u(r,t) = \frac{1}{4\pi cr}\delta(ct-r) = \frac{1}{4\pi rc^2}\delta\left(t - \frac{r}{c}\right). \tag{1.8.36}$$

1.9 Laplace Transforms and Initial Boundary-Value Problems

If $u(x,t)$ is any function defined in $a \le x \le b$ and $t > 0$, then its *Laplace transform* with respect to t is denoted by $\mathcal{L}\{u(x,t)\} = \bar{u}(x,s)$ and is defined by

$$\mathcal{L}\{u(x,t)\} = \bar{u}(x,s) = \int_0^{\infty} e^{-st} u(x,t)\, dt, \quad \operatorname{Re} s > 0, \tag{1.9.1}$$

where s is called the *transform variable,* which is a complex number. Under certain broad conditions on $u(x,t)$, its transform $\bar{u}(x,s)$ is an analytic function of s in the half plane Re $s > c$.

The *inverse Laplace transform* is denoted by $\mathcal{L}^{-1}\{\bar{u}(x,s)\} = u(x,t)$ and defined by the complex integral

$$\mathcal{L}^{-1}\{\bar{u}(x,s)\} = u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{u}(x,s)\,ds, \quad c > 0. \tag{1.9.2}$$

This integral is evaluated by using the Cauchy residue theorem for analytic functions. For more information about the basic properties of the Laplace transforms including the convolution, see Section B-4, pp. 758–766.

If $\mathcal{L}\{u(x,t)\} = \bar{u}(x,s)$, then (see Debnath 1995)

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = s\bar{u}(x,s) - u(x,0), \tag{1.9.3}$$

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^n}\right\} = s^n\bar{u}(x,s) - \sum_{r=1}^{n} s^{n-r}u^{(r-1)}(x,0), \tag{1.9.4}$$

and so on.

Similarly, it is easy to show that

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{d\bar{u}}{dx}, \qquad \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{\partial^2 \bar{u}}{\partial x^2}. \tag{1.9.5ab}$$

The following examples are useful for applications:

$$\mathcal{L}\{f(t-a)H(t-a)\} = \exp(-sa)\bar{f}(s), \quad a > 0, \tag{1.9.6}$$

$$\mathcal{L}\{\delta(t-a)\} = \exp(-as), \quad a > 0, \tag{1.9.7}$$

$$\mathcal{L}\left\{\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s}\exp\left(-a\sqrt{s}\right), \quad a \geq 0, \tag{1.9.8}$$

$$\mathcal{L}\{\exp(at)\operatorname{erf}(\sqrt{at})\} = \frac{\sqrt{a}}{\sqrt{s}(s-a)}, \quad a > 0. \tag{1.9.9}$$

The Laplace transforms are also very useful in finding solutions of initial-value problems described by linear partial differential equations. The following examples of applications illustrate the method of Laplace transforms.

Example 1.9.1 (The Inhomogeneous Cauchy Problem for the Wave Equation). We use the joint Laplace and Fourier transform method to solve the inhomogeneous Cauchy problem

$$u_{tt} - c^2 u_{xx} = q(x,t), \quad x \in \mathbb{R} \text{ (i.e., } -\infty < x < \infty), \ t > 0, \tag{1.9.10}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for all } x \in \mathbb{R}, \tag{1.9.11ab}$$

where $q(x,t)$ is a given function representing a source term.

We define the joint Laplace and Fourier transform of $u(x,t)$ by

$$\overline{U}(k,s) = \mathcal{L}\left[\mathcal{F}\{u(x,t)\}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,dx \int_0^{\infty} e^{-st}u(x,t)\,dt. \tag{1.9.12}$$

Application of the joint transform leads to the solution of the transformed problem in the form

$$\overline{U}(k,s) = \frac{sF(k) + G(k) + \overline{Q}(k,s)}{(s^2 + c^2k^2)}. \tag{1.9.13}$$

The inverse Laplace transform of (1.9.13) gives

$$\begin{aligned} U(k,t) &= F(k)\cos(ckt) + \frac{1}{ck}G(k)\sin(ckt) + \frac{1}{ck}\mathcal{L}^{-1}\left\{\frac{ck}{s^2+c^2k^2}\cdot\overline{Q}(k,s)\right\} \\ &= F(k)\cos(ckt) + \frac{G(k)}{ck}\sin(ckt) + \frac{1}{ck}\int_0^t \sin ck(t-\tau)Q(k,\tau)\,d\tau. \end{aligned} \tag{1.9.14}$$

The inverse Fourier transform leads to the solution

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{ickt}+e^{-ickt}\right)e^{ikx}F(k)\,dk \\ &\quad + \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{ickt}-e^{-ickt}\right)e^{ikx}\cdot\frac{G(k)}{ick}\,dk \\ &\quad + \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{2c}\int_0^t d\tau\int_{-\infty}^{\infty}\frac{Q(k,\tau)}{ik}\left[e^{ick(t-\tau)}+e^{-ick(t-\tau)}\right]e^{ikx}\,dk \\ &= \frac{1}{2}\left[f(x+ct)+f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \\ &\quad + \frac{1}{2c}\int_0^t d\tau\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}Q(k,\tau)\,dk\int_{x-c(t-\tau)}^{x+c(t-\tau)} e^{ik\xi}\,d\xi \\ &= \frac{1}{2}\left[f(x-ct)+f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \\ &\quad + \frac{1}{2c}\int_0^t d\tau\int_{x-c(t-\tau)}^{x+c(t-\tau)} q(\xi,\tau)\,d\xi. \end{aligned} \tag{1.9.15}$$

In the case of the homogeneous Cauchy problem, $q(x,t) \equiv 0$, the solution of (1.9.15) reduces to the famous d'Alembert solution (1.7.21).

Example 1.9.2 (The Heat Conduction Equation in a Semi-Infinite Medium and Fractional Derivatives). Solve the equation

$$u_t = \kappa u_{xx}, \quad x > 0,\ t > 0, \tag{1.9.16}$$

with the initial and boundary conditions

$$u(x,0) = 0, \quad x > 0, \tag{1.9.17}$$

$$u(0,t) = f(t), \quad t > 0, \tag{1.9.18}$$

$$u(x,t) \to 0, \quad \text{as } x \to \infty,\ t > 0. \tag{1.9.19}$$

Application of the Laplace transform with respect to t to (1.9.16) gives

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{\kappa}\bar{u} = 0. \tag{1.9.20}$$

The general solution of this equation is

$$\bar{u}(x,s) = A\exp\left(-x\sqrt{\frac{s}{\kappa}}\right) + B\exp\left(x\sqrt{\frac{s}{\kappa}}\right),$$

where A and B are integrating constants. For bounded solutions, $B \equiv 0$, and using $\bar{u}(0,s) = \bar{f}(s)$, we obtain the solution

$$\bar{u}(x,s) = \bar{f}(s)\exp\left(-x\sqrt{\frac{s}{\kappa}}\right). \tag{1.9.21}$$

The Laplace inversion theorem gives the solution

$$u(x,t) = \frac{x}{2\sqrt{\pi\kappa}}\int_0^t f(t-\tau)\tau^{-\frac{3}{2}}\exp\left(-\frac{x^2}{4\kappa\tau}\right)d\tau, \tag{1.9.22}$$

which is, by setting $\lambda = \frac{x}{2\sqrt{\kappa\tau}}$ or $d\lambda = -\frac{x}{4\sqrt{\kappa}}\tau^{-\frac{3}{2}}\,d\tau$,

$$= \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\kappa t}}}^{\infty} f\left(t - \frac{x^2}{4\kappa\lambda^2}\right)e^{-\lambda^2}\,d\lambda. \tag{1.9.23}$$

This is the formal solution of the heat conduction problem.

In particular, if $f(t) = T_0 =$ const., the solution (1.9.23) becomes

$$u(x,t) = \frac{2T_0}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{\kappa t}}}^{\infty} e^{-\lambda^2}\,d\lambda = T_0\,\text{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right). \tag{1.9.24}$$

Clearly, the temperature distribution tends asymptotically to the constant value T_0, as $t \to \infty$.

We consider another physical problem that is concerned with determining the temperature distribution of a semi-infinite solid when the rate of flow of heat is prescribed at the end $x = 0$. Thus, the problem is to solve diffusion equation (1.9.16) subject to conditions (1.9.17), (1.9.19), and

$$-k\left(\frac{\partial u}{\partial x}\right) = g(t) \quad \text{at } x = 0,\ t > 0, \tag{1.9.25}$$

where k is a constant called *thermal conductivity*.

Application of the Laplace transform gives the solution of the transformed problem

$$\bar{u}(x,s) = \frac{1}{k}\sqrt{\frac{\kappa}{s}}\bar{g}(s)\exp\left(-x\sqrt{\frac{s}{\kappa}}\right). \tag{1.9.26}$$

The inverse Laplace transform yields the solution

$$u(x,t) = \frac{1}{k}\sqrt{\frac{\kappa}{\pi}} \int_0^t g(t-\tau)\tau^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4\kappa\tau}\right) d\tau, \tag{1.9.27}$$

which, by the change of variable $\lambda = \frac{x}{2\sqrt{\kappa\tau}}$,

$$= \frac{x}{k\sqrt{\pi}} \int_{\frac{x}{2\sqrt{\kappa t}}}^{\infty} g\left(t - \frac{x^2}{4\kappa\lambda^2}\right) \lambda^{-2} e^{-\lambda^2}\, d\lambda. \tag{1.9.28}$$

In particular, if $g(t) = T_0 = \text{const.}$, this solution becomes

$$u(x,t) = \frac{T_0 x}{k\sqrt{\pi}} \int_{\frac{x}{2k\sqrt{\kappa t}}}^{\infty} \lambda^{-2} e^{-\lambda^2}\, d\lambda.$$

Integrating this result by parts gives

$$u(x,t) = \frac{T_0}{k}\left[2\sqrt{\frac{\kappa t}{\pi}} \exp\left(-\frac{x^2}{4\kappa t}\right) - x \operatorname{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]. \tag{1.9.29}$$

Alternatively, the heat conduction problem (1.9.16)–(1.9.19) can be solved by using fractional derivatives (Debnath 1995). We recall (1.9.21) and rewrite it as

$$\frac{\partial \bar{u}}{\partial x} = -\sqrt{\frac{s}{\kappa}}\, \bar{u}. \tag{1.9.30}$$

This can be expressed in terms of a fractional derivative of order $\frac{1}{2}$ as

$$\frac{\partial u}{\partial x} = -\frac{1}{\sqrt{\kappa}} \mathcal{L}^{-1}\left\{\sqrt{s}\,\bar{u}(x,s)\right\} = -\frac{1}{\sqrt{\kappa}}\, {}_0D_t^{\frac{1}{2}} u(x,t). \tag{1.9.31}$$

Thus, the heat flux is expressed in terms of the fractional derivative. In particular, when $u(0,t) = \text{const.} = T_0$, then the heat flux at the surface is given by

$$-k\left(\frac{\partial u}{\partial x}\right)_{x=0} = \frac{k}{\sqrt{\kappa}} D_t^{\frac{1}{2}} T_0 = \frac{kT_0}{\sqrt{\pi\kappa t}}. \tag{1.9.32}$$

Example 1.9.3 (Diffusion Equation in a Finite Medium). Solve the diffusion equation

$$u_t = \kappa u_{xx}, \quad 0 < x < a,\ t > 0, \tag{1.9.33}$$

with the initial and boundary conditions

$$u(x,0) = 0, \quad 0 < x < a, \tag{1.9.34}$$

$$u(0,t) = U, \quad t > 0, \tag{1.9.35}$$

$$u_x(a,t) = 0, \quad t > 0, \tag{1.9.36}$$

where U is a constant.

We introduce the Laplace transform of $u(x,t)$ with respect to t to obtain

$$\frac{d^2\bar{u}}{dx^2} - \frac{s}{\kappa}\bar{u} = 0, \quad 0 < x < a, \tag{1.9.37}$$

$$\bar{u}(0,s) = \frac{U}{s}, \qquad \left(\frac{d\bar{u}}{dx}\right)_{x=a} = 0. \tag{1.9.38ab}$$

The general solution of (1.9.37) is

$$\bar{u}(x,s) = A\cosh\left(x\sqrt{\frac{s}{\kappa}}\right) + B\sinh\left(x\sqrt{\frac{s}{\kappa}}\right), \tag{1.9.39}$$

where A and B are constants of integration. Using (1.9.38ab), we obtain the values of A and B, so that the solution (1.9.39) becomes

$$\bar{u}(x,s) = \frac{U}{s} \cdot \frac{\cosh[(a-x)\sqrt{\frac{s}{\kappa}}]}{\cosh(a\sqrt{\frac{s}{\kappa}})}. \tag{1.9.40}$$

The inverse Laplace transform gives the solution

$$u(x,t) = U\mathcal{L}^{-1}\left\{\frac{\cosh(a-x)\sqrt{\frac{s}{\kappa}}}{s\cosh\sqrt{a\frac{s}{\kappa}}}\right\}. \tag{1.9.41}$$

The inversion can be carried out by the Cauchy residue theorem to obtain the solution

$$u(x,t) = U\left[1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\cos\left\{\frac{(2n-1)(a-x)\pi}{2a}\right\} \times \exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2\kappa t\right\}\right]. \tag{1.9.42}$$

By expanding the cosine term, this becomes

$$u(x,t) = U\left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\left\{\left(\frac{2n-1}{2a}\right)\pi x\right\} \times \exp\left\{-(2n-1)^2\left(\frac{\pi}{2a}\right)^2\kappa t\right\}\right]. \tag{1.9.43}$$

This result can be obtained by solving the problem by the method of separation of variables.

Example 1.9.4 (Diffusion in a Finite Medium). Solve the one-dimensional diffusion equation in a finite medium $0 < z < a$, where the concentration function $C(z,t)$ satisfies the equation

$$C_t = \kappa C_{zz}, \quad 0 < z < a, t > 0, \tag{1.9.44}$$

and the initial and boundary data

$$C(z,0) = 0 \quad \text{for } 0 < z < a, \tag{1.9.45}$$

$$C(z,t) = C_0 \quad \text{for } z = a, t > 0, \tag{1.9.46}$$

$$\frac{\partial C}{\partial z} = 0 \quad \text{for } z = 0, t > 0, \tag{1.9.47}$$

where C_0 is a constant.

Application of the Laplace transform of $C(z,t)$ with respect to t gives

$$\frac{d^2\overline{C}}{dz^2} - \left(\frac{s}{\kappa}\right)\overline{C} = 0, \quad 0 < z < a,$$

$$\overline{C}(a,s) = \frac{C_0}{s}, \qquad \left(\frac{d\overline{C}}{dz}\right)_{z=0} = 0.$$

The solution of this differential equation system is

$$\overline{C}(z,s) = \frac{C_0 \cosh(z\sqrt{\frac{s}{\kappa}})}{s\cosh(a\sqrt{\frac{s}{\kappa}})}, \tag{1.9.48}$$

which, by writing $\alpha = \sqrt{\frac{s}{\kappa}}$,

$$\begin{aligned}
&= \frac{C_0}{s}\frac{(e^{\alpha z} + e^{-\alpha z})}{(e^{\alpha a} + e^{-\alpha a})} \\
&= \frac{C_0}{s}\left[\exp\{-\alpha(a-z)\} + \exp\{-\alpha(a+z)\}\right]\sum_{n=0}^{\infty}(-1)^n \exp(-2n\alpha a) \\
&= \frac{C_0}{s}\left[\sum_{n=0}^{\infty}(-1)^n \exp\left[-\alpha\{(2n+1)a - z\}\right]\right. \\
&\quad \left. + \sum_{n=0}^{\infty}(-1)^n \exp\left[-\alpha\{(2n+1)a + z\}\right]\right].
\end{aligned} \tag{1.9.49}$$

Using the result (1.9.8), we obtain the final solution

$$\begin{aligned}
C(z,t) = C_0\left\{\sum_{n=0}^{\infty}(-1)^n\left[\operatorname{erfc}\left\{\frac{(2n+1)a - z}{2\sqrt{\kappa t}}\right\}\right.\right. \\
\left.\left. + \operatorname{erfc}\left\{\frac{(2n+1)a + z}{2\sqrt{\kappa t}}\right\}\right]\right\}.
\end{aligned} \tag{1.9.50}$$

This solution represents an infinite series of complementary error functions. The successive terms of this series are, in fact, the concentrations at depth $a - z$, $a + z$, $3a - z$, $3a + z$, $\ldots$ in the medium. The series converges rapidly for all except large values of $\left(\frac{\kappa t}{a^2}\right)$.

Example 1.9.5 (The Wave Equation for the Transverse Vibration of a Semi-Infinite String). Find the displacement of a semi-infinite string, which is initially at rest in its equilibrium position. At time $t = 0$, the end $x = 0$ is constrained to move so that the displacement is $u(0,t) = Af(t)$ for $t \geq 0$, where A is a constant. The problem is to solve the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 \leq x < \infty, \ t > 0, \tag{1.9.51}$$

with the boundary and initial conditions

$$u(x,t) = Af(t) \quad \text{at } x = 0, t \geq 0, \tag{1.9.52}$$

$$u(x,t) \to 0 \quad \text{as } x \to \infty, t \geq 0, \tag{1.9.53}$$

$$u(x,t) = 0 = \frac{\partial u}{\partial t} \quad \text{at } t = 0 \text{ for } 0 < x < \infty. \tag{1.9.54ab}$$

Application of the Laplace transform of $u(x,t)$ with respect to t gives

$$\begin{aligned} \frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} &= 0, \quad \text{for } 0 \leq x < \infty, \\ \bar{u}(x,s) &= A\bar{f}(s) \quad \text{at } x = 0, \\ \bar{u}(x,s) &\to 0 \quad \text{as } x \to \infty. \end{aligned}$$

The solution of this differential equation system is

$$\bar{u}(x,s) = A\bar{f}(s) \exp\left(-\frac{xs}{c}\right). \tag{1.9.55}$$

Inversion gives the solution

$$u(x,t) = Af\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right). \tag{1.9.56}$$

In other words, the solution is

$$u(x,t) = \begin{cases} Af(t - \frac{x}{c}) & \text{if } t > \frac{x}{c}, \\ 0, & \text{if } t < \frac{x}{c}. \end{cases} \tag{1.9.57}$$

This solution represents a wave propagating at a velocity c with the characteristic $x = ct$.

Example 1.9.6 (The Cauchy–Poisson Wave Problem in Fluid Dynamics). We consider the two-dimensional Cauchy–Poisson problem (Debnath 1994) for an inviscid liquid of infinite depth with a horizontal free surface. We assume that the liquid has constant density ρ and negligible surface tension. Waves are generated on the free surface of water initially at rest for time $t < 0$ by the prescribed free surface displacement at $t = 0$.

In terms of the velocity potential $\phi(x, z, t)$ and the free surface elevation $\eta(x, t)$, the linearized surface wave motion in Cartesian coordinates (x, y, z) is governed by the following equation and free surface and boundary conditions:

$$\nabla^2\phi = \phi_{xx} + \phi_{zz} = 0, \quad -\infty < z \le 0,\ x \in \mathbb{R},\ t < 0, \tag{1.9.58}$$

$$\left.\begin{aligned} \phi_z - \eta_t &= 0, \\ \phi_t + g\eta &= 0 \end{aligned}\right\} \quad \text{on } z = 0, t > 0, \tag{1.9.59ab}$$

$$\phi_z \to 0 \quad \text{as } z \to -\infty. \tag{1.9.60}$$

The initial conditions are

$$\phi(x, 0, 0) = 0 \quad \text{and} \quad \eta(x, 0) = \eta_0(x), \tag{1.9.61}$$

where $\eta_0(x)$ is a given initial elevation with compact support.

We introduce the Laplace transform with respect to t and the Fourier transform with respect to x defined by

$$\left[\bar{\tilde{\phi}}(k, z, s), \bar{\tilde{\eta}}(k, s)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, dx \int_0^{\infty} e^{-st}[\phi, \eta]\, dt. \tag{1.9.62}$$

Application of the joint transform method to the above system gives

$$\bar{\tilde{\phi}}_{zz} - k^2\bar{\tilde{\phi}} = 0, \quad -\infty < z \le 0, \tag{1.9.63}$$

$$\left.\begin{aligned} \bar{\tilde{\phi}} &= s\bar{\tilde{\eta}} - \tilde{\eta}_0(k), \\ s\bar{\tilde{\phi}} + g\bar{\tilde{\eta}} &= 0 \end{aligned}\right\} \quad \text{on } z = 0, \tag{1.9.64ab}$$

$$\bar{\tilde{\phi}}_z \to 0 \quad \text{as } z \to -\infty, \tag{1.9.65}$$

where

$$\tilde{\eta}_0(k) = \mathcal{F}\{\eta_0(x)\}.$$

The bounded solution of equation (1.9.63) is

$$\bar{\tilde{\phi}}(k, s) = \overline{A}\exp\bigl(|k|z\bigr), \tag{1.9.66}$$

where $\overline{A} = \overline{A}(s)$ is an arbitrary function of s.

Substituting (1.9.66) into (1.9.64ab) and eliminating $\bar{\tilde{\eta}}$ from the resulting equations gives $\overline{A}$. Hence, the solutions for $\bar{\tilde{\phi}}$ and $\bar{\tilde{\eta}}$ are

$$\left[\bar{\tilde{\phi}}, \bar{\tilde{\eta}}\right] = \left[-\frac{g\tilde{\eta}_0 \exp(|k|z)}{s^2 + \omega^2}, \frac{s\tilde{\eta}_0}{s^2 + \omega^2}\right], \tag{1.9.67}$$

where the dispersion relation is

$$\omega^2 = g|k|. \tag{1.9.68}$$

The inverse Laplace and Fourier transforms give the solutions

$$\phi(x,z,t) = -\frac{g}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin \omega t}{\omega} \exp\bigl(ikx + |k|z\bigr)\tilde{\eta}_0(k)\, dk, \tag{1.9.69}$$

$$\begin{aligned} \eta(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\eta}_0(k) \cos \omega t e^{ikx}\, dk, \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \tilde{\eta}_0(k) \bigl[e^{i(kx-\omega t)} + e^{i(kx+\omega t)}\bigr]\, dk, \end{aligned} \tag{1.9.70}$$

in which $\tilde{\eta}_0(-k) = \tilde{\eta}_0(k)$ is assumed.

Physically, the first and second integrals of (1.9.70) represent waves traveling in the positive and negative directions of x, respectively, with phase velocity $\frac{\omega}{k}$. These integrals describe superposition of all such waves over the wavenumber spectrum $0 < k < \infty$.

For the classical Cauchy–Poisson wave problem, $\eta_0(x) = a\delta(x)$ where $\delta(x)$ is the Dirac delta function, so that $\tilde{\eta}_0(k) = a/\sqrt{2\pi}$. Thus, solution (1.9.70) becomes

$$\eta(x,t) = \frac{a}{2\pi} \int_0^{\infty} \bigl[e^{i(kx-\omega t)} + e^{i(kx+\omega t)}\bigr]\, dk. \tag{1.9.71}$$

The wave integrals (1.9.69) and (1.9.70) represent the exact solution for the velocity potential ϕ and the free surface elevation η for all x and $t > 0$. However, they do not lend any physical interpretations. In general, the exact evaluation of these integrals is almost a formidable task. So it is necessary to resort to asymptotic methods. It would be sufficient for the determination of the principal features of the wave motions to investigate (1.9.70) or (1.9.71) asymptotically for large time t and large distance x with (x,t) held fixed. The asymptotic solution for this kind of problem is available in many standard books; for example, see Debnath (1994, p. 85). We state the stationary phase approximation of a typical wave integral, for $t \to \infty$,

$$\eta(x,t) = \int_a^b f(k) \exp\bigl[itW(k)\bigr]\, dk \tag{1.9.72}$$

$$\sim f(k_1)\left[\frac{2\pi}{t|W''(k_1)|}\right]^{\frac{1}{2}} \exp\left[i\left\{tW(k_1) + \frac{\pi}{4}\operatorname{sgn} W''(k_1)\right\}\right], \tag{1.9.73}$$

where $W(k) = \frac{kx}{t} - \omega(k)$, $x > 0$, and $k = k_1$ is a stationary point that satisfies the equation

$$W'(k_1) = \frac{x}{t} - \omega'(k_1) = 0, \quad a < k_1 < b. \tag{1.9.74}$$

Application of (1.9.73) to (1.9.70) shows that only the first integral in (1.9.70) has a stationary point for $x > 0$. Hence, the stationary phase approximation gives the asymptotic solution, as $t \to \infty$, $x > 0$,

$$\eta(x,t) \sim \left[\frac{1}{t|\omega''(k_1)|}\right]^{\frac{1}{2}} \tilde{\eta}_0(k_1) \exp\left[i\left\{k_1 x - t\omega(k_1)\right\} + \frac{i\pi}{4}\operatorname{sgn}\left\{-\omega''(k_1)\right\}\right], \tag{1.9.75}$$

where $k_1 = (gt^2/4x^2)$ is the root of the equation $\omega'(k) = \frac{x}{t}$.

On the other hand, when $x < 0$, only the second integral of (1.9.70) has a stationary point $k_1 = (gt^2/4x^2)$, and hence, the same result (1.9.73) can be used to obtain the asymptotic solution for $t \to \infty$ and $x < 0$ as

$$\eta(x,t) \sim \left[\frac{1}{t|\omega''(k_1)|}\right]^{\frac{1}{2}} \tilde{\eta}_0(k_1) \exp\left[i\left\{t\omega(k_1) - k_1|x|\right\} + \frac{i\pi}{4}\operatorname{sgn}\omega''(k_1)\right]. \tag{1.9.76}$$

In particular, for the classical Cauchy–Poisson solution (1.9.71), the asymptotic representation for $\eta(x,t)$ follows from (1.9.76) in the form

$$\eta(x,t) \sim \frac{at}{2\sqrt{2\pi}} \frac{\sqrt{g}}{x^{3/2}} \cos\left(\frac{gt^2}{4x}\right), \quad gt^2 \gg 4x, \tag{1.9.77}$$

and gives a similar result for $\eta(x,t)$, when $x < 0$ and $t \to \infty$.

1.10 Hankel Transforms and Initial Boundary-Value Problems

The *Hankel transform* of a function $f(r)$ is defined formally by

$$\mathcal{H}_n\{f(r)\} = \tilde{f}_n(\kappa) = \int_0^\infty r J_n(\kappa r) f(r)\, dr, \tag{1.10.1}$$

where $J_n(\kappa r)$ is the Bessel function of order n and assuming the integral on the right-hand side is convergent.

The *inverse Hankel transform* is defined by

$$\mathcal{H}_n^{-1}\left[\tilde{f}_n(\kappa)\right] = f(r) = \int_0^\infty \kappa J_n(\kappa r) \tilde{f}_n(\kappa)\, d\kappa, \tag{1.10.2}$$

provided that the integral exists.

Integrals (1.10.1) and (1.10.2) exist for certain large classes of functions that usually occur in physical applications. In particular, the Hankel transforms of order zero $(n = 0)$ and of order one $(n = 1)$ are useful for solving initial-value and boundary-value problems involving Laplace's or Helmholtz's equations in an axisymmetric cylindrical geometry.

Example 1.10.1.

(a) $\tilde{f}(\kappa) = \mathcal{H}_0\{\frac{1}{r}\exp(-ar)\} = \int_0^\infty \exp(-ar) J_0(\kappa r)\, dr = \frac{1}{\sqrt{\kappa^2+a^2}}$,

(b) $\tilde{f}(\kappa) = \mathcal{H}_0\{\frac{\delta(r)}{r}\} = 1$,

(c) $\tilde{f}(\kappa) = \mathcal{H}_0\{H(a-r)\} = \int_0^a rJ_0(\kappa r)\,dr = \frac{a}{\kappa}J_1(a\kappa)$,
(d) $\tilde{f}(\kappa) = \mathcal{H}_1\{e^{-ar}\} = \int_0^\infty r\exp(-ar)J_1(\kappa r)\,dr = \frac{\kappa}{(a^2+\kappa^2)^{\frac{3}{2}}}$,
(e) $\tilde{f}(\kappa) = \mathcal{H}_1\{\frac{e^{-ar}}{r}\} = \int_0^\infty e^{-ar}J_1(\kappa r)\,dr = \frac{1}{\kappa}[1 - a(\kappa^2+a^2)^{-\frac{1}{2}}]$.

It can be shown (Debnath 1995) that

$$\mathcal{H}_n\left\{\left(\nabla^2 - \frac{n^2}{r^2}\right)f(r)\right\} = -\kappa^2\tilde{f}_n(\kappa), \tag{1.10.3}$$

where

$$\nabla^2 f = \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) = \frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}, \tag{1.10.4}$$

and $rf(r)$ and $rf'(r)$ vanish, as $r \to 0$ and $r \to \infty$.

In particular, when $n = 0$ and $n = 1$, (1.10.3) reduces to special results which are very useful for applications.

Example 1.10.2. Obtain the solution of the boundary-value problem

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad 0 \le r < \infty,\ z \ge 0, \tag{1.10.5}$$

$$u(r,0) = u_0 \quad \text{for } 0 \le r \le a,\ u_0 \text{ is a constant}, \tag{1.10.6}$$

$$u(r,z) \to 0 \quad \text{as } z \to \infty. \tag{1.10.7}$$

Application of the zero-order Hankel transform with respect to r to the problem gives

$$\frac{d^2\tilde{u}}{dz^2} - \kappa^2\tilde{u} = 0,$$

$$\tilde{u}(\kappa,0) = u_0\int_0^a rJ_0(\kappa r)\,dr = \frac{au_0}{\kappa}J_1(a\kappa).$$

Thus, the solution of the transformed problem is

$$\tilde{u}(\kappa,z) = \frac{au_0}{\kappa}J_1(a\kappa)\exp(-\kappa z).$$

Using the inverse Hankel transform gives the formal solution

$$u(r,z) = au_0\int_0^\infty J_0(r\kappa)J_1(a\kappa)\exp(-\kappa z)\,d\kappa. \tag{1.10.8}$$

Example 1.10.3 (Axisymmetric Wave Equation). Find the solution of the free vibration of a large circular membrane governed by the initial-value problem

$$c^2\left(u_{rr} + \frac{1}{r}u_r\right) = u_{tt}, \quad 0 \le r < \infty,\ t > 0, \tag{1.10.9}$$

$$u(r,0) = f(r), \quad u_t(r,0) = g(r), 0 \le r \le \infty, \tag{1.10.10}$$

where $c^2 = (T/\rho) =$ const., T is the tension in the membrane, ρ is the surface density of the membrane, and $f(r)$ and $g(r)$ are arbitrary functions.

Application of the zero-order Hankel transform $\tilde{u}(\kappa,t)$ of the displacement function $u(r,t)$ to (1.10.9), (1.10.10) gives the solution

$$\tilde{u}(\kappa,t) = \tilde{f}(\kappa)\cos(c\kappa t) + \frac{\tilde{g}(\kappa)}{c\kappa}\sin(c\kappa t). \tag{1.10.11}$$

The inverse Hankel transform leads to the solution

$$u(r,t) = \int_0^\infty \kappa \tilde{f}(\kappa)\cos(c\kappa t)J_0(\kappa r)\,d\kappa + \frac{1}{c}\int_0^\infty \tilde{g}(\kappa)\sin(c\kappa t)J_0(\kappa r)\,d\kappa. \tag{1.10.12}$$

In particular, we consider the initial data

$$u(r,0) = f(r) = Aa\left(r^2 + a^2\right)^{-\frac{1}{2}}, \qquad u_t(r,0) = g(r) = 0, \tag{1.10.13ab}$$

so that $\tilde{g}(\kappa) \equiv 0$ and

$$\tilde{f}(\kappa) = Aa\int_0^\infty r\left(a^2 + r^2\right)^{-\frac{1}{2}} J_0(\kappa r)\,dr = \frac{Aa}{\kappa}e^{-a\kappa}, \quad \text{by Example 1.10.1(a).}$$

Thus, the formal solution (1.10.12) becomes

$$\begin{aligned} u(r,t) &= Aa\int_0^\infty e^{-a\kappa}J_0(\kappa r)\cos(c\kappa t)\,d\kappa \\ &= Aa\,\text{Re}\int_0^\infty \exp\left[-\kappa(a+ict)\right]J_0(\kappa r)\,d\kappa \\ &= Aa\,\text{Re}\left\{r^2 + (a+ict)^2\right\}^{-\frac{1}{2}}, \quad \text{by Example 1.10.1(a).} \end{aligned} \tag{1.10.14}$$

Example 1.10.4 (Steady Temperature Distribution in a Semi-Infinite Solid with a Steady Heat Source). Find the solution of the Laplace equation for the steady temperature distribution $u(r,z)$ with a steady, symmetric heat source $Q_0 q(r)$:

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = -Q_0 q(r), \quad 0 \le r < \infty,\ 0 < z < \infty, \tag{1.10.15}$$

$$u(r,0) = 0, \quad 0 \le r < \infty, \tag{1.10.16}$$

where Q_0 is constant. This boundary condition represents zero temperature at the boundary $z = 0$.

Application of the zero-order Hankel transform to (1.10.15), (1.10.16) gives

$$\frac{d^2\tilde{u}}{dz^2} - \kappa^2\tilde{u} = -Q_0\tilde{q}(\kappa), \qquad \tilde{u}(\kappa,0) = 0.$$

The bounded general solution of this system is

$$\tilde{u}(\kappa, z) = A \exp(-\kappa z) + \frac{Q_0}{\kappa^2}\tilde{q}(\kappa),$$

where A is a constant to be determined from the transformed boundary condition. In this present case,

$$A = -\frac{Q_0}{\kappa^2}\tilde{q}(\kappa).$$

Thus, the formal solution is given by

$$\tilde{u}(\kappa, z) = \frac{Q_0 \tilde{q}(\kappa)}{\kappa^2}\left(1 - e^{-\kappa z}\right). \tag{1.10.17}$$

The inverse Hankel transform yields the exact integral solution

$$u(r, z) = Q_0 \int_0^\infty \frac{\tilde{q}(\kappa)}{\kappa}\left(1 - e^{-\kappa z}\right) J_0(\kappa r)\, d\kappa. \tag{1.10.18}$$

Example 1.10.5 (Axisymmetric Diffusion Equation). Find the solution of the axisymmetric diffusion equation

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r\right), \quad 0 \le r < \infty,\ t > 0, \tag{1.10.19}$$

where $\kappa (> 0)$ is a diffusivity constant, with the initial condition

$$u(r, 0) = f(r), \quad \text{for } 0 < r < \infty. \tag{1.10.20}$$

We apply the zero-order Hankel transform to obtain

$$\frac{d\tilde{u}}{dt} + k^2 \kappa \tilde{u} = 0, \qquad \tilde{u}(k, 0) = \tilde{f}(k),$$

where k is the Hankel transform variable. The solution of this transformed system is

$$\tilde{u}(k, t) = \tilde{f}(k) \exp\left(-\kappa k^2 t\right). \tag{1.10.21}$$

Application of the inverse Hankel transform gives

$$\begin{aligned} u(r, t) &= \int_0^\infty k \tilde{f}(k) J_0(kr) e^{-\kappa k^2 t}\, dk \\ &= \int_0^\infty k \left[\int_0^\infty l J_0(kl) f(l)\, dl\right] e^{-\kappa k^2 t} J_0(kr)\, dk, \end{aligned}$$

which, interchanging the order of integration,

$$= \int_0^\infty l f(l)\, dl \int_0^\infty k J_0(kl) J_0(kr) \exp\left(-\kappa k^2 t\right) dk. \tag{1.10.22}$$

Using a standard table of integrals involving Bessel functions, we state that

$$\int_0^\infty kJ_0(kl)J_0(kr)\exp\left(-\kappa k^2 t\right)dk = \frac{1}{2\kappa t}\exp\left[-\frac{(r^2+l^2)}{4\kappa t}\right]I_0\left(\frac{rl}{2\kappa t}\right), \tag{1.10.23}$$

where $I_0(x)$ is the modified Bessel function and $I_0(0)=1$. In particular, when $l=0$, $J_0(0)=1$, and integral (1.10.23) becomes

$$\int_0^\infty kJ_0(kr)\exp\left(-k^2\kappa t\right)dk = \frac{1}{2\kappa t}\exp\left(-\frac{r^2}{4\kappa t}\right). \tag{1.10.24}$$

We next use (1.10.23) to rewrite the solution (1.10.22) as

$$u(r,t) = \frac{1}{2\kappa t}\int_0^\infty lf(l)I_0\left(\frac{rl}{2\kappa t}\right)\exp\left[-\frac{(r^2+l^2)}{4\kappa t}\right]dl. \tag{1.10.25}$$

We now assume that $f(r)$ represents a heat source concentrated in a circle of radius a and allow $a\to 0$, so that the heat source is concentrated at $r=0$ and

$$\lim_{a\to 0} 2\pi\int_0^a rf(r)\,dr = 1.$$

Or equivalently,

$$f(r) = \frac{1}{2\pi}\frac{\delta(r)}{r},$$

where $\delta(r)$ is the Dirac delta function.

Thus, the final solution due to the concentrated heat source at $r=0$ is

$$\begin{aligned} u(r,t) &= \frac{1}{4\pi\kappa t}\int_0^\infty \delta(l)I_0\left(\frac{rl}{2\kappa t}\right)\exp\left[-\frac{r^2+l^2}{4\kappa t}\right]dl \\ &= \frac{1}{4\pi\kappa t}\exp\left(-\frac{r^2}{4\kappa t}\right). \end{aligned} \tag{1.10.26}$$

Example 1.10.6 (Axisymmetric Acoustic Radiation Problem). Obtain the solution of the wave equation

$$c^2\left(u_{rr}+\frac{1}{r}u_r+u_{zz}\right) = u_{tt}, \quad 0\le r<\infty,\ t>0, \tag{1.10.27}$$

$$u_z = F(r,t) \quad \text{on } z=0, \tag{1.10.28}$$

where $F(r,t)$ is a given function and c is a constant. We also assume that the solution is bounded and behaves as outgoing spherical waves.

We seek a steady-state solution for the acoustic radiation potential $u = e^{i\omega t}\times\phi(r,z)$ with $F(r,t)=e^{i\omega t}f(r)$, so that ϕ satisfies the Helmholtz equation

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} + \frac{\omega^2}{c^2}\phi = 0, \quad 0 \le r < \infty, z \ge 0, \tag{1.10.29}$$

with the boundary condition

$$\phi_z = f(r) \quad \text{on } z = 0, \tag{1.10.30}$$

where $f(r)$ is a given function of r.

Application of the Hankel transform $\mathcal{H}_0\{\phi(r,z)\} = \tilde{\phi}(k,z)$ to (1.10.29), (1.10.30) gives

$$\begin{aligned} \tilde{\phi}_{zz} &= \kappa^2 \tilde{\phi}, \quad z > 0, \\ \tilde{\phi}_z &= \tilde{f}(k), \quad \text{on } z = 0, \end{aligned}$$

where

$$\kappa = \left(k^2 - \frac{\omega^2}{c^2}\right)^{\frac{1}{2}}.$$

The solution of this differential equation system is

$$\tilde{\phi}(k,z) = -\frac{1}{\kappa}\tilde{f}(k)\exp(-\kappa z), \tag{1.10.31}$$

where κ is real and positive for $k > \omega/c$ and purely imaginary for $k < \omega/c$.

The inverse Hankel transform yields the formal solution

$$\phi(r,z) = -\int_0^\infty \frac{k}{\kappa}\tilde{f}(k)J_0(kr)\exp(-\kappa z)\,dk. \tag{1.10.32}$$

Since the exact evaluation of this integral is difficult for an arbitrary $\tilde{f}(k)$, we choose a simple form of $f(r)$ as

$$f(r) = AH(a - r), \tag{1.10.33}$$

where A is a constant, and hence, $\tilde{f}(k) = \frac{Aa}{k}J_1(ak)$.

Thus, the solution (1.10.32) takes the form

$$\phi(r,z) = -Aa\int_0^\infty \frac{1}{\kappa}J_1(ak)J_0(kr)\exp(-\kappa z)\,dk. \tag{1.10.34}$$

For an asymptotic evaluation of this integral, it is convenient to express (1.10.34) in terms of the distance R from the z-axis, so that $R^2 = (r^2 + z^2)$ and $z = R\cos\theta$. We use the asymptotic result for the Bessel function in the form

$$J_0(kr) \sim \left(\frac{2}{\pi k r}\right)^{\frac{1}{2}} \cos\left(kr - \frac{\pi}{4}\right) \quad \text{as } r \to \infty, \tag{1.10.35}$$

where $r = R\sin\theta$. Consequently, (1.10.34) combined with $u = \exp(i\omega t)\phi$ becomes

$$u \sim -\frac{Aa\sqrt{2}e^{i\omega t}}{\sqrt{\pi R \sin\theta}} \int_0^\infty \frac{1}{\kappa\sqrt{k}} J_1(ak) \cos\left(kR\sin\theta - \frac{\pi}{4}\right) \exp(-\kappa z)\, dk.$$

This integral can be evaluated asymptotically for $R \to \infty$, by using the stationary phase approximation formula to obtain the final result

$$u \sim -\frac{Aac}{\omega R \sin\theta} J_1(ak_1) \exp\left[i\left(\omega t - \frac{\omega R}{c}\right)\right], \tag{1.10.36}$$

where $k_1 = \omega/c \sin\theta$ is the stationary point. Physically, this solution represents outgoing spherical waves with constant velocity c and decaying amplitude, as $R \to \infty$.

Example 1.10.7 (Axisymmetric Biharmonic Equation). We solve the axisymmetric boundary-value problem

$$\nabla^4 u(r,z) = 0, \quad 0 \le r < \infty,\ z > 0 \tag{1.10.37}$$

with the boundary data

$$u(r,0) = f(r), \quad 0 \le r < \infty, \tag{1.10.38}$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{on } z = 0,\ 0 \le r < \infty, \tag{1.10.39}$$

$$u(r,z) \to 0 \qquad \text{as } r \to \infty, \tag{1.10.40}$$

where the axisymmetric biharmonic operator is

$$\nabla^4 = \nabla^2(\nabla^2) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right). \tag{1.10.41}$$

The use of the Hankel transform $\mathcal{H}_0\{u(r,z)\} = \tilde{u}(k,z)$ for this problem gives

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 \tilde{u}(k,z) = 0, \quad z > 0, \tag{1.10.42}$$

$$\tilde{u}(k,0) = \tilde{f}(k), \quad \frac{d\tilde{u}}{dz} = 0 \text{ on } z = 0. \tag{1.10.43}$$

The bounded solution of equation (1.10.42) is

$$\tilde{u}(k,z) = (A + zB)\exp(-kz), \tag{1.10.44}$$

where A and B are integration constants to be determined by (1.10.43) as $A = \tilde{f}(k)$ and $B = k\tilde{f}(k)$. Thus, solution (1.10.44) becomes

$$\tilde{u}(k,z) = (1 + kz)\tilde{f}(k)\exp(-kz). \tag{1.10.45}$$

The inverse Hankel transform gives the formal solution

$$u(r,z) = \int_0^\infty k(1 + kz)\tilde{f}(k) J_0(kr) \exp(-kz)\, dk. \tag{1.10.46}$$

Example 1.10.8 (The Axisymmetric Cauchy–Poisson Water Wave Problem). We consider the initial-value problem (Debnath 1994) for inviscid water of finite depth h with a free horizontal surface at $z = 0$ and the z-axis positive upward. We assume that the liquid has constant density ρ with no surface tension. The surface waves are generated in water which is initially at rest for $t < 0$ by the prescribed free surface elevation. In cylindrical polar coordinates (r, θ, z), the axisymmetric water wave equations for the velocity potential $\phi(r, z, t)$ and the free surface elevation $\eta(r, t)$ are

$$\nabla^2\phi = \phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \quad 0 \le r < \infty, -h \le z \le 0, t > 0, \tag{1.10.47}$$

$$\left.\begin{aligned} \phi_z - \eta_t = 0, \\ \phi_t + g\eta = 0 \end{aligned}\right\} \quad \text{on } z = 0,\ t > 0, \tag{1.10.48ab}$$

$$\phi_z = 0, \quad \text{on } z = -h,\ t > 0, \tag{1.10.49}$$

where g is the constant gravitational acceleration. The initial conditions are

$$\phi(r, 0, 0) = 0 \quad \text{and} \quad \eta(r, 0) = \eta_0(r), \quad \text{for } 0 \le r < \infty, \tag{1.10.50ab}$$

where $\eta_0(r)$ is the prescribed free surface elevation.

We apply the joint Laplace and the zero-order Hankel transform defined by

$$\tilde{\bar{\phi}}(k, z, s) = \int_0^\infty e^{-st}\,dt \int_0^\infty rJ_0(kr)\phi(r, z, t)\,dr, \tag{1.10.51}$$

to (1.10.47)–(1.10.49) so that these equations reduce to

$$\left(\frac{d^2}{dz^2} - k^2\right)\tilde{\bar{\phi}} = 0, \quad -h \le z \le 0,$$

$$\left.\begin{aligned} \tfrac{d\tilde{\bar{\phi}}}{dz} - s\tilde{\bar{\eta}} = -\tilde{\eta}_0(k), \\ s\tilde{\bar{\phi}} + g\tilde{\bar{\eta}} = 0 \end{aligned}\right\} \quad \text{on } z = 0,$$

$$\tilde{\bar{\phi}}_z = 0 \quad \text{on } z = -h,$$

where $\tilde{\eta}_0(k)$ is the zero-order Hankel transform of $\eta_0(r)$.

The solutions of this system are

$$\tilde{\bar{\phi}}(k, z, s) = -\frac{g\tilde{\eta}_0(k)}{(s^2 + \omega^2)}\,\frac{\cosh k(z + h)}{\cosh kh}, \tag{1.10.52}$$

$$\tilde{\bar{\eta}}(k, z, s) = \frac{s\tilde{\eta}_0(k)}{(s^2 + \omega^2)}, \tag{1.10.53}$$

where

$$\omega^2 = gk\tanh(kh) \tag{1.10.54}$$

is the famous *dispersion relation* between frequency ω and wavenumber k for water waves in a liquid of depth h. Physically, this dispersion relation describes the interaction between the inertial and gravitational forces.

Application of the inverse transforms gives the integral solutions

$$\phi(r,z,t) = -g\int_0^\infty kJ_0(kr)\tilde{\eta}_0(k)\left(\frac{\sin\omega t}{\omega}\right)\frac{\cosh k(z+h)}{\cosh kh}\,dk, \tag{1.10.55}$$

$$\eta(r,t) = \int_0^\infty kJ_0(kr)\tilde{\eta}_0(k)\cos\omega t\,dk. \tag{1.10.56}$$

These wave integrals represent exact solutions for ϕ and η for all r and t, but the physical features of the wave motions cannot be described by them. In general, the exact evaluation of the integrals is almost a formidable task. In order to resolve this difficulty, it is necessary and useful to resort to asymptotic methods. It will be sufficient for the determination of the basic features of the wave motions to evaluate (1.10.55) or (1.10.56) asymptotically for a large time and distance with (r/t) held fixed. We now replace $J_0(kr)$ by its asymptotic formula (1.10.35) for $kr\to\infty$, so that (1.10.56) gives

$$\begin{aligned}\eta(r,t) &\sim \left(\frac{2}{\pi r}\right)^{\frac{1}{2}}\int_0^\infty \sqrt{k}\tilde{\eta}_0(k)\cos\left(kr-\frac{\pi}{4}\right)\cos\omega t\,dk\\ &= (2\pi r)^{-\frac{1}{2}}\operatorname{Re}\int_0^\infty \sqrt{k}\tilde{\eta}_0(k)\exp\left[i\left(\omega t - kr + \frac{\pi}{4}\right)\right]dk. \end{aligned} \tag{1.10.57}$$

Application of the stationary phase method to (1.10.57) yields the solution

$$\eta(r,t) \sim \left[\frac{k_1}{rt|\omega''(k_1)|}\right]^{\frac{1}{2}}\tilde{\eta}_0(k_1)\cos\left[t\omega(k_1) - k_1 r\right], \tag{1.10.58}$$

where the stationary point $k_1 = (gt^2/4r^2)$ is the root of the equation

$$\omega'(k) = \frac{r}{t}. \tag{1.10.59}$$

For sufficiently deep water, $kr\to\infty$, the dispersion relation becomes

$$\omega^2 = gk. \tag{1.10.60}$$

The solution of the axisymmetric Cauchy–Poisson problem is obtained by using a prescribed initial displacement of unit volume concentrated at the origin, which means that $\eta_0(r) = (a/2\pi r)\delta(r)$, so that $\tilde{\eta}_0(k) = \frac{a}{2\pi}$. Thus, the asymptotic solution is obtained from (1.10.58) in the form

$$\eta(r,t) \sim \frac{agt^2}{4\pi\sqrt{2}r^3}\cos\left(\frac{gt^2}{4r}\right), \quad gt^2 \gg 4r. \tag{1.10.61}$$

It is noted that solution (1.10.58) is no longer valid when $\omega''(k_1) = 0$. This case can be handled by a modification of the asymptotic evaluation (see Debnath 1994, p. 91).

1.11 Green's Functions and Boundary-Value Problems

Many physical problems are described by second-order nonhomogeneous differential equations with homogeneous boundary conditions or by second-order homogeneous equations with nonhomogeneous boundary conditions. Such problems can be solved by a powerful method based on a device known as Green's functions.

We consider a nonhomogeneous partial differential equation of the form

$$L_{\mathbf{x}} u(\mathbf{x}) = f(\mathbf{x}), \tag{1.11.1}$$

where $\mathbf{x} = (x, y, z)$ is a vector in three (or higher) dimensions, $L_{\mathbf{x}}$ is a linear partial differential operator in three or more independent variables with constant coefficients, and $u(\mathbf{x})$ and $f(\mathbf{x})$ are functions of three or more independent variables. The Green function $G(\mathbf{x}, \boldsymbol{\xi})$ of this problem satisfies the equation

$$L_{\mathbf{x}} G(\mathbf{x}, \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}) \tag{1.11.2}$$

and represents the effect at the point $\mathbf{x}$ of the Dirac delta function source of the point $\boldsymbol{\xi} = (\xi, \eta, \zeta)$.

Multiplying (1.11.2) by $f(\boldsymbol{\xi})$ and integrating over the volume V of the $\boldsymbol{\xi}$ space, so that $dV = d\xi\, d\eta\, d\zeta$, we obtain

$$\int_V L_{\mathbf{x}} G(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \int_V \delta(\mathbf{x} - \boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi} = f(\mathbf{x}). \tag{1.11.3}$$

Interchanging the order of the operator $L_{\mathbf{x}}$ and integral sign in (1.11.3) gives

$$L_{\mathbf{x}} \left[\int_V G(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi} \right] = f(\mathbf{x}). \tag{1.11.4}$$

A simple comparison of (1.11.4) with (1.11.1) leads to the solution of (1.11.1) in the form

$$u(\mathbf{x}) = \int_V G(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{1.11.5}$$

Clearly, (1.11.5) is valid for any infinite number of components of $\mathbf{x}$. Accordingly, the Green's function method can be applied, in general, to any linear, constant coefficient, inhomogeneous partial differential equations in any number of independent variables.

Another way to approach the problem is by looking for the inverse operator $L_{\mathbf{x}}^{-1}$. If it is possible to find $L_{\mathbf{x}}^{-1}$, then the solution of (1.11.1) can be obtained as $u(\mathbf{x}) = L_{\mathbf{x}}^{-1}(f(\mathbf{x}))$. It turns out that, in many important cases, it is possible, and the inverse operator can be expressed as an integral operator of the form

$$u(\mathbf{x}) = L_{\mathbf{x}}^{-1}\big(f(\boldsymbol{\xi})\big) = \int_V G(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi}. \tag{1.11.6}$$

The kernel $G(\mathbf{x}, \boldsymbol{\xi})$ is called the *Green's function* which is, in fact, the characteristic of the operator $L_{\mathbf{x}}$ for any finite number of independent variables.

The main goal of this section is to develop a general method of Green's function from several examples of applications.

Example 1.11.1 (Green's Function for the One-Dimensional Diffusion Equation). We consider the inhomogeneous, one-dimensional, diffusion equation

$$u_t - \kappa u_{xx} = f(x)\delta(t), \quad x \in \mathbb{R},\ t > 0, \tag{1.11.7}$$

with the boundary conditions

$$u(x,0) = 0 \quad \text{for } x \in \mathbb{R} \quad \text{and} \quad u(x,t) \to 0 \quad \text{as } |x| \to \infty. \tag{1.11.8ab}$$

We take the Laplace transform with respect to t and the Fourier transform with respect to x to (1.11.7), (1.11.8ab), so that

$$\tilde{\bar{u}}(k,s) = \frac{\tilde{f}(k)}{(s + \kappa k^2)}, \tag{1.11.9}$$

$$\tilde{\bar{u}}(k,t) \to 0 \quad \text{as } |k| \to \infty. \tag{1.11.10}$$

The inverse Laplace transform gives

$$\tilde{u}(k,t) = \tilde{f}(k)\exp\big(-\kappa k^2 t\big) = \tilde{f}(k)\tilde{g}(k), \tag{1.11.11}$$

where $\tilde{g}(k) = \exp(-\kappa k^2 t)$, so that

$$g(x) = \mathcal{F}^{-1}\big\{\exp\big(-\kappa k^2 t\big)\big\} = \frac{1}{\sqrt{2\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right). \tag{1.11.12}$$

Application of the inverse Fourier transform combined with Convolution Theorem 1.7.1 gives

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\tilde{g}(k)\,d\kappa = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\xi)\tilde{g}(x-\xi)\,d\xi \\ &= \frac{1}{\sqrt{4\kappa\pi t}}\int_{-\infty}^{\infty}\tilde{f}(\xi)\exp\left(-\frac{(x-\xi)^2}{4\kappa t}\right)d\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\xi)G(x,t;\xi)\,d\xi, \end{aligned} \tag{1.11.13}$$

where the Green function $G(x,t;\xi)$ is given by

$$G(x,t;\xi) = \frac{1}{\sqrt{4\pi\kappa t}}\exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]. \tag{1.11.14}$$

Evidently, $G(x,t) = G(x,t;0)$ is an even function of x, and at any time t, the spatial distribution of $G(x,t)$ is Gaussian. The amplitude (or peak height) of $G(x,t)$ decreases inversely with $\sqrt{\kappa t}$, whereas the width of the peak increases with $\sqrt{\kappa t}$. The evolution of $G(x,t) = u(x,t)$ has already been plotted against x for different values of $\tau = 2\sqrt{\kappa t}$ in Figure 1.6.

Example 1.11.2 (Green's Function for the Two-Dimensional Diffusion Equation). We consider the two-dimensional diffusion equation

$$u_t - K\nabla^2 u = f(x,y)\delta(t), \quad -\infty < x, y < \infty,\ t > 0, \tag{1.11.15}$$

with the initial and boundary conditions

$$u(x,y,0) = 0, \quad \text{for all } (x,y) \in \mathbb{R}^2, \tag{1.11.16}$$

$$u(x,y,t) \to 0 \quad \text{as } r = \sqrt{x^2+y^2} \to \infty, \tag{1.11.17}$$

where K is the diffusivity constant.

Application of the Laplace transform and the double Fourier transform to the above system gives

$$\tilde{\bar{u}}(\kappa, s) = \frac{\tilde{f}(k,l)}{(s + K\kappa^2)}, \tag{1.11.18}$$

where $\kappa = (k, l)$.

The inverse Laplace transform gives

$$\tilde{u}(\kappa, t) = \tilde{f}(k,l)\exp\left(-K\kappa^2 t\right) = \tilde{f}(k,l)\tilde{g}(k,l), \tag{1.11.19}$$

where $\tilde{g}(k,l) = \exp(-K\kappa^2 t)$, so that

$$g(x,y) = \mathcal{F}^{-1}\left\{\exp\left(-K\kappa^2 t\right)\right\} = \frac{1}{2Kt}\exp\left[-\frac{(x^2+y^2)}{4Kt}\right]. \tag{1.11.20}$$

Finally, the convolution theorem of the Fourier transform gives the formal solution

$$u(x,y,t) = \frac{1}{4\pi Kt}\iint_{-\infty}^{\infty} f(\xi,\eta)\exp\left[-\frac{(x-\xi)^2+(y-\eta)^2}{4Kt}\right] d\xi\, d\eta, \tag{1.11.21}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{\xi})G(\mathbf{r},\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{1.11.22}$$

where $\mathbf{r} = (x,y)$ and $\boldsymbol{\xi} = (\xi, \eta)$ and Green's function $G(\mathbf{r},\boldsymbol{\xi})$ is given by

$$G(\mathbf{r},\boldsymbol{\xi}) = \frac{1}{(4\pi Kt)}\exp\left[-\frac{|\mathbf{r}-\boldsymbol{\xi}|^2}{4Kt}\right]. \tag{1.11.23}$$

Similarly, we can construct Green's function for the three-dimensional diffusion equation

$$u_t - K\nabla^2 u = f(\mathbf{r})\delta(t) \quad \text{for } -\infty < x, y, z < \infty, t > 0, \tag{1.11.24}$$

with the initial and boundary data

$$u(\mathbf{r},0) = 0 \quad \text{for } -\infty < x,y,z < \infty, \tag{1.11.25}$$

$$u(\mathbf{r},t) \to 0 \quad \text{as } r = \left(x^2+y^2+z^2\right)^{\frac{1}{2}} \to \infty, \tag{1.11.26}$$

where $\mathbf{r} = (x,y,z)$.

Application of the Laplace transform of $u(\mathbf{r},t)$ with respect to t and the three-dimensional Fourier transform with respect to x, y, z gives the solution

$$u(\mathbf{r},t) = \frac{1}{(4\pi Kt)^{\frac{3}{2}}} \iiint_{-\infty}^{\infty} f(\boldsymbol{\xi}) \exp\left[-\frac{|\mathbf{r}-\boldsymbol{\xi}|^2}{4Kt}\right] d\boldsymbol{\xi}, \tag{1.11.27}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{\xi}) G(\mathbf{r},\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{1.11.28}$$

where $\boldsymbol{\xi} = (\xi,\eta,\zeta)$, and Green's function is given by

$$G(\mathbf{r},\boldsymbol{\xi}) = \frac{1}{(4\pi Kt)^{\frac{3}{2}}} \exp\left[-\frac{|\mathbf{r}-\boldsymbol{\xi}|^2}{4Kt}\right]. \tag{1.11.29}$$

In fact, the same method of construction can be used to find the Green function of the n-dimensional diffusion equation

$$u_t - K\nabla_n^2 u = f(\mathbf{r})\delta(t), \quad \mathbf{r} \in R^n,\ t > 0, \tag{1.11.30}$$

$$u(\mathbf{r},0) = 0 \quad \text{for all } \mathbf{r} \in R^n, \tag{1.11.31}$$

$$u(\mathbf{r},t) \to 0 \quad \text{as } r = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}} \to \infty, \tag{1.11.32}$$

where $\mathbf{r} = (x_1, x_2, \ldots, x_n)$ and ∇_n^2 is the n-dimensional Laplacian given by

$$\nabla_n^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{1.11.33}$$

The solution of this problem is given by

$$u(\mathbf{r},t) = \frac{1}{(4\pi Kt)^{\frac{n}{2}}} \int_{-\infty}^{\infty} f(\boldsymbol{\xi}) G(\mathbf{r},\boldsymbol{\xi})\, d\boldsymbol{\xi}, \tag{1.11.34}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ and the n-dimensional Green function $G(\mathbf{r},\boldsymbol{\xi})$ is given by

$$G(\mathbf{r},\boldsymbol{\xi}) = \frac{1}{(4\pi Kt)^{\frac{n}{2}}} \exp\left[-\frac{|\mathbf{r}-\boldsymbol{\xi}|^2}{4Kt}\right]. \tag{1.11.35}$$

Example 1.11.3 (The Three-Dimensional Poisson Equation). We show that the solution of the Poisson equation

$$-\nabla^2 u = f(\mathbf{r}), \tag{1.11.36}$$

where $\mathbf{r} = (x,y,z)$ is given by

$$u(\mathbf{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{r},\boldsymbol{\xi})f(\boldsymbol{\xi})\,d\boldsymbol{\xi}, \tag{1.11.37}$$

where *Green's function* $G(\mathbf{r},\boldsymbol{\xi})$ of the operator, $-\nabla^2$, is given by

$$G(\mathbf{r},\boldsymbol{\xi}) = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\boldsymbol{\xi}|}. \tag{1.11.38}$$

To obtain the fundamental solution, we need to solve the equation

$$-\nabla^2 G(\mathbf{r}-\boldsymbol{\xi}) = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta), \quad \mathbf{r}\neq\boldsymbol{\xi}. \tag{1.11.39}$$

Application of the three-dimensional Fourier transform, defined by (1.8.5), to (1.11.39) gives

$$\kappa^2\widehat{G}(\boldsymbol{\kappa},\boldsymbol{\xi}) = \frac{1}{(2\pi)^{3/2}}\exp(-i\boldsymbol{\kappa}\cdot\boldsymbol{\xi}), \tag{1.11.40}$$

where $\widehat{G}(\boldsymbol{\kappa},\boldsymbol{\xi}) = \mathcal{F}\{G(\mathbf{r},\boldsymbol{\xi})\}$ and $\boldsymbol{\kappa} = (k,\ell,m)$.

The inverse Fourier transform gives the formal solution

$$\begin{aligned} G(\mathbf{r},\boldsymbol{\xi}) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i\boldsymbol{\kappa}\cdot(\mathbf{r}-\boldsymbol{\xi})\}\frac{d\boldsymbol{\kappa}}{\kappa^2} \\ &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(i\boldsymbol{\kappa}\cdot\mathbf{R})\frac{d\boldsymbol{\kappa}}{\kappa^2}, \end{aligned} \tag{1.11.41}$$

where $\mathbf{R} = \mathbf{r}-\boldsymbol{\xi}$.

We evaluate this integral using the spherical polar coordinates in the $\boldsymbol{\kappa}$-space with the axis along the $\mathbf{R}$-axis. In terms of spherical polar coordinates (κ,θ,ϕ), $\boldsymbol{\kappa}\cdot\mathbf{R} = \kappa R\cos\theta$ where $R = |\mathbf{r}-\boldsymbol{\xi}|$. Thus, (1.11.41) becomes

$$\begin{aligned} G(\mathbf{r},\boldsymbol{\xi}) &= \frac{1}{(2\pi)^3}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\int_0^{\infty}\exp(\kappa R\cos\theta)\kappa^2\sin\theta\cdot\frac{d\kappa}{\kappa^2} \\ &= \frac{1}{(2\pi)^2}\int_0^{\infty}2\frac{\sin(\kappa R)}{\kappa R}\,d\kappa = \frac{1}{4\pi R} = \frac{1}{4\pi|\mathbf{r}-\boldsymbol{\xi}|}, \end{aligned} \tag{1.11.42}$$

provided that $R > 0$.

In electrodynamics, the fundamental solution (1.11.42) has a well-known interpretation. Physically, it represents the potential at point $\mathbf{r}$ generated by the unit point charge distribution at point $\boldsymbol{\xi}$. This is what can be expected because $\delta(\mathbf{r}-\boldsymbol{\xi})$ is the charge density corresponding to a unit point charge at $\boldsymbol{\xi}$.

The solution of (1.11.36) is then given by

$$u(\mathbf{r}) = \iiint_{-\infty}^{\infty} G(\mathbf{r},\boldsymbol{\xi})f(\boldsymbol{\xi})\,d\boldsymbol{\xi} = \frac{1}{4\pi}\iiint_{-\infty}^{\infty}\frac{f(\boldsymbol{\xi})\,d\boldsymbol{\xi}}{|\mathbf{r}-\boldsymbol{\xi}|}. \tag{1.11.43}$$

The integrand in (1.11.43) consists of the given charge distribution $f(\mathbf{r})$ at $\mathbf{r} = \boldsymbol{\xi}$ and Green's function $G(\mathbf{r},\boldsymbol{\xi})$. Physically, $G(\mathbf{r},\boldsymbol{\xi})f(\boldsymbol{\xi})$ represents the resulting potentials due to elementary point charges, and the total potential due to a given charge distribution $f(\mathbf{r})$ is then obtained by the integral superposition of the resulting potentials. This is called the *principle of superposition*.

Example 1.11.4 (The Two-Dimensional Helmholtz Equation). Find the fundamental solution of the two-dimensional Helmholtz equation

$$-\nabla^2 G + \alpha^2 G = \delta(x-\xi)\delta(y-\eta), \quad -\infty < x, y < \infty. \tag{1.11.44}$$

It is convenient to change variables $x - \xi = x^*$, $y - \eta = y^*$. Consequently, dropping the asterisks, (1.11.44) reduces to the form

$$G_{xx} + G_{yy} - \alpha^2 G = -\delta(x)\delta(y). \tag{1.11.45}$$

Application of the double Fourier transform $\widehat{G}(\boldsymbol{\kappa}) = \mathcal{F}\{G(x,y)\}$ to (1.11.45) gives the solution as

$$\widehat{G}(\boldsymbol{\kappa}) = \frac{1}{2\pi}\frac{1}{(\kappa^2+\alpha^2)}, \tag{1.11.46}$$

where $\boldsymbol{\kappa} = (k, \ell)$ and $\kappa^2 = k^2 + \ell^2$.

The inverse Fourier transform yields the solution

$$G(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(i\boldsymbol{\kappa}\cdot\mathbf{x})\big(\kappa^2+\alpha^2\big)^{-1}\,dk\,d\ell. \tag{1.11.47}$$

In terms of polar coordinates $(x,y) = r(\cos\theta, \sin\theta)$, $(k,\ell) = \rho(\cos\phi, \sin\phi)$, the integral solution (1.11.47) becomes

$$G(x,y) = \frac{1}{4\pi^2}\int_0^{\infty}\frac{\rho\,d\rho}{(\rho^2+\alpha^2)}\int_0^{2\pi}\exp\big\{ir\rho\cos(\phi-\theta)\big\}\,d\phi,$$

which, replacing the second integral by $2\pi J_0(r\rho)$,

$$= \frac{1}{2\pi}\int_0^{\infty}\frac{\rho J_0(r\rho)\,d\rho}{(\rho^2+\alpha^2)}. \tag{1.11.48}$$

In terms of the original coordinates, the fundamental solution of (1.11.44) is given by

$$G(\mathbf{r},\boldsymbol{\xi}) = \frac{1}{2\pi}\int_0^{\infty}\frac{\rho J_0[\rho\{(x-\xi)^2+(y-\eta)^2\}^{\frac{1}{2}}]\,d\rho}{(\rho^2+\alpha^2)}. \tag{1.11.49}$$

Accordingly, the solution of the inhomogeneous equation

$$\big(\nabla^2 - \alpha^2\big)u = -f(x,y) \tag{1.11.50}$$

is given by

$$u(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{r},\boldsymbol{\xi})f(\boldsymbol{\xi})\,d\boldsymbol{\xi}, \tag{1.11.51}$$

where $G(\mathbf{r},\boldsymbol{\xi})$ is given by (1.11.49).

Since the integral solution (1.11.48) does not exist for $\alpha = 0$, Green's function for the two-dimensional Poisson equation (1.11.44) cannot be derived from (1.11.48). Instead, we differentiate (1.11.48) with respect to r to obtain

$$\frac{\partial G}{\partial r} = \frac{1}{2\pi}\int_0^\infty \frac{\rho J_0'(r\rho)\,d\rho}{(\rho^2+\alpha^2)},$$

which is, for $\alpha = 0$,

$$\frac{\partial G}{\partial r} = \frac{1}{2\pi}\int_0^\infty \frac{1}{\rho}J_0'(r\rho)\,d\rho = -\frac{1}{2\pi r}.$$

Integrating this result gives Green's function

$$G(r,\theta) = -\frac{1}{2\pi}\log r.$$

In terms of the original coordinates, Green's function becomes

$$G(\mathbf{r},\boldsymbol{\xi}) = \frac{1}{4\pi}\log\bigl[(x-\xi)^2+(y-\eta)^2\bigr]. \tag{1.11.52}$$

This is Green's function for the two-dimensional Poisson equation $\nabla^2 = -f(x,y)$. Thus, the solution of the Poisson equation is

$$u(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\mathbf{r},\boldsymbol{\xi})f(\boldsymbol{\xi})\,d\boldsymbol{\xi}, \tag{1.11.53}$$

where $G(\mathbf{r},\boldsymbol{\xi})$ is given by (1.11.52).

Example 1.11.5 (Green's Function of the Three-Dimensional Helmholtz Equation). We consider the three-dimensional wave equation

$$\bigl[u_{tt} - c^2\nabla^2 u\bigr] = q(\mathbf{r},t), \tag{1.11.54}$$

where $q(\mathbf{r},t)$ is a source. If $q(\mathbf{r},t) = q(\mathbf{r})\exp(-i\omega t)$ represents a source oscillating with a single frequency ω, then, as expected, at least after an initial transient period, the entire motion reduces to a wave motion with the same frequency ω so that we can write $u(\mathbf{r},t) = u(\mathbf{r})\exp(-i\omega t)$. Consequently, the wave equation (1.11.54) reduces to the three-dimensional Helmholtz equation

$$-\bigl(\nabla^2+k^2\bigr)u(\mathbf{r}) = f(\mathbf{r}), \tag{1.11.55}$$

where $k = \frac{\omega}{c}$ and $f(\mathbf{r}) = c^{-2}q(\mathbf{r})$. The function $u(\mathbf{r})$ satisfies this equation on some domain $D \subset \mathbb{R}^3$ with boundary ∂D, and it also satisfies some prescribed boundary conditions. We also assume that $u(\mathbf{r})$ satisfies the Sommerfeld radiation condition which simply states that the solution behaves like outgoing waves generated by the source. In the limit as $\omega \to 0$, so that $k \to 0$ and $f(\mathbf{r})$ can be interpreted as a heat source, equation (1.11.55) results in a three-dimensional Poisson equation. The solution $u(\mathbf{r})$ would represent the steady temperature distribution in D due to the heat source $f(\mathbf{r})$. However, in general, $u(\mathbf{r})$ can be interpreted as a function of physical interest.

We construct a Green function $G(\mathbf{r},\boldsymbol{\xi})$ for equation (1.11.55) so that $G(\mathbf{r},\boldsymbol{\xi})$ satisfies the equation

$$-\left(\nabla^2+k^2\right)G=\delta(x)\delta(y)\delta(z). \tag{1.11.56}$$

Using the spherical polar coordinates, the three-dimensional Laplacian can be expressed in terms of radial coordinate r only so that (1.11.56) assumes the form

$$-\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right)+k^2G\right]=\frac{\delta(r)}{4\pi r^2}, \quad 0<r<\infty, \tag{1.11.57}$$

with the radiation condition

$$\lim_{r\to\infty} r(G_r+ikG)=0. \tag{1.11.58}$$

For $r>0$, the function G satisfies the homogeneous equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right)+k^2G=0, \tag{1.11.59}$$

or equivalently,

$$\frac{\partial}{\partial r}(rG)+k^2(rG)=0. \tag{1.11.60}$$

This equation admits a solution of the form

$$rG(r)=Ae^{ikr}+Be^{-ikr}, \tag{1.11.61}$$

or

$$G(r)=A\frac{e^{ikr}}{r}+B\frac{e^{-ikr}}{r}, \tag{1.11.62}$$

where A and B are arbitrary constants. In order to satisfy the radiation condition, we need to set $A=0$, and hence, the solution (1.11.62) becomes

$$G(r)=B\frac{e^{-ikr}}{r}. \tag{1.11.63}$$

To determine B, we use the spherical surface S_ε of radius ε, so that

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon}\frac{\partial G}{\partial r}dS=-\lim_{\varepsilon\to 0}\int_{S_\varepsilon}\frac{B}{r}e^{-ikr}\left(\frac{1}{r}+ik\right)dS=1, \tag{1.11.64}$$

from which we find $B=-\frac{1}{4\pi}$ as $\varepsilon\to 0$. Consequently, Green's function takes the form

$$G(r) = -\frac{e^{-ikr}}{4\pi r}. \tag{1.11.65}$$

Physically, this represents outgoing spherical waves radiating away from the source at the origin. With a point source at a point $\boldsymbol{\xi}$, Green's function is represented by

$$G(\mathbf{r}, \boldsymbol{\xi}) = -\frac{\exp\{-ik|\mathbf{r}-\boldsymbol{\xi}|\}}{4\pi|\mathbf{r}-\boldsymbol{\xi}|}, \tag{1.11.66}$$

where $\mathbf{r}$ and $\boldsymbol{\xi}$ are position vectors in $\mathbb{R}^3$.

Finally, when $k = 0$, this result reduces exactly to Green's function for the three-dimensional Poisson equation (1.11.36).

Example 1.11.6 (One-Dimensional Wave Equation). We first consider the one-dimensional inhomogeneous wave equation

$$-\left[u_{xx} - \frac{1}{c^2}u_{tt}\right] = c^{-2}q(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{1.11.67}$$

with the initial and boundary conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0 \quad \text{for } x \in \mathbb{R}, \tag{1.11.68ab}$$

$$u(x,t) \to 0 \quad \text{as } |x| \to \infty. \tag{1.11.69}$$

The Green function $G(x,t)$ for this problem satisfies the equation

$$-\left[G_{xx} - \frac{1}{c^2}G_{tt}\right] = c^{-2}\delta(x)\delta(t) \tag{1.11.70}$$

and the same initial and boundary conditions (1.11.68ab), (1.11.69) satisfied by $u(x,t)$.

We apply the joint Laplace transform with respect to t and the Fourier transform with respect to x to equation (1.11.70), so that

$$\tilde{\overline{G}}(k,s) = \frac{1}{\sqrt{2\pi}} \frac{c^{-2}}{(k^2 + \frac{s^2}{c^2})}, \tag{1.11.71}$$

where k and s represent the Fourier and Laplace transform variables, respectively.

The inverse Fourier transform gives

$$\overline{G}(x,s) = \frac{1}{c^2}\mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}} \frac{1}{(k^2 + \frac{s^2}{c^2})}\right\} = \frac{1}{2sc}\exp\left(-\frac{s}{c}|x|\right). \tag{1.11.72}$$

Finally, the inverse Laplace transform yields Green's function with a source at the origin

$$G(x,t) = \frac{1}{2c}\mathcal{L}^{-1}\left\{\frac{1}{s}\exp\left(-\frac{s}{c}|x|\right)\right\} = \frac{1}{2c}H\left(t - \frac{|x|}{c}\right), \tag{1.11.73}$$

where $H(z)$ is the Heaviside unit step function.

With a point source at (ξ, τ), Green's function has the form

$$G(x,t;\xi,\tau) = \frac{1}{2c} H\left(t - \tau - \frac{|x-\xi|}{c}\right). \tag{1.11.74}$$

This function is also called the *Riemann function* for the wave equation. The result (1.11.74) shows that $G = 0$ unless the point (x,t) lies within the *characteristic cone* defined by the inequality $c(t-\tau) > |x-\xi|$.

The solution of equation (1.11.67) is

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} d\xi \int_0^t G(x,t;\xi,\tau) q(\xi,\tau)\, d\tau, \\ &= \frac{1}{2c} \int_{-\infty}^{\infty} d\xi \int_0^t H\left(t - \tau - \frac{|x-\xi|}{c}\right) q(\xi,\tau)\, d\tau, \end{aligned} \tag{1.11.75}$$

which, since $H = 1$ for $x - c(t-\tau) < \xi < x + c(t-\tau)$ and zero outside,

$$= \frac{1}{2c} \int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} q(\xi,\tau)\, d\xi = \frac{1}{2c} \iint_D q(\xi,\tau)\, d\tau\, d\xi, \tag{1.11.76}$$

where D is the triangular domain made up of two points $x \mp ct$ on the x-axis and another point (x,t) off the x-axis in the (x,t)-plane.

Thus, the solution of the general Cauchy problem described in Example 1.9.1 can be obtained by adding (1.11.75) to the d'Alembert solution (1.7.21), and hence, it reduces to (1.9.15).

Example 1.11.7 (Green's Function for an Axisymmetric Wave Equation). We consider the two-dimensional wave equation in polar coordinates

$$-\left[\nabla^2 u - \frac{1}{c^2} u_{tt}\right] = f(r,t), \quad 0 < r < \infty,\ t > 0, \tag{1.11.77}$$

where the Laplacian ∇^2 in cylindrical polar coordinates without θ-dependence is given by

$$\nabla^2 \equiv \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) \tag{1.11.78}$$

and $u = u(r,t)$ and $f(r,t)$ are the given functions of r and t. We also prescribe appropriate initial conditions

$$u(r,0) = 0, \qquad u_t(r,0) = 0. \tag{1.11.79}$$

Green's function $G(r,t)$ must satisfy the equation

$$-\left[\nabla^2 G - \frac{1}{c^2} G_{tt}\right] = \frac{1}{2\pi r} \delta(r)\delta(t), \tag{1.11.80}$$

with the same initial conditions (1.11.79).

We apply the joint zero-order Hankel transform with respect to r and the Laplace transform with respect to t to (1.11.80) to obtain

$$\bar{\tilde{G}}(\kappa, s) = \frac{c^2}{2\pi(s^2 + c^2\kappa^2)}, \tag{1.11.81}$$

where κ and s are the Hankel and Laplace transform variables, respectively. The inverse Laplace transform gives

$$\tilde{G}(\kappa, s) = \frac{c}{2\pi\kappa}\sin(c\kappa t). \tag{1.11.82}$$

Then, the inverse Hankel transform yields the solution for $G(r,t)$ as

$$\begin{aligned} G(r,t) &= \frac{c}{2\pi}\mathcal{H}_0^{-1}\left\{\frac{1}{\kappa}\sin(c\kappa t)\right\} \\ &= \frac{c\mathcal{H}(ct-r)}{2\pi(c^2t^2-r^2)^{\frac{1}{2}}} = \frac{c\mathcal{H}(t-\frac{r}{c})}{2\pi(t^2-\frac{r^2}{c^2})^{\frac{1}{2}}}. \end{aligned} \tag{1.11.83}$$

This represents the Green function for the two-dimensional wave equation with a source at $(0,0)$.

If this source is placed at the point $\mathbf{r} = \boldsymbol{\xi}$, the Green function satisfies the equation

$$-\left[\nabla^2 G - \frac{1}{c^2}G_{tt}\right] = \delta(x-\xi)\delta(y-\eta)\delta(t-\tau). \tag{1.11.84}$$

Introducing $R = |\mathbf{r} - \xi|$ and $T = t - \tau$, we can rewrite (1.11.84) in the form

$$-\left[\frac{1}{R}\cdot\frac{\partial}{\partial R}\left(R\frac{\partial G}{\partial R}\right) - \frac{1}{c^2}G_{TT}\right] = \frac{\delta(R)}{2\pi R}\delta(T). \tag{1.11.85}$$

This is identical with (1.11.80), and hence, Green's function is given by

$$G(R,T) = \frac{c\mathcal{H}(T-\frac{R}{c})}{2\pi(T^2-\frac{R^2}{c^2})^{\frac{1}{2}}} = \frac{c\mathcal{H}\{(t-\tau)-\frac{|\mathbf{r}-\xi|}{c}\}}{2\pi\{(t-\tau)^2-\frac{1}{c^2}|\mathbf{r}-\xi|^2\}^{\frac{1}{2}}}. \tag{1.11.86}$$

Example 1.11.8 (Green's Function for the Three-Dimensional Inhomogeneous Wave Equation). The three-dimensional inhomogeneous wave equation is given by

$$-\left[\nabla^2 u - \frac{1}{c^2}u_{tt}\right] = f(\mathbf{r},t), \quad -\infty < x,y,z < \infty,\ t>0, \tag{1.11.87}$$

where $\mathbf{r} = (x,y,z)$, and the Laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{1.11.88}$$

The initial and boundary conditions are

$$u(r,0) = 0, \quad u_t(\mathbf{r},t) = 0, \tag{1.11.89ab}$$

$$u_t(\mathbf{r},t) \to 0, \quad \text{as } r \to \infty. \tag{1.11.90}$$

Green's function $G(\mathbf{x},t)$ for this problem satisfies the equation

$$-\left[\nabla^2 G - \frac{1}{c^2} G_{tt}\right] = \delta(x)\delta(y)\delta(z)\delta(t), \quad -\infty < x,y,z < \infty, t > 0, \tag{1.11.91}$$

with the same initial and boundary data (1.11.89ab), (1.11.90).

Application of the joint Laplace and Fourier transform gives

$$\tilde{\overline{G}}(\boldsymbol{\kappa},s) = \frac{c^2}{(2\pi)^{3/2}} \cdot \frac{1}{(s^2 + c^2\kappa^2)}, \quad \boldsymbol{\kappa} = (k,\ell,m). \tag{1.11.92}$$

The joint inverse transform yields the integral solution

$$G(\mathbf{x},t) = \frac{c}{(2\pi)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin(c\kappa t)}{\kappa} \exp(i\boldsymbol{\kappa}\cdot\mathbf{x})\, d\boldsymbol{\kappa}. \tag{1.11.93}$$

In terms of the spherical polar coordinates with the polar axis along the vector $\mathbf{x}$, so that $\boldsymbol{\kappa}\cdot\mathbf{x} = \kappa r\cos\theta$, $\mathbf{r} = |x|$ and $\mathbf{d}\boldsymbol{\kappa} = \kappa^2\, d\kappa \sin\theta\, d\theta\, d\phi$, integral (1.11.93) assumes the form

$$G(\mathbf{x},t) = \frac{c}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\infty} \kappa\sin(c\kappa t)\, d\kappa \int_0^{\pi} \exp(i\kappa r\cos\theta)\sin\theta\, d\theta, \tag{1.11.94}$$

$$\begin{aligned}
&= \frac{c}{4\pi^2 ri} \int_0^{\infty} \left(e^{i\kappa r} - e^{-i\kappa r}\right)\sin(c\kappa t)\, d\kappa \\
&= -\frac{c}{8\pi^2 r} \int_0^{\infty} \left(e^{i\kappa r} - e^{-i\kappa r}\right)\left(e^{ic\kappa t} - e^{-ic\kappa t}\right) d\kappa \\
&= \frac{c}{8\pi^2 r}\left[\int_0^{\infty} \left\{e^{i\kappa(ct-r)} + e^{-i\kappa(ct-r)}\right\} d\kappa \right. \\
&\qquad \left. - \int_0^{\infty} \left\{e^{i\kappa(ct+r)} + e^{-i\kappa(ct+r)}\right\} d\kappa\right] \\
&= \frac{c}{8\pi^2 r}\left[\int_{-\infty}^{\infty} e^{i\kappa(ct-r)} d\kappa - \int_{-\infty}^{\infty} e^{i\kappa(ct+r)} d\kappa\right] \\
&= \frac{2\pi c}{8\pi^2 r}\left[\delta(ct-r) - \delta(ct+r)\right].
\end{aligned} \tag{1.11.95}$$

For $t > 0$, $ct + r > 0$, and hence, $\delta(ct + r) = 0$. Thus,

$$G(\mathbf{x},t) = \frac{1}{4\pi r}\delta\left(t - \frac{r}{c}\right), \tag{1.11.96}$$

in which the formula $\delta(ax) = \frac{1}{a}\delta(x)$ is used.

If the source is located at $(\xi, \eta, \zeta, \tau) = (\boldsymbol{\xi}, \tau)$, the desired Green function is given by

$$G(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = \frac{1}{4\pi|\mathbf{x} - \boldsymbol{\xi}|}\left[\delta\left(t - \tau - \frac{|\mathbf{x} - \boldsymbol{\xi}|}{c}\right) - \delta\left(t - \tau + \frac{|\mathbf{x} - \boldsymbol{\xi}|}{c}\right)\right]. \tag{1.11.97}$$

It should be noted that Green's function (1.11.96) for the hyperbolic equation is a generalized function, whereas in the other examples of Green's functions, it was always a piecewise analytic function. In general, Green's function for an elliptic function is always analytic, whereas Green's function for a hyperbolic equation is a generalized function.

1.12 Sturm–Liouville Systems and Some General Results

We can generalize the method of separation of variables and the associated eigenvalue problems by considering the classical wave equation with variable coefficients

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) + qu = \rho\frac{\partial^2 u}{\partial t^2}, \quad 0 < x < \ell, t > 0, \tag{1.12.1}$$

subject to the boundary condition for $t > 0$

$$a_1 u + a_2 u' = 0, \quad x = 0; \qquad b_1 u + b_2 u' = 0, \quad x = \ell, \tag{1.12.2ab}$$

and the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 < x < \ell, \tag{1.12.3ab}$$

where p, q, and ρ are assumed to be continuous functions of x in $0 \le x \le \ell$ and a_1, a_2, b_1, b_2 are real positive constants such that

$$a_1^2 + a_2^2 > 0 \quad \text{and} \quad b_1^2 + b_2^2 > 0.$$

Using the method of separation of variables as in Example 1.6.1 with $u(x, t) = X(x)T(t) \neq 0$ and $-\lambda$ as a separation constant, we obtain

$$\frac{d}{dx}\left(p(x)\frac{dX}{dx}\right) + (q + \lambda\rho)X = 0, \tag{1.12.4}$$

$$\frac{d^2 T}{dt^2} + \lambda T = 0, \tag{1.12.5}$$

with the boundary conditions

$$a_1 X + a_2 X' = 0, \quad x = 0; \qquad b_1 X + b_2 X' = 0, \quad x = \ell, \tag{1.12.6ab}$$

where the prime denotes the derivative with respect to x.

The eigenvalue problem defined by (1.12.4) and (1.12.6ab) is called the *Sturm–Liouville* (*SL*) *system*. The values of λ for which the Sturm–Liouville problem has a nontrivial solution are called the *eigenvalues*, and the corresponding solutions are called the *eigenfunctions*.

In terms of the operator

$$L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x), \tag{1.12.7}$$

we can write (1.12.4) in the form, $X(x) = u(x)$,

$$Lu + \lambda\rho u = 0. \tag{1.12.8}$$

The Sturm–Liouville equation (1.12.8) is called *regular* in a closed finite interval $[a, b]$ if the functions $p(x)$ and $\rho(x)$ are positive in $[a, b]$. Thus, for a given λ, there exist two linearly independent solutions of a regular Sturm–Liouville equation (1.12.8) in $[a, b]$.

The Sturm–Liouville equation (1.12.8) in $[a, b]$ together with two separated end conditions

$$a_1 u(a) + a_2 u'(a) = 0, \qquad b_1 u(b) + b_2 u'(b) = 0, \tag{1.12.9ab}$$

where a_1, a_2, b_1, b_2 are given real constants such that $a_1^2 + a_2^2 > 0$ and $b_1^2 + b_2^2 > 0$ is called a *regular Sturm–Liouville* (*RSL*) *system*.

The set of all eigenvalues λ of a regular Sturm–Liouville problem is called the *spectrum* of the problem.

Example 1.12.1. Consider the regular Sturm–Liouville problem

$$u'' + \lambda u = 0, \quad 0 \le x \le \pi, \tag{1.12.10}$$

$$u(0) = 0 = u(\pi). \tag{1.12.11ab}$$

It is easy to check that, for $\lambda \le 0$, this problem has no nonzero solutions. In other words, there are no negative ($\lambda < 0$) or zero ($\lambda = 0$) eigenvalues of the problem.

However, when $\lambda > 0$, then the solutions of the equation are

$$u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

The boundary conditions (1.12.11ab) give

$$A = 0 \quad \text{and} \quad B\sin\left(\pi\sqrt{\lambda}\right) = 0.$$

Since $\lambda \ne 0$, and $B = 0$ yields a trivial solution, we must have $B \ne 0$, and hence,

$$\sin\left(\pi\sqrt{\lambda}\right) = 0.$$

Thus, the eigenvalues are $\lambda_n = n^2$, $n = 1, 2, \ldots$, and the eigenfunctions are

$$u_n(x) = \sin nx.$$

Note that $\lambda_n \to \infty$ as $n \to \infty$, unlike the case of self-adjoint, compact operators when the eigenvalues converge to zero (see Debnath and Mikusinski 1999, Theorem 4.9.9 and Theorem 5.10.4).

Example 1.12.2. Consider the Cauchy–Euler equation

$$x^2u'' + xu' + \lambda u = 0, \quad 1 \le x \le e \tag{1.12.12}$$

with the end conditions

$$u(1) = 0 = u(e). \tag{1.12.13ab}$$

The Cauchy–Euler equation can be put into the Sturm–Liouville form

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \frac{1}{x}\lambda u = 0.$$

The general solution of this equation is

$$u(x) = C_1 x^{i\sqrt{\lambda}} + C_2 x^{-i\sqrt{\lambda}},$$

where C_1 and C_2 are arbitrary constants.

In view of the fact that

$$x^{ia} = \exp(ia \ln x) = \cos(a \ln x) + i \sin(a \ln x),$$

the solution becomes

$$u(x) = A\cos\left(\sqrt{\lambda}\ln x\right) + B\sin\left(\sqrt{\lambda}\ln x\right),$$

where A and B are new arbitrary constants related to C_1 and C_2. The end condition $u(1) = 0$ gives $A = 0$, and the end condition $u(e) = 0$ gives

$$\sin\sqrt{\lambda} = 0, \qquad B \ne 0,$$

which leads to the eigenvalues

$$\lambda_n = (n\pi)^2, \quad n = 1, 2, 3, \ldots,$$

and the corresponding eigenfunctions

$$u_n(x) = \sin(n\pi \ln x), \quad n = 1, 2, 3, \ldots.$$

A Sturm–Liouville equation (1.12.8) is called *singular* when it is given on a semi-infinite or infinite interval, or when the coefficient $p(x)$ or $\rho(x)$ vanishes, or when one of the coefficients becomes infinite at one end or both ends of a finite interval. A singular Sturm–Liouville equation together with appropriate linear homogeneous end conditions is called a *singular Sturm–Liouville system*. The conditions imposed in this case are not like the separated boundary conditions in the regular Sturm–Liouville problem.

Example 1.12.3. We consider the *singular Sturm–Liouville* problem involving Legendre's equation

$$\frac{d}{dx}\left[\left(1-x^2\right)u'\right]+\lambda u=0, \quad -1<x<1, \tag{1.12.14}$$

with the boundary conditions that u and u' are finite as $x \to \pm 1$.

In this case, $p(x)=1-x^2$ and $\rho(x)=1$, and $p(x)$ vanishes at $x=\pm 1$. The Legendre functions of the first kind $P_n(x)$, $n=0,1,2,\ldots$, are the eigenfunctions which are finite as $x \to \pm 1$. The corresponding eigenvalues are $\lambda_n=n(n+1)$ for $n=0,1,2,\ldots$. We note that the singular Sturm–Liouville problem has infinitely many eigenvalues, and the eigenfunctions $P_n(x)$ are orthogonal to each other with respect to the weight function $\rho(x)=1$.

Example 1.12.4. Another example of a singular Sturm–Liouville problem is the Bessel equation for fixed ν

$$\frac{d}{dx}\left(x\frac{du}{dx}\right)+\left(\lambda x-\frac{\nu^2}{x}\right)u=0, \quad 0<x<a, \tag{1.12.15}$$

with the end conditions that $u(a)=0$ and u, u' are finite as $x \to 0+$.

In this case, $p(x)=x$, $q(x)=-\frac{\nu^2}{x}$, and $\rho(x)=x$. Here $p(0)=0$, $q(x)$ is infinite as $x \to 0+$, and $\rho(0)=0$. Therefore, the problem is singular. If $\lambda=k^2$, the eigenfunctions are the Bessel functions $J_\nu(k_n x)$ of the first kind of order ν where $n=1,2,3,\ldots$, and $(k_n a)$ is the nth zero of J_ν. The eigenvalues are $\lambda_n=k_n^2$. The Bessel function J_ν and its derivative are both finite as $x \to 0+$. Thus, the problem has infinitely many eigenvalues and the eigenfunctions are orthogonal to each other with respect to the weight function $\rho(x)=x$.

In the preceding examples, we see that the eigenfunctions are orthogonal with respect to the weight function $\rho(x)$. In general, the eigenfunctions of a singular SL system are orthogonal with respect to the weight function $\rho(x)$, which will be proved later on.

Another type of problem that often arises in practice is the *periodic Sturm–Liouville system*:

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+(q+\lambda\rho)u=0, \quad a \le x \le b, \tag{1.12.16}$$

in which $p(a)=p(b)$, together with the periodic end conditions

$$u(a)=u(b), \qquad u'(a)=u'(b). \tag{1.12.17ab}$$

Example 1.12.5. Find the eigenvalues and eigenfunctions of the periodic Sturm–Liouville system:

$$u''+\lambda u=0, \quad -\pi \le x \le \pi, \tag{1.12.18}$$

$$u(-\pi)=u(\pi), \qquad u'(\pi)=u'(-\pi). \tag{1.12.19ab}$$

Note that here $p(x) = 1$ and hence $p(\pi) = p(-\pi)$. For $\lambda > 0$, the general solution of the equation is

$$u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

where A and B are arbitrary constants. Using the boundary conditions (1.12.19ab), we obtain

$$2B\sin\sqrt{\lambda}\pi = 0,$$
$$2A\sqrt{\lambda}\sin\sqrt{\lambda}\pi = 0.$$

Thus, for nontrivial solutions, we must have

$$\sin\sqrt{\lambda}\pi = 0, \qquad A \neq 0, \qquad B \neq 0.$$

Consequently,

$$\lambda_n = n^2, \quad n = 1, 2, 3, \ldots.$$

So, for every eigenvalue $\lambda_n = n^2$, there are two linearly independent solutions $\cos nx$ and $\sin nx$.

It can easily be checked that there are no negative eigenvalues of the system. However, $\lambda = 0$ is an eigenvalue and the associated eigenfunction is the constant function $u(x) = 1$. Thus, the eigenvalues are 0, $\{n^2\}$, and the corresponding eigenfunctions are 1, $\{\cos nx\}$, $\{\sin nx\}$, where n is a positive integer.

For the regular Sturm–Liouville problem, we denote the domain of L by $D(L)$, that is, $D(L)$ is the space of all complex-valued functions u defined on $[a, b]$ for which $u'' \in L^2([a, b])$ and which satisfy boundary conditions (1.12.9ab).

Theorem 1.12.1 (Lagrange's Identity). *For any $u, v \in D(L)$, we have*

$$uLv - vLu = \frac{d}{dx}\big[p(uv' - vu')\big]. \tag{1.12.20}$$

Proof. We have

$$\begin{aligned} uLv - vLu &= u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + quv - v\frac{d}{dx}\left(p\frac{du}{dx}\right) - quv \\ &= \frac{d}{dx}\big[p(uv' - vu')\big]. \end{aligned}$$

Theorem 1.12.2 (Abel's Formula). *If u and v are two solutions of the equation (1.12.8) in $[a, b]$, then*

$$p(x)W(u, v; x) = \text{const.}, \tag{1.12.21}$$

where W is the Wronskian defined by

$$W(u, v; x) = (uv' - u'v).$$

Proof. Since u, v are solutions of (1.12.8), we have

$$\frac{d}{dx}\left(pu'\right) + (q + \lambda\rho)u = 0,$$
$$\frac{d}{dx}\left(pv'\right) + (q + \lambda\rho)v = 0.$$

Multiplying the first equation by v and the second equation by u, and then subtracting gives

$$u\frac{d}{dx}\left(pv'\right) - v\frac{d}{dx}\left(pu'\right) = 0.$$

Integrating this equation from a to x yields

$$p(x)\left[u(x)v'(x) - u'(x)v(x)\right] - p(a)\left[u(a)v'(a) - u'(a)v(a)\right] = 0.$$

This is Abel's formula.

Theorem 1.12.3. *The Sturm–Liouville operator L is self-adjoint. In other words, for any u, $v \in D(L)$, we have*

$$\langle Lu, v\rangle = \langle u, Lv\rangle, \tag{1.12.22}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2([a, b])$ defined by

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx. \tag{1.12.23}$$

Proof. Since all constants involved in the boundary conditions of a Sturm–Liouville system are real, if $v \in D(L)$, then $\overline{v} \in D(L)$.

Also since p, q and ρ are real valued, $\overline{Lv} = L\overline{v}$. Consequently, we have

$$\begin{aligned} \langle Lu, v\rangle - \langle u, Lv\rangle &= \int_a^b \left(\overline{v}Lu - uL\overline{v}\right) dx \\ &= \left[p\left(\overline{v}u' - u\overline{v}'\right)\right]_a^b, \quad \text{by Lagrange's identity (1.12.20).} \end{aligned} \tag{1.12.24}$$

We shall show that the right-hand side of the above equality vanishes for both the regular and singular SL systems. If $p(a) = 0$, the result follows immediately. If $p(a) > 0$, then u and v satisfy the boundary conditions of the form (1.12.9ab) at $x = a$. That is,

$$\begin{bmatrix} u(a) & u'(a) \\ \overline{v}(a) & \overline{v}'(a) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0.$$

Since a_1 and a_2 are not both zero, we have

$$\overline{v}(a)u'(a) - u(a)\overline{v}'(a) = 0.$$

A similar argument can be used to the other end point $x = b$, so that the right-hand side of (1.12.24) vanishes. This proves the theorem.

Theorem 1.12.4. *All eigenvalues of a Sturm–Liouville system are real.*

Proof. Let λ be an eigenvalue of an SL system and let $u(x)$ be the corresponding eigenfunction. This means that $u \neq 0$ and $Lu = -\lambda \rho u$. Then

$$0 = \langle Lu, u \rangle - \langle u, Lu \rangle = (\overline{\lambda} - \lambda) \int_a^b \rho(x) \left| u(x) \right|^2 dx.$$

Since $\rho(x) > 0$ in $[a, b]$ and $u \neq 0$, the integral is a positive number. Thus $\overline{\lambda} = \lambda$. This completes the proof.

Remark. This theorem states that all eigenvalues of a regular SL system are real, but it does not guarantee that an eigenvalue exists. It has been shown by example that an SL system has an infinite sequence of eigenvalues. All preceding examples of the SL system suggest that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ with

$$\lim_{n \to \infty} \lambda_n = \infty.$$

All of these results can be stated in the form of a theorem as follows.

Theorem 1.12.5. *The eigenvalues λ_n of an SL system can be arranged in the form*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots ,$$

and

$$\lim_{n \to \infty} \lambda_n = \infty, \tag{1.12.25}$$

so that n refers to the number of zeros of the eigenfunctions $u_n(x)$ in $[a, b]$.

*The proof of this theorem is beyond the scope of this book, and we refer to Debnath and Mikusinski (*2005*).*

Theorem 1.12.6. *The eigenfunctions corresponding to distinct eigenvalues of a Sturm–Liouville system are orthogonal with respect to the inner product with the weight function $\rho(x)$.*

Proof. Suppose $u_1(x)$ and $u_2(x)$ are eigenfunctions corresponding to eigenvalues λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$.

Thus,

$$Lu_1 = -\lambda_1 \rho u_1 \quad \text{and} \quad Lu_2 = -\lambda_2 \rho u_2.$$

Hence

$$u_1 L u_2 - u_2 L u_1 = (\lambda_1 - \lambda_2) \rho u_1 u_2. \tag{1.12.26}$$

By Theorem 1.12.1, we have

$$u_1 L u_2 - u_2 L u_1 = \frac{d}{dx}\left[p\left(u_1 u_2' - u_2 u_1'\right)\right]. \tag{1.12.27}$$

Combining (1.12.26) and (1.12.27) and integrating from a to b gives

$$(\lambda_1 - \lambda_2)\int_a^b \rho(x)u_1(x)u_2(x)\,dx$$
$$= \left[p(x)\left\{u_1(x)u_2'(x) - u_2(x)u_1'(x)\right\}\right]_a^b = 0,$$

by boundary conditions (1.12.9ab).

Since $\lambda_1 \neq \lambda_2$, this equality shows that

$$\int_a^b \rho(x)u_1(x)u_2(x)\,dx = 0.$$

This proves the theorem.

We consider some general results about the eigenfunction expansions and their completeness property of an SL system.

Suppose $\{u_n(x)\}_{n=1}^{\infty}$ is a set of orthogonal eigenfunctions of an SL system in $[a, b]$. The inner product of these functions with respect to the weight function $\rho(x)$ is defined by

$$\langle u_n, u_m\rangle = \int_a^b \rho(x)u_n(x)u_m(x)\,dx, \tag{1.12.28}$$

so that the square of the norm is

$$\|u_n\|^2 = \langle u_n, u_n\rangle = \int_a^b \rho(x)u_n^2(x)\,dx. \tag{1.12.29}$$

The set of orthogonal eigenfunctions of an SL system is said to be *complete* if any arbitrary function $f \in L^2([a, b])$ can be expanded uniquely as

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x), \tag{1.12.30}$$

where the series converges to $f(x)$ in $L^2([a, b])$ and the coefficients a_n are given by

$$a_n = \frac{\langle f(x), u_n(x)\rangle}{\langle u_n(x), u_n(x)\rangle} = \frac{1}{\|u_n\|^2}\langle f, u_n\rangle, \tag{1.12.31}$$

where $n = 1, 2, 3, \ldots$.

The expansion (1.12.30) is called a *generalized Fourier series* of $f(x)$ and the associated scalars a_n are called the *generalized Fourier coefficients* of $f(x)$. The set of eigenfunctions $\{u_n\}_{n=1}^{\infty}$ is called *orthonormal* if $\|u_n\| = 1$. Obviously, the set of orthonormal eigenfunctions is said to be complete, if, for every $f \in L^2([a, b])$, the following expansion holds:

$$f(x) = \sum_{n=1}^{\infty} a_n u_n = \sum_{n=1}^{\infty} \langle f, u_n \rangle u_n. \tag{1.12.32}$$

A series of orthonormal eigenfunctions $\sum_{n=1}^{\infty} a_n u_n(x)$ is said to be convergent to $f(x)$ in $L^2([a,b])$ if

$$\lim_{n\to\infty} \|f(x) - s_n(x)\| = 0, \tag{1.12.33}$$

where $s_n(x) = \sum_{r=1}^{n} a_r u_r(x)$ is the nth partial sum of series (1.12.32). Equivalently, (1.12.33) reads as

$$\lim_{n\to\infty} \int_a^b \left| f(x) - \sum_{r=1}^{n} a_r u_r(x) \right|^2 \rho(x)\,dx = 0. \tag{1.12.34}$$

This type of convergence is called the *strong convergence* and is entirely different from pointwise or uniform convergence in analysis. In general, the strong convergence in $L^2([a,b])$ implies neither pointwise convergence nor uniform convergence. However, the uniform convergence implies both strong convergence and pointwise convergence.

We now determine the coefficients a_r such that the nth partial sum $s_n(x)$ of the series (1.12.32) represents the best approximation to $f(x)$ in the sense of least squares, that is, we seek to minimize the integral in (1.12.34)

$$\begin{aligned} I(a_r) &= \int_a^b \left[f(x) - \sum_{r=1}^{n} a_r u_r(x) \right]^2 \rho(x)\,dx \\ &= \int_a^b \rho(x) f^2(x)\,dx - 2\sum_{r=1}^{n} a_r \int_a^b \rho(x) f(x) u_r(x)\,dx \\ &\quad + \sum_{r=1}^{n} a_r^2 \int_a^b \rho(x) u_r^2(x)\,dx. \end{aligned} \tag{1.12.35}$$

This is an extremal problem. A necessary condition for $I(a_r)$ to be minimum is that the first partial derivatives of I with respect to the coefficients a_r vanish.

Thus, we obtain

$$\frac{\partial I}{\partial a_r} = \sum_{r=1}^{n} \left[-2\int_a^b \rho u_r f(x)\,dx + 2a_r \int_a^b \rho u_r^2\,dx \right] = 0. \tag{1.12.36}$$

Consequently,

$$a_r = \int_a^b f(x) u_r(x) \rho(x)\,dx = \langle f, u_r \rangle. \tag{1.12.37}$$

If we complete the square, the right-hand side of (1.12.35) becomes

$$I(a_r) = \int_a^b \rho f^2\,dx + \sum_{r=1}^{n}\left[a_r - \int_a^b \rho f u_r\,dx\right]^2 - \sum_{r=1}^{n}\left(\int_a^b \rho f u_r\,dx\right)^2. \tag{1.12.38}$$

The right-hand side shows that I is minimum if and only if a_r is given by (1.12.37). This choice of a_r gives the best approximation to $f(x)$ in the sense of least squares.

Substituting the values of a_r into (1.12.35) gives

$$\int_a^b \left[f(x) - \sum_{r=1}^{n} a_r u_r(x)\right]^2 \rho(x)\,dx = \int_a^b \rho(x) f^2(x)\,dx - \sum_{r=1}^{n} a_r^2. \tag{1.12.39}$$

If the series of orthonormal eigenfunctions converges to $f(x)$, then (1.12.34) is satisfied. Invoking the limit as $n \to \infty$ in (1.12.39) and using (1.12.34) gives the *Parseval relation*

$$\sum_{r=1}^{\infty} a_r^2 = \int_a^b \rho(x) f^2(x)\,dx = \|f\|^2, \tag{1.12.40}$$

or equivalently,

$$\sum_{r=1}^{\infty} \left|\langle f, u_r\rangle\right|^2 = \|f\|^2. \tag{1.12.41}$$

Since the left-hand side of (1.12.39) is nonnegative, it follows from (1.12.39) that

$$\sum_{r=1}^{n} a_r^2 \le \|f\|^2. \tag{1.12.42}$$

Since the right-hand side of (1.12.42) is finite, the left-hand side of (1.12.42) is bounded above for any n. Proceeding to the limit as $n \to \infty$ gives the inequality

$$\sum_{n=1}^{\infty} a_n^2 \le \|f\|^2, \tag{1.12.43}$$

or equivalently,

$$\sum_{n=1}^{\infty} \left|\langle f, u_n\rangle\right|^2 \le \|f\|^2. \tag{1.12.44}$$

This is called *Bessel's inequality*.

In Section 1.6, the method of separation of variables or the Fourier method or the method of eigenfunction expansions has been discussed with many examples. This is the basic method for solving partial differential equations in bounded special domains. We now illustrate the generalized Fourier method by solving more general Sturm–Liouville problems associated with a general wave equation and a general diffusion equation.

Example 1.12.6 (Solution of the Sturm–Liouville Problem Associated with the Wave Equation). We develop the generalized Fourier method by solving a more general Sturm–Liouville equation associated with the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho(x)}\left[\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + qu\right] + F(x,t), \quad a \le x \le b, t > 0, \tag{1.12.45}$$

with the boundary conditions (1.12.2ab) and the initial conditions (1.12.3ab), where $F(x,t)$ is the forcing (source) term.

In terms of the SL operator, equation (1.12.45) can be written as

$$u_{tt} = Lu + F(x,t), \quad a \le x \le b, t > 0, \tag{1.12.46}$$

where

$$Lu = \frac{1}{\rho(x)}\left[\frac{\partial}{\partial x}(pu_x) + qu\right]. \tag{1.12.47}$$

Following the method of separation of variables, we assume the solution of the wave equation (1.12.45) with $F = 0$ in the form $u(x,t) = \phi(x)\psi(t) \neq 0$ so that equation (1.12.45) reduces to

$$\frac{d^2\psi}{dt^2} = \lambda\psi, \quad t > 0, \tag{1.12.48}$$

$$L\phi = \lambda\phi, \quad a \le x \le b, \tag{1.12.49}$$

where λ is a separation constant.

The associated boundary conditions for $\phi(x)$ are

$$a_1\phi(a) + a_2\phi'(a) = 0, \qquad b_1\phi(b) + b_2\phi'(b) = 0. \tag{1.12.50ab}$$

Equation (1.12.49) with (1.12.50ab) is called the *associated SL problem*. In general, this problem can be solved by finding the eigenvalues λ_n and the orthonormal eigenfunctions $\phi_n(x)$, $n = 1, 2, 3, \ldots$. Using the principle of superposition, we can write the solution of the linear equation (1.12.46) in the form

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x)\psi_n(t), \tag{1.12.51}$$

where $\psi_n(t)$ are to be determined.

We further assume that the forcing term can also be expanded in terms of the eigenfunctions as

$$F(x,t) = \sum_{n=1}^{\infty} \phi_n(x) f_n(t), \tag{1.12.52}$$

where the generalized Fourier coefficients $f_n(t)$ of $F(x,t)$ are given by

$$f_n(t) = \langle F, \phi_n \rangle = \int_a^b F(x,t)\phi_n(x)\,dx. \tag{1.12.53}$$

Substituting (1.12.51) and (1.12.52) into (1.12.46) gives

$$\begin{aligned}\sum_{n=1}^{\infty} \ddot{\psi}_n(t)\phi_n(x) &= L\left[\sum_{n=1}^{\infty} \phi_n(x)\psi_n(t)\right] + \sum_{n=1}^{\infty} \phi_n(x) f_n(t) \\ &= \sum_{n=1}^{\infty} \psi_n(t) L\phi_n(x) + \sum_{n=1}^{\infty} \phi_n(x) f_n(t) \\ &= \sum_{n=1}^{\infty} \big[\lambda_n \psi_n(t) + f_n(t)\big]\phi_n(x).\end{aligned}$$

This leads to the ordinary differential equation

$$\ddot{\psi}_n(t) + \alpha_n^2 \psi_n(t) = f_n(t), \tag{1.12.54}$$

where $\lambda_n = -\alpha_n^2$.

Application of the Laplace transform method leads to the solution of (1.12.54) as

$$\begin{aligned}\psi_n(t) &= \psi_n(0)\cos(\alpha_n t) + \frac{1}{\alpha_n}\dot{\psi}(0)\sin(\alpha_n t) \\ &\quad + \frac{1}{\alpha_n}\int_0^t \sin\alpha_n(t-\tau) f_n(\tau)\,d\tau,\end{aligned} \tag{1.12.55}$$

where $\psi_n(0)$ and $\dot{\psi}_n(0)$ can be determined from the initial data (1.12.3ab) so that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} \phi_n(x)\psi_n(0), \tag{1.12.56}$$

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \dot{\psi}_n(0)\phi_n(x), \tag{1.12.57}$$

which give the generalized Fourier coefficients $\psi_n(0)$ and $\dot{\psi}_n(0)$ as follows:

$$\psi_n(0) = \langle f, \phi_n \rangle = \int_a^b f(\xi)\phi_n(\xi)\,d\xi, \tag{1.12.58}$$

$$\dot{\psi}_n(0) = \langle g, \phi_n \rangle = \int_a^b g(\xi)\phi_n(\xi)\,d\xi. \tag{1.12.59}$$

Therefore, the final solution is given by

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} \psi_n(t)\phi_n(x) \\ &= \sum_{n=1}^{\infty}\bigg[\langle f, \phi_n\rangle \cos\alpha_n t + \frac{1}{\alpha_n}\langle g, \phi_n\rangle \sin\alpha_n t \\ &\quad + \frac{1}{\alpha_n}\int_0^t \sin\alpha_n(t-\tau) f_n(\tau)\,d\tau\bigg]\phi_n(x).\end{aligned} \tag{1.12.60}$$

This represents an infinite series solution of the wave equation (1.12.45) with the boundary and initial data (1.12.2ab) and (1.12.3ab) under appropriate conditions on the initial data $f(x)$, $g(x)$ and the forcing term $F(x,t)$.

Replacing the inner products by the integrals (1.12.58) and (1.12.59), and $f_n(\tau)$ by (1.12.53) and interchanging the summation and integration, we obtain the solution in the form

$$\begin{aligned} u(x,t) = &\int_a^b \left[\sum_{n=1}^{\infty} \phi_n(x)\phi_n(\xi)\cos\alpha_n t\right] f(\xi)\,d\xi \\ &+ \int_a^b \left[\sum_{n=1}^{\infty} \frac{1}{\alpha_n}\phi_n(x)\phi_n(\xi)\sin\alpha_n t\right] g(\xi)\,d\xi \\ &+ \int_a^b \int_0^t \left[\sum_{n=1}^{\infty} \frac{1}{\alpha_n}\phi_n(x)\phi_n(\xi)\sin\alpha_n(t-\tau)\right] F(\xi,\tau)\,d\xi\,d\tau. \end{aligned} \tag{1.12.61}$$

Example 1.12.7 (Solution of the Sturm–Liouville Problem Associated with the Diffusion Equation). We consider the diffusion equation with a forcing (source) term $F(x,t)$ in the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}\right] + q(x)u + F(x,t), \quad a \le x \le b,\ t > 0, \tag{1.12.62}$$

with boundary conditions

$$a_1 u(a,t) + a_2 u_x(a,t) = 0, \quad b_1 u(b,t) + b_2 u_x(b,t) = 0, \quad t > 0, \tag{1.12.63ab}$$

and the initial condition

$$u(x,0) = f(x), \quad a < x < b. \tag{1.12.64}$$

In terms of the SL operator L, equation (1.12.62) takes the form

$$u_t = Lu + F. \tag{1.12.65}$$

We use the method of separation of variables to seek a solution of the equation (1.12.62) with $F = 0$ in the form $u(x,t) = \phi(x)\psi(t) \neq 0$ so that the equation (1.12.62) becomes

$$\frac{d\psi}{dt} = \lambda\psi, \quad t > 0, \tag{1.12.66}$$

$$L\phi = \lambda\phi, \quad a \le x \le b, \tag{1.12.67}$$

where λ is the separation constant.

The associated boundary conditions are

$$a_1\phi(a) + a_2\phi'(a) = 0, \qquad b_1\phi(b) + b_2\phi'(b) = 0. \tag{1.12.68ab}$$

Equation (1.12.67) with (1.12.68ab) is called the *associated SL problem*, which can easily be solved by finding the eigenvalues λ_n and the orthonormal eigenfunctions

$\phi_n(x)$, $n = 1, 2, 3, \ldots$. According to the linear superposition principle, we write the solution of (1.12.67) in the form

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x)\psi_n(t), \tag{1.12.69}$$

where $\psi_n(t)$ are to be determined.

We further assume that the forcing function can also be expanded in terms of the eigenfunctions as

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t)\phi_n(x), \tag{1.12.70}$$

where the Fourier coefficients $f_n(t)$ are given by

$$f_n(t) = \langle F, \phi \rangle = \int_a^b F(\xi,t)\phi_n(\xi)\, d\xi. \tag{1.12.71}$$

Putting (1.12.69) and (1.12.70) in equation (1.12.65) yields

$$\begin{aligned}
\sum_{n=1}^{\infty} \phi_n(x)\dot{\psi}_n(t) &= L\left[\sum_{n=1}^{\infty} \phi_n(x)\psi_n(t)\right] + \sum_{n=1}^{\infty} f_n(t)\phi_n(x) \\
&= \sum_{n=1}^{\infty} \big[\psi_n(t)L\phi_n(x) + f_n(t)\phi_n(x)\big] \\
&= \sum_{n=1}^{\infty} \big[\lambda_n\psi_n(t) + f_n(t)\big]\phi_n(x).
\end{aligned}$$

This gives an ordinary differential equation for $\psi_n(t)$ as

$$\dot{\psi}_n(t) = \lambda_n\psi_n(t) + f_n(t). \tag{1.12.72}$$

Applying the Laplace transform to this equation gives the solution

$$\psi_n(t) = \psi_n(0)\exp(\lambda_n t) + \int_0^t \exp\big\{\lambda_n(t-\tau)f_n(\tau)\, d\tau\big\}, \tag{1.12.73}$$

where $n = 1, 2, 3, \ldots$ and $\psi_n(0)$ can be determined from the initial condition

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} \phi_n(x)\psi_n(0), \tag{1.12.74}$$

where the Fourier coefficients $\psi_n(0)$ of the function $f(x)$ are given by

$$\psi_n(0) = \langle f, \phi_n \rangle = \int_a^b f(\xi)\phi_n(\xi)\, d\xi. \tag{1.12.75}$$

Substituting (1.12.73) in the solution (1.12.69) gives

$$u(x,t) = \sum_{n=1}^{\infty} \left[\langle f, \phi_n \rangle \exp(\lambda_n t) + \int_0^t \exp\{\lambda_n (t-\tau) f_n(\tau)\, d\tau\} \right] \phi_n(x). \tag{1.12.76}$$

We next replace the inner product in (1.12.76) by (1.12.75), $f_n(\tau)$ by (1.12.71) and interchange the summation and integration to obtain the final form of the solution in the form

$$\begin{aligned} u(x,t) = &\int_a^b \left[\sum_{n=1}^{\infty} \phi_n(\xi)\phi_n(x) \exp(\lambda_n t) \right] f(\xi)\, d\xi \\ &+ \int_0^t \int_a^b \sum_{n=1}^{\infty} \left[\phi_n(\xi)\phi_n(x) \exp\{\lambda_n (t-\tau)\} \right] F(\xi,\tau)\, d\xi\, d\tau. \end{aligned} \tag{1.12.77}$$

Introducing a new function G defined by

$$G(x,\xi,t) = \sum_{n=1}^{\infty} \phi_n(\xi)\phi_n(x) e^{\lambda_n t}, \tag{1.12.78}$$

we can write the solution in terms of G in the form

$$\begin{aligned} u(x,t) = &\int_a^b G(x,\xi,t) f(\xi)\, d\xi \\ &+ \int_0^t \int_a^b G(x,\xi,t-\tau) F(\xi,\tau)\, d\xi\, d\tau. \end{aligned} \tag{1.12.79}$$

It is noted that the first term of this solution represents the contribution from the initial condition, and the second term is due to the nonhomogeneous term of the equation (1.12.62).

A typical boundary-value problem for an ordinary differential equation can be written in the operator form as

$$Lu = f, \quad a \le x \le b. \tag{1.12.80}$$

Usually, we seek a solution of this equation with the given boundary conditions. One formal approach to the problem is to find the inverse operator L^{-1}. Then the solution of (1.12.80) can be found as $u = L^{-1}(f)$. It turns out that it is possible in many important cases, and the inverse operator is an integral operator of the form

$$u(x) = \left(L^{-1} f\right)(x) = \int_a^b G(x,t) f(t)\, dt. \tag{1.12.81}$$

The function G is called *Green's function* of the operator L. The existence of Green's function and its construction is not a simple problem in the case of the regular Sturm–Liouville system.

Theorem 1.12.7 (Green's Function for an SL System). *Suppose $\lambda = 0$ is not an eigenvalue of the following regular SL system*:

$$Lu \equiv \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x), \quad a \leq x \leq b, \tag{1.12.82}$$

with the homogeneous boundary conditions

$$a_1 u(a) + a_2 u'(a) = 0, \qquad b_1 u(b) + b_2 u'(b) = 0, \tag{1.12.83ab}$$

where p and q are continuous real-valued functions on $[a, b]$, p is positive in $[a, b]$, $p'(x)$ exists and is continuous in $[a, b]$, and a_1, a_2, b_1, b_2, are given real constants such that $a_1^2 + a_1^2 > 0$ and $b_1^2 + b_1^2 > 0$. Thus, for any $f \in C^2([a, b])$, the SL system has a unique solution

$$u(x) = \int_a^b G(x,t) f(t)\, dt, \tag{1.12.84}$$

where G is the Green function given by

$$G(x,t) = \begin{cases} \frac{u_2(x)u_1(t)}{p(t)W(t)} & \text{if } a \leq t < x, \\ \frac{u_1(x)u_2(t)}{p(t)W(t)} & \text{if } x < t \leq b, \end{cases} \tag{1.12.85}$$

where u_1 and u_2 are nonzero solutions of the homogeneous system $(f = 0)$ and W is the Wronskian given by

$$W(t) = \begin{vmatrix} u_1(t) & u_2(t) \\ u_1'(t) & u_2'(t) \end{vmatrix}.$$

Proof. According to the theory of ordinary differential equations, the general solution of (1.12.82) is of the form

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + u_p(x), \tag{1.12.86}$$

where c_1 and c_2 are arbitrary constants, u_1 and u_2 are two linearly independent solutions of the homogeneous equation $Lu = 0$, and $u_p(x)$ is any particular solution of (1.12.82).

Using the method of variation of parameters, we obtain the particular solution

$$u_p(x) = u_1(x)v_1(x) + u_2(x)v_2(x), \tag{1.12.87}$$

where $v_1(x)$ and $v_2(x)$ are given by

$$v_1(x) = -\int \frac{f(x)u_2(x)}{p(x)W(x)}\, dx, \qquad v_2(x) = \int \frac{f(x)u_1(x)}{p(x)W(x)}\, dx. \tag{1.12.88}$$

According to Abel's formula (see Theorem 1.12.2), $p(x)W(x)$ is a constant. Since $W(x) \neq 0$ in $[a, b]$ and $p(x)$ is assumed to be positive, the constant is nonzero. Denoting the constant by c so that

$$c = \frac{1}{p(x)W(x)},$$

then

$$v_1(x) = -\int cf(x)u_2(x)\,dx \quad \text{and} \quad v_2(x) = \int cf(x)u_1(x)\,dx.$$

Thus, the final form of $u_p(x)$ is

$$\begin{aligned} u_p(x) &= -cu_1(x)\int_b^x f(t)u_2(t)\,dt + cu_2(x)\int_a^x f(t)u_1(t)\,dt \\ &= \int_a^x cu_2(x)u_1(t)f(t)\,dt + \int_x^b cu_1(x)u_2(t)f(t)\,dt. \end{aligned} \tag{1.12.89}$$

Consequently, if we denote Green's function as

$$G(x,t) = \begin{cases} cu_2(x)u_1(t) & \text{if } a \le t < x, \\ cu_1(x)u_2(t) & \text{if } x < t \le b, \end{cases} \tag{1.12.90}$$

we can write

$$u_p(x) = \int_a^b G(x,t)f(t)\,dt, \tag{1.12.91}$$

provided the integral exists. This follows immediately from the continuity of G. The continuity of G is left as an exercise.

We denote the integral operator T given by (1.12.81), that is,

$$(Tf)(x) = \int_a^b G(x,t)f(t)\,dt. \tag{1.12.92}$$

Theorem 1.12.8. *The operator T defined by (1.12.92) is self-adjoint from $L^2([a,b])$ into $C([a,b])$ if $G(x,t) = \overline{G(t,x)}$.*

Proof. The function $G(x,t)$ defined on $[a,b]\times[a,b]$ is continuous if

$$\int_a^b\int_a^b \left|G(x,t)\right|^2 dx\,dt < \infty.$$

We have

$$\begin{aligned} \langle Tf, g\rangle &= \int_a^b\int_a^b G(x,t)f(t)\overline{g(x)}\,dx\,dt \\ &= \overline{\int_a^b\int_a^b \overline{G(x,t)f(t)}g(x)\,dx\,dt} \\ &= \int_a^b\int_a^b f(t)\,dt\,\overline{\int_a^b \overline{G(x,t)}g(x)\,dx} \\ &= \langle f, T^*g\rangle, \end{aligned}$$

which shows that

$$\big(T^* f\big)(x) = \int_a^b \overline{G(t,x)} f(t)\,dt.$$

Obviously, T is self-adjoint if its kernel satisfies the equality $G(x,t) = \overline{G(t,x)}$.

Theorem 1.12.9. *Under the assumptions of Theorem 1.12.7, λ is an eigenvalue of L if and only if $(1/\lambda)$ is an eigenvalue of T. Furthermore, if f is an eigenfunction of L corresponding to the eigenvalue λ, then f is an eigenfunction of T corresponding to the eigenvalue $(1/\lambda)$.*

Proof. Suppose $Lf = \lambda f$ for some nonzero f in the domain of L. In view of the definition of T, and Theorem 1.12.7, we have

$$f = L^{-1}(\lambda f) = T(\lambda f).$$

Or equivalently, since $\lambda \neq 0$,

$$Tf = \frac{1}{\lambda} f.$$

This means $(1/\lambda)$ is an eigenvalue of T with the corresponding eigenfunction f.

Conversely, if $(f \neq 0)$ is an eigenfunction of T corresponding to the eigenvalue $\lambda \neq 0$, then

$$Tf = \lambda f.$$

Since $T = L^{-1}$, one has

$$f = L(Tf) = L(\lambda f) = \lambda L(f).$$

Thus, $(1/\lambda)$ is an eigenvalue of L and the corresponding eigenfunction is f.

Theorem 1.12.10 (Bilinear Expansion of Green's Function). *If $G(x,t)$ is Green's function for the regular SL system (1.12.82), (1.12.83ab) and the associated eigenvalue problem*

$$L\phi = \lambda\phi, \quad a \le x \le b, \tag{1.12.93}$$

with (1.12.83ab) has infinitely many nonzero eigenvalues λ_n with the corresponding orthonormal eigenfunctions ϕ_n, then $G(x,t)$ can be expanded in terms of λ_n and ϕ_n as

$$G(x,t) = \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n}\right) \phi_n(x)\phi_n(t). \tag{1.12.94}$$

Proof. We assume that the solution $u(x)$ of (1.12.82), (1.12.83ab) is given in terms of the eigenfunctions as

$$u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x), \tag{1.12.95}$$

where the coefficients a_n are to be determined.

We next express the given forcing function f in terms of the eigenfunctions as

$$f(x) = \sum_{n=0}^{\infty} f_n \phi_n(x), \tag{1.12.96}$$

where the coefficients f_n are

$$f_n = \langle f, \phi_n \rangle = \int_a^b f(t)\phi_n(t)\, dt. \tag{1.12.97}$$

Substituting (1.12.95), (1.12.96) into (1.12.82) yields

$$L\left[\sum_{n=1}^{\infty} a_n \phi_n(x)\right] = \sum_{n=1}^{\infty} f_n \phi_n(x). \tag{1.12.98}$$

But the left-hand side of (1.12.98) is

$$L\left[\sum_{n=1}^{\infty} a_n \phi_n(x)\right] = \sum_{n=1}^{\infty} a_n L\big(\phi_n(x)\big) = \sum_{n=1}^{\infty} a_n \lambda_n \phi_n(x). \tag{1.12.99}$$

Equating the right-hand side of (1.12.98) and (1.12.99) yields

$$a_n = \frac{1}{\lambda_n} f_n = \frac{1}{\lambda_n}\langle f, \phi_n \rangle = \frac{1}{\lambda_n}\int_a^b f(t)\phi_n(t)\, dt. \tag{1.12.100}$$

Consequently, (1.12.95) leads to the result

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\left[\int_a^b f(t)\phi_n(t)\, dt\right]\phi_n(x),$$

which, by interchanging the summation and integration,

$$= \int_a^b \left[\sum_{n=1}^{\infty} \frac{1}{\lambda_n}\phi_n(t)\phi_n(x)\right] f(t)\, dt. \tag{1.12.101}$$

In view of (1.12.84), Green's function $G(x, t)$ is then given by

$$G(x, t) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\phi_n(t)\phi_n(x). \tag{1.12.102}$$

This is the desired result (1.12.94).

1.13 Energy Integrals and Higher Dimensional Equations

In this section, we discuss the energy integrals, the law of conservation of energy, uniqueness theorems, higher dimensional wave equations, and diffusion equations.

We first derive the energy integral associated with the $(1+1)$-dimensional wave equation problem

$$u_{tt} = c^2 u_{xx}, \quad a \le x \le b, \ t > 0, \tag{1.13.1}$$

$$u(a,t) = 0 = u(b,t) \quad t > 0, \tag{1.13.2ab}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad a \le x \le b. \tag{1.13.3ab}$$

Multiplying equation (1.13.1) by u_t, we can rewrite the result in the form

$$\frac{d}{dt}\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2 u_x^2\right) - c^2 \frac{\partial}{\partial x}(u_t u_x) = 0. \tag{1.13.4}$$

Integrating this equation with respect to x from a to b gives

$$\begin{aligned} \frac{d}{dt}\int_a^b \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right) dx &= c^2 [u_t u_x]_a^b \\ &= c^2 \big[u_t(b,t)u_x(b,t) - u_t(a,t)u_x(a,t)\big], \end{aligned} \tag{1.13.5}$$

which is zero because $u_t(a,t) = 0 = u_t(b,t)$, which follows from (1.13.2ab).

We introduce

$$E(t) = \int_a^b \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right) dx, \tag{1.13.6}$$

which is called the *energy integral* or the *total energy* of the system. It follows from (1.13.5) that

$$\frac{dE}{dt} = 0. \tag{1.13.7}$$

This implies that

$$E(t) = \text{const.} = E_0. \tag{1.13.8}$$

This means that the energy of the wave equation system is conserved.

The energy equation (1.13.8) can be used to prove the uniqueness theorem which states that the wave equation system (1.13.1)–(1.13.3ab) has a unique solution.

Suppose that there are two solutions $u(x,t)$ and $v(x,t)$ of the system and set $w(x,t) = u - v$.

Obviously, $w(x,t)$ satisfies the following equation:

$$w_{tt} = c^2 w_{xx}, \quad a \le x \le b, \ t > 0,$$

$$w(a,t) = 0 = w(b,t), \quad t > 0,$$
$$w(x,0) = 0 = w_t(x,0), \quad a < x < b.$$

In view of equation (1.13.5) with $u = w$ and the law of conservation of energy (1.13.8), we obtain

$$\frac{dE}{dt} = \frac{d}{dt}\int_a^b \left(w_t^2 + c^2 w_x^2\right) dx = 0.$$

Since the integrand is positive, $w_t = 0 = w_x$ for all x and t, hence $w \equiv 0$, which means that $u = v$. This proves the uniqueness.

We next consider the $(n+1)$-dimensional wave equation in the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}\right) = c^2 \nabla_n^2 u, \tag{1.13.9}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t > 0$, and ∇_n^2 is the n-dimensional Laplacian.

Suppose $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ is a unit vector in $\mathbb{R}^n$ so that

$$e_1^2 + e_2^2 + \cdots + e_n^2 = 1. \tag{1.13.10}$$

For a fixed t and a constant a, the equation

$$e_1 x_1 + e_2 x_2 + \cdots + e_n x_n - ct = a \tag{1.13.11}$$

represents a plane in the $\mathbf{x}$-space $\mathbb{R}^n$. The unit vector $\mathbf{e}$ is normal to the above plane. As t increases, the plane moves in the direction of $\mathbf{e}$ with constant speed c.

It is easy to verify that

$$u(\mathbf{x}, t) = F(\mathbf{x} \cdot \mathbf{e} - ct) \tag{1.13.12}$$

is a solution of the wave equation (1.13.9). The value of u on the plane (1.13.11) is equal to the constant $F(a)$. Usually, solutions (1.13.12) of the wave equation (1.13.9) are called *plane waves*. This idea of plane waves is consistent with that of the wave equation for $n = 1, 2, 3$.

For $n = 1$ with $x_1 = x$, equation (1.13.9) gives the $(1+1)$-dimensional wave equation

$$u_{tt} = c^2 u_{xx}.$$

Obviously, condition (1.13.10) becomes $e_1^2 = 1$ which gives only two possible values of $e_1 = \pm 1$. Thus, the corresponding plane wave solutions are $u(x,t) = F(x - ct)$ and $u(x,t) = G(x + ct)$, where F and G are arbitrary C^2 functions of a single variable. The former represents a wave traveling in the positive x-direction with constant speed c and the latter also represents a wave traveling in the negative x-direction with constant speed c.

For $n = 2$ with $x_1 = x$, and $x_2 = y$, (1.13.9) become the $(2+1)$-dimensional wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}).$$

In this case, there are infinitely many unit vectors (e_1, e_2) in $\mathbb{R}^2$. For example, $\mathbf{e} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is a unit vector and the corresponding plane wave solutions are

$$u(x, y, t) = F\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - ct\right), \tag{1.13.13}$$

where F is an arbitrary C^2 function of a single variable.

For $n = 3$ with $x_1 = x$, $x_2 = y$, $x_3 = z$, equation (1.13.9) represents the $(3+1)$-dimensional wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}).$$

Again there are infinitely many unit vectors $\mathbf{e} = (e_1, e_2, e_3)$. For example, one such vector is $\mathbf{e} = (1, 0, 0)$ and the corresponding plane wave solution is

$$u(x, y, z, t) = F(x - ct).$$

In cylindrical polar coordinates $(x_1 = r\cos\theta, x_2 = r\sin\theta, z = z)$, the $(3+1)$-dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right]. \tag{1.13.14}$$

If u is independent of θ and z, this wave equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]. \tag{1.13.15}$$

This is known as the *cylindrical wave equation*.

In spherical polar coordinates $(x = r\sin\phi\cos\theta, y = r\sin\phi\sin\theta, z = r\cos\phi)$ in $\mathbb{R}^3$, the $(3+1)$-dimensional wave equation (1.13.9) takes the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right], \tag{1.13.16}$$

where $0 \le r \le \infty$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi$. If u does not depend on θ and ϕ, then (1.13.16) reduces to the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right). \tag{1.13.17}$$

This is known as the *equation of spherical waves*.

Substituting $w = ru$ in (1.13.17) leads to the one-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2\frac{\partial^2 w}{\partial r^2}. \tag{1.13.18}$$

Every solution of this equation can be expressed as the sum of solutions of the form $F(r-ct)$ and $G(r+ct)$ where F and G are arbitrary C^2 functions of a single variable. Obviously, solutions of (1.13.17) can be written as the sum of solutions of the form

$$u(r,t)=\frac{1}{r}F(r-ct) \quad \text{and} \quad u(r,t)=\frac{1}{r}G(r+ct). \tag{1.13.19}$$

These are known as *spherical waves*.

In general, the wave equation (1.13.9) can be expressed in spherical coordinates in the $\mathbf{x}$-space $\mathbb{R}^n$ in the form

$$\frac{\partial^2 u}{\partial t^2}=c^2\left[\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\Delta_n u\right], \tag{1.13.20}$$

where Δ_n is a second-order partial differential operator involving derivatives with respect to the angular coordinates.

According to usual definitions, a spherical wave is a solution of (1.13.20) which depends only on r and t and does not depend on angular coordinates. Thus, the equation of spherical waves is

$$\frac{\partial^2 u}{\partial t^2}=c^2\left[\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right)\right]. \tag{1.13.21}$$

We can apply the method of separation of variables to the wave equation (1.13.9) by expressing a solution of the form $u(\mathbf{x},t)=v(\mathbf{x})T(t)$. Substituting this separable solution into (1.13.9) gives

$$\frac{1}{v(\mathbf{x})}\left[\frac{\partial^2 v}{\partial x_1^2}+\frac{\partial^2 v}{\partial x_2^2}+\cdots+\frac{\partial^2 v}{\partial x_n^2}\right]=\frac{\ddot{T}(t)}{c^2 T(t)}=-\lambda, \tag{1.13.22}$$

where $-\lambda$ is a separation constant. This leads to the pair of equations

$$\frac{\partial^2 v}{\partial x_1^2}+\frac{\partial^2 v}{\partial x_2^2}+\cdots+\frac{\partial^2 v}{\partial x_n^2}+\lambda v=0, \tag{1.13.23}$$

$$\ddot{T}+\lambda c^2 T=0. \tag{1.13.24}$$

Equation (1.13.23) is a famous equation of the elliptic type and is often known as the *reduced wave equation*.

We consider the first important case $\lambda=\omega^2(\omega>0)$. In this case, the general solution of (1.13.24) is

$$T(t)=A\cos(\omega ct)+B\sin(\omega ct). \tag{1.13.25}$$

The solutions of the reduced wave equation (1.13.23) can be obtained by further separation of variables. We illustrate the method for equation (1.13.16) which takes the form

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\Delta_3 v\right]+\omega^2 v=0, \tag{1.13.26}$$

where

$$\Delta_3 v\equiv\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial v}{\partial\phi}\right)+\frac{1}{\sin^2\phi}\frac{\partial^2 v}{\partial\theta^2}. \tag{1.13.27}$$

We seek separable solutions in the form

$$v(r,\theta,\phi)=R(r)Y(\theta,\phi). \tag{1.13.28}$$

Substituting (1.13.28) into (1.13.26) gives

$$\frac{\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR}{dr})+\omega^2 R}{R/r^2}=-\frac{\Delta_3 Y}{Y}=\mu, \tag{1.13.29}$$

where μ is a separation constant, and the above equation becomes

$$\Delta_3 Y+\mu Y=0, \tag{1.13.30}$$

$$r^2R''+2rR'+\big(\omega^2r^2-\mu\big)R=0. \tag{1.13.31}$$

The partial differential equation (1.13.30) is considerably more difficult to solve. It is useful to consider (θ,ϕ) as coordinates of a point on the surface of the unit sphere $S(0,1)$ with the center at the origin of $\mathbb{R}^3$. Instead of finding all solutions of (1.13.30), it is useful to determine only those solutions $Y(\theta,\phi)$ which are defined as C^2 functions on the whole of $S(0,1)$. Such solutions are periodic in θ with period 2π, and at the poles of the sphere (that is, at the points where $\phi=0$ and $\phi=\pi$) the solutions tend to limits independent of θ. It can be shown (see Courant and Hilbert 1953, Vol. I, Chapter VII, §5) that equation (1.13.30) has nontrivial smooth solutions satisfying these conditions only when μ is equal to one of the integral values $\mu_m = m(m+1)$, $m=0,1,2,\ldots$. For each such value of μ_m, there are $(2m+1)$ linearly independent solutions of (1.13.30) denoted by

$$Y_m^{(k)}(\theta,\phi),\quad k=1,2,3,\ldots,(2m+1).$$

These solutions are known as the *Laplace spherical harmonics*.

For each μ_m, the radial equation (1.13.31) becomes

$$r^2R''+2rR'+\big\{\omega^2r^2-m(m+1)\big\}R=0. \tag{1.13.32}$$

In terms of the new dependent variable $w=\sqrt{r}R$, equation (1.13.32) can be rewritten as

$$r^2w''+rw'+\left[\omega^2r^2-\left(m+\frac{1}{2}\right)^2\right]w=0. \tag{1.13.33}$$

This represents Bessel's equation of the first kind of the order $(m+\frac{1}{2})$ with ω as a parameter, and has two linearly independent Bessel's functions solutions

$$J_{m+\frac{1}{2}}(\omega r) \text{ and } J_{-(m+\frac{1}{2})}(\omega r), \tag{1.13.34}$$

where

$$J_\nu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{x}{2})^{\nu+2n}}{n!\Gamma(\nu+n+1)}, \tag{1.13.35}$$

with Γ representing the gamma function. The solutions (1.13.34) can be distinguished by their behavior at the origin, $J_{m+\frac{1}{2}}(\omega r)$ behaves like $r^{m+\frac{1}{2}}$ near $r = 0$, whereas $J_{-(m+\frac{1}{2})}(\omega r)$ behaves like $r^{-(m+\frac{1}{2})}$ near $r = 0$.

Consequently, for each $m = 0, 1, 2, \ldots$, the radial equation (1.13.33) has two linearly independent solutions,

$$\frac{1}{\sqrt{r}} J_{m+\frac{1}{2}}(\omega r) \quad \text{and} \quad \frac{1}{\sqrt{r}} J_{-(m+\frac{1}{2})}(\omega r). \tag{1.13.36}$$

These solutions behave like r^m and r^{-m-1}, respectively, in the vicinity of $r = 0$. Thus, the corresponding product solutions of the reduced wave equation (1.13.26) are given by

$$\frac{1}{\sqrt{r}} J_{m+\frac{1}{2}}(\omega r) Y_m^{(k)}(\theta, \phi) \quad \text{and} \quad \frac{1}{\sqrt{r}} J_{-(m+\frac{1}{2})}(\omega r) Y_m^{(k)}(\theta, \phi), \tag{1.13.37}$$

where $m = 0, 1, 2, 3, \ldots$, and $k = 1, 2, 3, \ldots, (2m + 1)$.

Thus, the method of separation of variables leads to the following solutions for the three-dimensional wave equation:

$$\frac{1}{\sqrt{r}} \begin{Bmatrix} J_{m+\frac{1}{2}}(\omega r) \\ J_{-(m+\frac{1}{2})}(\omega r) \end{Bmatrix} Y_m^{(k)}(\theta, \phi) \begin{Bmatrix} \cos(\omega ct) \\ \sin(\omega ct) \end{Bmatrix}, \tag{1.13.38}$$

where $m = 0, 1, 2, \ldots$, and $k = 1, 2, \ldots, (2m + 1)$. These solutions are oscillatory in nature because of the harmonic dependence on time t.

On the other hand, nonoscillatory solutions of (1.13.16) correspond to the case $\lambda = -\omega^2 < 0$ $(\omega > 0)$. Obviously, the two linearly independent solutions of the time equation (1.13.24) are $\exp(\omega ct)$ and $\exp(-\omega ct)$, and the equations corresponding to (1.13.26), (1.13.30), and (1.13.31) are found by replacing ω^2 by $-\omega^2$. The equation corresponding to (1.13.33) is

$$r^2 w'' + r w' - \left[\omega^2 r^2 + \left(m + \frac{1}{2}\right)^2\right] w = 0. \tag{1.13.39}$$

This represents Bessel's equation with purely imaginary argument and has two linearly independent Bessel's functions solutions,

$$I_{m+\frac{1}{2}}(\omega r) \quad \text{and} \quad I_{-(m+\frac{1}{2})}(\omega r), \tag{1.13.40}$$

where $I_\nu(x)$ is the modified Bessel function of the first kind of order ν defined by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{\nu+2n}}{n!\Gamma(\nu+n+1)}. \tag{1.13.41}$$

We next consider the one-dimensional diffusion equation (1.6.36)–(1.6.38) to establish an energy inequality.

Multiplying the diffusion equation (1.6.36) by $u(x,t)$ and integrating with respect to x over $[0,\ell]$ gives

$$\int_0^\ell (uu_t)\,dx = \kappa \int_0^\ell (uu_{xx})\,dx.$$

Integrating the right-hand side by parts yields

$$\begin{aligned}\int_0^\ell \left(\frac{1}{2}u^2\right)_t dx &= [\kappa u u_x]_0^\ell - \kappa \int_0^\ell u_x^2(x,t)\,dx \\ &= -\kappa \int_0^\ell u_x^2(x,t)\,dx, \quad \text{by (1.6.37ab)}.\end{aligned} \tag{1.13.42}$$

For mathematical reasons, we introduce another quantity $E(t)$ by

$$E(t) = \int_0^\ell \left(\frac{1}{2}u^2\right) dx, \tag{1.13.43}$$

which may be called the *energy* at time t. Evidently, for $\kappa > 0$,

$$\frac{dE}{dt} = -\kappa \int_0^\ell u_x^2(x,t)\,dx \le 0. \tag{1.13.44}$$

This result governing the energy is essentially a form of the entropy principle of thermodynamics. It states that the energy $E(t)$ is a decreasing function of time t.

Finally, integrating (1.13.44) with respect to time t over (t_0,t) gives

$$E(t) - E(t_0) = -\kappa \int_0^\ell \int_{t_0}^t u_x^2(x,t)\,dx\,dt \le 0.$$

This means that

$$\int_0^\ell u^2(x,t)\,dx \le \int_0^\ell u^2(x,t_0)\,dx. \tag{1.13.45}$$

This may be referred to as the *energy inequality*.

Theorem 1.13.1 (Uniqueness). *There exists a unique solution of the diffusion equation system (1.6.36)–(1.6.38).*

Proof. Suppose that there are two distinct solutions $u(x,t)$ and $v(x,t)$ of the diffusion equation system and $w(x,t) = u(x,t) - v(x,t)$. It is easy to check that $w(x,t)$ satisfies the diffusion equation (1.6.36) with $w(x,0) = 0$ and $w(0,t) = 0 = w(\ell,t)$. It then follows from (1.13.44) that $E(t) \le 0$. But, by definition, $E(t) \ge 0$. Evidently, $E(t) \equiv 0$ for $t \ge 0$. This means that $w(x,t) \equiv 0$ for $0 \le x \le \ell$ and $t \ge 0$. Thus, $u(x,t) = v(x,t)$. This proves the theorem.

It follows from the derivation of the diffusion equation that heat is conducted away from regions of high temperature. This suggests that a temperature attains its maximum only initially or on boundaries. This is the essence and the origin of the *maximum principle*, which states that the maximum value of the temperature occurs either initially or on the boundaries.

We now prove the maximum principle for the one-dimensional diffusion equation with the initial and boundary conditions

$$u(x,0) = f(x), \qquad u(0,t) = f_1(t), \qquad u(\ell,t) = f_2(t),$$

and prove that a maximum of $u(x,t)$ cannot occur at an interior point (x_0, t_0) of the domain where $0 < x_0 < \ell$, $0 < t_0 < T$.

Theorem 1.13.2 (Maximum–Minimum Principle). *If $u(x,t)$ is continuous on the closed rectangle $D = \{0 \le x \le \ell, 0 \le t \le T\}$ and satisfies the diffusion equation (1.6.36) in D, then $u(x,t)$ attains its maximum and minimum values on the lower base $t = 0$ or on the vertical sides $x = 0$, $x = \ell$ of D.*

Proof. We first prove the maximum principle. The proof is by contradiction. Let M be the maximum value of u in D. Contrary to the assertion of the theorem, assume that the maximum value of u on the lower base and the vertical sides of D is $M - \varepsilon$, where $\varepsilon > 0$. Suppose that u attains its maximum at a point (x_0, t_0) in D so that $u(x_0, t_0) = M$. We must have $0 < x_0 < \ell$ and $t_0 > 0$. Consider the auxiliary function

$$v(x,t) = u(x,t) + \frac{\varepsilon}{4\ell^2}(x - x_0)^2.$$

On the lower base and vertical sides of D,

$$v(x,t) \le M - \varepsilon + \frac{\varepsilon}{4} = M - \frac{3\varepsilon}{4},$$

where $v(x_0, t_0) = M$. Thus, the maximum value of v in D is not attained on the lower base and vertical sides of D. Let (x_1, t_1) be a point where v attains its maximum. We must have $0 < x_1 < \ell$ and $0 < t_1 < T$. At (x_1, t_1), v must satisfy the necessary condition for a maximum, that is, $v_t = 0$ if $t_1 < T$ or $v_t = 0$ if $t_1 = T$, and $v_{xx} \le 0$. Hence, at (x_1, t_1), $v_t - v_{xx} \ge 0$.

On the other hand,

$$v_t - v_{xx} = u_t - u_{xx} - \frac{\varepsilon}{2\ell^2} < 0.$$

This is a contradiction and the maximum part is proved.

The minimum assertion becomes the maximum assertion when $w = -u$ and w attains a maximum where u has a minimum. Since w satisfies all assumptions of the theorem, it must attain its maximum value on the lower base and vertical sides of D. Consequently, u must attain its minimum value there.

Finally, we consider the n-dimensional diffusion equation system

$$u_t = \kappa \nabla_n^2 u, \quad \mathbf{x} \in \mathbb{R}^n, \ t > 0, \tag{1.13.46}$$

$$u(\mathbf{x}, 0) = f(\mathbf{x}), \quad u(\mathbf{x}, t) \to 0 \quad \text{as } |\mathbf{x}| \to \infty, \tag{1.13.47}$$

where

$$\nabla_n^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{1.13.48}$$

is the n-dimensional Laplacian and

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \quad \text{and} \quad |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Application of the n-dimensional Fourier transform gives the solution

$$u(\mathbf{x}, t) = \frac{1}{(4\pi\kappa t)^{n/2}} \int_{-\infty}^{\infty} f(\boldsymbol{\xi}) \exp\left(-\frac{|\mathbf{x} - \boldsymbol{\xi}|^2}{4\kappa t}\right) d\boldsymbol{\xi}, \tag{1.13.49}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$.

Example 1.13.1 (Transverse Vibration of a Thin Elastic Circular Membrane). The transverse vibration of a thin elastic circular membrane of radius a stretched over the rim is governed by the two-dimensional wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad (x, y) \in \mathbb{R}^2, \ t > 0, \tag{1.13.50}$$

where c^2 is a constant.

In cylindrical polar coordinates, equation (1.13.50) is given by

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad 0 < r < a, 0 < \theta \leq 2\pi, \ t > 0. \tag{1.13.51}$$

For the membrane, the displacement vanishes along the rim, and hence, the boundary condition is

$$u(a, \theta) = 0. \tag{1.13.52}$$

The initial conditions on the displacement and velocity are

$$u(r, \theta, 0) = f(r, \theta), \qquad u_t(r, \theta, 0) = g(r, \theta). \tag{1.13.53}$$

Using the separation of variables width $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$ in (1.13.51) gives

$$\frac{1}{c^2}R\Theta T'' = \left(R'' + \frac{R'}{r}\right)\Theta T + \frac{R}{r^2}\Theta'' T, \tag{1.13.54}$$

where the primes denote differentiation with respect to the argument. Dividing by $R\Theta T$ yields

$$\frac{1}{c^2}\frac{T''}{T} = \frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{\Theta''}{r^2\Theta} = -k^2, \tag{1.13.55}$$

where $-k^2$ is a separation constant.

Thus, equation (1.13.55) leads to

$$T'' + c^2k^2T = 0, \tag{1.13.56}$$

$$\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) + \frac{\Theta''}{r^2\Theta} = -k^2. \tag{1.13.57}$$

Equation (1.13.57) can be further written as

$$\frac{r^2}{R}\left(R'' + \frac{1}{r}R'\right) + k^2r^2 = -\frac{\Theta''}{\Theta} = \alpha^2, \tag{1.13.58}$$

where α^2 is a another separation constant.

Consequently, we obtain

$$r^2R'' + rR' + \left(k^2r^2 - \alpha^2\right)R = 0, \tag{1.13.59}$$

$$\Theta'' + \alpha^2\Theta = 0. \tag{1.13.60}$$

The general solutions of (1.13.56) and (1.13.60) are

$$T(t) = A\cos(kct) + B\sin(kct), \tag{1.13.61}$$

$$\Theta''(\theta) = C\cos\alpha\theta + D\sin\alpha\theta, \tag{1.13.62}$$

where A, B, C, D are constants of integration.

Since the physical domain covers the entire circle, Θ and Θ' must be periodic in θ, and hence α must be an integer n so that $\Theta(\theta)$ is a linear combination of $\cos n\theta$ and $\sin n\theta$. The radial equation (1.13.59) becomes

$$r^2R'' + rR' + \left(k^2r^2 - n^2\right)R = 0. \tag{1.13.63}$$

If the displacement depends only on the radial coordinate r from the center of the membrane and time t, then θ-dependence is absent, and hence, $\alpha = n = 0$ and (1.13.63) reduces to the form

$$rR'' + R' + k^2rR = 0, \quad 0 \le r < a, \tag{1.13.64}$$

with the boundary condition $R(a) = 0$, $|R(0)| < \infty$.

The general solution of (1.13.64) is

$$R(r) = C_1J_0(kr) + C_2Y_0(kr), \tag{1.13.65}$$

where J_0 and Y_0 are Bessel functions of the first and second kinds of order zero. The condition $|R(0)| < \infty$ requires $C_2 = 0$. Since Y_0 becomes unbounded at $r = 0$, the boundary condition $R(a) = 0$ leads to the eigenvalue equation

$$J_0(ak) = 0, \qquad C_1 \neq 0, \tag{1.13.66}$$

which has an infinite number of positive roots at $ak_n = \lambda_n$, $n = 1, 2, 3, \ldots$, so that the eigenvalues are given by

$$k_n^2 = \left(\frac{\lambda_n}{a}\right)^2, \quad (n = 1, 2, 3, \ldots). \tag{1.13.67}$$

The corresponding eigenfunctions are

$$R_n(r) = C_{1n} J_0\left(\frac{\lambda_n r}{a}\right), \tag{1.13.68}$$

with the eigenvalues k_n^2. The solutions are

$$u_n(r,t) = \left[a_n \cos(ck_n t) + b_n \sin(ck_n t)\right] J_0\left(\frac{\lambda_n r}{a}\right), \tag{1.13.69}$$

where a_n and b_n are arbitrary constants. These solutions represent standing waves with fixed shape $J_n(k_n r)$ and amplitude $T_n(t)$. Thus, these waves are similar to those observed in the vibrating string problem where the zeros of the sine functions are evenly spaced. However, the zeros of the Bessel functions are not evenly spaced, and hence, the sound emitted from a drum is quite different from that of stringed musical instruments, where the frequencies are integral multiples of the fundamental frequency, and zeros of the solution are equally spaced. Indeed, the musical quality of sound produced by stringed instruments is much more melodious than that of drums.

According to the principle of superposition, the formal series solution is

$$u(r,t) = \sum_{n=1}^{\infty} (a_n \cos ck_n t + b_n \sin ck_n t) J_n(rk_n), \tag{1.13.70}$$

where constants a_n and b_n are determined by the initial conditions

$$f(r) = u(r,0) = \sum_{n=1}^{\infty} a_n J_n(rk_n), \tag{1.13.71}$$

$$g(r) = u_t(r,0) = \sum_{n=1}^{\infty} ck_n b_n J_0(rk_n). \tag{1.13.72}$$

These are called the *Fourier–Bessel series*. In particular, if $g(r) = 0$, (1.13.72) is satisfied when $b_n = 0$, $n = 1, 2, 3, \ldots$. In this case, the coefficients a_n are given by the formula (1.12.31), that is,

$$a_n = \frac{\langle f(r), u_n(r)\rangle}{\|u_n\|^2} = \frac{\int_0^a r f(r) J_0(rk_n)\, dr}{\int_0^a r J_0^2(rk_n)\, dr}, \tag{1.13.73}$$

$$= \frac{2}{a^2 J_1^2(ak_n)} \int_0^a r f(r) J_0(rk_n)\, dr. \tag{1.13.74}$$

Thus, the problem is completely solved.

Example 1.13.2 (Diffusion Equation in a Wedge-Shaped Region). The diffusion equation in a wedged-shaped region is

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad 0 \le r \le a, 0 \le \theta \le \alpha, \tag{1.13.75}$$

where $0 \le \alpha \le 2\pi$.

With the insulated boundary, the normal derivative of u on the boundary must vanish, that is,

$$u_r(a,\theta) = u_\theta(r,0) = u_\theta(r,\alpha) = 0. \tag{1.13.76}$$

The initial condition is

$$u(r,\theta,0) = f(r,\theta). \tag{1.13.77}$$

Using the separation of variables, we write $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ so that (1.13.75) leads to the following equations:

$$r^2R'' + rR' + \left(k^2r^2 - \nu^2\right)R = 0, \qquad R'(a) = 0, \tag{1.13.78}$$

$$\Theta'' + \nu^2\Theta = 0, \qquad \Theta'(0) = \Theta'(\alpha) = 0, \tag{1.13.79}$$

$$T'(t) + \kappa k^2 T(t) = 0, \quad t > 0. \tag{1.13.80}$$

We also assume that $R(r)$ is bounded at $r = 0$.

The equation (1.13.79) with $\Theta'(0) = 0$ implies that

$$\Theta(\theta) = A\cos(\nu\theta), \tag{1.13.81}$$

and the boundary condition at $\theta = \alpha$ yields $\nu = \frac{n\pi}{\alpha}$. This leads to the Fourier series in θ, Bessel functions of order ν in r, and exponential functions in t. Consequently, the eigenvalues for the Sturm–Liouville problem in r are

$$k^2 = \left(\frac{\lambda_{mn}}{a}\right)^2, \tag{1.13.82}$$

and the corresponding eigenfunctions are

$$R_m(r) = B_m J_\nu\left(\frac{\lambda_{mn}r}{a}\right), \quad m = 1,2,3,\ldots, \tag{1.13.83}$$

where λ_{mn} are the positive roots of $J_\nu'(x) = 0$.

For $n = 0$, we include the constant eigenfunction 1 corresponding to the eigenvalue $k = 0$ in the solution. Consequently, the general solution is

$$u(r,\theta,t) = a_{00} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm}J_\nu\left(\frac{\lambda_{mn}r}{a}\right)\cos\left(\frac{n\pi\theta}{\alpha}\right)\exp\left(-\frac{\kappa t}{a^2}\lambda_{mn}^2\right). \tag{1.13.84}$$

The initial condition (1.13.77) leads to the result

$$f(r,\theta) = a_{00} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm} J_\nu\left(\frac{r\lambda_{mn}}{a}\right)\cos\left(\frac{n\pi\theta}{\alpha}\right). \tag{1.13.85}$$

This Fourier–Bessel double series expansion allows us to find the coefficients

$$a_{00} = \frac{2}{\alpha a^2}\int_0^a\int_0^\alpha f(r,\theta) r\,dr\,d\theta, \tag{1.13.86}$$

$$a_{0m} = \frac{2\lambda_{m0}^2}{\alpha a^2 \lambda_{m0}^2 J_0^2(\lambda_{m0})}\int_0^a\int_0^\alpha f(r,\theta) J_0\left(\frac{r\lambda_{m0}}{a}\right) r\,dr\,d\theta, \tag{1.13.87}$$

and for $n, m \geq 1$

$$a_{nm} = \frac{4\lambda_{nm}^2(\lambda_{nm}^2 - \nu^2)^{-1}}{\alpha a^2 J_\nu^2(\lambda_{nm})}\int_0^a\int_0^\alpha f(r,\theta) J_\nu\left(\frac{r\lambda_{nm}}{a}\right)\cos\left(\frac{n\pi\theta}{\alpha}\right) r\,dr\,d\theta. \tag{1.13.88}$$

In particular, we consider the case of $\alpha = \frac{\pi}{2}$ and the initial temperature

$$u(r,\theta,0) = f(r,\theta) = r^2\cos 2\theta.$$

Consequently,

$$r^2\cos 2\theta = a_{00} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_{nm} J_\nu\left(\frac{\lambda_{nm} r}{a}\right)\cos 2n\theta, \tag{1.13.89}$$

where $\nu = \frac{n\pi}{\alpha} = 2n$. The left-hand side of (1.13.89) involves $\cos 2n\theta$ only for $n = 1$. Clearly, the orthogonality relations for $\cos 2n\theta$ in $[0, \frac{\pi}{2}]$ suggest that only the terms with $n = 1$ on the right-hand side will be nonzero. We cancel the $\cos 2n\theta$ from both sides of (1.13.89) to obtain

$$r^2 = \sum_{m=1}^{\infty} a_m J_2\left(\frac{\lambda_m r}{a}\right), \tag{1.13.90}$$

where $a_{1m} = a_m$ and $\lambda_{m1} = \lambda_m$. It follows from (1.13.90) that

$$a_m = \frac{2\lambda_m^2}{a^2(\lambda_m^2 - 4)J_2^2(\lambda_m)}\int_0^a r^3 J_2\left(\frac{\lambda_m r}{a}\right) dr, \tag{1.13.91}$$

where the integral on the right-hand side can be exactly evaluated so that

$$\int_0^a r^3 J_2\left(\frac{\lambda_m r}{a}\right) dr = \left(\frac{a}{\lambda_m}\right)^4 \int_0^{\lambda_m} x^3 J_2(x)\,dx = \left(\frac{a}{\lambda_m}\right)^4 \lambda_m^3 J_3(\lambda_m). \tag{1.13.92}$$

Consequently, the final solution is

$$u(r,\theta,t) = 2a^2\cos 2\theta \sum_{m=1}^{\infty} \frac{\lambda_m J_3(\lambda_m)}{(\lambda_m^2 - 4)J_2^2(\lambda_m)} J_2\left(\frac{\lambda_m r}{a}\right)\exp\left(-\frac{\kappa\lambda_m^2 t}{a^2}\right). \tag{1.13.93}$$

1.14 Fractional Partial Differential Equations

(a) Fractional Diffusion Equation

The fractional diffusion equation is given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0, \tag{1.14.1}$$

with the boundary and initial conditions

$$u(x,t) \to 0 \quad \text{as } |x| \to \infty, \tag{1.14.2}$$

$$\left[{}_0D_t^{\alpha-1} u(x,t)\right]_{t=0} = f(x) \quad \text{for } x \in \mathbb{R}, \tag{1.14.3}$$

where κ is a diffusivity constant and $0 < \alpha \le 1$.

Application of the Fourier transform to (1.14.1) with respect to x and using the boundary condition (1.14.2) yield

$${}_0D_t^\alpha \tilde{u}(k,t) = -\kappa k^2 \tilde{u}, \tag{1.14.4}$$

$$\left[{}_0D_t^{\alpha-1} \tilde{u}(k,t)\right]_{t=0} = \tilde{f}(k), \tag{1.14.5}$$

where $\tilde{u}(k,t)$ is the Fourier transform of $u(x,t)$ defined by (1.7.1).

The Laplace transform solution of (1.14.4) and (1.14.5) yields

$$\tilde{\bar{u}}(k,s) = \frac{\tilde{f}(k)}{(s^\alpha + \kappa k^2)}. \tag{1.14.6}$$

The inverse Laplace transform of (1.14.6) gives

$$\tilde{u}(k,t) = \tilde{f}(k) t^{\alpha-1} E_{\alpha,\alpha}\left(-\kappa k^2 t^\alpha\right), \tag{1.14.7}$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad \alpha > 0,\ \beta > 0. \tag{1.14.8}$$

Finally, the inverse Fourier transform leads to the solution

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t) f(\xi)\, d\xi, \tag{1.14.9}$$

where

$$G(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}\left(-\kappa k^2 t^\alpha\right) \cos kx\, dk. \tag{1.14.10}$$

This integral can be evaluated by using the Laplace transform of $G(x,t)$ as

$$\overline{G}(x,s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos kx\, dk}{s^{\alpha}+\kappa k^2} = \frac{1}{\sqrt{4\kappa}} s^{-\alpha/2} \exp\left[-\frac{|x|}{\sqrt{\kappa}} s^{\alpha/2}\right], \tag{1.14.11}$$

where

$$\mathcal{L}\left[t^{m\alpha+\beta-1}E_{\alpha,\beta}^{(m)}(\pm a t^{\alpha})\right] = \frac{m!\, s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{m+1}}, \tag{1.14.12}$$

and

$$E_{\alpha,\beta}^{(m)}(z) = \frac{d^m}{dz^m} E_{\alpha,\beta}(z). \tag{1.14.13}$$

The inverse Laplace transform of (1.14.11) gives the explicit solution

$$G(x,t) = \frac{1}{\sqrt{4\kappa}} t^{\frac{\alpha}{2}-1} W\left(-\xi, -\frac{\alpha}{2}, \frac{\alpha}{2}\right), \tag{1.14.14}$$

where $\xi = \frac{|x|}{\sqrt{\kappa} t^{\alpha/2}}$, and $W(z,\alpha,\beta)$ is the Wright function (see Erdélyi 1955, formula 18.1 (27)) defined by

$$W(z,\alpha,\beta) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\alpha n+\beta)}. \tag{1.14.15}$$

It is important to note that when $\alpha = 1$, the initial-value problem (1.14.1)–(1.14.3) reduces to the classical diffusion problem, and solution (1.14.9) reduces to the classical solution (1.7.63) because

$$G(x,t) = \frac{1}{\sqrt{4\kappa t}} W\left(-\frac{x}{\sqrt{\kappa t}}, -\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right). \tag{1.14.16}$$

The fractional diffusion equation (1.14.1) has also been solved by other authors including Schneider and Wyss (1989), Mainardi (1994, 1995), Debnath (2003a, 2003b), and Nigmatullin (1986) with a physical realistic initial condition

$$u(x,0) = f(x), \quad x \in \mathbb{R}. \tag{1.14.17}$$

The solutions obtained by these authors are in total agreement with (1.14.9).

It is noted that the order α of the derivative with respect to time t in equation (1.14.1) can be of arbitrary real order including $\alpha = 2$ so that it may be called the fractional diffusion–wave equation. For $\alpha = 2$, it becomes the classical wave equation. Equation (1.14.1) with $1 < \alpha \le 2$ will be solved next in some detail.

(b) Fractional Nonhomogeneous Wave Equation

The fractional nonhomogeneous wave equation is given by

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - c^2 \frac{\partial^2 u}{\partial x^2} = q(x,t), \quad x \in \mathbb{R},\ t > 0 \tag{1.14.18}$$

with the initial condition

$$u(x,0)=f(x),\quad u_t(x,0)=g(x),\quad x\in\mathbb{R},\tag{1.14.19}$$

where c is a constant and $1<\alpha\le 2$.

Application of the joint Laplace transform with respect to t and the Fourier transform with respect to x gives the transform solution

$$\tilde{\bar{u}}(k,s)=\frac{\tilde{f}(k)s^{\alpha-1}}{s^\alpha+c^2k^2}+\frac{\tilde{g}(k)s^{\alpha-2}}{s^\alpha+c^2k^2}+\frac{\tilde{\bar{q}}(k,s)}{s^\alpha+c^2k^2},\tag{1.14.20}$$

where k is the Fourier transform variable and s is the Laplace transform variable.

The inverse Laplace transform produces the following result:

$$\begin{aligned}\tilde{u}(k,t)=\tilde{f}(k)\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha+c^2k^2}\right\}+\tilde{g}(k)\mathcal{L}^{-1}\left\{\frac{s^{\alpha-2}}{s^\alpha+c^2k^2}\right\}\\+\mathcal{L}^{-1}\left\{\frac{\tilde{\bar{q}}(k,s)}{s^\alpha+c^2k^2}\right\},\end{aligned}\tag{1.14.21}$$

which, by (1.14.12), is

$$\begin{aligned}&=\tilde{f}(k)E_{\alpha,1}\left(-c^2k^2t^\alpha\right)+\tilde{g}(k)tE_{\alpha,2}\left(-c^2k^2t^\alpha\right)\\&\quad+\int_0^t\tilde{q}(k,t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}\left(-c^2k^2\tau^\alpha\right)d\tau.\end{aligned}\tag{1.14.22}$$

Finally, the inverse Fourier transform gives the formal solution

$$\begin{aligned}u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k)E_{\alpha,1}\left(-c^2k^2t^\alpha\right)e^{ikx}\,dk\\+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t\tilde{g}(k)E_{\alpha,2}\left(-c^2k^2\tau^\alpha\right)e^{ikx}\,dk\\+\frac{1}{\sqrt{2\pi}}\int_0^t\tau^{\alpha-1}\,d\tau\int_{-\infty}^{\infty}\tilde{q}(k,t-\tau)E_{\alpha,\alpha}\left(-c^2k^2\tau^\alpha\right)e^{ikx}\,dk.\end{aligned}\tag{1.14.23}$$

In particular, when $\alpha=2$, the fractional wave equation (1.14.18) reduces to the classical wave equation (1.9.10). In this particular case, we use

$$E_{2,1}\left(-c^2k^2t^2\right)=\cosh(ickt)=\cos(ckt),\tag{1.14.24}$$

$$tE_{2,2}\left(-c^2k^2t^2\right)=t\cdot\frac{\sinh(ickt)}{ickt}=\frac{1}{ck}\sin ckt.\tag{1.14.25}$$

Consequently, solution (1.14.23) reduces to solution (1.9.15) for $\alpha=2$

$$\begin{aligned}u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(k)\cos(ckt)e^{ikx}\,dk+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(k)\frac{\sin(ckt)}{ck}e^{ikx}\,dk\\+\frac{1}{\sqrt{2\pi}c}\int_0^t d\tau\int_{-\infty}^{\infty}\tilde{q}(k,\tau)\frac{\sin ck(t-\tau)}{k}e^{ikx}\,dk\end{aligned}\tag{1.14.26}$$

$$= \frac{1}{2}\big[f(x-ct)+f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi + \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} q(\xi,\tau)\,d\xi. \tag{1.14.27}$$

We now derive the solution of the inhomogeneous fractional diffusion equation (1.14.18) with $c^2 = \kappa$ and $g(x) = 0$. In this case, the joint transform solution (1.14.20) becomes

$$\tilde{\bar{u}}(k,s) = \frac{\tilde{f}(k)s^{\alpha-1}}{(s^\alpha+\kappa k^2)} + \frac{\tilde{\bar{q}}(k,s)}{(s^\alpha+\kappa k^2)} \tag{1.14.28}$$

which is inverted by (1.14.12) to obtain $\tilde{u}(k,t)$ in the form

$$= \tilde{f}(k)E_{\alpha,1}\big(-\kappa k^2 t^\alpha\big) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big\{-\kappa k^2(t-\tau)^\alpha\big\}\tilde{q}(k,\tau)\,d\tau. \tag{1.14.29}$$

Finally, the inverse Fourier transform gives the exact solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(k)E_{\alpha,1}\big(-\kappa k^2 t^\alpha\big)e^{ikx}\,dk + \frac{1}{\sqrt{2\pi}}\int_0^t d\tau \int_{-\infty}^{\infty} (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big\{-\kappa k^2(t-\tau)^\alpha\big\}\tilde{q}(k,\tau)e^{ikx}\,dk. \tag{1.14.30}$$

Application of the convolution theorem of the Fourier transform gives the final solution in the form

$$u(x,t) = \int_{-\infty}^{\infty} G_1(x-\xi,t)f(\xi)\,d\xi + \int_0^t (t-\tau)^{\alpha-1}\,d\tau \int_{-\infty}^{\infty} G_2(x-\xi,t-\tau)q(\xi,\tau)\,d\xi, \tag{1.14.31}$$

where

$$G_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}E_{\alpha,1}\big(-\kappa k^2 t^\alpha\big)\,dk \tag{1.14.32}$$

and

$$G_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}E_{\alpha,\alpha}\big(-\kappa k^2 t^\alpha\big)\,dk. \tag{1.14.33}$$

In particular, when $\alpha = 1$, the classical solution of the nonhomogeneous diffusion equation (1.14.18) is obtained in the form

$$u(x,t) = \int_{-\infty}^{\infty} G_1(x-\xi,t)f(\xi)\,d\xi + \int_0^t d\tau \int_{-\infty}^{\infty} G_2(x-\xi,t-\tau)q(\xi,\tau)\,d\xi, \tag{1.14.34}$$

where

$$G_1(x,t) = G_2(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right). \tag{1.14.35}$$

In the case of classical homogeneous diffusion equation (1.14.18), solutions (1.14.30) and (1.14.34) are in perfect agreement with those of Mainardi (1996a, 1996b), who obtained the solution by using the Laplace transform method together with complicated evaluation of the Laplace inversion integral and the auxiliary function $M(z,\alpha)$. However, Mainardi (1996a, 1996b) obtained the solution in terms of $M(z,\frac{\alpha}{2})$ and discussed the nature of the solution for different values of α. He made some comparison between the ordinary diffusion ($\alpha = 1$) and fractional diffusion ($\alpha = \frac{1}{2}$ and $\alpha = \frac{2}{3}$). For cases $\alpha = \frac{4}{3}$ and $\alpha = \frac{3}{2}$, the solution exhibits a striking difference from ordinary diffusion with a transition from the Gaussian function centered at $z = 0$ (ordinary diffusion) to the Dirac delta function centered at $z = 1$ (wave propagation). This indicates a possibility of an *intermediate process* between diffusion and wave propagation. A special difference is observed between the solutions of the fractional diffusion equation ($0 < \alpha \le 1$) and the fractional wave equation ($1 < \alpha \le 2$). In addition, the solution exhibits a slow process for the case with $0 < \alpha \le 1$ and an intermediate process for $1 < \alpha \le 2$.

(c) Fractional-Order Diffusion Equation in Semi-Infinite Medium

We consider the fractional-order diffusion equation in a semi-infinite medium $x > 0$, when the boundary is kept at a temperature $u_0 f(t)$ and the initial temperature is zero in the whole medium. Thus, the initial boundary-value problem is governed by the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty,\ t > 0, \tag{1.14.36}$$

with

$$u(x, t = 0) = 0, \quad x > 0, \tag{1.14.37}$$

$$u(x = 0, t) = u_0 f(t), \quad t > 0 \text{ and } u(x,t) \to 0 \text{ as } x \to \infty. \tag{1.14.38}$$

Application of the Laplace transform with respect to t gives

$$\frac{d^2\bar{u}}{dx^2} - \left(\frac{s^\alpha}{\kappa}\right)\bar{u}(x,s) = 0, \quad x > 0, \tag{1.14.39}$$

$$\bar{u}(x = 0, s) = u_0 \bar{f}(s), \quad \bar{u}(x,s) \to 0 \text{ as } x \to \infty. \tag{1.14.40}$$

Evidently, the solution of this transformed boundary-value problem is

$$\bar{u}(x,s) = u_0\bar{f}(s)\exp(-ax), \tag{1.14.41}$$

where $a = (s^\alpha/\kappa)^{\frac{1}{2}}$. Thus, the solution is given by

$$u(x,t) = u_0\int_0^t f(t-\tau)g(x,\tau)\,d\tau = u_0 f(t) * g(x,t), \tag{1.14.42}$$

where

$$g(x,t) = \mathcal{L}^{-1}\big\{\exp(-ax)\big\}.$$

In this case, $\alpha = 1$ and $f(t) = 1$, and the solution (1.14.41) becomes

$$\bar{u}(x,s) = \frac{u_0}{s}\exp\left(-x\sqrt{\frac{s}{\kappa}}\right), \tag{1.14.43}$$

which yields the classical solution in terms of the complementary error function (see Debnath 1995)

$$u(x,t) = u_0\text{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right). \tag{1.14.44}$$

In the classical case $(\alpha = 1)$, the more general solution is given by

$$u(x,t) = u_0\int_0^t f(t-\tau)g(x,\tau)\,d\tau = u_0 f(t) * g(x,t), \tag{1.14.45}$$

where

$$g(x,t) = \mathcal{L}^{-1}\left\{\exp\left(-x\sqrt{\frac{s}{\kappa}}\right)\right\} = \frac{x}{2\sqrt{\pi\kappa t^3}}\exp\left(-\frac{x^2}{4\kappa t}\right). \tag{1.14.46}$$

(d) The Fractional Stokes and Rayleigh Problems in Fluid Dynamics

The classical Stokes problem (see Debnath 1995) deals with the unsteady boundary layer flows induced in a semi-infinite viscous fluid bounded by an infinite horizontal disk at $z = 0$ due to nontorsional oscillations of the disk in its own plane with a given frequency ω. When $\omega = 0$, the Stokes problem reduces to the classical Rayleigh problem where the unsteady boundary layer flow is generated in the fluid from rest by moving the disk impulsively in its own plane with constant velocity U.

We consider the unsteady fractional boundary layer equation for the fluid velocity $u(z,t)$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \nu\frac{\partial^2 u}{\partial z^2}, \quad 0 < z < \infty, t > 0, \tag{1.14.47}$$

with the given boundary and initial conditions

$$u(0,t) = Uf(t), \quad u(z,t) \to 0 \quad \text{as } z \to \infty,\ t > 0, \tag{1.14.48}$$

$$u(z,0) = 0 \quad \text{for all } z > 0, \tag{1.14.49}$$

where ν is the kinematic viscosity, U is a constant velocity, and $f(t)$ is an arbitrary function of time t.

Application of the Laplace transform with respect to t gives

$$s^{\alpha}\bar{u}(z,s) = \nu \frac{d^2\bar{u}}{dz^2}, \quad 0 < z < \infty, \tag{1.14.50}$$

$$\bar{u}(0,s) = U\,\bar{f}(s), \quad \bar{u}(z,s) \to 0 \text{ as } z \to \infty. \tag{1.14.51}$$

Using the Fourier sine transform (see Debnath 1995) with respect to z yields

$$\overline{U}_s(k,s) = \left(\sqrt{\frac{2}{\pi}}\nu U\right)\frac{k\bar{f}(s)}{(s^{\alpha}+\nu k^2)}. \tag{1.14.52}$$

The inverse Fourier sine transform of (1.14.52) leads to the solution

$$\bar{u}(z,s) = \left(\frac{2}{\pi}\nu U\right)\bar{f}(s)\int_0^{\infty}\frac{k\sin kz}{(s^{\alpha}+\nu k^2)}\,dk, \tag{1.14.53}$$

and the inverse Laplace transform gives the solution for the velocity

$$u(z,t) = \left(\frac{2}{\pi}\nu U\right)\int_0^{\infty} k\sin kz\,dk\int_0^t f(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\nu k^2\tau^{\alpha}\right)d\tau. \tag{1.14.54}$$

When $f(t) = \exp(i\omega t)$, the solution of the fractional Stokes problem is

$$u(z,t) = \left(\frac{2\nu U}{\pi}\right)e^{i\omega t}\int_0^{\infty} k\sin kz\,dk\int_0^t e^{-i\omega\tau}\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\nu k^2\tau^{\alpha}\right)d\tau. \tag{1.14.55}$$

When $\alpha = 1$, solution (1.14.55) reduces to the classical Stokes solution

$$u(z,t) = \left(\frac{2\nu U}{\pi}\right)\int_0^{\infty}\left(1-e^{-\nu t k^2}\right)\frac{k\sin kz}{(i\omega+\nu k^2)}\,dk. \tag{1.14.56}$$

For the fractional Rayleigh problem, $f(t) = 1$ and the solution follows from (1.14.54) in the form

$$u(z,t) = \left(\frac{2\nu U}{\pi}\right)\int_0^{\infty} k\sin kz\,dk\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left(-\nu k^2\tau^{\alpha}\right)d\tau. \tag{1.14.57}$$

This solution reduces to the classical Rayleigh solution when $\alpha = 1$ as

$$\begin{aligned}
u(z,t) &= \left(\frac{2\nu U}{\pi}\right)\int_0^\infty k\sin kz\,dk\int_0^t E_{1,1}\left(-\nu\tau k^2\right)d\tau \\
&= \left(\frac{2\nu U}{\pi}\right)\int_0^\infty k\sin kz\,dk\int_0^t \exp\left(-\nu\tau k^2\right)d\tau \\
&= \left(\frac{2U}{\pi}\right)\int_0^\infty \left(1-e^{-\nu t k^2}\right)\frac{\sin kz}{k}\,dk,
\end{aligned}$$

which, by (2.10.10) of Debnath (1995),

$$= \left(\frac{2U}{\pi}\right)\left[\frac{\pi}{2}-\frac{\pi}{2}\operatorname{erf}\left(\frac{z}{2\sqrt{\nu t}}\right)\right] = U\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}}\right). \tag{1.14.58}$$

The above analysis is in full agreement with the classical solutions of the Stokes and Rayleigh problems (see Debnath 1995).

(e) The Fractional Unsteady Couette Flow

We consider the unsteady viscous fluid flow between the plate at $z = 0$ at rest and the plate $z = h$ in motion parallel to itself with a variable velocity $U(t)$ in the x-direction. The fluid velocity $u(z,t)$ satisfies the fractional equation of motion

$$\frac{\partial^\alpha u}{\partial t^\alpha} = P(t) + \nu\frac{\partial^2 u}{\partial z^2}, \quad 0 \le z \le h,\ t > 0, \tag{1.14.59}$$

with the boundary and initial conditions

$$u(0,t) = 0 \quad \text{and} \quad u(h,t) = U(t), \quad t > 0, \tag{1.14.60}$$

$$u(z,t) = 0 \quad \text{at } t \le 0 \text{ for } 0 \le z \le h, \tag{1.14.61}$$

where $-\frac{1}{\rho}p_x = P(t)$ and ν is the kinematic viscosity of the fluid.

We apply the joint Laplace transform with respect to t and the finite Fourier sine transform with respect to z defined by

$$\bar{\tilde{u}}_s(n,s) = \int_0^\infty e^{-st}\,dt\int_0^h u(z,t)\sin\left(\frac{n\pi z}{h}\right)dz \tag{1.14.62}$$

to the system (1.14.59)–(1.14.61) so that the transform solution is

$$\bar{\tilde{u}}_s(n,s) = \frac{\overline{P}(s)\frac{1}{a}[1-(-1)^n]}{(s^\alpha+\nu a^2)} + \frac{\nu a(-1)^{n+1}\overline{U}(s)}{(s^\alpha+\nu a^2)}, \tag{1.14.63}$$

where $a = \left(\frac{n\pi}{h}\right)$ and n is the finite Fourier sine transform variable.

Thus, the inverse Laplace transform yields

$$\begin{aligned}
\tilde{u}_s(n,t) &= \frac{1}{a}\left[1-(-1)^n\right]\int_0^t P(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\nu a^2\tau^\alpha\right)d\tau \\
&\quad + \nu a(-1)^{n+1}\int_0^t U(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\nu a^2\tau^\alpha\right)d\tau.
\end{aligned}\tag{1.14.64}$$

Finally, the inverse finite Fourier sine transform leads to the solution

$$u(z,t) = \frac{2}{h}\sum_{n=1}^{\infty} \tilde{u}_s(n,t)\sin\left(\frac{n\pi z}{h}\right). \tag{1.14.65}$$

If, in particular, when $\alpha = 1$, $P(t) =$ const., and $U(t) =$ const., then solution (1.14.65) reduces to the solution of the generalized Couette flow (see p. 277 Debnath 1995).

(f) Fractional Axisymmetric Wave–Diffusion Equation

The fractional axisymmetric equation in an infinite domain

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \quad 0 < r < \infty, t > 0, \tag{1.14.66}$$

is called the *diffusion* or *wave equation* accordingly as $a = \kappa$ or $a = c^2$.

For the fractional diffusion equation, we prescribe the initial condition

$$u(r,0) = f(r), \quad 0 < r < \infty. \tag{1.14.67}$$

Application of the joint Laplace transform with respect to t and the Hankel transform of zero order (see Debnath 1995) with respect to r to (1.14.66), (1.14.67) gives the transform solution

$$\bar{\tilde{u}}(k,s) = \frac{s^{\alpha-1}\tilde{f}(k)}{(s^\alpha + \kappa k^2)}, \tag{1.14.68}$$

where k, s are the Hankel and Laplace transform variables, respectively.

The joint inverse transform leads to the solution

$$u(r,t) = \int_0^\infty kJ_0(kr)\tilde{f}(k)E_{\alpha,1}\left(-\kappa k^2 t^\alpha\right) dk, \tag{1.14.69}$$

where $J_0(kr)$ is the Bessel function of the first kind of order zero and $\tilde{f}(k)$ is the Hankel transform of $f(r)$.

When $\alpha = 1$, solution (1.14.69) reduces to the classical solution (1.10.22).

On the other hand, we can solve the wave equation (1.14.66) with $a = c^2$ and the initial conditions

$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \quad \text{for} 0 < r < \infty, \tag{1.14.70}$$

provided the Hankel transforms of $f(r)$ and $g(r)$ exist.

Application of the joint Laplace and Hankel transform leads to the transform solution

$$\bar{\tilde{u}}(k,s) = \frac{s^{\alpha-1}\tilde{f}(k)}{(s^\alpha + c^2k^2)} + \frac{s^{\alpha-2}\tilde{g}(k)}{(s^\alpha + c^2k^2)}. \tag{1.14.71}$$

The joint inverse transformation gives the solution

$$u(r,t) = \int_0^\infty kJ_0(k,r)\tilde{f}(k)E_{\alpha,1}\left(-c^2k^2t^\alpha\right)dk + \int_0^\infty kJ_0(k,r)\tilde{g}(k)tE_{\alpha,2}\left(-c^2k^2t^\alpha\right)dk. \quad (1.14.72)$$

When $\alpha = 2$, (1.14.72) reduces to the classical solution (1.10.12).

In a finite domain $0 \le r \le a$, the fractional diffusion equation (1.14.66) can be solved by using the joint Laplace and finite Hankel transform with the boundary and initial data

$$u(r,t) = f(t) \quad \text{on } r = a,\ t > 0, \quad (1.14.73)$$

$$u(r,0) = 0 \quad \text{for all } r \text{ in } (0,a). \quad (1.14.74)$$

Application of the joint Laplace and finite Hankel transform of zero order (see pp. 317, 318, Debnath 1995) yields the solution

$$u(r,t) = \frac{2}{a^2}\sum_{i=1}^\infty \tilde{u}(k_i,t)\frac{J_0(rk_i)}{J_1^2(ak_i)}, \quad (1.14.75)$$

where

$$\tilde{u}(k_i,t) = (a\kappa k_i)J_1(ak_i)\int_0^t f(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\kappa k_i^2\tau^\alpha\right)d\tau. \quad (1.14.76)$$

When $\alpha = 1$, (1.14.75) reduces to (11.4.7) obtained by Debnath (1995).

Similarly, the fractional wave equation (1.14.66) with $a = c^2$ in a finite domain $0 \le r \le a$ with the boundary and initial conditions

$$u(r,t) = 0 \quad \text{on } r = a, t > 0, \quad (1.14.77)$$

$$u(r,0) = f(r) \quad \text{and} \quad u_t(r,0) = g(r) \quad \text{for } 0 < r < a, \quad (1.14.78)$$

can be solved by means of the joint Laplace and finite Hankel transforms.

The solution of this problem is

$$u(r,t) = \frac{2}{a^2}\sum_{i=1}^\infty \tilde{u}(k_i,t)\frac{J_0(rk_i)}{J_1^2(ak_i)}, \quad (1.14.79)$$

where

$$\tilde{u}(k_i,t) = \tilde{f}(k_i)E_{\alpha,1}\left(-c^2k_i^2t^\alpha\right) + \tilde{g}(k_i)tE_{\alpha,2}\left(-c^2k_i^2t^\alpha\right). \quad (1.14.80)$$

When $\alpha = 2$, solution (1.14.79) reduces to the solution (11.4.26) obtained by Debnath (1995).

(g) The Fractional Schrödinger Equation in Quantum Mechanics

The one-dimensional fractional Schrödinger equation for a free particle of mass m (see (1.7.71)) is

$$i\hbar\frac{\partial^\alpha\psi}{\partial t^\alpha} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}, \quad -\infty < x < \infty,\ t > 0, \tag{1.14.81}$$

$$\psi(x,0) = \psi_0(x), \quad -\infty < x < \infty, \tag{1.14.82}$$

$$\psi(x,t) \to 0 \quad \text{as } |x| \to \infty, \tag{1.14.83}$$

where $\psi(x,t)$ is the wave function, $h = 2\pi\hbar = 6.625 \times 10^{-27}$ erg sec $= 4.14 \times 10^{-21}$ MeV sec is the Planck constant, and $\psi_0(x)$ is an arbitrary function.

Application of the joint Laplace and Fourier transform to (1.14.81)–(1.14.83) gives the solution in the transform space in the form

$$\bar{\tilde{\psi}}(k,s) = \frac{s^{\alpha-1}\tilde{\psi}_0(k)}{s^\alpha + ak^2} \quad \left(a = \frac{i\hbar}{2m}\right), \tag{1.14.84}$$

where k, s represent the Fourier and the Laplace transform variables.

The use of the joint inverse transform yields the solution

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\tilde{\psi}_0(k)E_{\alpha,1}\left(-ak^2t^\alpha\right) dk \tag{1.14.85}$$

$$= \mathcal{F}^{-1}\left\{\tilde{\psi}_0(k)E_{\alpha,1}\left(-ak^2t^\alpha\right)\right\}, \tag{1.14.86}$$

which is, by the Convolution Theorem 1.7.1 of the Fourier transform,

$$= \int_{-\infty}^{\infty} G(x-\xi,t)\psi_0(\xi)\,d\xi, \tag{1.14.87}$$

where

$$G(x,t) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left\{E_{\alpha,1}\left(-ak^2t^\alpha\right)\right\}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}E_{\alpha,1}\left(-ak^2t^\alpha\right) dk. \tag{1.14.88}$$

When $\alpha = 1$, solution (1.14.87) becomes

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t)\psi_0(\xi)\,d\xi, \tag{1.14.89}$$

where Green's function $G(x,t)$ is given by

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}E_{1,1}\left(-ak^2t\right) dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(ikx - atk^2\right) dk$$
$$= \frac{1}{\sqrt{4\pi at}}\exp\left(-\frac{x^2}{4at}\right). \tag{1.14.90}$$

This solution (1.14.89) is in perfect agreement with the classical solution obtained by Debnath (1995).

1.15 Exercises

1. Classify each of the partial differential equations below as either hyperbolic, parabolic, or elliptic, determine the characteristics, and transform the equations to canonical form:

   (a) $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$, (b) $2u_{xx} - 3u_{xy} + u_{yy} = y$,
   (c) $yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$, (d) $u_{xx} + yu_{yy} = 0$,
   (e) $yu_{xx} - 2u_{xy} + e^x u_{yy} + x^2 u_x - u = 0$, (f) $u_{xx} + xu_{yy} = 0$,
   (g) $x^2 u_{xx} + 4yu_{xy} + u_{yy} + 2u_x = 0$, (h) $3yu_{xx} - xu_{yy} = 0$,
   (i) $u_{xx} + 2xu_{xy} + a^2 u_{yy} + u = 5$, (j) $y^2 u_{xx} + x^2 u_{yy} = 0$.

2. Determine the nature of the following equations and reduce them to canonical form:

   (a) $x^2 u_{xx} + 4xyu_{xy} + y^2 u_{yy} = 0$, (b) $u_{xx} - xu_{yy} = 0$,
   (c) $u_{xx} - 2u_{xy} + 3u_{yy} + 24u_y + 5u = 0$, (d) $u_{xx} + \text{sech}^4 x u_{yy} = 0$,
   (e) $u_{xx} + 6yu_{xy} + 9y^2 u_{yy} + 4u = 0$, (f) $u_{xx} - \text{sech}^4 x u_{yy} = 0$,
   (g) $u_{xx} + 2\text{cosec} y u_{xy} + \text{cosec}^2 y u_{yy} = 0$, (h) $u_{xx} - 5u_{xy} + 5u_{yy} = 0$.

3. For what values of m is $u_{xx} - m_x u_{xy} + 4x^2 u_{yy} = 0$ (a) hyperbolic, (b) parabolic, or (c) elliptic? For $m = 0$, reduce to canonical form.

4. (a) Show that the nonlinear equation

$$u^2 u_{xx} + 2u_x u_y u_{xy} - u^2 u_{yy} = 0$$

   is hyperbolic for every solution $u(x, y)$.
   (b) Show that the nonlinear equation for the velocity potential $u(x, y)$

$$\left(1 - u_x^2\right) u_{xx} - 2u_x u_y u_{xy} + \left(1 - u_y^2\right) u_{yy} = 0$$

   in certain kinds of compressible fluid flow is (i) elliptic, (ii) parabolic, or (iii) hyperbolic for those solutions such that $|\nabla u| < 1$, $|\nabla u| = 1$, or $|\nabla u| > 1$.

5. Solve Example 1.6.2 with the boundary conditions

$$u_x(0, t) = 0 = u_x(\ell, t) \quad \text{for } t > 0,$$

   leaving the initial condition (1.6.38) unchanged.

6. Use the separation of variables to solve the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a,\ 0 \le y \le b,$$

   with $u(0, y) = 0 = u(a, y)$ for $0 \le y \le b$, and $u(x, 0) = f(x)$ for $0 < x < a$; $u(x, b) = 0$ for $0 \le x \le a$.

7. Show that the eigenvalues of the eigenvalue problem

$$\begin{aligned} u_{tt} + c^2 u_{xxxx} &= 0, \quad 0 < x < \ell,\ t > 0, \\ u(0, t) = 0 &= u(\ell, t) \quad \text{for } t \ge 0, \\ u_x(0, t) = 0 &= u_x(\ell, t) \quad \text{for } t \ge 0, \end{aligned}$$

satisfy the equation

$$\cos(\lambda\ell)\cosh(\lambda\ell) = 1.$$

8. Solve the problem in Exercise 4 with the boundary conditions

$$\begin{aligned} u(x,0) &= f(x), \quad u_t(x,0) = g(x) \quad \text{for } 0 \le x \le \ell, \\ u(0,t) &= 0 = u(\ell,t) \quad \text{for } t > 0, \\ u_{xx}(0,t) &= 0 = u_{xx}(\ell,t) \quad \text{for } t > 0. \end{aligned}$$

9. Solve Example 1.6.2 with the initial data

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{\ell}{2}, \\ \ell - x & \text{if } \frac{\ell}{2} \le x \le \ell. \end{cases}$$

10. Solve Example 1.6.1 with the initial data

$$\text{(i) } f(x) = \begin{cases} \frac{hx}{a} & \text{if } 0 \le x \le a, \\ h(\ell - x)/(\ell - a) & \text{if } a \le x \le \ell, \end{cases} \quad \text{and } g(x) = 0.$$

$$\text{(ii) } f(x) = 0 \quad \text{and} \quad g(x) = \begin{cases} \frac{u_0 x}{a} & \text{if } 0 \le x \le a, \\ u_0(\ell - x)/(\ell - a) & \text{if } a \le x \le \ell. \end{cases}$$

11. Solve the biharmonic wave equation

$$\begin{aligned} u_{tt} + u_{xxxx} &= 0, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = 0 \quad \text{for } -\infty < x < \infty. \end{aligned}$$

12. Find the solution of the dissipative wave equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} + \alpha u_t &= 0, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x), \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \quad \text{for } -\infty < x < \infty, \end{aligned}$$

where $\alpha > 0$ is the dissipation parameter.

13. Solve the Cauchy problem for the linear Klein–Gordon equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} + a^2 u &= 0, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x), \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \quad \text{for } -\infty < x < \infty. \end{aligned}$$

14. Solve the telegraph equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} + u_t - a u_x &= 0, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x), \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \quad \text{for } -\infty < x < \infty. \end{aligned}$$

Show that the solution is unstable when $c^2 < a^2$. If $c^2 > a^2$, show that the bounded integral solution is given by

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \exp\left[-k^2\left(c^2 - a^2\right)t + ik(x + at)\right] dk,$$

where $A(k)$ is given in terms of the transformed functions of the initial data. Hence, deduce the asymptotic solution, as $t \to \infty$, in the form

$$u(x,t) = A(0)\sqrt{\frac{\pi}{2(c^2 - a^2)t}} \exp\left[-\frac{(x + at)^2}{4(c^2 - a^2)t}\right].$$

15. The transverse vibration of an infinite elastic beam of mass m per unit length and bending stiffness EI is governed by

$$u_{tt} + a^2 u_{xxxx} = 0, \quad a^2 = \frac{EI}{m}, \ -\infty < x < \infty, \ t > 0.$$

Solve this equation subject to the boundary and initial data

$$u(0,t) = 0 \quad \text{for all } t > 0,$$
$$u(x,0) = \phi(x), \quad \text{and} \quad u_t(x,0) = \psi''(x) \quad \text{for } 0 < x < \infty.$$

Show that the Fourier transform solution is

$$U(k,t) = \Phi(k)\cos\left(atk^2\right) - \left(\frac{1}{a}\right)\Psi(k)\sin\left(atk^2\right).$$

Find the integral solution for $u(x,t)$.

16. Solve the Lamb (1904) problem in geophysics that satisfies the Helmholtz equation in an infinite elastic half-space

$$u_{xx} + u_{zz} + \frac{\omega^2}{c_2^2} u = 0, \qquad -\infty < x < \infty, \quad z > 0,$$

where ω is the frequency and c_2 is the shear wave speed.
At the surface of the half-space ($z = 0$), the boundary condition relating the surface stress to the impulsive point load distribution is given by

$$\mu \frac{\partial u}{\partial z} = -P\delta(x) \quad \text{at } z = 0,$$

where μ is one of the Lamé constants, P is a constant, and

$$u(x,z) \to 0 \quad \text{as } z \to \infty \text{ for } -\infty < x < \infty.$$

Show that the solution in terms of polar coordinates is

$$u(x,z) = \frac{P}{2i\mu} H_0^{(2)}\left(\frac{\omega r}{c_2}\right)$$
$$\sim \frac{P}{2i\mu}\left(\frac{2c_2}{\pi\omega r}\right)^{\frac{1}{2}} \exp\left(\frac{\pi i}{4} - \frac{i\omega r}{c_2}\right) \quad \text{for } \omega r \gg c_2.$$

17. Find the solution of the Cauchy–Poisson problem (Debnath 1994, p. 83) in inviscid water of infinite depth which is governed by

$$\begin{aligned} &\phi_{xx} + \phi_{zz} = 0, \quad -\infty < x < \infty, -\infty < z \le 0,\ t > 0, \\ &\left.\begin{aligned} &\phi_z - \eta_t = 0, \\ &\phi_t + g\eta = 0 \end{aligned}\right\} \quad \text{on } z = 0,\ t > 0, \\ &\phi_z \to 0 \quad \text{as } z \to -\infty. \\ &\phi(x,0,0) = 0, \quad \text{and} \quad \eta(x,0) = P\delta(x), \end{aligned}$$

where $\phi = \phi(x,z,t)$ is the velocity potential, $\eta(x,t)$ is the free surface elevation, and P is a constant.
Derive the asymptotic solution for the free surface elevation as $t \to \infty$.

18. Obtain the solutions for the velocity potential $\phi(x,z,t)$ and the free surface elevation $\eta(x,t)$ involved in the two-dimensional surface waves in water of finite (or infinite) depth h. The governing equation, boundary, and free surface conditions and initial conditions (see Debnath 1994, p. 92) are

$$\begin{aligned} &\phi_{xx} + \phi_{zz} = 0, \quad -h \le z \le 0, -\infty < x < \infty,\ t > 0, \\ &\left.\begin{aligned} &\phi_t + g\eta = -\tfrac{P}{\rho} p(x) \exp(i\omega t), \\ &\phi_z - \eta_t = 0 \end{aligned}\right\} \quad z = 0, t > 0, \\ &\phi(x,z,0) = 0 = \eta(x,0) \quad \text{for all } x \text{ and } z. \end{aligned}$$

19. Solve the steady-state surface wave problem (Debnath 1994, p. 47) on a running stream of infinite depth due to an external steady pressure applied to the free surface. The governing equation and the free surface conditions are

$$\begin{aligned} &\phi_{xx} + \phi_{zz} = 0, \quad -\infty < x < \infty, -\infty < z < 0,\ t > 0, \\ &\left.\begin{aligned} &\phi_x + U\phi_x + g\eta = -\tfrac{P}{\rho}\delta(x) \exp(\varepsilon t), \\ &\eta_t + U\eta_x = \phi_z \end{aligned}\right\} \quad \text{on } z = 0\ (\varepsilon > 0), \\ &\phi_z \to 0 \quad \text{as } z \to -\infty. \end{aligned}$$

where U is the stream velocity, $\phi(x,z,t)$ is the velocity potential, and $\eta(x,t)$ is the free surface elevation.

20. Apply the Fourier transform to solve the initial-value problem for the dissipative wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + \alpha u_{xxt}, \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = \alpha f''(x) \quad \text{for } -\infty < x < \infty, \end{aligned}$$

where α is a positive constant.

21. Use the Fourier transform to solve the boundary-value problem

$$u_{xx} + u_{yy} = -x \exp\left(-x^2\right), \quad -\infty < x < \infty,\ 0 < y < \infty,$$

$u(x,0)=0$, for $-\infty<x<\infty$, u and its derivative vanish as $y\to\infty$.
Show that

$$u(x,y)=\frac{1}{\sqrt{4\pi}}\int_0^{\infty}\left[1-\exp(-ky)\right]\frac{\sin kx}{k}\exp\left(-\frac{k^2}{4}\right)dk.$$

22. Solve the *initial-value problem* (Debnath 1994, p. 115) for the two-dimensional surface waves at the free surface of a running stream of velocity U. The problem satisfies the following equation, boundary, and initial conditions:

$$\phi_{xx}+\phi_{zz}=0,\qquad -\infty<x<\infty, -h\le z\le 0,\ t>0,$$
$$\left.\begin{aligned}&\phi_x+U\phi_x+g\eta=-\tfrac{P}{\rho}\delta(x)\exp(i\omega t),\\ &\eta_t+U\eta_x-\phi_z=0\end{aligned}\right\}\quad \text{on } z=0, t>0,$$
$$\phi(x,z,0)=\eta(x,0)=0,\qquad \text{for all } x \text{ and } z.$$

23. Apply the Fourier transform to solve the equation

$$u_{xxxx}+u_{yy}=0,\qquad -\infty<x<\infty,\ y\ge 0,$$

satisfying the conditions

$$u(x,0)=f(x),\qquad u_y(x,0)=0\quad \text{for } -\infty<x<\infty,$$

and $u(x,y)$ and its partial derivatives vanish as $|x|\to\infty$.

24. The transverse vibration of a thin membrane of great extent satisfies the wave equation

$$c^2(u_{xx}+u_{yy})=u_{tt},\qquad -\infty<x,y<\infty,\ t>0,$$

with the initial and boundary conditions

$$u(x,y,t)\to 0\quad \text{as } |x|\to\infty, |y|\to\infty \text{ for all } t\ge 0,$$
$$u(x,y,0)=f(x,y),\qquad u_t(x,y,0)=0\quad \text{for all } x,y.$$

Apply the double Fourier transform method to solve this problem.

25. Solve the diffusion problem with a source function $q(x,t)$

$$u_t=\kappa u_{xx}+q(x,t),\qquad -\infty<x<\infty,\ t>0,$$
$$u(x,0)=0\quad \text{for } -\infty<x<\infty.$$

Show that the solution is

$$u(x,t)=\frac{1}{\sqrt{4\pi\kappa}}\int_0^t(t-\tau)^{-\frac{1}{2}}\,d\tau\int_{-\infty}^{\infty}q(k,\tau)\exp\left[-\frac{(x-k)^2}{4\kappa(t-\tau)}\right]dk.$$

26. Apply the triple Fourier transform to solve the initial-value problem

$$u_t=\kappa(u_{xx}+u_{yy}+u_{zz}),\qquad -\infty<x,y,z<\infty,\ t>0,$$
$$u(\mathbf{x},0)=f(\mathbf{x})\quad \text{for all } x,y,z,$$

where $\mathbf{x}=(x,y,z)$.

27. Use the double Fourier transform to solve the telegraph equation

$$u_{tt} + au_t + bu = c^2 u_{xx}, \quad -\infty < x, t < \infty,$$
$$u(0,t) = f(t), \quad u_x(0,t) = g(t), \quad \text{for } -\infty < t < \infty,$$

where a, b, c are constants and $f(t)$ and $g(t)$ are arbitrary functions of t.

28. Use the Fourier transform to solve the Rossby wave problem in an inviscid β-plane ocean bounded by walls at $y = 0$ and $y = 1$, where y and x represent vertical and horizontal directions. The fluid is initially at rest and then, at $t = 0+$, an arbitrary disturbance localized to the vicinity of $x = 0$ is applied to generate Rossby waves. This problem satisfies the Rossby wave equation

$$\frac{\partial}{\partial t}\left[\left(\nabla^2 - \kappa^2\right)\psi\right] + \beta\psi_x = 0, \quad -\infty < x < \infty, 0 \le y \le 1,\ t > 0,$$

with the boundary and initial conditions

$$\psi_x(x,y) = 0 \quad \text{for } 0 < x < \infty,\ y = 0 \text{ and } y = 1,$$
$$\psi(x,y,t) = \psi_0(x,y) \quad \text{at } t = 0 \text{ for all } x \text{ and } y.$$

Examine the case for $\psi_{0n}(x) = \frac{1}{\alpha\sqrt{2}} \exp\{ik_0 x - (\frac{x}{a})^2\}$.

29. The equations for the current $I(x,t)$ and potential $V(x,t)$ at a point x and time t of a transmission line containing resistance R, inductance L, capacitance C, and leakage inductance G are

$$LI_t + RI = -V_x, \quad \text{and} \quad CV_t + GV = -I_x.$$

Show that both I and V satisfy the telegraph equation

$$\frac{1}{c^2} u_{tt} - u_{xx} + au_t + bu = 0,$$

where $c^2 = (LC)^{-1}$, $a = LG + RC$, and $b = RG$.
Solve the telegraph equation for the following cases with $R = 0$ and $G = 0$:
(a) $V(x,t) = V_0 H(t)$ at $x = 0$, $t > 0$, $V(x,t) \to 0$ as $x \to \infty$, $t > 0$, where V_0 is constant.
(b) $V(x,t) = V_0 \cos\omega t$ at $x = 0$, $t > 0$, $V(x,t) \to 0$ as $x \to \infty$, $t > 0$.

30. Solve the telegraph equation in Exercise 29 with $V(x,0) = 0$ for (a) the Kelvin ideal cable line $(L = 0 = G)$ with the boundary data $V(0,t) = V_0 = \text{const.}$, $V(x,t) \to 0$ as $x \to \infty$ for $t > 0$.
(b) the noninductive leaky cable $(L = 0)$ with the boundary conditions $V(0,t) = H(t)$ and $V(x,t) \to 0$ as $x \to \infty$ for $t > 0$.

31. Solve the telegraph equation in Exercise 29 with $V(x,0) = 0 = V_t(x,0)$ for the Heaviside distortionless cable $(\frac{R}{L} = \frac{G}{C} = \text{const.} = k)$ with the boundary data $V(0,t) = V_0 f(t)$ and $V(x,t) \to 0$ as $x \to \infty$ for $t > 0$, where V_0 is constant and $f(t)$ is an arbitrary function of t. Explain the physical significance of the solution.

32. Solve the inhomogeneous partial differential equation

$$u_{xt} = -\omega \sin \omega t, \quad t > 0,$$
$$u(x, 0) = x, \qquad u(0, t) = 0.$$

33. Find the solution of the inhomogeneous equation

$$\frac{1}{c^2} u_{tt} - u_{xx} = k \sin\left(\frac{\pi x}{a}\right), \quad 0 < x < a, \ t > 0,$$
$$u(x, 0) = 0 = u_t(x, 0) \quad \text{for } 0 < x < a,$$
$$u(0, t) = 0 = u(a, t) \quad \text{for } t > 0.$$

34. Solve the Stokes problem which is concerned with the unsteady boundary layer flows induced in a semi-infinite viscous fluid bounded by an infinite horizontal disk at $z = 0$ due to nontorsional oscillations of the disk in its own plane with a given frequency ω. The equation of motion and the boundary and initial conditions are

$$\begin{aligned} u_t &= \nu u_{zz}, \quad z > 0, \ t > 0, \\ u(z, t) &= U e^{i\omega t} \quad \text{on } z = 0, \ t > 0, \\ u(z, t) &\to 0 \quad \text{as } z \to \infty \text{ for } t > 0, \\ u(z, 0) &= 0 \quad \text{for } t \le 0 \text{ and } z > 0, \end{aligned}$$

where $u(z, t)$ is the velocity of the fluid of kinematic viscosity ν and U is constant. Solve the Rayleigh problem ($\omega = 0$). Explain the physical significance of both the Stokes and Rayleigh solutions.

35. Solve the Blasius problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid enclosed by an infinite horizontal disk at $z = 0$. The governing equation and the boundary and initial conditions are

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial z^2}, \quad z > 0, \ t > 0, \\ u(z, t) &= U t \quad \text{on } z = 0, \ t > 0, \\ u(z, t) &\to 0 \quad \text{as } z \to \infty, \ t > 0, \\ u(z, t) &= 0 \quad \text{at } t \le 0, \ z > 0. \end{aligned}$$

Explain the significance of the solution.

36. Obtain the solution of the Stokes–Ekman problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid bounded by an infinite horizontal disk at $z = 0$ when both the fluid and the disk rotate with a uniform angular velocity Ω about the z-axis. The governing boundary layer equation and the boundary and the initial conditions are

$$\begin{aligned} \frac{\partial q}{\partial t} + 2\Omega i q &= \nu \frac{\partial^2 q}{\partial z^2}, \quad z > 0, \ t > 0, \\ q(z, t) &= a e^{i\omega t} + b e^{-i\omega t} \quad \text{on } z = 0, \ t > 0, \end{aligned}$$

$$\begin{aligned} q(z,t) &\to 0 \quad \text{as } z \to \infty\ t > 0, \\ q(z,t) &= 0 \quad \text{at } t \le 0,\ \text{for all } z > 0, \end{aligned}$$

where $q = u + iv$, ω is the frequency of oscillations of the disk, and a, b are complex constants. Hence, deduce the steady-state solution and determine the structure of the associated boundary layers.

37. Show that, when $\omega = 0$ in Exercise 36, the steady-flow field is given by

$$q(z,t) \sim (a+b)\exp\left\{\left(-\frac{2i\Omega}{\nu}\right)^{1/2} z\right\}.$$

Hence, determine the thickness of the Ekman layer.

38. Solve the telegraph equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} + 2a u_t = 0, &\quad -\infty < x < \infty,\ t > 0, \\ u(x,0) = 0, &\quad u_t(x,0) = g(x), \quad -\infty < x < \infty. \end{aligned}$$

39. Use the Laplace transform to solve the initial boundary-value problem

$$\begin{aligned} u_t &= c^2 u_{xx}, \quad 0 < x < a,\ t > 0, \\ u(x,0) &= x + \sin\left(\frac{3\pi x}{a}\right) \quad \text{for } 0 < x < a, \\ u(0,t) &= 0 = u(a,t) \quad \text{for } t > 0. \end{aligned}$$

40. Solve the diffusion equation

$$\begin{aligned} u_t &= \kappa u_{xx}, \quad -a < x < a,\ t > 0, \\ u(x,0) &= 1 \quad \text{for } -a < x < a, \\ u(-a,t) &= 0 = u(a,t) \quad \text{for } t > 0. \end{aligned}$$

41. Use the joint Laplace and Fourier transform to solve the initial-value problem for transient water waves which satisfies (see Debnath 1994, p. 92)

$$\begin{aligned} &\nabla^2\phi = \phi_{xx} + \phi_{zz} = 0, \quad -\infty < x < \infty, -\infty < z < 0,\ t > 0, \\ &\left.\begin{aligned} &\phi_z = \eta_t, \\ &\phi_t + g\eta = -\tfrac{P}{\rho} p(x) e^{i\omega t} \end{aligned}\right\} \quad \text{on } z = 0,\ t > 0, \\ &\phi(x,z,0) = 0 = \eta(x,0) \quad \text{for all } x \text{ and } z, \end{aligned}$$

where P and ρ are constants.

42. Show that the solution of the boundary-value problem

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, &\quad 0 < r < \infty,\ 0 < z < \infty, \\ u(r,z) = \frac{1}{\sqrt{a^2 + r^2}} &\quad \text{on } z = 0,\ 0 < r < \infty, \end{aligned}$$

is

$$u(r,z) = \int_0^\infty e^{-\kappa(z+a)} J_0(\kappa r)\, d\kappa = \frac{1}{\sqrt{(z+a)^2 + r^2}}.$$

43. (a) The axisymmetric initial-value problem is governed by

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r\right) + \delta(t) f(r), \quad 0 < r < \infty,\ t > 0,$$
$$u(r,0) = 0 \quad \text{for } 0 < r < \infty.$$

Show that the formal solution of this problem is

$$u(r,t) = \int_0^\infty k J_0(kr) \tilde{f}(k) \exp\left(-k^2 \kappa t\right) dk.$$

(b) When $f(r) = \frac{Q}{\pi a^2} H(a - r)$, show that the solution is

$$u(r,t) = \frac{Q}{\pi a} \int_0^\infty J_0(kr) J_1(ak) \exp\left(-k^2 \kappa t\right) dk.$$

44. If $f(r) = A(a^2 + r^2)^{-\frac{1}{2}}$, where A is a constant, show that the solution of the biharmonic equation described in Example 1.10.7 is

$$u(r,z) = A\frac{\{r^2 + (z+a)(2z+a)\}}{[r^2 + (z+a)^2]^{3/2}}.$$

45. Solve the axisymmetric biharmonic equation for the free vibration of an elastic disk

$$b^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 u + u_{tt} = 0, \quad 0 < r < \infty,\ t > 0,$$
$$u(r,0) = f(r), \quad u_t(r,0) = 0 \quad \text{for } 0 < r < \infty,$$

where $b^2 = \frac{D}{2\sigma h}$ is the ratio of the flexural rigidity of the disk and its mass $2h\sigma$ per unit area.

46. Show that the zero-order Hankel transform solution of the axisymmetric problem

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad 0 < r < \infty,\ -\infty < z < \infty,$$
$$\lim_{r\to 0}\left(r^2 u\right) = 0, \quad \lim_{r\to 0}(2\pi r) u_r = -f(z), \quad -\infty < z < \infty,$$

is

$$\tilde{u}(k,z) = \frac{1}{4\pi k}\int_{-\infty}^\infty \exp\{-k|z-\zeta|\} f(\zeta)\, d\zeta.$$

Hence, show that

$$u(r,z) = \frac{1}{4\pi}\int_{-\infty}^\infty \{r^2 + (z-\zeta)\}^{-\frac{1}{2}} f(\zeta)\, d\zeta.$$

47. Solve the nonhomogeneous diffusion problem

$$u_t = \kappa\left(u_{rr} + \frac{1}{r}u_r\right) + Q(r,t), \quad 0 < r < \infty,\ t > 0,$$
$$u(r,0) = f(r), \quad 0 < r < \infty,$$

where κ is a constant.

48. Solve the problem of the electrified unit disk in the (x,t)-plane with center at the origin. The electric potential $u(r,z)$ is axisymmetric and satisfies the boundary-value problem

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + u_{zz} &= 0, \quad 0 < r < \infty,\ 0 < z < \infty, \\ u(r,0) &= u_0, \quad 0 \le r \le a, \\ \frac{\partial u}{\partial z} &= 0, \quad \text{on } z = 0 \text{ for } a < r < \infty, \\ u(r,z) &\to 0 \quad \text{as } z \to \infty \text{ for all } r, \end{aligned}$$

where u_0 is constant. Show that the solution is

$$u(r,z) = \frac{2u_0}{\pi}\int_0^\infty J_0(kr)\frac{\sin ak}{k}e^{-kz}\,dk.$$

49. Solve the axisymmetric surface wave problem in deep water due to an oscillatory surface pressure. The governing equations are

$$\nabla^2\phi = \phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \quad 0 \le r < \infty,\ -\infty < z \le 0,$$
$$\left.\begin{aligned} &\phi_t + g\eta = -\tfrac{P}{\rho}p(r)\exp(i\omega t), \\ &\phi_z - \eta_t = 0 \end{aligned}\right\} \quad \text{on } z = 0, t > 0,$$
$$\phi(r,z,0) = 0 = \eta(r,0), \quad \text{for } 0 \le r < \infty, \text{ and } -\infty < z \le 0.$$

50. Solve the Neumann problem for the Laplace equation

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + u_{zz} &= 0, \quad 0 < r < \infty,\ 0 < z \le \infty, \\ u_z(r,0) &= -\frac{1}{\pi a^2}H(a-r), \quad 0 < r < \infty, \\ u(r,z) &\to 0 \quad \text{as } z \to \infty \text{ for } 0 < r < \infty. \end{aligned}$$

Show that

$$\lim_{a\to 0} u(r,z) = \frac{1}{2\pi}\left(r^2 + z^2\right)^{-\frac{1}{2}}.$$

51. Solve the Cauchy problem for the wave equation in a dissipating medium

$$u_{tt} + 2\kappa u_t = c^2\left(u_{rr} + \frac{1}{r}u_r\right), \quad 0 < r < \infty,\ t > 0,$$
$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \quad \text{for } 0 < r < \infty,$$

where κ is a constant.

52. Use the joint Laplace and Hankel transform to solve the initial boundary-value problem

$$\begin{aligned} c^2\left(u_{rr}+\frac{1}{r}u_r+u_{zz}\right) &= u_{tt}, \quad 0<r<\infty,\ 0<z<\infty,\ t>0, \\ u_z(r,0,t) &= H(a-r)H(t), \quad 0<r<\infty,\ t>0, \\ u(r,z,t) &\to 0 \quad \text{as } r\to\infty \quad \text{and} \quad u(r,z,t)\to 0 \quad \text{as } z\to\infty, \\ u(r,z,t) &= 0 = u_t(r,z,0), \end{aligned}$$

and show that

$$u_t(r,z,t) = -acH\left(t-\frac{z}{c}\right)\int_0^\infty J_1(ak)J_0\left\{ck\sqrt{t^2-\frac{z^2}{c^2}}\right\}J_0(kr)\,dk.$$

53. Find the steady temperature $u(r,z)$ in a beam $0\le r<\infty$, $0\le z\le a$, when the face $z=0$ is kept at temperature $u(r,0)=0$ and the face $z=a$ is insulated except that heat is supplied through a circular hole such that

$$u_z(r,a) = H(b-r).$$

The temperature $u(r,z)$ satisfies the axisymmetric equation

$$u_{rr}+\frac{1}{r}u_r+u_{zz}=0, \quad 0\le r<\infty,\ 0\le z\le a.$$

54. Find the integral solution of the initial boundary-value problem

$$\begin{aligned} u_{rr}+\frac{1}{r}u_r+u_{zz} &= u_t, \quad 0\le r<\infty,\ 0\le z<\infty,\ t>0, \\ u(r,z,0) &= 0, \quad \text{for all } r \text{ and } z, \\ \left(\frac{\partial u}{\partial r}\right)_{r=0} &= 0, \quad \text{for } 0\le z<\infty,\ t>0, \\ \left(\frac{\partial u}{\partial z}\right)_{z=0} &= -\frac{H(a-r)}{\sqrt{a^2-r^2}}, \quad \text{for } 0<r<\infty,\ 0<t<\infty, \\ u(r,z,t) &\to 0 \quad \text{as } r\to\infty \text{ or } z\to\infty. \end{aligned}$$

55. Solve the Cauchy–Poisson wave problem (Debnath 1989) for a viscous liquid of finite or infinite depth governed by the equations, free surface, boundary and initial conditions

$$\phi_{rr}+\frac{1}{r}\phi_r+\phi_{zz}=0,$$

$$\psi_t = \nu\left(\psi_{rr}+\frac{1}{r}\psi_r-\frac{1}{r^2}\psi+\psi_{zz}\right),$$

where $\phi(r,z,t)$ and $\psi(r,z,t)$ represent the potential and stream functions, respectively, $0\le r<\infty$, $-h\le z\le 0$ (or $-\infty<z\le 0$), and $t>0$.

The free surface conditions are

$$\left.\begin{aligned} &\eta_t - w = 0, \\ &\mu(u_z - w_r) = 0, \\ &\phi_t + g\eta + 2\nu w_z = 0 \end{aligned}\right\} \quad \text{on } z = 0,\ t > 0,$$

where $\eta = \eta(r,t)$ is the free surface elevation, $u = \phi_r + \psi_z$ and $w = \phi_z - \frac{\psi}{r} - \psi_r$ are the radial and vertical velocity components of liquid particles, $\mu = \rho\nu$ is the dynamic viscosity, ρ is the density, and ν is the kinematic viscosity of the liquid.

The boundary conditions at the rigid bottom are

$$\left.\begin{aligned} &u = \phi_r + \psi_z = 0, \\ &w = \phi_z - \tfrac{1}{r}(r\psi)_r = 0 \end{aligned}\right\} \quad \text{on } z = -h.$$

The initial conditions are

$$\eta = a\frac{\delta(r)}{r}, \qquad \phi = \psi = 0 \text{ at } t = 0,$$

where a is a constant and $\delta(r)$ is the Dirac delta function.

If the liquid is of infinite depth, the bottom boundary conditions are

$$(\phi, \psi) \to (0,0) \quad \text{as } z \to \infty.$$

56. Use the joint Hankel and Laplace transform method to solve the initial boundary-value problem

$$u_{rr} + \frac{1}{r}u_r + u_{tt} - 2\varepsilon u_t = a\frac{\delta(r)}{r}\delta(t), \quad 0 < r < \infty,\ t > 0,$$

$$u(r,t) \to 0 \quad \text{as } r \to \infty,$$

$$u(0,t) \text{ is finite for } t > 0,$$

$$u(r,0) = 0 = u_t(r,0) \quad \text{for } 0 < r < \infty.$$

57. Surface waves are generated in an inviscid liquid of infinite depth due to an explosion (Sen 1963) above it which generates the pressure field $p(r,t)$. The velocity potential $\phi(r,z,t)$ satisfies the Laplace equation

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad 0 < r < \infty,\ t > 0,$$

and the free surface condition

$$u_{tt} + g u_z = \frac{1}{\rho}\left(\frac{\partial p}{\partial t}\right)\left[H(r) - H\{r, r_0(t)\}\right] \quad \text{on } z = 0,$$

where ρ is the constant density of the liquid, $r_0(t)$ is the extent of the blast, and the liquid is initially at rest.

Solve this problem.

58. Use the joint Laplace and Fourier transform to show that the solution of the inhomogeneous diffusion problem

$$u_t - \kappa u_{xx} = q(x,t), \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = f(x) \quad \text{for all } x \in \mathbb{R},$$

can be expressed in terms of Green's function $G(x,t;\xi,\tau)$ as

$$u(x,t) = \int_0^t d\tau \int_{-\infty}^{\infty} q(\xi,\tau) G(x,t;\xi,\tau)\, d\xi + \int_{-\infty}^{\infty} f(\xi) G(x,t;\xi,0)\, d\xi,$$

where

$$G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi\kappa(t-\tau)}} \exp\left\{-\frac{(x-\xi)^2}{4\kappa(t-\tau)}\right\}.$$

59. Find Green's function $G(x,t)$ of the Bernoulli–Euler equation on an elastic foundation

$$EI\frac{\partial^4 G}{\partial x^4} + \kappa G + m\frac{\partial^2 G}{\partial t^2} = W\delta(x)\delta(t), \quad x \in \mathbb{R},\ t > 0,$$

with initial conditions

$$G(x,0) = 0 = G_t(x,0).$$

60. Solve the initial boundary-value problem

$$u_t - \kappa u_{xx} = q(t)\delta(x - Vt), \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = 0 \quad \text{and} \quad u(x,t) \to 0 \quad \text{as } |x| \to \infty,$$

where $q(t) = 0$ for $t < 0$ and V is constant.

61. Find the Green function satisfying the equation

$$G_{xx} + G_{yy} = \delta(x-\xi)\delta(y-\eta), \quad 0 < x,\xi < a,\ 0 < y,\eta < b$$
$$G(x,y) = 0 \quad \text{on } x = 0,\ \text{and } x = a; \qquad G(x,y) = 0 \quad \text{on } y = 0 \text{ and } y = b.$$

62. Show that the solution of the two-dimensional diffusion equation

$$u_t - \kappa(u_{xx} + u_{yy}) = q(x,y,t); -\infty < x,y < \infty, t > 0,$$

with

$$u(x,y;0) = 0$$

is

$$u(x,y,t) = \int_0^t d\tau \left[\int_{-\infty}^{\infty}\int \exp\left\{-\frac{(x-\xi)^2 + (y-\eta)^2}{4\kappa(t-\tau)}\right\} \times \frac{q(\xi,\eta,\tau)}{\sqrt{4\pi\kappa(t-\tau)}}\, d\xi\, d\eta\right].$$

63. Find the Green function for the one-dimensional Klein–Gordon equation

$$u_{tt} - c^2 u_{xx} + d^2 u = p(x,t), \quad x \in \mathbb{R},\ t > 0,$$

with the initial and boundary conditions

$$u(x,0) = 0 = u_t(x,0) \quad \text{for all } x \in \mathbb{R},$$

where c and d are constants.
Show that Green's function for this problem reduces to that of the wave equation in the limit as $d \to 0$.
Derive the Green functions for both two- and three-dimensional Klein–Gordon equations.

64. Use the Fourier series method to solve the equation for a diffusion model

$$u_t = \kappa u_{xx}, \quad -\ell < x < \ell,\ t > 0,$$

with the periodic boundary conditions

$$\left.\begin{aligned} u(-\ell,t) &= u(\ell,t), \\ u_x(-\ell,t) &= u_x(\ell,t) \end{aligned}\right\} \quad t > 0,$$

and the initial condition

$$u(x,0) = f(x), \quad -\ell \le x \le \ell.$$

65. (a) Verify that

$$u_n(x,y) = \exp\bigl(ny - \sqrt{n}\bigr) \sin nx,$$

where n is a positive integer, is the solution of the Cauchy problem for the Laplace equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad && x \in \mathbb{R},\ y > 0, \\ u(x,0) &= 0, && u_y(x,0) = n \exp\bigl(-\sqrt{n}\bigr) \sin nx. \end{aligned}$$

(b) Show that this Cauchy problem is not well posed.

66. Find the eigenvalues and eigenfunctions of the Sturm–Liouville problems

(a) $u'' + \lambda u = 0,\ 0 < x < 1,$
$u(0) = 0,\ u(1) + u'(1) = 0,$

(b) $u'' + \lambda u = 0,\ 0 < x < 1,$
$u'(0) + u'(1) = 0,$

(c) $u'' + \lambda u = 0,\ 0 < x < 1,$
$u(0) = u(1), u'(0) = u'(1),$

(d) $u'' + 2u' + 4\lambda u = 0,\ 0 < x < a,$
$u(0) = 0 = u(a).$

67. Show that the equation

$$a_2(x)u'' + a_1(x)u' + \bigl(a_0(x) + \lambda\bigr)u = 0,$$

can be reduced into the Sturm–Liouville form

$$\frac{d}{dx}\big[p(x)u'\big] + \big[q(x) + \lambda\rho(x)\big]u = 0,$$

where

$$p(x) = \exp\left[\int \frac{a_1(x)}{a_2(x)}\,dx\right], \qquad q(x) = \frac{p(x)a_0(x)}{a_2(x)}, \qquad \rho(x) = \frac{p(x)}{a_2(x)}.$$

68. Reduce the given equation into the Sturm–Liouville form

(a) $u'' - 2xu' + \lambda xu = 0,$ (b) $u'' + u' + \lambda u = 0,$
(c) $xu'' + (1-x)u' + \lambda u = 0\ (x > 0),$
(d) $x^2(x^2+1)u'' + 2x^3u' + \lambda u = 0, \quad x > 0.$

69. Determine the Euler load and the corresponding fundamental buckling mode of a simply supported beam of length a under an axial compressive force P which is governed by the eigenvalue problem

$$y^{(4)}(x) + \lambda y'' = 0, \quad 0 < x < a,$$
$$y(0) = 0 = y''(0), \qquad y(a) = 0 = y''(a),$$

where $\lambda = (\frac{P}{EI}) > 0$.

70. Use the Fourier method to solve the Klein–Gordon problem

$$u_{tt} - c^2u_{xx} + d^2u = 0, \quad 0 < x < a, t > 0,$$
$$u(0,t) = 0 = u(a,t), \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = 0, \quad 0 < x < a.$$

71. Use the Fourier method to solve the diffusion model

$$u_t = u_{xx} + 2bu_x, \quad 0 < x < a, t > 0,$$
$$u(0,t) = 0 = u(a,t), \quad t > 0,$$
$$u(x,0) = f(x), \quad 0 < x < a.$$

72. (a) Solve the vibration problem of a circular membrane governed by (1.13.56) and (1.13.59), (1.13.60) with the boundary and initial conditions (1.13.52), (1.13.53) when $g(r,\theta) = 0$.
(b) Obtain the solution for the problem (a) when $f(r,\theta) = a^2 - r^2$.

73. (a) Use the method of separation of variables to solve the Dirichlet problem in the cylinder for $u(r,z)$

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad 0 \le x \le a,\ 0 \le z \le h,$$
$$u(r,0) = 0 = u(a,z), \qquad u(r,h) = f(r).$$

(b) If $f(r) = 1$, obtain the solution.

74. (a) Derive the differential equality

$$2u_t\left(c^2\nabla^2 u - u_{tt}\right) = 2c^2\left[(u_t u_x)_x + (u_t u_y)_y\right] - \left[c^2\left(u_x^2 + u_y^2\right) + u_t^2\right]_t,$$

associated with the wave equation

$$c^2(u_{xx} + u_{yy}) - u_{tt} \equiv c^2\nabla^2 u - u_{tt} = 0.$$

(b) Generalize the above differential equality for the $(n+1)$-dimensional wave equation (1.13.9).
(c) Show that the differential equality for the $(n+1)$-dimensional wave equation can be written in the form

$$2u_t\left(c^2\nabla_n^2 - u_{tt}\right) = 2c^2\nabla_n \cdot (u_t\nabla_n u) - \left(c^2|\nabla_n u|^2 + u_t^2\right)_t,$$

where

$$\nabla_n = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right).$$

75. (a) Derive the energy integral

$$E(t) = \frac{1}{2}\int_a^b \left(u_t^2 + c^2u_x^2 + d^2u^2\right) dx,$$

for the Klein–Gordon equation model

$$\begin{aligned} u_{tt} - c^2u_{xx} + d^2u = 0, \quad & a \le x \le b,\ t > 0, \\ u(a,t) = 0 = u(b,t), & \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & \quad a \le x \le b. \end{aligned}$$

(b) Show that the energy is constant, that is,

$$E(t) = E_0 = \text{const.}$$

(c) Use the law of conservation of energy to prove that the Klein–Gordon equation has a unique solution.

76. (a) Use the method of separation of variables to solve the spherically symmetric wave equation (1.13.21) in three dimensions

$$\begin{aligned} u_{tt} = c^2\nabla^2 u \equiv c^2\left(u_{rr} + \frac{2}{r}u_r\right), \quad & 0 < r < a,\ t > 0, \\ u(a,t) = 0, \quad t > 0, & \\ u(r,0) = f(r), \quad u_t(r,0) = g(r), \quad & 0 < r < a. \end{aligned}$$

(b) Find the solution for $f(r) = 0$ and $g(r) = a - r$.

77. Solve the axisymmetric Dirichlet problem in a right circular cylinder

$$u_{rr} + r^{-1} u_r + u_{zz} \equiv 0, \quad 0 < r < a,\ 0 < z < h,$$
$$u(r, 0) = 0 = u(r, h), \qquad u(a, z) = f(z).$$

78. Use Example 1.6.4 to find the solution of equation (1.6.80) with the boundary conditions

$$\text{(a) } u(1, \theta) = 2\cos^2\theta, \qquad \text{(b) } u(1, \theta) = |2\theta|,$$
$$\text{(c) } u_r(1, \theta) = 2\cos 2\theta, \quad \text{(d) } u_r(1, \theta) = \cos\theta + \sin\theta.$$

79. Apply the method of separation of variables to solve the Laplace equation in an annular region

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 < a < r < b,\ 0 \le \theta \le 2\pi$$

with the following boundary conditions:
(a) $u(a, \theta) = f(\theta)$, $u(b, \theta) = g(\theta)$, $0 \le \theta \le 2\pi$.
(b) $u(1, \theta) = \frac{1}{2} + \sin\theta$, $u(2, \theta) = \frac{1}{2}(1 - 1n2 + 2\cos\theta)$, $0 \le \theta \le 2\pi$.
(c) $u_r(1, \theta) = 0$, $u_r(2, \theta) = \frac{3}{4}(\cos\theta - \sin\theta)$, $0 \le \theta \le 2\pi$.

80. In Example 1.6.1, take $c = 1$, $\ell = \pi$. If $u(x, t) = \frac{1}{2}[f(x - t) + f(x + t)] + \int_{x-t}^{x+t} g(\tau)\, d\tau$, then show that

$$\int_0^\pi \left(u_x^2 + u_t^2\right) dx = \int_0^\pi \left(f'^2 + g'^2\right) dx,$$

where f and g are real functions on $0 \le x \le \pi$ with continuous partial derivatives and $f(0) = f(\pi) = g(0) = g(\pi) = 0$.

81. Consider the boundary-value problem for the elliptic equation

$$\nabla^2 u + 1 = 0 \quad \text{in } D = \left\{ (x, y) \middle| \left(\frac{|x|}{a} + \frac{|y|}{b} \right) < 1 \right\},$$
$$u = 0 \quad \text{on } \partial D,$$

where $a > b > 0$.
Show that

$$\frac{a^2 b^2}{4(a^2 + b^2)} \le u(0, 0) \le \frac{a^2}{4}.$$

82. Show that the Dirichlet problem

$$\nabla^2 u = 0, \quad \mathbf{x} \in D,$$
$$u(\mathbf{x}) = f(\mathbf{x}) \quad \text{on } \partial D,$$

has a unique solution.

83. Consider the Dirichlet boundary-value problem

$$\nabla^2 u + 1 = 0, \quad \mathbf{x} \in \left\{ (x, y) \middle| \left(\frac{|x|}{a} + \frac{|y|}{b} \right) < 1 \right\},$$
$$u = 0 \quad \text{on } \partial D,$$

where $a > b > 0$. Use a suitable function $v(x, y) = Ax^2 + By^2$ satisfying $\nabla^2 v = 1$ with $(A, B) > (0, 0)$ to prove that

$$\frac{(ab)^2}{2(a+b)^2} \leq u(0, 0) \leq \frac{(ab)^2}{2(a^2 + b^2)}.$$

84. Solve the fractional Blasius problem as stated in Exercise 35 with the governing equation (see Debnath 2003a, 2003b)

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \nu \frac{\partial^2 u}{\partial z^2}, \quad x \in \mathbb{R}, \ t > 0.$$

85. Solve the fractional Stokes–Ekman problem as stated in Exercise 36 with the governing equation (see Debnath 2003a, 2003b)

$$\frac{\partial^\alpha q}{\partial t^\alpha} + 2i\Omega q = \nu \frac{\partial^2 q}{\partial z^2}, \quad x \in \mathbb{R}, \ t > 0.$$

86. Apply the method of separation of variables $u(x, t) = X(x)T(t)$ to solve the eigenvalue problem for the dissipation wave equation

$$u_{tt} - c^2 u_{xx} + \alpha u_t = 0, \quad 0 < x < \ell, \ t > 0,$$
$$u(0, t) = 0 = u(\ell, t), \quad t > 0,$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \ 0 < x < \ell.$$

Show that
(i) $X'' + \lambda^2 X = 0$, $\ddot{T} + \alpha \dot{T} + \lambda^2 c^2 T = 0$, where λ^2 is a separation constant.
(ii) The eigenvalues and eigenfunctions are $\lambda_n = \frac{n\pi}{\ell}$, $X_n(x) = B_n \sin(\frac{n\pi x}{\ell})$, $n = 1, 2, \ldots$.
(iii) The general solution is

$$u(x, t) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}\alpha t} \left[a_n \cos\left(\frac{1}{2} \delta_n t \right) + b_n \sin\left(\frac{1}{2} \delta_n t \right) \right],$$

where $\delta_n = \sqrt{4\lambda_n^2 c^2 - \alpha^2}$. Consider other cases $4\lambda_n^2 c^2 <$ or $= \alpha^2$.

87. Solve the above problem 86 with the same initial conditions and the following boundary conditions:
(i) $u_x(0, t) = 0 = u(\ell, t)$, $t > 0$,
(ii) $u(0, t) = 0 = u_x(\ell, t)$, $t > 0$,
Find the eigenvalues, eigenfunctions, and the general solution in each case.

88. Solve the eigenvalue problem for the telegraph equation

$$u_{tt} - c^2 u_{xx} + a u_t + b = 0, \quad 0 < x < 1,\ t > 0,$$

with the boundary and initial conditions

$$u(0,t) = 0, \quad u_x(1,t) + u(1,t) = 0, \quad t > 0,$$
$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x), \quad 0 < x < 1.$$

Show that the eigenvalue equation and the eigenfunctions are

$$\tan\lambda = \lambda \quad \text{and} \quad X_n(x) = A_n \sin\lambda x,$$

where $-\lambda^2$ is the separation constant.

89. Use the method of separation of variables to solve the problem

$$u_{tt} - c^2 u_{xx} + \alpha u = 0, \quad 0 < x < 1,\ t > 0,$$
$$u_x(0,t) = 0 = u(1,t), \quad t > 0,$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < 1.$$

Show that the eigenvalues and the eigenfunctions are

$$\lambda_n = \left[\frac{1}{4}(2n-1)^2\pi^2c^2 + \alpha\right]^{\frac{1}{2}}, \qquad X_n(x) = A_n \cos\left[(2n-1)\frac{\pi x}{2}\right],$$

where $n = 1, 2, 3, \ldots$, and $-\lambda^2$ is the separation constant.
Derive the general solution

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos\lambda_n t + b_n \sin\lambda_n t)\cos\left[(2n-1)\frac{\pi x}{2}\right].$$

Show that the solution for $f(x) = x$ and $g(x) = 0$ corresponds to

$$a_n = \frac{4}{(2n-1)\pi}\left[(-1)^{n-1} - \frac{2}{(2n-1)\pi}\right], \qquad b_n = 0.$$

90. Consider the eigenvalue problem

$$u_{tt} - c^2 u_{xx} + a u = 0, \quad 0 < x < 1,\ t > 0,$$

$u_x(0,t) = 0$, $u_x(1,t) + u(1,t) = 0$, $t > 0$, with the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x), \quad 0 < x < 1,$$

(a) Show that, for $u(x,t) = X(x)T(t)$,

$$c^2 X'' - aX + \lambda^2 X = 0, \qquad \ddot{T} + \lambda^2 T = 0;$$

and the eigenvalue equation is

$$\beta\tan\beta = 1, \quad \beta = \frac{1}{c}\left(\lambda^2 - a\right)^{\frac{1}{2}},$$

where $-\lambda^2$ is the separation constant.

91. Consider the Helmholtz equation

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + \varepsilon^2 u = 0 \quad \text{for } r \le 1,$$

where $\varepsilon = \frac{\omega}{c} \ll 1$, with the boundary condition

$$u(1, \theta) = \sin\theta.$$

Obtain the asymptotic solution in the form

$$u = u_0 + \varepsilon^2 u_2 + O(\varepsilon^4),$$

where $u = O(1)$ on the boundary.
Show that the two-term asymptotic solution is

$$u(r, \theta) = r \sin\theta + \frac{1}{8} \varepsilon^2 (r - r^3) + O(\varepsilon^4).$$

92. Consider the boundary value problem for the modified Helmholtz equation

$$\varepsilon^2 \nabla^2 u = u,$$

with $u(1, \theta) = 1$ and $u(r, \theta) \to 0$ as $r \to \infty$. Show that the asymptotic solution is given by

$$u = \exp\left(\frac{1 - r}{\varepsilon}\right).$$

93. (a) The temperature distribution $u(x, t)$ in a homogeneous rod of length ℓ with insulated endpoints is described by the initial boundary-value problem

$$\begin{aligned} u_t &= \kappa u_{xx}, \quad 0 < x < \ell,\ t > 0, \\ u_x(0, t) &= 0 = u_x(\ell, t), \quad t > 0, \\ u(x, 0) &= f(x), \quad 0 < x < \ell. \end{aligned}$$

(b) Obtain the solution from (a) for the case $\ell = 1$, $f(x) = x$.

94. (a) Show that the telegraph equation can be written in the form

$$u_{tt} - c^2 u_{xx} + (p + q) u_t + pqu = 0,$$

where $p = (G/C)$ and $q = (R/L)$.
(b) Apply the transformation $u = v \exp[-\frac{1}{2}(p + q)t]$ to transform the equation in the form

$$v_{tt} - c^2 v_{xx} = \frac{1}{4} (p - q)^2 v.$$

(c) When $p = q$, there exists an undistorted wave solution. Show that a progressive wave of the form

$$u(x,t) = \exp(-pt) f(x \pm ct)$$

propagate in either direction, where f is an arbitrary twice differentiable function of its argument.
(d) If $u(x,t) = A \exp[i(kx - \omega t)]$ is a solution of equation as stated in (a), show that the dispersion relation is

$$\omega^2 + i(p+q)\omega - \left(c^2 k^2 + pq\right) = 0.$$

95. Consider the telegraph equation problem

$$\begin{aligned} &u_t - c^2 u_{xx} + a u_t + b u = 0, \quad 0 < x < l, \ t > 0, \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for } 0 < x < l, \\ &u(0,t) = 0 = u(l,t) \quad \text{for } t \geq 0, \end{aligned}$$

where a and b are constants.
(a) Show that, for any $T > 0$

$$\int_0^t \left(u_t^2 + c^2 u_x^2 + b u^2\right)_{t=T} dx \leq \int_0^l \left(u_t^2 + c^2 u_x^2 + b u^2\right)_{t=0} dx.$$

(b) Use the above integral inequality from (a) to show that the initial boundary-value problem for the telegraph equation can have only one solution.

96. Use the solution (1.9.15) to obtain the solution of the nonhomogeneous wave equation problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \sin(kx - \omega t), \quad x \in \mathbb{R}, \ t > 0, \\ u(x,0) &= 0 = u_t(x,0), \quad x \in \mathbb{R}. \end{aligned}$$

Discuss the solution for cases
(a) $c \neq \frac{\omega}{k}$ and (b) $c = \frac{\omega}{k}$ (resonance).

97. Derive the Duhamel formula for the solution of Example 1.9.2 is

$$u(x,t) = f(t) * \frac{\partial}{\partial t} u_0(x,t) = \int_0^t f(t-\tau) \left(\frac{\partial u_0}{\partial \tau}\right) d\tau$$

where

$$u_0(x,t) = \mathcal{L}^{-1} \left\{ \frac{1}{s} \exp\left(-x\sqrt{\frac{s}{k}}\right) \right\} = \operatorname{erfc}\left(\frac{x}{\sqrt{4\kappa t}}\right).$$

98. Solve the axisymmetric unsteady viscous flow problem in a long rotating cylinder of radius a governed by

$$v_t = \nu \left(v_{rr} + \frac{1}{r} v_r - \frac{v}{r^2}\right), \quad 0 < r \leq a, \ t > 0,$$

where $v = v(r,t)$ is the tangential fluid velocity and ν is the kinematic viscosity of the fluid.

The cylinder is at rest until at $t = 0+$, it is caused to rotate so that the boundary and initial conditions are

$$v(r,t) = a\Omega f(t)H(t) \quad \text{on } r = a,$$
$$v(r,0) = 0 \quad \text{for } r < a,$$

where $f(t)$ is a physically realistic function of time t.
Find the solution when $f(t) = \cos \omega t$ and examine its feature as $t \to \infty$.

99. (a) Show that the solution of the Cauchy problem for the diffusion equation

$$u_t = u_{xx}, \quad x \in \mathbb{R},\ t > 0,$$
$$u(0,t) = \left(\frac{2}{n}\right) \sin\left(2n^2 t\right), \quad u_x(0,t) = 0, \quad t > 0,$$

is given by

$$u_n(x,t) = \frac{1}{n}\left[e^{nx} \sin\left(nx + 2n^2 t\right) - e^{-nx} \sin\left(nx - 2n^2 t\right)\right]$$

where n is an integer.
(b) Show that this Cauchy problem is ill-posed.

100. (a) Verify that

$$u_n(x,y) = \exp\left(ny - \sqrt{n}\right) \sin nx$$

is the solution of the Cauchy problem for the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R},\ y > 0,$$
$$u(x,0) = 0, \quad u_y(x,0) = n \exp\left(-\sqrt{n}\,\right) \sin nx,$$

where n is an integer.
(b) Show that this Cauchy problem is ill-posed.

101. (a) Verify that

$$u_n(x,y) = \frac{1}{n} e^{-\sqrt{n}} \sin nx \sinh ny$$

is the solution of the Cauchy problem for the Laplace equation in the upper half-strip

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le \pi,\ y > 0,$$
$$u(0,y) = 0 = u(\pi,y), \quad y > 0$$
$$u(x,0) = 0 \quad \text{and} \quad u_y(x,0) = e^{-\sqrt{n}} \sin nx.$$

(b) Show that this Cauchy problem is ill-posed.

102. Using the wave function $\psi(x,t) = a(x,t)\exp[\frac{i}{\hbar} S(x,t)]$, where a and S are real functions and the transformation $u(x,t) = m^{-1} S_x$ in the Schrödinger equation (1.7.66), derive

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[V(x)a\right] + \frac{\hbar^2}{2m}\left(\frac{1}{a}a_{xx}\right),$$
$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x} = 0.$$

(Hydrodynamic analogy of quantum mechanics).

103. Consider the two-dimensional boundary value problem for the Laplace equation in the upper half plane

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R},\ y > 0,$$
$$u(x,0) = f(x) \quad \text{and} \quad u_y(x,0) = g(x), \quad x \in \mathbb{R}.$$

(a) If $f(x) = 0$ and $g(x) = 0$, show that $u(x,y) \equiv 0$ is the solution.
(b) If the boundary data is changed to $u(x,0) = \frac{1}{n}\cos nx$ and $g(x) = 0$, $x \in \mathbb{R}$, show that $u(x,y) = \frac{1}{n}\cos nx \cosh ny$ is the solution. Examine the ill-posedness of the problem.

104. (a) Show that the solution of the Cauchy problem for the negative diffusion equation

$$u_t + \kappa u_{xx} = 0, \quad x \in \mathbb{R},\ t > 0,\ \kappa > 0,$$
$$u(x,0) = f(x), \quad x \in \mathbb{R}$$

is given by

$$u(x,t) = \frac{1}{n}\exp\left(\kappa n^2 t\right)\sin nx.$$

(b) Show that this negative diffusion problem is ill-posed.

105. (a) Show that the ill posed problem 104(a) can be made well posed by adding higher order diffusive terms, that is, the modified Cauchy problem

$$u_t + \kappa u_{xx} + \delta u_{xxxx} = 0, \quad x \in \mathbb{R},\ t > 0\ (\kappa, \delta > 0),$$
$$u(x,0) = \frac{1}{n}\sin nx, \quad x \in \mathbb{R}$$

is a well posed problem.
(b) Verify that

$$u(x,t) = \frac{1}{n}\sin nx \exp\left(\kappa n^2 - \delta n^4\right)t$$

is the solution of the modified Cauchy problem.

2

Nonlinear Model Equations and Variational Principles

True Laws of Nature cannot be linear.

Albert Einstein

... the progress of physics will to a large extent depend on the progress of nonlinear mathematics, of methods to solve nonlinear equations ... and therefore we can learn by comparing different nonlinear problems.

Werner Heisenberg

Our present analytical methods seem unsuitable for the solution of the important problems arising in connection with nonlinear partial differential equations and, in fact, with virtually all types of nonlinear problems in pure mathematics. The truth of this statement is particularly striking in the field of fluid dynamics...

John Von Neumann

2.1 Introduction

This chapter deals with the basic ideas and many major nonlinear model equations which arise in a wide variety of physical problems. Included are one-dimensional wave, Klein–Gordon (KG), sine–Gordon (SG), Burgers, Fisher, Korteweg–de Vries (KdV), Boussinesq, modified KdV, nonlinear Schrödinger (NLS), Benjamin–Ono (BO), Benjamin–Bona–Mahony (BBM), Ginzburg–Landau (GL), Burgers–Huxley (BH), KP, concentric KdV, Whitham, Davey–Stewartson, Toda lattice, Camassa–Holm (CH), and Degasperis–Procesi (DP) equations. This is followed by variational principles and the Euler–Lagrange equations. Also included are Plateau's problem, Hamilton's principle, Lagrange's equations, Hamilton's equations, the variational principle for nonlinear Klein–Gordon equations, and the variational principle for nonlinear water waves. Special attention is given to the Euler equation of motion,

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the continuity equation, the associated energy equation and energy flux, linear water wave problems and their solutions, nonlinear finite amplitude waves (the Stokes waves), gravity waves, gravity-capillary waves, and linear and nonlinear dispersion relations. Finally, the modern theory of nonlinear water waves is formulated.

2.2 Basic Concepts and Definitions

The most general first-order nonlinear partial differential equation in two independent variables x and y has the form

$$F(x, y, u, u_x, u_y) = 0. \tag{2.2.1}$$

The most general second-order nonlinear partial differential equation in two independent variables x and y has the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{2.2.2}$$

Similarly, the most general first-order and second-order nonlinear equations in more independent variables can be introduced.

More formally, it is possible to write these equations in the operator form

$$L_{\mathbf{x}} u(\mathbf{x}) = f(\mathbf{x}), \tag{2.2.3}$$

where $L_{\mathbf{x}}$ is a partial differential operator and $f(\mathbf{x})$ is a given function of two or more independent variables $\mathbf{x} = (x, y, \ldots)$. It has already been indicated in Section 1.2 that if $L_{\mathbf{x}}$ is *not* a linear operator, (2.2.3) is called a *nonlinear partial differential equation*. Equation (2.2.3) is called an *inhomogeneous nonlinear equation* if $f(\mathbf{x}) \neq 0$. On the other hand, (2.2.3) is called a *homogeneous nonlinear equation* if $f(\mathbf{x}) = 0$.

In general, the linear superposition principle can be applied to linear partial differential equations if certain convergence requirements are satisfied. This principle is usually used to find a new solution as a linear combination of a given set of solutions. For nonlinear partial differential equations, however, the linear superposition principle *cannot* be applied to generate a new solution. So, because most solution methods for linear equations cannot be applied to nonlinear equations, there is no general method of finding analytical solutions of nonlinear partial differential equations, and numerical techniques are usually required for their solution. A transformation of variables can sometimes be found that transforms a nonlinear equation into a linear equation, or some other ad hoc method can be used to find a solution of a particular nonlinear equation. In fact, new methods are usually required for finding solutions of nonlinear equations.

Methods of solution for nonlinear equations represent only one aspect of the theory of nonlinear partial differential equations. Like linear equations, questions of existence, uniqueness, and stability of solutions of nonlinear partial differential equations are of fundamental importance. These and other aspects of nonlinear equations have led the subject into one of the most diverse and active areas of modern mathematics.

2.3 Some Nonlinear Model Equations

Nonlinear partial differential equations arise frequently in formulating fundamental laws of nature and in the mathematical analysis of a wide variety of physical problems. Listed below are some important model equations of most common interest.

Example 2.3.1. The simplest first-order *nonlinear wave* (*or kinematic wave*) *equation* is

$$u_t + c(u)u_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.1}$$

where $c(u)$ is a given function of u. This equation describes the propagation of a nonlinear wave (or disturbance). A large number of nonlinear problems governed by equation (2.3.1) include waves in traffic flow on highways (Lighthill and Whitham 1955; Richards 1956), shock waves, flood waves, waves in glaciers (Nye 1960, 1963), chemical exchange processes in chromatography, sediment transport in rivers (Kynch 1952), and waves in plasmas.

Example 2.3.2. The *nonlinear Klein–Gordon equation* is

$$u_{tt} - c^2\nabla^2 u + V'(u) = 0, \tag{2.3.2}$$

where c is a constant, and $V'(u)$ is a nonlinear function of u usually chosen as the derivative of the potential energy $V(u)$. It arises in many physical problems including nonlinear dispersion (Scott 1969; Whitham 1974) and nonlinear meson theory (Schiff 1951).

Example 2.3.3. The *sine-Gordon equation*

$$u_{tt} - c^2 u_{xx} + \kappa \sin u = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.3}$$

where c and κ are constants, has arisen classically in the study of differential geometry, and in the propagation of a 'slip' dislocation in crystals (Frenkel and Kontorova 1939). More recently, it arises in a wide variety of physical problems including the propagation of magnetic flux in Josephson-type super conducting tunnel junctions, the phase jump of the wave function of superconducting electrons along long Josephson junctions (Josephson 1965; Scott 1969), a chain of rigid pendula connected by springs (Scott 1969), propagation of short optical pulses in resonant laser media (Arecchi et al. 1969; Lamb 1971), stability of fluid motions (Scott et al. 1973; Gibbon 1985), in ferromagnetism and ferroelectric materials, in the dynamics of certain molecular chains such as DNA (Barone et al. 1971), in elementary particle physics (Skyrme 1958, 1961; Enz 1963), and in weakly unstable baroclinic wave packets in a two-layer fluid (Gibbon et al. 1979).

Example 2.3.4. The Burgers equation is

$$u_t + uu_x = \nu u_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.4}$$

where ν is the kinematic viscosity. This is the simplest nonlinear model equation for diffusive waves in fluid dynamics. It was first introduced by Burgers (1948) to describe one-dimensional turbulence, and it also arises in many physical problems including sound waves in a viscous medium (Lighthill 1956), waves in fluid-filled viscous elastic tubes, and magnetohydrodynamic waves in a medium with finite electrical conductivity.

Example 2.3.5. The *Fisher equation*

$$u_t - \nu u_{xx} = k\left(u - \frac{u^2}{\kappa}\right), \quad x \in \mathbb{R},\ t > 0, \tag{2.3.5}$$

where ν, k, and κ are constants, is used as a nonlinear model equation to study wave propagation in a large number of biological and chemical systems. Fisher (1936) first introduced this equation to investigate wave propagation of a gene in a population. It is also used to study logistic growth–diffusion phenomena. In recent years, the Fisher equation has been used as a model equation for a large variety of problems which include gene-culture waves of advance (Aoki 1987), chemical wave propagation (Arnold et al. 1987), neutron population in a nuclear reactor (Canosa 1969, 1973), and spread of early farming in Europe (Ammerman and Cavalli-Sforva 1971). It also arises in the theory of combustion, nonlinear diffusion, and chemical kinetics (Kolmogorov et al. 1937; Aris 1975; and Fife 1979).

Example 2.3.6. The *Boussinesq equation*

$$u_{tt} - u_{xx} + \left(3u^2\right)_{xx} - u_{xxxx} = 0 \tag{2.3.6}$$

describes one-dimensional weakly nonlinear dispersive water waves propagating in both positive and negative x-directions (Peregrine 1967; Toda and Wadati 1973; Zakharov 1968a, 1968b; Ablowitz and Haberman 1975; and Prasad and Ravindran 1977). It also arises in one-dimensional lattice waves (Zabusky 1967) and ion-acoustic solitons (Kako and Yajima 1980). In recent years, considerable attention has been given to new forms of Boussinesq equations (Madsen et al. 1991; Madsen and Sorensen 1992, 1993) dealing with water wave propagation and to modified Boussinesq equations (Nwogu 1993; Chen and Liu 1995a, 1995b) in terms of a velocity potential on an arbitrary elevation and free surface displacement of water.

Example 2.3.7. The *Korteweg–de Vries* (*KdV*) *equation*

$$u_t + \alpha u u_x + \beta u_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.7}$$

where α and β are constants, is a simple and useful model for describing the long time evolution of dispersive wave phenomena in which the steepening effect of the nonlinear term is counterbalanced by the dispersion. It was originally introduced by Korteweg and de Vries (1895) to describe the propagation of unidirectional shallow water waves.

It admits the exact solution called the *soliton*. This equation arises in many physical problems including water waves (Johnson 1980, 1997; Debnath 1994), internal gravity waves in a stratified fluid (Benney 1966; Redekopp and Weidman 1968), ion-acoustic waves in a plasma (Washimi and Taniuti 1966), pressure waves in a liquid-gas bubble (Van Wijngaarden 1968), and rotating flow in a tube (Leibovich 1970). There are other physical systems to which the KdV equation applies as a long wave approximation, including acoustic-gravity waves in a compressible heavy liquid, axisymmetric waves in a nonuniformly rotating fluid, acoustic waves in anharmonic crystals, nonlinear waves in cold plasmas, axisymmetric magnetohydrodynamic waves, and longitudinal dispersive waves in elastic rods.

Example 2.3.8. The *modified KdV* (*mKdV*) *equation*

$$u_t - 6u^2 u_x + u_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.8}$$

describes nonlinear acoustic waves in an anharmonic lattice (Zabusky 1967) and Alfvén waves in a collisionless plasma (Kakutani and Ono 1969). It also arises in many other physical situations.

Example 2.3.9. The *nonlinear Schrödinger* (*NLS*) *equation*

$$iu_t + u_{xx} + \gamma |u|^2 u = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.9}$$

where γ is a constant, describes the evolution of water waves ((Benney and Roskes 1969; Hasimoto and Ono 1972; Davey 1972; Davey and Stewartson 1974; Peregrine 1983); Zakharov 1968a, 1968b; Chu and Mei 1970; Yuen and Lake 1975; Infeld et al. 1987; Johnson 1997). It also arises in some other physical systems which include nonlinear optics (Kelley 1965; Talanov 1965; Bespalov and Talanov 1966; Karpman and Krushkal 1969; Asano et al. 1969; Hasegawa and Tappert 1973), hydromagnetic and plasma waves (Ichikawa et al. 1972; Schimizu and Ichikawa 1972; Taniuti and Washimi 1968; Fulton 1972; Hasegawa 1990; Ichikawa 1979; Weiland and Wiljelmsson 1977; Weiland et al. 1978), the propagation of a heat pulse in a solid (Tappert and Varma 1970), nonlinear waves in a fluid-filled viscoelastic tube (Ravindran and Prasad 1979), nonlinear instability problems (Stewartson and Stuart 1971; Nayfeh and Saric 1971), and the propagation of solitary waves in piezoelectric semiconductors (Pawlik and Rowlands 1975).

Example 2.3.10. The *Benjamin–Ono* (*BO*) *equation* is

$$u_t + uu_x + \mathcal{H}\{u_{xx}\} = 0, \tag{2.3.10}$$

where $\mathcal{H}\{f(\xi,t)\} = \tilde{f}(x,t)$ is the Hilbert transform of $f(\xi,t)$ defined by

$$\mathcal{H}\{f(\xi,t)\} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\xi,t)\, d\xi}{\xi - x}, \tag{2.3.11}$$

where P stands for the Cauchy principal value. This equation arises in the study of weakly nonlinear long internal gravity waves (Benjamin 1967; Davis and Acrivos 1967; and Ono 1975) and belongs to the class of weakly nonlinear models.

Example 2.3.11. The *Benjamin–Bona–Mahony* (*BBM*) *equation*

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.12}$$

represents another nonlinear model for long water waves. The KdV equation can be written as

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0. \tag{2.3.13}$$

The basic mathematical difference between the BBM and KdV equations can readily be determined by comparing the approximate dispersion relations for the respective linearized equations. We seek a plane wave solution of both linearized equations of the form

$$u(x,t) \sim \exp\big[i(\omega t - kx)\big]. \tag{2.3.14}$$

The dispersion relation of the linearized KdV equation is then given by

$$\omega = k - k^3. \tag{2.3.15}$$

The phase and group velocities are given by

$$C_p = \frac{\omega}{k} = 1 - k^2 \quad \text{and} \quad C_g = \frac{d\omega}{dk} = 1 - 3k^2, \tag{2.3.16ab}$$

which become negative for $k^2 > 1$. This means that all waves of large wavenumbers (small wavelengths) propagate in the *negative* x-direction in contradiction to the original assumption that waves travel only in the positive x-direction. This is an undesirable physical feature of the KdV equation. To eliminate this unrealistic feature of the KdV equation, Benjamin et al. (1972) proposed equation (2.3.12). The dispersion relation of the linearized version of (2.3.12) is

$$\omega = \frac{k}{(1+k^2)}. \tag{2.3.17}$$

Thus the phase and group velocities of waves associated with this model are given by

$$C_p = \frac{\omega}{k} = \big(1+k^2\big)^{-1}, \qquad C_g = \big(1-k^2\big)\big(1+k^2\big)^{-2}. \tag{2.3.18ab}$$

Both C_p and C_g tend to zero, as $k \to \infty$, showing that short waves do not propagate. In other words, the BBM model has the approximate features of responding only significantly to short wave components introduced in the initial wave form. Thus, the BBM equation seems to be a preferable model. However, the fact that the BBM model is a better model than the KdV model has not been fully confirmed, yet.

Example 2.3.12. The *Ginzburg–Landau* (*GL*) *equation* is

$$A_t + aA_{xx} = bA + cA|A|^2, \tag{2.3.19}$$

where a and b are complex constants determined by the dispersion relation of linear waves, and c is determined by the weakly nonlinear interaction (Stewartson and Stuart 1971). This equation describes slightly unstable nonlinear waves and has arisen originally in the theories of superconductivity and phase transitions.

The complex Ginzburg–Landau equation simplifies significantly if all of the coefficients are real. The real Ginzburg–Landau equation has been extensively investigated in problems dealing with phase separation in condensed matter physics (Ben-Jacob et al. 1985; Van Saarloos 1989; Balmforth 1995).

Example 2.3.13. The *Burgers–Huxley* (*BH*) *equation*

$$u_t + \alpha u u_x - \nu u_{xx} = \beta(1-u)(u-\gamma)u, \quad x \in \mathbb{R},\ t>0, \tag{2.3.20}$$

where α, $\beta \geq 0$, γ $(0<\gamma<1)$, and ν are parameters, describes the interaction between convection, diffusion, and reaction. When $\alpha = 0$, equation (2.3.20) reduces to the Hodgkin and Huxley (1952) equation which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals (Scott 1977; Satsuma 1987a, 1987b; Wang 1985, 1986; Wang et al. 1990). Because of the complexity of the Huxley equation, the FitzHugh–Nagumo equations (FitzHugh 1961; Sleeman 1982; Nagumo et al. 1962) proposed simple, analytically tractable, and particularly useful model equations which contain the key features of the Huxley model. On the other hand, when $\beta = 0$, equation (2.3.20) reduces to the Burgers equation (2.3.4) describing diffusive waves in nonlinear dissipating systems. Satsuma (1987a, 1987b) obtained solitary wave solutions of (2.3.20) by using Hirota's method in soliton theory.

Example 2.3.14. The *Kadomtsev–Petviashvili* (*KP*) *equation*

$$(u_t - 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \tag{2.3.21}$$

is a two-dimensional generalization of the KdV equation. Kadomtsev and Petviashvili (1970) first introduced this equation to describe slowly varying nonlinear waves in a dispersive medium (Johnson 1980, 1997). Equation (2.3.21) with $\sigma^2 = +1$ arises in the study of weakly nonlinear dispersive waves in plasmas and also in the modulation of weakly nonlinear long water waves (Ablowitz and Segur 1979) which travel nearly in one dimension (that is, nearly in a vertical plane). Satsuma (1987a, 1987b) showed that the KP equation has N line-soliton solutions which describe the oblique interaction of solitons. The equation with $\sigma^2 = -1$ arises in acoustics and admits unstable soliton solutions, whereas for $\sigma^2 = +1$ the solitons are stable. Freeman (1980) presented an interesting review of soliton interactions in two dimensions. Recently, Chen and Liu (1995a, 1995b) have derived the unified KP (uKP) equation for surface and interfacial waves propagating in a rotating channel with varying topography and sidewalls. This new equation includes most of the existing KP-type equations in the literature as special cases.

Example 2.3.15. The *concentric KdV equation*

$$2u_R + \frac{1}{R}u + 3uu_\xi + \frac{1}{3}u_{\xi\xi\xi} = 0 \tag{2.3.22}$$

describes concentric waves on the free surface of water that have decreasing amplitude with increasing radius. This is also called the *cylindrical KdV equation* which was first derived in another context by Maxon and Viecelli (1974). The inverse scattering transform for equation (2.3.22) involves a linearly increasing potential which yields eigenfunctions based on the Airy function (see Calogero and Degasperis 1978). A discussion of this equation and its solution can also be found in Johnson (1997) and Freeman (1980).

Example 2.3.16. The *nearly concentric KdV equation* (or the *Johnson equation*)

$$\left(2u_R + \frac{1}{R}u + 3uu_\xi + \frac{1}{3}u_{\xi\xi\xi}\right)_\xi + \frac{1}{R^2}u_{\theta\theta} = 0 \tag{2.3.23}$$

describes the nearly concentric surface waves incorporating weak dependence on the angular coordinate θ. In the absence of θ-dependence, equation (2.3.23) reduces to (2.3.22). This equation was first derived by Johnson (1980) in his study of problems of nonlinear water waves.

Example 2.3.17. The *Davey–Stewartson* (*DS*) *equations*

$$-2ikc_pA_\tau + aA_{\zeta\zeta} - c_pc_gA_{yy} + bA|A|^2 + ck^2Af_\zeta = 0, \tag{2.3.24}$$

$$\left(1 - c_g^2\right)f_{\zeta\zeta} + f_{yy} = d\left(|A|^2\right)_\zeta, \tag{2.3.25}$$

where a, b, c, d are functions of δk (see Davey and Stewartson 1974; Johnson 1997), describe weakly nonlinear dispersive waves propagating in the x-direction with a slowly varying structure in both the x- and y-directions. In the absence of y-dependence with $f_\zeta \equiv 0$, the DS equations recover the NLS equation for water waves (see Hasimoto and Ono 1972) in the form

$$-2ikc_pA_\tau + aA_{\zeta\zeta} + bA|A|^2 = 0. \tag{2.3.26}$$

This is similar to (2.3.9).

Example 2.3.18. The *Whitham* (1974) *nonlinear nonlocal integrodifferential equation*

$$\eta_t + d\eta\eta_x + \int_{-\infty}^{\infty} K(x-\xi)\eta_\xi(\xi,t)\,d\xi = 0 \tag{2.3.27}$$

can describe symmetric waves that propagate without change of shape and peak at a critical height, as well as asymmetric waves that invariably break. The kernel $K(x)$ is given by the inverse Fourier transform of the phase velocity $c(k) = \frac{\omega}{k}$ in the form

$$K(x) = \mathcal{F}^{-1}\{c(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}c(k)\,dk. \tag{2.3.28}$$

It is a well known fact that the nonlinear shallow water equations which neglect dispersion altogether lead to breaking of the typical hyperbolic kind, with development of a vertical slope and a multivalued wave profile. It is clear that the third derivative dispersion term in the KdV equation (2.3.7) prevents wave breaking. Whitham formulated his equation (2.3.27) to describe the observed phenomena of solitary and periodic cnoidal waves as well as peaking and breaking of water waves. The Whitham equation is a kind of generalization of the KdV equation that takes $c(k) = c_0 - \gamma k^2$ and $K(x) = c_0\delta(x) + \delta''(x)$, $c_0^2 = gh_0$. The detailed analysis of Whitham's analysis is given in Section 7.8 in Chapter 7.

Example 2.3.19. The *Camassa and Holm* (*CH*) *Equation* for the free surface elevation $u(x,t)$ over a flat rigid bottom is

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xx}, \quad x \in \mathbb{R},\ t > 0. \tag{2.3.29}$$

It describes the propagation of nonlinear dispersive shallow water equation to capture the essential features of wave breaking. It is integrable in the sense that there exists a Lax pair, and has infinitely many conservation laws. The CH equation admits stable solitary wave solutions with a peak at their crests; these waves are called *peakons*. A more elaborate discussion of this equation (Camassa and Holm 1993) and its various extensions are presented in Section 9.13.

Example 2.3.20. The *Degasperis and Procesi* (*DP*) *equation* is

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R},\ t > 0. \tag{2.3.30}$$

It also describes the propagation of nonlinear dispersive shallow water waves. Its solutions are singular, leading to wave breaking. The DP equation admits a shock-peakon solution which is significantly different from the peakon solutions of the CH equation. Both the CH and DP equations have soliton solutions which develop singularities in finite time (or solutions blow-up in finite time). Both the CH and DP equations can be combined into a $(1+1)$-dimensional b-family equation for fluid velocity $u(x,t)$ in the form

$$m_t + um_x + bmu_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{2.3.31}$$

where $m = (u - u_{xx})$ and $u = g * m$ is the convolution product given by

$$u(x) = \int_{\mathbb{R}} g(x-\xi)m(\xi)\,d\xi, \qquad g(x) = \frac{1}{2}\exp\bigl(-|x|\bigr), \tag{2.3.32}$$

which determines the traveling wave shape and length scale for equation (2.3.31) and the constant b is a balance (or bifurcation) parameter. Degasperis and Procesi (1999) showed that (2.3.31) cannot be completely integrable unless $b = 2$ or $b = 3$. When $b = 2$, equation (2.3.31) reduces to the CH equation (2.3.29) and when $b = 3$, (2.3.31) becomes the DP equation (2.3.30). A more detailed discussion on these equations can be found in Section 9.13.

Example 2.3.21 (The Toda Lattice Equation *in a mass–spring system).* A mass–spring lattice is an infinite chain of identical masses m interconnected by nonlinear springs. We assume that the springs have potential $V(r)$, where r is the increase in distance between adjacent masses from the rest value at which the spring energy is minimum and its force $(F = -\frac{dV}{dr})$ is zero. If y_n is the longitudinal displacement of the nth mass from its equilibrium position, it follows from the Newton second law of motion that

$$m\frac{d^2y_n}{dt^2} = V'(y_{n+1} - y_n) - V'(y_n - y_{n-1}). \tag{2.3.33}$$

With $r_n = (y_{n+1} - y_n)$, this gives an infinite set of differential equations

$$m\ddot{r}_n = \left[\frac{dV(r_{n+1})}{dr_{n+1}} - \frac{dV(r_n)}{dr_n}\right] - \left[\frac{dV(r_n)}{dr_n} - \frac{dV(r_{n-1})}{dr_{n-1}}\right], \tag{2.3.34}$$

where $n \in \mathbb{N}$.

In his celebrated paper, Toda (1967a, 1967b) investigated a mass–spring lattice system with an anharmonic potential in the form

$$V(r) = \frac{a}{b}\left(e^{-br} + br - 1\right), \quad a, b > 0. \tag{2.3.35}$$

With unit masses $(m = 1)$, equation (2.3.34) reduces to the form

$$\ddot{r}_n = a\left(2e^{-br_n} - e^{-br_{n+1}} - e^{-br_{n-1}}\right). \tag{2.3.36}$$

This is known as the *Toda lattice equation*.

In the limit as $b \to 0$ with finite ab, equation (2.3.36) reduces to the linear differential-difference equation

$$\ddot{r}_n = ab(r_{n+1} - 2r_n + r_{n-1}). \tag{2.3.37}$$

This has solutions with a long wavelength velocity of $\sqrt{ab}$ lattice points per unit time.

When b is *not* small, the Toda lattice equation (2.3.36) admits exact solitary wave solutions of the form (see Section 11.13)

$$r_n = -\frac{1}{b}\log\left[1 + \sinh^2\kappa\,\mathrm{sech}^2\left\{\kappa\left(n \pm t\frac{\sqrt{ab}}{\kappa}\sinh\kappa\right)\right\}\right], \tag{2.3.38}$$

where the velocity of the lattice wave is expressed in terms of the amplitude parameter κ in the form

$$v = \frac{\sinh\kappa}{\kappa}\sqrt{ab}, \tag{2.3.39}$$

and the minus sign in (2.3.38) implies that the Toda lattice soliton (TLS) is a compression wave.

As the amplitude of the TLS is reduced to zero (by letting $\sinh\kappa$ approach zero), it reduces to a solution of the linear equation (2.3.37) traveling with velocity $v = \sqrt{ab}$.

We close this section by mentioning the *Yang–Mills field equations* which seem to be a useful model unifying electromagnetic and weak forces. They have solutions, called *instantons*, localized in space and time, which are interpreted as quantum-mechanical transitions between different states of a particle. Recently, it has been shown that the *self-dual Yang–Mills equations* are multidimensional integrable systems, and these equations admit reductions to well-known soliton equations in $(1+1)$ dimensions, that is, the sine-Gordon, NLS, KdV, and Toda lattice equations (Ward 1984, 1985, 1986).

2.4 Variational Principles and the Euler–Lagrange Equations

Many physical systems are often characterized by their extremum (minimum, maximum, or saddle point) property of some associated physical quantity that appears as an integral in a given domain, known as a *functional*. Such a characterization is a variational principle leading to the Euler–Lagrange equation which optimizes the related functional. For example, light rays travel along a path from one point to another in a minimum time. The shortest distance between two points on a plane curve is a straight line. A physical system is in equilibrium if its potential energy is minimum. So the main problem is to optimize a physical quantity (time, distance, or energy) in most real-world problems. These problems belong to the subject of the calculus of variations.

The classical Euler–Lagrange variational problem is to determine the extremum value of the functional

$$I(u) = \int_a^b F(x, u, u')\, dx, \quad u' = \frac{du}{dx}, \tag{2.4.1}$$

with the boundary conditions

$$u(a) = \alpha \quad \text{and} \quad u(b) = \beta, \tag{2.4.2ab}$$

where α and β are given numbers and $u(x)$ belongs to the class $C^2([a, b])$ of functions which have continuous derivatives up to the second order in $a \le x \le b$ and the integrand F has continuous second derivatives with respect to all of its arguments.

We assume that $I(u)$ has an extremum at some $u \in C^2([a, b])$. Then we consider the set of all variations $u + \varepsilon v$ for finite u, and arbitrary v belonging to $C^2([a, b])$ such that $v(a) = 0 = v(b)$. We next consider the variation δI of the functional $I(u)$

$$\begin{aligned} \delta I &= I(u + \varepsilon v) - I(u) \\ &= \int_a^b \big[F(x, u + \varepsilon v, u' + \varepsilon v') - F(x, u, u')\big]\, dx \end{aligned}$$

which, by the Taylor series expansion,

$$\begin{aligned} &= \int_a^b \left[F(x, u, u') + \varepsilon\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right) \right. \\ &\quad \left. + \frac{\varepsilon^2}{2!}\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right)^2 + \cdots - F(x, u, u')\right] dx \\ &= \int_a^b \varepsilon\left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right) dx + O\big(\varepsilon^2\big). \end{aligned} \tag{2.4.3}$$

Thus, a necessary condition for the functional $I(u)$ to have an extremum (or for $I(u)$ to be stationary) for an arbitrary ε is

$$0 = \delta I = \int_a^b \left(v\frac{\partial F}{\partial u} + v'\frac{\partial F}{\partial u'}\right) dx, \tag{2.4.4}$$

which, integrating the second term by parts, is

$$= \int_a^b v\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right] dx + \left[v\frac{\partial F}{\partial u'}\right]_a^b. \tag{2.4.5}$$

Since v is arbitrary with $v(a) = 0 = v(b)$, the last term of (2.4.5) vanishes and consequently, the integrand of the integral in (2.4.5) must vanish, that is,

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0. \tag{2.4.6}$$

This is called the *Euler–Lagrange equation* of the variational problem involving one independent variable. Using the result

$$d\left(\frac{\partial F}{\partial u'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right) dx + \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u'}\right) du + \frac{\partial}{\partial u'}\left(\frac{\partial F}{\partial u'}\right) du', \tag{2.4.7}$$

the Euler–Lagrange equations (2.4.6) can be written in the form

$$F_u - F_{xu'} - u'F_{uu'} - u''F_{u'u'} = 0. \tag{2.4.8}$$

This is a second-order nonlinear ordinary differential equation for u provided $F_{u'u'} \neq 0$ and, hence, there are two arbitrary constants involved in the solution. However, when F does not depend explicitly on one of its variables x, u, or u', the Euler–Lagrange equation assumes a simplified form. Evidently, there are three possible cases:

1. If $F = F(x, u)$, then (2.4.6) reduces to $F_u(x, u) = 0$, which is an algebraic equation.
2. If $F = F(x, u')$, then (2.4.6) becomes

$$\frac{\partial F}{\partial u'} = \text{const.} \tag{2.4.9}$$

3. If $F = F(u, u')$, then (2.4.6) takes the form

$$F - u'F_{u'} = \text{const.} \tag{2.4.10}$$

This follows from the fact that

$$\begin{aligned}\frac{d}{dx}(F - u'F_{u'}) &= \frac{dF}{dx} - u'\frac{d}{dx}F_{u'} - u''F_{u'} \\ &= F_x + u'F_u + u''F_{u'} - u'\frac{d}{dx}F_{u'} - u''F_{u'} \\ &= u'\left(F_u - \frac{d}{dx}F_{u'}\right) = 0 \quad \text{by (2.4.6).}\end{aligned}$$

The Euler–Lagrange variational problem involving two independent variables is to determine a function $u(x, y)$ in a domain $D \subset \mathbb{R}^2$ satisfying the boundary conditions prescribed on the boundary ∂D of D and extremizing the functional

$$I\big[u(x,y)\big] = \iint_D F(x,y,u,u_x,u_y)\,dx\,dy, \tag{2.4.11}$$

where the function F is defined over the domain D and assumed to have continuous second-order partial derivatives.

Similarly, for functionals depending on a function of two independent variables, the first variation δI of I is defined by

$$\delta I = I(u+\varepsilon v) - I(u). \tag{2.4.12}$$

In view of Taylor's expansion theorem, this reduces to

$$\delta I = \iint_D \big[\varepsilon(vF_u + v_x F_p + v_y F_q) + O\big(\varepsilon^2\big)\big]\,dx\,dy, \tag{2.4.13}$$

where $v = v(x,y)$ is assumed to vanish on ∂D and $p = u_x$ and $q = u_y$.

A necessary condition for the functional I to have an extremum is that the first variation of I vanishes, that is,

$$\begin{aligned} 0 = \delta I &= \iint_D (vF_u + v_x F_p + v_y F_q)\,dx\,dy \\ &= \iint_D v\left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)dx\,dy \\ &\quad + \iint_D \left[v\left(\frac{\partial}{\partial x}F_p + \frac{\partial}{\partial y}F_q\right) + (v_x F_p + v_y F_q)\right]dx\,dy \\ &= \iint_D v\left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)dx\,dy \\ &\quad + \iint_D \left[\frac{\partial}{\partial x}(vF_p) + \frac{\partial}{\partial y}(vF_q)\right]dx\,dy. \end{aligned} \tag{2.4.14}$$

We assume that the boundary curve ∂D has a piecewise, continuously moving tangent so that Green's theorem can be applied to the second double integral in (2.4.14). Consequently, (2.4.14) reduces to

$$\begin{aligned} 0 = \delta I &= \iint_D v\left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)dx\,dy \\ &\quad + \int_{\partial D} v(F_p\,dy - F_q\,dx). \end{aligned} \tag{2.4.15}$$

Since $v = 0$ on ∂D, the second integral in (2.4.15) vanishes. Moreover, since v is an arbitrary function, it follows that the integrand of the first integral in (2.4.15) must vanish. Thus, the function $u(x,y)$ extremizing the functional defined by (2.4.11) satisfies the partial differential equation

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q = 0. \tag{2.4.16}$$

This is called the *Euler–Lagrange equation* for the variational problem involving two independent variables.

The above variational formulation can readily be generalized for functionals depending on functions of three or more independent variables. Many physical problems require determining a function of several independent variables which will lead to an extremum of such functionals.

Example 2.4.1. Find $u(x,y)$ which extremizes the functional

$$I\big[u(x,y)\big] = \iint_D \left(u_x^2 + u_y^2\right) dx\,dy, \quad D \subset \mathbb{R}^2. \tag{2.4.17}$$

The Euler–Lagrange equation with $F = u_x^2 + u_y^2 = p^2 + q^2$ is

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0,$$

or

$$u_{xx} + u_{yy} = 0. \tag{2.4.18}$$

This is a two-dimensional Laplace equation. Similarly, the functional

$$I\big[u(x,y,z)\big] = \iiint_D \left(u_x^2 + u_y^2 + u_z^2\right) dx\,dy\,dz, \quad D \subset \mathbb{R}^3, \tag{2.4.19}$$

will lead to the three-dimensional Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{2.4.20}$$

In this way, we can derive the n-dimensional Laplace equation

$$\nabla^2 u = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} = 0. \tag{2.4.21}$$

Example 2.4.2 (Plateau's Problem). Find the surface S in the (x,y,z)-space of minimum area passing through a given plane curve C.

The direction cosine of the angle between the z-axis and the normal to the surface $z = u(x,y)$ is $(1 + u_x^2 + u_y^2)^{-\frac{1}{2}}$. The projection of the element dS of the area of the surface onto the (x,y)-plane is given by $(1 + u_x^2 + u_y^2)^{-\frac{1}{2}} dS = dx\,dy$. The area A of the surface S is given by

$$A = \iint_D \left(1 + u_x^2 + u_y^2\right)^{\frac{1}{2}} dx\,dy, \tag{2.4.22}$$

where D is the area of the (x,y)-plane bounded by the curve C.

The Euler–Lagrange equation with $F = (1 + p^2 + q^2)^{\frac{1}{2}}$ is given by

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+p^2+q^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+p^2+q^2}}\right) = 0. \tag{2.4.23}$$

This is the *equation of minimal surface*, which reduces to the nonlinear elliptic partial differential equation

$$\left(1+u_y^2\right)u_{xx} - 2u_x u_y u_{xy} + \left(1+u_x^2\right)u_{yy} = 0. \tag{2.4.24}$$

Therefore, the desired function $u(x,y)$ should be determined as the solution of the *nonlinear Dirichlet problem* for (2.4.24). This is difficult to solve. However, if the equation (2.4.23) is linearized around the zero solution, the square root term is replaced by one, and then the Laplace equation is obtained.

Example 2.4.3 (Lagrange's Equation in Mechanics). According to the Hamilton principle in mechanics, the first variation of the time integral of the Lagrangian $L = L(q_i, \dot{q}_i, t)$ of any dynamical system must be stationary, that is,

$$0 = \delta I = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\,dt, \tag{2.4.25}$$

where $L = T - V$ is the difference between the kinetic energy, T, and the potential energy, V. In coordinate space, there are infinitely many possible paths joining any two positions. From all these paths, which start at a point A at time t_1 and end at another point B at time t_2, nature selects the path $q_i = q_i(t)$ for which $\delta I = 0$. Consequently, in this case, the Euler–Lagrange equation (2.4.6) reduces to

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad i = 1, 2, \ldots, n. \tag{2.4.26}$$

In classical mechanics, these equations are universally known as the *Lagrange equations of motion*.

The *Hamilton function* (or simply *Hamiltonian*) H is defined in terms of the generalized coordinates q_i, generalized momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$, and L by

$$H = \sum_{i=1}^{n} (p_i \dot{q}_i - L) = \sum_{i=1}^{n} \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L(q_i, \dot{q}_i)\right). \tag{2.4.27}$$

It readily follows that

$$\frac{dH}{dt} = \frac{d}{dt}\left[\sum_{i=1}^{n}(p_i \dot{q}_i - L)\right] = \sum_{i=1}^{n} \dot{q}_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right) = 0. \tag{2.4.28}$$

Thus, H is a constant, and hence, the Hamiltonian is the constant of motion.

Example 2.4.4 (Hamilton's Equations in Mechanics). To derive Hamilton equations of motion, we use the concepts of generalized momentum p_i and generalized force F_i defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \text{and} \tag{2.4.29a}$$

$$F_i = \frac{\partial L}{\partial q_i}. \tag{2.4.29b}$$

Consequently, the Lagrange equations of motion (2.4.26) reduce to

$$\frac{\partial L}{\partial q_i} = \frac{dp_i}{dt} = \dot{p}_i. \tag{2.4.30}$$

In general, the Lagrangian $L = L(q_i, \dot{q}_i, t)$ is a function of q_i, $\dot{q}_i$, and t where $\dot{q}_i$ enters through the kinetic energy as a quadratic term. It then follows from the definition (2.4.27) of the Hamiltonian that $H = H(p_i, q_i, t)$, and hence, its differential is

$$dH = \sum \frac{\partial H}{\partial p_i}\, dp_i + \sum \frac{\partial H}{\partial q_i}\, dq_i + \frac{\partial H}{\partial t}\, dt. \tag{2.4.31}$$

Differentiating (2.4.27) with respect to t gives

$$\frac{dH}{dt} = \sum p_i \frac{d}{dt}\dot{q}_i + \sum \dot{q}_i \frac{d}{dt} p_i - \sum \frac{\partial L}{\partial q_i}\frac{d}{dt} q_i - \sum \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\dot{q}_i - \frac{\partial L}{\partial t},$$

or equivalently,

$$dH = \sum p_i\, d\dot{q}_i + \sum \dot{q}_i\, dp_i - \sum \frac{\partial L}{\partial q_i}\, dq_i - \sum \frac{\partial L}{\partial \dot{q}_i}\, d\dot{q}_i - \frac{\partial L}{\partial t}\, dt,$$

which, due to equation (2.4.29a), is

$$= \sum \dot{q}_i\, dp_i - \sum \frac{\partial L}{\partial q_i}\, dq_i - \frac{\partial L}{\partial t}\, dt. \tag{2.4.32}$$

We next equate the coefficients of the two identical expressions (2.4.31) and (2.4.32) to obtain

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad -\frac{\partial L}{\partial q_i} = \frac{\partial H}{\partial q_i}, \qquad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \tag{2.4.33}$$

Using the Lagrange equations (2.4.30), the first two equations in (2.4.33) give

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{2.4.34ab}$$

These are universally known as the *Hamilton canonical equations* of motion.

Example 2.4.5 (Law of Conservation of Energy). The kinetic energy of a mechanical system described by a set of generalized coordinates q_i is defined by

$$T = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2} a_{ij}\dot{q}_i\dot{q}_j, \tag{2.4.35}$$

where a_{ij} are known functions of q_i, and $\dot{q}_i$ is the generalized velocity.

In general, the potential energy $V = V(q_i, \dot{q}_i, t)$ is a function of q_i, $\dot{q}_i$, and t. We assume here that V is independent of $\dot{q}_i$. For such a mechanical system, the Lagrangian is defined by $L = T - V$.

Using the above definitions, the Hamilton principle states that, between any two points t_1 and t_2, the actual motion takes place along the path $q_i = q_i(t)$ such that the functional

$$I\big(q_i(t)\big) = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} (T - V)\,dt, \tag{2.4.36}$$

is stationary (that is, the functional is an extremum). Or equivalently, the Hamilton principle can be stated as

$$\delta I = \delta \int_{t_1}^{t_2} (T - V)\,dt = 0. \tag{2.4.37}$$

The integral I defined by (2.4.36) is often called the *action integral* of the system.

Since the potential energy V does not depend on $\dot{q}_i$, it follows that

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} a_{ij}\dot{q}_j,$$

and the Hamiltonian H defined by (2.4.27) becomes

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L = \sum_{i=1}^{n} \dot{q}_i \left(\sum_{j=1}^{n} a_{ij}\dot{q}_j \right) - L = 2T - L = T + V. \tag{2.4.38}$$

This proves that the Hamiltonian H is equal to the total energy. By (2.4.28), H is a constant, thus, the total energy of the system is constant. This is the celebrated *law of conservation of energy*.

Example 2.4.6 (Motion of a Particle Under the Action of a Central Force). Consider the motion of a particle of mass m under the action of a central force $-mF(r)$ where r is the distance of the particle from the center of force. The kinetic energy T is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big),$$

which, in terms of polar coordinates,

$$= \frac{1}{2}m\left[\left\{\frac{d}{dt}(r\cos\theta)\right\}^2 + \left\{\frac{d}{dt}(r\sin\theta)\right\}^2\right] = \frac{1}{2}m\big(\dot{r}^2 + r^2\dot{\theta}^2\big). \tag{2.4.39}$$

Since $\mathbf{F} = \nabla V$, the potential is given by

$$V(r) = \int^{r} F(r)\,dr. \tag{2.4.40}$$

Then the Lagrangian L is given by

$$L = T - V = \frac{1}{2}m\left[\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - 2\int^r F(r)\,dr\right]. \tag{2.4.41}$$

Thus the Hamilton principle requires that the functional

$$I(r,\theta) = \int_{t_1}^{t_2} L\,dr = \int_{t_1}^{t_2} (T - V)\,dt \tag{2.4.42}$$

be stationary, that is, $\delta I = 0$. Consequently, the Euler–Lagrange equations are given by

$$L_r - \frac{d}{dt}L_{\dot{r}} = 0 \quad \text{and} \quad L_\theta - \frac{d}{dt}L_{\dot{\theta}} = 0, \tag{2.4.43}$$

or equivalently,

$$\ddot{r} - r\dot{\theta}^2 = -F(r) \quad \text{and} \quad \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0. \tag{2.4.44}$$

These equations describe the planar motion of the particle.

It follows immediately from the second equation of (2.4.44) that

$$r^2\dot{\theta} = \text{const.} = h. \tag{2.4.45}$$

In this case, $r\dot{\theta}$ represents the transverse velocity of the particle and $mr^2\dot{\theta} = mh$ is the constant angular momentum of the particle about the center of force.

Introducing $r = \frac{1}{u}$, we find

$$\dot{r} = \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt} = -h\frac{du}{d\theta},$$
$$\ddot{r} = \frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting these into the first equation of (2.4.44) gives

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2u^2}F\left(\frac{1}{u}\right). \tag{2.4.46}$$

This is the differential equation of the central orbit, and it can be solved by standard methods.

In particular, if the law of force is the attractive inverse square $F(r) = \mu/r^2$ so that the potential $V(r) = -\mu/r$, the differential equation (2.4.46) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}. \tag{2.4.47}$$

If the particle is projected initially from the distance a with velocity v at an angle α that the direction of motion makes with the outward radius vector, then the constant h in (2.4.45) is $h = av\sin\alpha$.

The angle ϕ between the tangent and radius vector of the orbit at any point is given by

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = u\frac{d}{d\theta}\left(\frac{1}{u}\right) = -\frac{1}{u}\frac{du}{d\theta}.$$

The initial conditions at $t = 0$ are

$$u = \frac{1}{a}, \qquad \frac{du}{d\theta} = -\frac{1}{a}\cot\alpha, \quad \text{when } \theta = 0. \tag{2.4.48}$$

The general solution of (2.4.47) is

$$u = \frac{\mu}{h^2}\big[1 + e\cos(\theta + \varepsilon)\big], \tag{2.4.49}$$

where e and ε are constants to be determined by the initial data.

Finally, the solution can be written as

$$\frac{\ell}{r} = 1 + e\cos(\theta + \varepsilon), \tag{2.4.50}$$

where

$$\ell = \frac{h^2}{\mu} = \frac{1}{\mu}(av\sin\alpha)^2. \tag{2.4.51}$$

This represents a conic section of semilatus rectum ℓ and eccentricity e with its axis inclined at the point of projection.

The initial conditions (2.4.48) give

$$\frac{\ell}{a} = 1 + e\cos\varepsilon, \qquad -\frac{\ell}{a}\cot\alpha = -e\sin\varepsilon,$$

so that

$$\begin{aligned}
\tan\varepsilon &= \left(\frac{\ell}{\ell - a}\right)\cot\alpha, \\
e^2 &= \left(\frac{\ell}{a} - 1\right)^2 + \frac{\ell^2}{a^2}\cot^2\alpha = 1 - \frac{2\ell}{a} + \frac{\ell^2}{a^2}\operatorname{cosec}^2\alpha \\
&= 1 - \frac{1}{\mu}\big(2av^2\sin^2\alpha\big) + \frac{1}{\mu^2}\big(a^2v^4\sin^2\alpha\big) \\
&= 1 + \left(\frac{av\sin\alpha}{\mu}\right)^2\left(v^2 - \frac{2\mu}{a}\right).
\end{aligned} \tag{2.4.52}$$

Thus, the central orbit is an ellipse, parabola, or hyperbola accordingly as $e < 1$, $= 1$, or > 1, that is, $v^2 < (2\mu/a)$, $= (2\mu/a)$, or $> (2\mu/a)$.

Example 2.4.7 (The Wave Equation of a Vibrating String). We assume that, initially, the string of length ℓ and line density ρ is stretched along the x-axis from $x = 0$ to $x = \ell$. The string will be given a small lateral displacement, which is denoted by $u(x,t)$ at each point along the x-axis at time t. The kinetic energy T of the string is given by

$$T = \frac{1}{2} \int_0^\ell \rho u_t^2 \, dx, \tag{2.4.53}$$

and the potential energy is given by

$$T = \frac{T^*}{2} \int_0^\ell u_x^2 \, dx, \tag{2.4.54}$$

where T^* is the constant tension of the string.

According to the Hamilton principle

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} L \, dt = \delta \int_{t_1}^{t_2} (T - V) \, dt \\ &= \delta \int_{t_1}^{t_2} \frac{1}{2} \int_0^\ell \left(\rho u_t^2 - T^* u_x^2\right) dx \, dt. \end{aligned} \tag{2.4.55}$$

In this case, $L = \frac{1}{2}(\rho u_t^2 - T^* u_x^2)$ which does not depend explicitly on x, t, or u, and hence, the Euler–Lagrange equation is given by

$$\frac{\partial}{\partial t}(\rho u_t) - \frac{\partial}{\partial x}(T^* u_x) = 0,$$

or

$$u_{tt} - c^2 u_{xx} = 0, \tag{2.4.56}$$

where $c^2 = (T^*/\rho)$. This is the wave equation of the vibrating string.

Example 2.4.8 (Two-Dimensional Wave Equation of Motion for Vibrating Membrane). We consider the motion of a vibrating membrane occupying the domain D under the action of a prescribed lateral force $f(x, y, t)$ and subject to the homogeneous boundary conditions $u = 0$ on the boundary ∂D.

The kinetic energy T and the potential energy V are given by

$$T = \frac{1}{2}\rho \iint_D u_t^2 \, dx \, dy, \qquad V = \frac{1}{2}\mu \iint_D \left(u_x^2 + u_y^2\right) dx \, dy, \tag{2.4.57}$$

where ρ is the surface density, μ is the elastic modulus of the membrane, and $u = u(x, y, t)$ is the displacement function. The Lagrangian functional is of the form

$$L = \iint_D \mathcal{L} \, dx \, dy, \tag{2.4.58}$$

where the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\rho u_t^2 - \frac{1}{2}\mu\left(u_x^2 + u_y^2\right) - u f(x, y, t). \tag{2.4.59}$$

According to the Hamilton principle, the first variation of the Lagrangian L must be stationary, that is,

$$0 = \delta \int_{t_1}^{t_2} L\,dt = \delta \int_{t_1}^{t_2} \iint_D \left[\frac{1}{2}\rho u_t^2 - \frac{1}{2}\mu\left(u_x^2 + u_y^2\right) - uf\right] dx\,dy. \tag{2.4.60}$$

The Euler–Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\mathcal{L}_{u_x} - \frac{\partial}{\partial y}\mathcal{L}_{u_y} - \frac{\partial}{\partial t}\mathcal{L}_{u_t} = 0, \tag{2.4.61}$$

or equivalently,

$$-f + \mu(u_{xx} + u_{yy}) - \rho u_{tt} = 0. \tag{2.4.62}$$

This leads to the nonhomogeneous wave equation

$$\mu \nabla^2 u - \rho u_{tt} = f(x, y, t). \tag{2.4.63}$$

This is the two-dimensional nonhomogeneous wave equation that can be solved with the initial conditions

$$u(x, y, t = 0) = \phi(x, y) \quad \text{and} \quad u_t(x, y, t = 0) = \psi(x, y) \quad \text{at } t = 0. \tag{2.4.64}$$

Example 2.4.9 (Three-Dimensional Nonhomogeneous Wave Equation). In three-dimensional wave propagation in elastic media, the traveling waves exhibit various modes of vibration including longitudinal and transverse waves. To derive the appropriate equations of motion in continuous media, we need to extend the Hamilton principle by considering the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$. We use symmetric motion given by $\mathbf{u} = \mathbf{u}(u_1, u_2, u_3)$ where $u_i = u_i(x_1, x_2, x_3, t)$, $i = 1, 2, 3$, and denote the particle velocity by $u_t = (u_{1,t}, u_{2,t}, u_{3,t})$. Using this notation and tensor summation convention, the kinetic energy T and the potential energy V are given by

$$T = \frac{1}{2}\rho u_{i,t} u_{i,t} \quad \text{and} \quad V = \frac{1}{2}\mu u_{i,j} u_{i,j}, \tag{2.4.65}$$

where $u_{i,t} = \frac{\partial u_i}{\partial t}$ and $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

Introducing an external force term $f(x_i, t)$ so that the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}\rho u_{i,t} u_{i,t} - \frac{1}{2}\mu u_{i,j} u_{i,j} - u_i f(x_i, t), \tag{2.4.66}$$

the Lagrangian functional is of the form

$$L = \iiint_D \mathcal{L}\,dx_j. \tag{2.4.67}$$

The generalized Hamilton principle for a three-dimensional continuum for various modes of wave propagation described by $\mathbf{u}(\mathbf{x}_j, t)$ takes the form

$$0 = \delta I(\mathbf{u}) = \delta \int_{t_1}^{t_2} dt \iiint_D \mathcal{L}\,dx_j = \int_{t_1}^{t_2} dt \iiint_D \delta\mathcal{L}\,dx_j. \tag{2.4.68}$$

This means that the function $\mathbf{u} = \mathbf{u}(x_j, t)$ makes the functional $I(\mathbf{u})$ an extremum.

Since $\mathcal{L}$ is a function of u_i and $u_{i,t}$, and the operator δ acts on the function u_i and $u_{i,t}$, we expand $\mathcal{L}$ to obtain

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial u_i}\delta u_i + \frac{\partial \mathcal{L}}{\partial u_{i,t}}\delta u_{i,t} + \frac{\partial \mathcal{L}}{\partial u_{i,j}}\delta u_{i,j}. \tag{2.4.69}$$

We next substitute (2.4.69) into (2.4.68) and then integrate by parts with respect to t to obtain

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial u_{i,t}}\delta u_{i,t}\, dt = -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_{i,t}}\right)\delta u_{i,t}\, dt. \tag{2.4.70}$$

Interchanging $\frac{\partial}{\partial x_j}$ and the δ variations in the integrals involving the spatial derivatives of u_i, it turns out that

$$\iiint_D \frac{\partial \mathcal{L}}{\partial u_{i,j}}\delta u_{i,j}\, dx_j = \iiint_D \frac{\partial \mathcal{L}}{\partial u_{i,j}}\left(\frac{\partial \delta u_i}{\partial x_j}\right) dx_j,$$

which, by integrating by parts, is

$$= \frac{\partial \mathcal{L}}{\partial u_{i,t}}\delta u_i - \iiint_D \frac{d}{dx_j}\left(\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right)\delta u_i\, dx_j. \tag{2.4.71}$$

Since u_i vanishes at t_1 and t_2, the integrated term also vanishes. Using (2.4.69)–(2.4.71) in (2.4.68) gives

$$0 = \delta I(\mathbf{u}) = \delta \int_{t_1}^{t_2} dt \iiint_D \delta u_i \left[\frac{\partial \mathcal{L}}{\partial u_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_{i,t}}\right) - \frac{d}{dx_j}\left(\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right)\right] dx_j. \tag{2.4.72}$$

This is true only if the coefficients of each of the linearly independent displacements δu_i vanish. Consequently, (2.4.72) leads to the Euler–Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial u_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_{i,t}}\right) - \frac{d}{dx_j}\left(\frac{\partial \mathcal{L}}{\partial u_{i,j}}\right) = 0, \tag{2.4.73}$$

where the summation over j is used.

In particular, if the Lagrangian $\mathcal{L}$ is of the form (2.4.66), (2.4.73) gives the non-homogeneous wave equations

$$\mu\nabla^2 u_i - \rho u_{i,tt} = f(x_i, t). \tag{2.4.74}$$

In the case of equilibrium, the Euler–Lagrange equations (2.4.74) reduce to the Poisson equation

$$\mu\nabla^2 u_i = f(x_i, t). \tag{2.4.75}$$

We close this section by adding an important comment. Many equations in applied mathematics and mathematical physics can be derived from the Euler–Lagrange variational principle, the Hamilton principle, or from some appropriate variational principle.

2.5 The Variational Principle for Nonlinear Klein–Gordon Equations

The nonlinear Klein–Gordon equation is

$$u_{tt} - c^2 u_{xx} + V'(u) = 0, \tag{2.5.1}$$

where $V'(u)$ is some nonlinear function of u chosen as the derivative of the potential energy $V(u)$.

The variational principle for equation (2.5.1) is given by

$$\delta \iint L(u, u_t, u_x)\, dt\, dx = 0, \tag{2.5.2}$$

where L is the associated Lagrangian density

$$L(u, u_t, u_x) = \frac{1}{2}\left(u_t^2 - c^2 u_x^2\right) - V(u). \tag{2.5.3}$$

The Euler–Lagrange equation associated with (2.5.2) is

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) = 0, \tag{2.5.4}$$

which can be simplified to obtain the Klein–Gordon equation (2.5.1).

We consider the variational principle

$$\delta \iint L\, dx\, dt = 0, \tag{2.5.5}$$

with the Lagrangian L given by

$$L \equiv \frac{1}{2}\left(u_t^2 - c^2 u_x^2 - d^2 u^2\right) - \gamma u^4, \tag{2.5.6}$$

where γ is a constant. The Euler–Lagrange equation associated with (2.5.5) gives the special case of the Klein–Gordon equation

$$u_{tt} - c^2 u_{xx} + d^2 u + 4\gamma u^3 = 0. \tag{2.5.7}$$

2.6 The Variational Principle for Nonlinear Water Waves

In his pioneering work, Whitham (1965a, 1965b) first developed a general approach to linear and nonlinear dispersive waves using a Lagrangian. It is well known that most of the general ideas about dispersive waves have originated from the classical problems of water waves. So it is important to have a variational principle for water waves. Luke (1967) first explicitly formulated a variational principle for two-dimensional water waves and showed that the basic equations and boundary and free surface conditions can be derived from the Hamilton principle.

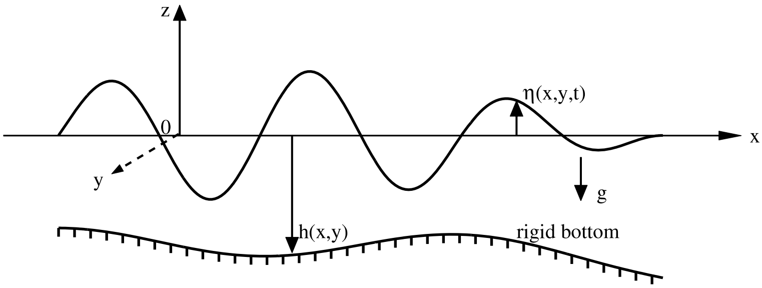

Fig. 2.1 A general surface gravity wave problem.

We now formulate the *variational principle* for three-dimensional water waves in the form

$$\delta I = \delta \iint_D L \, d\mathbf{x} \, dt = 0, \tag{2.6.1}$$

where the *Lagrangian* L is assumed to be equal to the pressure, so that

$$L = -\rho \int_{-h(x,y)}^{\eta(x,t)} \left[\phi_t + \frac{1}{2}(\nabla \phi)^2 + gz \right] dz, \tag{2.6.2}$$

where D is an arbitrary region in the $(\mathbf{x}, t)$ space, ρ is the density of water, g is the gravitational acceleration, and $\phi(\mathbf{x}, z, t)$ is the velocity potential in an unbounded fluid lying between the rigid bottom at $z = -h(x, y)$ and the free surface $z = \eta(x, y, t)$ as shown in Figure 2.1. The functions $\phi(\mathbf{x}, z, t)$ and $\eta(\mathbf{x}, t)$ are allowed to vary subject to the restrictions $\delta\phi = 0$ and $\delta\eta = 0$ at x_1, x_2, y_1, y_2, t_1, and t_2.

Using the standard procedure in the calculus of variations, (2.6.1) becomes

$$\begin{aligned} 0 &= -\delta \iint_D \frac{L}{\rho} \, d\mathbf{x} \, dt \\ &= \iint_D \left\{ \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz \right]_{z=\eta} \delta\eta \right. \\ &\quad \left. + \int_{-h}^{\eta} \left[\phi_x \delta\phi_x + \phi_y \delta\phi_y + \phi_z \delta\phi_z + \delta\phi_t \right] dz \right\} d\mathbf{x} \, dt, \end{aligned} \tag{2.6.3}$$

which, integrating the z-integral by parts, is

$$\begin{aligned} &= \iint_D \left\{ \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz \right]_{z=\eta} \delta\eta \right. \\ &\quad + \left[\frac{\partial}{\partial t} \int_{-h}^{\eta} \delta\phi \, dz + \frac{\partial}{\partial x} \int_{-h}^{\eta} \phi_x \delta\phi \, dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} \phi_y \delta\phi \, dz \right] \\ &\quad - \int_{-h}^{\eta} (\phi_{xx} + \phi_{yy} + \phi_{zz}) \delta\phi \, dz - \left[(\eta_t + \eta_x \phi_x + \eta_y \phi_y - \phi_z) \delta\phi \right]_{z=\eta} \\ &\quad \left. + \left[(\phi_x h_x + \phi_y h_y + \phi_z) \delta\phi \right]_{z=-h} \right\} d\mathbf{x} \, dt. \end{aligned} \tag{2.6.4}$$

The second term within the square brackets integrates out to the boundaries ∂D of D and vanishes if $\delta\phi$ is chosen to be zero on ∂D. If we take $\delta\eta = 0$, $[\delta\phi]_{z=\eta} = [\delta\phi]_{z=-h} = 0$, since $\delta\phi$ is otherwise arbitrary; it turns out that

$$\nabla^2\phi = 0, \quad -\infty < x, y < \infty, \ -h < z < \eta. \tag{2.6.5}$$

Since $\delta\eta$, $[\delta\phi]_{z=\eta}$, $[\delta\phi]_{z=-h}$ may be given arbitrary independent values, it follows that

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0 \quad \text{on } z = \eta, \tag{2.6.6}$$

$$\eta_t + \eta_x\phi_x + \eta_y\phi_y - \phi_z = 0 \quad \text{on } z = \eta, \tag{2.6.7}$$

$$\phi_x h_x + \phi_y h_y + \phi_z = 0 \quad \text{on } z = -h. \tag{2.6.8}$$

Thus, (2.6.5)–(2.6.8) represent the well-known nonlinear system of equations for classical water waves. Finally, this analysis is in perfect agreement with that of Luke (1967) and Whitham (1965a, 1965b, 1974) for two-dimensional waves on water of arbitrary but uniform depth h.

It may be relevant to mention Zakharov's (1968a, 1968b) Hamiltonian formulation. The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{\infty}\left(g\eta^2 + \int_{-h}^{\eta}(\nabla\phi)^2\,dz\right)dx. \tag{2.6.9}$$

On the other hand, Benjamin and Olver (1982) have described the Hamiltonian structure, symmetrics, and conservation laws for water waves. Olver (1984a, 1984b) has discussed Hamiltonian and non-Hamiltonian models for water waves, and Hamiltonian perturbation theory and nonlinear water waves.

2.7 The Euler Equation of Motion and Water Wave Problems

The Euler equation of motion and the equation of continuity have provided the fundamental basis of the study of modern theories of water waves, which are the most common observable phenomena in nature. Water wave motions are of great importance as they range from waves generated by wind or solar heating at the surface of oceans to flood waves in rivers, from waves caused by a moving ship in a channel to tsunami (tidal waves) generated by earthquakes, and from solitary waves on the surface of a channel generated by a disturbance to waves generated by underwater explosions, to mention only a few.

Making reference to Debnath's book (1994), *Nonlinear Water Waves*, the Euler equation of motion in an inviscid and incompressible fluid of constant density ρ under the action of body force $\mathbf{F} = (0, 0, -g)$ where g is the constant acceleration of gravity and the equation of continuity are given by

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{F}, \tag{2.7.1}$$

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.7.2}$$

where $\mathbf{x} = (x, y, z)$ is the rectangular Cartesian coordinates and $\mathbf{u} = (u, v, w)$ is the velocity vector, p is the pressure field, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{2.7.3}$$

These equations constitute a closed system of four *nonlinear* partial differential equations for four unknowns u, v, w, and p. So these equations with appropriate initial and boundary conditions are sufficient to determine the velocity field $\mathbf{u}$ and pressure p uniquely.

In the study of water waves, the body force is always the acceleration due to gravity, that is, $\mathbf{F} = (0, 0, -g)$. It is convenient to write the three components of the Euler equation in the form

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.7.4}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.7.5}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.7.6}$$

and the continuity equation (2.7.2).

In cylindrical polar coordinates $\mathbf{x} = (r, \theta, z)$ with the velocity vector $\mathbf{u} = (u, v, w)$, the Euler equations and the continuity equation are given by

$$\frac{Du}{Dt} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.7.7}$$

$$\frac{Dv}{Dt} + \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{2.7.8}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2.7.9}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \tag{2.7.10}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}. \tag{2.7.11}$$

One of the fundamental properties of a fluid flow is called the *vorticity*, which is defined by the *curl* of the velocity field so that $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u} = \nabla \times \mathbf{u}$. The vorticity vector $\boldsymbol{\omega}$ measures the local spin or rotation of individual fluid particles. Evidently, fluid flows in which $\boldsymbol{\omega} = \mathbf{0}$ are called *irrotational*. In the real world, fluid flows are hardly irrotational anywhere; however, for many flows the vorticity is very small almost everywhere and the fluid motion may be treated as irrotational. In problems of water waves, the motion of fluid is considered unsteady and irrotational which implies that the vorticity $\boldsymbol{\omega} = \text{curl}\,\mathbf{u} = \nabla \times \mathbf{u} = \mathbf{0}$. So there exists a single-valued velocity potential ϕ so that $\mathbf{u} = \nabla\phi$. The continuity equation (2.7.2) then reduces to the Laplace equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{2.7.12}$$

Using the vector identity, $(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla u^2 - \mathbf{u}\times\boldsymbol{\omega}$ combined with $\boldsymbol{\omega} = \mathbf{0}$ and $\mathbf{u} = \nabla\phi$, the Euler equation (2.7.1) may be written in the form

$$\nabla\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz\right] = 0. \tag{2.7.13}$$

This can be integrated with respect to the space variables to obtain the equation

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz = c(t), \tag{2.7.14}$$

where $c(t)$ is an arbitrary function of time only $(\nabla c = 0)$ determined by the pressure imposed at the boundaries of the fluid flow. Since only the pressure gradient affects the flow, a function of t alone added to the pressure field p has virtually no effect on the motion. So, without loss of generality, we can set $c(t) \equiv 0$ in (2.7.14). Consequently, equation (2.7.14) becomes

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz = 0. \tag{2.7.15}$$

This equation is known as *Bernoulli's equation* (or the *pressure equation*), which completely determines the pressure in terms of the velocity potential ϕ. Thus, the Laplace equation (2.7.12) and (2.7.15) are used to determine ϕ and p, and hence the velocity components u, v, w, and the pressure p.

In cylindrical polar coordinates $\mathbf{x} = (r, \theta, z)$ with the velocity field $u = (\phi_r, \frac{1}{r}\phi_\theta, \phi_z)$, the Laplace equation becomes

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\phi_r) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{2.7.16}$$

We assume that the fluid occupies the region $-h \le z \le 0$ with the plane $z = -h$ as the rigid *bottom boundary* and the plane $z = 0$ as the *upper* (*free surface*) *boundary* in the undisturbed state. We suppose that the upper boundary is the surface exposed to a constant atmospheric pressure p_a. Since the free surface is exposed to the constant atmospheric pressure p_a, we have $p = p_a$ on this surface. After the motion is set up, we denote this surface by S with the equation $z = \eta(x, y, t)$ where η is an unknown function of x, y, and t that tends to zero as $t \to 0$. The function $\eta(x, y, t)$ is referred to as the *free surface elevation*.

The rate of change of η, following a fluid particle, is equal to the vertical component of $\nabla\phi$ at the surface, that is,

$$\eta_t + \mathbf{u} \cdot \nabla \eta = \phi_z \quad \text{on } z = \eta.$$

Or equivalently, this free surface condition reads as

$$\eta_t + \phi_x \eta_x + \phi_y \eta_y - \phi_z = 0 \quad \text{on } z = \eta. \tag{2.7.17}$$

This is called the *kinematic free surface condition*.

Since $p = p_a$ on S, after absorbing $\frac{p_a}{\rho}$ and $c(t)$ into ϕ_t, equation (2.7.14) can be rewritten as

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + gz = 0 \quad \text{on } S \text{ for } t \geq 0. \tag{2.7.18}$$

Since S is a free boundary surface, it contains the same fluid particles for all times, that is, S is a material surface. Hence, it follows from (2.7.18) that

$$\frac{D}{Dt}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + gz\right] = 0 \quad \text{on } S \text{ for } t \geq 0.$$

Or equivalently, on S for $t \geq 0$,

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \nabla \phi \cdot \nabla\right)\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + gz\right] \\ &\quad = \phi_{tt} + 2\nabla \phi \cdot \nabla(\phi_t) + \frac{1}{2}\nabla \phi \cdot \nabla(\nabla \phi)^2 + g\phi_z = 0. \end{aligned} \tag{2.7.19}$$

Since the bottom boundary $z = -h$ is a rigid solid surface at rest, the condition to be satisfied at this boundary is

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = -h,\ t \geq 0. \tag{2.7.20}$$

Thus, Laplace's equation (2.7.12) together with the free surface boundary conditions (2.7.17), (2.7.19) and the bottom boundary condition (2.7.20) determine the velocity potential ϕ and the free surface elevation η. Because of the presence of the nonlinear terms in the free surface boundary conditions (2.7.17) and (2.7.19), the determination of ϕ and η in the general case is a difficult task. We restrict our discussion to two particular cases because of the great importance of water wave motions.

Example 2.7.1 (Small Amplitude Water Waves). We consider plane waves propagating in the x-direction whose amplitude varies in the z-direction with the gravitational force as the only body force. We first consider the case where the motion is linear so that nonlinear terms in velocity components may be neglected. In this case, no distinction is made between the initial and the current states of the free surface boundary, and the boundary conditions (2.7.18) and (2.7.19) are given in the linearized forms

$$\phi_t + g\eta = 0 \quad \text{on } z = 0,\ t > 0, \tag{2.7.21}$$

$$\phi_{tt} + g\phi_z = 0 \quad \text{on } z = 0,\ t > 0. \tag{2.7.22}$$

These conditions yield

$$\eta_t = \phi_z \quad \text{on } z = 0,\ t > 0. \tag{2.7.23}$$

For a plane wave propagating in the x-direction with frequency ω and wavenumber k, we seek a solution for $\phi(x, z, t)$ in the form

$$\phi = \Phi(z)\exp\big[i(\omega t - kx)\big], \tag{2.7.24}$$

where $\Phi(z)$ is a function to be determined.

Substituting (2.7.24) in the Laplace equation (2.7.12) with no y dependence gives an equation for Φ as

$$\Phi_{zz} = k^2\Phi. \tag{2.7.25}$$

The general solution of this equation is

$$\Phi(z) = Ae^{kz} + Be^{-kz}, \tag{2.7.26}$$

where A and B are arbitrary constants. Using the boundary condition (2.7.20), we find $A\exp(-kh) = B\exp(kh)$ so that the solution (2.7.26) takes the form

$$\Phi = C\cosh k(z + h), \tag{2.7.27}$$

where $C = 2A\exp(-kh) = 2B\exp(kh)$ is an arbitrary constant so that the solution (2.7.24) becomes

$$\phi = C\cosh k(z + h)\exp\big[i(\omega t - kx)\big]. \tag{2.7.28}$$

Using (2.7.28) in (2.7.21) yields

$$\eta = a\exp\big[i(\omega t - kx)\big], \tag{2.7.29}$$

where $a = (C\omega/ig)\cosh kh = \max|\eta|$ is the *amplitude*. Thus, the solution (2.7.28) assumes the final form

$$\phi = \left(\frac{iag}{\omega}\right)\frac{\cosh k(z + h)}{\cosh kh}\exp\big[i(\omega t - kx)\big]. \tag{2.7.30}$$

Substituting (2.7.30) into (2.7.22) gives the following *dispersion relation* between the frequency and wavenumber:

$$\omega^2 = gk\tanh kh. \tag{2.7.31}$$

Thus, the phase velocity, $c_p = (\frac{\omega}{k})$, can be obtained from (2.7.31) as

$$c_p^2 = \frac{\omega^2}{k^2} = \frac{g}{k}\tanh(kh). \tag{2.7.32}$$

This result shows that the phase velocity c_p depends on the wavenumber k, depth h, and the gravity g. Hence, water waves are dispersive in nature. This means that, as the time passes, the waves would disperse (spread out) into different groups such that each group would consist of waves having approximately the same wavelength. The quantity $\frac{d\omega}{dk}$ represents the velocity of such a group in the direction of propagation and is called the *group velocity*, c_g. It follows from (2.7.31) that

$$c_g = \frac{d\omega}{dk} = \left(\frac{g}{2\omega}\right)\left(\tanh kh + kh \sec h^2 kh\right), \tag{2.7.33}$$

which, by using (2.7.32), is

$$= \frac{1}{2} c_p \left[1 + \frac{2kh}{\sinh 2kh}\right]. \tag{2.7.34}$$

Evidently, the group velocity is different from the phase velocity.

In the case where wavelength $2\pi/k$ is large compared with the depth h, such waves are called *long waves* (or *shallow water waves*), $kh \ll 1$ so that $\tanh kh \approx kh$, and hence, $\sinh 2kh \approx 2kh$. In such a situation, results (2.7.32) and (2.7.34) give

$$c_g = c_p \approx \sqrt{gh} = c. \tag{2.7.35}$$

Thus, shallow water waves are nondispersive and their speed varies as the square root of the depth.

In the other limiting case, where the wavelength is very small compared with the depth, such waves are called *short waves* (or *deep water waves*), $kh \gg 1$. In the limit $kh \to \infty$, $[\cosh k(z+h)/\cosh kh] \to \exp(kz)$, and the corresponding solutions for ϕ and η become

$$\begin{aligned} \phi &= \operatorname{Re}\left(\frac{iag}{\omega}\right) \exp\left[kz + i(\omega t - kx)\right] \\ &= \left(\frac{ag}{\omega}\right) \exp(kz) \sin(kx - \omega t), \end{aligned} \tag{2.7.36}$$

$$\eta = \operatorname{Re} a \exp\left[i(\omega t - kx)\right] = a\cos(\omega t - kx). \tag{2.7.37}$$

In the limit $kh \to \infty$, $\tanh kh \to 1$ so that the dispersion relation becomes

$$\omega^2 = gk. \tag{2.7.38}$$

Consequently,

$$c_p = \left(\frac{g}{k}\right)^{\frac{1}{2}} = \left(\frac{g\lambda}{k\pi}\right)^{\frac{1}{2}}, \tag{2.7.39}$$

$$c_g = \frac{1}{2} c_p. \tag{2.7.40}$$

Evidently, deep water waves are dispersive and the phase velocity is proportional to the square root of their wavelength. Also the group velocity is equal to one-half of the phase velocity.

Example 2.7.2 (The Stokes' Waves or Nonlinear Finite Amplitude Waves). We consider the Stokes' waves where the motion is nonlinear and the amplitude of the waves is *not* small. We recall Bernoulli's equation (2.7.18) and (2.7.19) and write them for ready reference in the form

$$\eta = -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]_{z=\eta}, \tag{2.7.41}$$

$$[\phi_{tt} + g\phi_z]_{z=\eta} + 2[\nabla\phi\cdot\nabla\phi_t]_{z=\eta} + \frac{1}{2}\left[\nabla\phi\cdot\nabla(\nabla\phi)^2\right]_{z=\eta} = 0. \tag{2.7.42}$$

A systematic procedure can be employed to rewrite these boundary conditions by using Taylor's series expansions of the potential ϕ and its derivatives in the form

$$\phi(x,y,z=\eta,t) = [\phi]_{z=0} + \eta[\phi_z]_{z=0} + \frac{1}{2}\eta^2[\phi_{zz}]_{z=0} + \cdots, \tag{2.7.43}$$

$$\phi_z(x,y,z=\eta,t) = [\phi_z]_{z=0} + \eta[\phi_{zz}]_{z=0} + \frac{1}{2}\eta^2[\phi_{zzz}]_{z=0} + \cdots. \tag{2.7.44}$$

Substituting these and similar Taylor's expansions into (2.7.41) gives

$$\begin{aligned}\eta &= -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]_{z=0} + \eta\left[-\frac{1}{g}\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2\right\}_z\right]_{z=0} + \cdots\\ &= -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right]_{z=0}\\ &\quad + \frac{1}{g^2}\left[\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2\right\}\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2\right\}_z\right]_{z=0} + \cdots\\ &= -\frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right]_{z=0} + O\big(\phi^3\big).\end{aligned} \tag{2.7.45}$$

Similarly, condition (2.7.42) gives

$$\begin{aligned}&[\phi_{tt} + g\phi_z]_{z=0} + \eta\big[(\phi_{tt} + g\phi_z)_z\big] + \frac{1}{2}\eta^2\big[(\phi_{tt} + g\phi_z)_{zz}\big]_{z=0}\\ &\quad + \cdots + 2[\nabla\phi\cdot\nabla\phi_t]_{z=0} + 2\eta\big[\{\nabla\phi\cdot\nabla\phi_t\}_z\big]_{z=0} + \eta^2\big[\{\nabla\phi\cdot\nabla\phi_t\}_{zz}\big]_{z=0}\\ &\quad + \cdots + \frac{1}{2}\big[\{\nabla\phi\cdot\nabla(\nabla\phi)^2\}\big]_{z=0} + \frac{1}{2}\eta\big[\{\nabla\phi\cdot\nabla(\nabla\phi)^2\}_z\big]_{z=0}\\ &\quad + \frac{1}{4}\eta^2\big[\{\nabla\phi\cdot\nabla(\nabla\phi)^2\}_{zz}\big]_{z=0} + \cdots = 0.\end{aligned} \tag{2.7.46}$$

We substitute (2.7.45) for η into (2.7.46) to obtain

$$\begin{aligned}&[\phi_{tt} + g\phi_z]_{z=0} - \frac{1}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right]_{z=0}\big[(\phi_{tt} + g\phi_z)_z\big]_{z=0}\\ &\quad + \frac{1}{2g^2}\left[\left\{\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right\}^2\right]_{z=0}\big[(\phi_{tt} + g\phi_z)_{zz}\big]_{z=0}\end{aligned}$$

$$+ 2\big[(\nabla\phi)\cdot\nabla\phi_t\big]_{z=0} - \frac{2}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right]_{z=0}\big[(\nabla\phi\cdot\nabla\phi_t)_z\big]_{z=0}$$
$$+ \frac{1}{2}\big[\nabla\phi\cdot\nabla(\nabla\phi)^2\big]_{z=0} - \frac{2}{g}\left[\phi_t + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{g}\phi_t\phi_{zt}\right]_{z=0}$$
$$\times\big[\big\{\nabla\phi\cdot\nabla(\nabla\phi)^2\big\}_z\big]_{z=0} = 0. \tag{2.7.47}$$

The first-, second-, and third-order boundary conditions on $z = 0$ are respectively given by

$$(\phi_{tt} + g\phi_z) = 0 + O\big(\phi^2\big), \tag{2.7.48}$$

$$(\phi_{tt} + g\phi_z) + 2[\nabla\phi\cdot\nabla\phi_t] - \frac{1}{g}\phi_t(\phi_{tt} + g\phi_z)_z = 0 + O\big(\phi^3\big), \tag{2.7.49}$$

$$(\phi_{tt} + g\phi_z) + 2[\nabla\phi\cdot\nabla\phi_t] + \frac{1}{2}\big[\nabla\phi\cdot\nabla(\nabla\phi)^2\big]$$
$$- \frac{1}{g}\phi_t\big[\phi_{tt} + g\phi_z + 2(\nabla\phi\cdot\nabla\phi_t)\big]_z - \frac{1}{g}\left[\frac{1}{2}\big(\nabla\phi^2\big) - \frac{1}{g}\phi_t\phi_{zt}\right][\phi_{tt} + g\phi_z]_z$$
$$+ \frac{1}{2g^2}[\phi_t]^2\big[(\phi_{tt} + g\phi_z)_{zz}\big] = 0 + O\big(\phi^4\big), \tag{2.7.50}$$

where $O\,(\cdot)$ indicates the order of magnitude of the neglected terms. These results can be determined by the third-order expansion of plane progressive surface waves.

As indicated before, the first-order plane wave potential ϕ in deep water is given by (2.7.36). Direct substitution of the first-order velocity potential (2.7.36) in the second-order boundary condition (2.7.49) reveals that the second-order terms in (2.7.49) vanish. Thus, the first-order potential is a solution of the second-order boundary-value problem, and we can state that

$$\phi = \left(\frac{ga}{\omega}\right)e^{kz}\sin(kx - \omega t) + O\big(a^3\big). \tag{2.7.51}$$

Substitution of this result into (2.7.45) leads to the second-order result for η in the form

$$\eta = a\cos(kx - \omega t) - \frac{1}{2}ka^2 + ka^2\cos^2(kx - \omega t) + \cdots$$
$$= a\cos(kx - \omega t) + \frac{1}{2}ka^2\cos\big\{2(kx - \omega t)\big\} + \cdots. \tag{2.7.52}$$

The second term in (2.7.52), which represents the second-order correction to the surface profile, is positive at the *crests* $kx - \omega t = 0, 2\pi, 4\pi, \ldots$, and negative at the *troughs* $kx - \omega t = \pi, 3\pi, 5\pi, \ldots$. But the crests are steeper, and the troughs flatter as a result of the nonlinear effect. The notable feature of solution (2.7.52) is that the wave profile is no longer sinusoidal. The actual shape of the wave profile is a curve known as a *trochoid* (see Figure 2.2), whose crests are steeper and troughs are flatter than those of the sinusoidal wave.

Substituting the wave potential (2.7.51) in the third-order boundary condition (2.7.50) reveals that all nonlinear terms vanish identically except one term, $(\frac{1}{2})\nabla\phi\,\cdot$

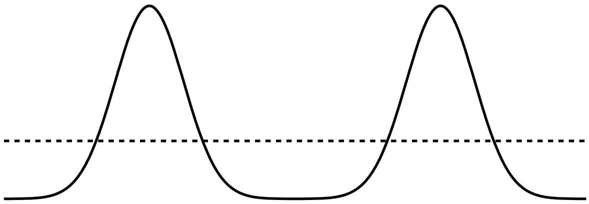

Fig. 2.2 The surface wave profile.

$\nabla(\nabla\phi)^2$. Thus the boundary condition for the third-order plane wave solution is given by

$$\phi_{tt} + g\phi_z + \frac{1}{2}\nabla\phi \cdot \nabla(\nabla\phi)^2 = 0 + O\big(\phi^4\big). \tag{2.7.53}$$

If the first-order solution (2.7.51) is substituted into the third-order boundary condition on $z = 0$, the dispersion relation with second-order effect is obtained in the form

$$\omega^2 = gk\big(1 + a^2k^2\big) + O\big(k^3a^3\big). \tag{2.7.54}$$

Note that this relation involves the amplitude in addition to frequency and wavenumber. This is called the *nonlinear dispersion relation* and it can be expressed in terms of the phase velocity as

$$c_p = \frac{\omega}{k} = \left(\frac{g}{k}\right)^{\frac{1}{2}} \big(1 + k^2a^2\big)^{\frac{1}{2}} \approx \left(\frac{g}{k}\right)^{\frac{1}{2}} \left(1 + \frac{1}{2}a^2k^2\right). \tag{2.7.55}$$

Thus the phase velocity depends on the wave amplitude, and waves of large amplitude travel faster than smaller ones. The dependence of c_p on amplitude is known as the *amplitude dispersion* in contrast to the *frequency dispersion* as given by (2.7.38).

It may be noted that Stokes' results (2.7.52), (2.7.54), and (2.7.55) can easily be approximated further to obtain solutions for long waves (or shallow water) and for short waves (or deep water).

We conclude this example by discussing the phenomenon of breaking of water waves, which is one of the most common observable phenomena on an ocean beach. A wave coming from deep ocean changes shape as it moves across a shallow beach. Its amplitude and wavelength also are modified. The wavetrain is very smooth some distance offshore, but as it moves inshore, the front of the wave steepens noticeably until, finally, it breaks. After breaking, waves continue to move inshore as a series of bores or hydraulic jumps, whose energy is gradually dissipated by means of the water turbulence. Of the phenomena common to waves on beaches, breaking is the physically most significant and mathematically least known. In fact, it is one of the most intriguing longstanding problems of water waves theory.

For waves of small amplitude in deep water, the maximum particle velocity is $v = a\omega = ack$. But the basic assumption of small amplitude theory implies that $\frac{v}{c} = ak \ll 1$. Therefore, wave breaking can never be predicted by the small amplitude wave theory. That possibility arises only in the theory of finite amplitude waves. It is

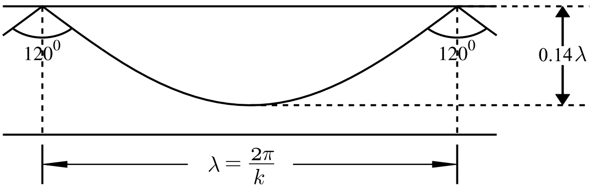

Fig. 2.3 The steepest wave profile.

to be noted that the Stokes' expansions are limited to relatively small amplitude and cannot predict the wavetrain of maximum height at which the crests are found to be very sharp. For a wave profile of constant shape moving at a uniform velocity, it can be shown that the maximum total crest angle as the wave begins to break is 120°; see Figure 2.3.

The upshot of the Stokes' analysis reveals that the inclusion of higher-order terms in the representation of the surface wave profile distorts its shape away from the linear sinusoidal curve. The effects of nonlinearity are likely to make crests narrower (sharper) and the troughs flatter as depicted in Figure 2.7 of Debnath (1994). The resulting wave profile more accurately portrays the water waves that are observed in nature. Finally, the sharp crest angle of 120° was first found by Stokes.

On the other hand, in 1865, Rankine conjectured that there exists a wave of extreme height. In a moving reference frame, the Euler equations are Galilean invariant, and the Bernoulli equation (2.7.14) on the free surface of water with $\rho = 1$ becomes

$$\frac{1}{2}|\nabla\phi|^2 + gz = E.$$

Thus, this equation represents the conservation of local energy, where the first term is the kinetic energy of the fluid and the second term is the potential energy due to gravity. For the wave of maximum height, $E = gz_{\text{max}}$, where z_{max} is the maximum height of the fluid. Thus, the velocity is zero at the maximum height so that there will be a stagnation point in the fluid flow. Rankine conjectured that a cusp is developed at the peak of the free surface with a vertical slope so that the angle subtended at the peak is 120° as also conjectured by Stokes (1847). Toland (1978) and Amick et al. (1982) have proved rigorously the existence of a wave of greatest height and the Stokes' conjecture for the wave of extreme form. However, Toland (1978) also proved that if the singularity at the peak is *not* a cusp, that is, if there is no vertical slope at the peak of the free surface, then the Stokes' remarkable conjecture of the crest angle of 120° is true. Subsequently, Amick et al. (1982) confirmed that the singularity at the peak is *not* a cusp. Therefore, the full Euler equations exhibit singularities, and there is a limiting amplitude to the periodic waves.

We next formulate the modern mathematical theory of nonlinear water waves. It is convenient to take the free surface elevation above the undisturbed mean depth h as $z = \eta(x, y, t)$ so that the free surface of water is at $z = H = h + \eta$ and the horizontal rigid bottom is at $z = 0$ where the z-axis is vertical positive upwards.

It is also convenient to introduce nondimensional flow variables based on a typical horizontal length scale ℓ (which may be wavelength λ), typical vertical length scale h, typical horizontal velocity scale, $c = \sqrt{gh}$ (shallow water wave speed), typical time scale $(\frac{c}{\ell})$, typical vertical velocity scale $(\frac{hc}{\ell})$, and the typical pressure scale ρc^2. Using asterisks to denote nondimensional flow variables, we write

$$(x, y) = \ell(x^*, y^*), \qquad z = hz^*, \qquad t = \left(\frac{\ell}{c}\right)t^*, \tag{2.7.56}$$

$$(u, v) = c(u^*, v^*), \qquad w = \left(\frac{hc}{\ell}\right)w^*, \quad \text{and} \quad p = \rho c^2 p^*. \tag{2.7.57}$$

We next introduce two fundamental parameters δ and ε defined by

$$\delta = \frac{h^2}{\ell^2} \quad \text{and} \quad \varepsilon = \frac{a}{h}, \tag{2.7.58}$$

where δ is called the *long wavelength* (or *shallowness*) *parameter*, ε is called the *amplitude parameter*, and a is the typical wave amplitude. These two parameters play a crucial role in the modern theory of water waves.

In terms of the amplitude parameter, the free surface at $z = H = h + \eta$ and the bottom boundary surface at $z = 0$ of the fluid can be written as the nondimensional form

$$z = 1 + \varepsilon\eta \quad \text{and} \quad z = 0, \tag{2.7.59}$$

where $(\frac{\eta}{a})$ is replaced by the nondimensional value η.

The variable pressure field P representing the deviation from the hydrostatic pressure $g\rho(h - z)$ is given by

$$p = p_a + g\rho(z - h) + g\rho h P, \tag{2.7.60}$$

where p_a is the constant atmospheric pressure and the scale $g\rho h$ of pressure is based on the pressure at the depth $z = h$.

In terms of the nondimensional variables, the Euler equations (2.7.4)–(2.7.6) and the continuity equation (2.7.2) can be written in the form, dropping the asterisks and replacing P by p,

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \qquad \delta\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \tag{2.7.61}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.7.62}$$

The kinematic free surface and the dynamic free surface conditions (see Debnath 1994, pp. 6–7) are expressed in the nondimensional form, dropping the asterisks,

$$w = \varepsilon(\eta_t + u\eta_x + v\eta_y) \quad \text{on } z = 1 + \varepsilon\eta, \tag{2.7.63}$$

$$p = \varepsilon\eta \quad \text{on } z = 1 + \varepsilon\eta. \tag{2.7.64}$$

The bottom boundary condition is

$$w = 0 \quad \text{on } z = 0. \tag{2.7.65}$$

It follows from the free surface conditions that both w and p are proportional to the amplitude parameter ε. In the limit as $\varepsilon \to 0$, both w and p tend to zero, indicating that there is no disturbance at the free surface.

Consistent with the governing equations and the boundary conditions, we introduce a set of scaled flow variables

$$(u, v, w, p) \to \varepsilon(u, v, w, p) \tag{2.7.66}$$

so that the governing equations (2.7.61) and (2.7.62) and the boundary conditions (2.7.63)–(2.7.65) become

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \qquad \delta\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \tag{2.7.67}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.7.68}$$

$$w = \eta_t + \varepsilon(u\eta_x + v\eta_y), \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta, \tag{2.7.69}$$

$$w = 0 \quad \text{on } z = 0, \tag{2.7.70}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right), \tag{2.7.71}$$

and parameters δ and ε are given by (2.7.58).

In general, there are two most commonly adopted and useful approximations: (i) $\varepsilon \to 0$, that is, small amplitude water waves governed by the linearized equations, and (ii) $\delta \to 0$, that is, shallow water wave equations (or long water waves). These approximate models and their solutions constitute the classical theory of water waves (see Debnath 1994).

In the first case, the linearized equations of water waves are obtained from (2.7.67)–(2.7.71) in the limit as $\varepsilon \to 0$ in the form

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \qquad \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}, \qquad \delta\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z}, \tag{2.7.72}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.7.73}$$

$$w = \eta_t \quad \text{and} \quad p = \eta \quad \text{on } z = 1, \tag{2.7.74}$$

$$w = 0 \quad \text{on } z = 0. \tag{2.7.75}$$

In the second case, the shallow water equations (long water waves) are described in the sense that $\sqrt{\delta} = (\frac{h}{\ell})$ is small so that $\delta \to 0$ with fixed amplitude parameter ε. Consequently, the governing equations and the boundary conditions are obtained from (2.7.67)–(2.7.71) in the limit as $\delta \to 0$ in the form

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \qquad \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \qquad \frac{\partial p}{\partial z} = 0, \tag{2.7.76}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{2.7.77}$$

$$w=\eta_t+\varepsilon(u\eta_x+v\eta_y) \quad \text{and} \quad p=\eta \quad \text{on } z=1+\varepsilon\eta, \tag{2.7.78}$$

$$w=0 \quad \text{on } z=0, \tag{2.7.79}$$

where $\frac{D}{Dt}$ is given by (2.7.71).

Finally, equations that describe small amplitude waves $\varepsilon \to 0$ and long waves $(\delta \to 0)$ are obviously consistent with (2.7.72)–(2.7.75) for the first case, and also with equations (2.7.76)–(2.7.79) with

$$\frac{\partial p}{\partial z}=0; \quad p=\eta \quad \text{on } z=1 \tag{2.7.80}$$

or (2.7.76)–(2.7.79) with $\varepsilon \to 0$.

The solutions of these various approximate governing equations describe the classical water waves (see Debnath 1994).

Example 2.7.3 (Solution of a Linearized Water Wave Problem). We consider the propagation of a plane harmonic water wave in the x-direction in a fluid of constant depth. With no y-dependence, the governing equations and the boundary conditions are obtained from (2.7.72)–(2.7.75) in the form

$$u_t=-p_x, \qquad \delta w_t=-p_z,\ u_x+w_z=0, \tag{2.7.81}$$

$$w=\eta_t \quad \text{and} \quad p=\eta \quad \text{on } z=1, \tag{2.7.82}$$

$$w=0 \quad \text{on } z=0. \tag{2.7.83}$$

We assume a plane wave solution in the form

$$(u,w,p)=\big[u^*(z),w^*(z),p^*(z)\big]\exp\big[i(\omega t-kx)\big]. \tag{2.7.84}$$

The free surface elevation is given by

$$\eta(x,t)=a\exp\big[i(\omega t-kx)\big]+c.c., \tag{2.7.85}$$

where a is a constant wave amplitude and $c.c.$ denotes the complex conjugate. Obviously, (2.7.85) represents a plane harmonic wave whose initial form at $t=0$ is given by $\eta(x,0)$.

Substituting the solution (2.7.84) into (2.7.81) gives, dropping asterisks,

$$u=\frac{k}{\omega}p, \qquad p'=-i\omega\delta w, \qquad w'=iku, \tag{2.7.86}$$

where the prime denotes the derivative with respect to z.

It readily follows from (2.7.86) that

$$w''=iku'=\frac{ik^2}{\omega}p'=\big(\delta k^2\big)w. \tag{2.7.87}$$

Thus, the general solution of (2.7.87) is

$$w = A\exp\big(\sqrt{\delta}kz\big) + B\exp\big(-\sqrt{\delta}kz\big), \tag{2.7.88}$$

where A and B are arbitrary constants to be determined from (2.7.82), (2.7.83) which give

$$w(1) = i\omega a, \qquad p(1) = a, \qquad w(0) = 0. \tag{2.7.89}$$

Consequently, the solution (2.7.88) assumes the final form

$$w(z) = \text{Re}(i\omega a)\frac{\sinh(\sqrt{\delta}kz)}{\sinh(\sqrt{\delta}k)}. \tag{2.7.90}$$

It follows from the boundary conditions that

$$a = p(1) = \frac{\omega}{k}u(1) = \frac{\omega}{ik^2}w'(1) = \left(\frac{a\omega^2}{k}\right)\sqrt{\delta}\coth\big(k\sqrt{\delta}\big). \tag{2.7.91}$$

This leads to the dispersion relation

$$\omega^2 = \left(\frac{k}{\sqrt{\delta}}\right)\tanh\big(k\sqrt{\delta}\big), \tag{2.7.92}$$

where $\sqrt{\delta} = h\lambda^{-1}$ which is equal to dimensional (physical) quantity $(\frac{h}{\ell})$ and $k\sqrt{\delta}$ is equal to dimensional quantity kh.

As before, the dispersion relation determines the frequency $\omega = \omega(k)$ and the phase velocity

$$c_p^2 = \left(\frac{\omega}{k}\right)^2 = \big(k\sqrt{\delta}\big)^{-1}\tanh\big(k\sqrt{\delta}\big). \tag{2.7.93}$$

The group velocity c_g is given by

$$c_g = \frac{d\omega}{dk} = \frac{1}{2\omega\sqrt{\delta}}\left[\tanh\big(k\sqrt{\delta}\big) + k\sqrt{\delta}\sec h^2\big(k\sqrt{\delta}\big)\right]$$

which, by (2.7.93), is

$$= \frac{1}{2}c_p\left[1 + \frac{2k\sqrt{\delta}}{\sinh(2k\sqrt{\delta})}\right]. \tag{2.7.94}$$

In the case of shallow water waves, $k\sqrt{\delta} \to 0$ so that $\tanh(k\sqrt{\delta}) \approx k\sqrt{\delta}$. Hence, results (2.7.93), (2.7.94) lead to $c_p = c_g = 1$. Both the phase and group velocities are independent of the wavelength. So, the shallow water waves are nondispersive. In terms of the dimensional variables, the phase velocity is

$$c_p = \pm c = \pm\sqrt{gh}. \tag{2.7.95}$$

This confirms the choice of the velocity scale c adopted before.

For deep water waves, $k\sqrt{\delta} \to \infty$ so that $\tanh(k\sqrt{\delta}) \to 1$. Consequently,

$$\omega^2 = \frac{k}{\sqrt{\delta}}, \qquad c_p^2 = \big(k\sqrt{\delta}\big)^{-1}, \quad \text{and} \quad c_g = \frac{1}{2}c_p. \tag{2.7.96}$$

Example 2.7.4 (Small Amplitude Gravity-Capillary Surface Waves on Water of depth h). The governing equation for the two-dimensional linearized gravity-capillary surface waves on water of constant depth h with the free surface at $z = 0$ are given by

$$\nabla^2 \phi = \phi_{xx} + \phi_{zz} = 0, \quad -h \le z < 0,\ t > 0, \tag{2.7.97}$$

where $\phi(x, z, t)$ is the velocity potential.

Representing the free surface elevation function by $\eta = \eta(x, t)$, the linearized kinematic and dynamic free surface conditions are

$$\eta_t - \phi_z = 0 \quad \text{on } z = 0,\ t > 0, \tag{2.7.98}$$

$$\phi_t + g\eta - \frac{T}{\rho}\eta_{xx} = 0 \quad \text{on } z = 0,\ t > 0, \tag{2.7.99}$$

where g is the acceleration of gravity and T is the surface tension, and ρ is the constant density of water.

The boundary condition at the horizontal rigid bottom at $z = -h$

$$\phi_z = 0 \quad \text{at } z = -h. \tag{2.7.100}$$

We seek the same solution (2.7.28) for $\phi(z, x, t)$ and (2.7.29) for $\eta(x, t)$ so that $\phi(x, z, t)$ assumes the same form (2.7.30). In view of (2.7.98), (2.7.99) reduces to the form

$$\phi_{tt} + g\phi_z - \frac{T}{\rho}\phi_{zxx} = 0 \quad \text{on } z = 0,\ t > 0. \tag{2.7.101}$$

Substituting (2.7.30) into (2.7.101) gives the *dispersion relation for the gravity-capillary waves* in the form

$$\omega^2 = gk\left(1 + \frac{Tk^2}{\rho g}\right)\tanh kh. \tag{2.7.102}$$

Or equivalently, this gives the phase velocity c_p of the surface gravity-capillary waves on water of finite depth h

$$c_p^2 = \frac{\omega^2}{k^2} = \left(\frac{g}{k} + \frac{T}{\rho}k\right)\tanh kh. \tag{2.7.103}$$

It can easily be recognized that result (2.7.102) or (2.7.103) is *exactly the same* as result (2.7.31) or (2.7.32) with g replaced by $(g + \rho^{-1}Tk^2)$. This means that all the properties of gravity-capillary waves can be described correctly when this replacement is made in the results of pure gravity waves.

Introducing the *Froude number* F and the *Bond number* τ by

$$F = \frac{c_p}{\sqrt{gh}} \quad \text{and} \quad \tau = \frac{T}{g\rho h^2}, \tag{2.7.104}$$

the dimensionless form of the dispersion relation (2.7.103) is

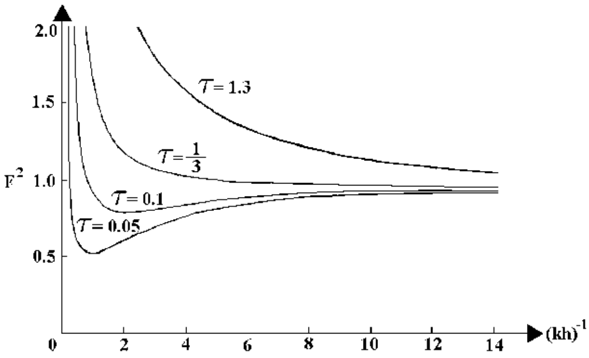

Fig. 2.4 The square of the Froude number, F^2, against $(kh)^{-1}$.

$$F^2 = \left(\frac{1}{kh} + \tau kh\right) \tanh kh. \tag{2.7.105}$$

The square of the Froude number is plotted against $(kh)^{-1} = (\frac{\lambda}{2\pi h})$ for four values of the Bond number $\tau = 1.3, \frac{1}{3}, 0.1$, and 0.05 in Figure 2.4. This figure shows that F^2 decreases monotonically with $(kh)^{-1}$ when $\tau > \frac{1}{3}$, and it has a minimum for $\tau < \frac{1}{3}$. As $kh = 2\pi(\frac{h}{\lambda}) \to 0$ (or $\frac{\lambda}{h} \to \infty$), $F \to 1$.

In case of water of infinite depth ($kh \to \infty, \tanh kh \to 1$), (2.7.102) or (2.7.103) leads to

$$\omega^2 = gk\left(1 + \frac{Tk^2}{g\rho}\right) \quad \text{or} \quad c_p^2 = \left(\frac{g}{k} + \frac{Tk}{\rho}\right). \tag{2.7.106}$$

Thus, for pure surface gravity waves ($T = 0,\ g \neq 0$) in deep water, (2.7.106) reduces to (2.7.38). Similarly, for pure surface capillary waves ($g = 0,\ T \neq 0$), the dispersion relation is

$$\omega^2 = \frac{Tk^3}{\rho} \quad \text{or} \quad c_p^2 = \frac{Tk}{\rho}. \tag{2.7.107}$$

It is convenient to write (2.7.106) as

$$\omega^2 = gk(1 + T^*) \quad \text{or} \quad c_p^2 = \frac{g}{k}(1 + T^*), \tag{2.7.108}$$

where the parameter $T^* = (Tk^2/g\rho)$ represents the relative importance of surface tension and gravity.

It also follows from (2.7.106) that the phase velocity c_p has a minimum value at $k = k_m = \sqrt{g\rho/T}$ (or $T^* = 1$) with the corresponding minimum value for c_p is

$$(c_p)_m = \left(\frac{4Tg}{\rho}\right)^{\frac{1}{4}} \tag{2.7.109}$$

at the wavelength $\lambda = \lambda_m = 2\pi(T/g\rho)^{\frac{1}{2}}$.

The inequality $k \ll k_m$ is the condition for the waves to be effectively pure gravity waves with negligible surface tension. This is equivalent to large wavelength

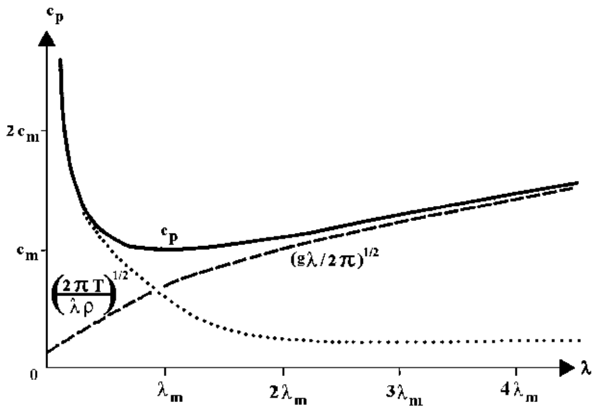

Fig. 2.5 The solid curve represents the phase velocity c_p for capillary-gravity waves against λ. (From Lighthill 1978.)

$\lambda > \lambda_m = (\frac{2\pi}{k_m}) = 2\pi(\frac{T}{\rho g})^{\frac{1}{2}}$. The phase velocity c_p in (2.7.106) for gravity-capillary waves in deep water is shown by the solid curve in Figure 2.5 against λ with minimum $(c_p)_m$ attained at $\lambda = \lambda_m$. The dotted curve corresponds to (2.7.107) for pure capillary waves (or *ripples*) in deep water dominated by surface tension for small $\lambda < \lambda_m$. The dashed curve corresponds to $c_p = (g/k)^{\frac{1}{2}}$ for pure gravity waves for large wavelengths $\lambda > \lambda_m$.

The group velocity for gravity-capillary waves can be calculated from the dispersion relation (2.7.102) and is given by

$$c_g = \frac{g}{2\omega}\left[\left(1 + \frac{3Tk^2}{g\rho}\right)\tanh kh + \left(1 + \frac{Tk^2}{g\rho}\right)kh\,\text{sech}^2\,kh\right] \tag{2.7.110}$$

$$= \frac{g}{2\omega}\big[(1 + 3T^*)\tanh kh + (1 + T^*)kh\,\text{sech}^2\,kh\big]. \tag{2.7.111}$$

We next multiply the numerator by kc_p and the denominator by $\omega(= kc_p)$, then replace ω^2 by (2.7.106) to obtain

$$c_g = \frac{1}{2}c_p\left(\frac{1 + 3T^*}{1 + T^*} + \frac{2kh}{\sinh 2kh}\right). \tag{2.7.112}$$

In the deep water limit, $kh \to \infty$, the second term in (2.7.112) tends to zero, and hence, the group velocity of gravity-capillary waves is

$$c_g = \frac{1}{2}c_p\left(\frac{1 + 3T^*}{1 + T^*}\right). \tag{2.7.113}$$

This reduces to the result (2.7.96) for pure gravity waves $(T^* = 0)$ in deep water

$$c_g = \frac{1}{2}c_p = \frac{1}{2}\sqrt{\frac{g}{k}}, \tag{2.7.114}$$

and for pure capillary waves in deep water ($g \to 0$ and $T^* \to \infty$), the group velocity is

$$c_g = \frac{3}{2}c_p = \frac{3}{2}\left(\frac{Tk}{\rho}\right)^{\frac{1}{2}}. \tag{2.7.115}$$

It follows from the definition of group velocity (2.7.33) that the group and phase velocities are related by a simple formula

$$c_g = \frac{d\omega}{dk} = \frac{d}{dk}(kc_p) = c_p + k\frac{dc_p}{dk} = c_p - \lambda\frac{dc_p}{d\lambda}. \tag{2.7.116}$$

It is clear from (2.7.116) that $c_g \neq c_p$. However, if c_p does not depend on the wavelength, λ (or wavenumber, k), then $c_g = c_p$. If $\frac{dc_p}{dk} > 0$, then $c_g > c_p$, and if $\frac{dc_p}{dk} < 0$, then $c_g < c_p$. If c_p is minimum for some k, then $\frac{dc_p}{dk} = 0$, and hence, $c_g = c_p$. For shallow water waves, $\omega^2 = (gh)k^2$ or $c_p = \sqrt{gh}$, and then $c_g = c_p$.

In case of gravity-capillary waves in deep water, c_p has a minimum for $k = k_{\min} = \sqrt{g\rho/T}$, and hence, $c_g = (c_p)_m$. Figure 2.5 reveals that on the left of the minimum, $\frac{dc_p}{dk} > 0$, and hence, result (2.7.116) confirms that $c_g > c_p$, whereas on the right of the minimum, $\frac{dc_p}{dk} < 0$, and hence, $c_g < c_p$.

Finally, formula (2.7.116) can also be written in the form

$$c_g = c_p\left(1 - k\frac{dc_p}{d\omega}\right)^{-1} = c_p\left(1 - \frac{\omega}{c_p}\frac{dc_p}{d\omega}\right)^{-1}. \tag{2.7.117}$$

This is known as the *Rayleigh formula* for one-dimensional dispersive waves. The general theory of dispersive waves was developed by Whitham in 1960s that will be discussed in Chapter 7.

Example 2.7.5 (Total Energy of Pure Gravity Waves on Water of Constant Depth). We calculate the potential energy and the kinetic energy of pure gravity waves on water constant depth h. The potential energy over a single wavelength λ is given by

$$V = \frac{g\rho}{2}\int_0^\lambda \eta^2\, dx = \frac{1}{4}g\rho a^2, \tag{2.7.118}$$

where the free surface elevation η given by (2.7.37) is used to obtain the above value V.

The kinetic energy T is given by

$$T = \frac{1}{2}\rho\int_0^\lambda dx\int_{-h}^{n}\left(\phi_x^2 + \phi_z^2\right) dz$$

which can be transformed into

$$T = -\frac{1}{2}\rho\int \phi\frac{\partial\phi}{\partial n}\, dS = \frac{1}{2}\rho\int_0^\lambda \left(\phi\frac{\partial\phi}{\partial z}\right)_{z=0} dx$$

which is, by using (2.7.30),

$$= \frac{1}{2}\rho g a^2 \int_0^\lambda \sin^2(kx - \omega t)\, dx = \frac{1}{4} g\rho a^2 \lambda. \tag{2.7.119}$$

Hence, the total energy per unit wavelength is

$$E = T + V = \frac{1}{2} g\rho a^2 \lambda. \tag{2.7.120}$$

Thus, the total energy is half kinetic and half potential.

The horizontal and vertical velocity components of water particles are

$$u = \phi_x = \left(\frac{agk}{\omega}\right) \frac{\cosh k(z+h)}{\cosh kh} \exp\bigl[i(\omega t - kx)\bigr], \tag{2.7.121}$$

$$v = \phi_z = i\left(\frac{agk}{\omega}\right) \frac{\sinh k(z+h)}{\cosh kh} \exp\bigl[i(\omega t - kx)\bigr]. \tag{2.7.122}$$

So, it is possible to determine the actual path of a fluid particle in motion from (2.7.121)–(2.7.122). In terms of particle displacements X and Z of a particle whose mean motion is (x, z), we get $\dot{X} = u$ and $\dot{Z} = v$ in which terms of the second order are neglected. So integration gives

$$X = \left(\frac{agk}{\omega^2}\right) \frac{\cosh k(z+h)}{\cosh kh} \sin(\omega t - kx) + X_0, \tag{2.7.123}$$

$$Z = \left(\frac{agk}{\omega^2}\right) \frac{\sinh k(z+h)}{\cosh kh} \cos(\omega t - kx) + Z_0, \tag{2.7.124}$$

where X_0 and Z_0 are constants of integration, and they move the origin of X and Z. Eliminating $(\omega t - kx)$ from (2.7.123)–(2.7.124) gives the equation of a particle path as

$$\frac{(X - X_0)^2}{\cosh^2 k(z+h)} + \frac{(Z - Z_0)^2}{\sinh^2 k(z+h)} = \frac{a^2}{\sinh^2 kh}. \tag{2.7.125}$$

This represents an ellipse with the semi-major axis in the x-direction of magnitude $a \operatorname{cosech} kh \cosh k(z+h)$ and with semi-minor axis in the z-direction of magnitude $a \operatorname{cosech} kh \cosh k(z+h)$. Both semi-axes decrease with depth. When $X_0 = Z_0 = 0$ and $z = -h$, $X \neq 0$, $Z = 0$, and particles oscillate along the bottom. But for a real liquid, viscosity would prevent such oscillations.

In deep water $(kh \to \infty)$, both $\cosh k(z+h)/\cosh kh$ and $\cosh k(z+h)/\sinh kh$ tend to $\exp(kz)$; hence, (2.7.123)–(2.7.124) give

$$X - X_0 = ae^{kz} \sin(\omega t - kx), \tag{2.7.126}$$

$$Z - Z_0 = ae^{kz} \cos(\omega t - kx). \tag{2.7.127}$$

These results show that the paths of the fluid particles are circles of radius ae^{kz}. Clearly, the radius decreases exponentially with increasing depth.

2.8 The Energy Equation and Energy Flux

In dealing with surface gravity waves, it is important and useful to derive an equation that describes the flow of energy in the fluid. Thus, an energy equation is obtained by taking the scalar product of the velocity vector $\mathbf{u}$ with the respective terms of the momentum equation (2.7.13), with $\nabla\phi$ replaced by $\mathbf{u}$ so that

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho} + gz\right) = 0. \tag{2.8.1}$$

We take the scalar product of $\mathbf{u}$ with (2.8.1) and use $\mathbf{u}\cdot\mathbf{u} = u^2$ to obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \mathbf{u}\cdot\nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho} + gz\right) = 0. \tag{2.8.2}$$

Since $\operatorname{div}\mathbf{u} = 0$, we can add $\left(\frac{1}{2}u^2 + p/\rho + gz\right)\operatorname{div}\mathbf{u}$ to (2.8.2) and use $\partial z/\partial t = 0$ and the formula for $\operatorname{div}(a\mathbf{u})$ with any scalar a to derive, multiplying by ρ,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho g z\right) + \operatorname{div}\left[\mathbf{u}\left(\frac{1}{2}\rho u^2 + p + g\rho z\right)\right] = 0. \tag{2.8.3}$$

The terms $\frac{1}{2}\rho u^2$ and $\rho g z$ represent the kinetic and potential energies, respectively, and equation (2.8.3) describes a balance between the rate of change of energy and energy flux terms, including convection by the velocity and the rate of working of the pressure. In fact, the rate of change of energy per unit volume is described in terms of the divergence of the *energy flux* $\mathfrak{F}$

$$\mathfrak{F} = \mathbf{u}\left(\frac{1}{2}\rho u^2 + p + g\rho z\right). \tag{2.8.4}$$

Equation (2.8.3) gives, writing $E = \frac{1}{2}\rho u^2 + \rho g z$,

$$\frac{\partial E}{\partial t} + \operatorname{div}\mathfrak{F} = 0. \tag{2.8.5}$$

This is usually called the *law of conservation of energy*.

In order to see some physical meaning of (2.8.3), we integrate it over some volume V enclosed by a closed surface S. By using the Gauss divergence theorem, we can transform the volume integral over V to a surface integral over S. Consequently,

$$\begin{aligned}\frac{\partial}{\partial t}\int_V \rho\left(\frac{1}{2}u^2 + gz\right) dV &= -\int_V \operatorname{div}\left[\rho\mathbf{u}\left(\frac{1}{2}u^2 + \frac{p}{\rho} + gz\right)\right] dV \\ &= -\int_S \left(\frac{1}{2}\rho u^2 + p + \rho g z\right)\mathbf{u}\cdot\mathbf{n}\, dS. \end{aligned}\tag{2.8.6}$$

This represents the rate of change of the total energy in a volume V that is equal to the amount of energy flowing out of this volume across the surface S per unit time.

For this reason $\mathfrak{F}$ is called the *energy flux density vector*. Its magnitude represents the amount of energy passing across a unit surface area normal to the velocity field $\mathbf{u}$ per unit time. We may rewrite the right-hand side of (2.8.6) as

$$-\int_S \mathbf{u}\left(\frac{1}{2}\rho u^2\right)\cdot\mathbf{n}\,dS - \int_S p\mathbf{u}\cdot\mathbf{n}\,dS - \int_S \rho g z\mathbf{u}\cdot\mathbf{n}\,dS. \tag{2.8.7}$$

The first term is the kinetic energy transported across S per unit time by the fluid; the second term is the work done by the pressure forces on the fluid within the surface, and the third term is the work done by the gravitational force acting on the system.

2.9 Exercises

1. Use the Hamilton principle to derive
   (i) the Newton second law of motion, and
   (ii) the equation for a simple harmonic oscillator.
2. Derive the equation of motion of an elastic beam of length ℓ, line density ρ, cross-sectional moment of inertia I, and modulus of elasticity E which is fixed at each end and performs small transverse oscillations in the horizontal (x,t)-plane.
3. Apply the variational principle

   $$\delta\iint L\,dx\,dt = 0,$$

   where the Lagrangian $L = \frac{1}{2}(u_{xx}^2 + u_t u_x) + u_x^3$, to derive the equation

   $$u_{xt} + 6u_x u_{xx} + u_{xxxx} = 0.$$

   Show that this equation leads to the KdV equation when $\eta = u_x$.
4. Use the variational principle

   $$\delta\iint \left(1 - u_t^2 + u_x^2\right)^{\frac{1}{2}}\,dx\,dt = 0$$

   to derive the Born and Infeld (1934) equation

   $$\left(1 - u_t^2\right)u_{xx} + 2u_x u_t u_{xt} - \left(1 + u_x^2\right)u_{tt} = 0.$$

5. Show that the variational principle (Whitham 1967a, 1967b)

   $$\delta\iint\left\{\frac{1}{2}\psi_x\psi_t + \frac{1}{2}c_0\psi_x^2 + \frac{1}{6}c_0\psi_x^3 + \frac{1}{12}c_0 h_0^2\left(\chi^2 + 2\chi_x\psi_x\right)\right\}dx\,dt = 0$$

   gives the coupled equations

   $$\psi_{xt} + c_0(1+\psi_x)\psi_{xx} + \frac{1}{6}c_0 h_0^2\chi_{xx} = 0, \qquad \psi_{xx} - \chi = 0,$$

   where c_0 and h_0 are constants.

6. Show that the variational principle (Whitham 1967a, 1967b)

$$\delta \iiint \left\{ \phi_t + \alpha\beta_t + \frac{1}{2}\left(u^2 + v^2\right) \right\} dx\, dy\, dt = 0$$

leads to the equations

$$u_x + v_y = 0, \qquad \frac{D\alpha}{Dt} + fv = 0, \qquad \frac{D\beta}{Dt} - u = 0,$$

where $u = \phi_x + \alpha\beta_x - \alpha$, $v = \phi_y + \alpha\beta_y - f\beta$, and

$$-p = \phi_t + \alpha\beta_t + \frac{1}{2}\left(u^2 + v^2\right).$$

7. If $\mathcal{L} = \mathcal{L}(\omega, k)$ where $\omega = -\theta_t$ and $k = \theta_x$, show that the variational principle (Whitham 1965a, 1965b; Lighthill 1967)

$$\delta \iint \mathcal{L}(\omega, k)\, dt\, dx = 0$$

gives the Euler–Lagrange equation

$$\frac{\partial}{\partial t}(\mathcal{L}_\omega) = \frac{\partial}{\partial x}(\mathcal{L}_k).$$

Show also that this equation reduces to a second-order quasi-linear partial differential equation for $\theta(x, t)$

$$\mathcal{L}_{\omega\omega}\theta_{tt} - 2\mathcal{L}_{\omega k}\theta_{xt} + \mathcal{L}_{kk}\theta_{xx} = 0.$$

8. Derive the Boussinesq equation for water waves

$$u_{tt} - c^2 u_{xx} - \mu u_{xxtt} = \frac{1}{2}\left(u^2\right)_{xx}$$

from the variational principle

$$\delta \iint L\, dx\, dt = 0,$$

where $L \equiv \frac{1}{2}\phi_t^2 - \frac{1}{2}c^2\phi_x^2 + \frac{1}{2}\mu\phi_{xt}^2 - \frac{1}{6}\phi_x^3$ and ϕ is the potential for u where $u = \phi_x$.

9. Show that the Euler equation of the variational principle

$$\delta I\left[u(x, y)\right] = \delta \iint_D F(x, y, u, p, q, l, m, n)\, dx\, dy = 0$$

is

$$F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q + \frac{\partial^2}{\partial x^2}F_l + \frac{\partial^2}{\partial x \partial y}F_m + \frac{\partial^2}{\partial y^2}F_n = 0,$$

where

$$p = u_x, \qquad q = u_y, \qquad l = u_{xx}, \qquad m = u_{xy}, \qquad \text{and} \qquad n = u_{yy}.$$

10. In each of the following cases, apply the variational principle or its simple extension with appropriate boundary conditions to derive the corresponding equations:
    (a) $F = u_{xx}^2 + u_{yy}^2 + u_{xy}^2$,
    (b) $F = \frac{1}{2}[u_t^2 - \alpha^2(u_x^2 + u_y^2) - \beta^2 u^2]$,
    (c) $F = \frac{1}{2}(u_t u_x + \alpha u_x^2 + \beta u_{xx}^2)$,
    (d) $F = \frac{1}{2}(u_t^2 + \alpha^2 u_{xx}^2)$,
    (e) $F = p(x)u'^2 + \frac{d}{dx}(q(x)u^2) - [r(x) + \lambda s(x)]u^2$,

    where p, q, r, and s are given functions of x, and α and β are constants.
11. Derive the Schrödinger equation from the variational principle

$$\delta \iiint_D \left[\frac{\hbar^2}{2m}(\psi_x^2 + \psi_y^2 + \psi_z^2) + (V - E)\psi^2\right] dx\, dy\, dz = 0,$$

    where $h = 2\pi\hbar$ is the Planck constant, m is the mass of a particle moving under the action of a force field described by the potential $V(x, y, z)$, and E is the total energy of the particle.
12. Derive the Poisson equation $\nabla^2 u = F(x, y)$ from the variational principle with the functional

$$I(u) = \iint_D \left[u_x^2 + u_y^2 + 2uF(x, y)\right] dx\, dy,$$

    where $u = u(x, y)$ is given on the boundary ∂D of D.
13. Prove that the Euler–Lagrange equation for the functional

$$I = \iiint_D f(x, y, z, u, p, q, r, l, m, n, a, b, c)\, dx\, dy\, dz$$

    is

$$F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q - \frac{\partial}{\partial z}F_r + \frac{\partial^2}{\partial x^2}F_l + \frac{\partial^2}{\partial y^2}F_m + \frac{\partial^2}{\partial z^2}F_n + \frac{\partial^2}{\partial x \partial y}F_a + \frac{\partial^2}{\partial y \partial z}F_b + \frac{\partial^2}{\partial z \partial x}F_c = 0,$$

    where $(p, q, r) = (u_x, u_y, u_z)$, $(l, m, n) = (u_{xx}, u_{yy}, u_{zz})$, and $(a, b, c) = (u_{xy}, u_{yz}, u_{zx})$.
14. Derive the equation of motion of a vibrating string of length l under the action of an external force $F(x, t)$ from the variational principle

$$\delta \int_{t_1}^{t_2} \int_0^l \left[\left(\frac{1}{2}\rho u_t^2 - T^* u_x^2\right) + \rho u F(x, t)\right] dx\, dt = 0,$$

    where ρ is the line density and T^* is the constant tension of the string.

15. The kinetic and potential energies associated with the transverse vibration of a thin elastic plate of constant thickness h are

$$T = \frac{1}{2}\rho \iint_D \dot{u}^2 \, dx \, dy,$$
$$V = \frac{1}{2}\mu_0 \iint_D \left[(\nabla u)^2 - 2(1-\sigma)\left(u_{xx}u_{yy} - u_{xy}^2\right)\right] dx \, dy,$$

where ρ is the surface density and $\mu_0 = 2h^3E/3(1-\sigma^2)$.
Use the variational principle

$$\delta \int_{t_1}^{t_2} \iint_D \left[(T-V) + fu\right] dx \, dy \, dt = 0$$

to derive the equation of motion of the plate

$$\rho \ddot{u} + \mu_0 \nabla^4 u = f(x,y,t),$$

where f is the transverse force per unit area acting on the plate.

16. The kinetic and potential energies associated with wave motion in elastic solids are

$$T = \frac{1}{2} \iiint_D \rho\left(u_t^2 + v_t^2 + w_t^2\right) dx \, dy \, dz,$$
$$V = \frac{1}{2} \iiint_D \left[\lambda(u_x + v_y + w_z)^2 + 2\mu\left(u_x^2 + v_y^2 + w_z^2\right)\right.$$
$$\left. + \mu\left\{(v_x + u_y)^2 + (w_y + v_z)^2 + (u_z + w_x)^2\right\}\right] dx \, dy \, dz.$$

Use the variational principle

$$\delta \int_{t_1}^{t_2} \iiint_D (T-V) \, dx \, dy \, dz \, dt = 0$$

to derive the equation of wave motion in an elastic medium

$$(\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mu \nabla^2 \mathbf{u} = \rho \mathbf{u}_{tt},$$

where $\mathbf{u} = (u, v, w)$ is the displacement vector.

17. Apply the variational principle (2.5.2) with the Lagrangian $L = \frac{1}{2}(u_t u_x - u_{xx}^2 - 2u_x^3)$ to derive $u_{xt} - 6u_x u_{xx} + u_{xxxx} = 0$. Show that this equation leads to the KdV equation when $u_x = \eta$.

18. An inviscid and incompressible fluid flow under the conservative force field $\mathbf{F} = -\nabla \Omega$ with a potential $\Omega = gz$, where g is the constant acceleration due to gravity, is governed by the Euler equation (2.7.1).

(a) Show that

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{p}{\rho} + \Omega\right) - \mathbf{u} \times \boldsymbol{\omega} = 0,$$

where $\nabla \times \mathbf{u} = \boldsymbol{\omega}$ is the vorticity vector.

(b) Taking the scalar product of the above equation with $\mathbf{u}$, derive the result

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)+(\mathbf{u}\cdot\nabla)\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}+\frac{p}{\rho}+\Omega\right)=0.$$

(c) Derive the energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+\rho\Omega\right)+\nabla\cdot\left[\mathbf{u}\left(\frac{1}{2}\rho\mathbf{u}\cdot\mathbf{u}+p+\rho\Omega\right)\right]=0.$$

Explain the significance of each term of the energy equation.

19. The three-dimensional Plateau problem is governed by the functional

$$I\big[u(x,y,z)\big]=\iiint_D\left(1+p^2+q^2+r^2\right)^{\frac{1}{2}}\,dx\,dy\,dz$$

where $p=u_x$, $q=u_y$, and $r=u_z$.
Find the Euler–Lagrange equation of this functional.

20. (a) Show that the Euler–Lagrange equation for the functional

$$I(\mathbf{u})=\int_a^b F(x,\mathbf{u},\mathbf{u}')\,dx,$$

where $u=(u_1,u_2,\ldots,u_n)$, $u_i\in C^2[a,b]$, $u_i(a)=a_i$, and $u_i(b)=b_i$, $i=1,2,\ldots,n$, is a system of n ordinary differential equations

$$F_{u_i}-\frac{d}{dx}F_{u_i'}=0,\quad i=1,2,\ldots,n.$$

(b) If F in (a) does not depend explicitly on x, then show that

$$F-\sum_{i=1}^{n}u_i'F_{u_i'}=\text{const}.$$

21. Consider a simple pendulum of length ℓ with a bob of mass m suspended from a frictionless support. Apply the Hamilton principle to the functional

$$I[\theta(t)]=\int_{t_1}^{t_2}(T-V)\,dt,$$

where $T=\frac{1}{2}m\ell^2\dot{\theta}^2$ and $V=mg(\ell-\ell\cos\theta)$ to derive the equation of the simple pendulum

$$\ddot{\theta}+\omega^2\sin\theta=0,\quad \omega^2=\frac{g}{\ell}.$$

22. Derive the Euler–Lagrange equation for the functional

$$I(y(x))=\int_a^b F(x,y,y')\,dx,$$

where

(a) $F(x, y, y') = u(x, y)\sqrt{1 + y'^2}$,
(b) $F(x, y, y') = \frac{1}{\sqrt{2g}}(\frac{1+y'^2}{y_1 - y})^{\frac{1}{2}}$ with $y(a) = y_1$, $y(b) = y_2 < y_1$ (Brachistochrone problem).

23. The Fermat principle in optics states that light travels from one point $A(x_1, y_1)$ to another point $B(x_2, y_2)$ in an optically homogeneous medium along a path in a minimum (least) time. The time taken for the light beam to travel from A to B is

$$I\big(y(x)\big) = \int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \left(\frac{dt}{ds}\right)\left(\frac{ds}{dx}\right) dx = \frac{1}{c}\int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx,$$

where $c = \frac{ds}{dt}$ is the constant velocity of light.
Apply the variational principle

$$\delta I = \frac{1}{c}\delta \int \sqrt{1 + y'^2}\, dx = 0$$

to derive the *Snell law of refraction of light*, $n \sin\phi =$ const., where $n = \frac{1}{c}$ is the refractive index of the medium and ϕ is the angle made by the tangent to the minimum path with the vertical y-axis.

24. (a) Derive the *principle of least action* for a conservative system

$$\delta \int_{t_1}^{t_2} 2T\, dt = 0,$$

where the time integral of $2T$ is called the *action* of the system.
(b) Explain the significance of this principle.

25. Show that the Euler–Lagrange equation of the variational principle

$$\delta I = \delta \int_a^b F\big(x, y, y', y'', \ldots, y^{(n)}\big)\, dx = 0$$

is an ordinary differential equation of order $2n$

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \cdots + (-1)^n \frac{d^n}{dx^n}F_{y^{(n)}} = 0.$$

26. The electrostatic potential $\phi(x, y, z)$ is defined in terms of the electrostatic field $\mathbf{E}$ so that $\mathbf{E} = -\nabla\phi$ in a domain D of volume V, where ϕ is specified on ∂D. Show that ϕ that minimizes the electric energy functional

$$I[\phi] = \frac{e_0}{2}\iiint_V E^2 dv = \frac{e_0}{2}\iiint_V (\nabla\phi)^2\, dv$$

satisfies the Laplace equation.

27. Use the Lagrangian $L = T - V$ and the Lagrange equation to derive the Newton laws of motion of a particle of mass m moving under a force, $F = -\nabla V$, in

(a) one dimension,
(b) a two dimensional plane in Cartesian coordinates,
(c) a two dimensional plane in polar coordinates.

28. Seek a traveling wave solution

$$r_n = A\cos\theta = A\cos(\omega t - kn)$$

of the linearized Toda lattice equation

$$m\ddot{r}_n = (ab)(r_{n+1} - 2r_n + r_{n-1}),$$

where $r_n = (y_{n+1} - y_n)$ and y_n is the displacement of the nth mass. Show that the dispersion relation is

$$\omega^2 = \left(\frac{4ab}{m}\right)\sin^2\left(\frac{k}{2}\right).$$

29. Consider the Ablowitz and Ladik (AL) equation for the lattice system (1976a, 1976b)

$$i\frac{d\phi_n}{dt} + (\phi_{n+1} + \phi_{n-1})\left(1 + \frac{\gamma}{2}|\phi_n|^2\right) = 0.$$

(a) Using $\phi_n = e^{2it}\psi_n$, show that the solution of the AL equation reduces to that of the NLS equation

$$i\psi_t + \psi_{xx} + \gamma|\psi|^2\psi = 0,$$

as the ratio of anharmoncity to dispersion (γ) tends to zero.
(b) Show that the solution of the above AL equation is

$$\phi_n(t) = A cn\big[\beta(n - vt); k\big]\exp\big[-i(\omega t + \alpha n + \phi_0)\big],$$

selecting the units of ϕ_n so that $\frac{\gamma}{2} = 1$, the parameters A, ω and v are given by

$$A = \frac{k sn(\beta; k)}{dn(\beta; k)}, \qquad \omega = -\frac{2cn(\beta; k)\cos\alpha}{dn^2(\beta; k)}, \qquad v = -\frac{2sn(\beta; k)\sin\alpha}{\beta\, dn(\beta; k)},$$

where $0 < \beta < \infty$, $-\pi \le \alpha \le \pi$, and $0 < k < 1$.

3

First-Order, Quasi-linear Equations and Method of Characteristics

As long as a branch of knowledge offers an abundance of problems, it is full of vitality.

David Hilbert

The advance of analysis is, at this moment, stagnant along the entire front of nonlinear problems. That this phenomenon is not of a transient nature but that we are up against an important conceptual difficulty ... yet no decisive progress has been made against them ... which could be rated as important by the criteria that are applied in other, more successful (linear!) parts of mathematical physics.
It is important to avoid a misunderstanding at this point. One may be tempted to qualify these (shock wave and turbulence) problems as problems in physics, rather than in applied mathematics, or even pure mathematics. We wish to emphasize that it is our conviction that such an interpretation is wholly erroneous.

John Von Neumann

3.1 Introduction

Many problems in mathematical, physical, and engineering sciences deal with the formulation and the solution of first-order partial differential equations. From a mathematical point of view, first-order equations have the advantage of providing a conceptual basis that can be utilized for second-, third-, and higher-order equations.

This chapter is concerned with first-order, quasi-linear and linear partial differential equations and their solutions by using the Lagrange method of characteristics and its generalizations.

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3.2 The Classification of First-Order Equations

The most general, first-order, partial differential equation in two independent variables x and y is of the form

$$F(x, y, u, u_x, u_y) = 0, \quad (x, y) \in D \subset \mathbb{R}^2, \tag{3.2.1}$$

where F is a given function of its arguments, and $u = u(x, y)$ is an unknown function of the independent variables x and y which lie in some given domain D in $\mathbb{R}^2$, $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$. Equation (3.2.1) is often written in terms of standard notation $p = u_x$ and $q = u_y$ so that (3.2.1) takes the form

$$F(x, y, u, p, q) = 0. \tag{3.2.2}$$

Similarly, the most general, first-order, partial differential equation in three independent variables x, y, z can be written as

$$F(x, y, z, u, u_x, u_y, u_z) = 0. \tag{3.2.3}$$

Equation (3.2.1) or (3.2.2) is called a *quasi-linear partial differential* equation if it is linear in first-partial derivatives of the unknown function $u(x, y)$. So, the most general quasi-linear equation must be of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \tag{3.2.4}$$

where its coefficients a, b, and c are functions of x, y, and u.

The following are examples of quasi-linear equations:

$$x\left(y^2 + u\right)u_x - y\left(x^2 + u\right)u_y = \left(x^2 - y^2\right)u, \tag{3.2.5}$$

$$uu_x + u_t + nu^2 = 0, \tag{3.2.6}$$

$$\left(y^2 - u^2\right)u_x - xyu_y = xu. \tag{3.2.7}$$

Equation (3.2.4) is called a *semilinear* partial differential equation if its coefficients a and b are independent of u, and hence, the semilinear equation can be expressed in the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u). \tag{3.2.8}$$

Examples of semilinear equations are

$$xu_x + yu_y = u^2 + x^2, \tag{3.2.9}$$

$$(x + 1)^2u_x + (y - 1)^2u_y = (x + y)u^2, \tag{3.2.10}$$

$$u_t + au_x + u^2 = 0, \tag{3.2.11}$$

where a is a constant.

Equation (3.2.1) is said to be *linear* if F is linear in each of the variables u, u_x, and u_y, and the coefficients of these variables are functions only of the independent variables x and y. The most general, first-order, *linear* partial differential equation has the form

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y), \tag{3.2.12}$$

where the coefficients a, b, and c, in general, are functions of x and y, and $d(x,y)$ is a given function. Unless stated otherwise, these functions are assumed to be continuously differentiable. Equations of the form (3.2.12) are called *homogeneous* if $d(x,y) \equiv 0$ or *inhomogeneous* if $d(x,y) \neq 0$.

Obviously, linear equations are a special kind of the quasi-linear equation (3.2.4) if a, b are independent of u and c is a linear function in u. Similarly, semilinear equation (3.2.8) reduces to a linear equation if c is linear in u.

Examples of linear equations are

$$xu_x + yu_y - nu = 0, \tag{3.2.13}$$

$$nu_x + (x+y)u_y - u = e^x, \tag{3.2.14}$$

$$yu_x + xu_y = xy, \tag{3.2.15}$$

$$(y-z)u_x + (z-x)u_y + (x-y)u_z = 0. \tag{3.2.16}$$

An equation which is *not* linear is often called a *nonlinear equation*. So, the first-order equations are often classified as linear and nonlinear.

3.3 The Construction of a First-Order Equation

We consider a system of geometrical surfaces described by the equation

$$f(x,y,z,a,b) = 0, \tag{3.3.1}$$

where a and b are arbitrary parameters. We differentiate (3.3.1) with respect to x and y to obtain

$$f_x + pf_z = 0, \qquad f_y + qf_z = 0, \tag{3.3.2}$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

The set of three equations (3.3.1) and (3.3.2) involves two arbitrary parameters a and b. In general, these two parameters can be eliminated from this set to obtain a first-order equation of the form

$$F(x,y,z,p,q) = 0. \tag{3.3.3}$$

Thus the system of surfaces (3.3.1) gives rise to a first-order partial differential equation (3.3.3). In other words, an equation of the form (3.3.1) containing *two* arbitrary parameters is called a *complete solution* or a *complete integral* of equation (3.3.3). Its role is somewhat similar to that of a general solution for the case of an ordinary differential equation.

On the other hand, any relationship of the form

$$f(\phi,\psi) = 0, \tag{3.3.4}$$

which involves an *arbitrary function* f of two known functions $\phi = \phi(x,y,z)$ and $\psi = \psi(x,y,z)$ and provides a solution of a first-order partial differential equation is called a *general solution* or *general integral* of this equation. Clearly, the general solution of a first-order partial differential equation depends on an *arbitrary function*. This is in striking contrast to the situation for ordinary differential equations where the general solution of a first-order ordinary differential equation depends on one arbitrary constant. The general solution of a partial differential equation can be obtained from its complete integral. We obtain the general solution of (3.3.3) from its complete integral (3.3.1) as follows.

First, we prescribe the second parameter b as an arbitrary function of the first parameter a in the complete solution (3.3.1) of (3.3.3), that is, $b = b(a)$. We then consider the envelope of the one-parameter family of solutions so defined. This envelope is represented by the two simultaneous equations

$$f\big(x,y,z,a,b(a)\big) = 0, \tag{3.3.5}$$

$$f_a\big(x,y,z,a,b(a)\big) + f_b\big(x,y,z,b(a)\big)b'(a) = 0, \tag{3.3.6}$$

where the second equation (3.3.6) is obtained from the first equation (3.3.5) by partial differentiation with respect to a. In principle, equation (3.3.5) can be solved for $a = a(x,y,z)$ as a function of x, y, and z. We substitute this result back in (3.3.5) to obtain

$$f\big\{x,y,z,a(x,y,z),b\big(a(x,y,z)\big)\big\} = 0, \tag{3.3.7}$$

where b is an arbitrary function. Indeed, the two equations (3.3.5) and (3.3.6) together define the general solution of (3.3.3). When a definite $b(a)$ is prescribed, we obtain a *particular solution* from the general solution. Since the general solution depends on an arbitrary function, there are infinitely many solutions. In practice, only one solution satisfying prescribed conditions is required for a physical problem. Such a solution may be called a *particular solution*.

In addition to the general and particular solutions of (3.3.3), if the envelope of the two-parameter system (3.3.1) of surfaces exists, it also represents a solution of the given equation (3.3.3); the envelope is called the *singular solution* of equation (3.3.3). The singular solution can easily be constructed from the complete solution (3.3.1) representing a two-parameter family of surfaces. The envelope of this family is given by the system of three equations

$$f(x,y,z,a,b) = 0, \qquad f_a(x,y,z,a,b) = 0, \qquad f_b(x,y,z,a,b) = 0. \tag{3.3.8}$$

In general, it is possible to eliminate a and b from (3.3.8) to obtain the equation of the envelope which gives the singular solution. It may be pointed out that the singular solution *cannot* be obtained from the general solution. Its nature is similar to that of the singular solution of a first-order ordinary differential equation.

Finally, it is important to note that solutions of a partial differential equation are expected to be represented by smooth functions. A function is called *smooth* if all of its derivatives exist and are continuous. However, in general, solutions *are not always smooth*. A solution which is *not* everywhere differentiable is called a *weak solution*.

The most common weak solution is the one that has discontinuities in its first partial derivatives across a curve, so that the solution can be represented by shock waves as surfaces of discontinuity. In the case of a first-order partial differential equation, there are discontinuous solutions where z itself and *not* merely $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ are discontinuous. In fact, this kind of discontinuity is usually known as a *shock wave*. An important feature of quasi-linear and nonlinear partial differential equations is that their solutions may develop discontinuities as they move away from the initial state. We close this section by considering some examples.

Example 3.3.1. Show that a family of spheres

$$x^2 + y^2 + (z - c)^2 = r^2, \tag{3.3.9}$$

satisfies the first-order linear partial differential equation

$$yp - xq = 0. \tag{3.3.10}$$

Differentiating the equation (3.3.9) with respect to x and y gives

$$x + p(z - c) = 0 \quad \text{and} \quad y + q(z - c) = 0.$$

Eliminating the arbitrary constant c from these equations, we obtain the first-order, partial differential equation

$$yp - xq = 0.$$

Example 3.3.2. Show that the family of spheres

$$(x - a)^2 + (y - b)^2 + z^2 = r^2 \tag{3.3.11}$$

satisfies the first-order, nonlinear, partial differential equation

$$z^2\left(p^2 + q^2 + 1\right) = r^2. \tag{3.3.12}$$

We differentiate the equation of the family of spheres with respect to x and y to obtain

$$(x - a) + zp = 0, \qquad (y - b) + z\,q = 0.$$

Eliminating the two arbitrary constants a and b, we find the nonlinear partial differential equation

$$z^2\left(p^2 + q^2 + 1\right) = r^2.$$

All surfaces of revolution with the z-axis as the axis of symmetry satisfy the equation

$$z = f\left(x^2 + y^2\right), \tag{3.3.13}$$

where f is an arbitrary function. Writing $u = x^2 + y^2$ and differentiating (3.3.13) with respect to x and y, respectively, we obtain

$$p = 2xf'(u), \qquad q = 2yf'(u).$$

Eliminating the arbitrary function $f(u)$ from these results, we find the equation

$$yp - xq = 0.$$

Theorem 3.3.1. *If* $\phi = \phi(x, y, z)$ *and* $\psi = \psi(x, y, z)$ *are two given functions of* x, y, *and* z *and if* $f(\phi, \psi) = 0$, *where* f *is an arbitrary function of* ϕ *and* ψ, *then* $z = z(x, y)$ *satisfies a first-order, partial differential equation*

$$p\frac{\partial(\phi, \psi)}{\partial(y, z)} + q\frac{\partial(\phi, \psi)}{\partial(z, x)} = \frac{\partial(\phi, \psi)}{\partial(x, y)}, \tag{3.3.14}$$

where

$$\frac{\partial(\phi, \psi)}{\partial(x, y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}. \tag{3.3.15}$$

Proof. We differentiate $f(\phi, \psi) = 0$ with respect to x and y, respectively, to obtain the following equations:

$$\frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial x} + p\frac{\partial \psi}{\partial z}\right) = 0, \tag{3.3.16}$$

$$\frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z}\right) + \frac{\partial f}{\partial \psi}\left(\frac{\partial \psi}{\partial y} + q\frac{\partial \psi}{\partial z}\right) = 0. \tag{3.3.17}$$

Nontrivial solutions for $\frac{\partial f}{\partial \phi}$ and $\frac{\partial f}{\partial \psi}$ can be found if the determinant of the coefficients of these equations vanishes, that is,

$$\begin{vmatrix} \phi_x + p\phi_z & \psi_x + p\psi_z \\ \phi_y + q\phi_z & \psi_y + q\psi_z \end{vmatrix} = 0. \tag{3.3.18}$$

Expanding this determinant gives the first-order, quasi-linear equation (3.3.14).

3.4 The Geometrical Interpretation of a First-Order Equation

To investigate the geometrical content of a first-order, partial differential equation, we begin with a general, quasi-linear equation

$$a(x, y, u)u_x + b(x, y, u)u_y - c(x, y, u) = 0. \tag{3.4.1}$$

We assume that the possible solution of (3.4.1) in the form $u = u(x, y)$ or in an implicit form

$$f(x, y, u) \equiv u(x, y) - u = 0 \tag{3.4.2}$$

represents a possible *solution surface* in the (x, y, u)-space. This is often called an *integral surface* of the equation (3.4.1). At any point (x, y, u) on the solution surface, the gradient vector $\nabla f = (f_x, f_y, f_u) = (u_x, u_y, -1)$ is normal to the solution surface. Clearly, equation (3.4.1) can be written as the dot product of two vectors

$$au_x + bu_y - c = (a, b, c) \cdot (u_x, u_y, -1) = 0. \tag{3.4.3}$$

This clearly shows that the vector (a, b, c) must be a tangent vector of the integral surface (3.4.2) at the point (x, y, u), and hence, it determines a direction field called

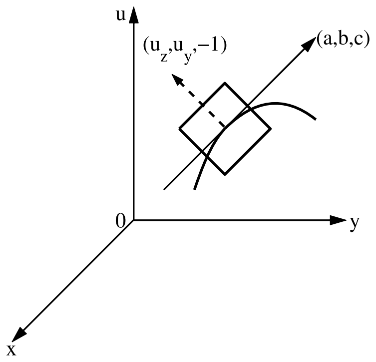

Fig. 3.1 Tangent and normal vector fields of solution surface at a point (x, y, u).

the *characteristic direction* or *Monge axis*. This direction is of fundamental importance in determining a solution of equation (3.4.1). To summarize, we have shown that $f(x, y, u) = u(x, y) - u = 0$, as a surface in the (x, y, u)-space, is a solution of (3.4.1) if and only if the direction vector field (a, b, c) lies in the tangent planes of the integral surface $f(x, y, u) = 0$ at each point (x, y, u), where $\nabla f \neq 0$, as shown in Figure 3.1.

A curve in the (x, y, u)-space whose tangent at every point coincides with the characteristic direction field (a, b, c) is called a *characteristic curve*. If the parametric equations of this characteristic curve are

$$x = x(t), \qquad y = y(t), \qquad u = u(t), \tag{3.4.4}$$

then the tangent vector to this curve is $(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt})$ which must be equal to (a, b, c). Therefore, the system of ordinary differential equations of the characteristic curve is given by

$$\frac{dx}{dt} = a(x, y, u), \qquad \frac{dy}{dt} = b(x, y, u), \qquad \frac{du}{dt} = c(x, y, u). \tag{3.4.5}$$

These are called the *characteristic equations* of the quasi-linear equation (3.4.1).

In fact, there are only two independent ordinary differential equations in the system (3.4.5); therefore, its solutions form a two-parameter family of curves in the (x, y, u)-space.

The projection on $u = 0$ of a characteristic curve on the (x, t)-plane is called a *characteristic base curve*, or simply a *characteristic*.

Equivalently, the characteristic equations (3.4.5) in the nonparametric form are

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{3.4.6}$$

The typical problem of solving equation (3.4.1) with a prescribed u on a given plane curve C is equivalent to finding an integral surface in the (x, y, u)-space, satisfying equation (3.4.1) and containing the three-dimensional space curve Γ defined by the values of u on C, which is the projection of Γ on $u = 0$.

Remark 1. The above geometrical interpretation can be generalized for higher-order partial differential equations. However, it is not easy to visualize geometrical arguments that have been described for the case of three space dimensions.

Remark 2. The geometrical interpretation is more complicated for the case of nonlinear partial differential equations because the normals to possible solution surfaces through a point do not lie in a plane. The tangent planes no longer intersect along one straight line, but instead, they envelope along a curved surface known as the *Monge cone*. Any further discussion is beyond the scope of this book.

We conclude this section by adding an important observation regarding the nature of the characteristics in the (x, t)-plane. For a quasi-linear equation, characteristics are determined by the first two equations in (3.4.5) with their slopes

$$\frac{dy}{dx} = \frac{b(x, y, u)}{a(x, y, u)}. \tag{3.4.7}$$

If (3.4.1) is a linear equation, then a and b are independent of u, and the characteristics of (3.4.1) are *plane curves* with slopes

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}. \tag{3.4.8}$$

By integrating this equation, we can determine the characteristics which represent a one-parameter family of *curves* in the (x, t)-plane. However, if a and b are constant, the characteristics of equation (3.4.1) are straight lines.

3.5 The Method of Characteristics and General Solutions

We can use the geometrical interpretation of the first-order, partial differential equation and the properties of characteristic curves to develop a method for finding the general solution of quasi-linear equations. This is usually referred to as *the method of characteristics* due to Lagrange. This method of solution of quasi-linear equations can be described by the following result.

Theorem 3.5.1. *The general solution of a first-order, quasi-linear partial differential equation*

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{3.5.1}$$

is

$$f(\phi, \psi) = 0, \tag{3.5.2}$$

where f is an arbitrary function of $\phi(x, y, u)$ and $\psi(x, y, u)$ and $\phi = const. = c_1$ and $\psi = const. = c_2$ are solution curves of the characteristic equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{3.5.3}$$

The solution curves defined by $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ are called the families of characteristic curves of equation (3.5.1).

Proof. Since $\phi(x,y,u) = c_1$ and $\psi(x,y,u) = c_2$ satisfy equations (3.5.3), these equations must be compatible with the equation

$$d\phi = \phi_x\,dx + \phi_y\,dy + \phi_u\,du = 0. \tag{3.5.4}$$

This is equivalent to the equation

$$a\,\phi_x + b\,\phi_y + c\,\phi_u = 0. \tag{3.5.5}$$

Similarly, equation (3.5.3) is also compatible with

$$a\,\psi_x + b\,\psi_y + c\,\psi_u = 0. \tag{3.5.6}$$

We now solve (3.5.5), (3.5.6) for a, b, and c to obtain

$$\frac{a}{\frac{\partial(\phi,\psi)}{\partial(y,u)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{c}{\frac{\partial(\phi,\psi)}{\partial(x,y)}}. \tag{3.5.7}$$

It has been shown earlier that $f(\phi,\psi) = 0$ satisfies an equation similar to (3.3.14), that is,

$$p\frac{\partial(\phi,\psi)}{\partial(y,u)} + q\frac{\partial(\phi,\psi)}{\partial(u,x)} = \frac{\partial(\phi,\psi)}{\partial(x,y)}. \tag{3.5.8}$$

Substituting (3.5.7) into (3.5.8), we find that $f(\phi,\psi) = 0$ is a solution of (3.5.1). This completes the proof.

Note that an analytical method has been used to prove Theorem 3.5.1. Alternatively, a geometrical argument can be used to prove this theorem. The geometrical method of proof is left to the reader as an exercise.

Many problems in applied mathematics, science, and engineering involve partial differential equations. We seldom try to find or discuss the properties of a solution to these equations in its most general form. In most cases of interest, we deal with those solutions of partial differential equations which satisfy certain supplementary conditions. In the case of a first-order partial differential equation, we determine the specific solution by formulating an *initial-value problem* or a *Cauchy problem*.

Theorem 3.5.2 (The Cauchy Problem for a First-Order Partial Differential Equation). *Suppose that C is a given curve in the (x,y)-plane with its parametric equations*

$$x = x_0(t), \qquad y = y_0(t), \tag{3.5.9}$$

where t belongs to an interval $I \subset \mathbb{R}$, and the derivatives $x_0'(t)$ and $y_0'(t)$ are piecewise continuous functions such that $x_0'^2 + y_0'^2 \neq 0$. Also, suppose that $u = u_0(t)$ is a given function on the curve C. Then, there exists a solution $u = u(x,y)$ of the equation

$$F(x,y,u,u_x,u_y) = 0 \tag{3.5.10}$$

in a domain D of $\mathbb{R}^2$ containing the curve C for all $t \in I$, and the solution $u(x,y)$ satisfies the given initial data, that is,

$$u\big(x_0(t), y_0(t)\big) = u_0(t) \tag{3.5.11}$$

for all values of $t \in I$.

In short, the Cauchy problem is to determine a solution of equation (3.5.10) in a neighborhood of C, such that the solution $u = u(x, y)$ takes a prescribed value $u_0(t)$ on C. The curve C is called the initial curve of the problem, and $u_0(t)$ is called the initial data. Equation (3.5.11) is called the initial condition of the problem.

The solution of the Cauchy problem also deals with such questions as the conditions on the functions F, $x_0(t)$, $y_0(t)$, and $u_0(t)$ under which a solution exists and is unique.

We next discuss a method for solving a Cauchy problem for the first-order, quasi-linear equation (3.5.1). We first observe that geometrically $x = x_0(t)$, $y = y_0(t)$, and $u = u_0(t)$ represent an initial curve Γ in the (x, y, u)-space. The curve C, on which the Cauchy data is prescribed, is the projection of Γ on the (x, y)-plane. We now present a precise formulation of a Cauchy problem for the first-order, quasi-linear equation (3.5.1).

Theorem 3.5.3 (The Cauchy Problem for a Quasi-linear Equation). *Suppose that $x_0(t)$, $y_0(t)$, and $u_0(t)$ are continuously differentiable functions of t in a closed interval, $0 \le t \le 1$, and that a, b, and c are functions of x, y, and u with continuous first-order partial derivatives with respect to their arguments in some domain D of the (x, y, u)-space containing the initial curve*

$$\Gamma : x = x_0(t), \quad y = y_0(t), \quad u = u_0(t), \tag{3.5.12}$$

where $0 \le t \le 1$, and satisfying the condition

$$y_0'(t)a\big(x_0(t), y_0(t), u_0(t)\big) - x_0'(t)b\big(x_0(t), y_0(t), u_0(t)\big) \neq 0. \tag{3.5.13}$$

Then there exists a unique solution $u = u(x, y)$ of the quasi-linear equation (3.5.1) in the neighborhood of $C : x = x_0(t)$, $y = y_0(t)$, and the solution satisfies the initial condition

$$u_0(t) = u\big(x_0(t), y_0(t)\big) \quad \text{for } 0 \le t \le 1. \tag{3.5.14}$$

Note: Condition (3.5.13) excludes the possibility of C being a characteristic.

Example 3.5.1. Find the general solution of the first-order linear partial differential equation

$$xu_x + yu_y = u. \tag{3.5.15}$$

The characteristic curves of this equation are the solutions of the characteristic equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}. \tag{3.5.16}$$

This system of equations gives the integral surfaces

$$\phi = \frac{y}{x} = C_1 \quad \text{and} \quad \psi = \frac{u}{x} = C_2,$$

where C_1 and C_2 are arbitrary constants. Thus, the general solution of (3.5.15) is

$$f\left(\frac{y}{x}, \frac{u}{x}\right) = 0, \tag{3.5.17}$$

where f is an arbitrary function. This general solution can also be written as

$$u(x,y) = xg\left(\frac{y}{x}\right), \tag{3.5.18}$$

where g is an arbitrary function.

Example 3.5.2. Obtain the general solution of the linear Euler equation

$$xu_x + yu_y = nu. \tag{3.5.19}$$

The integral surfaces are the solutions of the characteristic equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{nu}. \tag{3.5.20}$$

From these equations, we get

$$\frac{y}{x} = C_1, \qquad \frac{u}{x^n} = C_2,$$

where C_1 and C_2 are arbitrary constants. Hence, the general solution of (3.5.19) is

$$f\left(\frac{y}{x}, \frac{u}{x^n}\right) = 0. \tag{3.5.21}$$

This can also be written as

$$\frac{u}{x^n} = g\left(\frac{y}{x}\right),$$

or equivalently,

$$u(x,y) = x^n g\left(\frac{y}{x}\right). \tag{3.5.22}$$

This shows that the solution $u(x,y)$ is a homogeneous function of x and y of degree n.

Example 3.5.3. Find the general solution of the linear equation

$$x^2 u_x + y^2 u_y = (x+y)u. \tag{3.5.23}$$

The characteristic equations associated with (3.5.23) are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u}. \tag{3.5.24}$$

From the first two of these equations, we find

$$x^{-1} - y^{-1} = C_1, \tag{3.5.25}$$

where C_1 is an arbitrary constant.

It follows from (3.5.24) that

$$\frac{dx - dy}{x^2 - y^2} = \frac{du}{(x + y)u},$$

or equivalently,

$$\frac{d(x - y)}{x - y} = \frac{du}{u}.$$

This gives

$$\frac{x - y}{u} = C_2, \tag{3.5.26}$$

where C_2 is a constant. Furthermore, (3.5.25) and (3.5.26) also give

$$\frac{xy}{u} = C_3, \tag{3.5.27}$$

where C_3 is a constant.

Thus, the general solution (3.5.23) is given by

$$f\left(\frac{xy}{u}, \frac{x - y}{u}\right) = 0, \tag{3.5.28}$$

where f is an arbitrary function. This general solution representing the integral surface can also be written as

$$u(x, y) = xyg\left(\frac{x - y}{u}\right), \tag{3.5.29}$$

where g is an arbitrary function, or equivalently,

$$u(x, y) = xyh\left(\frac{x - y}{xy}\right), \tag{3.5.30}$$

where h is an arbitrary function.

Example 3.5.4. Show that the general solution of the linear equation

$$(y - z)u_x + (z - x)u_y + (x - y)u_z = 0 \tag{3.5.31}$$

is

$$u(x, y, z) = f\left(x + y + z, x^2 + y^2 + z^2\right), \tag{3.5.32}$$

where f is an arbitrary function.

The characteristic curves satisfy the characteristic equations

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y} = \frac{du}{0}, \tag{3.5.33}$$

or equivalently,

$$du = 0, \qquad dx + dy + dz = 0, \qquad x\,dx + y\,dy + z\,dz = 0.$$

Integration of these equations gives

$$u = C_1, \qquad x + y + z = C_2, \quad \text{and} \quad x^2 + y^2 + z^2 = C_3,$$

where C_1, C_2 and C_3 are arbitrary constants.

Thus, the general solution can be written in terms of an arbitrary function f in the form

$$u(x,y,z) = f\big(x + y + z, x^2 + y^2 + z^2\big).$$

We next verify that this is a general solution by introducing three independent variables ξ, η, ζ defined in terms of x, y, and z as

$$\xi = x + y + z, \qquad \eta = x^2 + y^2 + z^2, \quad \text{and} \quad \zeta = y + z, \tag{3.5.34}$$

where ζ is an arbitrary combination of y and z. Clearly, the general solution becomes

$$u = f(\xi, \eta),$$

and hence,

$$u_\zeta = u_x \frac{\partial x}{\partial \zeta} + u_y \frac{\partial y}{\partial \zeta} + u_z \frac{\partial z}{\partial \zeta}. \tag{3.5.35}$$

It follows from (3.5.34) that

$$0 = \frac{\partial x}{\partial \zeta} + \frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \zeta}, \qquad 0 = 2\left(x\frac{\partial x}{\partial \zeta} + y\frac{\partial y}{\partial \zeta} + z\frac{\partial z}{\partial \zeta}\right), \qquad \frac{\partial y}{\partial \zeta} + \frac{\partial z}{\partial \zeta} = 1.$$

It follows from the first and the third results that $\frac{\partial x}{\partial \zeta} = -1$ and, therefore,

$$x = y\frac{\partial y}{\partial \zeta} + z\frac{\partial z}{\partial \zeta}, \qquad y = y\frac{\partial y}{\partial \zeta} + y\frac{\partial z}{\partial \zeta}, \qquad z = z\frac{\partial y}{\partial \zeta} + z\frac{\partial z}{\partial \zeta}.$$

Clearly, it follows by subtracting that

$$x - y = (z - y)\frac{\partial z}{\partial \zeta}, \qquad x - z = (y - z)\frac{\partial y}{\partial \zeta}.$$

Using the values for $\frac{\partial x}{\partial \zeta}$, $\frac{\partial z}{\partial \zeta}$, and $\frac{\partial y}{\partial \zeta}$ in (3.5.35), we obtain

$$(z - y)\frac{\partial u}{\partial \zeta} = (y - z)\frac{\partial u}{\partial x} + (z - x)\frac{\partial u}{\partial y} + (x - y)\frac{\partial u}{\partial z}. \tag{3.5.36}$$

If $u = u(\xi, \eta)$ satisfies (3.5.31), then $\frac{\partial u}{\partial \zeta} = 0$, and hence, (3.5.36) reduces to (3.5.31). This shows that the general solution (3.5.32) satisfies equation (3.5.31).

Example 3.5.5. Find the solution of the equation

$$u(x+y)u_x + u(x-y)u_y = x^2 + y^2, \tag{3.5.37}$$

with the Cauchy data $u = 0$ on $y = 2x$.

The characteristic equations are

$$\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2+y^2} = \frac{y\,dx + x\,dy - u\,du}{0} = \frac{x\,dx - y\,dy - u\,du}{0}.$$

Consequently,

$$d\left[\left(xy - \frac{1}{2}u^2\right)\right] = 0 \quad \text{and} \quad d\left[\frac{1}{2}\left(x^2 - y^2 - u^2\right)\right] = 0. \tag{3.5.38}$$

These give two integrals

$$u^2 - x^2 + y^2 = C_1 \quad \text{and} \quad 2xy - u^2 = C_2, \tag{3.5.39ab}$$

where C_1 and C_2 are constants. Hence, the general solution is

$$f\left(x^2 - y^2 - u^2, 2xy - u^2\right) = 0,$$

where f is an arbitrary function.

Using the Cauchy data in (3.5.39ab), we obtain $4C_1 = 3C_2$. Therefore,

$$4\left(u^2 - x^2 + y^2\right) = 3\left(2xy - u^2\right).$$

Thus, the solution of equation (3.5.37) is given by

$$7u^2 = 6xy + 4\left(x^2 - y^2\right). \tag{3.5.40}$$

Example 3.5.6. Obtain the solution of the linear equation

$$u_x - u_y = 1, \tag{3.5.41}$$

with the Cauchy data

$$u(x,0) = x^2.$$

The characteristic equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{1}. \tag{3.5.42}$$

Obviously,

$$\frac{dy}{dx} = -1 \quad \text{and} \quad \frac{du}{dx} = 1.$$

Clearly,

$$x + y = \text{const.} = C_1 \quad \text{and} \quad u - x = \text{const.} = C_2.$$

Thus, the general solution is given by

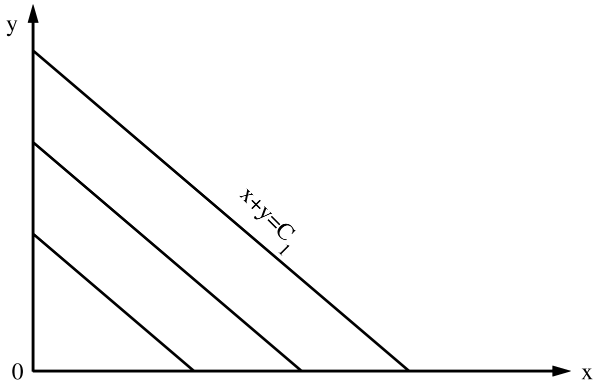

Fig. 3.2 Characteristics of equation (3.5.41).

$$u - x = f(x + y), \tag{3.5.43}$$

where f is an arbitrary function.

We now use the Cauchy data to find $f(x) = x^2 - x$, and hence, the solution is

$$u(x, y) = (x + y)^2 - y. \tag{3.5.44}$$

The characteristics $x+y = C_1$ are drawn in Figure 3.2. The value of u must be given at one point on each characteristic which intersects the line $y = 0$ only at one point, as shown in Figure 3.2.

Example 3.5.7. Obtain the solution of the equation

$$(y - u)u_x + (u - x)u_y = x - y, \tag{3.5.45}$$

with the condition $u = 0$ on $xy = 1$.

The characteristic equations for equation (3.5.45) are

$$\frac{dx}{y - u} = \frac{dy}{u - x} = \frac{du}{x - y}. \tag{3.5.46}$$

The parametric forms of these equations are

$$\frac{dx}{dt} = y - u, \qquad \frac{dy}{dt} = u - x, \qquad \frac{du}{dt} = x - y.$$

These lead to the following equations:

$$\dot{x} + \dot{y} + \dot{u} = 0 \quad \text{and} \quad x\dot{x} + y\dot{y} + u\dot{u} = 0, \tag{3.5.47ab}$$

where the dot denotes the derivative with respect to t.

Integrating (3.5.47ab), we obtain

$$x + y + u = \text{const.} = C_1 \quad \text{and} \quad x^2 + y^2 + u^2 = \text{const.} = C_2. \tag{3.5.48ab}$$

These equations represent circles.

Using the Cauchy data, we find that

$$C_1^2 = (x+y)^2 = x^2 + y^2 + 2xy = C_2 + 2.$$

Thus, the integral surface is described by

$$(x+y+u)^2 = x^2 + y^2 + u^2 + 2.$$

Hence, the solution is given by

$$u(x,y) = \frac{1-xy}{x+y}. \tag{3.5.49}$$

Example 3.5.8. Solve the linear equation

$$yu_x + xu_y = u, \tag{3.5.50}$$

with the Cauchy data

$$u(x,0) = x^3 \quad \text{and} \quad u(0,y) = y^3. \tag{3.5.51ab}$$

The characteristic equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{u},$$

or equivalently,

$$\frac{du}{u} = \frac{dx-dy}{y-x} = \frac{dx+dy}{y+x}.$$

Solving these equations, we obtain

$$u = \frac{C_1}{x-y} = C_2(x+y),$$

or

$$u = C_2(x+y), \qquad x^2 - y^2 = \frac{C_1}{C_2} = \text{const.} = C.$$

So the characteristics are rectangular hyperbolas for $C > 0$ or $C < 0$.

Thus, the general solution is given by

$$f\left(\frac{u}{x+y}, x^2 - y^2\right) = 0.$$

Or equivalently,

$$u(x,y) = (x+y)g\left(x^2 - y^2\right). \tag{3.5.52}$$

Using the Cauchy data, we find that $g(x^2) = x^2$, that is, $g(x) = x$.

Consequently, the solution becomes

$$u(x,y) = (x+y)\left(x^2-y^2\right) \quad \text{on } x^2-y^2=C>0.$$

Similarly,

$$u(x,y) = (x+y)\left(y^2-x^2\right) \quad \text{on } y^2-x^2=C>0.$$

It follows from these results that $u \to 0$ in all regions, as $x \to \pm y$ (or $y \to \pm x$), and hence, u is continuous across $y = \pm x$ which represent asymptotes of the rectangular hyperbolas $x^2 - y^2 = C$. However, u_x and u_y are *not* continuous, as $y \to \pm x$. For $x^2 - y^2 = C > 0$,

$$\begin{aligned} u_x &= 3x^2 + 2xy - y^2 = (x+y)(3x-y) \to 0, \quad \text{as } y \to -x, \\ u_y &= -3y^2 - 2xy + x^2 = (x+y)(x-3y) \to 0, \quad \text{as } y \to -x. \end{aligned}$$

Hence, both u_x and u_y are continuous as $y \to -x$. On the other hand,

$$u_x \to 4x^2, \qquad u_y \to -4x^2 \quad \text{as } y \to x.$$

This implies that u_x and u_y are discontinuous across $y = x$.

Combining all these results, we conclude that $u(x,y)$ is continuous everywhere in the (x,t)-plane, and u_x, u_y are continuous everywhere in the (x,t)-plane except on the line $y = x$. Hence, the partial derivatives u_x, u_y are discontinuous on $y = x$. Thus, the development of *discontinuities* across characteristics is a significant feature of the solutions of partial differential equations.

Example 3.5.9. Determine the integral surfaces of the equation

$$x\left(y^2+u\right)u_x - y\left(x^2+u\right)u_y = \left(x^2-y^2\right)u, \tag{3.5.53}$$

with the data

$$x + y = 0, \qquad u = 1.$$

The characteristic equations are

$$\frac{dx}{x(y^2+u)} = \frac{dy}{-y(x^2+u)} = \frac{du}{(x^2-y^2)u}, \tag{3.5.54}$$

or equivalently,

$$\frac{\frac{dx}{x}}{(y^2+u)} = \frac{\frac{dy}{y}}{-(x^2+u)} = \frac{\frac{du}{u}}{(x^2-y^2)} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{du}{u}}{0}.$$

Consequently,

$$\log(xyu) = \log C_1,$$

or

$$xyu = C_1.$$

From (3.5.54), we obtain

$$\frac{x\,dx}{x^2(y^2+u)} = \frac{y\,dy}{-y^2(x^2+u)} = \frac{du}{(x^2-y^2)u} = \frac{x\,dx + y\,dy - du}{0},$$

whence we find that

$$x^2 + y^2 - 2u = C_2.$$

Using the given data, we obtain

$$C_1 = -x^2 \quad \text{and} \quad C_2 = 2x^2 - 2,$$

so that

$$C_2 = -2(C_1 + 1).$$

Thus the integral surface is given by

$$x^2 + y^2 - 2u = -2 - 2xyu,$$

or

$$2xyu + x^2 + y^2 - 2u + 2 = 0. \tag{3.5.55}$$

Example 3.5.10. Obtain the solution of the equation

$$xu_x + yu_y = x\exp(-u) \tag{3.5.56}$$

with the data

$$u = 0 \quad \text{on } y = x^2.$$

The characteristic equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{x\exp(-u)}, \tag{3.5.57}$$

or

$$\frac{y}{x} = C_1.$$

We also obtain from (3.5.57) that $dx = e^u\,du$ which can be integrated to find

$$e^u = x + C_2.$$

Thus, the general solution is given by

$$f\left(e^u - x, \frac{y}{x}\right) = 0$$

or equivalently,

$$e^u = x + g\left(\frac{y}{x}\right). \tag{3.5.58}$$

Applying the Cauchy data, we obtain $g(x) = 1 - x$. Thus, the solution of (3.5.56) is given by

$$e^u = x + 1 - \frac{y}{x},$$

or

$$u = \log\left(x + 1 - \frac{y}{x}\right). \tag{3.5.59}$$

Example 3.5.11. Solve the initial-value problem

$$u_t + u\,u_x = x, \qquad u(x,0) = f(x), \tag{3.5.60}$$

where (a) $f(x) = 1$ and (b) $f(x) = x$.

The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{x} = \frac{d(x+u)}{x+u}. \tag{3.5.61}$$

Integration gives

$$t = \log(x+u) - \log C_1,$$

or

$$(u+x)e^{-t} = C_1.$$

Similarly, we get

$$u^2 - x^2 = C_2.$$

For case (a), we obtain

$$1 + x = C_1 \quad \text{and} \quad 1 - x^2 = C_2, \quad \text{and hence,} \quad C_2 = 2C_1 - C_1^2.$$

Thus,

$$\left(u^2 - x^2\right) = 2(u+x)e^{-t} - (u+x)^2 e^{-2t},$$

or equivalently,

$$u - x = 2e^{-t} - (u+x)e^{-2t}.$$

A simple manipulation gives the solution

$$u(x,t) = x\tanh t + \operatorname{sech} t. \tag{3.5.62}$$

Case (b) is left to the reader as an exercise.

Example 3.5.12. Find the integral surface of the equation

$$u\,u_x + u_y = 1, \tag{3.5.63}$$

so that the surface passes through an initial curve represented parametrically by

$$x = x_0(s), \qquad y = y_0(s), \qquad u = u_0(s), \tag{3.5.64}$$

where s is a parameter.

The characteristic equations for the given equations are

$$\frac{dx}{u} = \frac{dy}{1} = \frac{du}{1},$$

which are, in the parametric form,

$$\frac{dx}{d\tau} = u, \qquad \frac{dy}{d\tau} = 1, \qquad \frac{du}{d\tau} = 1, \tag{3.5.65}$$

where τ is a parameter. Thus the solutions of this parametric system in general depend on two parameters s and τ. We solve this system (3.5.65) with the initial data

$$x(s,0) = x_0(s), \qquad y(s,0) = y_0(s), \qquad u(s,0) = u_0(s).$$

The solutions of (3.5.65) with the given initial data are

$$\left.\begin{aligned} x(s,\tau) &= \tfrac{\tau^2}{2} + \tau\, u_0(s) + x_0(s), \\ y(s,\tau) &= \tau + y_0(s), \\ u(s,\tau) &= \tau + u_0(s). \end{aligned}\right\} \tag{3.5.66}$$

We choose a particular set of values for the initial data as

$$x(s,0) = 2s^2, \quad y(s,0) = 2s, \quad u(s,0) = 0, \quad s > 0.$$

Therefore, the solutions are given by

$$x = \frac{1}{2}\tau^2 + 2s^2, \qquad y = \tau + 2s, \qquad u = \tau. \tag{3.5.67}$$

Eliminating τ and s from (3.5.67) gives the integral surface

$$(u-y)^2 + u^2 = 2x,$$

or

$$2u = y \pm \left(4x - y^2\right)^{\frac{1}{2}}. \tag{3.5.68}$$

The solution surface satisfying the data $u = 0$ on $y^2 = 2x$ is given by

$$2u = y - \left(4x - y^2\right)^{\frac{1}{2}}. \tag{3.5.69}$$

This represents the solution surface only when $y^2 < 4x$. Thus, the solution does not exist for $y^2 > 4x$ and is *not* differentiable when $y^2 = 4x$. We verify that $y^2 = 4x$ represents the *envelope* of the family of characteristics in the (x,t)-plane given by the τ-eliminant of the first two equations in (3.5.67), that is,

$$F(x,y,s) = 2x - (y-2s)^2 - 4s^2 = 0. \tag{3.5.70}$$

This represents a family of parabolas for different values of the parameter s. Thus, the envelope is obtained by eliminating s from equations $\frac{\partial F}{\partial s} = 0$ and $F = 0$. This gives $y^2 = 4x$, which is the envelope of the characteristics for different s, as shown in Figure 3.3.

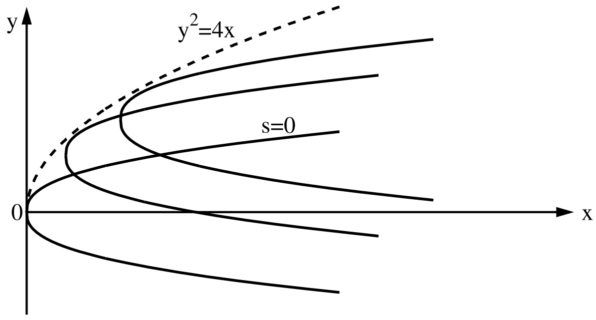

Fig. 3.3 Dotted curve is the envelope of the characteristics.

3.6 Exercises

1. (a) Show that the family of right circular cones whose axis coincides with the z-axis

$$x^2 + y^2 = (z - c)^2 \tan^2 \alpha$$

satisfies the first-order, partial differential equation

$$yp - xq = 0.$$

(b) Show that all the surfaces of revolution, $z = f(x^2+y^2)$, with the z-axis as the axis of symmetry, where f is an arbitrary function, satisfy the partial differential equation

$$yp - xq = 0.$$

(c) Show that the two-parameter family of curves $u - ax - by - ab = 0$ satisfies the nonlinear equation

$$xp + yq + pq = u.$$

2. Find the partial differential equation arising from each of the following surfaces:

(a) $z = x + y + f(xy)$, (b) $z = f(x - y)$,
(c) $z = xy + f(x^2 + y^2)$, (d) $2z = (\alpha x + y)^2 + \beta$.

3. Find the general solution of each of the following equations:

(a) $u_x = 0$, (b) $au_x + b\,u_y = 0$, where a, b, are constant,
(c) $u_x + y\,u_y = 0$, (d) $(1 + x^2)u_x + u_y = 0$,
(e) $2xy\,u_x + (x^2 + y^2)u_y = 0$, (f) $(y + u)u_x + y\,u_y = x - y$,
(g) $y^2 u_x - xy\,u_y = x(u - 2y)$, (h) $yu_y - xu_x = 1$,
(i) $y^2 up + u^2 xq = -xy^2$, (j) $(y - xu)p + (x + yu)q = x^2 + y^2$.

4. Show that the general solution of the equation

$$u_x + 2xy^2 u_y = 0$$

is given by

$$u = f\left(x^2 + \frac{1}{y}\right),$$

where f is an arbitrary function. Verify by differentiation that u satisfies the original equation.

5. Find the solution of the following Cauchy problems:
   (a) $3u_x + 2u_y = 0$, with $u(x,0) = \sin x$,
   (b) $y\,u_x + x\,u_y = 0$, with $u(0,y) = \exp(-y^2)$,
   (c) $x\,u_x + y\,u_y = 2xy$, with $u = 2$ on $y = x^2$,
   (d) $u_x + x\,u_y = 0$, with $u(0,y) = \sin y$,
   (e) $y\,u_x + x\,u_y = xy$, $x \geq 0$, $y \geq 0$ with $u(0,y) = \exp(-y^2)$ for $y > 0$ and $u(x,0) = \exp(-x^2)$ for $x > 0$,
   (f) $u_x + x\,u_y = (y - \frac{1}{2}x^2)^2$, with $u(0,y) = \exp(y)$,
   (g) $x\,u_x + y\,u_y = u + 1$, with $u(x,y) = x^2$ on $y = x^2$,
   (h) $u\,u_x - u\,u_y = u^2 + (x+y)^2$, with $u = 1$ on $y = 0$,
   (i) $x\,u_x + (x+y)u_y = u + 1$, with $u(x,y) = x^2$ on $y = 0$.
6. Solve the initial-value problem

$$u_t + u\,u_x = 0$$

with the initial curve

$$x = \frac{1}{2}\tau^2, \quad t = \tau, \; u = \tau.$$

7. Find the solution of the Cauchy problem

$$2xy\,u_x + \left(x^2 + y^2\right)u_y = 0, \quad \text{with } u = \exp\left(\frac{x}{x-y}\right) \text{ on } x + y = 1.$$

8. Solve the following equations:
   (a) $x\,u_x + y\,u_y + z\,u_z = 0$,
   (b) $x^2\,u_x + y^2\,u_y + z(x+y)u_z = 0$,
   (c) $x(y-z)u_x + y(z-x)u_y + z(x-y)u_z = 0$,
   (d) $yz\,u_x - xz\,u_y + xy(x^2+y^2)u_z = 0$,
   (e) $x(y^2 - z^2)u_x + y(z^2 - y^2)u_y + z(x^2 - y^2)u_z = 0$.
9. Solve the equation

$$u_x + x\,u_y = y$$

with the Cauchy data

$$\text{(a) } u(0,y) = y^2, \qquad \text{(b) } u(1,y) = 2y.$$

10. Show that $u_1 = e^x$ and $u_2 = e^{-y}$ are solutions of the nonlinear equation

$$(u_x + u_y)^2 - u^2 = 0,$$

but that their sum $(e^x + e^{-y})$ is not a solution of the equation.

11. Solve the Cauchy problem

$$(y + u)u_x + y\,u_y = (x - y), \quad \text{with } u = 1 + x \text{ on } y = 1.$$

12. Find the integral surfaces of the equation $u\,u_x + u_y = 1$ for each of the following initial data:
(a) $x(s,0) = s$, $y(s,0) = 2s$, $u(s,0) = s$,
(b) $x(s,0) = s^2$, $y(s,0) = 2s$, $u(s,0) = s$,
(c) $x(s,0) = \frac{s^2}{2}$, $y(s,0) = s$, $u(s,0) = s$.
Draw characteristics in each case.
13. Show that the solution of the equation

$$yu_x - x\,u_y = 0$$

containing the curve $x^2 + y^2 = a^2$, $u = y$, does not exist.
14. Solve the following Cauchy problems:
(a) $x^2u_x - y^2u_y = 0$, $u \to e^x$ as $y \to \infty$,
(b) $y\,u_x + x\,u_y = 0$, $u = \sin x$ on $x^2 + y^2 = 1$,
(c) $x\,u_x - y\,u_y = -1$ for $0 < x < y$, $u = 2x$ on $y = 3x$,
(d) $2x\,u_x + (x+1)u_y = y$ for $x > 0$, $u = 2y$ on $x = 1$,
(e) $x\,u_x + y\,u_y = x^2 + y^2$ for $x > 0$, $y > 0$, $u = x^2$ on $y = 1$,
(f) $y^2u_x + (xy)u_y = x$, $u = x^2$ when $y = 1$,
(g) $x\,u_x + y\,u_y = xy$, $u = \frac{1}{2}x^2$ when $y = x$.
15. Find the solution surface of the equation

$$\left(u^2 - y^2\right)u_x + xy\,u_y + xu = 0, \quad \text{with } u = y = x,\ x > 0.$$

16. (a) Solve the Cauchy problem

$$u_x + uu_y = 1, \qquad u(0,y) = ay, \quad \text{where } a \text{ is a constant.}$$

(b) Find the solution of the equation in (a) with the data

$$x(s,0) = 2s, \qquad y(s,0) = s^2, \qquad u\left(0,s^2\right) = s.$$

17. Solve the following equations:
(a) $(y+u)u_x + (x+u)u_y = x + y$,
(b) $x\,u(u^2 + xy)u_x - y\,u(u^2 + xy)u_y = x^4$,
(c) $(x+y)u_x + (x-y)u_y = 0$,
(d) $y\,u_x + x\,u_y = xy(x^2 - y^2)$,
(e) $(cy - bz)z_x + (az - cx)z_y = bx - ay$.
18. Solve the equation

$$x\,z_x + y\,z_y = z,$$

and find the curves which satisfy the associated characteristic equations and intersect the helix $x + y^2 = a^2$, $z = b\ \tan^{-1}(\frac{y}{x})$.
19. Obtain the family of curves which represent the general solution of the partial differential equation

$$(2x - 4y + 3u)u_x + (x - 2y - 3u)u_y = -3(x - 2y).$$

Determine the particular member of the family which contains the line $u = x$ and $y = 0$.

20. Find the solution of the equation

$$y\,u_x - 2xy\,u_y = 2xu$$

with the condition $u(0,y) = y^3$.

21. Obtain the general solution of the equation

$$(x+y+5z)p + 4zq + (x+y+z) = 0 \quad (p = z_x,\ q = z_y),$$

and find the particular solution which passes through the circle

$$z = 0, \qquad x^2 + y^2 = a^2.$$

22. Obtain the general solution of the equation

$$\left(z^2 - 2yz - y^2\right)p + x(y+z)q = x(y-z) \quad (p = z_x,\ q = z_y).$$

Find the integral surfaces of this equation passing through (a) the x-axis, (b) the y-axis, and (c) the z-axis.

23. Solve the Cauchy problem

$$(x+y)u_x + (x-y)u_y = 1, \qquad u(1,y) = \frac{1}{\sqrt{2}}.$$

24. Solve the following Cauchy problems:
    (a) $3\,u_x + 2\,u_y = 0$, $u(x,0) = f(x)$,
    (b) $a\,u_x + b\,u_y = c\,u$, $u(x,0) = f(x)$, where a, b, c are constants,
    (c) $x\,u_x + y\,u_y = c\,u$, $u(x,0) = f(x)$,
    (d) $u\,u_x + u_y = 1$, $u(s,0) = \alpha s$, $x(s,0) = s$, $y(s,0) = 0$.
25. Apply the method of separation of variables $u(x,y) = f(x)g(y)$ to solve the following equations:
    (a) $u_x u_y = u^2$,
    (b) $u_x + u = u_y$, $u(x,0) = 4e^{-3x}$,
    (c) $u_x + 2u_y = 0$, $u(0,y) = 3e^{-2y}$,
    (d) $y^2 u_x^2 + x^2 u_y^2 = (xyu)^2$,
    (e) $x^2 u_{xy} + 9y^2 u = 0$, $u(x,0) = \exp(\frac{1}{x})$.
26. Use a separable solution $u(x,y) = f(x) + g(y)$ to solve the following equations:

    (a) $u_x^2 + u_y^2 = 1$, (b) $u_x^2 + u_y^2 = u$,
    (c) $u_x^2 + u_y + x^2 = 0$, (d) $x^2 u_x^2 + y^2 u_y^2 = 1$,
    (e) $y\,u_x + x\,u_y = 0$, $u(0,y) = y^2$.

27. Apply $v = \ln u$ and then $v(x,y) = f(x) + g(y)$ to solve the following equations:
    (a) $x^2 u_x^2 + y^2 u_y^2 = u^2$,
    (b) $x^2 u_x^2 + y^2 u_y^2 = (xyu)^2$.
28. Apply $\sqrt{u} = v$ and $v(x,y) = f(x) + g(y)$ to solve the equation

$$x^4 u_x^2 + y^2 u_y^2 = 4u.$$

29. Using $v = \ln u$ and $v = f(x) + g(y)$, show that the solution of the Cauchy problem

$$y^2 u_x^2 + x^2 u_y^2 = (xyu)^2, \qquad u(x,0) = e^{x^2}$$

is

$$u(x,y) = \exp\left(x^2 + i\frac{\sqrt{3}}{2}\, y^2\right).$$

30. Consider the eigenvalue problem for the Klein–Gordon equation

$$u_{xx} = \frac{1}{c^2}\, u_{tt} + a^2 u, \quad 0 < x < 1,\ t > 0,$$
$$u(0,t) = 0 = u(1,t), \quad t > 0,$$
$$u(x,0) = 0, \quad \text{and} \quad u_t(x,0) = 1, \quad 0 < x < 1.$$

(a) Using $u(x,t) = X(x)T(t)$, show that

$$X'' + \lambda^2 X = 0, \qquad \ddot{T} + \left(a^2 + \lambda^2 c^2\right) T = 0,$$

where $-\lambda^2$ is a separation constant.
(b) Show that the eigenvalues and eigenfunctions are

$$\lambda = \lambda_n = n\pi, \quad X_n(x) = \beta_n \sin(n\pi x), \quad n = 1, 2, \ldots.$$

(c) Use the Fourier series to obtain the final solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{n\pi\omega_n}\right)\left\{1 - (-1)^{n+1}\right\} \sin(\omega_n t)\sin(n\pi x),$$

where $\omega_n = \sqrt{a^2 + \lambda_n^2 c^2}$.

31. Solve the following system of equations with initial data:
(a) $u_t + 3u u_x = v - x$, $v_t - c\, v_x = 0$, $u(x,0) = x$, and $v(x,0) = x$,
(b) $u_t + 2u u_x = v - x$, $v_t - c\, v_x = 0$, $u(x,0) = x$, and $v(x,0) = x$.

32. Find the solution of the system of equations with initial data:
(a) $u_t + u u_x = v\, e^{-x}$, $v_t - a\, v_x = 0$,
$u(x,0) = x$, and $v(x,0) = e^x$,
(b) $u_t - 2u u_x = v - x$, $v_t + c\, v_x = 0$,
$u(x,0) = x$, and $v(x,0) = x$.

33. Solve the following Cauchy problems with initial data:
(a) $\sqrt{x}\, u_x + u u_y = -u^2$, $u(x,0) = 1$, $0 < x < \infty$,
(b) $(x^2 u) u_x + e^{-y} u_y = -u^2$, $u(x,0) = 1$, $0 < x < \infty$.

34. Use $\xi(x,t) = x - ct$ and $\tau = \mu t$ to transform the equation

$$u_t + c u_x = d\, u^n u_x$$

in the form

$$\frac{\partial u}{\partial \tau} + \alpha u^n \frac{\partial u}{\partial \xi} = 0, \quad \alpha = (d/\mu).$$

Show that the solution of the equation with $u(\xi, 0) = u_0 \sin k\xi$ is

$$u(\xi, \tau) = u_0 \sin\left[k\xi - \left(\frac{u}{u_0}\right)^n (\alpha k u_0^n \tau)\right].$$

35. Use the method of characteristics to solve the equations:
    (a) $(x + y)u_x + (x - y)u_y = 0$,
    (b) $u_x - (ax + by)u_y = 0$,
    (c) $x^{-1}u_x + y^{-1}u_y = x^2 - y^2$,
    (d) $x \sin x \, u_x + \frac{1}{y}u_y = u$,
    (e) $yu_x - xu_y = 0$, $u = y$ when $x = a$,
    (f) $xu_x + (x + y)u_y = \frac{1}{u}, \; u = 0, \; \text{when } y = x^2$.
36. Solve the first order Fokker–Planck equation with initial condition:

$$u_t - xu_x = u, \qquad u(x, 0) = f(x).$$

4

First-Order Nonlinear Equations and Their Applications

Physics can't exist without mathematics which provides it with the only language in which it can speak. Thus, services are continuously exchanged between pure mathematical analysis and physics. It is really remarkable that among works of analysis most useful for physics were those cultivated for their own beauty. In exchange, physics, exposing new problems, is as useful for mathematics as it is a model for an artist.

Henri Poincaré

Our present analytical methods seem unsuitable for the solution of the important problems arising in connection with nonlinear partial differential equations and, in fact, with virtually all types of nonlinear problems in pure mathematics. The truth of this statement is particularly striking in the field of fluid dynamics. Only the most elementary problems have been solved analytically in this field...

John Von Neumann

4.1 Introduction

First-order, nonlinear, partial differential equations arise in various areas of physical sciences which include geometrical optics, fluid dynamics, and analytical dynamics. An important example of such equations is the Hamilton–Jacobi equation used to describe dynamical systems. Another famous example of the first-order nonlinear equations is the eikonal equation which arises in nonlinear optics and also describes the propagation of wave fronts and discontinuities for acoustic wave equations, Maxwell's equations, and equations of elastic wave propagation. Evidently, first-order, nonlinear equations play an important role in the development of these diverse areas.

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This chapter deals with the theory of the first-order nonlinear equations and their applications. The generalized method of characteristics is developed to solve these nonlinear equations. This is followed by several examples, as well as examples of applications in analytical dynamics, quantum mechanics, and nonlinear optics.

4.2 The Generalized Method of Characteristics

The most general, first-order, nonlinear partial differential equation in two independent variables x and y has the form

$$F(x,y,u,p,q) = 0, \tag{4.2.1}$$

where $u = u(x,y)$, $p = u_x$, and $q = u_y$.

It has been shown in Section 3.3 that the complete solution (integral) of (4.2.1) is a two-parameter family of surfaces of the form

$$f(x,y,u,a,b) = 0, \tag{4.2.2}$$

where a and b are parameters. Specifying a space curve, through which the complete integral must pass, generates a solution surface. The equation of the tangent plane at each point (x,y,u) of the solution surface is

$$p(x-\xi) + q(y-\eta) - (u-\zeta) = 0, \tag{4.2.3}$$

where (ξ,η,ζ) are running coordinates on the tangent plane. Since the given equation (4.2.1) is a relation between p and q for *any* solution surface, there is a family of tangent planes corresponding to different values of p and q. We can find an equation that represents the envelope of these planes by considering the intersection of (4.2.3) and the tangent plane at the same point (x,y,u) corresponding to neighboring values $p+dp$ and $q+dq$ of p and q,

$$(p+dp)(x-\xi) + (q+dq)(y-\eta) - (u-\zeta) = 0. \tag{4.2.4}$$

Thus, the intersection of (4.2.3) and (4.2.4) leads to the result

$$(x-\xi)dp + (y-\eta)dq = 0. \tag{4.2.5}$$

We can rewrite this equation in terms of quantities specific to the given equation by calculating the differential of (4.2.1) with fixed x, y, and u, that is,

$$F_p dp + F_q dq = 0. \tag{4.2.6}$$

Eliminating dp and dq from (4.2.5) and (4.2.6) gives

$$(x-\xi)F_q - (y-\eta)F_p = 0,$$

or

$$\frac{x-\xi}{F_p} = \frac{y-\eta}{F_q}. \tag{4.2.7}$$

The equation of the surface for which the tangent planes are the envelopes is thus determined by eliminating p and q from (4.2.1), (4.2.3), and (4.2.7). This surface represents a cone for nonlinear equations, called the *Monge cone*. We next combine (4.2.7) with (4.2.3) to obtain

$$\frac{x-\xi}{F_p} = \frac{y-\eta}{F_q} = \frac{u-\zeta}{pF_p + qF_q}. \tag{4.2.8}$$

Both (x, y, u) and (ξ, η, ζ) lie on the tangent plane, and hence, (4.2.8) represents a line on the tangent plane to the solution surface. Further, $x - \xi$, $y - \eta$, and $u - \zeta$ are direction numbers of a line on the tangent plane, and hence, a fixed direction on each tangent plane (for particular values of p and q) is determined by (4.2.8). This direction is determined by the vector $(F_p, F_q, pF_p + qF_q)$. As p and q change, these directions, known as the *characteristic directions*, determine the family of lines (4.2.8) that generate a Monge cone at (x, y, u). To find differential equations for the characteristics as this point moves along a solution surface, we replace $x - \xi$, $y - \eta$, $u - \zeta$ by dx, dy, du, respectively, so that (4.2.8) becomes

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q}. \tag{4.2.9}$$

The system (4.2.9) represents equations for the characteristics curves of (4.2.1). As in the case of the quasi-linear equation, equations (4.2.9) depend on the unknown solution $u(x, y)$, but unlike the quasi-linear case, they also depend on p and q. This means that the characteristic curves also depend on the *orientation* of the tangent planes on the Monge cone at each point. As this geometrical analysis suggests, there is a whole (Monge) cone of characteristics, not just *one* characteristic, as in the quasi-linear case. By equating the ratios in (4.2.9) with dt, we obtain the differential equations for the characteristic curves in the parametric form

$$\frac{dx}{dt} = F_p, \qquad \frac{dy}{dt} = F_q, \qquad \frac{du}{dt} = pF_p + qF_q. \tag{4.2.10}$$

This is a system of three equations for five unknowns x, y, u, p, and q. Thus, for the nonlinear equation (4.2.1), the system (4.2.10) is not a closed set. We need two more equations to close the system, and hence, it is natural to look for equations for $\frac{dp}{dt}$ and $\frac{dq}{dt}$. Since $p = p[x(t), y(t)]$ and $q = q[x(t), y(t)]$, we have, by the chain rule and (4.2.10),

$$\frac{dp}{dt} = p_x \frac{dx}{dt} + p_y \frac{dy}{dt} = p_x F_p + p_y F_q, \tag{4.2.11}$$

$$\frac{dq}{dt} = q_x \frac{dx}{dt} + q_y \frac{dy}{dt} = q_x F_p + q_y F_q. \tag{4.2.12}$$

Further, $u = u(x, y)$ is a solution of the equation (4.2.1) which gives

$$\frac{dF}{dx} = F_x + pF_u + p_x F_p + q_x F_q = 0, \tag{4.2.13}$$

$$\frac{dF}{dy} = F_y + qF_u + p_y F_p + q_y F_q = 0. \tag{4.2.14}$$

We next use the fact that $p_y = q_x$, which can be considered an *integrability condition* for the solution surface, and then utilize (4.2.13), (4.2.14) in (4.2.11), (4.2.12) to obtain

$$\frac{dp}{dt} = -(F_x + pF_u), \qquad \frac{dq}{dt} = -(F_y + qF_u). \tag{4.2.15}$$

Equations (4.2.10) and (4.2.15) form a closed set of five ordinary differential equations for five unknown functions x, y, u, p, and q. They are known as *characteristic equations* for the nonlinear equation (4.2.1). In principle, these can be solved provided that all five unknown functions are prescribed at $t = 0$. Usually, as in the case of quasi-linear equations, the Cauchy data are sufficient to specify x, y, and u at $t = 0$ in terms of some parameter s on an initial curve C. It is also necessary to determine initial values for p and q in terms of s at $t = 0$.

Eliminating dt from (4.2.10) and (4.2.15) gives the Charpit equations

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = -\frac{dp}{F_x + pF_u} = -\frac{dq}{F_y + qF_u}. \tag{4.2.16}$$

We formulate the Cauchy problem for (4.2.1) to determine the family of curves in the parametric form

$$x = x(t,s), \qquad y = y(t,s), \quad \text{and} \quad u = u(t,s), \tag{4.2.17}$$

and the values of p and q as

$$p = p(t,s) \quad \text{and} \quad q = q(t,s), \tag{4.2.18}$$

where t is a running variable along a particular curve and s is a parameter which specifies a member of the family.

We seek a solution surface $u = u(x,y)$ with the given initial data at $t = 0$ as

$$C\colon \quad x(0,s) = x_0(s), \qquad y(0,s) = y_0(s), \quad \text{and} \quad u(0,s) = u_0(s), \tag{4.2.19}$$

and determine the initial conditions for $p(t,s)$ and $q(t,s)$ at $t = 0$ as

$$p(0,s) = p_0(s) \quad \text{and} \quad q(0,s) = q_0(s), \tag{4.2.20}$$

where $p_0(s)$ and $q_0(s)$ are to be determined.

A set of five quantities $x_0(s)$, $y_0(s)$, $u_0(s)$, $p_0(s)$, and $q_0(s)$ cannot all be chosen independently, since we must observe the relation

$$\frac{du_0}{ds} = \frac{du_0}{dx_0}\frac{dx_0}{ds} + \frac{du_0}{dy_0}\frac{dy_0}{ds} = p_0(s)x_0'(s) + q_0(s)y_0'(s), \tag{4.2.21}$$

where $p_0(s)$ and $q_0(s)$ are the initial directions which determine the normal to the solution surface and $u_0(s) = u(x_0(s), y_0(s))$. The relation (4.2.21) is called the *strip condition*. The initial curve $x_0(s)$, $y_0(s)$, $u_0(s)$, and the orientations of the tangent planes $p_0(s)$ and $q_0(s)$ are referred to as the *initial strip*, since the initial conditions can be viewed as the initial curve combined with the initial tangent planes attached.

Since $p_0(s)$ and $q_0(s)$ are the initial values of $p(t,s)$ and $q(t,s)$ at $t = 0$ on the curve C, these values must satisfy the original equation, that is,

$$F(x_0, y_0, u_0, p_0, q_0) = 0. \tag{4.2.22}$$

Evidently, equations (4.2.21) and (4.2.22) determine the initial values $p_0(s)$ and $q_0(s)$. Thus the set of five equations (4.2.10) and (4.2.15) with the initial data (4.2.19), (4.2.20) can be solved to obtain solutions (4.2.17) and (4.2.18).

It is necessary to use the first two equations of the system (4.2.17) for finding the parameters t and s in terms of x and y, and the result is substituted into $u = u(t,s)$ to obtain a solution surface $u = u(x,y)$. The sufficient condition for expressing t and s in terms of x and y from the first two equations of the system (4.2.17) is that the Jacobian J is nonzero along the initial strip, that is,

$$J = \frac{\partial(x,y)}{\partial(t,s)} = \begin{vmatrix} x_t & y_t \\ x_s & y_s \end{vmatrix} = \begin{vmatrix} F_p & F_q \\ x_s & y_s \end{vmatrix} \neq 0, \tag{4.2.23}$$

in which the first two equations of the system (4.2.10) are used, where F_p and F_q in (4.2.23) are computed at the initial strip x_0, y_0, u_0, p_0, and q_0.

Thus, the above analysis shows that if the Jacobian $J \neq 0$ along the initial strip, there exists a *unique* solution of the equation (4.2.1) passing through the given initial curve (4.2.19). However, in general, the solution is *not unique*, but it is unique for a fixed determination of the roots p and q of equations (4.2.21) and (4.2.22). However, if $J = 0$ along the initial strip, then the initial-value problem has a solution if it is a characteristic strip. Indeed, the problem may have infinitely many solutions.

The solutions $x(t,s)$, $y(t,s)$, and $u(t,s)$ of the problem for fixed s give a space curve, and $p(t,s)$ and $q(t,s)$ determine a tangent plane with normal vector $(p, q, -1)$ at each point of the space curve. The space curve together with its tangent plane is called the *characteristic strip*. Similarly, the Charpit equations are known as the *characteristic strip equations*.

4.3 Complete Integrals of Certain Special Nonlinear Equations

Case (i): Nonlinear equations (4.2.1) involving only p and q.

Such equations must be of the form $F(p,q) = 0$. Thus, the Charpit equations (4.2.16) become

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = \frac{dp}{0} = \frac{dq}{0} = dt. \tag{4.3.1}$$

The solutions of the last two equations of (4.3.1) are $p = \text{const.} = a$, and $q = \text{const.}$ on the characteristic curves. Thus, $F(a, q) = 0$, which leads to $q = f(a)$.

It follows from the first two equations of the system (4.3.1) that

$$du = p\, dx + q\, dy,$$

which gives that complete solution by integration as

$$u = ax + f(a)y + b, \tag{4.3.2}$$

where b is a constant of integration.

Case (ii): Equations not involving independent variables.

Such equations are of the form

$$F(u, p, q) = 0. \tag{4.3.3}$$

Thus, the Charpit equations are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qF_q} = \frac{dp}{-pF_u} = \frac{dq}{-qF_u}. \tag{4.3.4}$$

The last equation gives $q = ap$ where a is a constant. Substituting this result into (4.3.3) gives $F(u, p, ap) = 0$ which can be solved for $p = f(u, a)$, where f is an arbitrary function. Hence, $q = af(u, a)$ and the differential equation for u is

$$du = p\, dx + q\, dy = f(u, a)\, dx + af(u, a)\, dy.$$

Integrating this equation gives the complete integral

$$\int^u \frac{du}{f(u, a)} = x + ay + b, \tag{4.3.5}$$

where b is a constant of integration.

Case (iii): Separable equations.

Such equations are of the form

$$F = G(x, p) - H(y, q) = 0. \tag{4.3.6}$$

Thus, the Charpit equations are

$$\frac{dx}{G_p} = \frac{dy}{H_q} = \frac{du}{pG_p + qH_q} = \frac{dp}{-G_x} = \frac{dq}{H_y} = dt.$$

It follows from the first and fourth of these equations that

$$G_x\, dx + G_p\, dp = 0.$$

This gives $G(x,p) = \text{const.} = a = H(y,q)$, and hence, p and q can be solved in terms of x and a, and y and a, respectively, to obtain $p = g(x,a)$ and $q = h(y,a)$, so that

$$du = p\,dx + q\,dy = g(x,a)\,dx + h(y,a)\,dy.$$

This can be integrated to obtain the complete integral

$$u = \int g(x,a)\,dx + \int h(y,a)\,dy + b, \tag{4.3.7}$$

where b is a constant of integration.

Case (iv): Clairaut's equation.

A first-order partial differential equation of the form

$$u = px + qy + f(p,q) \tag{4.3.8}$$

is called *Clairaut's* equation, where f is an arbitrary function of p and q.

The associated Charpit equations are

$$\frac{dx}{x + f_p} = \frac{dy}{y + f_q} = \frac{du}{xp + yq + pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0} = dt.$$

The last two equations imply that $p = \text{const.} = a$ and $q = \text{const.} = b$. Substituting these values in (4.3.8) gives the complete integral of (4.3.8) as

$$u = ax + by + f(a,b). \tag{4.3.9}$$

This represents the two-parameter family of planes. It can be verified by direct differentiation that (4.3.9) satisfies Clairaut's equation. Finally, it can also be shown that the characteristic strips are all straight lines.

We illustrate the above cases by the following examples.

Example 4.3.1. Find the complete integral of the nonlinear equation

$$p^2 + qy - u = 0. \tag{4.3.10}$$

The associated Charpit equations are

$$\frac{dx}{dt} = F_p = 2p, \qquad \frac{dy}{dt} = F_q = y, \qquad \frac{du}{dt} = pF_p + qF_q = 2p^2 + qy,$$
$$\frac{dp}{dt} = -(F_x + pF_u) = p, \qquad \frac{dq}{dt} = -(F_y + qF_u) = 0.$$

The last equation gives $q = \text{const.} = a$. Clearly, the given equation can be solved for p as

$$p = (u - ay)^{\frac{1}{2}}.$$

Also, we have

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{du}{2p^2 + qy} = \frac{p\,dx + q\,dy}{2p^2 + qy},$$

and hence,

$$du = p\,dx + q\,dy = (u - ay)^{\frac{1}{2}}\,dx + a\,dy,$$

or equivalently,

$$\frac{du - a\,dy}{\sqrt{u - ay}} = dx.$$

Integrating this equation gives

$$2(u - ay)^{\frac{1}{2}} = x + b,$$

where b is a constant of integration.

Thus, the complete integral is given by

$$u(x, y) = \frac{1}{4}(x + b)^2 + ay. \tag{4.3.11}$$

Example 4.3.2. Solve the Cauchy problem

$$p^2 q = 1, \qquad u(x, 0) = x. \tag{4.3.12}$$

This equation corresponds to Case (i) with $F = p^2 q - 1$. Hence, $dp = 0$ and $dq = 0$ which gives $p =$ const. $= a$ and $q =$ const. $= c$. Thus, the given equation yields $c = \frac{1}{a^2}$. Consequently, the complete integral is

$$u = ax + \frac{1}{a^2}y + b. \tag{4.3.13}$$

Using the initial data, we obtain

$$x = ax + b,$$

which gives $a = 1$ and $b = 0$. Thus, the solution surface is

$$u(x, y) = x + y. \tag{4.3.14}$$

Example 4.3.3. Solve the Cauchy problem

$$p^2 + q + u = 0, \qquad u(x, 0) = x. \tag{4.3.15}$$

The parametric forms of the Charpit equations are

$$\frac{dx}{dt} = 2p, \qquad \frac{dy}{dt} = 1, \qquad \frac{du}{dt} = 2p^2 + q, \qquad \frac{dp}{dt} = -p, \qquad \frac{dq}{dt} = -q.$$

The parametric forms of the initial data are

$$x(s, 0) = s, \qquad y(s, 0) = 0, \quad \text{and} \quad u(s, 0) = s.$$

Differentiating $u(s,0) = s$ with respect to s gives $p(s,0) = 1$, and then substituting in the given equation shows that $q(s,0) = -(s+1)$. The equations for p and q can be integrated with these initial conditions to obtain

$$p = \exp(-t), \qquad q = -(s+1)\exp(-t). \tag{4.3.16}$$

Substituting these values of p and q into the first three Charpit equations gives

$$\frac{dx}{dt} = 2\exp(-t), \qquad \frac{dy}{dt} = 1, \qquad \frac{du}{dt} = 2\exp(-2t) - (s+1)\exp(-t).$$

Integrating these equations with the initial data gives

$$x = (s+2) - 2\exp(-t), \qquad y = t, \qquad u = (s+1)\exp(-t) - \exp(-2t). \tag{4.3.17}$$

Eliminating t and s gives the solution surface

$$u(x,y) = (x-1)e^{-y} + e^{-2y}. \tag{4.3.18}$$

Example 4.3.4. Find the solution of the initial-value problem

$$p^2 x + qy = u, \qquad u(s,1) = -s. \tag{4.3.19}$$

The parametric forms of the Charpit equations are

$$\frac{dx}{dt} = 2px, \qquad \frac{dy}{dt} = y, \qquad \frac{du}{dt} = 2p^2 x + qy, \qquad \frac{dp}{dt} = p - p^2, \qquad \frac{dq}{dt} = 0.$$

We differentiate $u(s,1) = -s$ with respect to s to obtain $p(s,0) = -1$. Substituting this in the given equation gives $q(s,0) = -2s$. Integrating the last equation of the Charpit system gives $q(s,t) = -2s$.

The equation for p can be integrated to obtain $Ap(1-p)^{-1} = \exp(t)$, where A is a constant of integration. Using the initial data $p(s,0) = -1$, we find that $A = -2$, and hence,

$$p(s,t) = \frac{e^t}{(e^t - 2)}. \tag{4.3.20}$$

Substituting this solution for p in the first equation of the Charpit set and integrating with the condition $x(s,0) = s$, we obtain

$$x(s,t) = s\left(e^t - 2\right)^2. \tag{4.3.21}$$

The equation for y can be solved with $y(s,0) = 1$ and the solution is $y(s,0) = \exp(t)$, so that the equations of characteristics are $x = s(y-2)^2$. They will intersect at $(0,2)$.

Finally, the equation for u can be expressed in terms of t and s as

$$\frac{du}{dt} = 2s\left(e^{2t} - e^t\right).$$

Integrating this equation with $u(s,0) = -s$ gives the solution

$$u(s,t) = s\exp(t)\{\exp(t) - 2\}. \tag{4.3.22}$$

In terms of x and y, the solution takes the form

$$u(x,y) = \frac{xy}{y-2}. \tag{4.3.23}$$

This shows that the solution $u(x,y)$ is singular at $y = 2$.

Example 4.3.5. Solve the initial-value problem

$$p^2 - 3q^2 - u = 0 \quad \text{with } u(x,0) = x^2. \tag{4.3.24}$$

The characteristic strip equations are

$$\frac{dx}{dt} = F_p = 2p, \qquad \frac{dy}{dt} = F_q = -6q, \qquad \frac{du}{dt} = 2\left(p^2 - 3q^2\right) = 2u, \tag{4.3.25}$$

$$\frac{dp}{dt} = -(F_x + pF_u) = p, \qquad \frac{dq}{dt} = -(F_y + qF_u) = q. \tag{4.3.26}$$

The initial data in the parametric form are given by

$$x(0,s) = x_0(s) = s, \qquad y(0,s) = y_0(s) = 0, \qquad u(0,s) = u_0(s) = s^2, \tag{4.3.27}$$

and $p_0(s)$ and $q_0(s)$ are the solutions of equations (4.2.21) and (4.2.22) which become

$$p(0,s) = p_0(s) = 2s, \quad \text{and} \quad p_0^2 - 3q_0^2 = s^2, \tag{4.3.28}$$

and hence,

$$q_0(s) = \mp s. \tag{4.3.29}$$

Next we use $p_0(s) = 2s$ and $q_0(s) = -s$ to solve (4.3.26). It turns out that

$$p(t,s) = 2se^t \quad \text{and} \quad q(t,s) = -se^t. \tag{4.3.30}$$

Substituting these results in (4.3.25) and solving the resulting equations with the initial data (4.3.27) gives

$$x(t,s) = 4s\left(e^t - 1\right) + s, \qquad y(t,s) = 6s\left(e^t - 1\right), \quad \text{and} \quad u(t,s) = s^2e^{2t}. \tag{4.3.31}$$

We use the first two results in (4.3.31) to find s and t in terms of x and y as

$$s = \frac{1}{3}(3x - 2y) \quad \text{and} \quad \exp(t) = \frac{3}{2}\frac{(2x-y)}{(3x-2y)}. \tag{4.3.32}$$

Substituting these results into the third equation of the system (4.3.31) gives the solution surface

$$u(x,y) = \left(x - \frac{y}{2}\right)^2. \tag{4.3.33}$$

If we use the initial data $p_0(s) = 2s$ and $q_0(s) = s$ and solve the problem in a similar way, we obtain a different solution surface

$$u(x,y) = \left(x + \frac{y}{2}\right)^2. \tag{4.3.34}$$

Example 4.3.6. Solve the Clairaut equation

$$z = px + qy + f(p,q) = px + qy + \left(1 + p^2 + q^2\right)^{\frac{1}{2}}, \tag{4.3.35}$$

with the initial data

$$\begin{aligned} &x(0,s) = x_0(s) = a\cos s, \qquad y(0,s) = y_0(s) = a\sin s, \\ &z(0,s) = z_0(s) = 0. \end{aligned} \tag{4.3.36}$$

The characteristic strip equations are

$$\frac{dx}{dt} = x + \frac{p}{f}, \qquad \frac{dy}{dt} = y + \frac{q}{f}, \qquad \frac{dz}{dt} = z - \frac{1}{f}, \tag{4.3.37}$$

$$\frac{dp}{dt} = 0, \quad \text{and} \quad \frac{dq}{dt} = 0. \tag{4.3.38}$$

We have to find the initial data $p(0,s) = p_0(s)$ and $q(0,s) = q_0(s)$ which must satisfy the original equation and the strip condition, that is,

$$ap_0(s)\cos s + aq_0(s)\sin s + \left(1 + p_0^2 + q_0^2\right)^{\frac{1}{2}} = 0, \tag{4.3.39}$$

$$-ap_0(s)\sin s + aq_0(s)\cos s = 0 \quad (a \neq 0). \tag{4.3.40}$$

Clearly, equation (4.3.40) is satisfied if

$$p_0(s) = a_0(s)\cos s \quad \text{and} \quad q_0(s) = a_0(s)\sin s. \tag{4.3.41}$$

Putting these results into (4.3.39) gives $(a^2 - 1)a_0^2(s) = 1$ which leads to

$$a_0(s) = \pm\left(a^2 - 1\right)^{-\frac{1}{2}} = \text{const.} \quad (a > 1). \tag{4.3.42}$$

We take the negative sign for $a_0(s)$ to solve equations (4.3.37) and (4.3.38). In fact, the latter equation gives the solutions

$$\begin{aligned} p(t,s) &= \text{const.} = -(a^2 - 1)^{-\frac{1}{2}}\cos s, \\ q(t,s) &= \text{const.} = -(a^2 - 1)^{-\frac{1}{2}}\sin s. \end{aligned} \tag{4.3.43ab}$$

Then,

$$f = \left(1 + p^2 + q^2\right)^{\frac{1}{2}} = a\left(a^2 - 1\right)^{-\frac{1}{2}}, \tag{4.3.44}$$

which is used to rewrite the first two equations in (4.3.37) as

$$\frac{dx}{dt} = x + \frac{p}{f} = x - \frac{\cos s}{a}, \qquad \frac{dy}{dt} = y - \frac{\sin s}{a}. \tag{4.3.45}$$

The solution of the first equation in (4.3.45) with $x(0, s) = a \cos s$ is given by

$$x(t, s) = \left[\left(a - \frac{1}{a}\right)e^t + \frac{1}{a}\right] \cos s = \left(\frac{a^2 - 1}{a}\right)\left(e^t + \frac{1}{a^2 - 1}\right) \cos s. \tag{4.3.46}$$

Similarly,

$$y(t, s) = \left[\left(a - \frac{1}{a}\right)e^t + \frac{1}{a}\right] \sin s = \left(\frac{a^2 - 1}{a}\right)\left(e^t + \frac{1}{a^2 - 1}\right) \sin s, \tag{4.3.47}$$

$$z(t, s) = -\frac{(a^2 - 1)^{\frac{1}{2}}}{a}\left(e^t - 1\right). \tag{4.3.48}$$

Or equivalently,

$$z - \frac{a}{\sqrt{a^2 - 1}} = -\frac{\sqrt{a^2 - 1}}{a}\left(e^t + \frac{1}{a^2 - 1}\right). \tag{4.3.49}$$

Thus, the equation of the characteristic strip is

$$x^2 + y^2 = \left(a^2 - 1\right)\left[z - \frac{a}{\sqrt{a^2 - 1}}\right]^2. \tag{4.3.50}$$

This represents a *cone*. In general, any solution of the Clairaut equation represents a *developable surface*, that is, a *ruled surface* which can be deformed into a plane without stretching or tearing.

4.4 The Hamilton–Jacobi Equation and Its Applications

In analytical dynamics, the *Hamilton principle function* $S(q_i, t)$ characterizes the dynamical system and satisfies the celebrated *Hamilton–Jacobi equation*

$$\frac{\partial S}{\partial t} + H\left(q_i, t, \frac{\partial S}{\partial q_i}\right) = 0, \quad i = 1, 2, \ldots, n, \tag{4.4.1}$$

where q_i are the generalized coordinates, t is the time variable, and H is the Hamiltonian of a dynamical system. This equation is a first-order, nonlinear, partial differential equation with $n+1$ independent variables (q_i, t), and it plays a fundamental role in the development of analytical dynamics. It follows from (4.4.1) that S is equal to the time integral of the Lagrangian L. We have

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \dot{q}_i \frac{\partial S}{\partial q_i} = -H + \dot{q}_i p_i = L, \tag{4.4.2}$$

where $p_i = \frac{\partial S}{\partial q_i}$.

Integrating (4.4.2) from t_1 and t_2 gives

$$S = \int_{t_1}^{t_2} L\,dt. \tag{4.4.3}$$

This shows that S is a functional which satisfies the Hamilton principle. Hamilton realized that S is a solution of the Hamilton–Jacobi equation.

Using the conventional notation $S(q_i,t) = u(q_i,t)$, $\frac{\partial S}{\partial q_i} = \frac{\partial u}{\partial q_i} = p_i$, and $p = S_t$, we rewrite equation (4.4.1) in the form

$$F(q_i,t,p_i,p) = p + H(q_i,t,p_i) = 0, \tag{4.4.4}$$

where the dependent variable $u(q_i,t)$ does not occur in the equation.

In terms of the parameter τ, the Charpit equations associated with (4.4.4) are given by

$$\frac{dq_i}{d\tau} = F_{p_i} = \frac{\partial H}{\partial p_i}, \tag{4.4.5}$$

$$\frac{dt}{d\tau} = F_p = 1, \tag{4.4.6}$$

$$\frac{du}{d\tau} = \sum_{i=1}^{n} p_i F_{p_i} + pF_p = \sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} + p, \tag{4.4.7}$$

$$\frac{dp_i}{d\tau} = -(F_{q_i} + p_i F_u) = -\frac{\partial H}{\partial q_i}, \tag{4.4.8}$$

$$\frac{dp}{d\tau} = -(F_t + pF_u) = -\frac{\partial H}{\partial t}. \tag{4.4.9}$$

The second equation (4.4.6) with the given initial condition $t(\tau) = 0$ at $\tau = 0$ gives $t = \tau$. Thus, the independent time variable t can be used as the parameter of the characteristics. Thus, the above system of equations reduces to the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \tag{4.4.10}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{4.4.11}$$

$$\frac{du}{dt} = \sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} + p, \tag{4.4.12}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial t}. \tag{4.4.13}$$

The first two equations (4.4.10), (4.4.11) constitute a set of $2n$ coupled, first-order, ordinary differential equations, precisely the Hamilton canonical equations of motion, which reflect symmetry except for a negative sign. The solutions of these equations represent the characteristics of the Hamilton–Jacobi equation. They also represent the generalized coordinates and generalized momenta of a dynamical system whose Hamiltonian is $H(q_i, p_i, t)$. The last equation (4.4.13) gives $p = -H$ which is used to rewrite (4.4.12) in the form

$$\frac{du}{dt} = \sum p_i \frac{\partial H}{\partial p_i} - H. \tag{4.4.14}$$

In principle, the Hamilton system of $2n$ equations can be solved for $q_i(t)$ and $p_i(t)$. If we substitute these solutions in $(\frac{\partial H}{\partial p_i})$ and H in (4.4.12), the right-hand side of (4.4.12) gives a known function of t, and then $u(q_i, t)$ can be found by integration. A similar argument can be used to find p by integrating (4.4.13). Thus, the upshot of this analysis is that the characteristics of the Hamilton–Jacobi equation are the solutions of the Hamilton equations.

By using Hamilton's equations (4.4.10), (4.4.11), the equations of motion for any canonical function $F(q_i, p_i, t)$ can be expressed in the form

$$\begin{aligned}\frac{dF}{dt} &= \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i\right) + \frac{\partial F}{\partial t} \\ &= \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial F}{\partial t} \\ &= \{F, H\} + \frac{\partial F}{\partial t},\end{aligned} \tag{4.4.15}$$

where $\{F, H\}$ is called the *Poisson bracket* of two functions F and H.

If the canonical function F does not explicitly depend on time t, then $F_t = 0$ and hence (4.4.15) becomes

$$\frac{dF}{dt} = \{F, H\}. \tag{4.4.16}$$

In addition, if $\{F, H\} = 0$, then F is a constant of motion. In fact, equation (4.4.16) really includes the Hamilton equations, which can easily be verified by setting $F = q_i$ and $F = p_i$. Further, if $F = H$, (4.4.16) implies that H is constant.

More generally, the *Poisson bracket* of any two functions F and G is defined by

$$\{F, G\} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right). \tag{4.4.17}$$

Obviously, the Poisson bracket is antisymmetric. It also readily follows from the definition of the Poisson bracket that

$$\{q_i, p_j\} = \delta_{ij}, \tag{4.4.18}$$

$$\{q_i, q_j\} = 0 = \{p_i, p_j\}, \tag{4.4.19ab}$$

where δ_{ij} is the *Kronecker delta* notation. Results (4.4.18), (4.4.19ab) represent the fundamental Poisson brackets for the canonically conjugate variables q_i and p_i.

It is important to point out that the sum in a Poisson bracket is taken over all the *independent variables*. Partial derivatives of *only two dependent variables* occur in a given Poisson bracket, and these two variables are used in the notation for that bracket.

To treat time t similarly to the other independent variables, we replace t by q_{n+1} and write (4.4.4) in the form

$$G(q_i, q_{n+1}, p_i, p_{n+1}) = p_{n+1} + H(q_i, t, p_i). \tag{4.4.20}$$

This allows us to write the Hamilton equations in the form

$$\frac{dq_i}{dt} = \frac{\partial G}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial G}{\partial q_i}, \tag{4.4.21ab}$$

where $G = G(q_i, p_i)$ and $i = 1, 2, \ldots, n+1$. Evidently, if G is given, equations (4.4.21ab) represent a system of $(2n+2)$ first-order, ordinary differential equations for the unknown functions q_i and p_i. The main problem is to find a suitable transformation under which the symmetric form of the Hamilton equations is preserved. Such a transformation is called a *canonical transformation,* defined as follows.

A transformation of the form

$$q_i = q_i(\tilde{q}_i, \tilde{p}_i), \qquad p_i = p_i(\tilde{q}_i, \tilde{p}_i), \tag{4.4.22ab}$$

where $i = 1, 2, \ldots, n+1$, is called *canonical* if there exists a function $\tilde{G}(\tilde{q}_i, \tilde{p}_i)$ such that equations (4.4.21ab) transform into

$$\frac{d\tilde{q}_i}{dt} = \frac{\partial \tilde{G}}{\partial \tilde{p}_i}, \qquad \frac{d\tilde{p}_i}{dt} = -\frac{\partial \tilde{G}}{\partial \tilde{q}_i}. \tag{4.4.23ab}$$

There are other mathematical expressions invariant under canonical transformations of the form (4.4.22ab). One such set is the integral invariants of Poincaré. A theorem of Poincaré states that the integral

$$J_1 = \iint_S \sum_i dq_i \, dp_i \tag{4.4.24}$$

is invariant under the canonical transformation, where S indicates that the integral is to be evaluated over any arbitrary two-dimensional surface in the phase space formed by coordinates $q_1, q_2, \ldots, q_n$, and $p_1, p_2, \ldots, p_n$. It is noted that the position of a point on any two-dimensional surface is completely specified by not more than two parameters. We assume that u and v are two such parameters appropriate to the surface S, so that $q_i = q_i(u, v)$ and $p_i = p_i(u, v)$ on this surface. It is well known that the elementary area $dq_i \, dp_i$ transforms to the element of area $du \, dv$ according to the relation

$$dq_i \, dp_i = \frac{\partial(q_i, p_i)}{\partial(u, v)} \, du \, dv, \tag{4.4.25}$$

where the Jacobian determinant is given by

$$\frac{\partial(q_i, p_i)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix} \neq 0. \tag{4.4.26}$$

The integral J_1 has the same value for all canonical coordinates, that is,

$$\iint_S \Big(\sum_i dq_i \, dp_i\Big) = \iint_S \Big(\sum_k d\tilde{q}_k \, d\tilde{p}_k\Big), \tag{4.4.27}$$

which can also be expressed as

$$\iint_S \sum_i \frac{\partial(q_i, p_i)}{\partial(u, v)} \, du \, dv = \iint_S \sum_k \frac{\partial(\tilde{q}_k, \tilde{p}_k)}{\partial(u, v)} \, du \, dv. \tag{4.4.28}$$

Since the region of integration is arbitrary, the two integrals in (4.4.28) are equal, provided that the two integrands are equal, that is,

$$\sum_i \frac{\partial(q_i, p_i)}{\partial(u, v)} = \sum_k \frac{\partial(\tilde{q}_k, \tilde{p}_k)}{\partial(u, v)}. \tag{4.4.29}$$

This means that the sum of the Jacobian determinants is invariant. More explicitly, (4.4.29) can be written as

$$\sum_i \left(\frac{\partial q_i}{\partial u} \cdot \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \cdot \frac{\partial q_i}{\partial v}\right) = \sum_k \left(\frac{\partial \tilde{q}_k}{\partial u} \cdot \frac{\partial \tilde{p}_k}{\partial v} - \frac{\partial \tilde{p}_k}{\partial u} \cdot \frac{\partial \tilde{q}_k}{\partial v}\right). \tag{4.4.30}$$

Each side of this equation is in the form of what is called the *Lagrange bracket* of two independent variables u and v defined by

$$(u, v) = \sum_i \left(\frac{\partial q_i}{\partial u} \cdot \frac{\partial p_i}{\partial v} - \frac{\partial p_i}{\partial u} \cdot \frac{\partial q_i}{\partial v}\right). \tag{4.4.31}$$

More generally, if F_k and G_k, $k = 1, 2, \ldots, n$ are a set of functions of a number of independent variables, and if u and v are any two of these variables, then the *Lagrange bracket* is defined by

$$(u, v) = \sum_k \left(\frac{\partial F_k}{\partial u} \cdot \frac{\partial G_k}{\partial v} - \frac{\partial F_k}{\partial u} \cdot \frac{\partial G_k}{\partial v}\right). \tag{4.4.32}$$

Obviously, the Lagrange bracket is antisymmetric.

We now consider the Lagrange bracket of the generalized coordinates

$$(q_i, q_j) = \sum_k \left(\frac{\partial q_k}{\partial q_i} \cdot \frac{\partial p_k}{\partial q_j} - \frac{\partial q_k}{\partial q_j} \cdot \frac{\partial p_k}{\partial q_i}\right) = 0. \tag{4.4.33}$$

Since the q's and p's are independent coordinates, $\frac{\partial p_k}{\partial q_i} = 0$ and $\frac{\partial p_k}{\partial q_j} = 0$.

Similarly, we can show that

$$(p_i, p_j) = 0. \tag{4.4.34}$$

Finally, it can be shown that

$$(q_i, p_j) = \sum_k \left(\frac{\partial q_k}{\partial q_i} \cdot \frac{\partial p_k}{\partial p_j} - \frac{\partial q_k}{\partial p_j} \cdot \frac{\partial p_k}{\partial q_i} \right) = \delta_{ij}. \tag{4.4.35}$$

The second term in the above sum vanishes because the q's and p's are independent, but the first term is not zero because

$$\frac{\partial q_k}{\partial q_i} = \delta_{ki}, \quad \text{and} \quad \frac{\partial p_k}{\partial p_j} = \delta_{kj}, \tag{4.4.36ab}$$

and hence,

$$(q_i, p_j) = \sum_k \delta_{jk}\delta_{ki} = \delta_{ij}.$$

Thus, (4.4.33)–(4.4.35) represent the Lagrange brackets of canonical variables and are often referred to as the *fundamental Lagrange brackets*.

In contrast to Poisson brackets, the sum in a Lagrange bracket is taken over all the *dependent variables*. Partial derivatives with respect to *two* of the *independent variables* occur in a given Lagrange bracket, and these two independent variables are utilized in the notation for the bracket.

The following observations are in order. First, the condition that the transformation be canonical can be expressed either in terms of Poisson brackets or in terms of Lagrange brackets. Second, there exists a mathematical relation between the Poisson and Lagrange brackets which can be found in Goldstein (1965). Third, both Poisson and Lagrange brackets are found useful in the transition from classical mechanics to quantum mechanics. Fourth, for more information on Hamilton–Jacobi equations, see Qiao (2001, 2002).

Example 4.4.1 (The Hamilton–Jacobi Equation for a Single Particle). We consider a conservative dynamical system, and write $S = u - Et$ where u is independent of time t and E is an arbitrary constant. We see that S is a solution of the Hamilton–Jacobi equation (4.4.1) if u satisfies the time-independent Hamilton–Jacobi equation in the form

$$H\left(q_i, \frac{\partial u}{\partial q_i} \right) = E, \tag{4.4.37}$$

where u is called *Hamilton's characteristic function*.

We now derive the Hamilton–Jacobi equation for a single particle of mass m moving under the influence of a conservative force field. The Hamiltonian of this system is given by

$$H = T + V = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) + V(x, y, z), \tag{4.4.38}$$

where $V(x, y, z)$ is the potential energy of the particle and

$$p_i = \frac{\partial S}{\partial q_i} = \frac{\partial u}{\partial q_i}, \quad i = 1, 2, 3. \tag{4.4.39}$$

We may use Cartesian coordinates as the generalized coordinates, that is, $q_1 = x$, $q_2 = y$, and $q_3 = z$, to reduce the Hamilton–Jacobi equation (4.4.38) in the form

$$\frac{1}{2m}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] + V(x,y,z) = E, \tag{4.4.40}$$

or equivalently,

$$u_x^2 + u_y^2 + u_z^2 = f(x,y,z), \tag{4.4.41}$$

where $f(x,y,z) = 2m(E - V)$. This is a first-order nonlinear equation. In the next section, we discuss the solution of this equation in the context of nonlinear geometrical optics.

Example 4.4.2 (Simple Harmonic Oscillator). The Hamilton–Jacobi equation can readily be applied to solve the problem of a simple harmonic oscillator. In terms of generalized coordinates (q, p), the Hamiltonian of this conservative problem is given by

$$H = T + V = E = \frac{p^2}{2m} + \frac{k}{2}q^2, \tag{4.4.42}$$

where m is the mass and k is a positive constant.

Setting $p = \frac{\partial S}{\partial q}$, the Hamilton–Jacobi equation (4.4.1) becomes

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{k}{2}q^2 = 0. \tag{4.4.43}$$

For this conservative system, the time-independent part of S can be separated by introducing a function u so that $S(q,t) = u(q) - Et$. Consequently, (4.4.43) reduces to the form

$$\frac{1}{2m}\left(\frac{\partial u}{\partial q}\right)^2 + \frac{k}{2}q^2 = E. \tag{4.4.44}$$

Integrating this equation gives

$$u = \sqrt{mk}\int \sqrt{\left(\frac{2E}{k}\right) - q^2}\, dq = S + Et. \tag{4.4.45}$$

Since S is independent of E, we differentiate (4.4.45) with respect to E to obtain

$$\frac{\partial u}{\partial E} = \sqrt{\frac{m}{k}}\int \frac{dq}{\sqrt{\left(\frac{2E}{k}\right) - q^2}} = -\sqrt{\frac{m}{k}}\cos^{-1}\left(q\sqrt{\frac{k}{2m}}\right) + \tau = t, \tag{4.4.46}$$

where τ is a constant of integration. Thus, solving (4.4.46) for q yields

$$q(t) = \sqrt{\frac{2E}{k}}\cos[\omega(t-\tau)] \quad \left(\omega^2 = \frac{k}{m}\right). \tag{4.4.47}$$

The momentum conjugate to q is given by

$$p = m\dot{q} = -\sqrt{2mE}\sin\big[\omega(t-\tau)\big]. \tag{4.4.48}$$

A simple calculation of $L = T - V$ can be used to verify that S is equal to the time integral of L.

Example 4.4.3 (Jacobi's Principle of Least Action). For a conservative dynamical system $(H = E)$, the principle of least action is given by

$$\begin{aligned} 0 = \delta A &= \delta \int_{t_1}^{t_2} 2T\,dt \quad \left(T = \frac{m}{2}\dot{s}^2\right) \\ &= \delta \int_{s_1}^{s_2} \sqrt{2mT}\,ds = \delta \int_{s_1}^{s_2} \sqrt{2m(E-V)}\,ds. \end{aligned} \tag{4.4.49}$$

This is known as the *Jacobi principle of least action*, and it is concerned with the path of the system in the configuration space, rather than the motion in time. Indeed, the time does not appear in the integrand of the integral in (4.4.49) because $V = V(q)$ does not depend on time. This principle in the same form as (4.4.49) is also valid for a conservative system of particles.

Example 4.4.4 (Wave Propagation in Continuous Media). The Hamilton–Jacobi theory can also be applied to problems of wave propagation in continuous media. In a conservative dynamical system, the Hamiltonian is a constant of motion and is identified with the total energy. For an n-dimensional configuration space where $q = (q_1, q_2, \ldots, q_n)$, the Hamilton function S can be expressed in terms of the characteristic function u as

$$S(\mathbf{q}, t) = u(\mathbf{q}) - Et, \tag{4.4.50}$$

where u is independent of time t, and the surface of constant u is fixed in the configuration space. However, the surface of constant S moves with time according to (4.4.50).

We suppose that at some time t the surface of constant S corresponds to the surface of constant u so that at time $t + dt$ that surface coincides with the surface for which $u = S + Edt$. Obviously, during the small time interval dt, the surface of constant S moves to a new surface $u + du$. Thus, the family of surfaces, $S =$ const., can be interpreted as the family of *wave fronts* propagating in the space. The outward normal vector at each point on the surface $S =$ const. represents the direction of the phase (wave) velocity $c_p = \frac{ds}{dt}$, where ds is the distance normal to the surface, and $S =$ a constant that moves from u to a new position $U + dU$ in time dt. The definition of du and (4.4.50) reveals that

$$\frac{du}{dt} = \nabla u \cdot \mathbf{c} = E \tag{4.4.51}$$

so that the phase velocity is

$$c_p = \frac{ds}{dt} = \frac{E}{|\nabla u|}. \tag{4.4.52}$$

For a single particle of mass m in the configuration space $\mathbf{q} = (x, y, z)$, the Hamiltonian is $H = T + V = E$ so that the Hamilton–Jacobi equation for S is

$$E - V = \frac{1}{2m}\left(\frac{\partial u}{\partial q_i}\right)^2 = \frac{1}{2m}|\nabla u|^2,$$

so that

$$|\nabla u|^2 = 2m(E - V). \tag{4.4.53}$$

This result can be used to rewrite the phase speed (4.4.52) in the form

$$c = \frac{E}{|\nabla u|} = \frac{E}{\sqrt{2m(E - V)}} = \frac{E}{p}. \tag{4.4.54}$$

In an n-dimensional configuration space, $p_i = \frac{\partial u}{\partial q_i}$ which is, in vector notation,

$$\mathbf{p} = \nabla u. \tag{4.4.55}$$

Evidently, ∇u determines the normal to the surface of the constant S or u, and hence, represents the direction of wave propagation. It also follows from (4.4.54) that $pc = E = \text{const.}$, which asserts that when the surfaces move slower, particles move faster, and vice versa.

4.5 Applications to Nonlinear Optics

Light waves are electromagnetic waves which are transverse waves that can be described by both scalar and vector potentials. For simplicity, we consider the scalar wave equation for the electromagnetic potential ϕ

$$\nabla^2\phi - \frac{n^2}{c_0^2}\phi_{tt} = 0, \tag{4.5.1}$$

where c_0 is the speed of light in a vacuum, $n = \frac{c_0}{c}$ is called the *refractive index* of the medium, and c is the speed of light in an optical medium. In general, n depends on the optical density of the medium and is a function of space variables.

For constant refractive index n, equation (4.5.1) admits a plane wave solution

$$\phi = A_0 \exp\big[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\big], \tag{4.5.2}$$

where A_0 is a complex wave amplitude, provided the following dispersion relation holds:

$$\left(k^2 + l^2 + m^2\right) = \frac{n^2\omega^2}{c_0^2}, \qquad \mathbf{k} = (k, l, m). \tag{4.5.3}$$

In the case of one-dimensional wave propagation problems, the wave number $k = \frac{2\pi}{\lambda}$ is given by

$$k = \frac{n\omega}{c_0} = \frac{\omega}{c}. \tag{4.5.4}$$

In general, n depends on $\mathbf{x}$, and the wave form is likely to be changed so that the exact plane wave solutions may not be possible. We adopt a geometrical optics approximation where n varies slowly with distance which is assumed to be the order of the wavelength λ. In geometrical optics, λ is small (or k is large) where (4.5.4) holds. Thus, the geometrical optics approximation deals with a small wavelength or high frequency of waves. It is then natural to seek a nearly plane wave solution in the form

$$\phi = \exp\big[A(\mathbf{x}) + ik_0\{\psi(\mathbf{x}) - c_0 t\}\big], \tag{4.5.5}$$

where the wave amplitude A and the phase ψ of the wave are slowly varying functions of $\mathbf{x}$, and k_0 is the wavenumber in the vacuum $(n = 1)$ with $k_0 = \frac{n\omega}{c_0}$.

Application of the gradient operator to (4.5.5) gives

$$\nabla\phi = \phi\nabla(A + ik_0\psi). \tag{4.5.6}$$

$$\begin{aligned}\nabla^2\phi &= \phi\big[\nabla^2(A + ik_0\psi) + \big\{\nabla(A + ik_0\psi)\big\}^2\big] \\ &= \phi\big[\nabla^2 A + ik_0\nabla^2\psi + (\nabla A)^2 - k_0^2(\nabla\psi)^2 + 2ik_0\nabla A\cdot\nabla\psi\big].\end{aligned} \tag{4.5.7}$$

Consequently, the wave equation (4.5.1) becomes

$$\nabla^2 A + (\nabla A)^2 + k_0^2\big\{n^2 - (\nabla\psi)^2\big\}\phi + ik_0\big[2\nabla A\cdot\nabla\psi + \nabla^2\psi\big]\phi = 0. \tag{4.5.8}$$

Both A and ψ are real, and hence, the real and imaginary parts of (4.5.8) must vanish so that

$$2\nabla A\cdot\nabla\psi + \nabla^2\psi = 0, \tag{4.5.9}$$

$$\nabla^2 A + (\nabla A)^2 + k_0^2\big\{n^2 - (\nabla\psi)^2\big\} = 0. \tag{4.5.10}$$

The geometrical optics approximation has no effect on equation (4.5.9), but equation (4.5.10) is modified in the limit as $k_0 \to \infty$ so that it becomes

$$(\nabla\psi)^2 = n^2. \tag{4.5.11}$$

This is known as the *eikonal equation* in optics, and the phase function ψ is called the *eikonal* (eikon is a Greek word for image or figure). This is a first-order nonlinear partial differential equation which plays a very important role in optics and in wave propagation in continuous media. Physically, equation (4.5.11) determines a surface of constant phase as *wave fronts*. There is a remarkable correspondence between the eikonal equation (4.5.11) in optics and the Hamilton–Jacobi equation (4.4.40) in classical mechanics, where n in the former equation plays a role similar to that of $\sqrt{2m(E-V)}$ in the latter. Thus, there is a striking similarity between the eikonal ψ and the characteristic function u in classical mechanics. It is also evident from the Hamilton–Jacobi theory that classical mechanics may be regarded as the small wavelength (or high frequency) limit of the geometrical optics. This relationship between classical mechanics and geometrical optics can be reconfirmed from the fact that Fermat's principle of least time follows from Jacobi's principle of least action as seen below:

$$0=\delta\int_{s_1}^{s_2}\sqrt{2m(E-V)}\,ds=\delta\int_{s_1}^{s_2}n\,ds=\delta\int_{s_1}^{s_2}(nc)\,dt=\delta\int_{s_1}^{s_2}c_0\,dt. \tag{4.5.12}$$

Example 4.5.1 (Eikonal Equation in Nonlinear Geometrical Optics). In optics, light waves propagate in a medium along *rays* with a given speed $c=c(x,y,z)$ depending on space variables. The surfaces of constant optical phase, $x_0(\tau)=\tau$, $y_0(\tau)=\tau$, and $u(x,y,z)=\text{const.}$, are called the *wave fronts,* which are orthogonal to the *light rays*. The function $u(x,y,z)$ satisfies the classical, nonlinear, first-order equation

$$u_x^2+u_y^2+u_z^2=n^2(x,y,z), \tag{4.5.13}$$

where $n=\left(\frac{c_0}{c}\right)$ is called the *index of refraction.*

In two space dimensions, the eikonal equation (4.5.13) takes the form

$$u_x^2+u_y^2=n^2(x,y). \tag{4.5.14}$$

This describes the propagation of cylindrical waves in optics, acoustics, elasticity, and electromagnetic theory. The classical and simplest problem of wave propagation deals with shallow water waves where wave crests or troughs, or any curve of constant phase, are the level curves expressed as

$$u(x,y)-at=\text{const.}, \tag{4.5.15}$$

with t as the time variable. As time progresses, waves propagate in the (x,y)-plane.

When n is a constant, we can apply Case (i) in Section 4.3 to find the general solution of (4.5.14). It follows from (4.3.2) that the complete solution of (4.5.14) can be written as

$$u(x,y)=anx+n\sqrt{1-a^2}y+b, \tag{4.5.16}$$

where a and n are constants. Or equivalently,

$$u(x,y)=nx\cos\theta+ny\sin\theta+b, \tag{4.5.17}$$

where $a=\cos\theta$.

With $n(x,y)=\text{const.}=n$, we solve (4.5.14) as the initial-value problem for $x(s,\tau)$, $y(s,\tau)$, $u(s,\tau)$, $p(s,\tau)$, and $q(s,\tau)$ with the Cauchy data

$$x(0,\tau)=x_0(\tau),\qquad y(0,\tau)=y_0(\tau),\qquad u(0,\tau)=u_0(\tau), \tag{4.5.18}$$

$$p(0,\tau)=p_0(\tau),\quad\text{and}\quad q(0,\tau)=q_0(\tau), \tag{4.5.19}$$

where s and τ are parameters.

According to the general theory of first-order equations, the Charpit equations associated with (4.5.14) are

$$\begin{aligned} &\frac{dx}{ds} = 2p, \qquad \frac{dy}{ds} = 2q, \qquad \frac{du}{ds} = 2\left(p^2 + q^2\right), \\ &\frac{dp}{ds} = 0, \qquad \frac{dq}{ds} = 0. \end{aligned} \tag{4.5.20}$$

Evidently, the last two equations in (4.5.20) give $p(s,\tau)$ and $q(s,\tau)$ which are constant on the characteristics, that is,

$$p(s,\tau) = \text{const.} = p_0(t) \quad \text{and} \quad q(s,\tau) = \text{const.} = q_0(t). \tag{4.5.21}$$

Since p and q are constant for fixed τ, the first two equations in (4.5.20) imply that characteristics (light rays) are straight lines which can be obtained by direct integration in the form

$$x(s,\tau) = 2p_0 s + x_0(\tau), \qquad y(s,\tau) = 2q_0 s + y_0(\tau). \tag{4.5.22}$$

By integration, the third equation in (4.5.20) gives

$$u(s,\tau) = 2\left(p_0^2 + q_0^2\right)s + u_0(\tau) = 2n^2 s + u_0(\tau). \tag{4.5.23}$$

Along the characteristics (or light rays), equation (4.5.23) has the same form as (4.5.15) with s playing the role of a scaled-time variable and where $u_0(\tau)$ is independent of time like variable s.

Eliminating p_0, q_0, and s from (4.5.22) and (4.5.23) gives the solution

$$(u - u_0)^2 = n^2\left\{(x - x_0)^2 + (y - y_0)^2\right\}. \tag{4.5.24}$$

This represents a cone with its vertex at (x_0, y_0, u_0). Or equivalently, taking the positive square root, we have

$$u(x,y) = u_0 + n\left\{(x - x_0)^2 + (y - y_0)^2\right\}^{\frac{1}{2}}. \tag{4.5.25}$$

This is a three-parameter family of characteristic strips of (4.5.14), and hence, it represents the solution of the eikonal equation (4.5.14). However, the solution (4.5.25) satisfies (4.5.14) *everywhere* except at the initial point (x_0, y_0), where u_x and u_y are singular. Thus, (4.5.25) is called the *singular solution* which determines a cone with its vertex at (x_0, y_0, u_0), identical with the Monge cone through that point.

Physically, the level curves $u(x,y) = \text{const.}$ characterize the *wave fronts*, where $\nabla u = (u_x, u_y) = (p, q)$ represents the normal vector to the wave fronts. It follows from (4.5.20) that $2(p,q)$ represents the tangent vector to the characteristics or light rays. Thus, the light rays are *normal* to the wave fronts. This is true even for a variable index of refraction.

Now, we discuss several special solutions of eikonal equation (4.5.14) with constant n and the following three sets of initial data:

$$\text{(i) } x_0(\tau) = 0, \qquad y_0(\tau) = 0, \quad \text{and} \quad u_0(\tau) = 0.$$

In this case, solutions (4.5.22) and (4.5.23) reduce to the form

$$x = 2p_0 s, \qquad y = 2q_0 s, \quad \text{and} \quad u = 2n^2 s. \tag{4.5.26}$$

Eliminating s from the first two results gives

$$x^2 + y^2 = 4\left(p_0^2 + q_0^2\right)s^2 = 4n^2 s^2. \tag{4.5.27}$$

Therefore, the solution is

$$u(x, y) = n\left(x^2 + y^2\right)^{\frac{1}{2}}. \tag{4.5.28}$$

The level curves representing the wave fronts are $u(x, y) = \text{const.}$, which represents a family of concentric circles. The strip condition (4.2.21) is automatically satisfied, the given equation leads to $p_0^2 + q_0^2 = n^2$, and hence, p_0 and q_0 are constant. Thus, the characteristics representing the rays are given by a family of straight lines $q_0 x - p_0 y = 0$ which passes through the origin. This example also confirms the fact that rays are normal to the wave fronts.

$$\text{(ii) } x_0(\tau) = \tau, \qquad y_0(\tau) = \tau, \qquad u_0(\tau) = a\tau.$$

It follows from equation (4.5.14) that

$$p_0(\tau) = n\cos\theta \quad \text{and} \quad q_0(\tau) = n\sin\theta,$$

for some real values of θ. In order to satisfy the strip condition (4.2.21) in which s is replaced by τ, the result

$$n(\cos\theta + \sin\theta) = a \tag{4.5.29}$$

must hold for some values of θ.

Consequently, solutions (4.5.22) and (4.5.23) become

$$x(s, \tau) = 2ns\cos\theta + \tau, \qquad y(s, \tau) = 2ns\sin\theta + \tau, \tag{4.5.30}$$

$$u(s, \tau) = 2n^2 s + n\tau(\cos\theta + \sin\theta). \tag{4.5.31}$$

Eliminating s and τ from (4.5.30) gives

$$s = \frac{(x - y)}{2n(\cos\theta - \sin\theta)} \quad \text{and} \quad \tau = \frac{(y\cos\theta - x\sin\theta)}{(\cos\theta - \sin\theta)}. \tag{4.5.32}$$

Substituting the values for s and τ in (4.5.31) leads to the solution

$$u(x, y) = n(x\cos\theta + y\sin\theta). \tag{4.5.33}$$

In particular, when $a = n$, then $\cos\theta + \sin\theta = 1$, which implies that either $\theta = 0$ or $\theta = \frac{\pi}{2}$. Therefore, solution (4.5.33) leads to two continuously differentiable solutions given by

$$u(x,y) = nx \quad \text{or} \quad ny. \tag{4.5.34}$$

$$\text{(iii)}\ x_0(\tau) = \frac{\cos\tau}{1-\cos\tau}, \qquad y_0(\tau) = \frac{\sin\tau}{1-\cos\tau}, \quad \text{and} \quad u_0(\tau) = \frac{n\cos\tau}{1-\cos\tau}. \tag{4.5.35}$$

The first two initial conditions in (4.5.35) determine the initial curve $y^2 = 2x+1$, which is a parabola with its vertex at $\left(-\frac{1}{2}, 0\right)$ and its focus at $(0,0)$. To determine the initial data $p_0(\tau)$and $q_0(\tau)$, we need to satisfy the given equation (4.5.14) and the strip condition as

$$p_0^2(\tau) + q_0^2(\tau) = n^2, \tag{4.5.36}$$

$$p_0(\tau)x_0'(\tau) + q_0(\tau)y_0'(\tau) = u_0'(\tau). \tag{4.5.37}$$

These equations give two sets of solutions:

$$\text{(a)}\ p_0 = n, q_0 = 0; \qquad \text{(b)}\ p_0 = n\cos\tau, q_0 = n\sin\tau.$$

For case (a), the solution can be obtained from (4.5.22) and (4.5.23) as

$$x(s,\tau) = 2ns + x_0(\tau), \qquad y(s,\tau) = y_0(\tau), \qquad u(s,\tau) = 2n^2 s + u_0(\tau). \tag{4.5.38}$$

From the first and the third results with the data (4.5.35), we obtain the solution

$$u(x,y) = nx. \tag{4.5.39}$$

The level curves $u = $ const. give $x = $ const., which is a family of straight lines parallel to the y-axis as shown in Figure 4.1.

For case (b), the solutions are found from (4.5.22) and (4.5.23) as

$$x = 2ns\cos\tau + x_0(\tau), \qquad y = 2ns\sin\tau + y_0(\tau), \qquad u(s,\tau) = 2n^2 s + u_0(\tau).$$

After some algebraic simplification, we obtain the solution

$$u(x,y) = n\left[\left(x^2+y^2\right)^{\frac{1}{2}} - 1\right]. \tag{4.5.40}$$

Thus, the wave fronts represented by the level curves $u(x,y) = $ const. are

$$x^2 + y^2 = a^2. \tag{4.5.41}$$

They represent concentric circles as shown in Figure 4.1.

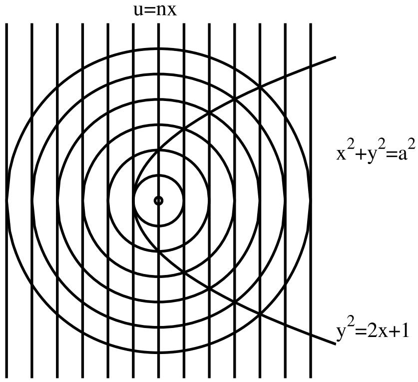

Fig. 4.1 The level curves (wave fronts) for solutions (4.5.39) and (4.5.41).

Example 4.5.2 (Eikonal Equation with a Point Source). We consider the initial-value problem for the eikonal equation with a point source which moves along a straight line in the (x,t)-plane with constant speed v. Without loss of generality, we assume that motion is set up along the x-axis and passes through the origin when the eikonal $u = 0$. We solve the equation

$$p^2 + q^2 = \frac{1}{c^2}, \tag{4.5.42}$$

with the Cauchy data

$$x(0,\tau) = x_0(\tau) = \tau, \qquad y(0,\tau) = y_0(\tau) = 0, \qquad u(0,\tau) = u_0(\tau) = \frac{\tau}{v}, \tag{4.5.43}$$

where c is a constant. The conditions (4.5.36), (4.5.37) give

$$p_0^2(\tau) + q_0^2(\tau) = \frac{1}{c^2} \quad \text{and} \tag{4.5.44a}$$

$$p_0^2(\tau) = \frac{1}{v}, \tag{4.5.44b}$$

and hence,

$$q_0^2(\tau) = \pm\frac{1}{v}\left(M^2 - 1\right)^{\frac{1}{2}}, \tag{4.5.45}$$

where $M = \frac{v}{c}$ is the Mach number.

The condition (4.5.44a) is satisfied if we set $p_0(\tau) = \frac{\cos\theta}{c}$ and $q_0(\tau) = \frac{\sin\theta}{c}$, where the new parameter θ is defined by $\theta = \cos^{-1}(\frac{1}{M})$ which has real solutions for θ in $(0, 2\pi)$ provided the Mach number $M > 1$. Thus, the solutions of the present problem can be obtained from (4.5.22) and (4.5.23) as

$$x(s,\tau) = \frac{2s}{v} + \tau, \qquad y(s,\tau) = \frac{2s}{c}\sin\theta = \frac{2s}{cM}\left(M^2 - 1\right)^{\frac{1}{2}}, \tag{4.5.46}$$

$$u(x,\tau) = \frac{2s}{c^2} + \frac{\tau}{v}, \qquad p(s,\tau) = Mc, \qquad q(s,\tau) = \frac{1}{cM}\left(M^2 - 1\right)^{\frac{1}{2}}. \tag{4.5.47}$$

Eliminating s and τ from the first two results in (4.5.46) and substituting them into the first equation in (4.5.47) gives the solution

$$u(x,y) = \frac{1}{v}\left[x + \frac{y(M^2-1)}{M\sin\theta}\right] = \frac{1}{v}\left[x + y\left(M^2-1\right)^{\frac{1}{2}}\right]. \tag{4.5.48}$$

The other choice of θ gives a negative q_0, and hence, the solution is given by

$$u(x,y) = \frac{1}{v}\left[x - y\left(M^2-1\right)^{\frac{1}{2}}\right]. \tag{4.5.49}$$

Example 4.5.3 (The Schrödinger Equation in Quantum Mechanics). The striking feature of this example is that the optical wave equation (4.5.1) for the electromagnetic potential reduces to the Schrödinger equation in quantum mechanics. As indicated earlier, the eikonal function ψ is proportional to the characteristic function u. If u corresponds to ψ, then $S = u - Et$ must be equal to a constant times the total phase of the optical wave which is given by the imaginary part of the exponent in (4.5.5) so that

$$u - Et = \left(\frac{h}{2\pi}\right)k_0\left[\psi(\mathbf{x}) - c_0 t\right], \tag{4.5.50}$$

where $\left(\frac{h}{2\pi}\right)$ is constant. This shows that the total energy E is given by

$$E = \left(\frac{h}{2\pi}\right)k_0 c_0 = \left(\frac{h}{2\pi}\right)\omega = \hbar\omega = h\nu, \tag{4.5.51}$$

where $\omega = 2\pi\nu$ and $h = 2\pi\hbar$.

Equation (4.5.51) is the fundamental equation in quantum mechanics as h is the Planck constant. On the other hand, the wave speed c, frequency ν, and wavelength λ are related by

$$c = \lambda\nu = \frac{\lambda\omega}{2\pi}. \tag{4.5.52}$$

According to the Hamilton–Jacobi theory, equation (4.4.54) relates c, E, and p. Consequently, (4.4.54) and (4.5.52) give another fundamental result in quantum mechanics:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m(E-V)}}. \tag{4.5.53}$$

Thus, the optical wave equation for the potential ϕ with $c = \frac{c_0}{n}$ takes the form

$$\phi_{tt} - c^2\nabla^2\phi = 0. \tag{4.5.54}$$

If $\phi = \psi(\mathbf{x})\exp(i\omega t)$ with (4.5.52), then (4.5.54) reduces to the form

$$\nabla^2\psi + \frac{\omega^2}{c^2}\psi \equiv \nabla^2\psi + \frac{4\pi^2}{\lambda^2}\psi = 0. \tag{4.5.55}$$

Using (4.5.53) and (4.4.54), we can replace the factor $\left(\frac{4\pi^2}{\lambda^2}\right)$ by using (4.5.53) so that (4.5.55) becomes

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0. \tag{4.5.56}$$

This is the celebrated Schrödinger equation in quantum mechanics.

4.6 Exercises

1. Solve the following first-order partial differential equations:
   (a) $pq = u$ with $u(0, y) = y^2$,
   (b) $p + q = pq$ with $u(x, 0) = \alpha x$, where α is a constant,
   (c) $xp^2 + yq = u$ with $u(s, 1) = -2s$, where s is a parameter,
   (d) $pq = xy$ with $u(x, y) = -y$ at $x = 0$.
2. Obtain the complete integral of the following equations:
   (a) $pq - u^2 = 0$, (b) $p^2 + q^2 = u$,
   (c) $p^2 + q + x^2 = 0$, (d) $p^2 + qy - u = 0$,
   (e) $pq = xy$, (f) $x^2p^2 + y^2q^2 = u^2$,
   (g) $x^2p^2 + y^2q^2 = 4$, (h) $yp^2(1 + x^2) - qx^2 = 0$,
   (i) $2q(u - px - qy) = 1 - q^2$, (j) $(p^2 + q^2)x - pu = 0$.
3. Solve the following Cauchy problems:
   (a) $p^2 - qu = 0$, $u(x, y) = 1$ on $y = 1 - x$,
   (b) $px + qy - p^2q - u = 0$, $u(x, y) = x + 1$ on $y = 2$,
   (c) $2p^2x + qy - u = 0$, $u(x, 1) = -\frac{1}{2}x$,
   (d) $2pq - u = 0$, $u(x, y) = \frac{1}{2}y^2$ at $x = 0$,
   (e) $pq = 1$, $u(x, 0) = x$,
   (f) $pq = u$, $u(x, y) = 1$ on $y = -x$,
   (g) $u_x^2 + u_y = 0$, $u(x, 0) = x$,
   (h) $u - p^2 + 3q^2 = 0$, $u(x, 0) = x^2$,
   (i) $cu_x + u_t + acu = 0$, $u(x, 0) = f(x)$.
4. Find the solution of the equation in the parametric form

$$xp + qy - p^2q - u = 0,$$

   with the initial data

$$x(t, s) = s, \quad y(t, s) = 2, \quad u(t, s) = s + 1, \quad \text{at } t = 0.$$

5. Show that the complete integrals of the equation

$$xpq + yq^2 = 1,$$

   are (a) $(u + b)^2 = 4(ax + y)$ and (b) $cx(u + d) = x^2 + c^2y$, where a, b, c, and d are constants.
6. Show that the integral surfaces of the nonlinear equation

$$2q(u - xp) - 2y(q + x) = 0,$$

   which are developable surfaces, are cones

$$(u + ax)^2 = 2y(x + b).$$

7. Show that the complete integrals of the equation

$$2xu + q^2 = x(xp + yq)$$

   are (a) $u + a^2x = axy + bx^2$ and (b) $x(y + cx)^2 = 4(u - dx^2)$, where a, b, c, and d are constants.

8. Find the solution of the equation in the parametric form

$$pq = xy, \quad u(x, y) = -y \quad \text{when } x = 0.$$

9. Show that no solution exists for the Cauchy problem

$$pq = u, \qquad u(x, y) = -1, \qquad y = -x.$$

10. Show that the solution of the Cauchy problem

$$u_x^2 + u_t = 0, \qquad u(x, 0) = cx$$

is

$$u(x, t) = cx - c^2 t.$$

11. Find the solution of the following Cauchy problems:
(a) $xp + yq = xy$, $u(x, y) = \frac{1}{2}x^2$ on $y = x$.
(b) $-xp + yq = a$ $(0 < x < y)$, $u = 2x$ on $y = 3x$, where a is a constant.
(c) $(p^2 + q^2)x = pu$, $u(0, s^2) = 2s$.
(d) $u_x^2 + u_y = 0$, $u(x, 0) = x$.
(e) $4u_t - u_x^2 = 4x^2$, $u(x, 0) = 0$.
12. Obtain the solution of the equation

$$z\left(p^2 + q^2\right)^{\frac{1}{2}} - \left(1 - z^2\right)^{\frac{1}{2}} = 0 \quad (p = z_x \text{and} q = z_y),$$

so that the solution surface contains the line

$$x(0, s) = s, \quad y(0, s) = \sin\theta, \quad z_0(s) = \cos\theta, \quad 0 < \theta < \frac{\pi}{2}.$$

13. Assuming $u = u(x + ct) = u(\xi)$, show that the equation

$$F\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

reduces to the ordinary differential equation

$$F\left(u, \frac{du}{d\xi}, c\frac{du}{d\xi}\right) = 0.$$

14. Show that the complete solution of the equation

$$z^2\left(p^2 + q^2 + 1\right) = 1 \quad (p = z_x, q = z_y)$$

is the family of cylinders of unit radius whose axis lies on the (x, y)-plane, that is,

$$z^2 + (x\cos\theta + y\sin\theta + b)^2 = 1,$$

where $a = \tan\theta$ and b is a constant of integration.

15. Solve the first-order nonlinear equations:
    (a) $F(u, p, q, a) = (u - a)^2(p^2 + q^2 + 1) - 1 = 0$,
    (b) $u^2(p^2 + q^2 + 1) - 1 = 0$,
    (c) $p^2 + q^2 = 1$.
16. Solve the following Cauchy problems:
    (a) $uu_x + u_y = 0$, $u(x, 0) = x^2$,
    (b) $u^2u_x + u_y = 0$, $u(x, 0) = x$.
    Show that the solution $u(x, y)$ as $y \to 0$ agrees with the initial condition in each case.
17. Show that the complete integral of the equation

$$u^2\left(p^2 + q^2 + 1\right) = 1$$

is

$$(x - a)^2 + (y - b)^2 + u^2 = 1,$$

where a and b are constants.

5

Conservation Laws and Shock Waves

The strides that have been made recently, in the theory of nonlinear partial differential equations, are as great as in the linear theory. Unlike the linear case, no wholesale liquidation of broad classes of problems has taken place; rather, it is steady progress on old fronts and on some new ones, the complete solution of some special problems, and the discovery of some brand new phenomena. The old tools—variational methods, fixed-point theorems, mapping degree, and other topological tools have been augmented by some new ones. Pre-eminent for discovering new phenomena is numerical experimentation; but it is likely that in the future numerical calculations will be parts of proofs.

Peter Lax

5.1 Introduction

Conservation laws describe the conservation of some basic physical quantities of a system and they arise in all branches of science and engineering. In this chapter, we study first-order, quasi-linear, partial differential equations which become conservation laws. We discuss the fundamental role of characteristics in the study of quasi-linear equations and then solve the nonlinear, initial-value problems with both continuous and discontinuous initial data. Special attention is given to discontinuous (or weak) solutions, development of shock waves, and breaking phenomena. As we have observed, quasi-linear equations arise from integral conservation laws which may be satisfied by functions which are not differentiable, and not even continuous, but simply bounded and measurable. These functions are called *weak or generalized solutions*, in contrast to classical solutions, which are smooth (differentiable) functions. It is shown that the integral conservation law can be used to derive the jump condition, which allows us to determine the speed of discontinuity or shock waves. Finally, a formal definition of a shock wave is given.

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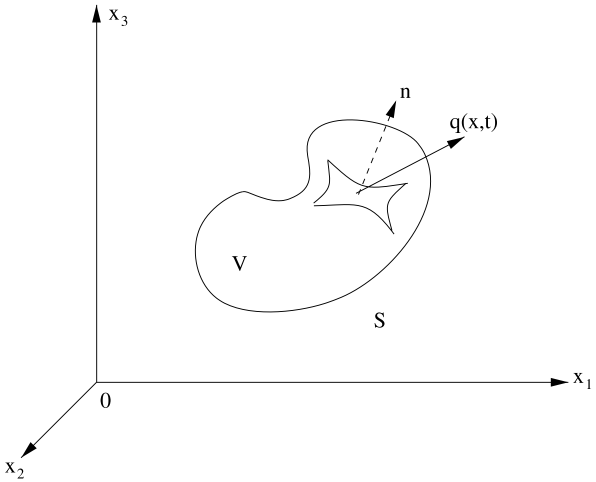

Fig. 5.1 Volume V of a closed domain bounded by a surface S with surface element dS and outward normal vector $\mathbf{n}$.

5.2 Conservation Laws

A conservation law states that the rate of change of the total amount of material contained in a fixed domain of volume V is equal to the flux of that material across the closed bounding surface S of the domain. If we denote the density of the material by $\rho(\mathbf{x}, t)$ and the flux vector by $\mathbf{q}(\mathbf{x}, t)$, then, the conservation law is given by

$$\frac{d}{dt}\int_V \rho\, dV = -\int_S (\mathbf{q}\cdot\mathbf{n})\, dS, \tag{5.2.1}$$

where dV is the volume element and dS is the surface element of the boundary surface S, $\mathbf{n}$ denotes the outward unit normal vector to S as shown in Figure 5.1, and the right-hand side measures the *outward* flux—hence, the minus sign is used.

Applying the Gauss divergence theorem and taking $\frac{d}{dt}$ inside the integral sign, we obtain

$$\int_V \left(\frac{\partial\rho}{\partial t} + \operatorname{div}\mathbf{q}\right) dV = 0. \tag{5.2.2}$$

This result is true for any arbitrary volume V, and, if the integrand is continuous, it must vanish everywhere in the domain. Thus, we obtain the differential form of the conservation law

$$\rho_t + \operatorname{div}\mathbf{q} = 0. \tag{5.2.3}$$

In the presence of a source (or sink) function $f(\mathbf{x}, t, \rho)$, the total rate at which ρ is created (or destroyed) in the given domain is

$$\int_V f(\mathbf{x}, t, \rho)\, dV. \tag{5.2.4}$$

Inserting this term in (5.2.2), we obtain the integral form of the conservation law

$$\frac{d}{dt}\int_V \rho\, dV + \int_V \operatorname{div}\mathbf{q}\, dV = \int_V f(\mathbf{x}, t, \rho)\, dV. \tag{5.2.5}$$

An argument similar to that used before gives the differential form of the conservation law corresponding to (5.2.3) as

$$\rho_t + \operatorname{div}\mathbf{q} = f(\mathbf{x}, t, \rho), \tag{5.2.6}$$

where the flux term, div $\mathbf{q}$, usually arises from the transport of ρ in the domain, and the source term f is often called a *growth term* in biological problems or a *reaction term* in chemistry.

The one-dimensional version of the conservation law is

$$\rho_t + q_x = f(x, t, \rho). \tag{5.2.7}$$

From a mathematical or an empirical point of view, it is reasonable to assume a functional relation between q and ρ as a constitutive equation

$$q = Q(\rho). \tag{5.2.8}$$

Thus, equation (5.2.7) and (5.2.8) form a closed system for ρ and q. Substituting (5.2.8) in (5.2.7) gives

$$\rho_t + c(\rho)\rho_x = f(x, t, \rho), \tag{5.2.9}$$

where $c(\rho) = \frac{dQ}{d\rho}$.

Equation (5.2.9) is universally considered the most fundamental, first-order, quasi-linear inhomogeneous wave equation. In particular, when $f \equiv 0$, (5.2.9) reduces to what is called the *kinematic wave equation*

$$\rho_t + c(\rho)\rho_x = 0. \tag{5.2.10}$$

This equation often arises in nonlinear wave phenomena when the effects of dissipation, such as viscosity and diffusion, are neglected. We next investigate the development of shocks from the *initial-value problem* for $u(x, t)$

$$u_t + c(u)u_x = 0, \quad -\infty < x < \infty,\ t > 0, \tag{5.2.11}$$

$$u(x, 0) = f(x), \quad -\infty < x < \infty, \tag{5.2.12}$$

where $c(u)$ and $f(x)$ are $C^1(\mathbb{R})$ functions of their arguments, that is, they are smooth functions.

The characteristic equations associated with (5.2.11) are

$$\frac{dt}{1} = \frac{dx}{c(u)} = \frac{du}{0}.$$

These equations give

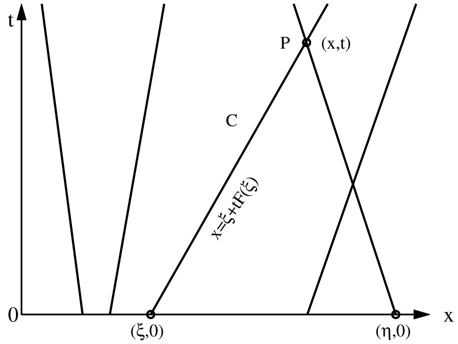

Fig. 5.2 Characteristic lines of different slopes.

$$\frac{du}{dt} = 0 \quad \text{and} \tag{5.2.13a}$$

$$\frac{dx}{dt} = c(u). \tag{5.2.13b}$$

Clearly, the solution of (5.2.13b) represents characteristics of equation (5.2.11). Along these characteristics,

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_t = c(u)u_x + u_t = 0.$$

This means that u is constant on the characteristics which propagate with speed $c(u)$. The dependence of c on u produces a gradual nonlinear distortion of the wave profile as it propagates in the medium. It also follows that $c(u)$ is constant on the characteristics, and therefore, the characteristics must be straight lines in the (x,t)-plane with constant slope $1/c(u)$. Equations of these lines are given by

$$x - tc(u) = \text{const.} = A, \tag{5.2.14}$$

where A is a constant, that is, $x = x(t) = tc(u) + A$.

If any one of these characteristics intersects that x-axis $(t = 0)$ at $x(0) = \xi$, then, by the initial condition, $u(\xi, 0) = f(\xi)$. Thus, the equation of a typical characteristic line (see Figure 5.2) joining two points $(\xi, 0)$ and (x,t) is

$$x = \xi + tF(\xi), \tag{5.2.15}$$

where $F(\xi) = c(f(\xi))$.

Since $u(x,t)$ is constant on the characteristics, it follows from (5.2.15) and the initial condition that

$$u(x,t) = u\big(\xi + tF(\xi), t\big) = u(\xi, 0) = f(\xi).$$

Thus, if a solution of the initial-value problem exists for $t > 0$, then the solution can be written in the parametric form

$$\left.\begin{aligned} u(x,t) &= f(\xi), \\ \xi &= x - tF(\xi), \end{aligned}\right\} \tag{5.2.16ab}$$

where $F(\xi) = c(f(\xi))$.

If there are two points $(\xi, 0)$ and $(\eta, 0)$, with $\xi < \eta$ and

$$m_1 = \frac{1}{F(\xi)} < \frac{1}{F(\eta)} = m_2,$$

then the characteristics starting at $(\xi, 0)$ and $(\eta, 0)$ will intersect at the point $P(x,t)$ for $t > 0$. At the point of intersection $P(x,t)$, the solution $u(x,t)$ has two different values $f(\xi)$ and $f(\eta)$. This means that u is double valued, and hence, the solution is *not unique* at the point of intersection of the characteristics. Thus, the solution *must* be *discontinuous* at the point of intersection. The conclusion is that if no two characteristic lines intersect in the half plane $t > 0$, there exists a solution of the initial-value problem (5.2.11), (5.2.12) as a differentiable function for all $t > 0$. This can happen only if the reciprocal of the slope is an increasing function of the intercept, that is,

$$F(\xi) \le F(\eta) \quad \text{for } \xi \le \eta. \tag{5.2.17}$$

In other words, the family of characteristics spreads only for $t > 0$ and generates a solution of the problem that is at least as smooth as $f(x)$. Such a solution is called an *expansive* (or *refractive*) *wave*.

We now verify that (5.2.16ab) represents an analytical solution of the problem. Differentiating (5.2.16ab) with respect to x and t, we obtain $u_x = f'(\xi)\xi_x$, $u_t = f'(\xi)\xi_t$, $1 = \{1 + tF'(\xi)\}\xi_x$, $0 = F(\xi) + \{1 + tF'(\xi)\}\xi_t$.

Eliminating ξ_x and ξ_t gives

$$u_x = \frac{f'(\xi)}{1 + tF'(\xi)}, \qquad u_t = -\frac{F(\xi)f'(\xi)}{1 + tF'(\xi)}. \tag{5.2.18ab}$$

Substituting u_x and u_t, equation (5.2.11) is satisfied provided $\{1 + tF'(\xi)\} \neq 0$. The solution (5.2.16ab) also satisfies the initial condition at $t = 0$ since $\xi = x$, and hence, it is unique.

Suppose that $u(x,t)$ and $v(x,t)$ are two solutions. Then, on $x = \xi + tF(\xi)$,

$$u(x,t) = u(\xi, 0) = f(\xi) = v(x,t).$$

Thus, we proved the following.

Theorem 5.2.1. *The nonlinear initial-value problem given by* (5.2.11), (5.2.12) *has a unique solution provided that* $\{1 + tF'(\xi)\} \neq 0$ *and* f *and* c *are* $C^1(\mathbb{R})$ *functions where* $F(\xi) = c(f(\xi))$. *The solution is given by the parametric form* (5.2.16ab).

Remark. When $c(u) = \text{const.} = c > 0$, we obtain the linear initial-value problem

$$u_t + cu_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{5.2.19}$$

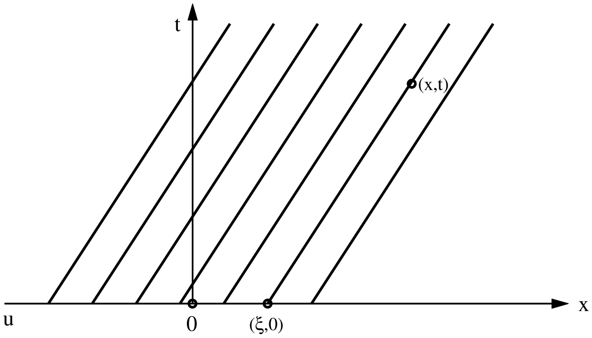

Fig. 5.3 Parallel characteristic lines.

$$u(x,0) = f(x), \quad x \in \mathbb{R}. \tag{5.2.20}$$

This linear problem has a unique solution given by

$$u(x,t) = f(x - ct). \tag{5.2.21}$$

This solution represents a traveling wave moving with constant velocity c in the positive direction of the x-axis *without* any change of shape.

In this case, the characteristics $x = \xi + ct$ form a family of parallel straight lines in the (x,t)-plane as shown in Figure 5.3.

- *Physical Significance of Solution (5.2.16ab).*

For the general nonlinear problem, the dependence of the wave speed c on u produces a gradual nonlinear distortion of the wave as it propagates in the medium. This means that some parts of the wave travel faster than others. When $c'(u) > 0$, $c(u)$ is an increasing function of u. In this case, higher values of u propagate faster than lower ones. On the other hand, when $c'(u) < 0$, $c(u)$ is a decreasing function of u, and higher values of u travel slower than the lower ones. This means that the wave profile progressively distorts itself, leading to a multi-valued solution with a vertical slope, and hence, it breaks. In the linear case, c is constant, there is no such distortion of the wave, and hence, it propagates *without* any change of shape. Thus, there is a striking difference between the linear and nonlinear solutions.

At any *compressive* part of the wave, the wave speed is a decreasing function of x, as shown in Figure 5.4. The wave profile distorts progressively to produce a triple-valued solution for $u(x,t)$, and hence, it ultimately breaks.

It follows from Theorem 5.2.1 that the solution of the nonlinear initial-value problem exists provided that $1 + tF'(\xi) \neq 0$. This condition is always satisfied for a sufficiently small time t. It also follows from (5.2.18ab) that both u_x and u_t tend to infinity as $1 + tF'(\xi) \to 0$. This means that the solution develops a *discontinuity* (*singularity*) when $1 + tF'(\xi) = 0$. Thus, on any characteristic for which $F'(\xi) < 0$, a discontinuity occurs at time t given by

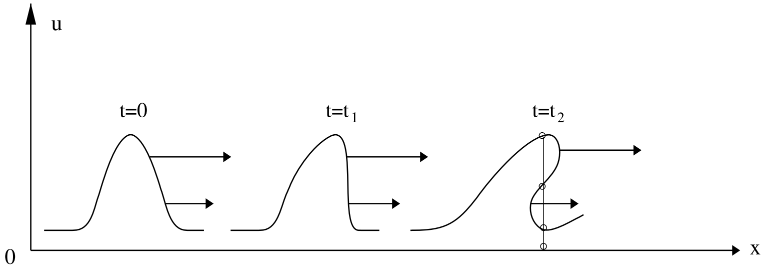

Fig. 5.4 Distortion of wave profile with increasing time $t_2 \gg t_1 > 0$.

$$t = -\frac{1}{F'(\xi)}, \tag{5.2.22}$$

which is positive because $F'(\xi) = c'(f)f'(\xi) < 0$. If we assume $c'(f) > 0$, this inequality implies that $f'(\xi) < 0$. Hence, the solution (5.2.16ab) ceases to exist for all time if the initial data is such that $f'(\xi) < 0$ for some value of ξ. Suppose that $t = \tau$ is the time when the solution first develops a discontinuity (singularity) for some value of ξ. Then,

$$\tau = -\frac{1}{\min_{-\infty<\xi<\infty}\{c'(f)f'(\xi)\}} > 0. \tag{5.2.23}$$

Thus, the shape of the initial curve for $u(x,t)$ changes continuously with increasing values of t, and the solution becomes multi-valued with a vertical slope for $t \geq \tau$. Therefore, the solution breaks down when $F'(\xi) < 0$ for some ξ, and such breaking is a strikingly nonlinear phenomenon. Indeed, Whitham (1974) emphasized that: "This breaking phenomenon is one of the most intriguing long-standing problems of water wave theory." In the linear theory, such breaking will *never* occur.

More precisely, the development of a discontinuity in the solution for $t \geq \tau$ can also be seen in the (x,t)-plane. If $f'(\xi) < 0$, then we can find two values of $\xi = \xi_1, \xi_2$ $(\xi_1 < \xi_2)$ on the initial line $(t = 0)$ such that the characteristics through them have different slopes $1/c(u_1)$ and $1/c(u_2)$, where $u_1 = f(\xi_1)$, $u_2 = f(\xi_2)$, and $c(u_2) > c(u_1)$. These two characteristics will intersect at a point in the (x,t)-plane for some $t > 0$. Since the characteristics carry constant values of u, the solution ceases to be single-valued at the point of intersection. As Figure 5.4 shows, the solution $u(x,t)$ progressively distorts itself, and, at any instant of time, there exists an interval on the x-axis where u becomes triple-valued for a given x. The end result is the development of multi-valued solutions, and hence, it leads to breaking. This is exactly the situation always observed on beaches when water waves break. Finally, we conclude the above discussion by stating the remarkable fact that both the distortion of the wave profile and the development of a discontinuity or a shock are typical nonlinear phenomena.

Therefore, when $1 + tF'(\xi) = 0$, the solution develops a discontinuity known as a *shock*. The analysis of a shock involves an extension of a solution to allow for discontinuities. It is also necessary to impose certain restrictions on the solution to be satisfied across its discontinuity.

Example 5.2.1. Solve the nonlinear initial-value problem

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{5.2.24}$$

$$u(x,0) = \begin{cases} (a^2 - x^2) & \text{if } |x| \le a, \\ 0 & \text{if } |x| \ge a. \end{cases} \tag{5.2.25}$$

In this case, $c(u) = u$, and the solution follows from (5.2.16ab) as

$$u(x,t) = \begin{cases} (a^2 - \xi^2) & \text{if } |x| \le a, \\ 0 & \text{if } |x| \ge a, \end{cases}$$

where

$$\xi = x - t\big(a^2 - \xi^2\big).$$

This is a quadratic equation in ξ giving

$$\xi = \frac{1}{2t}\Big[1 \pm \big\{1 - 4t\big(x - ta^2\big)\big\}^{\frac{1}{2}}\Big], \quad t \ne 0.$$

The solution of (5.2.24) becomes, for $|\xi| \le a$,

$$\begin{aligned} u(x,t) &= \big(a^2 - \xi^2\big), \\ &= \frac{1}{2t^2}\Big[2xt - 1 \pm \big\{1 - 4xt + 4a^2t^2\big\}^{\frac{1}{2}}\Big], \quad t \ne 0, \end{aligned} \tag{5.2.26}$$

and, for $|\xi| \ge a$,

$$u(x,t) = 0.$$

For small values of t $(t \to 0)$, only the positive sign before the radical in (5.2.26) is acceptable so that the initial condition is satisfied. On the other hand, when $t > T$, both signs are admissible for $x > a$.

We next draw the characteristics in the (x,t)-plane for $a = 1$ in Figure 5.5.

As stated before, characteristics are straight lines with speed u. Several characteristic lines intersect. At the point where two characteristics intersect, the solution becomes double-valued, and the slope in the (x,t)-plane becomes infinite. From this point onward, the solution is a discontinuous function of position, and it corresponds to the onset of a shock wave. Moreover, it follows that all the characteristics, originating from $(x,0)$ where $x > -1$, intersect the characteristics starting from the point x, where $x \ge 1$, at some point or another. In particular, the two characteristics initially at $x = 0$ and $x = 1$ intersect at the point $(x,t) = (1,1)$. The solution would be double-valued at $(1,1)$. Figure 5.6 represents the propagation of the initial parabolic pulse with $a = 1$.

As t increases, the initial pulse distorts progressively. This progressive change in the initial wave pulse is the result of the nonlinear term in the equation. In the linear case $(u_t + cu_x = 0, c = \text{const.})$ with the same initial data (5.2.25) for $a = 1$, the initial parabolic pulse propagates with constant velocity c in the positive direction of the x-axis *without change of shape*.

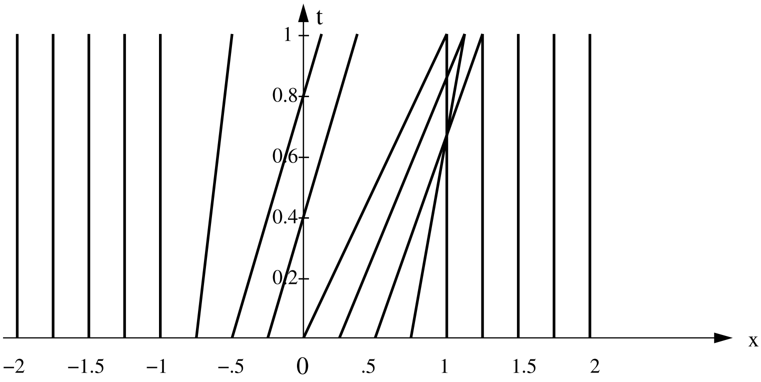

Fig. 5.5 Characteristics of (5.2.24) with the condition for $a = 1$.

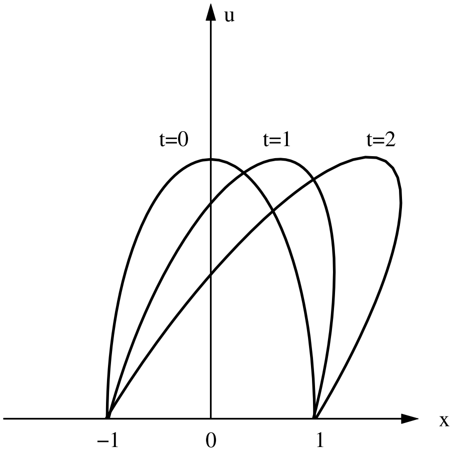

Fig. 5.6 Propagation of a parabolic pulse for $a = 1$.

Example 5.2.2. Solve the initial-value problem

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = x^2, \quad x \in \mathbb{R}.$$

According to (5.2.16ab), the solution is given by

$$u(x,t) = f(\xi) = \xi^2, \qquad \xi = x - t\xi^2.$$

Hence, the second equation becomes

$$t\xi^2 + \xi - x = 0.$$

Solving this quadratic equation in ξ, we obtain

$$\xi = \frac{1}{2t}\left[-1 \pm \sqrt{1+4xt}\,\right] \quad \text{for } t \neq 0. \tag{5.2.27}$$

Thus, the solution is

$$\begin{aligned} u(x,t) = \xi^2 &= \left(\frac{1}{4t^2}\right)\left[-1 \pm \sqrt{1+4xt}\,\right]^2 \\ &= \frac{1}{2t^2}\left[1+2xt \pm \sqrt{1+4xt}\,\right], \quad t \neq 0. \end{aligned} \tag{5.2.28}$$

This solution must satisfy the initial data

$$x^2 = u(x,0) = \lim_{t\to 0} u(x,t).$$

With the positive sign before the radical in the solution (5.2.28), this limit does not exist. With the negative sign before the radical, the limit exists by using the L'Hospital rule twice with fixed x. Thus, the solution is

$$u(x,t) = \frac{1}{2t^2}\left[1+2xt - \sqrt{1+4xt}\,\right], \quad t \neq 0. \tag{5.2.29}$$

This represents a solution only in the region $1+4xt \geq 0$, which is the region between two branches of the hyperbola $xt = -\frac{1}{4}$ in the (x,t)-plane.

Example 5.2.3. Solve the initial-value problem

$$u_t + uu_x = 1, \quad x \in \mathbb{R},\ t > 0, \tag{5.2.30}$$

$$u(s,2s) = s, \quad s \text{ is a parameter.} \tag{5.2.31}$$

The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{1} = d\tau. \tag{5.2.32}$$

The initial data at $\tau = 0$ are

$$x(0,s) = s, \qquad t(0,s) = 2s, \qquad u(0,s) = s. \tag{5.2.33}$$

Thus, the solutions of this system (5.2.32), (5.2.33) are

$$\left.\begin{aligned} x(\tau,s) &= \tfrac{\tau^2}{2} + s\tau + s, \\ t(\tau,s) &= \tau + 2s, \\ u(\tau,s) &= \tau + s. \end{aligned}\right\} \tag{5.2.34}$$

Eliminating τ from the first two results gives the characteristics in the (x,t)-plane as

$$(t-s)^2 = 2(x-s) + s^2, \tag{5.2.35}$$

where s is a parameter. This represents a family of parabolas for $s = 0, \pm 1, \pm 2, \ldots$. We leave the construction of the diagram for parabolas as an exercise for the reader.

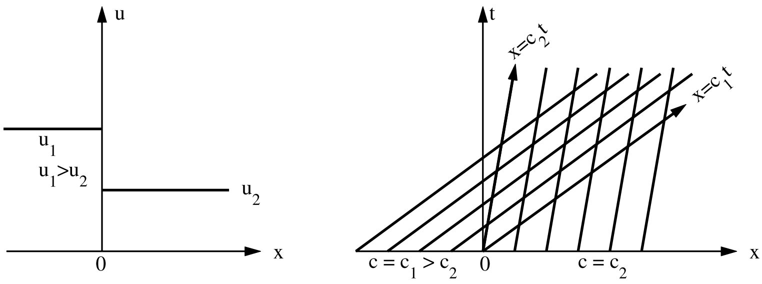

Fig. 5.7 Compression wave with overlapping characteristics.

We first use (5.2.35) to find s and then use the second result of (5.2.34) to find τ and s in terms of x and t. This gives

$$\tau = \left(\frac{2x-t}{t-1}\right), \qquad s = \frac{(t^2-2x)}{2(t-1)}. \tag{5.2.36}$$

Substituting these results into the third equation in (5.2.34) gives the integral surface of the equation (5.2.30) as

$$u(x,t) = \frac{(2x-2t+t^2)}{2(t-1)}. \tag{5.2.37}$$

Thus, the initial-value problem has the solution (5.2.37) *everywhere* in the (x,t)-plane except on the straight line $t = 1$.

Example 5.2.4 (Initial-Value Problem with Discontinuous Initial Data). We consider the initial-value problem for equation (5.2.11) with discontinuous initial data at $t = 0$,

$$u(x,0) = \begin{cases} u_1 & \text{if } x < 0, \\ u_2 & \text{if } x > 0, \end{cases} \tag{5.2.38}$$

and

$$F(x) = \begin{cases} c_1 = c(u_1) & \text{if } x < 0, \\ c_2 = c(u_2) & \text{if } x > 0, \end{cases} \tag{5.2.39}$$

where u_1 and u_2 are constants.

There are two cases: (i) $u_1 > u_2$ and (ii) $u_1 < u_2$.

Case (i) $u_1 > u_2$ with $c_1 > c_2$.

In this case, breaking will occur almost immediately, and this can be seen from Figure 5.7. The multi-valued region starts right at the origin and is bounded by the characteristics $x = c_2 t$ and $x = c_1 t$. This corresponds to a centered compression wave with overlapping characteristics in the (x,t)-plane.

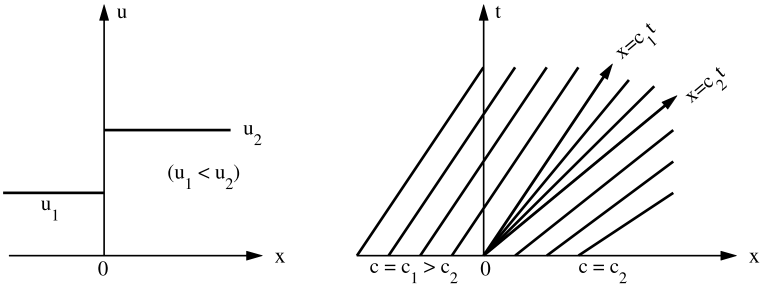

Fig. 5.8 Centered expansive wave.

Case (ii) $u_1 < u_2$ with $c_1 < c_2$.

In this case, the initial condition is expansive with $c_1 < c_2$. A continuous solution can be found from (5.2.16ab) in which all values of $F(x)$ in $[c_1, c_2]$ are taken on the characteristics through the origin $\xi = 0$. This corresponds to a centered fan of characteristics $x = ct$ where $c_1 \leq c \leq c_2$ in the (x,t)-plane so that the solution has the explicit form (see Figure 5.8)

$$c = \frac{x}{t}, \qquad c_1 < \frac{x}{t} < c_2. \tag{5.2.40}$$

Thus, the complete solution is given by

$$c = \begin{cases} c_1 & \text{if } x \leq c_1 t, \\ \frac{x}{t} & \text{if } c_1 t < x < c_2 t, \\ c_2 & \text{if } x \geq c_2 t. \end{cases} \tag{5.2.41}$$

Example 5.2.5. Solve the initial-value problem with discontinuous initial data

$$u_t + u u_x = 0, \quad x \in \mathbb{R},\ t \geq 0; \tag{5.2.42}$$

$$u(x,0) = \begin{cases} 1 & \text{if } x \leq 0, \\ 1 - x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{if } x \geq 1. \end{cases} \tag{5.2.43}$$

According to the parametric solution (5.2.16ab), we obtain the solution

$$u(x,t) = f(\xi) = \begin{cases} 1 & \text{if } \xi \leq 0, \\ 1 - \xi & \text{if } 0 \leq \xi \leq 1, \\ 0 & \text{if } \xi \geq 1, \end{cases} \tag{5.2.44}$$

$$u(\xi,t) = \xi + t f(\xi) = \begin{cases} \xi + t & \text{if } \xi \leq 0, \\ \xi(1-t) + t & \text{if } 0 \leq \xi < 1, \\ \xi & \text{if } \xi \geq 1. \end{cases} \tag{5.2.45}$$

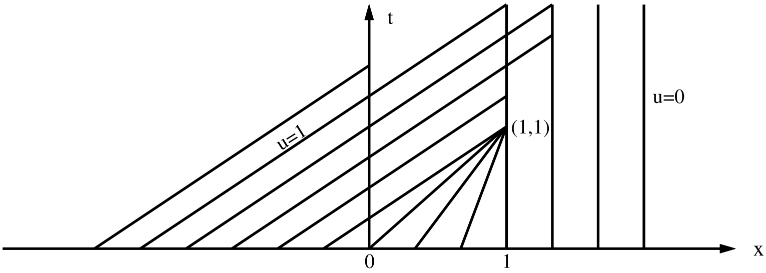

Fig. 5.9 Characteristics and graphical representation of solution.

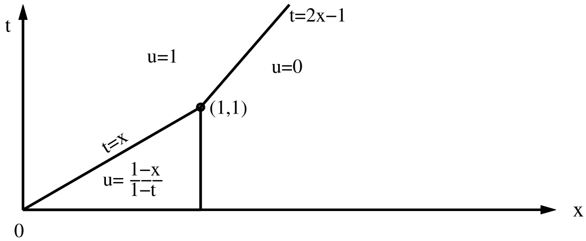

Fig. 5.10 Weak or generalized solution.

The solution and the characteristics are drawn in various regions, as shown in Figure 5.9. In $0 \leq \xi \leq 1$, the solution is $u(x,t) = \frac{1-x}{1-t}$. This is not defined at $t = 1$.

The characteristics are straight lines intersecting the x-axis at $(\xi, 0)$, and u is constant on a given characteristic line. The values of u on $t = 0$ are propagated along the characteristics. As shown in Figure 5.9, the characteristics are drawn in various regions in the (x,t)-plane. The value of u on a characteristic line originating from a point $(\xi, 0)$ is $1 - \xi$, where $0 \leq \xi \leq 1$, and these characteristics all intersect at $(1,1)$. This means that u is multi-valued at $(1,1)$, and hence, the solution breaks down at this point. It also follows from Figure 5.9 that characteristics originating from $(\xi, 0)$, with $\xi < 0$, intersect those originating from $(\xi, 0)$, with $\xi > 1$. As u has different values on these characteristics, it follows that it is not defined in the quadrant $x \geq 1, t \geq 1$.

Figure 5.9 also suggests that the most likely weak solution is one which is discontinuous across some curve originating from $(1,1)$ with $u = 1$ to its left and $u = 0$ to its right. In conservation form, the given equation can be written as

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \tag{5.2.46}$$

where the flux is $\frac{1}{2}u^2$. It follows from this equation combined with the shock condition that the slope of the shock path is 2. Thus the equation of the shock path is $t = 2x - 1$, as shown in Figure 5.10.

5.3 Discontinuous Solutions and Shock Waves

It should be pointed out that the nonlinear conservation equation

$$\rho_t + q_x = 0 \tag{5.3.1}$$

has been solved under two basic assumptions: (i) there exists a functional relation $q = Q(\rho)$, and (ii) ρ and q are continuously differentiable. When breaking occurs, questions arise about the validity of these assumptions. To examine the development of discontinuities or shocks, we still assume that $q = Q(\rho)$, but will allow jump discontinuity at $x = s(t)$. We also assume x_1 and x_2 such that $x_1 < s(t) < x_2$ and $\frac{ds}{dt} = U(t)$.

Without any source (or sink), we assume that the integral form of the conservation equation (5.2.5) holds for the one-dimensional case and has the form

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho\, dx = q(x_1,t) - q(x_2,t), \tag{5.3.2}$$

or

$$\frac{d}{dt}\left[\int_{x_1}^{s^-} \rho\, dx + \int_{s^+}^{x_2} \rho\, dx\right] + q(x_2,t) - q(x_1,t) = 0. \tag{5.3.3}$$

This implies that

$$\int_{x_1}^{s^-} \rho_t dx + \dot{s}\rho(s^-,t) + \int_{s^+}^{x_2} \rho_t dx - \dot{s}\rho(s^+,t) + q(x_2,t) - q(x_1,t) = 0, \tag{5.3.4}$$

where $\rho(s^-,t)$, $\rho(s^+,t)$ are the values of $\rho(x,t)$, as $x \to s$ from below and above, respectively. Since ρ_t is bounded on $x_1 < x < s^-$ and $s^+ < x < x_2$, respectively, the integrals in (5.3.4) must vanish as $x_1 \to s^-$ and $x_2 \to s^+$. Consequently, in the limit,

$$q(s^-,t) - q(s^+,t) = U\{\rho(s^-,t) - \rho(s^+,t)\}. \tag{5.3.5}$$

In terms of the conventional notation of shock dynamics, condition (5.3.5) can be written as

$$[q] = U[\rho], \tag{5.3.6}$$

where $[q] = (q^- - q^+)$ and $[\rho] = (\rho^- - \rho^+)$ denote the *jump* in q and ρ, respectively, across the discontinuity $x = s(t)$, as shown in Figure 5.11.

Result (5.3.6) is usually called the *jump (shock) condition*, which gives a relation ahead (right) of and behind (left) of the discontinuity and the speed of the discontinuity $U(t)$ (reciprocal of the slope). The discontinuity in ρ that propagates along the curve $x = s(t)$ is known as the *shock wave*, and $x = s(t)$ is simply called the *shock*. In gas dynamics, (5.3.6) is known as the *Rankine–Hugoniot condition* for the speed of the shock wave.

Thus, the basic problem can be written as

$$\frac{d\rho}{dt} + \frac{dq}{dx} = 0 \quad \text{at points of continuity,} \tag{5.3.7}$$

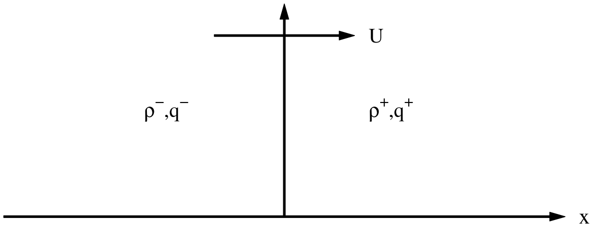

Fig. 5.11 Discontinuity at $x = s(t)$.

$$-U[\rho] + [q] = 0 \quad \text{at points of discontinuity.} \tag{5.3.8}$$

This leads to a nice correspondence:

$$\frac{d}{dt} \leftrightarrow -U[\,], \qquad \frac{d}{dx} \leftrightarrow [\,] \tag{5.3.9ab}$$

between the differential equation and the shock condition.

We can now generalize the concept of the solution for equation (5.3.1) to include discontinuous solutions. This kind of extension of a solution is not a purely mathematical exercise, as the notion of a shock is of great significance in applied areas, such as fluid dynamics, gas dynamics, and plasma physics. From a physical point of view, a shock represents an idealization of a thin transition region separating two regions where solutions of the basic flow equations can readily be determined.

Now, it is possible to find discontinuous solutions of equation (5.3.1). In any continuous part of the solution, equation (5.3.1) is still satisfied, and the assumption $q = Q(\rho)$ can be retained. Thus, $q^- = Q(\rho^-)$ and $q^+ = Q(\rho^+)$ hold on the two sides of a shock, and the shock condition (5.3.6) can be written as

$$U(\rho^- - \rho^+) = Q(\rho^-) - Q(\rho^+). \tag{5.3.10}$$

This leads to a problem of fitting shock discontinuities into the solution (5.2.16ab) so that (5.3.10) is satisfied and multi-valued solutions can be avoided.

5.4 Weak or Generalized Solutions

We formally examine the possibility of generalizing the concept of a classical solution so as to include discontinuous and nondifferentiable functions. We now consider the general procedure necessary to define what is usually called a *weak solution* of the nonlinear, initial-value problem

$$\rho_t + q_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{5.4.1}$$

$$\rho(x, 0) = f(x), \quad x \in \mathbb{R}, \tag{5.4.2}$$

where $q(x, t)$ is a continuously differentiable function on $\mathbb{R}$.

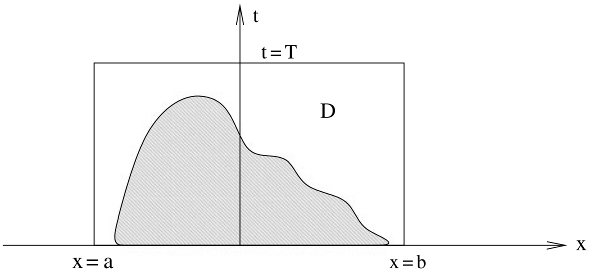

Fig. 5.12 Compression wave with overlapping characteristics.

A classical solution of the partial differential equation (5.4.1) is a smooth function $\rho = \rho(x,t)$ which satisfies (5.4.1). We assume that $\rho(x,t)$ is a classical solution of (5.4.1). We consider a class of test functions $\phi = \phi(x,t)$ such that $\phi \in C^{\infty}$ and it has a compact support in the (x,t)-plane. We can choose an arbitrary rectangle $D = \{(x,t) : a \le x \le b,\ 0 \le t \le T\}$ where $\phi = 0$ outside D, and on the boundary lines $x = a$, $x = b$, and $t = T$, as shown in Figure 5.12.

Equation (5.4.1) is multiplied by $\phi(x,t)$ and integrated over D to obtain

$$\iint_D (\rho_t + q_x)\phi\, dx\, dt = 0. \tag{5.4.3}$$

Integrating both terms by parts gives

$$\int_a^b \left\{ [\phi\rho]_0^T - \int_0^T \rho\phi_t\, dt \right\} dx + \int_0^T \left\{ [\phi q]_a^b - \int_a^b q\phi_x\, dx \right\} dt = 0,$$

or

$$-\int_a^b \phi(x,0) f(x)\, dx - \int_a^b \int_0^T \rho\phi_t\, dt\, dx - \int_a^b \int_0^T q\phi_x\, dt\, dx = 0.$$

Thus the final form of the above equation is

$$\iint_{DUt\ge 0} (\rho\phi_t + q\phi_x)\, dx\, dt + \int_{\mathbb{R}} \phi(x,0) f(x)\, dx = 0. \tag{5.4.4}$$

This holds for all test functions $\phi(x,t)$ in the half plane. Result (5.4.4) does not involve any derivatives of ρ or q. Indeed, it remains valid even if ρ and q or their derivatives are discontinuous.

Thus, we have proved that, if $\rho(x,t)$ is a classical solution of the problem (5.4.1), (5.4.2), then (5.4.4) holds for all test functions ϕ with compact support in the half (x,t)-plane. The functions $\rho(x,t)$, which satisfy (5.4.4) for all test functions ϕ, are called *weak* or *generalized solutions* of the problem. This leads to the following.

Definition 5.4.1 ***(Weak Solution).*** A bounded measurable function $\rho(x,t)$ is called a weak *solution* of the initial-value problem (5.4.1), (5.4.2) with bounded and measurable initial data $f(x)$, provided that (5.4.4) holds for all test functions $\phi(x,t)$ with compact support in the half plane.

This is a more general definition of a solution that does not require the smoothness property. To show that the concept of a weak solution is, indeed, a generalization of a classical solution, we prove the following.

Theorem 5.4.1. *If* (5.4.4) *holds for all test functions* ϕ *with compact support for* $t > 0$ *and if* u *is smooth, then* ρ *is a classical solution of the initial-value problems* (5.4.1) *and* (5.4.2).

Proof. Since ϕ has compact support for $t > 0$, then we integrate (5.4.4) by parts to obtain

$$\iint_{\mathbb{R}U\ t>0} (\rho_t + q_x)\phi\, dx\, dt = 0. \tag{5.4.5}$$

This is true for all test functions ϕ, and hence,

$$\rho_t + q_x = 0 \quad \text{for } x \in \mathbb{R},\ t > 0. \tag{5.4.6}$$

We next multiply (5.4.6) by ϕ and integrate the resulting equation by parts over the region D to find

$$\iint_{\mathbb{R}U\ t>0} (\rho\phi_t + q\phi_x)\, dx\, dt + \int_{\mathbb{R}} \phi(x,0)\rho(x,0)\, dx = 0. \tag{5.4.7}$$

Subtracting (5.4.7) from (5.4.4) gives

$$\int_{\mathbb{R}} \big[\rho(x,0) - f(x)\big]\phi(x,0)\, dx = 0. \tag{5.4.8}$$

This is true for all test functions $\phi(x,0)$. Since $f(x)$ is continuous, (5.4.8) leads to $\rho(x,0) = f(x)$. This shows that $\rho(x,t)$ is the classical solution of the initial-value problem.

We next consider only those weak solutions $\rho(x,t)$ which are continuously differentiable in two parts D_1 and D_2 of a domain D, but with a jump discontinuity (shock) along the dividing smooth curve Γ between D_1 and D_2, as shown in Figure 5.13. Suppose that the equation of Γ is $x = s(t)$. Since $\rho(x,t)$ has a jump discontinuity along this curve, $x = s(t)$, $\rho(x,t)$ is smooth away from Γ, and limits $\rho^-(t) = \rho(s-,t)$ and $\rho^+(t) = \rho(s+,t)$ from the left and the right exist. The speed of the shock wave is $\dot{s}(t)$, which is the reciprocal of the slope in Figure 5.13. For any test function ϕ, result (5.4.4) becomes

$$\begin{aligned} 0 = \int_D \int (\rho\phi_t + q\phi_x)\, dx\, dt &= \int_{D_1} \int (\rho\phi_t + q\phi_x)\, dx\, dt \\ &\quad + \int_{D_2} \int (\rho\phi_t + q\phi_x)\, dx\, dt. \end{aligned} \tag{5.4.9}$$

Since $\rho(x,t)$ is smooth in D_1 and in D_2, for $n = 1, 2$, the divergence theorem gives

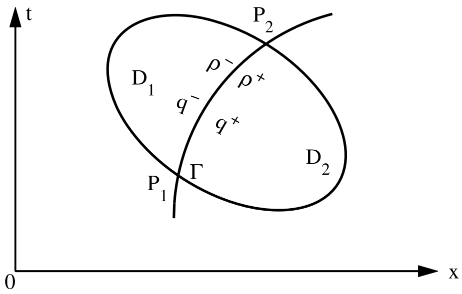

Fig. 5.13 Smooth curve $\varGamma$ across which $\rho(x,t)$ has a jump discontinuity.

$$\int_{D_n}\int (\rho\phi_t + q\phi_x)\,dx\,dt = \int_{D_n}\int \big[(\rho\phi)_t + (q\phi)_x\big]\,dx\,dt$$
$$= \int_{\partial D_n} \phi(-\rho\,dx + q\,dt).$$

Because $\phi = 0$ on the boundary ∂D, these line integrals vanish everywhere, except on $\varGamma$. Thus,

$$\int_{D_1} \phi(-\rho\,dx + q\,dt) = \int_{p_1}^{p_2} \phi\big(-\rho^-\,dx + q^-\,dt\big)$$
$$\int_{D_2} \phi(-\rho\,dx + q\,dt) = -\int_{p_1}^{p_2} \phi\big(-\rho^+\,dx + q^+\,dt\big).$$

Therefore, equation (5.4.9) gives

$$\int_{\varGamma} \phi\big(-[\rho]\,dx + [q]\,dt\big) = 0. \tag{5.4.10}$$

This result is true for all test functions ϕ, and therefore, the integrand must vanish, that is, on each point on $\varGamma$,

$$\dot{s}[\rho] = [q]. \tag{5.4.11}$$

This is the same jump condition (5.3.6). Thus, a weak solution leads to the jump condition across a jump discontinuity.

Example 5.4.1. Solve the nonlinear Cauchy problem

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{5.4.12}$$

with two different sets of discontinuous initial data

$$\text{(i)} \quad u(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases}$$

$$\text{(ii)} \quad u(x,0) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

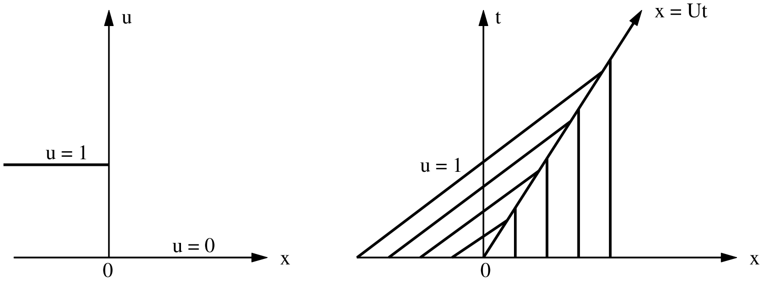

Fig. 5.14 Initial data, characteristics, and a discontinuous solution.

The conservation law associated with equation (5.4.12) is

$$u_t + \left(\frac{u^2}{2}\right)_x = 0. \tag{5.4.13}$$

In case (i), $c(u) = u$, $c(u(x,0))$ is a decreasing function of x, and condition (5.2.17) is violated. Thus, there is *no continuous solution* of the initial-value problem. However, there exists a *discontinuous solution* given by

$$u(x,t) = \begin{cases} 0 & \text{if } x < Ut, \\ 1 & \text{if } x > Ut, \end{cases} \tag{5.4.14}$$

which satisfies (5.4.12) for $x \neq Ut$ and (5.4.11) on the line $x = Ut$. Hence, (5.4.14) is called a *generalized* or *weak* solution. This solution has a jump discontinuity along the line $x = Ut$, where U is the speed of the shock wave given by

$$U = \frac{(q^- - q^+)}{(u^- - u^+)} = \frac{(\frac{1}{2}\cdot 0^2 - \frac{1}{2}\cdot 1^2)}{(0-1)} = \frac{1}{2}.$$

The initial data, characteristics, and the weak solution are shown in Figure 5.14.

In case (ii), $c(u) = u$, $c(u(x,0))$ is an increasing function of x, and condition (5.2.17) is satisfied. Hence, there is a continuous solution of (5.4.12) (except at $x = t = 0$) with the initial data (ii) given by

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x}{t} & \text{if } 0 < x < t, \\ 1 & \text{if } x > t. \end{cases} \tag{5.4.15}$$

A solution of the form (5.4.15) is called an *expansive or refractive wave*. Often this solution in the sector $0 < x < t$ is referred to as a *centered simple wave*. The boundary lines $x = 0$ and $x = t$ of the centered simple wave are weak waves which propagate along the characteristics, as shown in Figure 5.15. This represents a solution of (5.4.12) for $t \geq 0$ which can be verified by direct substitution,

$$\left(\frac{x}{t}\right)_t + \left(\frac{x}{t}\right)\left(\frac{x}{t}\right)_x = -\frac{x}{t^2} + \frac{x}{t^2} = 0.$$

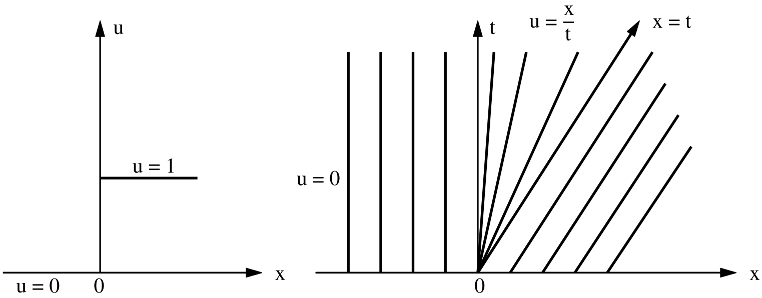

Fig. 5.15 Initial data, characteristics, and the continuous solution.

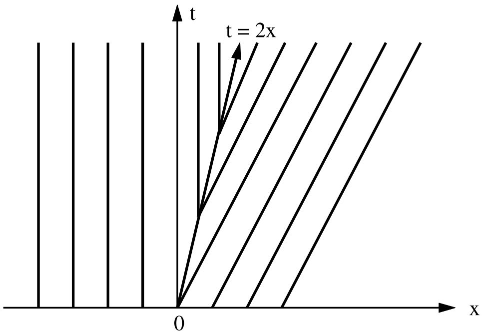

Fig. 5.16 Characteristics and the discontinuous solution.

The initial data, characteristics, and the solution are shown in Figure 5.15.

On the other hand, a remarkable fact involved in this nonlinear Cauchy problem with discontinuous data (ii) is that there is also a *discontinuous solution* given by

$$u(x,t) = \begin{cases} 0 & \text{if } 2x < t, \\ 1 & \text{if } 2x > t. \end{cases} \tag{5.4.16}$$

Obviously, this represents a solution everywhere in the (x,t)-plane, except on the line of discontinuity $t = 2x$. The jump condition gives the shock speed

$$U = \frac{\left(\frac{1}{2}\cdot 0^2 - \frac{1}{2}\cdot 1^2\right)}{(0-1)} = \frac{1}{2}.$$

The characteristics and discontinuous solution are drawn in Figure 5.16.

Thus, there are at least *two solutions* of equation (5.4.12) with the initial data (ii), and hence, the solution is *not unique*. The main question is: Which one of these solutions represents a physically meaningful solution? Obviously, another mathematical criterion is required for determining a unique weak solution. It seems that the criterion for determining a physically meaningful weak solution is closely related to the search for a criterion for an acceptable discontinuous solution. It is also clear from the above analysis that, in general, a discontinuous solution arises whenever characteristics of a nonlinear equation intersect in the (x,t)-plane. It can be proved that there is a unique weak discontinuous solution of the Cauchy problem which satisfies the following inequality on the curve of discontinuity:

$$a(u^-) > U > a(u^+). \tag{5.4.17}$$

This is called the Lax *entropy criterion*. Mathematically, this criterion means that the wave speed just behind the shock is greater than the wave speed just ahead of it. In other words, the wave behind the shock catches up to the wave ahead of it. This entropy criterion is a special case of the second law of thermodynamics: entropy *increases across a shock*. Geometrically, the criterion for a unique weak discontinuous solution can be stated as follows:

The characteristics originating on either side of the discontinuity curve, when continued in the direction of increasing t, intersect the curve of discontinuity.

Clearly, solution (5.4.14) satisfies the entropy inequality (5.4.17), whereas solution (5.4.16) does not satisfy (5.4.17). Therefore, the former is an acceptable discontinuous solution, but the latter is not. This leads to a formal definition of a shock wave. A discontinuity which satisfies both the jump condition (5.4.11) and the inequality (5.4.17) on its curve of discontinuity is called a *shock*. The main problem is to investigate whether every nonlinear initial-value problem has a unique weak solution defined for all $t \geq 0$ with only shock as a discontinuity. The proof of the existence and uniqueness of a weak solution is fairly difficult and is beyond the scope of this book, but is provided by Lax (1973).

5.5 Exercises

1. Find the solution of the initial-value problems
   (a) $u_t + uu_x = 0$, $x \in \mathbb{R}$, $t > 0$, $u(x,0) = \sin x$, $x \in \mathbb{R}$,
   (b) $u_t + uu_x = 0$, $x \in \mathbb{R}$, $t > 0$, $u(x,0) = -x$, $x \in \mathbb{R}$.
2. Show that the equation

$$u_t + uu_x = 0$$

gives an infinite number of conservation laws

$$\frac{\partial}{\partial t}u^n + \frac{\partial}{\partial x}\left(\frac{n}{n+1}u^{n+1}\right) = 0,$$

where $n = 1, 2, 3, \ldots$.
Hence, deduce the integral conservation law

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} u^n dx + \left(\frac{n}{n+1}\right)\left[u^{n+1}(x_1) - u^{n+1}(x_2)\right] = 0,$$

where x_1 and x_2 are two fixed points.

3. Show that the solution of the initial-value problem

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0, \qquad u(x,0) = bx, \quad x \in \mathbb{R},$$

where a and b are constants, is

$$u(x,t) = \frac{abx \exp(-at)}{(a+b) - b\exp(-at)}.$$

4. Find the solution of the initial-value problem

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \begin{cases} 0 & \text{if } x \le 0, \\ \exp(-\frac{1}{x}) & \text{if } x > 0. \end{cases}$$

5. Solve the initial-value problem

$$u_t + cu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \left(a^2 + x^2\right)^{-1}, \quad x \in \mathbb{R},$$

where c and a are constants.

6. Show that the solution of the Cauchy problem

$$u_t - 2axtu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = f(x), \quad x \in \mathbb{R},$$

is

$$u(x,t) = f\left(x \exp\left(at^2\right)\right).$$

Draw the characteristics $x = \xi \exp(-at^2)$, where ξ is a constant.

7. Show that the equation

$$u_t + uu_x = 1, \quad x \in \mathbb{R},\ t > 0,$$

has no solution such that $u = \frac{1}{2}t$, when $t^2 = 4x$, and has *no unique* solution, such that $u = t$ when $t^2 = 2x$.

8. Find a weak solution of the initial-value problem in $t \ge 0$

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 1, \\ 1 & \text{if } 0 \le x \le 1. \end{cases}$$

9. Find the solution of the initial-value problem in $t > 0$

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \begin{cases} -1 & \text{if } x \le 0, \\ 2x - 1 & \text{if } 0 \le x \le 1, \\ 1 & \text{if } x \ge 0. \end{cases}$$

10. Solve the initial-value problem in $t > 0$

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \begin{cases} 1 & \text{if } x \le 0, \\ 1 - 2x & \text{if } 0 \le x \le 1, \\ -1 & \text{if } x \ge 0. \end{cases}$$

11. Find the weak solution for $t \ge 0$ of the initial-value problem

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \begin{cases} -a & \text{if } x \le 0, \\ 2a & \text{if } x > 0, \end{cases}$$

where $a < 0$ and $a > 0$.

12. Show that the solution of the initial-value problem

$$u_t + u^2 u_x = 0, \quad x \in \mathbb{R},\ t > 0, \qquad u(x,0) = x, \quad x \in \mathbb{R},$$

is

$$u(x,t) = \begin{cases} x & \text{if } t = 0, \\ \frac{1}{2t}\left(\sqrt{1 + 4xt} - 1\right) & \text{if } t \ne 0 \text{ and } 1 + 4xt > 0. \end{cases}$$

13. Solve the equation

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$

for two cases:

$$\text{(a)} \quad u(x,0) = \begin{cases} 1 & \text{if } x \le 0, \\ \left(1 - \frac{x}{a}\right) & \text{if } 0 < x < a, \\ 0 & \text{if } x \ge a; \end{cases}$$

$$\text{(b)} \quad u(x,0) = \begin{cases} 0 & \text{if } x \le 0, \\ \frac{x}{a} & \text{if } 0 < x < a, \\ 1 & \text{if } x \ge a. \end{cases}$$

Examine both cases in the limit, as $a \to 0$.

14. Solve the initial-value problem

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

15. Solve the initial-value problem for $u = u(x,t)$

$$u_t + u^2 u_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \sqrt{x}, \quad x \in \mathbb{R}.$$

16. Show that the solution of the initial-value problem

$$\begin{aligned} u_t + u^3 u_x &= 0, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= x^{\frac{1}{3}}, \quad x \in \mathbb{R}, \end{aligned}$$

is

$$u(x,t) = \left(\frac{x}{1+t}\right)^{\frac{1}{3}}.$$

Examine the solution as $t \to \infty$.

17. Show that the solution of the initial-value problem

$$\begin{aligned} u_t + u u_x &= 2t, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= x, \quad x \in \mathbb{R}, \end{aligned}$$

is

$$u(x,t) = t^2 + \frac{3x - t^3}{3(1+t)}.$$

18. Show that the solution of the initial-value problem

$$\begin{aligned} u_t + 3t u_x &= u, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \cos x, \quad x \in \mathbb{R}, \end{aligned}$$

is

$$u(x,t) = e^t \cos\left(x - \frac{3}{2}t^2\right).$$

19. Show that the solution of the initial-value problem

$$\begin{aligned} 2u_t + u_x &= 0, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \begin{cases} \sin x & \text{if } 0 \le x \le \pi, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

is

$$u(x,t) = \begin{cases} \sin(x - \frac{1}{2}t) & \text{if } \frac{1}{2}t \le x \le \frac{1}{2}t + \pi, \\ 0 & \text{otherwise.} \end{cases}$$

20. Use the method of characteristics to solve the initial boundary-value problem

$$\begin{aligned} u_t + u_x &= x, \quad x > 0,\ t > 0, \\ u(x,0) &= \sin x, \quad x > 0, \\ u(0,t) &= t, \quad t > 0. \end{aligned}$$

Show that the solution is

$$u(x,t) = \begin{cases} \frac{1}{2}x^2 - (x-t) & \text{if } x \le t, \\ t(x - \frac{1}{2}t) + \sin(x-t) & \text{if } x > t. \end{cases}$$

21. Show that the implicit solution of the equation

$$x^2 u_x + u u_t = 1,$$
$$u = 0 \quad \text{on } x + t = 1,\ x > 0,$$

is

$$\left(\frac{1}{2}u^2 + 1 - t\right)(1 + ux) = x.$$

22. The conservation form of the Buckley–Leverett (1942) equation for saturation, s, for propagation of non-aqueous phase liquids (typically, hydrocarbons) and water in porous media is

$$s_t + F_x = 0, \quad \text{or} \quad s_t + c s_x = 0,$$

where $F = V_d s^2/[s^2 + b(1-s)^2]$, V_d is the Darcy velocity, $c = \frac{\partial F}{\partial s}$ is the propagation speed, and b is the shape parameter. Show that s is conserved. Explain the features of the speed c.

6

Kinematic Waves and Real-World Nonlinear Problems

... as Sir Cyril Hinshelwood has observed ... fluid dynamicists were divided into hydraulic engineers who observed things that could not be explained and mathematicians who explained things that could not be observed.

James Lighthill

In every mathematical investigation, the question will arise whether we can apply our mathematical results to the real world.

V.I. Arnold

6.1 Introduction

This chapter deals with the theory and applications of kinematic waves to several real-world problems, which include traffic flow on highways, flood waves in rivers, glacier flow, roll waves in an inclined channel, chromatographic models, and sediment transport in rivers. The general ideas and essential features of these problems are of wide applicability. Other applications of conservation laws include various chromatographic models in chemistry and the movement of pollutants in waterways. The propagation of traffic jams is almost similar to the shock waves that cause noise pollution near airports and spaceports. Kinematic wave phenomena also play an important role in traveling detonation and combustion fronts, the wetting water fronts observed in soils after rainfall, and the clanking of shunting trains. All of these problems are essentially based on the theory of kinematic waves developed by Lighthill and Whitham (1955). Many basic ideas and important features of hyperbolic waves and kinematic shock waves are found to originate from gas dynamics, so specific nonlinear models which describe Riemann's simple waves with Riemann's invariants and shock waves in gas dynamics are discussed. Considerable attention is also given to nonlinear hyperbolic systems and Riemann's invariants, generalized simple waves, and generalized Riemann's invariants.

L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers*,
DOI 10.1007/978-0-8176-8265-1_6,

6.2 Kinematic Waves

Classical wave motions are described by Newton's second law of motion together with some reasonable assumptions relating stress to displacement (as in gravity waves), to strain (as in nondispersive longitudinal and transverse waves), or to curvature (as in capillary waves and flexural waves). In contrast with the case of dynamic waves, a class of waves is called *kinematic waves* when an appropriate functional relation exists between the density and the flux of some physically observed quantity. Kinematic waves are not at all waves in the classical sense, and they are physically quite different from the classical wave motions involved in dynamical systems. They describe, approximately, many important real-world problems including traffic flows on highways, flood waves in rivers, roll waves in an inclined channel, and chromatographic models in chemistry. Lighthill and Whitham (1955) first gave a general and systematic treatment of kinematic waves and applications.

In many problems of one-dimensional wave propagation where there is a continuous distribution of either material or some state of the medium, we can define a *density* $\rho(x,t)$ per unit length and a flux $q(x,t)$ per unit time. Then, we can introduce a flow velocity $u(x,t)$ by $u = q/\rho$. Assuming that the material (or state) is conserved, we can stipulate that the time rate of change of the total amount in any arbitrary interval $x_1 \le x \le x_2$ must be balanced by the net influx across x_1 and x_2. Physically, this states that the quantity in a small length segment changes at a rate equal to the difference between inflow and outflow. Mathematically, this can be formulated as

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = q(x_1,t) - q(x_2,t) = -\int_{x_1}^{x_2}\left(\frac{\partial q}{\partial x}\right)dx, \tag{6.2.1}$$

or equivalently,

$$\int_{x_1}^{x_2}\left(\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x}\right)dx = 0. \tag{6.2.2}$$

If this result is to hold for any arbitrary interval $x_1 \le x \le x_2$, the integrand must vanish identically, so that

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{6.2.3}$$

provided ρ and q are sufficiently smooth functions. As stated in Chapter 5, (6.2.3) is called the *conservation law*, *kinematic wave equation*, or the *equation of continuity*.

Based on theoretical or empirical grounds, we assume that there exists a relation between q and ρ, so that we can write

$$q = q(\rho). \tag{6.2.4}$$

Thus, equations (6.2.3) and (6.2.4) form a closed system since there are two equations with two unknown functions. Substituting (6.2.4) in (6.2.3) gives

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{6.2.5}$$

where

$$c(\rho) = q'(\rho) = \frac{\partial q}{\partial \rho}. \tag{6.2.6}$$

Similarly, multiplying (6.2.3) by $c(\rho)$ leads to the equation

$$\frac{\partial q}{\partial t} + c(\rho)\frac{\partial q}{\partial x} = 0. \tag{6.2.7}$$

This means that q or ρ is constant in waves propagating with velocity $c(\rho)$ given by (6.2.6), and hence, $c(\rho)$ is called the *wave propagation velocity*. Mathematically, equation (6.2.7) has only one system of characteristics given by

$$\frac{dx}{dt} = c(\rho), \tag{6.2.8}$$

and along each of these characteristics the flow q or the density ρ is constant. The wave velocity c as given by (6.2.6) is the slope of the flow-density curve for fixed x. In terms of the mean flow velocity $u = q/\rho$, the wave propagation velocity is given by

$$c = q'(\rho) = \frac{d}{d\rho}(u\rho) = u + \rho\frac{du}{d\rho}. \tag{6.2.9}$$

Thus, $c < u$ when $\frac{du}{d\rho} > 0$, that is, the flow velocity increases with density as in flood waves in rivers, and $c < u$ when $\frac{du}{d\rho} < 0$, that is, it decreases with density as in traffic flows on highways.

Further, the following observations are in order. First, there is one important difference between kinematic waves and dynamic waves. A kinematic wave has only one wave velocity at each point, while dynamic waves possess at least two velocities (forward and backward relative to the medium). Second, kinematic waves are nondispersive, but they suffer from a change in form due to nonlinearity (dependence of the wave speed c on the flow q carried by the wave) exactly as do traveling sound waves of finite amplitude. Consequently, continuous wave forms may develop discontinuities because the faster waves overtake the slower ones. These discontinuities can be described as shock waves because their process of development is identical to that of shock waves in gas dynamics.

The law of motion of kinematic shock waves can be derived from the conservation laws, as was the law governing continuous kinematic waves. If the density and flow assume the values ρ_1 and q_1 on the one side, and ρ_2 and q_2 on the other side of the shock wave which propagates with velocity U, then the quantity crossing it per unit time can be written either as $q_1 - U\rho_1$ or as $q_2 - U\rho_2$. This gives the velocity of the shock waves as

$$U = \frac{(q_2 - q_1)}{(\rho_2 - \rho_1)}. \tag{6.2.10}$$

This represents the slope of the chord joining the two points (ρ_1, q_1) and (ρ_2, q_2) on the density-flow curve which corresponds to the states behind and ahead of the shock wave when it reaches a given point x. In the limit, when the shock wave become a continuous wave, the slope of the chord becomes the slope of the tangent, and hence (6.2.10) reduces to (6.2.6). The development of kinematic shock waves was described by examples with figures in Chapter 5. To avoid duplication of a similar analysis, we simply state appropriate results whenever needed.

Another important effect, known as *diffusion*, is confined entirely to the interior of kinematic shock waves, where its effect is crucial in smoothing the nonlinear steeping of the kinematic waves. The effect of diffusion is certainly small outside the shock regions, but inside the shock regions, the effect of diffusion is very important as a second-order approximation. Mathematically, diffusion corresponds to the inclusion in (6.2.7) of an additional term proportional to a second derivative of q. This will happen if the flow-density relation involves some dependence on a derivative of q or ρ in addition to q, ρ, and x. We assume that, for each x, q is a function of ρ and ρ_t; alternatively, ρ is a function of q and q_x because $\rho_t = -q_x$. Substituting for ρ in (6.2.3), we obtain

$$\frac{\partial \rho}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial \rho}{\partial q_x}\frac{\partial^2 q}{\partial x \partial t} + \frac{\partial q}{\partial x} = 0. \tag{6.2.11}$$

Further, if the coefficients of the derivatives of q in (6.2.11) are approximated as functions of q and x alone, then equation (6.2.11) becomes

$$\frac{\partial q}{\partial t} + c(\rho)\frac{\partial q}{\partial x} + \nu\frac{\partial^2 q}{\partial x \partial t} = 0, \tag{6.2.12}$$

where $c = \frac{\partial q}{\partial \rho}$ is the kinematic wave velocity as before and $\nu = c(\frac{\partial \rho}{\partial q_x})$. This is a typical nonlinear equation representing diffusive kinematic waves. Invoking the first approximation $q_t \sim -cq_x$, (6.2.12) may be rewritten as

$$\frac{\partial q}{\partial t} + c(\rho)\frac{\partial q}{\partial x} = \nu c\frac{\partial^2 q}{\partial x^2}. \tag{6.2.13}$$

Physically, there is no difference between the two equations (6.2.12) and (6.2.13). However, further details of the diffusion equation will be pursued in Chapter 8.

The rest of this chapter is devoted to a detailed treatment of several real-world problems as applications of kinematic waves, Riemann's simple waves, and Riemann's invariants and their extensions.

6.3 Traffic Flow Problems

Traffic flow on a highway is one of the most common real-world problems. Based on the works of Lighthill and Whitham (1955) and Richards (1956), we consider the traffic flow on a long highway under the assumptions that cars do not enter or exit the highway at any one of its points and that individual cars are replaced by a continuous density function. We take the x-axis along the highway and assume the traffic flows in the positive direction. This problem can be described by *three* fundamental traffic variables: the traffic density $\rho(x,t)$, which is equal to the number of cars per unit length at position x and at time t, the traffic flow $q(x,t)$, the number of cars passing a

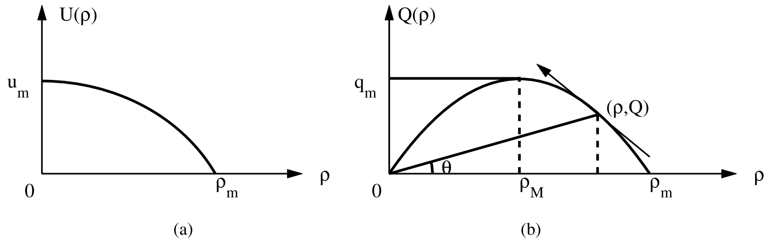

Fig. 6.1 (a) Velocity–density curve and (b) flow–density curve.

fixed point in unit time, and the traffic velocity $u(x,t)$, which represents the velocity of a car. We use the theory of kinematic waves to formulate the problem in terms of a first-order, nonlinear, partial differential equation on the basis of conservation of cars and experimental relationships between the car velocity and traffic density. So, the equations governing the traffic flow are

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{6.3.1}$$

$$q = q(\rho), \tag{6.3.2}$$

where q is some function of ρ as determined by theoretical or experimental findings of traffic flow phenomena. The functional relation (6.3.2) seems to be reasonable in the sense that the density of cars surrounding a given car indeed controls the speed of that car. In fact, this relation depends on other factors including speed limits, weather conditions, and road conditions. Several specific relations have been suggested by Haight (1963).

On the basis of observations of traffic flow, we make a basic simplifying assumption that the velocity of a car at any point along the highway depends only on the traffic density, that is, $u = U(\rho)$, and hence, the traffic flow $q = \rho U(\rho)$. Clearly, $u(\rho)$ must be a monotonically decreasing function of density ρ. If there are no cars on the highway (corresponding to very low densities), then the car would travel at a finite maximum speed $u_{\max}$, that is, $U(0) = u_{\max} = u_m$. As the density of cars increases, the velocities of the cars would continue to decrease, and hence, $U(\rho) = 0$ as $\rho \to \rho_{\max} = \rho_m$, where ρ_m is the maximum traffic density corresponding to what is called *bumper-to-bumper* traffic. Also, it follows that $\frac{du}{d\rho} = U'(\rho) \leq 0$. The decreasing feature of traffic velocity is shown in Figure 6.1(a).

On the other hand, the important feature of traffic flow $Q(\rho) = \rho U(\rho)$ can be inferred from $U(\rho)$. Clearly, $Q(\rho) \to 0$, as $\rho \to 0$, and $\rho \to \rho_{\max} = \rho_m (U(\rho_m) = 0)$. This means that the traffic flow $Q(\rho)$ is an increasing function of density ρ until it attains a maximum value $Q_{\max} = q_M$ for some $\rho = \rho_M$ in $0 < \rho < \rho_m$. In general, $Q(\rho)$ assumes a parabolic form, concave downward, $Q''(\rho) < 0$. All of these features of the flow–density curve are shown in Figure 6.1(b).

It is important to point out that, corresponding to any point on the flow–density curve, the mean speed $U = Q/\rho = \tan\theta$ of cars represents the slope of the chord from the origin as shown in Figure 6.1(b).

The propagation speed $c = \frac{dq}{d\rho}$ of density waves carrying continuous changes of flow through the streams of cars is the slope of the tangent to the curve at that point. This slope is smaller provided that the mean speed *decreases* with increased density. Hence, we can write

$$c = \frac{dq}{d\rho} = \frac{d}{d\rho}(\rho u) = u + \rho\frac{du}{d\rho}. \tag{6.3.3}$$

Evidently, $c < u$ if $\frac{du}{d\rho} < 0$, and $c = u$ only at a very low density.

Usually, under light traffic conditions, there are few cars on the highway traveling at high speed. On the other hand, under heavy traffic conditions, there are many cars on the road, and these cars travel slowly. So again, traffic flow would be small.

Substituting $q = U(\rho)$ into (6.3.1) gives the first-order, nonlinear, partial differential equation for traffic flow in the form

$$\frac{\partial\rho}{\partial t} + c(\rho)\frac{\partial\rho}{\partial x} = 0, \tag{6.3.4}$$

where

$$c(\rho) = Q'(\rho) = U(\rho) + \rho\, U'(\rho) \tag{6.3.5}$$

represents the velocity of the density waves. Since $U'(\rho) < 0$, $c(\rho) < U(\rho)$, that is, the velocity of these waves is always less than that of cars, and drivers experience such waves and are warned of disturbances ahead. Figure 6.1(b) shows that $c(\rho) = Q'(\rho) > 0$ for all ρ in $[0, \rho_M)$, is zero at $\rho = \rho_M$, and then, is negative in $(\rho_M, \rho_m]$. All these cases mean that waves propagate forward relative to the highway in $[0, \rho_M]$, are stationary at $\rho = \rho_M$, and then, travel backward in $[\rho_M, \rho_m]$. Further, discontinuous waves are likely to occur on any segment of highway when the traffic is very heavy in front and light behind. For waves on which the traffic flow is less dense, cars travel forward faster than, and hence tend to catch up with, those on which the flow is more dense. When this happens, a group of continuous waves can coalesce into discontinuous (or shock) waves.

According to the theory of characteristics of the first-order nonlinear equation, the characteristic equations for (6.3.4) are

$$\frac{dt}{1} = \frac{dx}{c(\rho)} = \frac{d\rho}{0}, \tag{6.3.6}$$

with the solution

$$\rho = \text{const.} \quad \text{on } \frac{dx}{dt} = c(\rho). \tag{6.3.7}$$

Since $c(\rho)$ is constant when ρ is constant, the characteristics are straight lines in the (x,t)-plane, and the characteristic velocity $c(\rho) = Q'(\rho)$ is the slope of the tangent to the curve at a point.

Actual observational data of traffic flow indicate that typical results on a single lane highway are $\rho_m \approx 225$ cars per mile, $\rho_M \approx 80$ cars per mile, and $q_M \approx 1590$ cars per hour. Thus, the maximum traffic flow q_M occurs at a low speed $u = q_M/\rho_M \approx 20$ miles per hour.

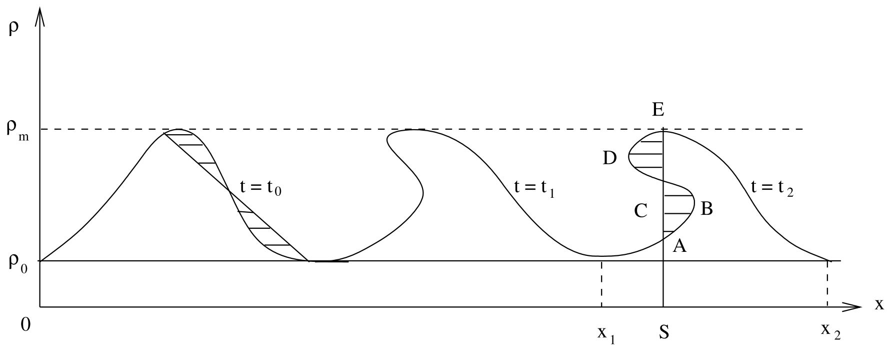

Fig. 6.2 Multi-valued density profile at $t = t_2 \gg t_1 > t_0 > 0$.

Next, we solve the nonlinear traffic flow equation (6.3.4) with the initial and boundary conditions

$$\rho(x, 0) = f(x), \quad x \in \mathbb{R}, \tag{6.3.8}$$

where $f(x)$ vanishes outside a finite domain of x and $\rho(x, t) \to 0$, as $|x| \to \infty$.

This initial-value problem has already been solved in Section 5.2, and hence, we state the result without proof. The solution is given by

$$\rho(x, t) = f(\xi), \quad x = \xi + t\, F(\xi), \tag{6.3.9ab}$$

where

$$F(\xi) = c\{f(\xi)\} = c(\rho). \tag{6.3.10}$$

The velocity $c(\rho) = Q'(\rho)$ is the slope of the traffic flow curve drawn in Figure 6.1(b), so that the density waves move forward or backward relative to the highway depending on whether $\rho < \rho_M$ or $\rho > \rho_M$.

We consider an initial density curve $\rho(x, 0) = f(x)$ in the form of a hump which has a maximum ρ_m at some point x, as shown in Figure 6.2 so that $\rho_m > \rho_0$ and $c(\rho_0) < c(\rho_m)$. Thus, the point on the hump with density ρ_m travels slower than the point with density ρ_0. As time progresses, the density profile continues to steepen at the back and flatten at the front, eventually leading to a multi-valued solution which is physically inadmissible. In other words, a vertical segment develops at the back for $t_2 \gg t_1$. The vertical segment corresponds to a point in the (x, t)-plane at which the density is discontinuous and, therefore, at which at least one of the partial derivatives ρ_x and ρ_t no longer exists. Thus the distortion of the initial density profile and the development of discontinuity as a *shock* are remarkable features of nonlinear traffic flow phenomena. However, the solution incorporating the discontinuity must satisfy conservation laws, and hence, the total number of cars in any interval (x_1, x_2) represented by the integral of ρ with respect to x from x_1 to $x_2 (> x_1)$ must be unchanged. Thus, the shock must be inserted at that value of x which leaves the total area under the curve unchanged, that is, at the point S such that the area $ABC =$ area CDE. This simple result that the two areas cut off by the vertical line through

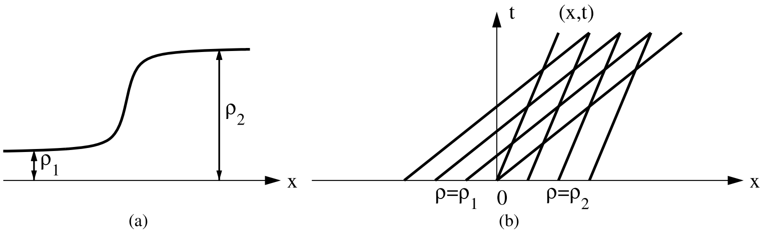

Fig. 6.3 (a) Initially heavier traffic ahead $(\rho_2 > \rho_1)$ and (b) intersecting characteristics.

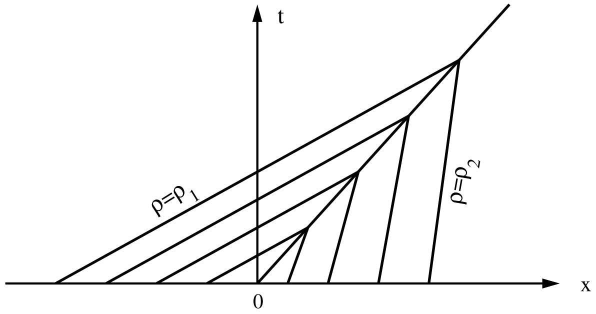

Fig. 6.4 Graphical representation of solution.

S are equal is known as *Whitham's rule*. So, the Whitham rule of equal area can be used to determine the position of the shock path for certain nonlinear equations.

We consider a situation in which traffic initially becomes *heavy* further along the highway. As already indicated in Section 5.2, the solution represents a compression wave. It is convenient to assume the initial data, as shown in Figure 6.3(a),

$$\rho(x,0) = \begin{cases} \rho_1 & \text{if } x < 0, \\ \rho_2 & \text{if } x > 0, \end{cases} \tag{6.3.11}$$

where ρ_1 and ρ_2 are constants, $0 \le \rho_1 < \rho_2 \le \rho_{\max}$, and hence, $U_1 > U_2$. The density wave for the lighter travel with velocity $c(\rho_1) = Q'(\rho_1)$, which is greater than the velocity $c(\rho_2) = Q'(\rho_2)$ of the heavier traffic density wave, that is, $c(\rho_1) > c(\rho_2)$. A set of characteristics using the initial data (6.3.11) is shown in Figure 6.3(b). In any situation where the traffic becomes denser further along the road, characteristics intersect. At any point where two characteristics intersect, $\rho = \rho_1$ and $\rho = \rho_2$, but it is physically impossible for the traffic density to be multi-valued. A jump occurs across a curve as a shock. The behavior of the 'jump' in the density ρ in the (x,t)-plane just treated indicates that a shock should be inserted between two regions of constant density. Since the overlapping starts at the origin, we look for a solution of the form shown in Figure 6.4, two regions of constant density ρ separated by a constant shock velocity given by (5.3.6). It can be shown that equation (6.3.4) and the initial data (6.3.11) are satisfied on both sides of the shock and the condition (5.3.6) is satisfied on the shock. Evidently, $\rho = \rho_1 =$ const. satisfies equation (6.3.4)

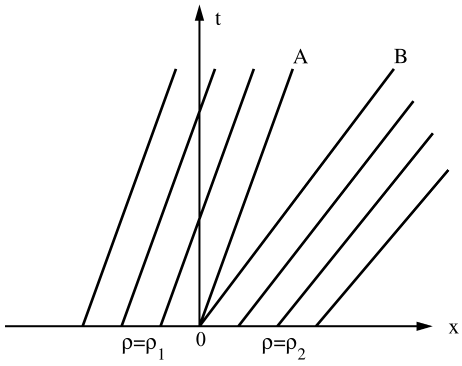

Fig. 6.5 The solution with a fan centered at the origin.

and (6.3.11) for $x < 0$ at $t = 0$; similarly, $\rho = \rho_2 =$ const. for $x > 0$ at $t = 0$. The shock condition (5.3.6) is satisfied if

$$U = \left(\frac{Q_1 - Q_2}{\rho_1 - \rho_2}\right), \tag{6.3.12}$$

that is, if the shock propagates at the constant velocity given by (6.3.12). The shock path is a straight line through the origin in the (x, t)-plane. Thus the traffic density $\rho(x, t)$ satisfies all equations and initial data and is single-valued for all x and t except across the shock.

There is another case associated with the initial data which corresponds to $\rho_1 > \rho_2$, and hence, $U_1 < U_2$, and $c(\rho_1) < c(\rho_2)$. Physically, this case represents a situation in which the traffic initially becomes *light* further along the highway. Using the initial conditions (6.3.11), characteristics are drawn in Figure 6.5. It is important to point out that there are no characteristics within the infinite sector AOB, and hence, no solution for the traffic density $\rho(x, t)$. According to Section 5.4, there are two regions, each of constant density ρ, separated by a *fan centered* at the origin. The solution is given by

$$\rho(x,t) = \begin{cases} \rho_1 & \text{to the left of OA,} \\ \rho_2 & \text{to the right of OB,} \\ \rho_0, \text{const.}, \ \rho_2 < \rho_0 < \rho_1 \text{ on } x = q'(\rho_0) & \text{in the sector AOB.} \end{cases} \tag{6.3.13}$$

The solution in the sector AOB is the characteristic form, equation (6.3.7), of the solution of equation (6.3.4). Across OA and OB, $\rho(x, t)$ is continuous, but ρ_x and ρ_t are discontinuous (ρ_x and ρ_t are nonzero in the sector AOB but zero elsewhere).

Thus, the problem is solved for both cases: (i) $\rho_1 < \rho_2$ and (ii) $\rho_1 > \rho_2$. However, a remarkable fact involved in this problem with case (ii) is that there exists another discontinuous solution which satisfies all equations and initial conditions. This solution consists of a *shock* dividing two regions of constant density ρ. All characteristics on the left of the shock are parallel, and so are those to the right of the shock as shown in Figure 6.6.

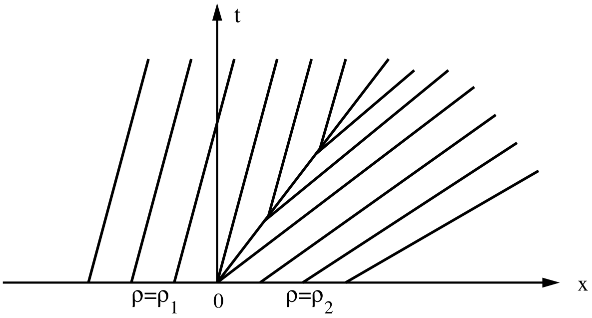

Fig. 6.6 Characteristics and the discontinuous solution.

For case (ii), there are at least two solutions: (a) the fan-centered solution and (b) the shock solution. Both satisfy all equations and the initial conditions, but the solution is *not unique*. The main question is to find which one of these solutions represents a physically meaningful solution. This question was answered in Section 5.4 by introducing the Lax entropy criterion (5.4.17).

Example 6.3.1 The Green Signal Problem in Front of the Traffic. We consider a stream of cars stopped by a red signal at $x = 0$. The road is jammed initially behind the signal and there is no traffic ahead of the signal. As soon as the signal turns green, the stream of cars starts moving across $x = 0$. The initial state of the traffic is given by

$$\rho(x,0) = \rho_m H(-x) = \begin{cases} \rho_m & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases} \tag{6.3.14}$$

where ρ_m is the maximum density and $H(x)$ is the Heaviside unit step function.

We assume that the traffic flow $q(\rho)$ is quadratic in the region $0 < \rho < \rho_m$ and zero otherwise, that is,

$$q = \frac{q_m}{\rho_m}\left(1 - \frac{\rho}{\rho_m}\right)\rho, \quad 0 < \rho < \rho_m. \tag{6.3.15}$$

It is noted that the maximum flow rate is $\frac{1}{4}q_m$ which occurs at $\rho = \frac{1}{2}\rho_m$. The velocity of the traffic flow is $u = \frac{q}{\rho}$, that is,

$$u = u_m\left(1 - \frac{\rho}{\rho_m}\right), \tag{6.3.16}$$

where $u_m = (q_m/\rho_m)$ is the maximum speed when $\rho = 0$. The traffic flow equation for the density $\rho(x,t)$ is given by

$$\rho_t + c(\rho)\rho_x = 0, \tag{6.3.17}$$

where

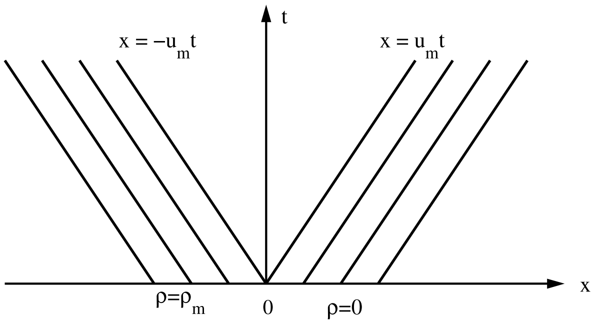

Fig. 6.7 Characteristics due to a discontinuous initial density.

$$c(\rho) = q'(\rho) = u_m\left(1 - \frac{2\rho}{\rho_m}\right). \tag{6.3.18}$$

Thus, the characteristic equations for (6.3.17) are

$$\frac{dt}{1} = \frac{dx}{c(\rho)} = \frac{d\rho}{0},$$

where

$$d\rho = 0, \qquad \frac{dx}{dt} = c(\rho) = u_m\left(1 - \frac{2\rho}{\rho_m}\right). \tag{6.3.19ab}$$

Thus the density ρ remains constant along a characteristic given by

$$\frac{dx}{dt} = u_m\left(1 - \frac{2\rho}{\rho_m}\right). \tag{6.3.20}$$

Any characteristic that intersects the positive x-axis at $x = x_0 > 0$ has the slope

$$\frac{dx}{dt} = u_m\left(1 - \frac{2\rho}{\rho_m}\right) = u_m\left[1 - \frac{2\rho(x,0)}{\rho_m}\right] = u_m. \tag{6.3.21}$$

This gives a family of characteristics as the right-leaning straight lines, as shown in Figure 6.7,

$$x = u_m t + x_0. \tag{6.3.22}$$

On the other hand, any characteristic that intersects the negative x-axis at $x = x_0 < 0$ has the slope

$$\frac{dx}{dt} = u_m\left[1 - \frac{2u_m}{u_m}\right] = -u_m. \tag{6.3.23}$$

As shown in Figure 6.7, equation (6.3.23) gives a family of characteristics representing the left-leaning straight lines

$$x = -u_m t + x_0. \tag{6.3.24}$$

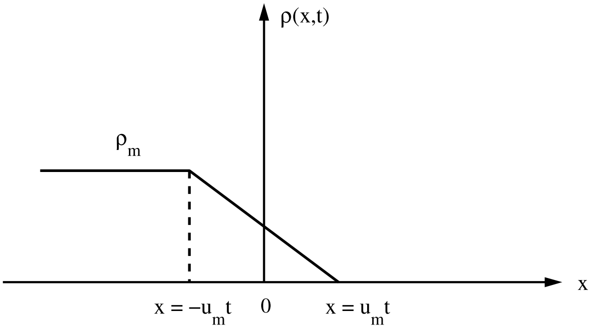

Fig. 6.8 Traffic density after the signal turns green.

In the fan-like region $-u_m t < x < u_m t$, all characteristics must pass through the origin and must be straight lines. Hence, $\frac{dx}{dt} = \frac{x}{t}$. It follows from (6.3.20) that

$$\frac{x}{t} = u_m \left(1 - \frac{2\rho}{\rho_m}\right). \tag{6.3.25}$$

Solving for ρ gives

$$\rho = \frac{1}{2}\rho_m \left(1 - \frac{x}{u_m t}\right), \quad -u_m t < x < u_m t. \tag{6.3.26}$$

At any instant, the density varies linearly in x from ρ_m at $x = -u_m t$ to zero at $x = u_m t$. Thus, the solution (6.3.26) for $\rho(x,t)$ for all x at any time t is drawn in Figure 6.8.

Example 6.3.2 The Traffic Signal Problem. We consider the same problem as in Example 6.3.1 with the traffic flow $q = \frac{3}{10}(200 - \rho)\rho$, and hence, the velocity of the traffic density waves is $c(\rho) = \frac{3}{5}(100 - \rho)$. We have to find the traffic density when the traffic signal turns green at $t = 0$, so that the initial density is given by (6.3.14). The typical value of ρ_m varies from 200 to 225.

According to the theory described above, the solution of the initial-value problem is

$$f(x,t) = f(\xi) = \rho_m H(-\xi), \qquad x = \xi + c\big(f(\xi)\big) = \xi + \frac{3}{5}\{100 - f(\xi)\}t. \tag{6.3.27ab}$$

Using $\rho_m = 200$, we obtain the solution as

$$f(x,t) = \begin{cases} 200 & \text{if } \xi < 0, \\ 0 & \text{if } \xi > 0, \end{cases} \qquad x = \begin{cases} -|\xi| - 60t & \text{if } \xi < 0, \\ \xi + 60t & \text{if } \xi > 0. \end{cases} \tag{6.3.28ab}$$

These two results can be combined to obtain the traffic density for any $t > 0$

$$\rho(x,t) = \begin{cases} 200 & \text{if } x < -60t, \\ 0 & \text{if } x > 60t. \end{cases} \tag{6.3.29}$$

However, the solution for $\rho(x,t)$ is not known in the interval $-60t < x < 60t$. The nonexistence of the solution in this interval is somewhat unusual. It is possibly due to the fact that the initial condition of the traffic flow is discontinuous at $\xi = 0$, but the use of the conservation law (6.3.4) requires that $\rho(x,t)$ be differentiable.

Example 6.3.3. Obtain the traffic density $\rho(x,t)$ governed by (6.3.4) with the given traffic flow $q(\rho) = \frac{3}{10}(200 - \rho)\rho$ and with the initial state of the traffic flow as

$$\rho(x,0) = f(x) = 1 - x. \tag{6.3.30}$$

In this problem, the velocity of the density waves is given by

$$c(\rho) = \frac{dq}{d\rho} = \frac{3}{5}(100 - \rho). \tag{6.3.31}$$

The solution of the initial-value problem is given by (6.3.9ab), that is,

$$\rho(x,t) = f(\xi), \tag{6.3.32a}$$

$$x = \xi + t\,c\big(f(\xi)\big), \tag{6.3.32b}$$

where $c\{f(\xi)\} = \frac{3}{5}(100 - 1 + \xi) = \frac{3}{5}(99 + \xi)$. Solving for ξ from (6.3.32b) gives

$$\xi = \frac{(x - \frac{3}{5}\cdot 99t)}{(1 + \frac{3}{5}t)}. \tag{6.3.33}$$

This result can be used to eliminate ξ from (6.3.32a) so that the solution becomes

$$\rho(x,t) = (1 - \xi) = \frac{(1 - x + 60t)}{(1 + \frac{3t}{5})}. \tag{6.3.34}$$

This traffic flow model is based on the first-order approximation, and hence, the original assumptions that $q = Q(\rho)$ and $u = U(\rho)$ are not good approximations. For a better approximation, it is reasonable to assume that q and u are functions of both ρ and its gradient ρ_x. Hence, we can assume that

$$q = Q(\rho) - \nu\rho_x, \tag{6.3.35a}$$

$$u = U(\rho) - \nu\left(\frac{\rho_x}{\rho}\right), \tag{6.3.35b}$$

where ν is a positive constant in the context of traffic flow because drivers are supposed to reduce their speed to account for heavy traffic ahead. This introduces two additional effects: (i) diffusion of the waves and (ii) response time defined by the time lag in the response of the driver and his or her car to any changes in traffic conditions.

To incorporate the first effect of diffusion produced by the driver's response to conditions ahead, we substitute (6.3.35a) in the traffic flow equation (6.3.1), which gives a new parabolic equation in the form

$$\rho_t + c(\rho)\rho_x = \nu\rho_{xx}. \tag{6.3.36}$$

This equation is known as the Burgers equation*,* which will be discussed in great detail in Chapter 8. Physically, the second-order term introduced by the diffusion will smooth out the traffic flow. In other words, the effect of diffusion tends to eliminate any discontinuity involved in the solution of the original hyperbolic traffic flow equation (6.3.4).

To incorporate the second effect, the following equation:

$$\frac{Du}{Dt} = u_t + uu_x = -\frac{1}{\tau}\left[u - U(\rho) + \nu\left(\frac{\rho_x}{\rho}\right)\right] \tag{6.3.37}$$

can be introduced for the acceleration of the car where the coefficient τ is a measure of the response time. Equation (6.3.37) is to be solved combined with the conservation law

$$\rho_t + (\rho u)_x = 0. \tag{6.3.38}$$

For the case of small ν and τ, (6.3.37) can be approximated by the original first-order result, that is, by $u = U(\rho)$.

Before we solve the nonlinear problem, it would be helpful to examine the linearized version of equations (6.3.37) and (6.3.38). These equations can be linearized for small perturbations about the equilibrium state $\rho = \rho_0$ and $u = u_0 = U(\rho_0)$, so that we can introduce $\rho = \rho_0 + \tilde{\rho}(x,t)$ and $u = u_0 + \tilde{u}(x,t)$. We substitute these results in (6.3.37) and (6.3.38) and retain only first-order terms to obtain

$$\tau[\tilde{u}_t + u_0\tilde{u}_x] = -\left[\tilde{u} - U'(\rho_0)\tilde{\rho} + \left(\frac{\nu}{\rho_0}\right)\tilde{\rho}\right], \tag{6.3.39}$$

$$\tilde{\rho}_t + u_0\tilde{\rho}_x + \rho_0\tilde{u}_x = 0. \tag{6.3.40}$$

The propagation velocity of the density wave is represented by

$$c_0 = \rho_0 U'(\rho_0) + U(\rho_0), \tag{6.3.41}$$

and hence, $U'(\rho_0) = -(u_0 - c_0)/\rho_0$ which is used in (6.3.39), (6.3.40) to eliminate $\tilde{u}$, and the final form of the equation for $\tilde{\rho}(x,t)$ is

$$(\tilde{\rho}_t + c_0\tilde{\rho}_x) = \nu\tilde{\rho}_{xx} - \tau\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x}\right)\tilde{\rho}. \tag{6.3.42}$$

It is noted that the first term on the right-hand side represents diffusion, whereas the second term arises from the effect of the response time τ.

If ν and τ are very small, (6.3.42) reduces to a simple first-order equation.

$$\tilde{\rho}_t + c_0\tilde{\rho}_x = 0. \tag{6.3.43}$$

The solution of this equation is $\tilde{\rho} = f(x - c_0 t)$ representing waves traveling with speed c_0. Equation (6.3.43) suggests that

$$\frac{\partial}{\partial t} \approx -c_0 \frac{\partial}{\partial x}. \tag{6.3.44}$$

This approximation is utilized in the right-hand side of (6.3.42), so that it reduces to the form

$$(\tilde{\rho}_t + c_0 \tilde{\rho}_x) = \left[\nu - \tau(u_0 - c_0)^2\right] \frac{\partial^2 \tilde{\rho}}{\partial x^2}. \tag{6.3.45}$$

It can be inferred from this equation that the solution would be stable or unstable depending on whether

$$\nu > \tau(u_0 - c_0)^2 \quad \text{or} \quad \nu < \tau(u_0 - c_0)^2. \tag{6.3.46ab}$$

We examine the stability of the plane wave solutions of the original equation (6.3.42) without any approximation. This equation admits solutions of the form

$$\tilde{\rho}(x,t) = a \exp\left[i(kx - \omega t)\right], \tag{6.3.47}$$

provided the equation

$$\nu(\omega - u_0 k)^2 + i(\omega - c_0 k) - \nu k^2 = 0 \tag{6.3.48}$$

is satisfied. Thus, the solutions would be stable if $\mathrm{I}(\omega) < 0$ for both roots of (6.3.48).

Finally, we observe that the right-hand side of (6.3.42) is a wave operator so that it corresponds to factoring as

$$\frac{\partial \tilde{\rho}}{\partial t} + c_0 \frac{\partial \tilde{\rho}}{\partial x} = -\tau \left(\frac{\partial}{\partial t} + c_+ \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c_- \frac{\partial}{\partial x}\right)\tilde{\rho}, \tag{6.3.49}$$

where

$$c_+ = u_0 + \sqrt{\frac{\nu}{\tau}}, \qquad c_- = u_0 - \sqrt{\frac{\nu}{\tau}}. \tag{6.3.50ab}$$

Evidently, one wave propagates with the speed c_+ and the other with the speed $c_- < c_+$. Thus, for small τ, the equation (6.3.49) can be approximated by

$$\tilde{\rho}_t + c_0 \tilde{\rho}_x = 0, \tag{6.3.51}$$

provided $c_- < c_0 < c_+$, which is exactly the stability criterion (6.3.46ab). Thus, the conclusion is that the solution is stable provided that the condition $c_- < c_0 < c_+$ holds.

6.4 Flood Waves in Long Rivers

Another application of the theory of kinematic waves dealing with flood waves in long rivers will be presented here in some detail. We consider the propagation of flood waves in a long rectangular river of constant breadth. We take the x-axis along the river which flows in the positive x-direction and assume that the disturbance is approximately the same across the breadth. In this problem, the depth $h(x,t)$ of the river plays the role of density involved in the traffic flow model discussed in Section 6.3. We denote the volume flow per unit breadth and per unit time by $q(x,t)$. An argument similar to that of Section 6.2 gives the kinematic wave equation for the flow in a river as

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{6.4.1}$$

Although the flow in a river is fairly complicated, we assume a simple functional relation $q = Q(h)$ as a first approximation to increased flow as the water level rises. Thus, equation (6.4.1) becomes

$$\frac{\partial h}{\partial t} + c(h)\frac{\partial h}{\partial x} = 0, \tag{6.4.2}$$

where $c(h) = Q'(h)$ and $Q(h)$ is essentially determined from the balance between the gravitational force and the frictional force of the river bed. One such result for $Q(h)$ is the famous *Chézy formula* given by

$$Q(h) = h\,u, \tag{6.4.3}$$

where u is the average velocity of the river flow and it is proportional to $\sqrt{h}$, so that $u = \alpha\sqrt{h}$, where α is a constant. The propagation velocity of flood waves, by (6.2.6), is given by

$$c(h) = Q'(h) = \frac{d}{dh}(uh) = u + h\frac{du}{dh} = \frac{3}{2}\,\alpha\,\sqrt{h} = \frac{3}{2}u. \tag{6.4.4}$$

This clearly shows that the flood waves propagate one and a half times faster than the stream velocity.

A more general result is $u = \alpha h^n$ for a constant α so that the volume flow is given by

$$Q(h) = hu = \alpha h^{n+1}. \tag{6.4.5}$$

Thus the propagation velocity of flood waves is

$$c(h) = Q'(h) = (n+1)u. \tag{6.4.6}$$

This also confirms the fact that flood waves propagate faster than the fluid velocity for all positive n. When $n = \frac{2}{3}$, (6.4.5) is called the *Manning formula*.

When the kinematic wave theory described in Section 5.3 is applied to flood waves in rivers, kinematic shock waves play a major role in the forward regions of flood waves. The general solution may be taken from Section 5.3 with shocks fitted in as discontinuities which satisfy the condition

$$U = \frac{(u_2 h_2 - u_1 h_1)}{(h_2 - h_1)}. \tag{6.4.7}$$

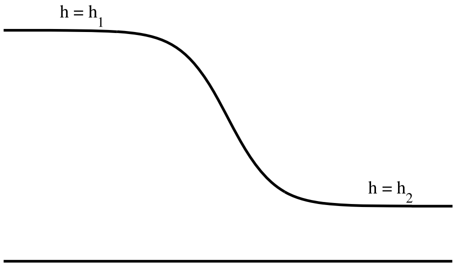

Fig. 6.9 Monoclinal flood wave.

The process by which kinematic waves may steepen into shock waves with a considerable change in flow in a relatively short distance has not been very clearly described in the literature. However, the possibility that such a wave propagates down a river, with different, uniform flows upstream and downstream of it, has been visualized as a flood wave model. Such a wave is called the *monoclinal flood wave* (or the *steady wave profile*). In fact, this is a progressing wave whose profile tends to different constant states upstream and downstream, with lower depth downstream, joined by a steadily falling region, as indicated schematically in Figure 6.9.

It has been suggested by Whitham (1974) that discontinuities described by the jump conditions in hydraulic theory are in reality the turbulent bores familiar in water wave phenomena as *hydraulic jumps* or *breakers* on a beach.

Although the above nonlinear model based on a first-order approximation, gives a fairly good description of the flood wave phenomena, Seddon (1900) raised a number of questions about the formulation of any river flow model based solely on the Chézy relation or its extensions. First, rivers do *not* have a uniformly sloping bed, nor do they, in any way, approximate this situation. Second, the river bed constantly varies with time, since its material is readily handled by the flow. To improve the above model, it is reasonable to assume that the volume flow depends on both h and x so that $q = q(h, x)$. This relation can be combined with (6.4.1) to give a first approximation for unsteady flows which vary slowly. Then, $h(x, t)$ satisfies the equation

$$\frac{\partial h}{\partial x} + \frac{\partial q}{\partial h} \cdot \frac{\partial h}{\partial x} = \frac{\partial q}{\partial x}. \tag{6.4.8}$$

Multiplying (6.4.1) by $c = (\frac{\partial q}{\partial h})_{x \text{ constant}} = c(h, x)$, we obtain

$$\frac{\partial q}{\partial t} + c(h, x)\left(\frac{\partial q}{\partial h}\right) = 0. \tag{6.4.9}$$

This means that q is constant for waves traveling past the point with velocity $c(h, x)$. Mathematically, the equation has only one system of characteristics given by $\frac{dx}{dt} = c$, and the flow q is constant along each of these characteristics. The wave velocity $c = c(h, x)$ is the slope of the density-flow curve for fixed x. This result for flood waves is known as the *Kleitz–Seddon law*.

6.5 Chromatographic Models and Sediment Transport in Rivers

In chemistry, chromatography is described as an exchange process between a solid and a fluid. In general, the exchange process may involve particles or ions of some substance or it may be heat exchange between the solid bed and a fluid. The general problem is extremely complicated. However, it is possible to develop a simple one-dimensional mathematical model for single-solute chromatography based on the assumption of a local equilibrium. The concentration of solute per unit volume in the fluid phase is denoted by $c(x,t)$, and the concentration of solute in the solid phase by $n(x,t)$. A simple argument of the balance of influx and outflux leads to the basic conservation equation

$$\varepsilon \frac{\partial c}{\partial t} + \varepsilon\, u \frac{\partial c}{\partial x} + (1-\varepsilon) \frac{\partial n}{\partial t} = 0, \tag{6.5.1}$$

where ε is a constant fractional void volume in the bed and u is a constant interstitial velocity of fluid through the bed. The equilibrium equation is

$$n = f(c), \tag{6.5.2}$$

where $f(c)$ is assumed to be positive for physical considerations, but $f(c)$ in general depends on temperature. Often, $f(c)$ is called an *absorption isotherm*. Substituting (6.5.2) in (6.5.1) gives

$$\left[\varepsilon + (1-\varepsilon) f'(c)\right] \frac{\partial c}{\partial t} + \varepsilon\, u \frac{\partial c}{\partial x} = 0. \tag{6.5.3}$$

Introducing $\nu = \frac{(1-\varepsilon)}{\varepsilon}$ in (6.5.3) yields

$$\left[1 + \nu\, f'(c)\right] \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = 0. \tag{6.5.4}$$

Finally, in terms of a new function defined by $g(c) = c + \nu\, f(c)$, equation (6.5.4) assumes the form

$$g'(c) \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = 0. \tag{6.5.5}$$

The function $g(c)$ is called the *column isotherm*. This equation is somewhat similar to that of the traffic flow equation. So, we may skip the detailed method of solution and state the salient features of the solution.

It is convenient to introduce nondimensional variables defined by

$$x^* = \frac{x}{\ell}, \qquad t^* = \frac{ut}{\ell}, \tag{6.5.6ab}$$

where ℓ is a characteristic length for the column. Dropping the asterisks, equation (6.5.5) becomes

$$g'(c) \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} = 0. \tag{6.5.7}$$

This nonlinear chromatographic equation can be solved subject to the initial and boundary data

$$c(x,0) = \phi(x), \qquad c(0,t) = \psi(t), \tag{6.5.8ab}$$

where ϕ and ψ are arbitrary functions.

It is often convenient to express the initial and boundary data in the parametric form

$$x = \xi, \quad c = \phi(\xi) \quad \text{at } t = 0, \tag{6.5.9}$$

$$t = \tau, \quad c = \psi(\tau) \quad \text{at } x = 0. \tag{6.5.10}$$

The characteristic equation of (6.5.7) is

$$\frac{dt}{dx} = g'(c), \tag{6.5.11}$$

so that $g''(c) = \nu f''(c)$. For convex isotherms, $f''(c) < 0$, and hence, $g''(c) < 0$.

Next, we solve a particular model, called the *Langmuir isotherm model*, described by

$$n = \frac{N\kappa c}{(1+\kappa c)} = \frac{\alpha c}{(1+\kappa c)} = f(c), \tag{6.5.12}$$

where N is the saturation concentration, κ is a rate constant, and $\alpha = N\kappa$. In this case, the characteristic equation is

$$\frac{dt}{dx} = g'(c) = 1 + \nu\, f'(c) = 1 + \frac{\nu\alpha}{(1+\kappa c)^2}. \tag{6.5.13}$$

We consider a particular case of (6.5.9), (6.5.10) representing a linear distribution of solute over a finite segment of the x-axis so that

$$x = \xi, \quad c = \begin{cases} \frac{c_0}{\alpha}\xi & \text{if } 0 \le \xi \le a, \\ c_0 & \text{if } \xi \ge a \end{cases} \quad \text{at } t = 0, \tag{6.5.14}$$

$$t = \tau, \quad c = 0 \quad \text{at } x = 0, \tag{6.5.15}$$

where c_0 is constant.

The graphs of the Langmuir isotherm and the characteristics in the (x,t)-plane are drawn in Figure 6.10(a) and (b).

It follows from Figure 6.10(b) that all characteristics originating from the t-axis have slope $g'(0) = 1 + \nu\alpha$, whereas those from the x-axis for $\xi \ge a$ are less steep with a slope $g'(c_0)$. These two sets of characteristics yield two constant states, $c = 0$ and $c = c_0$, respectively. On the other hand, the characteristics originating from the interval $0 \le \xi \le a$ on the x-axis have slope between $g'(0)$ and $g'(c_0)$, and hence, they represent a *fan or expansive wave*. Thus, the solution consists of two constant states separated by the expansion wave. For any point in the expansion wave region, we obtain

$$t = \{x - \xi(c)\}\left[1 + \frac{\nu\alpha}{(1+\kappa c)^2}\right], \qquad \xi(c) = \left(\frac{a}{c_0}\right)c. \tag{6.5.16ab}$$

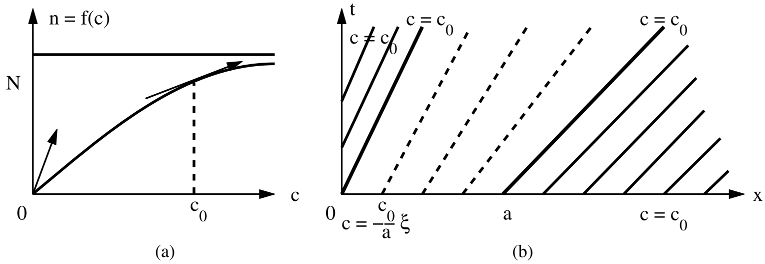

Fig. 6.10 (a) The Langmuir isotherm and (b) characteristics in (x,t)-plane.

In the limit as $a \to 0$, the characteristics originating from the interval $0 \le \xi \le a$ tend to translate to the left, but their slopes remain unchanged. In fact, these characteristics all originate from the origin, that is, the simple waves become centered there. In reality, this represents the elution of a uniformly absorbed column, which is a Riemann problem. In this case $(a \to 0)$, we obtain

$$\frac{t}{x} = g'(c) = 1 + \frac{\nu\alpha}{(1+\kappa c)^2} \tag{6.5.17}$$

from any point in the simple wave region, so that (κc) can be expressed as a function of $(\frac{t}{x})$ only:

$$(\kappa c) = -1 + \sqrt{\nu\alpha}\left(\frac{t}{x} - 1\right)^{-\frac{1}{2}}. \tag{6.5.18}$$

This result was recognized in the theory of chromatography as early as 1945.

We next consider the saturation of a clean column with the conditions

$$\left.\begin{aligned} c(x,t) &= 0 \quad \text{at } t = 0 \text{ for } x > 0, \\ c(x,t) &= c_0 \quad \text{at } x = 0 \text{ for } t > 0. \end{aligned}\right\} \tag{6.5.19ab}$$

Obviously, the characteristics originating from the x-axis $(t = 0)$ have slopes $g'(0) = (1+\nu\alpha)$, and they are steeper than those originating from the t-axis $(x = 0)$, since the slopes of the latter have slopes $g'(c_0) = 1 + (\nu\alpha)/(1+\kappa c_0)^2$. The characteristics are shown in Figure 6.11. It is important to point out that there is a characteristic corresponding to every concentration between 0 and c_0 generating from the origin. Three characteristics *PA*, *PO*, and *PB* corresponding to concentrations 0, $\frac{1}{\kappa}[-1 + \{\nu\alpha/(\frac{t}{x} - 1)\}^{\frac{1}{2}}]$, and c_0, respectively, intersect at P. Hence, there are three values of c at P showing the multi-valuedness of the concentration function c. This physically unacceptable situation arises from the fact that the higher concentrations move faster than the lower ones. The *concentration wave propagating with a sloping front leads to a compression wave* which gets progressively steeper as the higher concentration overtakes the lower, until the wave overtops itself as shown in Figure 6.12. Ultimately, the wave form breaks to give a multi-valued solution for $c(x,t)$. In other words, the solution develops a discontinuity. In contrast to linear waves, the breaking phenomenon is typical of nonlinear waves.

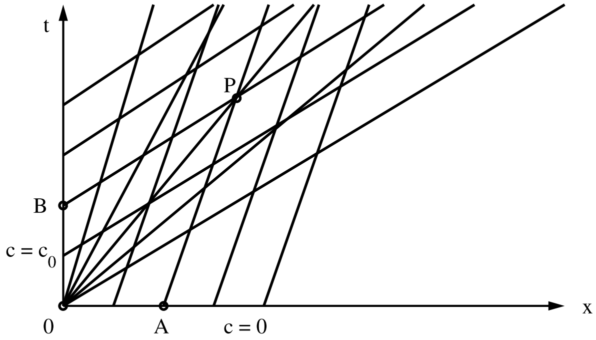

Fig. 6.11 Overlapping characteristic lines.

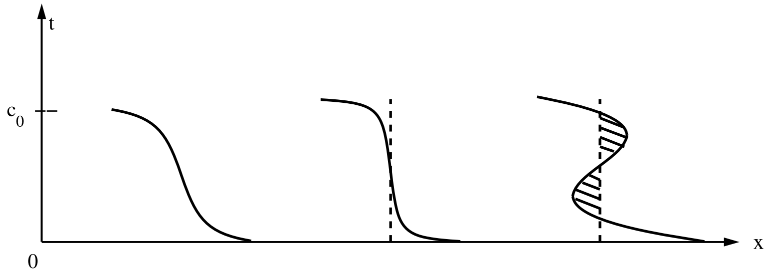

Fig. 6.12 Distortion of the waveform.

Another example of the application of kinematic waves was given by Kynch (1952). The process of sedimentation of solid particles in a fluid is one of great practical importance. It is based on an assumption that the velocity u of a particle is a function of only the local concentration ρ of particles in its immediate neighborhood. It is also assumed that the particles have the same size and shape. The particle flux $S = \rho u$ represents the number of particles crossing a horizontal section per unit area per unit time. Therefore, the particle flux S at any level determines or is determined by the particle concentration. As ρ increases from zero to its maximum value ρ_m, the velocity u decreases continuously from a finite value u to zero. The variation of S is very complicated, but a simple model is useful for understanding the key features of the problem. We denote the height of any level above the bottom of the column of dispersed particles by x. Based on the argument that the accumulation of particles between two adjacent layers at x and $x + dx$ is the difference between the inflow and the outflow, the continuity equation is given by

$$\frac{\partial \rho}{\partial t} = \frac{\partial S}{\partial x}. \tag{6.5.20}$$

Or equivalently,

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{6.5.21}$$

where

$$c(\rho) = -S'(\rho) = -\big[u(\rho) + \rho\, u'(\rho)\big]. \tag{6.5.22}$$

Equation (6.5.21) is the kinematic wave equation, and its mathematical analysis would be similar to that of Section 6.2.

Clearly, the characteristics of (6.5.21) are given by

$$\frac{dx}{dt} = c(\rho). \tag{6.5.23}$$

As ρ and, hence, $c(\rho)$ are constant, the characteristics are straight lines.

The basic equation for the sediment transport of solid particles is similar to the kinematic wave equation. So the results of the kinematic wave theory developed in Section 6.2 can be applied to sediment transport phenomena. Without any further discussion of mathematical analysis, we simply mention some key features of this problem. Any discontinuous changes in the particle concentration can occur in two ways: (i) a discontinuity of the first kind in the particle concentration is due to a sudden *finite* change of concentration at a certain level, and (ii) a discontinuity of the second kind is due to a very small change in the particle concentration. If ρ_1 and u_1 represent the concentration and the velocity of particles above the discontinuity and ρ_2 and u_2 denote the same quantities below the discontinuity, the velocity of the shock waves U is obtained from the equation

$$\rho_1(u_1 + U) = \rho_2(u_2 + U), \tag{6.5.24}$$

so that

$$U = \frac{(u_1\rho_1 - u_2\rho_2)}{(\rho_2 - \rho_1)} = \frac{(S_1 - S_2)}{(\rho_2 - \rho_1)}. \tag{6.5.25}$$

In a diagram in the (ρ, S)-plane, velocity U represents the slope of the chord joining the points (ρ_1, S_1) and (ρ_2, S_2). On the other hand, the discontinuity of the second kind gives $\rho_2 - \rho_1 = d\rho$ where $d\rho$ is small so that (6.5.25) becomes $U = -\frac{dS}{d\rho} = c(\rho)$. This means that the wave velocity c is equal to the shock wave velocity between concentrations ρ and $\rho + d\rho$. Evidently, a small change $d\rho$, if maintained, is propagated through a dispersion of concentration ρ with velocity c just as sound waves propagate through air with a definite velocity. Finally, for dispersions where the concentration increases downward toward the bottom, the condition for the formation of a first-order discontinuity can be expressed in the following equivalent ways:

(a) the characteristics in the (x, t)-plane would intersect,
(b) the velocity of propagation $c(\rho)$ increases with concentration, or
(c) the flow-concentration curve represented by $S = S(\rho)$ in the sediment transport phenomenon is concave to the ρ-axis.

If $c(\rho)$ increases with ρ, small concentration changes from denser regions below move faster upward than those in the less dense regions above and, subsequently, overtake them. This means that the concentration gradient increases until a first-order discontinuity is developed. On the other hand, if $c(\rho)$ decreases with ρ, the concentration gradient decreases, and hence, any discontinuity will be disseminated.

6.6 Glacier Flow

A glacier is a huge mass of ice that flows over the land under its own weight. Mathematically, a glacier can be treated, essentially, as a one-dimensional flow system, which continuously either gains new material by snowfall or loses it by melting and evaporation. The following discussion on waves in glaciers is based on papers by Nye (1960, 1963), who developed a modern theoretical analysis of glacier flow.

We consider the model of a glacier (or a large ice sheet) which rests on an inclined plane of slope α and flows down the plane in the x-direction. We also assume that the glacier is of unlimited extent in the direction normal to x, and $\alpha = \alpha(x)$ is a slowly varying function of x. The thickness of the glacier is assumed as $h = h(x,t)$ such that $(h_x)_t \ll 1$. Then, the surface slope $\beta(x,t)$ downhill in the x-direction is given by

$$\beta(x,t) = \alpha - h_x. \tag{6.6.1}$$

We denote the volume of ice by $q(x,t)$ at a given point x, in unit time, and in unit breadth. This quantity $q(x,t)$ is called a *flow*. The conservation of ice volume gives the equation

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a(x,t), \tag{6.6.2}$$

where $a(x,t)$ is the rate of accumulation of ice at the surface, that is, the rate of addition of ice to the upper surface by snowfall and avalanching, measured as thickness of ice per unit time. The negative value of a corresponds to melting or evaporation of ice from the upper or lower surface of the glacier. Assuming that q is a given function of x and h and, then, following Lighthill and Whitham (1955), we multiply equation (6.6.2) by $c = (\frac{\partial q}{\partial h})_x$ to obtain the equation for the glacier flow

$$\frac{\partial q}{\partial t} + c\,\frac{\partial q}{\partial x} = ac. \tag{6.6.3}$$

If $a = 0$, q would be constant in the characteristics, that is, there exist kinematic waves of constant flow q traveling with velocity c. In general, c differs from the average velocity of the ice itself, which is given by $u = \frac{q}{h}$. To examine the instability of the compression region of a glacier flow, we assume that $q = q(x,h,\alpha)$. Multiplying equation (6.6.2) by $c = (\frac{\partial q}{\partial h})_{x,\alpha}$ gives

$$c\left(\frac{\partial q}{\partial x}\right) + \left(\frac{\partial q}{\partial h}\right)_{x,\alpha} \cdot \left(\frac{\partial h}{\partial t}\right) = ac, \tag{6.6.4}$$

or equivalently,

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} - \left(\frac{\partial q}{\partial \alpha}\right)_{x,h} \cdot \left(\frac{\partial \alpha}{\partial t}\right) = ac. \tag{6.6.5}$$

Using (6.6.1), that is, replacing α_t by $-h_{xt}$, we obtain

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} + \kappa\frac{\partial^2 h}{\partial x \partial t} = ac, \tag{6.6.6}$$

where $\kappa = (\frac{\partial q}{\partial \alpha})_{x,h}$ is called the *diffusion coefficient*. Substituting h_t from (6.6.2) gives the equation for $q(x,t)$ in the form

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} - \kappa\left(\frac{\partial^2 q}{\partial x^2} - \frac{\partial a}{\partial x}\right) = ac. \tag{6.6.7}$$

In terms of perturbations q_1 and a_1, we write $q = q_0(x) + q_1(x,t)$ and $a = a_0(x) + a_1(x,t)$ where q_0 and a_0 are the steady-state values. Since $\frac{dq_0}{dx} = a_0$, we obtain the equation

$$\frac{\partial q_1}{\partial t} + c\frac{\partial q_1}{\partial x} - \kappa\left(\frac{\partial^2 q_1}{\partial x^2} - \frac{\partial a_1}{\partial x}\right) = a_1. \tag{6.6.8}$$

This is the equation for the finite perturbation $q_1(x,t)$ produced by a change $a_1(x,t)$ in the rate of accumulation. When $\kappa = 0$, the equation represents the equation for $q_1(x,t)$. In view of the difficulties in collecting data on flow curves for glaciers, we consider the shearing motion of the two-dimensional steady flow down a constant slope. We suppose that $u(y)$ is the velocity of the layer at a height y from the ground, and $\tau(y)$ is the shearing stress. The appropriate stress–strain relation for ice is assumed as

$$\mu\left(\frac{du}{dy}\right) = \tau^n, \tag{6.6.9}$$

where $n \approx 3$ or 4. Further, the appropriate law for ice slipping in its bed is

$$\nu u(0) = \tau^m(0), \tag{6.6.10}$$

where $m = \frac{1}{2}(n+1) \approx 2$ when $n = 3$, and $m \approx \frac{5}{2}$ when $n = 4$. If α is the angle of the slope and ρ is the density of ice, the shearing stress satisfies the equation

$$\frac{d\tau}{dy} = -g\rho \sin\alpha \tag{6.6.11}$$

with the surface condition $\tau(y = h) = 0$. Thus, the solution of equation (6.6.11) is

$$\tau(y) = (h - y)g\rho \sin\alpha. \tag{6.6.12}$$

Using this value of $\tau(y)$, we next solve the equation (6.6.9) with (6.6.10) to obtain

$$u(y) = \frac{1}{\nu}(g\rho \sin\alpha)^m h^m + \frac{1}{\mu} \cdot \frac{(q\rho \sin\alpha)^n}{(n+1)}\left[h^{n+1} - (h-y)^{n+1}\right]. \tag{6.6.13}$$

The steady flow q_s per unit breadth is given by

$$q_s(h) = \int_0^h u(y)\,dy = \frac{1}{\nu}(g\rho \sin\alpha)^m h^{m+1} + \frac{(q\rho \sin\alpha)^n h^{n+2}}{(n+2)}. \tag{6.6.14}$$

It is convenient to take $q_s(h) = A\,h^N$ where A is a constant and N is approximately in the range $3 \le N \le 5$. Thus, it turns out that the wave propagation velocity is

$$c = \frac{dq_s}{dh} = N\left(\frac{q_s}{h}\right) = Nu_s, \tag{6.6.15}$$

where $u_s = \frac{q_s}{h}$ is the average steady flow velocity. This shows that the kinematic waves travel about three to five times faster than the average steady flow speed.

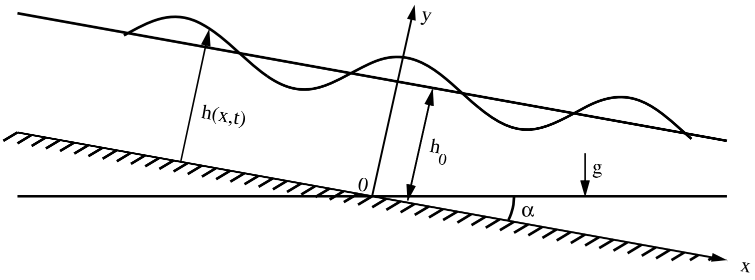

Fig. 6.13 Flow configuration in an inclined channel.

6.7 Roll Waves and Their Stability Analysis

We consider an incompressible viscous fluid flowing down an inclined channel that makes an angle α with the horizontal under the action of gravitational field g. We consider only two-dimensional unsteady flow described by the velocity field (u, v). Figure 6.13 exhibits the primary flow configuration.

The unsteady two-dimensional flow is governed by the Navier–Stokes equations and the continuity equation

$$u_t + uu_x + vu_y = -\frac{1}{\rho}p_x + g\sin\alpha + \nu\nabla^2 u, \tag{6.7.1}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho}p_y - g\cos\alpha + \nu\nabla^2 v, \tag{6.7.2}$$

$$u_x + v_y = 0, \tag{6.7.3}$$

where ρ and ν are the density and the kinematic viscosity of the fluid.

With the free surface elevation $y = h(x,t)$, the kinematic free surface condition is given by

$$h_t + uh_x - v = 0 \quad \text{on } y = h(x,t). \tag{6.7.4}$$

The dynamic free surface conditions described by the tangential stress and the normal stress with pressure on the free surface are given by

$$\sigma_{xy} = \mu(u_y + v_x) = 0 \quad \text{on } y = h(x,t), \tag{6.7.5}$$

$$-\frac{p}{\rho} + 2\,\nu\, v_y + g\,h\cos\alpha = 0 \quad \text{on } y = h(x,t). \tag{6.7.6}$$

The bottom boundary conditions are

$$u = v = 0 \quad \text{on } y = 0. \tag{6.7.7}$$

It can be shown that the primary flow field satisfying (6.7.1)–(6.7.7) exists and is given by

$$u = U(y) = U_0\left(2 - \frac{y}{h}\right)\left(\frac{y}{h}\right), \tag{6.7.8a}$$

$$v = 0, \tag{6.7.8b}$$

$$u = p_0(y) = g\rho(h-y)\cos\alpha, \quad h = h_0, \tag{6.7.9}$$

where

$$U_0 = \frac{1}{2\nu}\big(gh^2 \sin\alpha\big). \tag{6.7.10}$$

Based on the hydraulic approximation (that is, v is small), we integrate the continuity equation (6.7.3) with respect to y from $y=0$ to $y=h(x,t)$ and, then, combine the result with (6.7.4) to derive

$$\int_0^{h(x,t)} u_x\,dx + uh_x + h_t = 0. \tag{6.7.11}$$

Or equivalently,

$$h_t + \frac{\partial}{\partial x}\int_0^{h(x,t)} u(x,y,t)\,dy = 0. \tag{6.7.12}$$

Similarly, integrating (6.7.1) with respect to y from $y=0$ to $y=h$ and using (6.7.3), (6.7.4), and (6.7.7) gives

$$\frac{\partial}{\partial t}\int_0^h u\,dy + \frac{\partial}{\partial x}\int_0^h u^2\,dy = -\frac{1}{\rho}\frac{\partial}{\partial x}\int_0^h p\,dy + gh\sin\alpha - F, \tag{6.7.13}$$

where F represents the frictional force term given by

$$F = \nu\big[u_y(x,0,t) - u_y(x,h,t) + v_x(x,h,t)\big] - \frac{1}{\rho}p(x,h,t)h_x. \tag{6.7.14}$$

In view of the hydraulic approximation, the velocity component v is small, and hence, equation (6.7.2) has the approximate form

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} - g\cos\alpha = 0. \tag{6.7.15}$$

This can be integrated from $y=0$ to $y=h$ to obtain

$$p(x,y,t) = -g\rho(y-h)\cos\alpha, \tag{6.7.16}$$

where $p(x,h,t)=0$. Consequently, (6.7.13) reduces to the form

$$\frac{\partial}{\partial t}\int_0^h u\,dy + \frac{\partial}{\partial x}\int_0^h u^2\,dy = -\frac{1}{2}g\cos\alpha\frac{\partial}{\partial x}\big(h^2\big) + gh\sin\alpha - F. \tag{6.7.17}$$

Using the primary flow field (6.7.9), the flow rate q is given by

$$q = \int_0^h u\,dy = \frac{1}{3\nu}gh^3\sin\alpha. \tag{6.7.18}$$

We define $\overline{U}$ as the average value of u so that $q = \overline{U}\,h$, and therefore,

$$\overline{U} = \frac{1}{3\nu}(g\sin\alpha)h^2. \tag{6.7.19}$$

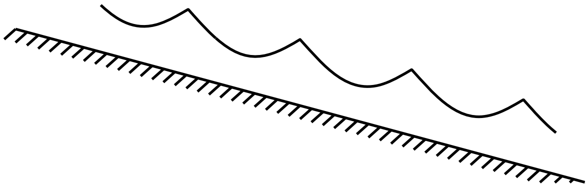

Fig. 6.14 Roll waves in a steep channel.

We next write the integral involved in the second term of the left-hand side of (6.7.17) as

$$\int_0^h u^2\,dy = a\overline{U}h = aq, \tag{6.7.20}$$

where a is a constant parameter. Using the primary flow field (6.7.8a) in (6.7.20) gives $a = \frac{6}{5} \approx 1$. The primary steady flow is simply caused by a balance between F and $gh\sin\alpha$, that is, $F = gh\sin\alpha$ which gives, by (6.7.19),

$$F = 3\nu\overline{U}h^{-1}. \tag{6.7.21}$$

In view of (6.7.18), (6.7.20), and (6.7.21), equations (6.7.12) and (6.7.17) reduce to

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{6.7.22}$$

$$\left(\frac{\partial q}{\partial t} + a\frac{\partial q}{\partial x}\right) + \frac{1}{2}g\cos\alpha\frac{\partial}{\partial x}\left(h^2\right) = gh\sin\alpha - \frac{3\nu\overline{U}}{h}, \tag{6.7.23}$$

where $q = h\overline{U}$. These two equations are known as the *roll wave equations* in hydraulics. Neglecting the left-hand side of (6.7.23), so that $\overline{U} = \frac{1}{3\nu}(gh^2\sin\alpha)$, equation (6.7.22) becomes

$$\frac{\partial h}{\partial t} + \overline{U}(h)\frac{\partial h}{\partial x} = 0. \tag{6.7.24}$$

It follows from this analysis that roll waves can be described by the kinematic wave equation (6.7.24), so that these waves propagate with the velocity $\overline{U}(h)$. A famous example of such waves is shown in Figure 6.14, and a beautiful photograph of this phenomenon (see Figure 6.15) was taken from a book by Cornish (1934). Cornish described his observations of roll waves with numerical data and several interesting photographs of roll waves in a long rectangular open conduit at Merligen which feeds water from a mountain to Lake Thun in the Alps. These photographs clearly show the exact periodic structure of the waves, although in this particular situation, the waves seem to be unusually long and shallow. The photograph in Figure 6.15 confirms the fact that these waves exhibit a periodic structure of a series of discontinuous bores separated by smooth profiles. Such waves frequently occur in sufficiently steep channels, for example, spillways in dams or in open channels, such as that of Figure 6.14. Observational data reported by Cornish also reveals that roll waves travel faster than the mean flow velocity.

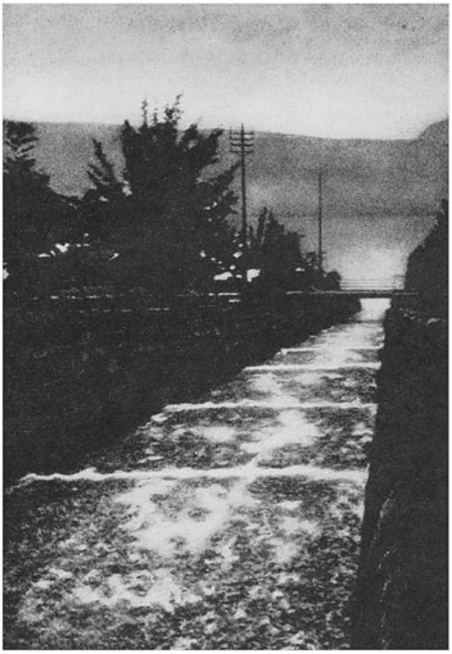

Fig. 6.15 Roll waves, looking downstream (The Grünbach, Switzerland). From Stoker (1957).

The Froude number $F = (U^2/gh)$ associated with this phenomenon always exceeds the critical value 2. Subsequently, Dressler (1949) made an important contribution to the subject of roll waves, and investigated how to construct nonlinear solutions of roll wave equations with appropriate jump conditions to describe the roll wave patterns. He has shown that smooth solutions can be pieced together through discontinuous bores so that the conditions describing the continuity of mass and momentum across the discontinuity are satisfied. For more details, the reader is referred to Dressler's paper (1949).

We close this section by adding the stability analysis of roll waves. We introduce small perturbations to the primary flow field as

$$h = h_0 + \eta \exp\big[ik(x-ct)\big], \tag{6.7.25}$$

$$\overline{U} = \overline{U}_0 + \overline{u} \exp\big[ik(x-ct)\big], \tag{6.7.26}$$

where

$$\overline{U}_0 = \frac{g h_0^2 \sin\alpha}{3\nu} = \frac{2}{3} U_0. \tag{6.7.27}$$

Substituting (6.7.25), (6.7.26) into (6.7.22) and (6.7.23) and retaining only terms linear in η and $\overline{u}$ leads to the following equations:

$$\big(\overline{U}_0 - c\big)\eta + h_0\,\overline{u} = 0, \tag{6.7.28}$$

$$\begin{aligned} &-ikc\big(\overline{U}_0\eta + h_0\overline{u}\big) + ika\big(\overline{U}_0^2\eta + 2\overline{U}_0 h_0 \overline{u}\big) + ikg\,h_0\eta\cos\alpha \\ &\quad = g\eta\sin\alpha - \frac{3\nu}{h_0}\overline{u} + \frac{3\nu\overline{U}_0}{h_0^2}\eta. \end{aligned} \tag{6.7.29}$$

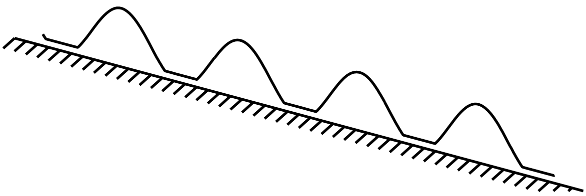

Fig. 6.16 A typical continuous roll wave solution.

We next use (6.7.28) to express $\overline{u}$ in terms of η, so that equation (6.7.29) can be written as

$$ik\left[-c\left\{\overline{U}_0 - \left(\overline{U}_0 - c\right)\right\} + a\left\{\overline{U}_0^2 - 2\overline{U}_0\left(\overline{U}_0 - c\right)\right\} + gh_0 \cos\alpha\right]$$
$$= g\sin\alpha + \frac{3\nu}{h_0^2}\left[\left(\overline{U}_0 - c\right) + \overline{U}_0\right]. \tag{6.7.30}$$

Or equivalently,

$$ik\left[-c^2 + a\left(2c - \overline{U}_0\right)\overline{U}_0 + gh_0\cos\alpha\right] = \left(3\nu h_0^{-2}\right)\left(3\overline{U}_0 - c\right). \tag{6.7.31}$$

The critical state exists for values of c which are given by

$$c = 3\overline{U}_0 = 2U_0 \quad \text{and} \tag{6.7.32a}$$
$$c^2 - a\left(2c - \overline{U}_0\right)\overline{U}_0 - (g\,h_0)\cos\alpha = 0. \tag{6.7.32b}$$

The former gives $\left(\frac{c}{U_0}\right) = 2$, and this result can be used to express (6.7.32b) in terms of the critical Froude number $F_c = \left(U_0^2/gh_0\right)$ as

$$4\left(1 - \frac{5a}{9}\right)F_c = \cos\alpha \tag{6.7.33}$$

for various values of a. In particular, when $a = \frac{27}{25}$, the growth rate of the instability can be obtained as

$$F > F_c = \frac{5}{8}\cos\alpha. \tag{6.7.34}$$

This result can also be derived from the stability analysis of the original equations (6.7.1)–(6.7.3).

In addition to the preceding discontinuous roll wave solutions, we conclude this section by mentioning Dressler's continuous roll wave solutions which describe periodic, progressive waves flowing over a sloping bottom in terms of cnoidal waves, as shown in Figure 6.16. For further mathematical details, the reader is referred to Dressler's (1949) paper.

6.8 Simple Waves and Riemann's Invariants

In one of his famous papers, Riemann (1858) made a remarkable discovery which laid the foundation for all subsequent work on the theory and applications of nonlinear plane sound waves and simple waves. Based on his general formulation, we discuss the mathematical analysis of the theory of simple waves and Riemann invariants.

To develop the governing equations for finite-amplitude wave motions in gas dynamics, we consider the continuity equation and momentum equations of an unsteady compressible fluid without body forces in the form

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) = 0, \tag{6.8.1}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p, \tag{6.8.2}$$

where ρ is the density, $\mathbf{u} = (u, v, w)$ is the fluid velocity vector, p is the pressure, and the total derivative is

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla. \tag{6.8.3}$$

It follows from thermodynamics that density is a function of pressure and entropy, that is, $\rho = \rho(p, s)$, where s is the entropy. Hence,

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp + \left(\frac{\partial \rho}{\partial s}\right)_p ds. \tag{6.8.4}$$

For isentropic flow, s is constant, that is, $ds = 0$. Thus, equation (6.8.4), written in terms of the total derivative following a fluid element, becomes

$$\frac{D\rho}{Dt} = \frac{1}{c^2}\frac{Dp}{Dt}, \tag{6.8.5}$$

where c is the *velocity of sound*, so that

$$c^2 = \frac{dp}{d\rho}. \tag{6.8.6}$$

Substituting (6.8.5) into (6.8.1) gives

$$\frac{1}{c^2}\frac{Dp}{Dt} + \rho\nabla \cdot \mathbf{u} = 0. \tag{6.8.7}$$

For one-dimensional unsteady isentropic flow, equations (6.8.7) and (6.8.2) become

$$\frac{1}{c^2}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) + \rho\frac{\partial u}{\partial x} = 0, \tag{6.8.8}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} = 0. \tag{6.8.9}$$

Adding (6.8.8) and (6.8.9) gives

$$\left[\frac{\partial u}{\partial t} + (u + c)\frac{\partial u}{\partial x}\right] + \frac{1}{\rho c}\left[\frac{\partial p}{\partial t} + (u + c)\frac{\partial p}{\partial x}\right] = 0. \tag{6.8.10}$$

Subtracting (6.8.8) from (6.8.9) yields

$$\left[\frac{\partial u}{\partial t}+(u-c)\frac{\partial u}{\partial x}\right]-\frac{1}{\rho c}\left[\frac{\partial p}{\partial t}+(u-c)\frac{\partial p}{\partial x}\right]=0. \tag{6.8.11}$$

In general and in principle, a solution of these equations gives $u = u(x,t)$ and $p = p(x,t)$. However, introducing $P = \frac{p}{\rho c}$, we rewrite (6.8.10) and (6.8.11) to obtain

$$\frac{\partial}{\partial t}(u+P)+(u+c)\frac{\partial}{\partial x}(u+P)=0, \tag{6.8.12}$$

$$\frac{\partial}{\partial t}(u-P)+(u-c)\frac{\partial}{\partial x}(u-P)=0. \tag{6.8.13}$$

According to the theory of characteristics for first-order, quasi-linear equations, equations (6.8.12), (6.8.13) signify that

$$u+P=\text{const.} \quad \text{along a curve } C_+ \text{ such that } \frac{dx}{dt}=u+c, \tag{6.8.14}$$

$$u-P=\text{const.} \quad \text{along a curve } C_- \text{ such that } \frac{dx}{dt}=u-c. \tag{6.8.15}$$

These are truly remarkable results first obtained by Riemann (1858). Physically, the curves C_+ and C_- in the (x,t)-plane are referred to as positive and negative *wavelets,* respectively. In general, these wavelets do not travel at constant speed, but, everywhere, they move with the local speed of sound relative to the fluid velocity. More precisely, a curve C_+ is the locus of a point (x,t) which always moves forward (that is, in the *positive* x-direction) at the local wave speed c relative to the local fluid velocity u. Similarly, a curve C_- is the locus of a point in the (x,t)-plane which always moves backward, that is, in the *negative* x-direction.

To gain further physical understanding of the Riemann theory, we examine an *initial-value problem*. If, at time $t = 0$, all the fluid except that in a certain slab of the x-axis is undisturbed, that is, $u = 0$ and $P = 0$ at $t = 0$, while the disturbance within the slab may be large but, of course, with constant entropy. The main question is how a disturbance propagates ahead of and behind the slab in the subsequent motion. We draw Figure 6.17 in the (x,t)-plane. At $t = 0$, the disturbance is confined to the slab between the points B (for back) and F (for front). To investigate the subsequent development of the flow, we consider a large number of C_+ curves shown on the figure including those designated as C_+^B and C_+^F, which originate at $t = 0$ from the points B and F, and, similarly, with the C_- curves. The slope of these curves in not known in advance because it depends on how u and c change. However, some important information about them is described below. First, equation (6.8.15) indicates that $u+P$ takes a constant value along each C_- curve, but different for different members of the C_- family. However, C_-^F and all curves ahead of it originate from the region ahead of F, where $u = P = 0$. The constant value of $u - P$ on each of these can only be zero. Thus, we conclude that $u = P$ ahead of C_-^F. Throughout this region ahead of C_-^F, which increases progressively wider than the region ahead of the slab,

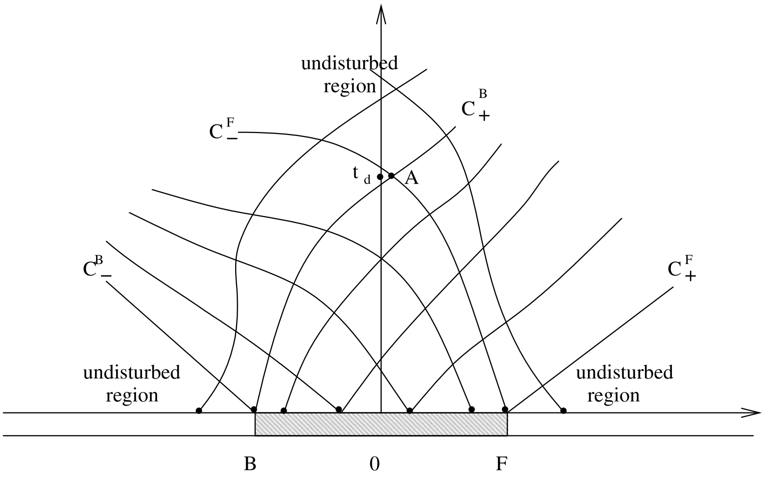

Fig. 6.17 Riemann's initial-value problem.

we examine how disturbances propagate. Similar arguments reveal that $u = P$ behind C_-^B. Analogous arguments for the C_+ curves indicate that $u = -P$ behind C_+^B and also ahead of C_+^F. It follows from the above analysis that C_+^F must lie ahead of C_-^F, and the region ahead of C_+^F remains undisturbed with $u = P = 0$ because there are no disturbances in this region. Similar comments apply to the region behind C_-^B, that is, this region is also undisturbed with $u = P = 0$. Figure 6.17 also indicates that, after some time $t = t_d$, C_-^F intersects C_+^B at a point A and the region $C_-^F A C_+^B$ remains undisturbed with $u = P = 0$. Evidently, during the time interval $0 < t < t_d$, the disturbances become disentangled, and thereafter, they propagate as two *simple waves*—one forward and one backward with an undisturbed region in between. So, the upshot of the Riemann analysis is that the disturbance initially confined to *BF* becomes disentangled when $t = t_d$ into a pair of simple waves propagating in opposite directions (one traveling to the right and the other traveling to the left), separated by an undisturbed region.

To give a more general formulation of the propagation of simple waves, we rewrite the equations (6.8.8), (6.8.9) for one-dimensional, unsteady, isentropic flow in terms of density ρ as

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \tag{6.8.16}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\left(\frac{\partial \rho}{\partial x}\right) = 0. \tag{6.8.17}$$

In matrix form, this system of equations is

$$A\frac{\partial U}{\partial x} + I\frac{\partial U}{\partial t} = 0, \tag{6.8.18}$$

where U, A, and I are matrices given by

$$U = \begin{pmatrix} \rho \\ u \end{pmatrix}, \qquad A = \begin{pmatrix} u & \rho \\ c^2/\rho & u \end{pmatrix}, \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.8.19}$$

It is of interest to determine how a solution evolves with time t. Hence, we leave the time variable unchanged and replace the space variable x by some arbitrary curvilinear coordinate ξ so that the semicurvilinear coordinate transformation from (x,t) to (ξ,τ) can be introduced by

$$\xi = \xi(x,t), \qquad \tau = t. \tag{6.8.20}$$

If the Jacobian of this transformation is nonzero, we can transform (6.8.19) by the following correspondence rule:

$$\begin{aligned} \frac{\partial}{\partial t} &\equiv \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t}\cdot\frac{\partial}{\partial \tau} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau}, \\ \frac{\partial}{\partial x} &\equiv \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial x}\cdot\frac{\partial}{\partial \tau} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}. \end{aligned} \tag{6.8.21ab}$$

This rule transforms equation (6.8.18) into the form

$$I\frac{\partial U}{\partial \tau} + \left(\frac{\partial \xi}{\partial t}I + \frac{\partial \xi}{\partial x}A\right)\frac{\partial U}{\partial \xi} = 0. \tag{6.8.22}$$

This equation can be used to determine $\partial U/\partial \xi$ provided that the determinant of its coefficient matrix is nonzero. Obviously, this condition depends on the nature of the curvilinear coordinate curves $\xi(x,t) = \text{const.}$, which has been kept arbitrary. We assume now that the determinant vanishes for the particular choice $\xi = \eta$, so that

$$\left|\frac{\partial \eta}{\partial t}I + \frac{\partial \eta}{\partial x}A\right| = 0. \tag{6.8.23}$$

In view of this, $\partial U/\partial \eta$ will become indeterminate on the family of curves $\eta = \text{const.}$, and hence, $\partial U/\partial \eta$ may be discontinuous across any of the curves $\eta = \text{const.}$ This implies that each element of $\partial U/\partial \eta$ will be discontinuous across any of the curves $\eta = \text{const.}$ Then, it is necessary to find how these discontinuities in the elements of $\partial U/\partial \eta$ are related across the curve $\eta = \text{const.}$ We next consider the solutions U which are everywhere continuous with discontinuous derivatives $\partial U/\partial \eta$ across the particular curve $\eta = \text{const.} = \eta_0$. Since U is continuous, elements of the matrix A are not discontinuous across $\eta = \eta_0$, so that A can be determined in the neighborhood of a point P on $\eta = \eta_0$. And since $\partial/\partial \tau$ represents differentiation along the curves $\eta = \text{const.}$, $\partial U/\partial \tau$ is continuous everywhere, and hence, it is continuous across the curve $\eta = \eta_0$ at P.

In view of these facts, it follows that differential equation (6.8.22) across the curve $\xi = \eta = \eta_0$ at P gives

$$\left(\frac{\partial \eta}{\partial t}I + \frac{\partial \eta}{\partial x}A\right)_P \left[\frac{\partial U}{\partial \eta}\right]_p = 0, \tag{6.8.24}$$

where $[f]_p = f(P+) - f(P-)$ denotes the discontinuous jump in the quantity f across the curve $\eta = \eta_0$, and $f(P-)$ and $f(P+)$ represent the values to the immediate left and immediate right of the curve at P. Since P is any arbitrary point on the curve, $\partial/\partial\eta$ denotes the differentiation normal to the curves $\eta =$ const., so that equation (6.8.24) can be regarded as the compatibility condition satisfied by $\partial U/\partial\eta$ on either side of and normal to these curves in the (x, t)-plane.

Obviously, (6.8.24) is a homogeneous system of equations for the two jump quantities $[\partial U/\partial\eta]$. Therefore, for the existence of a nontrivial solution, the coefficient determinant must vanish, that is,

$$\left|\frac{\partial\eta}{\partial t}I + \frac{\partial\eta}{\partial x}A\right| = 0. \tag{6.8.25}$$

However, along the curves $\eta =$ const., we have

$$0 = d\eta = \eta_t + \left(\frac{dx}{dt}\right)\eta_x, \tag{6.8.26}$$

so that these curves have the slope

$$\frac{dx}{dt} = -\frac{\eta_t}{\eta_x} = \lambda \quad (\text{say}). \tag{6.8.27}$$

Consequently, equations (6.8.24) and (6.8.25) can be expressed in terms of λ in the form

$$(A - \lambda I)\left[\frac{\partial U}{\partial\eta}\right] = 0, \tag{6.8.28}$$

$$|A - \lambda I| = 0, \tag{6.8.29}$$

where λ represents the eigenvalues of the matrix A and $[\partial U/\partial\eta]$ are proportional to the corresponding right eigenvectors of A.

Since A is a 2×2 matrix, it must have two eigenvalues. If these are real and distinct, integration of (6.8.27) leads to two distinct families of real curves Γ_1 and Γ_2 in the (x, t)-plane:

$$\Gamma_1 : \frac{dx}{dt} = \lambda_r, \qquad r = 1, 2. \tag{6.8.30ab}$$

The families of curves Γ_r are called the *characteristic curves* of the system (6.8.18). Any one of these families of curves Γ_r may be chosen for the curvilinear coordinate curves $\eta =$ const. The eigenvalues λ_r have the dimensions of velocity, and the λ_r associated with each family will then be the velocity of propagation of the matrix column vector $[\partial U/\partial\eta]$ along the curves Γ_r belonging to that family.

In this particular case, the eigenvalues λ of the matrix A are determined by (6.8.29), that is, by the equation

$$\begin{vmatrix} u - \lambda & \rho \\ c^2/\rho & u - \lambda \end{vmatrix} = 0, \tag{6.8.31}$$

so that

$$\lambda = \lambda_r = u \pm c, \quad r = 1, 2. \tag{6.8.32ab}$$

Consequently, the families of the characteristic curves $\Gamma_r (r = 1, 2)$ defined by (6.8.30ab) become

$$\Gamma_1 : \frac{dx}{dt} = u + c, \quad \text{and} \quad \Gamma_2 : \frac{dx}{dt} = u - c. \tag{6.8.33ab}$$

In physical terms, these results indicate that disturbances propagate with the sum of the velocities of the fluid and sound along the family of curves Γ_1. In the second family Γ_2, they propagate with the difference of the fluid velocity u and the sound velocity c.

The right eigenvectors $R_r \equiv \binom{R_r^{(1)}}{R_r^{(2)}}$ are solutions of the equations

$$(A - \lambda_r I) R_r = 0, \quad r = 1, 2, \tag{6.8.34}$$

or equivalently,

$$\begin{pmatrix} u - \lambda_r & \rho \\ c^2/\rho & u - \lambda_r \end{pmatrix} \begin{pmatrix} R_r^{(1)} \\ R_r^{(2)} \end{pmatrix} = 0, \quad r = 1, 2. \tag{6.8.35ab}$$

This result combined with (6.8.29) gives

$$\begin{pmatrix} [\rho_\eta] \\ [R_\eta] \end{pmatrix} = \begin{pmatrix} R_r^{(1)} \\ R_r^{(2)} \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ \pm c/\rho \end{pmatrix}, \quad r = 1, 2, \tag{6.8.36}$$

where α is a constant. In other words, across a wavefront in the Γ_1 family of characteristic curves,

$$\frac{[\partial \rho / \partial \eta]}{1} = \frac{[\partial u / \partial \eta]}{c/\rho}, \tag{6.8.37}$$

and across a wavefront in the Γ_2 family of characteristic curves,

$$\frac{[\partial \rho / \partial \eta]}{1} = \frac{[\partial u / \partial \eta]}{-c/\rho}, \tag{6.8.38}$$

where c and ρ have values appropriate to the wavefront.

To introduce the Riemann invariants explicitly, we form the linear combination of the eigenvectors $(\pm c/\rho, 1)$ with equations (6.8.16), (6.8.17) to obtain

$$\pm \frac{c}{\rho} (\rho_t + \rho u_x + u \rho_x) + \left(u_t + u u_x + \frac{c^2}{\rho} \rho_x \right) = 0. \tag{6.8.39}$$

We use $\partial u / \partial \rho = \pm c/\rho$ from (6.8.37), (6.8.38) and rewrite (6.8.39) as

$$\pm \frac{c}{\rho} \left[\rho_t + (u \pm c) \rho_x \right] + \left[u_t + (u \pm c) u_x \right] = 0. \tag{6.8.40}$$

In view of (6.8.33ab), equation (6.8.40) becomes

$$du \pm \frac{c}{\rho} d\rho = 0 \quad \text{on } \Gamma_r,\ r = 1, 2, \tag{6.8.41}$$

or equivalently,

$$d\big[F(\rho) \pm u\big] = 0 \quad \text{on } \Gamma_r, \tag{6.8.42}$$

where

$$F(\rho) = \int_{\rho_0}^{\rho} \frac{c(\rho)}{\rho}\, d\rho. \tag{6.8.43}$$

Integration of (6.8.42) gives

$$F(\rho) + u = 2r \quad \text{on } \Gamma_1 \quad \text{and} \quad F(\rho) - u = 2s \quad \text{on } \Gamma_2, \tag{6.8.44ab}$$

where $2r$ and $2s$ are constants of integration on Γ_1 and Γ_2, respectively. The quantities r and s are called the *Riemann invariants*. As stated above, r is an arbitrary constant on characteristics Γ_1, and hence, in general, r will vary on each Γ_2. Similarly, s is constant on each Γ_2 but will vary on Γ_1. It is natural to introduce r and s as new curvilinear coordinates. Since r is constant on Γ_1, s can be treated as the parameter on Γ_1. Similarly, r can be regarded as the parameter on Γ_2. Then, $dx = (u \pm c)\, dt$ on Γ_r implies that

$$\frac{dx}{ds} = (u + c)\frac{dt}{ds} \quad \text{on } \Gamma_1, \tag{6.8.45}$$

$$\frac{dx}{dr} = (u - c)\frac{dt}{dr} \quad \text{on } \Gamma_2. \tag{6.8.46}$$

In fact, r is constant on Γ_1, and s is constant on Γ_2. Therefore, the derivatives in equations (6.8.45), (6.8.46) are partial derivatives, with respect to s and r, so that we can rewrite them as

$$\frac{\partial x}{\partial r} = (u + c)\frac{\partial t}{\partial s}, \tag{6.8.47}$$

$$\frac{\partial x}{\partial r} = (u - c)\frac{\partial t}{\partial r}. \tag{6.8.48}$$

These two first-order partial differential equations, in general, can be solved for $x = x(r, s)$ and $t = t(r, s)$, and then, by inversion, r and s can be obtained as functions of x and t. Once this is done, we use (6.8.44ab) to determine $u(x, t)$ and $\rho(x, t)$ in terms of r and s as

$$u(x, t) = r - s, \tag{6.8.49a}$$

$$F(\rho) = r + s. \tag{6.8.49b}$$

When one of the Riemann invariants r and s is constant throughout the flow, the corresponding solution is tremendously simplified. The solutions are known as *simple wave motions* representing simple waves in one direction only. The generating mechanisms of simple waves with their propagation laws can be illustrated by the *piston problem* in gas dynamics (see Example 6.8.2).

We close this section by reducing the two first-order equations (6.8.47) and (6.8.48) to a single second-order linear hyperbolic equation. This can be done by equating x_{rs} and x_{sr} which eliminates x from the system (6.8.47), (6.8.48), so that we obtain the single equation

$$(u+c)t_{rs} + (u_r + c_r)t_s = (u-c)t_{rs} + (u_s - c_s)t_r. \tag{6.8.50}$$

The coefficients of this equation are functions of r and s, and they can be determined explicitly. First, we find that $u_r = 1$ and $u_s = -1$ from (6.8.49a). Second, from (6.8.49b), we calculate $F'(\rho)\rho_r = F'(\rho)\rho_s = 1$, and hence, from result (6.8.43), $F'(\rho) = \frac{c}{\rho}$ so that $\rho_r = \rho_s = \frac{\rho}{c}$. Finally, we determine that

$$c_r = c'(\rho)\rho_r = \frac{\rho}{c}\cdot c'(\rho), \qquad c_s = c'(\rho)\rho_s = \frac{\rho}{c}\cdot c'(\rho). \tag{6.8.51ab}$$

These results are used to simplify (6.8.50) further as

$$2ct_{rs} + \left(1 + \frac{\rho}{c}\frac{dc}{d\rho}\right)t_s = -\left(1 + \frac{\rho}{c}\frac{dc}{d\rho}\right)t_r, \tag{6.8.52}$$

or equivalently,

$$t_{rs} + \frac{1}{2c}\left(1 + \frac{\rho}{c}\frac{dc}{d\rho}\right)(t_r + t_s) = 0. \tag{6.8.53}$$

The coefficient of $(t_r + t_s)$ depends only on ρ, and hence, by (6.8.49b), it is a function of $(r+s)$, so that we can write it as $f(r+s)$. More explicitly, we write

$$\frac{1}{2c}\left(1 + \frac{\rho}{c}\frac{dc}{d\rho}\right) = f(r+s). \tag{6.8.54}$$

Consequently, equation (6.8.53) reduces to the second-order hyperbolic equation

$$t_{rs} + f(r+s)(t_r + t_s) = 0. \tag{6.8.55}$$

Once we can solve this equation for $t = t(r,s)$, it can be used to solve the system (6.8.47), (6.8.48) for $x = x(r,s)$.

Example 6.8.1 Riemann's Invariants for a Polytropic Gas. A polytropic gas is usually characterized by the pressure–density law $p = k\rho^\gamma$ where k and γ are constants. In this case, the velocity of sound c is calculated by using the formula $c^2 = k\gamma\rho^{\gamma-1}$. It follows from this result that

$$\rho_t = \left(\frac{2\rho}{\gamma-1}\right)\frac{c_t}{c} \quad \text{and} \quad \left(\frac{2}{r-1}\right)cc_x = \frac{c^2}{\rho}\rho_x. \tag{6.8.56}$$

In view of these results, equations of motion (6.8.16), (6.8.17) assume the form

$$c_t + uc_x + \left(\frac{\gamma - 1}{2}\right) cu_x = 0, \tag{6.8.57}$$

$$u_t + uu_x + \left(\frac{2}{\gamma - 1}\right) cc_x = 0. \tag{6.8.58}$$

We divide (6.8.57) by $(\gamma - 1)$ and (6.8.58) by 2, and then add them and subtract the former from the latter to obtain the following equations (Riemann 1858):

$$\frac{\partial r}{\partial x} + (c + u)\frac{\partial r}{\partial x} = 0, \qquad \frac{\partial s}{\partial t} - (c - u)\frac{\partial s}{\partial x} = 0, \tag{6.8.59ab}$$

where r and s are the *Riemann invariants* given by

$$r = \frac{c}{\gamma - 1} + \frac{1}{2}u, \tag{6.8.60a}$$

$$s = \frac{c}{\gamma - 1} - \frac{1}{2}u. \tag{6.8.60b}$$

Equations (6.8.60a), (6.8.60b) signify that

$$r = \frac{c}{\gamma - 1} + \frac{1}{2}u = \text{const.} \quad \text{on } \frac{dx}{dt} = c + u, \tag{6.8.61}$$

$$s = \frac{c}{\gamma - 1} - \frac{1}{2}u = \text{const.} \quad \text{on } \frac{dx}{dt} = c - u. \tag{6.8.62}$$

Furthermore, $u = r - s = \text{const.}$, and $c = \frac{1}{2}(\gamma - 1)(r + s) = \text{const.}$ confirming the fact that u and c are separately constant. Consequently, the characteristics $\frac{dx}{dt} = c \pm u = \text{const.}$, which represents two families of straight lines in the (x, t)-plane. Mathematically, the characteristics exist because the governing quasi-linear equations (6.8.57) and (6.8.58) are hyperbolic. Physically, they represent the *wavelets* propagating in both directions with speed c while also connected by the fluid with local speed u. If, as often happens, one of the Riemann invariants is constant everywhere (that is, it takes the *same* constant value on every characteristic line of its family), the resulting solution is said to form a *simple wave*.

This example simply illustrates Riemann's original general approach to the problem. However, it should be pointed out that the Riemann invariant can be derived directly from (6.8.44ab) by using (6.8.43) combined with the sound speed $c = \sqrt{\frac{dp}{d\rho}} = \sqrt{k\gamma}\rho^{\frac{1}{2}(\gamma-1)}$.

The solution of the equations of motion in the above simple wave assumes the form

$$u(x, t) = f(\xi), \quad \xi = x - (u + c)t, \tag{6.8.63}$$

where $u(x, 0) = f(x)$ represents the initial data. If $f(x) \geq 0$, $u + c$ will be everywhere positive, and hence, the family of characteristics carrying the simple wave has positive slope in the (x, t)-plane, and the wave is said to be *forward progressing*. The speed of propagation of the forward progressing wave is given by

$$u + c = \frac{1}{2}(\gamma + 1)u + c_0, \tag{6.8.64}$$

where a slightly different form of (6.8.61), that is, $c = \frac{1}{2}(\gamma + 1)u + \text{const.}$, is used, and c_0 is the sound speed of a one-dimensional compressive wave which cannot remain continuous indefinitely, whereas a similar expansion wave can. Differentiating (6.8.63) combined with (6.8.64) gives

$$\frac{\partial u}{\partial x} = f'(\xi)\xi_x = \frac{f'(\xi)}{1 + \frac{1}{2}(\gamma + 1)t\, f'(\xi)}, \tag{6.8.65}$$

$$\frac{\partial u}{\partial t} = f'(\xi)\xi_t = -\frac{1}{2}(\gamma + 1)\left[\frac{f'(\xi)f(\xi)}{1 + \frac{1}{2}(\gamma + 1)t\, f'(\xi)}\right]. \tag{6.8.66}$$

Evidently, both u_x and u_t tend to infinity as

$$t \to -\frac{2}{(\gamma + 1)f'(\xi)}, \tag{6.8.67}$$

which is positive provided that $f'(\xi) < 0$, corresponding to the compressive part of the wave profile. The discontinuity occurs first at the instant

$$t = \frac{2}{(\gamma + 1)\{-f'(\xi)\}_{\max}}. \tag{6.8.68}$$

The solution develops a discontinuity in the form of *shock waves*. The gas is at rest. From the constant value in this simple wave of the Riemann invariant, it is evident that the greater the speed u, the greater the value of c and, therefore, of the pressure p. Hence the largest velocities and pressures move most rapidly through the gas. The peaks of velocity or pressure tend to overtake the troughs, thus, leading to the breakdown of the uniqueness of the solution whenever the initial profile contains a compressive part. This has already been illustrated in Chapter 5 (see Figure 5.4) for a similar situation observed in the case of a general, nonlinear initial-value problem. As time increases, the initial velocity (or pressure) profile is progressively distorted, and the part of the wave profile that was initially expansive (velocity increases in the direction of motion) tends to elongate. Ultimately, the compressive part of the wave profile develops multiple-valuedness and then breaks at the point when it becomes vertical. This simple analysis confirms that, in inviscid flow, a shock wave develops in one-dimensional unsteady flow as exhibited in the photographs included in Figure 6.18, which is due to Dr. W. Bleakney and Dr. Wayland Griffith at the Palmer Physical Laboratory, Princeton University. This photograph shows that in (a) the density gradient is moderate, in (b) it becomes steeper, and in (c) it is clearly discontinuous. The steepening effect of the waveform for the case of one-dimensional flow is equally true for flows with cylindrical and spherical symmetry. This fundamental result was obtained long ago by Burton (1893) by intuitive arguments and has been confirmed by Pack (1960) by the method of characteristics.

We complete this example by adding Riemann's original approach to an explicit solution for the problem of polytropic gas. First, we can calculate the unknown function $f(r + s)$ in equation (6.8.55) to find the explicit form of this equation. First, we use $c = \sqrt{k\gamma}\rho^{\frac{1}{2}(\gamma-1)}$ to obtain $c'(\rho)$ and then (6.8.60a), (6.8.60b) to find $F(\rho)$

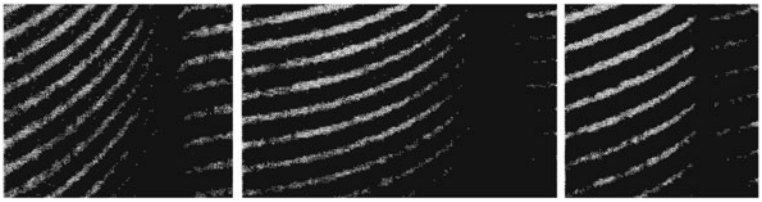

Fig. 6.18 Development of shock waves at successive instants. From Lighthill (1956).

$$c'(\rho) = \frac{1}{2}(\gamma - 1)\frac{c}{\rho} \quad \text{and} \quad F(\rho) = \frac{2c}{(\gamma - 1)}. \tag{6.8.69ab}$$

Thus, it follows from (6.8.54) that

$$f(r+s) = \frac{1}{4c}(\gamma + 1) = \frac{1}{2}\left(\frac{\gamma+1}{\gamma-1}\right)\frac{1}{(r+s)} = \frac{n}{(r+s)}, \tag{6.8.70}$$

where $n = \frac{1}{2}(\frac{\gamma+1}{\gamma-1}) = \text{const}$. Consequently, equation (6.8.55) reduces to the explicit form

$$t_{rs} + \left(\frac{n}{r+s}\right)(t_r + t_s) = 0, \tag{6.8.71}$$

where r and s are involved in this equation symmetrically. This allows us to extend the method to (6.8.71) by introducing a new dependent variable $w(r, s)$ so that

$$t(r, s) = g(r+s)w(r, s), \tag{6.8.72}$$

where g is to be determined so that w_r and w_s vanish. We substitute (6.8.72) into (6.8.71) to derive

$$g\,w_{rs} + \left(g' + \frac{n\,g}{r+s}\right)(w_r + w_s) + g'\left(1 + \frac{2n}{r+s}\right)w = 0. \tag{6.8.73}$$

To simplify the problem further, we assume that $g(r+s) = (r+s)^{-n}$, so that (6.8.73) reduces to the canonical hyperbolic equation

$$w_{rs} + \frac{n(1-n)}{(r+s)^2}w = 0. \tag{6.8.74}$$

Indeed, this is a *telegraph* equation with a variable coefficient. Following Riemann analysis (1858), we introduce the *Riemann function* $R(r, s, \rho, \sigma)$, which must be a function of (r, s), and look for the Riemann function as a function of a single variable

$$z = \frac{(r-\rho)(s-\sigma)}{(r+s)(\rho+\sigma)}. \tag{6.8.75}$$

More explicitly, we set $R(r, s, \rho, \sigma) = g(z)$ so that R satisfies (6.8.74), and hence, $g(z)$ can be determined. We also assume that R satisfies some additional conditions

$$\frac{\partial R}{\partial s} = 0 \quad \text{for } r = \rho, \qquad \frac{\partial R}{\partial r} = 0 \quad \text{for } s = \sigma, \quad \text{and} \quad R(\rho, \sigma, \rho, \sigma) = 1. \tag{6.8.76}$$

We next substitute $R(r, s, \rho, \sigma) = g(z)$ in (6.8.74) to generate the ordinary differential equation for $g(z)$ in the form

$$z(1-z)g''(z) + (1-2z)g'(z) - n(1-n)g(z) = 0. \tag{6.8.77}$$

This is the *hypergeometric equation*

$$z(1-z)g'' + \big[c - (1+a+b)z\big]g' - abg = 0 \tag{6.8.78}$$

with $a = 1-n$, $b = n$, and $c = 1$.

The first two conditions in (6.8.76) are automatically satisfied by $R(r, s, \rho, \sigma) = g(z)$, and the last condition in (6.8.76) needs $g(0) = 1$. The solution of (6.8.78) is the standard hypergeometric function

$$R(r, s, \rho, \sigma) = F(1-n, n, 1, -z), \tag{6.8.79}$$

where $F(a, b, c, z)$ is the hypergeometric function defined by

$$F(a, b, c, z) = 1 + \left(\frac{ab}{c}\right)z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots. \tag{6.8.80}$$

For the special case in which n is an integer, a transformation

$$\tau = (r+s)^{-1}(t_r + t_s) \tag{6.8.81}$$

can be used to solve the original equation (6.8.71) which becomes

$$\tau_{rs} + \left(\frac{n+1}{r+s}\right)(\tau_r + \tau_s) = 0. \tag{6.8.82}$$

This shows that, if a solution of (6.8.71) exists for any n, then the use of the transformation (6.8.81) leads us to a solution for $n+1$. In particular, when $n = 0$, equation (6.8.71) reduces to the canonical hyperbolic equation

$$t_{rs} = 0, \tag{6.8.83}$$

which gives a solution by direct integration as

$$t(r, s) = \phi(r) + \psi(s), \tag{6.8.84}$$

where ϕ and ψ are arbitrary functions. Using (6.8.81), the general solution of (6.8.82) with $n = 0$, that is, of (6.8.71) with $n = 1$, is given by

$$\tau(r, s) = \frac{\phi'(r) + \psi'(s)}{(r+s)}. \tag{6.8.85}$$

Similarly, for $n = 2$, we obtain

$$\tau(r,s) = (r+s)^{-1}\{\phi''(r)+\psi''(s)\} - 2(r+s)^{-2}\{\phi'(r)+\psi'(s)\}. \quad (6.8.86)$$

In general, when n is an integer, the solution of (6.8.71) is given by

$$t(r,s) = \frac{\partial^{n-1}}{\partial r^{n-1}}\left[\frac{\phi(r)}{(r+s)^n}\right] + \frac{\partial^{n-1}}{\partial s^{n-1}}\left[\frac{\psi(s)}{(r+s)^n}\right], \quad (6.8.87)$$

where $\phi(r)$ and $\psi(s)$ are arbitrary functions. This result is also valid for fractional values of n and can be expressed in terms of fractional derivatives with $n = m + \alpha$, m a nonnegative integer, and $0 \le \alpha < 1$. Consequently, (6.8.87) takes the form

$$t(r,s) = \frac{\partial^m}{\partial r^m}\cdot {}_0D_r{}^{-(1-\alpha)}\left[\frac{\phi(r)}{(r+s)^{m+\alpha}}\right] + \frac{\partial^m}{\partial s^m}\cdot {}_0D_s{}^{-(1-\alpha)}\left[\frac{\psi(s)}{(r+s)^{m+\alpha}}\right], \quad (6.8.88)$$

where the fractional integrals (Debnath 1995) are defined by

$${}_0D_r^{-(1-\alpha)} f(r) = \frac{1}{\Gamma(1+\alpha)}\int_0^r (r-x)^{-\alpha} f(x)\, dx. \quad (6.8.89)$$

Result (6.8.88) is valid for $n \ge 0$ and for $n < 0$. Then (6.8.88) also holds if m is set equal to zero and $\alpha = n$.

We consider another special case in which n is half-integer. It is convenient to handle this case with u and c instead of r and s as independent variables, where

$$u = r - s \quad \text{and} \quad c = \frac{\gamma-1}{2}(r+s). \quad (6.8.90ab)$$

Using these results, we change the variables from r, s to u, c so that equation (6.8.71) reduces to

$$t_{uu} - \left(\frac{\gamma-1}{2}\right)^2\left(t_{cc} + \frac{2n}{c}t_c\right) = 0. \quad (6.8.91)$$

This is known as the *Euler–Poisson–Darboux* equation. If m is an integer and $2n = m - 1$, then equation (6.8.91) becomes

$$t_{uu} - \left(\frac{\gamma-1}{2}\right)^2\left(t_{cc} + \frac{m-1}{c}t_c\right) = 0. \quad (6.8.92)$$

This can be treated as a wave equation in m space variables for a problem with spherical symmetry. To check this situation, we take $c = \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2}$, and write

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_m^2}, \quad (6.8.93)$$

and then assume that $t = t(x_1, x_2, \ldots, x_m, u) = t(c, u)$. With these assumptions, we obtain

$$\nabla^2 t = t_{cc} + \left(\frac{m-1}{c}\right)t_c. \quad (6.8.94)$$

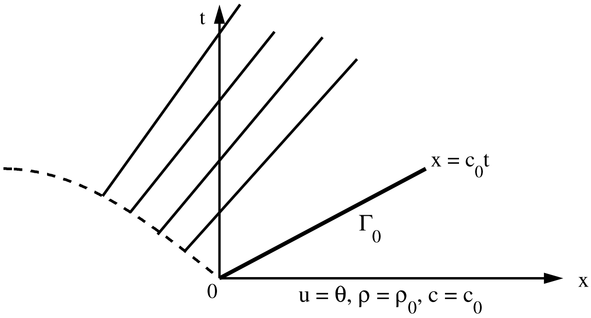

Fig. 6.19 Simple waves generated by the motion of a piston.

Then, equation (6.8.92) reduces to the form

$$t_{uu} - \left(\frac{\gamma-1}{2}\right)^2 \nabla^2 t = 0. \tag{6.8.95}$$

This is a wave equation where the waves travel with the speed $\frac{1}{2}(\gamma - 1)$.

Example 6.8.2 The Piston Problem in a Polytropic Gas. The problem is to determine how a simple wave is produced by the prescribed motion of a piston in the closed end of a semi-infinite tube filled with gas.

This is a one-dimensional unsteady problem in gas dynamics. We assume that the gas is initially at rest with a uniform state $u = 0$, $\rho = \rho_0$, and $c = c_0$. The piston starts from rest at the origin and is allowed to withdraw from the tube with a variable velocity for a time t_1, after which the velocity of withdrawal remains constant. The piston path is shown by a dotted curve in Figure 6.19. In the (x,t)-plane, the path of the piston is given by $x = X(t)$ with $X(0) = 0$. The fluid velocity u is equal to the piston velocity $\dot{X}(t)$ on the piston $x = X(t)$, which will be used as the boundary condition on the piston.

The initial state of the gas is given by $u = u_0$, $\rho = \rho_0$, and $c = c_0$ at $t = 0$ for $x \geq 0$. The characteristic line Γ_0 that bounds it and passes through the origin is determined by the equation

$$\frac{dx}{dt} = (u+c)_{t=0} = c_0,$$

so that the equation of the characteristic line Γ_0 is $x = c_0 t$.

In view of the uniform initial state, all of the Γ_2 characteristics start on the x-axis, so that the Riemann invariants s in (6.8.60b) must be constant and have the form

$$\frac{2c}{\gamma-1} - u = \frac{2c_0}{\gamma-1}, \tag{6.8.96}$$

or equivalently,

$$u = \frac{2(c - c_0)}{\gamma - 1}, \tag{6.8.97a}$$

$$c = c_0 + \frac{\gamma - 1}{2} u. \tag{6.8.97b}$$

The characteristics Γ_1 meeting the piston are given by

$$\frac{2c}{\gamma - 1} + u = 2r \quad \text{on each } \Gamma_1 : \frac{dx}{dt} = u + c, \tag{6.8.98}$$

which is, by (6.8.97a), (6.8.97b) which holds everywhere,

$$u = \text{const.} \quad \text{on } \Gamma_1 : \frac{dx}{dt} = c_0 + \frac{1}{2}(\gamma + 1)u. \tag{6.8.99}$$

Since the flow is continuous with no shocks, $u = 0$ and $c = c_0$ ahead of and on Γ_0, which separates those Γ_1 meeting the x-axis from those meeting the piston. The family of lines Γ_1 through the origin has the equation $(dx/dt) = \xi$, where ξ is a parameter with $\xi = c_0$ on Γ_0. The Γ_1 characteristics are also defined by $(dx/dt) = u + c$, so that $\xi = u + c$. Hence, eliminating c from (6.8.97b) gives

$$u = \left(\frac{2}{\gamma - 1}\right)(\xi - c_0). \tag{6.8.100}$$

Substituting this value of u in (6.8.97b), we obtain

$$c = \left(\frac{\gamma - 1}{\gamma - 1}\right)\xi + \frac{2c_0}{\gamma - 1}. \tag{6.8.101}$$

If follows from $c^2 = \gamma k \rho^{\gamma - 1}$ and (6.8.97b) with the initial data $\rho = \rho_0$ and $c = c_0$ that

$$\rho = \rho_0 \left[1 + \frac{\gamma - 1}{2c_0} u\right]^{2/(\gamma - 1)}. \tag{6.8.102}$$

With $\xi = x/t$, results (6.8.100) through (6.8.102) give the complete solution of the piston problem in terms of x and t.

Finally, the equation of the characteristic line Γ_1 is found by integrating the second equation of (6.8.99) and using the boundary condition on the piston. When a line Γ_1 intersects the piston path at time $t = \tau$, then $u = \dot{X}(\tau)$ along it, and the equation becomes

$$x = X(\tau) + \left\{c_0 + \frac{\gamma + 1}{2} \dot{X}(\tau)\right\}(t - \tau). \tag{6.8.103}$$

Note that the family Γ_1 represents straight lines with slope (dx/dt) increasing with velocity u. Consequently, the characteristics are likely to overlap on the piston, that is, $\dot{X}(\tau) > 0$ for any τ. If u increases, so do c, ρ, and p, so that instability develops. This shows that shocks will be formed in the compressive part of the disturbance.

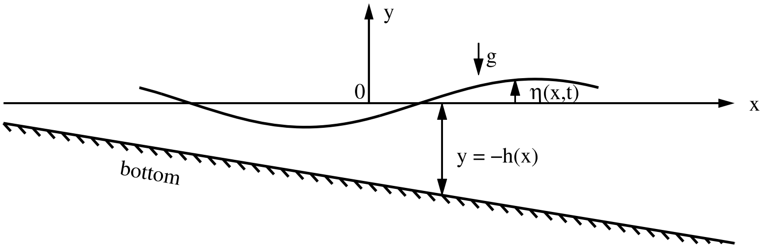

Fig. 6.20 Long surface waves in shallow water.

Example 6.8.3 Shallow Water Waves. We consider the propagation of surface waves whose wavelength is large compared with the depth $h(x)$, as shown in Figure 6.20. In terms of the horizontal velocity component $u = u(x,t)$ and the free surface elevation $\eta = \eta(x,t)$, shallow water equations are given by

$$u_t + uu_x = -g\eta_x, \tag{6.8.104}$$

$$\big[u(\eta + h)\big]_x = -\eta_t, \tag{6.8.105}$$

where g is the acceleration due to gravity. Introducing the propagation speed $c = \sqrt{g(\eta + h)}$, so that $c_x = \frac{g}{2c}(\eta_x + h_x)$ and $c_t = (g\eta_t/2c)$, we reformulate the preceding equations in terms of c to obtain

$$u_t + uu_x + 2cc_x - H_x = 0, \tag{6.8.106}$$

$$2(c_t + u\,c_x) + c\,u_x = 0, \tag{6.8.107}$$

where $H = g\,h(x)$. For the particular case in which $H_x = m = \text{const.}$, the slope of the bottom is constant.

Adding (6.8.106) and (6.8.107) leads to the equation

$$\left[\frac{\partial}{\partial t} + (u + c)\frac{\partial}{\partial x}\right](u + 2c - mt) = 0. \tag{6.8.108}$$

Similarly, subtracting (6.8.107) from (6.8.106) gives

$$\left[\frac{\partial}{\partial t} + (u - c)\frac{\partial}{\partial x}\right](u - 2c - mt) = 0. \tag{6.8.109}$$

This system of equations (6.8.108) and (6.8.109) is very similar to that of the equations for one-dimensional sound waves (6.8.12) and (6.8.13). So, we may follow the same analysis to investigate this problem. Equations (6.8.106), (6.8.107) state that the functions $(u \pm 2c - mt)$ are constant on the two sets of characteristics $C_\pm : \frac{dx}{dt} = u \pm c$, respectively. It may also be observed that the two families of characteristics $C_\pm$ are really distinct because $c = \sqrt{g(\eta + h)} \neq 0$, since we assume that $\eta > -h$, that is, the free surface never coincides with the bottom.

In particular, for a fluid of constant depth h, $H_x = 0$, and the quantity m in (6.8.108) and (6.8.109) is zero. Thus the functions $(u \pm 2c)$ are constants on the two families of characteristics $C_\pm : \frac{dx}{dt} = u \pm c$. Further, one of the two families of characteristics $C_\pm$ consists of straight lines along each of which u and c are constant. Consequently, the corresponding motion can be referred to as a *simple wave*. As in the case of the initial-value problem for sound waves treated at the beginning of this Section 6.8, the shallow water wave problem can easily be handled. The only difference is that, in the present situation, the solution will develop what is called a *bore* or *hydraulic jump*. In other words, the compressive part of the water waves will always break, that is, the development of a bore or hydraulic jump in water is very similar to the development of shock waves in gas dynamics.

Example 6.8.4 Nonlinear Wave Equation. We consider the nonlinear wave equation in the form

$$u_{tt} - c^2(u_x)u_{xx} = 0, \tag{6.8.110}$$

where, as indicated, the velocity c depends on the slope of u. We transform this equation into a system of two equations by introducing $u_t = v$, and $u_x = w$, so that (6.8.110) reduces to the system of coupled equations

$$v_x - w_t = 0, \tag{6.8.111}$$

$$v_t - c^2(w)w_x = 0. \tag{6.8.112}$$

We next rearrange the terms of the second equation to write the system in the matrix form

$$\begin{pmatrix} 1 & 0 \\ 0 & -c^2(w) \end{pmatrix} \begin{pmatrix} v_x \\ w_x \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_t \\ w_t \end{pmatrix} = 0. \tag{6.8.113}$$

According to (6.8.29), the eigenvalue equation for λ is

$$|A - \lambda I| = \begin{vmatrix} 1 & -\lambda \\ \lambda & -c^2 \end{vmatrix} = 0. \tag{6.8.114}$$

So, the eigenvalues λ are

$$\lambda = \pm c, \tag{6.8.115}$$

and the corresponding eigenvectors are $(c \mp 1)$. Consistent with the notation used before, we obtain two families of characteristics as

$$C_+ : \lambda = \frac{dx}{dt} = c; \qquad C_- : \lambda = \frac{dx}{dt} = -c. \tag{6.8.116}$$

We multiply (6.8.111) by c and then add the result with (6.8.112) to obtain

$$(v_t + cv_x) - c(w_t + cw_x) = 0. \tag{6.8.117}$$

Similarly, we multiply (6.8.111) by c and subtract the result from (6.8.112) to get

$$(v_t - cv_x) + c(w_t - cw_x) = 0. \tag{6.8.118}$$

We replace $\pm c$ by $\frac{dx}{dt}$ in (6.8.117) and (6.8.118) to find

$$dv \pm c(w)\,dw = 0 \quad \text{on } C_{\pm}. \tag{6.8.119}$$

We then write $c(w)\,dw = dF(w)$ in the last equation, so that

$$d\big[v \pm F(w)\big] = 0 \quad \text{on } C_{\pm}. \tag{6.8.120}$$

This enables us to introduce Riemann invariants r and s as before by

$$2r = u + F(w), \qquad 2s = v - F(w). \tag{6.8.121ab}$$

We can now continue the same mathematical analysis following equations (6.8.44ab), and so the rest is left to the reader as an exercise.

6.9 The Nonlinear Hyperbolic System and Riemann's Invariants

One of the simplest nonlinear (or quasi-linear) hyperbolic systems of evolution equations describing wave propagation has the form

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0, \tag{6.9.1}$$

where

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{with } a_{ij} = a_{ij}(u_1, u_2).$$

In this system, the elements a_{ij} of the matrix A depend explicitly on u_1 and u_2, but only implicitly on x and t through u_1 and u_2.

Following the discussion of Section 6.8, the real eigenvalues are the solutions of the equation

$$|A - \lambda I| = 0. \tag{6.9.2}$$

The system will be called *strictly hyperbolic* if the eigenvalues λ_1 and λ_2 are real and distinct. We denote the left eigenvectors L_1 and L_2 corresponding to the eigenvalues λ_1 and λ_2, so that

$$L_i A_i = \lambda_i L_i, \quad i = 1, 2. \tag{6.9.3}$$

Multiplication of (6.9.1) by L_i from the left combined with (6.9.3) gives

$$L_i\left(\frac{\partial}{\partial t} + \lambda_i \frac{\partial}{\partial x}\right)U = 0, \quad i = 1, 2. \tag{6.9.4}$$

This shows that the operators $(\frac{\partial}{\partial t} + \lambda_i \frac{\partial}{\partial x})$ represent differentiation along the respective families of characteristic curves $C_{\pm}$ defined by

$$C_{\pm} : \frac{dx}{dt} = \lambda_i. \tag{6.9.5ab}$$

We represent the differentiation along C_+ by $\frac{d}{d\xi}$ and along C_- by $\frac{d}{d\eta}$ to transform (6.9.4) into the form

$$L_1 \frac{dU}{d\xi} = 0 \quad \text{along } C_+, \tag{6.9.6}$$

$$L_2 \frac{dU}{d\eta} = 0 \quad \text{along } C_-. \tag{6.9.7}$$

We write $L_1 = (L_{11}, L_{12})$ and $L_2 = (L_{21}, L_{22})$ to reduce (6.9.6) and (6.9.7) to the form

$$L_{11}\frac{du_1}{d\xi} + L_{12}\frac{du_2}{d\xi} = 0 \quad \text{along } C_+, \tag{6.9.8}$$

$$L_{21}\frac{du_1}{d\eta} + L_{22}\frac{du_2}{d\eta} = 0 \quad \text{along } C_-. \tag{6.9.9}$$

We note that the eigenvectors L_1 and L_2 depend on u_1 and u_2 through the matrix elements $a_{ij} = a_{ij}(u_1, u_2)$ of A. If equations (6.9.8) and (6.9.9) are *not* exact, they can always be made exact equations by multiplying by the integrating factor μ. We next multiply by μ and integrate them to obtain

$$\int \mu L_{11}\, du_1 + \int \mu L_{12}\, du_2 = r(\eta) \quad \text{along } C_+, \tag{6.9.10}$$

$$\int \mu L_{21}\, du_1 + \int \mu L_{22}\, du_2 = s(\xi) \quad \text{along } C_-, \tag{6.9.11}$$

where r and s are arbitrary functions. These functions $r(\eta)$ and $s(\xi)$ are called *Riemann invariants* of the hyperbolic system (6.9.1). However, it is not necessarily the same constant on each C_+. In other words, r, in general, will vary on C_-. Similarly, s is constant on each C_-, but will usually vary on each C_+. In fact, the two families of characteristics are given by the equations

$$C_+ : \frac{dx}{dt} = \lambda_1; \qquad C_- : \frac{dx}{dt} = \lambda_2. \tag{6.9.12ab}$$

Furthermore, $r(\eta) =$ const. and $s(\xi) =$ const. are shown in Figure 6.21 in which the initial data for u_1 and u_2 are given as follows:

$$u_1(x_0, 0) = \tilde{u}_1(x_0), \qquad u_2(x_0, 0) = \tilde{u}_2(x_0), \tag{6.9.13}$$

$$u_1(x_1, 0) = \tilde{u}_1(x_1), \qquad u_2(x_1, 0) = \tilde{u}_2(x_1). \tag{6.9.14}$$

It can be shown that, in general, a solution can be found everywhere in the (x,t)-plane at which the solution remains smooth. However, the families of characteristics $C_\pm$ themselves depend on the solution; hence, it is necessary to determine the solution and the characteristics simultaneously.

An interesting special case exists when one of the Riemann invariants $s(\xi)$ is identically constant, that is, $s(\xi) \equiv s_0$. In this case, equations (6.9.10) and (6.9.11) can be integrated to obtain

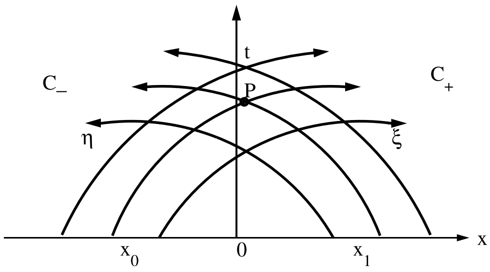

Fig. 6.21 Characteristics $C_{\pm}$.

$$f_1^+(u_1) + f_2^+(u_2) = r(\eta) \quad \text{along } C_+ \tag{6.9.15}$$

and

$$f_1^-(u_1) + f_2^-(u_2) = s_0 \quad \text{along } C_-, \tag{6.9.16}$$

where $f_i^+(u_i) = \int \mu L_{1i}\, du_i$ and $f_i^-(u_i) = \int \mu L_{2i}\, du_i$.

This demonstrates the fact that everywhere along a characteristic C_+ specified by $\eta = \eta_0 = \text{const.}$, u_1 and u_2 must be constant because they are the solutions of the nonlinear system of simultaneous equations

$$f_1^+(u_1) + f_2^+(u_2) = r(\eta_0), \tag{6.9.17}$$
$$f_1^-(u_1) + f_2^-(u_2) = s_0. \tag{6.9.18}$$

However, solutions of this system may *not* be *unique* due to nonlinearity, and this fact is associated with possible discontinuous solutions as shock waves. This is a typical feature of nonlinear partial differential equations as explained earlier in various other problems.

The solutions u_1 and u_2 are thus constant along the C_+ characteristic specified by setting $\eta = \eta_0$. Hence, $\lambda_1 = \lambda_1(u_1, u_2)$, and it follows that the characteristic must be a straight line. When one of the Riemann invariants r or s is identically constant, the corresponding solution is said to form a *simple wave*. Simple waves represent the simplest nonconstant solutions of a nonlinear hyperbolic system (6.9.1). The important fact that the system contains neither dispersive nor dissipative terms means that no physical mechanism exists which can prevent the formation of a discontinuous solution or that can permit the propagation of traveling waves.

Example 6.9.1 Riemann's Invariants of a Linear Wave Equation. We solve the linear initial-value problem

$$\phi_{tt} - c^2\phi_{xx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{6.9.19}$$

with the Cauchy data

$$\phi(x,0) = f(x), \quad \phi_t(x,0) = g(x) \quad \text{for } x \in \mathbb{R}. \tag{6.9.20ab}$$

We can factorize (6.9.19) in the form

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\phi = 0. \tag{6.9.21}$$

Introducing two new functions $u(x,t)$ and $v(x,t)$ defined by

$$u = \phi_t \quad \text{and} \quad v = c\phi_x, \tag{6.9.22ab}$$

we can combine (6.9.21) with (6.9.22ab) to obtain two first-order equations

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)(u+v) = 0, \tag{6.9.23}$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)(u-v) = 0. \tag{6.9.24}$$

These equations imply that the quantities $u \pm v$ remain constant along two families of characteristics given by $C_\pm : \frac{dx}{dt} = \pm c$. Further, $u+v$ and $u-v$ are two Riemann's invariants so that

$$u + v = \text{const.} \quad \text{along a curve } C_+ \text{ such that } \frac{dx}{dt} = c,$$

$$u - v = \text{const.} \quad \text{along a curve } C_- \text{ such that } \frac{dx}{dt} = -c.$$

In other words,

$$u + v = r(\eta) \quad \text{along } C_+ : \eta = x - ct, \tag{6.9.25}$$

$$u - v = s(\xi) \quad \text{along } C_- : \xi = x + ct, \tag{6.9.26}$$

where $r(\eta)$ and $s(\xi)$ are arbitrary functions of η and ξ, respectively, to be determined from the given initial data $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$. Along the characteristics $C_+ : x - ct = \eta$ and $C_+ : x + ct = \xi$, we obtain

$$u + v = r(\eta) = u_0(\eta) + v_0(\eta), \tag{6.9.27}$$

$$u - v = s(\xi) = u_0(\xi) - v_0(\xi). \tag{6.9.28}$$

Solving these equations gives the point of intersection of the lines C_+ and C_- on the (x,t)-plane. Clearly, both u and v are functions of ξ and η only, and hence, ϕ is a function of ξ and η only, by definition (6.9.22ab). Consequently, we can write

$$\phi(x,t) = \Phi(\xi) + \Psi(\eta) = \Phi(x+ct) + \Psi(x-ct), \tag{6.9.29}$$

where Φ and Ψ are arbitrary functions to be determined from the given Cauchy data (6.9.20ab). It turns out that

$$\phi(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau, \tag{6.9.30}$$

where $g(x) = \phi_t(x,0) = u_0(x)$. This is the well-known d'Alembert solution of the wave equation. As indicated in Chapter 3, this method of solving the wave equation by using the characteristic curves $C_\pm$ is called the *method of characteristics*. This method has also been used to solve quasi-linear or nonlinear equations.

The wave equation (6.9.19) with (6.9.22ab) can be expressed in the matrix form

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0, \tag{6.9.31}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}.$$

It has been shown that the values of u and v at the point of intersection of the two characteristics C_+ and C_- can be found in terms of their initial values $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$. Furthermore, the phase velocities of the wave $\frac{dx}{dt} = \pm c$ are found to determine the families of characteristics $C_\pm$. It turns out that these phase speeds are just the eigenvalues of the matrix A. To prove this, we denote the eigenvalue by λ and the corresponding right-eigenvector by R so that

$$AR = \lambda R. \tag{6.9.32}$$

It follows from this equation that $\lambda = \lambda_r = \pm c$ $(r = 1, 2)$ and the corresponding right-eigenvectors are given by

$$R_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad R_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.9.33}$$

To investigate the eigenstates represented by $R_\pm$, we write $U_\pm = \phi R_\pm$, where $\phi = \phi(x,t)$ is an arbitrary scalar function of x and t. Substituting this in equation (6.9.31) gives the equation for $\phi(x,t)$ in the form

$$\frac{\partial \phi}{\partial t} + \lambda\frac{\partial \phi}{\partial x} = 0. \tag{6.9.34}$$

Thus, the solutions of (6.9.31) corresponding to the eigenvalues $\lambda = \pm c$ are

$$U_+ = \phi_+(x - ct)R_+ \quad \text{and} \quad U_- = \phi_-(x + ct)R_-. \tag{6.9.35ab}$$

They represent waves propagating to the right and left with constant velocities $\pm c$, respectively. According to the superposition principle, the general solution of the linear wave equation is

$$U(x,t) = U_+ + U_- = \phi_+(x - ct)R_+ + \phi_-(x + ct)R_-. \tag{6.9.36}$$

When the initial condition $U(x,0) = U_0(x)$ at $t = 0$ is specified, equation (6.9.36) leads to

$$\phi_+(x)R_+ + \phi_-(x)R_- = U_0(x). \tag{6.9.37}$$

In principle, this equation can be solved, and hence, $\phi_\pm(x)$ can be uniquely determined, since $R_\pm$ are linearly independent. Note that the solution represents the superposition of two suitably combined eigenstates of the matrix A involved in the equation.

To obtain equations (6.9.25) and (6.9.26) involving the Riemann invariants of the system (6.9.31), we introduce the left eigenvectors L which satisfy the equation

$$LA = \lambda L \tag{6.9.38}$$

so that $L_\pm = (1, \mp 1)$. Multiplying the equation (6.9.31) by $L_\pm$ from the left gives

$$L_\pm \left(\frac{\partial}{\partial t} + \lambda_\pm \frac{\partial}{\partial x} \right) U_\pm = 0. \tag{6.9.39}$$

The differential operators in (6.9.39) represent the total derivative along the characteristics $C_+ : x - ct = \xi$ and the characteristics $C_- : x + ct = \eta$, corresponding to the eigenvalues $\lambda_+ = c$ and $\lambda_- = -c$, respectively. Thus, along the characteristics $\xi = x - ct =$ const. and $\eta = x + ct =$ const.,

$$L_\pm U_\pm = 0. \tag{6.9.40}$$

Integrating these equations and using the eigenvectors $L_\pm = (1, \mp 1)$, we obtain the same equations as (6.9.25) and (6.9.26).

Example 6.9.2 Steady Two-Dimensional Supersonic Fluid Flow. The equations of motion for two-dimensional, irrotational, isentropic flow are

$$\left(c^2 - u^2\right) u_x - u\, v (u_y + v_x) + \left(c^2 - v^2\right) v_y = 0, \tag{6.9.41}$$

$$v_x - u_y = 0, \tag{6.9.42}$$

where u and v are fluid velocities along the x-axis and y-axis, respectively, and c is the local sound speed.

In matrix notation, the preceding equations can be expressed in the form

$$A_1 U_x + B_1 U_y = 0, \tag{6.9.43}$$

where

$$A_1 = \begin{pmatrix} 0 & 1 \\ c^2 - u^2 & -uv \end{pmatrix}, \qquad B_1 = \begin{pmatrix} -1 & 0 \\ -uv & c^2 - v^2 \end{pmatrix}, \quad \text{and} \quad U = \begin{pmatrix} u \\ v \end{pmatrix}. \tag{6.9.44}$$

Since A_1^{-1} exists provided $c^2 - u^2 \neq 0$, we multiply (6.9.43) by A_1^{-1} from the left to obtain

$$U_x + A\, U_y = 0, \tag{6.9.45}$$

where

$$A = \begin{pmatrix} -\frac{2uv}{c^2-u^2} & \frac{c^2-v^2}{c^2-u^2} \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of the problem are the solutions of the equation $|A-\lambda I| = 0$, which has the explicit form

$$\begin{vmatrix} -\frac{2uv}{c^2-u^2} & \frac{c^2-v^2}{c^2-u^2} \\ -1 & -\lambda \end{vmatrix} = 0,$$

that is,

$$\lambda^2 + \frac{2uv}{c^2-u^2}\lambda + \frac{c^2-v^2}{c^2-u^2} = 0, \tag{6.9.46}$$

which gives two roots

$$\lambda_{1,2} = \frac{1}{(c^2-u^2)}\left[-uv \pm c\sqrt{u^2+v^2-c^2}\right]. \tag{6.9.47}$$

Clearly, the roots are real provided that $q^2 = u^2+v^2 > c^2$, and hence, the equations (6.9.41) and (6.9.42) are hyperbolic. The *local Mach number* is defined by $M = q/c$ which is greater than unity ensuring that the flow is *supersonic*. In supersonic flow, the characteristics in the (x, y)-plane are defined by the equations

$$\frac{dy}{dx} = \lambda_r, \quad r = 1, 2. \tag{6.9.48}$$

These characteristics are called the *Mach lines* and can be determined provided the matrix A_1^{-1} is nonsingular. This simply means that $|A_1| \neq 0$ or $c^2 - u^2 \neq 0$. If this matrix is singular, then $u = c$, that is, the x-component of the fluid velocity is equal to the local sound speed.

The left eigenvalues corresponding to the eigenvalues $\lambda_{1,2}$ given by (6.9.47) are

$$L_1 = (1, \lambda_2) \quad \text{and} \quad L_2 = (1, \lambda_1). \tag{6.9.49ab}$$

It follows from equations (6.9.8), (6.9.9) that the characteristics in the (u, v)-plane are given by

$$\Gamma_+ : \frac{\partial u}{\partial \xi} + \lambda_2 \frac{\partial v}{\partial \xi} = 0; \tag{6.9.50a}$$

$$\Gamma_- : \frac{\partial u}{\partial \eta} + \lambda_1 \frac{\partial v}{\partial \eta} = 0, \tag{6.9.50b}$$

where ξ and η are parameters. In terms of these parameters, equation (6.9.48) can be written as

$$C_+ : \frac{\partial y}{\partial \xi} - \lambda_1 \frac{\partial x}{\partial \xi} = 0; \tag{6.9.51a}$$

$$C_- : \frac{\partial y}{\partial \eta} - \lambda_2 \frac{\partial x}{\partial \eta} = 0. \tag{6.9.51b}$$

Combining (6.9.50a) and (6.9.51a) or (6.9.50b) and (6.9.51b) gives

$$\frac{du}{dv} = -\frac{dy}{dx}, \tag{6.9.52}$$

or equivalently,

$$\frac{dy}{dx} \cdot \frac{dv}{du} = -1. \tag{6.9.53}$$

This is the condition of orthogonality of two curves in the (x, y)-plane and (u, v)-plane when they are represented in the same coordinate plane. More precisely, C_- and Γ_+ characteristics and C_+ and Γ_- characteristics are mutually orthogonal.

The system of equation (6.9.1) or (6.9.45) is called *reducible* provided the elements of the coefficient matrix A depend explicitly only on the elements of U. The name originates from the fact that the system can be reduced to a linear system in the new independent variables u and v by interchanging the dependent and independent variables. Such a transformation exists provided the Jacobian $J = u_x\, v_y - u_y\, v_x$ is nonzero. Thus it turns out that

$$\frac{\partial u}{\partial x} = J\frac{\partial y}{\partial v}, \qquad \frac{\partial v}{\partial y} = J\frac{\partial x}{\partial u}, \qquad \frac{\partial v}{\partial x} = -J\frac{\partial y}{\partial u}, \qquad \frac{\partial u}{\partial y} = -J\frac{\partial x}{\partial v}. \tag{6.9.54}$$

This transformation is called the *hodograph transformation*. For more details, the reader is referred to Courant and Friedrichs (1948).

Example 6.9.3 Equations for the Current and Potential in a Transmission Line. We recall the equations for the current $I(x, t)$ and potential $V(x, t)$ from Problem 29 in 1.15 Exercise as

$$\begin{aligned} I_t + V_x + RI &= 0, \\ CV_t + I_x + GV &= 0, \end{aligned} \tag{6.9.55}$$

In matrix notation, these become

$$A_1 U_t + A_2 U_x + A_3 U = 0, \tag{6.9.56}$$

where

$$U = \begin{pmatrix} I \\ V \end{pmatrix}, \qquad A_1 = \begin{pmatrix} L & 0 \\ 0 & C \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} R & 0 \\ 0 & G \end{pmatrix}.$$

We multiply the equation (6.9.56) by A_1^{-1} to obtain

$$U_t + AU_x + BU = 0, \tag{6.9.57}$$

where

$$A = \begin{pmatrix} 0 & \frac{1}{L} \\ \frac{1}{C} & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{R}{L} & 0 \\ 0 & \frac{G}{C} \end{pmatrix}.$$

The eigenvalue equation $|A - \lambda I| = 0$ gives two eigenvalues $\lambda_{1,2} = \pm c$, where $c = (LC)^{-\frac{1}{2}}$. Clearly, the velocities of the two waves are

$$C_+ : \frac{dx}{dt} = \lambda_1 = c, \qquad C_- : \frac{dx}{dt} = \lambda_2 = -c. \tag{6.9.58ab}$$

The left eigenvectors $L_{1,2}$ corresponding to the eigenvalues $\lambda_{1,2}$ are given by

$$L_1 = (cL, 1) \quad \text{and} \quad L_2 = (cL, -1). \tag{6.9.59ab}$$

For a lossless transmission line, $R = G = 0$, equation (6.9.57) becomes the classical wave equation, and the Riemann invariants become

$$cI + V = r(\eta) \quad \text{along } C_+, \tag{6.9.60a}$$

$$cI - V = s(\xi) \quad \text{along } C_-. \tag{6.9.60b}$$

However, in general, R and G are nonzero quantities, and the equations corresponding to (6.9.8), (6.9.9), which are obtained by multiplying (6.9.57) by L_1 and L_2, are

$$(cL)I_\xi + V_\xi = -\left(RcI + \frac{G}{C}V\right) \quad \text{along } C_+, \tag{6.9.61a}$$

$$(cL)I_\eta - V_\eta = -\left(RcI - \frac{G}{C}V\right) \quad \text{along } C_-. \tag{6.9.61b}$$

In general, these equations are not integrable, but they can be integrated in some special cases. For the Heaviside distortionless cable, $\frac{R}{L} = \frac{G}{C} = \text{const.} = k$, equations (6.9.61a), (6.9.61b) become

$$\frac{\partial}{\partial \xi}(cLI + V) = -\frac{R}{L}(cLI + V) \quad \text{along } C_+, \tag{6.9.62a}$$

$$\frac{\partial}{\partial \eta}(cLI - V) = -\frac{R}{L}(cLI - V) \quad \text{along } C_-. \tag{6.9.62b}$$

These equations can be integrated to obtain

$$cLI + V = A(\eta) \exp\left(-\frac{R}{L}\xi\right) \quad \text{along } C_+, \tag{6.9.63a}$$

$$cLI - V = B(\xi) \exp\left(-\frac{R}{L}\eta\right) \quad \text{along } C_-. \tag{6.9.63b}$$

Adding and subtracting leads to the results

$$I = \frac{1}{2cL}\left[A(\eta)\exp\left(-\frac{R}{L}\,\xi\right) + B(\xi)\exp\left(-\frac{R}{L}\,\eta\right)\right], \tag{6.9.64}$$

$$V = \frac{1}{2}\left[A(\eta)\exp\left(-\frac{R}{L}\,\xi\right) - B(\xi)\exp\left(-\frac{R}{L}\,\eta\right)\right]. \tag{6.9.65}$$

We can identify the equations for the characteristics as $\xi = x + ct$ and $\eta = x - ct$, and these results can be used to reorganize the exponential factors in (6.9.64) and (6.9.65), so that the final form of the solutions becomes

$$I = \frac{1}{2cL} \exp\left(-\frac{2R}{L}x\right)\left[f(x-ct) + g(x+ct)\right], \tag{6.9.66}$$

$$V = \frac{1}{2} \exp\left(-\frac{2R}{L}x\right)\left[f(x-ct) - g(x+ct)\right], \tag{6.9.67}$$

where f and g are arbitrary functions of their arguments.

Evidently, the current and the potential represent the waveforms $f+g$ and $f-g$, respectively, which propagate along the Heaviside transmission line *without change of shape* except for the attentuation factor $\exp(-\frac{2R}{L}x)$. However, if the condition $\frac{R}{L} = \frac{G}{C}$ is not satisfied, the original shape of the waveform would suffer from distortion.

We close this section by adding the following comment. For the case of a homogeneous system (6.9.1) with analytic initial data, in general, a smooth solution is possible for a finite time. However, the problem for the case of inhomogeneous system (6.9.57) is more complicated. In general, a smooth solution of (6.9.57) does not exist for all time. Lax (1954a, 1954b) proved the existence of solutions for a short time and generalized the result to allow for Lipschitz continuous initial conditions which arise frequently in many physical problems, including the case of a wave propagating into a constant state.

6.10 Generalized Simple Waves and Generalized Riemann's Invariants

The preceding analysis for a two-element vector U can be extended to a hyperbolic system of the general form

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0, \tag{6.10.1}$$

where U is an $n \times 1$ column vector with elements $u_1, u_2, \ldots, u_n$, and A is an $n \times n$ matrix with elements $a_{ij} = a_{ij}(u_1, u_2, \ldots, u_n)$ which depends on the elements of U, and on x and t.

As a generalization of a simple wave, we take a solution U of (6.10.1) that depends on only one of the elements u_1, so that $U = U(u_1)$. Substituting this solution in (6.10.1) yields

$$\left(\frac{\partial u_1}{\partial t}I + \frac{\partial u_1}{\partial x}A\right)\left(\frac{dU}{du_1}\right) = 0. \tag{6.10.2}$$

This homogeneous algebraic system for the n elements of $(\frac{dU}{du_1})$ can have only a nontrivial solution provided that the determinant equation

$$|A - \lambda I| = 0 \tag{6.10.3}$$

holds, where $\lambda = -(\frac{\partial u_1}{\partial t})/(\frac{\partial u_1}{\partial x})$. The n solutions λ_r $(r = 1, 2, 3, \ldots, n)$ of (6.10.3) are simply the eigenvalues of A. We denote the right eigenvectors of A by R_r, corresponding to the eigenvalues λ_r. If the system (6.10.1) is totally hyperbolic, then all the eigenvalues λ_r are real and distinct, and there are n linearly independent eigenvalues R_r. Hence, the family of characteristics C_r is given by

$$C_r : \frac{dx}{dt} = \lambda_r = -\left(\frac{\partial u_1}{\partial t}\right) \Big/ \left(\frac{\partial u_1}{\partial x}\right), \quad r = 1, 2, \ldots, n. \tag{6.10.4}$$

Or equivalently,

$$\frac{\partial u_1}{\partial t}\, dt + \frac{\partial u_1}{\partial x}\, dx = 0 \quad \text{along each } C_r,$$

so that $du_1 = 0$, and hence,

$$u_1(x, t) = \text{const.} \quad \text{along each } C_r. \tag{6.10.5}$$

Since $\lambda_r = \lambda_r(u_1)$, it follows that each member of the n families of characteristics represents straight lines along which $U = \text{const.}$ On the other hand, if (6.10.4) is written in the form

$$\frac{\partial u_1}{\partial t} + \lambda_r(u_1)\frac{\partial u_1}{\partial x} = 0, \tag{6.10.6}$$

it follows that u_1 is the solution of the nonlinear equation (6.10.6), and this solution and hence $U = U(u_1)$ may or may not evolve as a discontinuous solution at a point depending on the initial data and on the form of $\lambda_r(u_1)$.

In a linear problem in which the matrix A is independent of U, λ and R become specific functions of x and t, and there are n wave modes whose phase velocities are given by the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. In particular, if A is a constant matrix, the linear superposition principle enables us to obtain the general solution in terms of an arbitrary set of functions $\phi_r(x - \lambda_r t)$ in the form

$$U(x, t) = \sum_{r=1}^{n} \phi_r(x - \lambda_r t) R_r, \tag{6.10.7}$$

where ϕ_r can be uniquely determined from the initial condition $U(x, 0) = f(x)$ for all x.

If we now set $r = k$ and consider the kth generalized simple wave, equation (6.10.2) must be proportional to the corresponding kth right-eigenvector of A with elements $R_1^{(k)}, R_2^{(k)}, \ldots, R_n^{(k)}$ so that

$$\frac{du_1}{R_1^{(k)}} = \frac{du_2}{R_2^{(k)}} = \cdots = \frac{du_n}{R_n^{(k)}}. \tag{6.10.8}$$

These equations represent a set of n first-order, ordinary differential equations which determines the nature of the solution U across what is called the *generalized* λ_k-*simple wave*. Integrating (6.10.8) gives $(n - 1)$ linearly independent relations between the n elements of U. These $(n - 1)$ invariant relations along members of the kth family of characteristics C_k are called the *generalized* $\lambda_{(k)}$-*Riemann invariants* and are denoted by

$$J_{(k)}(U) = \text{const.} \quad \text{for } r = 1, 2, \ldots, (n - 1). \tag{6.10.9}$$

These relations hold throughout the generalized λ_k-simple wave, and they can be used to determine the nature of the solution provided it remains differentiable.

Each generalized $\lambda_{(k)}$-Riemann invariant defines a manifold in the $(u_1, u_2, \ldots, u_n)$-space on the mth of which $\lambda^m_{(k)}$ must satisfy the condition $d\,J^{(i)}_{(k)} = 0$, which can be expressed as

$$\frac{\partial J^{(i)}_{(k)}}{\partial u_1}\,du_1 + \frac{\partial J^{(i)}_{(k)}}{\partial u_2}\,du_2 + \cdots + \frac{\partial J^{(i)}_{(k)}}{\partial u_n}\,du_n = 0. \tag{6.10.10}$$

We consider the generalized λ_k-simple waves with a parameter ξ to rewrite (6.10.8) as

$$\frac{du_1}{R^{(k)}_1} = \frac{du_2}{R^{(k)}_2} = \cdots = \frac{du_n}{R^{(k)}_n} = d\xi. \tag{6.10.11}$$

These lead to the results

$$\frac{du_i}{d\xi} = R^{(k)}_i \quad \text{for } i = 1, 2, \ldots, n. \tag{6.10.12}$$

Substituting these results into (6.10.9) gives an equivalent condition

$$\left(\nabla_u J^i_{(k)}\right) R^{(k)} = 0, \tag{6.10.13}$$

where $i = 1, 2, \ldots, (n-1)$ and ∇_u represents the gradient operator with respect to $u_1, u_2, \ldots, u_n$. It is noted here that Lax (1957, 1973) utilized condition (6.10.13), which requires the mutual orthogonality of $\nabla_u J^i_{(k)}$ and $R^{(k)}$, first to introduce the idea of generalized $\lambda_{(k)}$-simple waves.

In the rest of this section, we follow Jeffrey (1976) to discuss briefly the quasi-linear hyperbolic system of equations with spatial derivative in the form of a one-dimensional divergence

$$\frac{\partial F(U)}{\partial t} + \frac{\partial G(U)}{\partial x} = H(U), \tag{6.10.14}$$

where U is an $n \times 1$ column vector with elements $u_1, u_2, \ldots, u_n$, and F, G, and H are $n \times 1$ column vectors whose elements are functions of $u_1, u_2, \ldots, u_n$. If the rth element of $G(U)$ is $g_r(u_1, u_2, \ldots, u_n)$, the system of equations (6.10.14) is hyperbolic if the eigenvalues of the matrix

$$A = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_n} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \cdots & \frac{\partial g_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial u_1} & \frac{\partial g_n}{\partial u_2} & \cdots & \frac{\partial g_n}{\partial u_n} \end{bmatrix} \tag{6.10.15}$$

are real, and it has a complete set of linearly independent eigenvectors.

According to the general theory of Riemann simple waves and the development of shocks, it can be shown that the system (6.10.14) has a discontinuous solution provided that the *generalized Rankine–Hugoniot shock condition*

$$\tilde{\lambda}[F] = [G], \tag{6.10.16}$$

is satisfied, where $[X]$ denotes the discontinuous jump in the quantity X with $\tilde{\lambda}$ as the velocity of propagation. Clearly, (6.10.16) is a nonlinear system of n algebraic equations relating $\tilde{\lambda}$, the n elements of U_- on the one side of the jump discontinuity, and the n elements U_+ on the other side. Due to nonlinearity, a solution of this system may *not* be unique. In order to establish a unique solution of physical interest from the set of mathematically possible solutions, Lax (1957) generalized the selection principle for shock waves, which states that if the kth characteristic field is *genuinely nonlinear*, a discontinuous jump solution is admissible only if the *Lax entropy criterion*

$$\lambda_{(k)}(U_-) > \tilde{\lambda} > \lambda_{(k)}(U_+) \tag{6.10.17}$$

is satisfied. Physically, this is equivalent to the requirement that entropy cannot decrease. In fluid dynamics, this corresponds to the fact that a shock is associated with a flow which is *supersonic* at the front of the shock and *subsonic* at the rear. This leads to the physical fact that a hydrodynamical shock is a compressive wave, and it also implies an irreversible condition of the system. It has already been indicated that, in gas dynamics, this criterion is equivalent to the second law of thermodynamics which states that the entropy cannot decrease. Another important fact is that, only in very special cases, $\tilde{\lambda}$ is equal to an eigenvalue of the matrix A, and then the jump discontinuous solution is referred to as a *characteristic shock*.

In conclusion, we recall the generalization of a classical solution (see Chapter 5, Section 5.4) to a weak solution to resolve the difficulty involved in the breakdown of differentiability in quasi-linear hyperbolic systems. In the simplest physical problem described by the nonlinear scalar equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big[f(u)\big] = 0, \tag{6.10.18}$$

the function $u = u(x,t)$ is called the *weak solution* if, for $t > 0$,

$$\int_0^\infty \int_{-\infty}^\infty \big[u\phi_t + f(u)\phi_x\big]\,dx\,dt = 0 \tag{6.10.19}$$

for every *test* function $\phi(x,t) \in C^2$ which vanishes outside some finite region of the half plane $t > 0$. It has already been shown in Chapter 5, Section 5.4 that both discontinuous classical and weak solutions satisfy the generalized Rankine–Hugoniot condition, and a piecewise C^1 weak solution is also a piecewise C^1 classical solution. However, a weak solution is *not* uniquely determined by the initial data. So, the Lax entropy criterion is used to ascertain a unique and physically realistic solution. It is important to point out that weak solutions suffer from a major weakness that a variable change in the original conservation equation leads to different weak solutions. For more information about weak solutions and shock waves, we refer to Smoller (1994) and Le Veque (1990).

6.11 The Lorenz System of Nonlinear Differential Equations and Deterministic Chaos

In an effort to make an accurate prediction of complicated weather forecasting on Earth, Edward N. Lorenz (1917–) published a seminal paper (1963a, 1963b, 1963c) and discovered a system of three nonlinear ordinary differential equations for the three state variables $(x(t), y(t), z(t))$ that can exhibit chaotic behavior. Historically, the Lorenz system is one of the earliest examples of chaos realized on an electronic computer. The following are the dynamical equations of the Lorenz system

$$\frac{dx}{dt} = \sigma(y - x), \qquad \frac{dy}{dt} = rx - (y + xz), \qquad \frac{dz}{dt} = xy - bz, \tag{6.11.1}$$

where σ, r, and b are parameters. The right-hand sides of these equations do not include time t, and the time derivatives on the left-hand sides are determined solely by the state (x, y, z). Integrating the above system (6.11.1) numerically, the trajectory $(x(t), y(t), z(t))$ of state can be determined. A set of point in the phase space (x, y, z) where a family of trajectories for a set of initial conditions accumulates asymptotically as $t \to \infty$ is known as the *Lorenz attractor*.

Obviously, $\dot{x}$, $\dot{y}$, $\dot{z}$ can be considered as three components of a velocity vector $\mathbf{u}$ defined at the point $\mathbf{x} = (x, y, z)$ so that we can regard this as a dynamical system like a fluid motion with velocity field $\mathbf{u}(\mathbf{x})$. One of the important features of such a motion in the phase space is that the phase volume consisting of points moving with velocity $\mathbf{u} = (\dot{x}, \dot{y}, \dot{z})$ decreases steadily because the divergence of the velocity field is negative, that is,

$$\operatorname{div} \mathbf{u} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(\sigma + 1 + b) < 0. \tag{6.11.2}$$

This means that the phase volume of an attractor where the trajectories are approaching decreases indefinitely. This does not necessarily imply that the attractor is made of discrete points, but means only that the dimension of the attractor is less than three. Thus, the dimension of the attractor becomes non-integral or fractal.

Lorenz made a serious attempt to solve an initial value problem of the system (6.11.1) with $\sigma = 10$, $r = 28$, $b = \frac{8}{3}$, and the initial data $(x(0), y(0), z(0)) = (0, 1, 0)$. His numerical experiments show that the Lorenz system (6.11.1) captured a certain new but intrinsic chaotic nature of the weather phenomena, and the long-term weather prediction is almost impossible. It also revealed that a very small change in initial conditions leads to a significantly large change in the solution. It is now a well known fact that the temporal behavior of the Lorenz system is stochastic in a true sense. In fact, its short time behavior is *deterministic*, but its long time evolution is *stochastic*. This phenomenon is known as the *deterministic chaos*. Indeed, many nonlinear dynamical systems have property similar to that of the Lorenz attractor. Included is Figure 6.22 for a Lorenz attractor which is a representative example of the strange attractor in nonlinear dynamics.

In stability analysis, steady state plays a fundamental role because the decay or growth of perturbations of a steady state is usually investigated for a dynamical

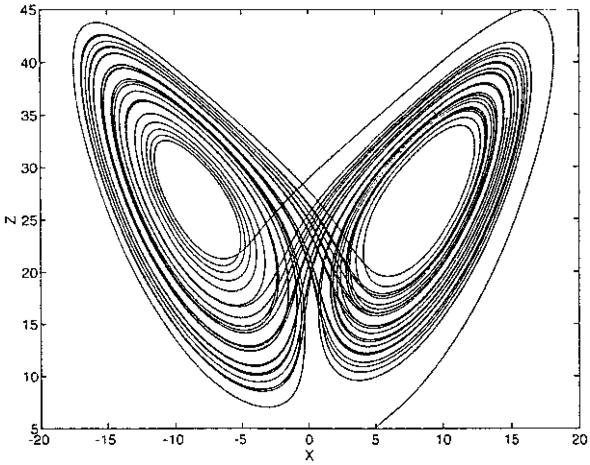

Fig. 6.22 A numerically computed solution of the Lorenz equations with a plot of z against x with $r = 28$, $(x(0), y(0), z(0) = (5, 5, 5))$.

system. A steady state corresponds to a fixed point of a dynamical system. In case of the Lorenz system (6.11.1), a fixed point is obtained from $(\dot{x}, \dot{y}, \dot{z}) = (0, 0, 0)$. Thus, it follows from system of equations (6.11.1) that $x = y$, $rx - (y + xz) = 0$, $xy - bz = 0$. For $r < 1$, there is only one fixed point which is the origin $O : (x, y, z) = (0, 0, 0)$. The origin O corresponds to the static state. This is in full agreement with the analysis of thermal convection problem in fluid dynamics where we consider a bifurcation problem from the static state to a thermal convection-cell state. However, for $r > 1$, it can be shown that there exist three fixed points: the origin O and two points given by $C_1(q, q, r - 1)$ and $C_2(-q, -q, r - 1)$ where $q = \sqrt{b(r - 1)}$, and C_1 and C_2 are located at points of mirror symmetry with respect to the vertical plane $x + y = 0$.

When $r > r_c \, (r_c > 1)$, all three fixed points are unstable, that is, there are no stable fixed points to which the trajectory can approach. This is really a strange phenomenon.

In order to study the stability of the origin O, we first linearize the Lorenz system for points in the neighborhood of O so that

$$\frac{dx'}{dt} = \sigma(y' - x'), \qquad \frac{dy'}{dt} = rx' - y', \qquad \frac{dz'}{dt} = -bz', \tag{6.11.3}$$

where x', y', z' are small perturbations to the fixed point $(0, 0, 0)$, which are represented as $(x', y', z') = (x_0, y_0, z_0)e^{\lambda t}$, where λ is the growth rate. Substituting these perturbations in (6.11.3) leads to a system of linear algebraic equations in the matrix form for the amplitude (x_0, y_0, z_0):

$$\begin{pmatrix} \lambda + \sigma & -\sigma & 0 \\ -r & \lambda + 1 & 0 \\ 0 & 0 & \lambda + b \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = 0. \tag{6.11.4}$$

For a nontrivial solution of (x_0, y_0, z_0), the determinant of the coefficient matrix must vanish. Thus, it gives the eigenvalue equation for the growth rate λ, that is,

$$\begin{vmatrix} \lambda+\sigma & -\sigma & 0 \\ -r & \lambda+1 & 0 \\ 0 & 0 & \lambda+b \end{vmatrix} = (\lambda+b)\left[\lambda^2+(\sigma+1)\lambda+\sigma(1-r)\right] = 0. \tag{6.11.5}$$

Consequently, the eigenvalues are given by

$$\lambda = -b, \qquad \lambda_\pm = \frac{1}{2}\left[-(\sigma+1) \pm \sqrt{(\sigma+1)^2+4\sigma(r-1)}\right]. \tag{6.11.6}$$

For positive σ and b, we draw the following conclusions:

For $0 < r \leq 1$, all three eigenvalues are real and non-positive. Hence, the fixed (or equilibrium) point O is stable. For $r > 1$, two eigenvalues are real and negative, but one eigenvalue is real and positive. Thus, O is unstable, in general.

When $r = 1$, it is a transition (or bifurcation) point. The fixed point O serves as the transition point from stability to instability, and at the same transition point, new equilibrium points begin to appear. These results are both qualitatively and quantitatively in agreement with the linear stability analysis of the Rayleigh–Benard convection problem in fluid mechanics.

It has been shown that the situation $r > r_H = 27.74$ is quite complicated. It can be shown that there exists a stable periodic orbit when r is very large. We introduce a small parameter $\varepsilon = \frac{1}{\sqrt{r}}$ and time scale $t = \varepsilon t'$. We next substitute

$$x = \frac{x'}{\varepsilon}, \qquad y = \left(\frac{1}{\varepsilon^2\sigma}\right)y', \qquad z = \frac{1}{\varepsilon^2}\left(\frac{z'}{\sigma}+1\right), \tag{6.11.7}$$

in the Lorenz system (6.11.1) so that it becomes, dropping the primes,

$$\frac{dx}{dt} = y - \varepsilon\sigma x, \qquad \frac{dy}{dt} = -(xz+\varepsilon y), \qquad \frac{dz}{dt} = xy - \varepsilon b(z+\sigma). \tag{6.11.8}$$

In the limit as $r \to \infty$, $\varepsilon \to 0$ so that (6.11.8) reduces to

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -xz, \qquad \frac{dz}{dt} = xy. \tag{6.11.9}$$

Consequently,

$$\frac{dx}{y} = \frac{dy}{-xz} = \frac{dz}{xy} = dt \tag{6.11.10}$$

lead to two integrals representing a periodic orbit in the form

$$x^2 - 2z = 2A, \qquad y^2 + z^2 = B^2, \tag{6.11.11}$$

where A and B are constants. Thus, the trajectories lie on the cylinder $y^2 + z^2 = B^2$ and, in general, it is a closed orbit. It can be shown based on perturbation expansion for small ε that there exists a stable periodic orbit when $(\sigma+1) > (b+1)$. The orbit winds once around the z-axis for large r, this is the only orbit which persists for all r. In the case of the Lorenz system for $\sigma = 10$ and $b = \frac{8}{3}$, the above condition $(\sigma + 1) > (b + 1)$ is satisfied.

Numerical computations reveal that as r decreases, period-doubling bifurcation will begin to occur at $r = 313$. Successive stages of period-doubling bifurcations will occur as r decreases and reaches the value 214.364. At each period-doubling bifurcation, the original periodic orbit becomes unstable, and new stable orbits which wind twice as much as the original will appear. For such phenomena of period-doubling, if r_n represents the value of r at the nth bifurcation, then, in the limit as $n \to \infty$, the celebrated *Feigenbaum constant*, F, appears. More precisely,

$$F = \lim_{n\to\infty} \left(\frac{r_{n-1} - r_n}{r_n - r_{n+1}} \right) = 4.6692016609\ldots. \tag{6.11.12}$$

This is known as a *universal* (probably transcendental) *constant* which often arises in many problems in nonlinear dynamics.

In the numerical experiment of the Lorenz system with $\sigma = 10$, $r = 28$, and $b = \frac{8}{3}$, two orbits beginning from two neighboring points near Lorenz attractor lose their correlation after some time. Such a property of sensitive dependence on initial conditions is a characteristic feature of chaotic orbits. After some time, both orbits trace a path on the same attractor without mutual correlation. The Lyapunov dimension (or fractal dimension), D_L, of the Lorenz attractor for the above parameter values is found as $2.00 < D_L < 2.401$. In view of the fractal dimension, the Lorenz attractor is called a *strange attractor*. Historically, chaos is the revelation of the Lorenz (strange) attractor. Chaotic flow is irregular and appears to be random in nature, but it is deterministic. In fact, the *deterministic chaos* arises in many different nonlinear physical systems. Chaos may represent a state leading to turbulence. From a modern view point, turbulence is regarded as a highly irregular fluctuating flow field which develops autonomously by a nonlinear mechanism of field dynamics. This change in view of turbulence is mainly due to the recent development of the theory of chaos. From 1960s, numerical experiments have confirmed the existence of phenomena such as chaos, intermittency and period-doubling process.

As far as the Lorenz nonlinear system of equations is concerned, it represents at best a mathematical model for the Rayleigh–Benard convection flow problem when the Reynolds number R is only sufficiently large compared to the critical value R_c.

On the other hand, it follows from the numerical studies of the Lorenz system of equations that a geometric model can be constructed, which can be defined by a nonlinear map on an interval. If we follow a trajectory given by the Lorenz equations and consider the plane $z = r - 1$, the trajectory will cross from the side $z > r - 1$ to the other side at various points on $z = r - 1$. The successive points on the plane section $z = r - 1$ define a return map which is known as the *Poincaré return map*. This shows that there is an interesting relationship between the Lorenz system of differential equations and the nonlinear map on an interval as can be illustrated by the famous logistic map. The *logistic map* is defined by the following recurrence relation for an integer n:

$$x_{n+1} = \lambda f(x_n) = \lambda x_n(1 - x_n), \tag{6.11.13}$$

where $x_n \in [0,1]$ and $\lambda \in (0,4]$. It is to be noted that the logistic map illustrates general features of nonlinear maps. This map is one of the simple models for the growth of the population of a single species. With an initial population x_0, equation (6.11.13) represents a measure of the population in subsequent generations. The state $x_n = 0$ gives the complete absence of the species, and when $x_n \ll 1$, $x_{n+1} \sim \lambda\, x_n$ so that the generation grows by a factor of λ. If x_n is not small, the term $(1-x_n)$ is not close to one and the full nonlinear equation (6.11.13) determines the size of the next generation. If $x_0 \in [0,1]$, then $0 \le x_n \le 1$ for $n \ge 0$. Thus, the interval $[0,1]$ is the physically meaningful range of the map. The fixed points of (6.11.13) satisfy $x_{n+1} = x_n$ so that $x_n = \lambda x_n(1 - x_n)$. So, it has two fixed points at $x = 0$ and $x = (\frac{\lambda-1}{\lambda})$. We next determine whether there are any solutions of period 2 (or 2-cycles) so that they satisfy $x_{n+1} = \lambda x_n(1 - x_n)$ and $x_{n+2} = x_n = \lambda x_{n+1}(1 - x_{n+1})$. Thus, the elimination of x_{n+1} leads to the equation for x_n,

$$x_n\left(x_n - \frac{\lambda-1}{\lambda}\right)\left[\lambda^2 x_n^2 - \lambda(1+\lambda)x_n + (1+\lambda)\right] = 0. \tag{6.11.14}$$

This equation must have two fixed points. The discriminant of the quadratic factor in (6.11.14) is $\lambda^2(\lambda+1)(\lambda-3)$ which is positive for $\lambda > 3$. So, there are fixed points of period 2 (or 2-cycle) for $\lambda > 3$. In fact, it can be shown that the nontrivial fixed point is stable for $\lambda < 3$, but loses stability in a bifurcation at $\lambda = 3$, where the points of period 2 emerge and are stable. As λ increases, the points of period 2 eventually lose stability, and a stable of 4-cycle emerges. This process is known as *period-doubling*, and as λ increases, it eventually leads to *chaotic solutions*. This phenomenon of period-doubling has become famous in recent years. It is possible to find the successive values $\lambda_1, \lambda_2, \lambda_3, \ldots$ of the growth rate parameter at which a bifurcation or qualitative change in the iteration $x_{n+1} = \lambda x_n(1 - x_n)$ occurs as the value of λ increased further. These are the discrete values of λ at which any sufficiently small increase in λ doubles the period of the iteration. This also leads to a certain order underlying this period-doubling process toward chaos with the same *Feigenbaum constant* F in (6.11.12).

This map takes its name from the corresponding differential equation

$$\frac{dx}{dt} = \mu x(1 - x), \quad \mu > 0. \tag{6.11.15}$$

This equation was originally suggested by Pierre-Francois Verhulst (1804–1849) in the study of population dynamics. It is worth noting that there are two equilibrium (or fixed) points of (6.11.15), one at $x = 0$ which is unstable, and another at $x = 1$ which is stable. However, the behavior of (6.11.13) which is the difference equation version of differential equation (6.11.15) is quite different.

6.12 Exercises

1. If $h(x,t)$ is the depth of the water level above impermeable rock and if the horizontal seepage velocity u and the slope of the water level satisfy the Darcy

law, $u = -\kappa\, h_x$, where κ is a constant, show that the mass conservation equation is given by

$$h_t = \kappa h_x^2 + \kappa h h_{xx}.$$

2. In Example 6.3.2, the discontinuous initial data can be approximated by continuous data in the form

$$\rho(x,0) = f(x) = \begin{cases} 200 & \text{if } x < -\varepsilon, \\ 100(1 - \frac{x}{\varepsilon}) & \text{if } -\varepsilon < x < \varepsilon, \\ 0 & \text{if } x > \varepsilon. \end{cases}$$

Find the solution for $\rho(x,t) = f(\xi)$, with the value of x in terms of ξ and t. Express the solution ρ in the terms of x and t, and draw the graph of $\rho(x,0)$.

3. (a) Solve the traffic flow equation (6.3.4) with the flow $q(\rho) = \frac{3}{10}(200 - \rho)\rho$, and with data representing a triangular hump whose center is at the origin given by

$$\rho(x,0) = f(x) = \begin{cases} 25(3 - |x|), & \text{if } |x| \le 1, \\ 50 & \text{if } |x| > 1. \end{cases}$$

(b) Examine the time evolution of the density profile at three different times $t = \frac{1}{15}(1-\varepsilon)$, $\frac{1}{15}$, and $\frac{1}{15}(1+\varepsilon)$, for $0 < \varepsilon < 1$. Show that the solution breaks down for $t = \frac{1}{15}$. Draw the graph of the solution $\rho(x, \frac{1}{15})$.

4. Show that any solution of the traffic flow equation

$$\rho_t + c_0 \rho_x = a\rho,$$

where c_0 and a are constants, has the form $\rho(x,t) = \exp(at) f(x - c_0 t)$.

5. Find the solution of the traffic flow model

$$\rho_t + c(\rho)\rho_x = 0,$$

where $c(\rho) = 60 - \frac{3}{5}\rho$, with initial data

$$\rho(x,0) = \begin{cases} 150\{1 + \frac{1}{5}(1 - |x|)\} & \text{if } |x| \le 1, \\ 150 & \text{if } |x| > 1 \end{cases}$$

6. Obtain the solution of the traffic flow equation

$$\rho_t + c(\rho)\rho_x = 0, \qquad \rho(x,0) = 50 H(-x),$$

where $c(\rho) = 60 + \frac{3}{5}\rho$.

7. For a traffic flow model with flow $q = A\rho(\rho_m - \rho) - \kappa\rho_x$, where A, ρ_m, and κ are constants, show that the traffic density satisfies the equation

$$\rho_t + A\big[\rho(\rho_m - \rho)\big]_x = \kappa \rho_{xx}.$$

Using the nondimensional variables

$$x^* = \ell^{-1} x, \qquad t^* = \left(\frac{\ell}{A\rho_m}\right)^{-1} t, \quad \text{and} \quad u = \left(\frac{1}{2}\rho_m\right)^{-1}\left(\frac{1}{2}\rho_m - \rho\right),$$

where ℓ is a characteristic length scale associated with the initial data, show that the Burgers equation, dropping the asterisks, is given by

$$u_t + uu_x = vu_{xx},$$

where $v = (\frac{\kappa}{A\rho_m})/\ell$ is the dimensionless parameter which is the ratio of the *diffusion length* and the characteristic length ℓ.

8. For a traffic flow model with the quadratic flow $q(\rho)$in the region $0 < \rho < \rho_m$ and zero otherwise, that is,

$$\frac{q}{q_m} = \frac{\rho}{\rho_m}\left(1 - \frac{\rho}{\rho_m}\right), \qquad 0 < \rho < \rho_m,$$

where ρ_m is the maximum traffic density, show that $\rho(x,t)$ satisfies the equation

$$\rho_t + c(\rho)\rho_x = 0,$$

where $c(\rho) = u_m(1 - \frac{2\rho}{\rho_m})$ and $u_m = \frac{q_m}{\rho_m}$.

9. The concentration $C(x,t)$ of a cloud of uniform sedimenting particles in a viscous fluid satisfies the equation (Kynch 1952)

$$\frac{\partial C}{\partial t} + v_0 \frac{\partial}{\partial x}\big[C(1 - \alpha C)\big] = 0,$$

where v_0 and α are constants. If the bottom at $x = 0$ is impermeable and the initial concentration is C_0 for all $x < 0$, where C_0 is a constant, discuss the changes of $C(x,t)$ for all $x < 0$ and $t > 0$.

10. Apply the method of characteristics to solve the system of equations

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with the initial data

$$u(x,0) = e^x \quad \text{and} \quad v(x,0) = e^{-x}.$$

Show that the Riemann invariants are $2r(\xi) = 2\cosh\xi$ and $2s(\eta) = 2\sinh\eta$. Hence, show that the solutions are

$$(u,v) = \cosh(x-t) \pm \sinh(x+t).$$

11. The shallow water equations are

$$\begin{aligned} u_t + uu_x + gh_x &= 0, \\ h_t + uh_x + hu_x &= 0, \end{aligned}$$

where $u = u(x,t)$ is the fluid velocity and $h = h(x,t)$ is the height of the free surface above the horizontal bottom. If $u = u(h)$, show that

$$\frac{\partial u}{\partial h} = \pm\sqrt{\frac{g}{h}}, \quad u = \pm 2(c - c_0),$$

where $c = \sqrt{gh}$, $c_0 = \sqrt{gh_0}$, and h_0 is the equilibrium value of h. Find the Riemann invariants of this system.

12. For one-dimensional anisentropic flow, the Euler equations of motion are

$$\begin{aligned} p_t + up_x + c^2\rho u_x &= 0, \\ u_t + uu_x + \frac{1}{\rho}p_x &= 0, \\ S_t + uS_x &= 0, \end{aligned}$$

where $p = f(\rho, S)$, u is the velocity, ρ is the density, p is the pressure, and S is the entropy. Show that this system has three families of characteristics given by

$$C_0 : \frac{dx}{dt} = u, \qquad C_+ : \frac{dx}{dt} = u + c, \quad \text{and} \quad C_- : \frac{dx}{dt} = u - c,$$

where $c^2 = (\frac{\partial p}{\partial \rho})_x = \text{const}$.

13. Show that the second-order equation

$$a\phi_{tt} + 2b\phi_{xt} + c\phi_{xx} = 0,$$

where a, b, c are real functions of ϕ_x and ϕ_t, can be reduced to the system in the matrix form (6.9.31) where $A = \begin{pmatrix} \frac{2b}{a} & \frac{c}{a} \\ -1 & 0 \end{pmatrix}$.

Show that eigenvalues λ are the roots of the quadratic equation $a\,\lambda^2 - 2b\lambda + c = 0$. Hence, show that the left eigenvectors are

$$L_{1,2} = (1, \lambda_{2,1}),$$

where the indices 1 and 2 in L correspond to the indices 2 and 1 in λ, respectively.

14. If a, b, and c are constants and $a^2 = 4b$, then the linear telegraph equation

$$u_{tt} - c^2 u_{xx} + au_t + bu = 0$$

can be factorized into two first-order equations. Discuss the characteristics of the two first-order equations.

15. An observer at a position $x = x(t)$ moves with the traffic. Show that the rate of the change of the traffic density measured by the observer is given by

$$\frac{d\rho}{dt} = \left(u - \frac{dq}{d\rho}\right)\frac{\partial \rho}{\partial x},$$

where u is the velocity of the car.

16. Show that equations (6.8.106) and (6.8.107) can be written as

$$U_t + A\,U_x = B,$$

where

$$U = \begin{pmatrix} u \\ c \end{pmatrix}, \qquad A = \begin{pmatrix} u & 2c \\ \frac{c}{2} & u \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} H_x \\ 0 \end{pmatrix}.$$

Show that the eigenvalues are $\lambda_{1,2} = u \pm c$ and the corresponding left eigenvectors are $L_1 = (1, 2)$ and $L_2 = (1, -2)$. Derive the equations for the directional derivatives of $(u \pm 2c)$ along the $C_{\pm}$ characteristics. Hence, find these equations when $H_x = gh = \text{const.} = m$.

17. Show that the nonlinear wave equation

$$c^2(\phi_x)\phi_{xx} - \phi_{tt} = 0,$$

where c is an even function of ϕ_x, can be written as

$$U_t + A\,U_x = 0,$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \phi_x \\ \phi_t \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}, \quad \text{and} \quad c = c(u).$$

Hence, show that the eigenvalues $\lambda_{1,2}$ and the corresponding left eigenvectors $L_{1,2}$ are given by

$$\lambda_{1,2} = \mp c \quad \text{and} \quad L_{1,2} = (c, \pm 1).$$

Show that the Riemann invariants are $r(\eta)$ and $s(\xi)$ given by $m(u) + v = r(\eta)$ along $\frac{dx}{dt} = -c$, and $m(u) - v = s(\xi)$ along $\frac{dx}{dt} = +c$, where $m(u)$ is defined by $m(u) = \int c(u)\,du$.

18. Express (Jeffrey and Taniuti 1964) equations (6.8.16) and (6.8.17) in the matrix form

$$U_t + AU_x = 0,$$

where U and A are given by (6.8.19). Show that the eigenvalues $\lambda_{1,2}$ and the corresponding left eigenvectors are given by $\lambda_{1,2} = u \pm c$ and $L_{1,2} = (\frac{c}{\rho}, \pm 1)$. Hence, show that

$$u + m(\rho) = r(\eta) \quad \text{along } C_+ : \frac{dx}{dt} = u + c,$$

$$u - m(\rho) = -s(\xi) \quad \text{along } C_- : \frac{dx}{dt} = u - c,$$

where $m(\rho)$ is defined by

$$m(\rho) = \int \frac{c(\rho)}{\rho}\,d\rho.$$

Discuss the simple wave solution and discontinuity at the cusp of the envelope of characteristics.

19. Using Example 6.8.3 in a fluid of constant depth ($H_x = 0$), show that the Riemann invariants r and s are

$$u + 2c = r(\eta) \quad \text{along characteristics } C_+ : \frac{dx}{dt} = u + c,$$
$$u - 2c = s(\xi) \quad \text{along characteristics } C_- : \frac{dx}{dt} = u - c.$$

Hence, derive

$$u = \frac{1}{2}(r+s), \qquad c = \frac{1}{4}(r-s),$$
$$\frac{dx}{d\xi} = (u+c)\frac{dt}{d\xi}, \qquad \frac{dx}{d\eta} = (u-c)\frac{dt}{d\eta},$$
$$C_+ : \frac{\partial x}{\partial s} = \frac{1}{4}(3r+s)\frac{\partial t}{\partial s} \quad \text{and} \quad C_- : \frac{\partial x}{\partial r} = \frac{1}{4}(r+3s)\frac{\partial t}{\partial r}.$$

Use the last two results to obtain the *Euler–Poisson–Darboux* equation for $t(r, s)$ in the form

$$2(r-s)t_{rs} + 3(t_s - t_r) = 0.$$

20. Use Exercise 16 to show that equations (6.8.106) and (6.8.107) can be written in the conservation form

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ c^2 \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \frac{1}{2}u^2 + c^2 - H(x) \\ u\,c^2 \end{pmatrix} = 0.$$

21. Show that the Tricomi equation

$$u_{xx} + x u_{yy} = 0$$

has the matrix form

$$U_x + AU_y = 0,$$

where

$$u = \begin{pmatrix} v \\ w \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & x \\ -1 & 0 \end{pmatrix}.$$

Solve the eigenvalue equation $|A - \lambda I| = 0$ to show that the Tricomi equation is hyperbolic, parabolic, or elliptic accordingly as $x < 0$, $x = 0$, or $x > 0$.

22. The equations of motion for the unsteady isentropic compressible fluid flows are

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho\,\mathbf{u}) = 0$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = 0,$$

where $\mathbf{u} = (u, v, w)$ is the flow field, $p = p(\rho)$, and $c^2 = \frac{dp}{d\rho}$ is the square of the local sound speed. Express these equations in the matrix form

$$U_t + AU_x + BU_y + CU_z = 0,$$

where

$$U = \begin{pmatrix} \rho \\ u \\ v \\ w \end{pmatrix}, \qquad A = \begin{pmatrix} u & \rho & 0 & 0 \\ c^2/\rho & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix}, \qquad B = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ c^2/\rho & 0 & v & 0 \\ 0 & 0 & 0 & v \end{pmatrix},$$

and

$$C = \begin{pmatrix} w & 0 & 0 & \rho \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ c^2/\rho & 0 & 0 & w \end{pmatrix}.$$

Show that the equation of motion has the conservation form

$$\frac{\partial F}{\partial t} + \operatorname{div} G = 0,$$

where

$$F = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} \rho u & \rho v & \rho w \\ p + \rho u^2 & \rho uv & \rho uw \\ \rho uv & p + \rho v^2 & \rho vw \\ \rho uw & \rho vw & p + \rho w^2 \end{pmatrix}.$$

23. In the one-dimensional case of the system of equations in Exercise 22, show that the matrix form of the system is

$$U_t + A\,U_x = 0,$$

where

$$U = \begin{pmatrix} \rho \\ u \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} u & \rho \\ c^2/\rho & u \end{pmatrix}.$$

24. Reduce the second-order nonlinear wave equation for the displacement $y(x,t)$ of a string

$$y_{tt} = c^2 \left(1 + y_x^2\right)^{-2} y_{xx} \qquad \left(c^2 = T^*/\rho\right)$$

to the first-order system of equations

$$u_t = c^2 \left(1 + v^2\right)^{-2} v_x, \qquad v_t = u_x,$$

where $u = y_t$ and $v = y_x$.
Express this system in the matrix form

$$U_t + AU_x = 0,$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & -c^2(1+v^2)^{-2} \\ -1 & 0 \end{pmatrix}.$$

25. Show that the nonuniqueness of the method of reduction can be seen for the equation
$$au_{tt} + cu_{xx} + u = 0$$
to the system
$$U_t + AU_x + B = 0,$$
when the two following equivalent substitutions are used:
$$\text{(i) } v = u_t, \quad w = u_x \quad \text{and} \quad \text{(ii) } v = u_t, \quad w = u_x + u_x.$$
Show also that the equation $|A - \lambda I| = 0$ gives in both cases the genuine eigenvalues $\lambda = \pm\sqrt{-c/a}$, for (i) the redundant eigenvalue $\lambda = 1$, and for (ii) the redundant eigenvalue $\lambda = \frac{1}{2}$.

26. Traffic velocity is related to the traffic density by $u = u(\rho) = u_m \exp(-\rho/\rho_0)$, where u_m is the maximum traffic velocity and ρ_0 is a reference traffic density. Solve the initial-value problem
$$\rho_t + c(\rho)\rho_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$\rho(x, 0) = \begin{cases} \rho_0 & \text{if } x < 0, \\ \rho_0(a - x) & \text{if } 0 < x < a, \\ 0 & \text{if } x > a, \end{cases}$$
where $c(\rho) = q'(\rho) = \frac{d}{d\rho}[\rho\, u(\rho)]$. Plot x against (ρ/ρ_0) for $0 < \frac{\rho}{\rho_0} < a$ and a fixed t.

27. For the one-dimensional case of the system of equations in Exercise 22, show that
(a) the scalar conservation form of the system is
$$(\rho u)_t + \left(\rho u^2 + p\right)_x = 0,$$
(b) the matrix conservation form is
$$U_t + \left[F(U)\right]_x = 0,$$
where
$$U = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} \quad \text{and} \quad F(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \end{bmatrix}.$$
(c) If $\rho = \rho(u)$, the equations in Exercise 22 in 1D can be written in the form
$$\frac{d\rho}{du}\frac{\partial u}{\partial t} + \left(\rho + u\frac{\partial \rho}{\partial v}\right)\frac{\partial u}{\partial x} = 0,$$
$$\frac{\partial u}{\partial t} + \left(u + \frac{c^2(\rho)}{\rho}\frac{d\rho}{du}\right)\frac{\partial u}{\partial x} = 0.$$

28. Use $u_t = v$, $u_x = w$ to transform the Tricomi equation

$$xu_{tt} + u_{xx} = 0,$$

in the matrix form $U_t + AU_x = 0$, where

$$U = \begin{bmatrix} v \\ w \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & \frac{1}{x} \\ -1 & 0 \end{bmatrix}.$$

29. The equations for long wave approximation for nonlinear shallow water waves are

$$h_t + uh_x + hu_x = 0 \quad \text{and} \quad u_t + uu_x + gh_x = 0,$$

where h is the depth of water, and u is the horizontal fluid velocity and g is the acceleration due to gravity. Express these equations in the conservation form

$$(uh)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0.$$

7

Nonlinear Dispersive Waves and Whitham's Equations

The tool which serves as intermediary between theory and practice, between thought and observation, is mathematics; it is mathematics which builds the linking bridges and gives the ever more reliable forms. From this it has come about that our entire contemporary culture, in as much as it is based on the intellectual penetration and the exploitation of nature, has its foundations in mathematics. Already Galileo said: one can understand nature only when one has learned the language and the signs in which it speaks to us; but this language is mathematics and these signs are mathematical figures... Without mathematics, the astronomy and physics of today would be impossible; these sciences, in their theoretical branches, virtually dissolve into mathematics.

David Hilbert

7.1 Introduction

Historically, the study of nonlinear dispersive waves started with the pioneering work of Stokes in (1847) on water waves. Stokes first proved the existence of periodic wavetrains which are possible in nonlinear dispersive wave systems. He also determined that the dispersion relation on the amplitude produces significant qualitative changes in the behavior of nonlinear waves. It also introduces many new phenomena in the theory of dispersive waves, not merely the correction of linear results. These fundamental ideas and the results of Stokes have provided a tremendous impact on the subject of nonlinear water waves, in particular, and on nonlinear dispersive wave phenomena, in general. Stokes' profound investigations on water waves can be considered as the starting point for the modern theory of nonlinear dispersive waves. In fact, most of the fundamental concepts and results on nonlinear dispersive waves originated in the investigation of water waves. The study of nonlinear dispersive waves has proceeded at a very rapid pace with remarkable developments over the past three decades.

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This chapter is devoted to a general treatment of linear and nonlinear dispersive waves. The initial-value problems of linear dispersive waves and their asymptotic analysis are briefly described. Included are Whitham's theory of nonlinear dispersive waves, Whitham's averaged variational principle, Whitham's equation, the peaking and breaking of waves, and Whitham's nonlinear instability analysis and its application to water waves.

7.2 Linear Dispersive Waves

We consider a dynamical problem so that a small disturbance $\phi(x,t)$ with reference to the undisturbed stable state is governed by a linear partial differential equation with constant coefficients:

$$P\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)\phi(\mathbf{x},t)=F(\mathbf{x},t), \tag{7.2.1}$$

where P is a polynomial, and t and $\mathbf{x}=(x_1,x_2,x_3)=(x,y,z)$ are independent time and space variables. The term $F(\mathbf{x},t)$ represents the action of external forces on the dynamical system and is usually referred to as a given *forcing term.*

We seek a plane wave solution of the homogeneous equation

$$P\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x_i}\right)\phi(\mathbf{x},t)=0 \tag{7.2.2}$$

in the form

$$\phi(\mathbf{x},t)=A\exp\big[i(\boldsymbol{\kappa}\cdot\mathbf{x}-\omega t)\big], \tag{7.2.3}$$

where A is the amplitude, $\boldsymbol{\kappa}=(k_1,k_2,k_3)\equiv(k,l,m)$ is the wavenumber vector, and ω is the frequency. Such a solution exists provided that the dispersion relation

$$P(-i\omega,ik_1,ik_2,ik_3)\equiv D(\omega,\boldsymbol{\kappa})=0 \tag{7.2.4}$$

is satisfied.

Thus, there exists a direct relationship between the governing equation and the dispersion relation through the obvious correspondence

$$\frac{\partial}{\partial t}\leftrightarrow -i\omega,\qquad \frac{\partial}{\partial x_j}\leftrightarrow ik_j. \tag{7.2.5ab}$$

This correspondence allows us to obtain the dispersion relation from the governing equation (7.2.2), and conversely, equation (7.2.2) can be constructed from the given dispersion relation (7.2.4).

If, in particular, the dispersion relation (7.2.4) can be solved explicitly in terms of real roots given by

$$\omega=\Omega(\boldsymbol{\kappa})=\Omega(k_i), \tag{7.2.6}$$

then there may be several possible roots for the frequency, corresponding to different modes of propagation.

The phase velocity of the waves is the velocity at which a surface of constant phase moves. It is defined by the relation

$$\mathbf{c} = \frac{\omega}{\kappa}\widehat{\boldsymbol{\kappa}}, \tag{7.2.7}$$

where $\widehat{\boldsymbol{\kappa}}$ is the unit vector in the direction of the wavenumber vector $\boldsymbol{\kappa}$.

Thus, for any particular wave mode $\omega = \Omega(\boldsymbol{\kappa})$, the phase velocity depends on the wavenumbers k_1, k_2, and k_3. In other words, different waves propagate with different phase velocities, and such waves are called *dispersive* provided the determinant $|\partial^2\Omega/\partial k_i \partial k_j| \neq 0$. On the other hand, waves are called *nondispersive* if the phase velocity does not depend on the wavenumber. For one-dimensional dispersive waves, the determinant reduces to $\Omega''(k) \neq 0$. However, in general, the governing dispersion relation for one-dimensional waves is $\omega = \Omega(k)$, which may give a complex ω for a real k. In such a case, the phase velocity depends not only on wavenumber, but also on the amplitude of the waves. This means that the amplitude will grow or decay in time accordingly as Im $(\omega) > 0$ or < 0. So, the former case leads to instability.

The group velocity vector for the three-dimensional wave motion is defined by the result

$$\mathbf{C}(\boldsymbol{\kappa}) = \nabla_\kappa\, \omega = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right). \tag{7.2.8}$$

For one-dimensional waves, the group velocity $C(k) = (d\omega/dk)$.

It is noted that the group velocity plays a fundamental role in the theory of dispersive waves and is much more important than the phase velocity.

The following are examples of linear dispersive waves.

1. The governing dispersion relation for surface waves on water of constant depth h:

$$\omega^2 = gk \tanh kh. \tag{7.2.9}$$

2. The Korteweg–de Vries (KdV) equation for long water waves:

$$\phi_t + c_0\phi_x + \alpha\phi_{xxx} = 0 \quad \left(c_0^2 = gh\right), \qquad \omega = c_0 k - \alpha k^3. \tag{7.2.10ab}$$

3. The Boussinesq equation for long water waves:

$$\phi_{tt} - c_0^2\phi_{xx} - \beta^2\phi_{xxtt} = 0, \tag{7.2.11a}$$
$$\omega^2 = c_0^2 k^2\left(1 + \beta^2 k^2\right)^{-1}. \tag{7.2.11b}$$

4. The Benjamin–Bona–Mahony (BBM) equation:

$$\phi_t - c_0\phi_x - \alpha\phi_{xxt} = 0, \qquad \omega = -c_0 k\left(1 + \alpha k^2\right)^{-1}. \tag{7.2.12ab}$$

5. The Klein–Gordon equation:

$$\phi_{tt} - c^2\nabla^2\phi + d^2\phi = 0, \qquad \omega = \pm\left(c^2\kappa^2 + d^2\right)^{1/2}. \tag{7.2.13ab}$$

6. The internal waves in a stratified ocean, governed by the equation

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}\right)+N^2\frac{\partial^2\psi}{\partial x^2}=0,$$
$$\omega=Uk\pm\frac{Nk}{\sqrt{k^2+m^2}}, \tag{7.2.14ab}$$

where $\psi(x,z,t)$ represents the stream function and

$$N=\left\{-\frac{g}{\rho_0}\frac{\partial\bar{\rho}}{\partial z}\right\}^{1/2}$$

is a constant *Brunt–Väisälä frequency*.

7. The inertial waves in a rotating liquid, governed by the equation

$$\left(\frac{\partial^2}{\partial t}+U\frac{\partial}{\partial z}\right)\nabla^2\chi+f^2\frac{\partial^2\chi}{\partial z^2}=0\quad(f=2\Omega),\qquad \omega=iUm\pm\frac{fm}{\kappa}. \tag{7.2.15ab}$$

8. The internal-inertial waves in a rotating stratified ocean, governed by the equation

$$\left(\frac{\partial^2}{\partial t^2}+N^2\right)\nabla_h^2\chi+\left(\frac{\partial^2}{\partial t^2}+f^2\right)\chi_{zz}=0,\qquad \omega^2=\frac{f^2m^2+N^2(k^2+\ell^2)}{\kappa^2}, \tag{7.2.16ab}$$

where

$$\nabla_h^2=\partial^2/\partial x^2+\partial^2/\partial y^2.$$

9. The Rossby waves in a β-plane ocean, governed by the equation

$$(\nabla_h^2-a^2)\psi_t+\beta\psi_x=0\quad\left(a^2=\frac{f^2}{c_0^2}\right),\qquad \omega=-\frac{\beta k}{k^2+\ell^2+a^2}. \tag{7.2.17ab}$$

10. The dispersion relation for the Alfvén gravity waves in an electrically conducting liquid of constant depth h:

$$\omega^2=\left(gk+a^2k^2\right)\tanh kh, \tag{7.2.18}$$

where $a=B_0/\sqrt{4\pi\rho}$ is the Alfvén wave velocity.

11. The dispersion relation for electromagnetic waves in plasmas:

$$\omega^2=\omega_p^2+c^2k^2, \tag{7.2.19}$$

where $\omega_p=(4\pi ne^2/m)^{1/2}$ is the plasma frequency, c is the velocity of light, and n is the number of electrons (per unit volume) of mass m and charge $-e$.

12. The Schrödinger equation and de Broglie waves:

$$i\hbar\psi_t=V\psi-\frac{\hbar^2}{2m}\nabla^2\psi,\qquad \omega=\frac{\hbar\kappa^2}{2m}+\frac{V}{\hbar}, \tag{7.2.20ab}$$

where $h(=2\pi\hbar)$ is the Planck constant and V is a constant potential.

The phase and the group velocities associated with these examples can readily be calculated. For dispersive waves, the former are different from the latter.

The general solution for any dispersive wave system can then be obtained by the Fourier superposition of the plane wave solution (7.2.3) for different wavenumbers (k, l, m) with the corresponding frequencies ω to satisfy the dispersion relation (7.2.6). The solution is represented by

$$\phi(\mathbf{x}, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\boldsymbol{\kappa}) \exp\big[i\big\{\boldsymbol{\kappa}\cdot\mathbf{x} - \Omega(\boldsymbol{\kappa})t\big\}\big]\, d\boldsymbol{\kappa}, \tag{7.2.21}$$

where the spectrum function $F(\boldsymbol{\kappa})$ is determined from appropriate initial or boundary conditions.

Although (7.2.21) is the exact solution, the principal features of the dispersive waves cannot be described without an exact or approximate evaluation of the integral in (7.2.21). We shall discuss this point in the next section, which is concerned with initial-value problems.

It is relevant to mention here that, in general, the governing equations for dispersive waves are inherently nonlinear, and the corresponding dispersion relation is also nonlinear in the sense that the frequency ω is not only a function of $\boldsymbol{\kappa}$, but also of other parameters, such as amplitude and local properties of the medium. The theory of water waves provides an excellent example of nonlinear dispersion. In the study of linear dispersive systems, the preceding parameters are assumed to be small, and the governing equations for such systems are obtained by linearizing the original nonlinear equations. The classical theory of water waves satisfying Laplace's equation and the linearized free surface conditions can again be cited as an excellent example of the process of linearization. However, in his pioneering work based upon averaged variational principles, Whitham (1974) gave a new description of nonlinear water waves and a completely different approach to nonlinear dispersive waves, in general. It has become clear from Whitham's theory that nonlinear dispersion is much more important than the corresponding linearized concept and plays a significant role in the general theory of dispersive wave systems.

7.3 Initial-Value Problems and Asymptotic Solutions

We consider the *initial-value problem* of wave propagation in which the disturbance $\phi(\mathbf{x}, t)$ is given by $f(\mathbf{x})$ at time $t = 0$. The general solution for any particular mode of propagation with the known dispersion relation $\omega = \Omega(\boldsymbol{\kappa})$ is given by the three-dimensional integral

$$\phi(x, t) = \int_{-\infty}^{\infty} F(\boldsymbol{\kappa}) \exp\big[i\big\{\boldsymbol{\kappa}\cdot\mathbf{x} - \Omega(\boldsymbol{\kappa})t\big\}\big]\, d\boldsymbol{\kappa}, \tag{7.3.1}$$

where $\int d\boldsymbol{\kappa}$ represents a line, area, or volume integral, depending on the number of dimensions $n = 1, 2$, or 3. At $t = 0$, result (7.3.1) gives

$$f(\mathbf{x}) = \phi(\mathbf{x}, 0) = \int_{-\infty}^{\infty} F(\boldsymbol{\kappa}) \exp\big[i(\boldsymbol{\kappa} \cdot \mathbf{x})\big]\, d\boldsymbol{\kappa}, \tag{7.3.2}$$

where $F(\boldsymbol{\kappa})$ is the generalized Fourier transform of the initial disturbance $f(\mathbf{x})$, so that

$$F(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} f(\mathbf{x}) e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}\, d\mathbf{x}. \tag{7.3.3}$$

Thus, the spectrum function $F(\boldsymbol{\kappa})$ can be determined from the given initial data. Whatever the particular initial-value problem may be, the general solution can always be represented by the integral (7.3.1). The number of wave modes of type (7.3.1) depends on the particular problem. So, the complete solution will be the sum of terms such as (7.3.1) with one integral for each of the solutions $\omega = \Omega(\boldsymbol{\kappa})$.

Note that the physical interpretation of the integral solution (7.3.1) is not at all obvious from its present form. On the other hand, the exact evaluation of (7.3.1) is a difficult task. To investigate the principal features of dispersive waves, it is important to consider the asymptotic evaluation of (7.3.1) for large x and t. It is then necessary to resort to asymptotic methods. We next turn our attention to asymptotic evaluation of (7.3.1) by using the method of stationary phase (Jones 1966). According to this method, the main contribution to the integral (7.3.1) for large t is from the terms associated with the stationary points, if any, where the phase function $\boldsymbol{\kappa} \cdot \mathbf{x} - t\Omega(\boldsymbol{\kappa})$ is stationary, that is, where

$$\frac{\partial \Omega(\boldsymbol{\kappa})}{\partial k_i} = \frac{x_i}{t}. \tag{7.3.4}$$

It can be shown that the dominant contribution to the integral (7.3.1) will come from those values of $k_i = k_i(x_i, t)$ that satisfy (7.3.4). Thus, the asymptotic solution from one such stationary point k_i is obtained in the standard form

$$\phi(\mathbf{x}, t) \sim \frac{(2\pi)^{n/2} F(k_i)}{\{t^n \det |\frac{\partial^2 \Omega}{\partial k_i \partial k_j}|\}^{1/2}} \exp\left[i\left\{\boldsymbol{\kappa} \cdot \mathbf{x} - t\Omega(\boldsymbol{\kappa}) - \frac{\pi}{4} \operatorname{sgn}\left(\frac{\partial^2 \Omega}{\partial k_i^2}\right)\right\}\right], \tag{7.3.5}$$

where the summation convention is used. The complete asymptotic solution for $\phi(\mathbf{x}, t)$ will be the sum of terms like (7.3.5) with one term for each stationary point k_i of equation (7.3.4).

The asymptotic solution (7.3.5) can be expressed in the form of the elementary plane wave solution

$$\phi(\mathbf{x}, t) \sim \operatorname{Re}\big[A(x_i, t) \exp\{i\theta(x_i, t)\}\big], \tag{7.3.6}$$

where Re denotes the real part, and the complex amplitude $A(x_i, t)$ and the phase function $\theta(x_i, t)$ are given by

$$A(x_i, t) = \frac{(2\pi)^{n/2} F(k_i)}{[t^n \det |\frac{\partial^2 \Omega}{\partial k_i \partial k_j}|]^{1/2}} \exp\left[-\frac{i\pi}{4} \operatorname{sgn}\left(\frac{\partial^2 \Omega}{\partial k_i^2}\right)\right], \tag{7.3.7}$$

$$\theta(x_i, t) = \boldsymbol{\kappa} \cdot \mathbf{x} - t\Omega(\boldsymbol{\kappa}), \tag{7.3.8}$$

with the fact that $k_i(x_i, t)$ is the solution of equation (7.3.4).

The asymptotic solution (7.3.6) has the same form as the elementary plane wave solution, and also represents an oscillatory wavetrain. But, in contrast to the elementary solution, (7.3.6) is not a uniform wavetrain because k_i is a function of x_i and t. Moreover, the quantities A and $\omega = \Omega(\kappa)$ involved in (7.3.6) are no longer constants, but are functions of x_i and t.

It follows from (7.3.7) combined with stationary phase equation (7.3.4) that

$$\frac{\partial \theta}{\partial x_i} = k_i + \left\{ x_i - t \frac{\partial \Omega}{\partial k_i} \right\} \frac{\partial k_i}{\partial x_i} = k_i(x_i, t), \tag{7.3.9}$$

$$-\frac{\partial \theta}{\partial t} = \Omega(\kappa_i) - \left\{ x_i - t \frac{\partial \Omega}{\partial k_i} \right\} \frac{\partial k_i}{\partial t} = \Omega(\kappa_i) = \omega(x_i, t). \tag{7.3.10}$$

This means that $\partial\theta/\partial x_i$ and $-\partial\theta/\partial t$ still have the significance of a wavenumber and a frequency. They are no longer constants, but functions of x_i and t. Thus, $k_i = \partial\theta/\partial x_i$ and $\omega = -\partial\theta/\partial t$ represent the *local wavenumber* and the *local frequency* and are still governed by the dispersion relation (7.2.6). The most remarkable difference between the asymptotic solution (7.3.6) and the elementary plane wave solution (7.2.3) is that the former represents an oscillatory nonuniform wavetrain, but the parameters A, k_i, and ω are no longer constants. Indeed, they are slowly varying functions of x_i and t in the sense that $\Delta A \ll A$, $\Delta\kappa \ll \kappa$, and $\Delta\omega \ll \omega$ over a length scale $2\pi/\kappa$ and a time scale $2\pi/\omega$. In other words, the relative changes of these quantities in one wavelength and in one period are very small. This point can readily be verified by using (7.3.4) and (7.3.7).

Finally, the present asymptotic analysis reveals two remarkable consequences concerning the dual role of the group velocity. First, a careful consideration of result (7.3.4) reveals that the local wavenumber propagates with the group velocity $\partial\Omega/\partial k_i$. Second, expression (7.3.7) indicates that $|A|^2$ is an energy-like quantity and also propagates with the group velocity.

7.4 Nonlinear Dispersive Waves and Whitham's Equations

To describe a slowly varying nonlinear and nonuniform oscillatory wavetrain in a dispersive medium, we assume the existence of a one-dimensional solution in the form (7.3.6), so that

$$\phi(x, t) = a(x, t) \exp\{i\theta(x, t)\} + c.c., \tag{7.4.1}$$

where $c.c.$ stands for the complex conjugate and $a(x, t)$ is the complex amplitude given by (7.3.7) with $n = 1$. The phase function $\theta(x, t)$ is given by

$$\theta(x, t) = xk(x, t) - t\omega(x, t), \tag{7.4.2}$$

and k, ω, and a are slowly varying functions of space variable x and time t.

Because of the slow variations of k and ω, it is reasonable to assume that these quantities still satisfy the dispersion relation of the form

$$\omega = W(k). \tag{7.4.3}$$

Differentiating (7.4.2) with respect to x and t, respectively, we obtain

$$\theta_x = k + \{x - tW'(k)\}k_x, \tag{7.4.4}$$

$$\theta_t = -W(k) + \{x - tW'(k)\}k_t. \tag{7.4.5}$$

In the neighborhood of stationary points defined by $W'(k) = (x/t) > 0$, these equations become

$$\theta_x = k(x,t) \quad \text{and} \quad \theta_t = -\omega(x,t). \tag{7.4.6ab}$$

These results can be used as a definition of *local wavenumber* and *local frequency* of a slowly varying nonlinear wavetrain.

In view of (7.4.6ab), relation (7.4.3) gives a nonlinear partial differential equation for the phase $\theta(x,t)$ in the form

$$\frac{\partial \theta}{\partial t} + W\left(\frac{\partial \theta}{\partial x}\right) = 0. \tag{7.4.7}$$

The solution of this equation determines the geometry of the wave pattern.

We eliminate θ from (7.4.6ab) to obtain the equation

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{7.4.8}$$

This is known as the *Whitham equation* for the conservation of waves, where k represents the *density* of waves and ω is the *flux* of waves.

Using the dispersion relation (7.4.3), equation (7.4.8) gives

$$\frac{\partial k}{\partial t} + C(k)\frac{\partial k}{\partial x} = 0, \tag{7.4.9}$$

where $C(k) = W'(k)$ is the group velocity. This represents the simplest nonlinear wave (hyperbolic) equation for the propagation of wavenumber k with group velocity $C(k)$.

Since equation (7.4.9) is similar to (5.2.11), we can use the analysis of Section 5.2 to find the general solution of (7.4.9) with the initial condition $k = f(x)$ at $t = 0$. In this case, the solution takes the form

$$k(x,t) = f(\xi), \qquad x = \xi + tF(\xi), \tag{7.4.10ab}$$

where $F(\xi) = C(f(\xi))$. This further confirms the propagation of wavenumber k with group velocity C. A physical interpretation of this kind of solution has already been discussed in Section 5.2.

Equations (7.4.9) and (7.4.3) reveal that ω also satisfies the first-order, nonlinear wave (hyperbolic) equation

$$\frac{\partial \omega}{\partial t} + W'(k)\frac{\partial \omega}{\partial x} = 0. \tag{7.4.11}$$

It follows from equations (7.4.9) and (7.4.11) that both k and ω remain constant on the characteristic curves defined by

$$\frac{dx}{dt} = W'(k) = C(k) \tag{7.4.12}$$

in the (x,t)-plane. Since k or ω is constant on each characteristic, the characteristics are straight lines with slope $C(k)$. The solution for $k(x,t)$ is given by (7.4.10ab).

Finally, it follows from the preceding analysis that any constant value of the phase ϕ propagates according to $\theta(x,t) = \text{const.}$, and hence,

$$\theta_t + \left(\frac{dx}{dt}\right)\theta_x = 0, \tag{7.4.13}$$

which gives, by (7.4.6ab),

$$\frac{dx}{dt} = -\frac{\theta_t}{\theta_x} = \frac{\omega}{k} = c. \tag{7.4.14}$$

Thus, the phase of the waves propagates with the phase speed c. On the other hand, equation (7.4.9) ensures that the wavenumber k propagates with group velocity $C(k) = (d\omega/dk) = W'(k)$.

We next investigate how wave energy propagates in a dispersive medium. We consider the following integral involving the square of the wave amplitude (energy) between any two points $x = x_1$ and $x = x_2$ $(0 < x_1 < x_2)$:

$$Q(t) = \int_{x_1}^{x_2} |A|^2\, dx = \int_{x_1}^{x_2} AA^*\, dx \tag{7.4.15}$$

$$= 2\pi \int_{x_1}^{x_2} \frac{F(k)F^*(k)}{t|W''(k)|}\, dx, \tag{7.4.16}$$

which, due to a change of variable $x = tW'(k)$, is

$$= 2\pi \int_{k_1}^{k_2} F(k)F^*(k)\, dk, \tag{7.4.17}$$

where $x_r = tW'(k_r)$, $r = 1, 2$.

When k_r is kept fixed as t varies, $Q(t)$ remains constant so that

$$0 = \frac{dQ}{dt} = \frac{d}{dt}\int_{x_1}^{x_2} |A|^2\, dx = \int_{x_1}^{x_2} \frac{\partial}{\partial t}|A|^2\, dx + |A|_2^2 W'(k_2) - |A|_1^2 W'(k_1). \tag{7.4.18}$$

In the limit, as $x_2 - x_1 \to 0$, this result reduces to the first-order, partial differential equation

$$\frac{\partial}{\partial t}|A|^2 + \frac{\partial}{\partial t}\left[W'(k)|A|^2\right] = 0. \tag{7.4.19}$$

This represents the equation for the conservation of wave energy where $|A|^2$ and $|A|^2 W'(k)$ are the energy density and energy flux, respectively. It also follows that the energy propagates with group velocity $W'(k)$. It has been shown that the wavenumber k also propagates with the group velocity. Thus, the group velocity plays a double role.

The preceding analysis reveals another important fact that (7.4.3), (7.4.8), and (7.4.19) constitute a closed set of equations for the three functions k, ω, and A. Indeed, these are the fundamental equations for nonlinear dispersive waves and are known as *Whitham's equations*.

7.5 Whitham's Theory of Nonlinear Dispersive Waves

The theory of linear dispersive waves is essentially based on the Fourier superposition principle. However, in nonlinear problems, the superposition principle is no longer applicable to construct a more general solution for a wavetrain. It follows from the asymptotic solution (7.3.6) that the form of the elementary solution $\phi = A\exp(i\theta)$, $\theta = \boldsymbol{\kappa}\cdot\mathbf{x} - \omega t$, can still be used to describe nonlinear wavetrains provided that A, ω, and $\boldsymbol{\kappa}$ are no longer constants but slowly varying functions of $\mathbf{x}$ and t corresponding to the slow modulation of the wavetrains. For slowly varying wavetrains, there exists a phase function $\theta(\mathbf{x},t)$ so that the local frequency $\omega(\mathbf{x},t)$ and the wavenumber vector $\boldsymbol{\kappa}(\mathbf{x},t) = k_i$ are defined in terms of θ by

$$\omega(\mathbf{x},t) = -\theta_t, \qquad k_i = \theta_{x_1}. \tag{7.5.1ab}$$

In view of the slow variation of k_i and ω, it seems reasonable to assume the linear dispersion relation in the form

$$\omega = \Omega(k_i). \tag{7.5.2}$$

We note that, for nonlinear wavetrains, Ω may also depend on amplitude and other parameters.

Elimination of θ from (7.5.1ab) immediately gives

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x} = 0, \tag{7.5.3a}$$

$$\frac{\partial k_i}{\partial x_j} - \frac{\partial k_j}{\partial x_i} = 0, \tag{7.5.3b}$$

where the former is a relationship between the wavenumbers and frequency. This can be regarded as an equation of continuity for the phase.

Substitution of (7.5.2) into (7.5.3a) and use of (7.5.3b) imply

$$\frac{\partial k_i}{\partial t} + C_j(k)\frac{\partial k_i}{\partial x_j} = -\frac{\partial \Omega}{\partial x_i}, \tag{7.5.4}$$

where $C_j = C_j(\mathbf{k})$ is called the *three-dimensional group velocity*, defined by

$$C_j = \frac{\partial \Omega}{\partial k_j}. \tag{7.5.5}$$

This represents the propagation velocity for the wavenumber k_i.

An observer traveling with the local group velocity $C_j(k)$ of the wavetrain moves along a path in (x_i, t)-space known as a *ray*. From the existence of the phase function $\theta(x_i, t)$ and the dispersion relation (7.5.2), it turns out that changes along rays are given by

$$\frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}, \tag{7.5.6a}$$

$$\frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t}. \tag{7.5.6b}$$

Equation (7.5.4) can also be written in the characteristic form

$$\frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}, \tag{7.5.7a}$$

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i} = C_i. \tag{7.5.7b}$$

These are identical with Hamilton's equations of motion for a particle with position x_i and generalized momentum k_i, where Ω plays the role of the Hamiltonian. If Ω is independent of time, that is, if $\omega = \Omega(k_i, x_i)$, then equation (7.5.6b) implies that the frequency ω remains constant for the wavepacket. This is identical with the important fact that, for a dynamical system with a time-independent Hamiltonian, every motion of the system carries a constant value of the Hamiltonian equal to the total energy. Thus, the frequency behaves like energy and the wavenumber like momentum. This remarkable analogy between classical (particle) mechanics and quantum (wave) mechanics leads to the well-known duality exploited in quantum physics.

In a uniform medium, $\partial\Omega/\partial x_i = 0$, equation (7.5.6a) or (7.5.7a) implies that the wavenumber k_i is constant on each characteristic, and the characteristics are straight lines in the (x_i, t)-space. Also, each k_i propagates along the characteristics with the constant group velocity C_i. Further, if the medium is independent of time t, $\partial\Omega/\partial t = 0$, and then the frequency ω is constant on each characteristic. In a nonuniform medium, equations (7.5.7a) and (7.5.7b) show significant differences. First, k_i is no longer constant, but varies at the rate $-\partial\Omega/\partial x_i$. Second, it propagates along the characteristics with the variable group velocity $C_i(k_i, x_i, t)$, and third, the characteristics are no longer straight lines.

It is interesting to observe that the local dispersion relation (7.5.2) with (7.5.1ab) gives the partial differential equation for phase $\theta(x_i, t)$ in the form

$$\frac{\partial \theta}{\partial t} + \Omega\left(\frac{\partial \theta}{\partial x_i}, x_i, t\right) = 0. \tag{7.5.8}$$

This is the well-known *Hamilton–Jacobi equation*. The solution of this equation determines the geometry (or the kinematics) of the wavetrain.

The preceding kinematic theory provides no indication of changes in the amplitude or energy of a wavetrain or wavepacket. Whitham (1974) initiated further study of the amplitude function (7.3.7) and the related energy density to derive a differential equation for the amplitude and to determine the second role of the group velocity.

For the one-dimensional case, the energy density described in (7.3.7) is proportional to $|A|^2$, so that

$$A^2 = \frac{2\pi F(k)F^*(k)}{t|\Omega''(k)|}, \tag{7.5.9}$$

where $F^*(k)$ is the complex conjugate of $F(k)$.

The amount of energy $Q(t)$ between two points x_1 and x_2 is given by

$$Q(t) = \int_{x_1}^{x_2} g(k)A^2\,dx, \tag{7.5.10}$$

where $g(k)$ is an arbitrary proportionality factor associated with the square of the amplitude and energy.

In a new coordinate system moving with the group velocity, that is, along the rays $x = C(k)t$, equation (7.5.10) reduces to the form

$$Q(t) = 2\pi \int_{x_1}^{x_2} g(k)F(k)F^*(k)\,dk, \tag{7.5.11}$$

where $\Omega''(k) > 0$ and k_1 and k_2 are defined by $x_1 = C(k_1)t$ and $x_2 = C(k_2)t$, respectively.

Using the principle of conservation of energy, that is, stating that the energy between the points x_1 and x_2 traveling with the group velocities $C(k_1)$ and $C(k_2)$ remains invariant, it turns out from (7.5.10) that

$$\begin{aligned}\frac{dQ}{dt} = \int_{x_1}^{x_2} \frac{\partial}{\partial t}\{g(k)A^2\}\,dx + g(k_2)C(k_2)A^2(x_2,t)\\ - g(k_1)C(k_1)A^2(x_1,t) = 0,\end{aligned} \tag{7.5.12}$$

which is, in the limit as $x_2 - x_1 \to 0$,

$$\frac{\partial}{\partial t}\{g(k)A^2\} + \frac{\partial}{\partial t}\{g(k)C(k)A^2\} = 0. \tag{7.5.13}$$

This may be treated as the energy equation. Quantities $g(k)A^2$ and $g(k)C(k)A^2$ represent the *energy density* and the *energy flux*, so that they are proportional to $|A|^2$ and $C(k)|A|^2$, respectively. The flux of energy propagates with the group velocity $C(k)$. Hence, the group velocity has a double role: it is the propagation velocity for the wavenumber k and for the energy $g(k)|A|^2$.

Thus, (7.5.3a) and (7.5.13) are known as Whitham's conservation equations for nonlinear dispersive waves. The former represents the conservation of wave-number k and the latter is the conservation of energy (or more generally, the conservation of wave action). It is important to observe that, even in a uniform medium

$(\partial \Omega/\partial x_i = 0)$, equation (7.5.3a) or its equivalent form given by (7.5.4) is the most fundamental hyperbolic equation, even if the original equation (7.2.1) for ϕ is linear or nonhyperbolic. The conservation equations can be derived more rigorously from a general and extremely powerful approach that is now known as *Whitham's averaged variational principle*.

7.6 Whitham's Averaged Variational Principle

In all dynamical problems where the governing equations admit uniform, periodic, wavetrain solutions, it is generally true that the system can be described by the Hamilton variational principle

$$\delta \iint L(u_t, u_{x_i}, u)\, d\mathbf{x}\, dt = 0, \tag{7.6.1}$$

where L is the Lagrangian and the dependent variable is $u = u(x_i, t)$.

The Euler–Lagrange equation for (7.6.1) is

$$\frac{\partial L_1}{\partial t} + \frac{\partial L_2}{\partial x_i} - L_3 = 0, \tag{7.6.2}$$

where

$$L_1 = \frac{\partial L}{\partial u_t}, \qquad L_2 = \frac{\partial L}{\partial u_{x_i}}, \quad \text{and} \quad L_3 = \frac{\partial L}{\partial u}. \tag{7.6.3abc}$$

Equation (7.6.2) is a second-order, partial differential equation for $u(x_i, t)$. We assume that this equation has periodic wavetrain solutions in the form

$$u = \Phi(\theta), \quad \theta = \boldsymbol{\kappa} \cdot \mathbf{x} - \omega t, \tag{7.6.4ab}$$

where $\boldsymbol{\kappa}$ and ω are constants and represent the wavenumber vector and the frequency, respectively. Since (7.6.2) is a second-order equation, its solution depends on two arbitrary constants of integration. One is the amplitude a and the other is the phase shift. Omitting the latter constant, it turns out that a solution of (7.6.2) exists provided that the three parameters ω, $\boldsymbol{\kappa}$, and a are connected by a dispersion relation

$$D(\omega, \boldsymbol{\kappa}, a) = 0. \tag{7.6.5}$$

In linear problems with a wavetrain solution in the form $u = \Phi(\theta) = ae^{i\theta}$, the dispersion relation (7.6.5) does not involve the amplitude a.

For slowly varying dispersive wavetrains, the solution maintains the elementary form $u = \Phi(\theta, a)$, but ω, $\boldsymbol{\kappa}$, and a are no longer constants, so that θ is not a linear function of x_i and t. The local wavenumber and local frequency are defined by

$$k_i = \frac{\partial \theta}{\partial x_i}, \quad \omega = -\frac{\partial \theta}{\partial t}. \tag{7.6.6ab}$$

The quantities ω, k_i, and a are slowly varying functions of x_i and t corresponding to the slow modulation of the wavetrain.

The *Whitham averaged Lagrangian* over the phase of the integral of the Lagrangian L is defined by

$$\mathcal{L}(\omega, \boldsymbol{\kappa}, a, \mathbf{x}, t) = \frac{1}{2\pi} \int_0^{2\pi} L \, d\theta \tag{7.6.7}$$

and is calculated by assuming the uniform periodic solution $u = \Phi(\theta, a)$ in L. Whitham first formulated the *averaged variational principle* in the form

$$\delta \iint \mathcal{L} \, d\mathbf{x} \, dt = 0, \tag{7.6.8}$$

to derive the equations for ω, $\boldsymbol{\kappa}$, and a.

It is noted that the dependence of $\mathcal{L}$ on $\mathbf{x}$ and t reflects possible nonuniformity of the medium supporting the wave motion. In a uniform medium, $\mathcal{L}$ is independent of $\mathbf{x}$ and t, so that the Whitham function $\mathcal{L} \equiv \mathcal{L}(\omega, \boldsymbol{\kappa}, a)$. However, in a uniform medium, some additional variables also appear only through their derivatives. They represent potentials whose derivatives are important physical quantities.

The Euler equations resulting from the independent variations of δa and $\delta\theta$ in (7.6.8) with $\mathcal{L} = \mathcal{L}(\omega, \boldsymbol{\kappa}, a)$ are

$$\delta a : \mathcal{L}_a(\omega, \boldsymbol{\kappa}, a) = 0, \tag{7.6.9}$$

$$\delta\theta : \frac{\partial}{\partial t}\mathcal{L}_\omega - \frac{\partial}{\partial x_i}\mathcal{L}_{k_i} = 0. \tag{7.6.10}$$

The θ-eliminant of (7.6.6ab) gives the consistency equations

$$\frac{\partial k_i}{\partial t} + \frac{\partial \omega}{\partial x_i} = 0, \qquad \frac{\partial k_i}{\partial x_j} - \frac{\partial k_j}{\partial x_i} = 0. \tag{7.6.11ab}$$

Thus, (7.6.9)–(7.6.11ab) represent the *Whitham equations* for describing the slowly varying wavetrain in a nonuniform medium and constitute a closed set from which the triad ω, $\boldsymbol{\kappa}$, and a can be determined.

In linear problems, the Lagrangian L, in general, is a quadratic in u and its derivatives. Hence, if $\Phi(\theta) = a\cos\theta$ is substituted in (7.6.7), $\mathcal{L}$ must always take the form

$$\mathcal{L}(\omega, \boldsymbol{\kappa}, a) = D(\omega, \boldsymbol{\kappa})a^2, \tag{7.6.12}$$

so that the dispersion relation $(\mathcal{L}_a = 0)$ must take the form

$$D(\omega, \boldsymbol{\kappa}) = 0. \tag{7.6.13}$$

We note that the stationary value of $\mathcal{L}$ is, in fact, zero for linear problems. In the simple case, L equals the difference between kinetic and potential energy. This proves the well-known principle of equipartition of energy, stating that average potential and kinetic energies are equal.

7.7 Whitham's Instability Analysis of Water Waves

Section 7.6 dealt with Whitham's new remarkable variational approach to the theory of slowly varying, nonlinear, dispersive waves. Based upon Whitham's ideas and, especially, Whitham's fundamental dispersion equation (7.6.10), Lighthill (1965, 1967) developed an elegant and remarkably simple result determining whether very gradual—not necessarily small—variations in the properties of a wavetrain are governed by hyperbolic or elliptic partial differential equations. A general account of Lighthill's work with special reference to the instability of wavetrains on deep water was described by Debnath (1994). This section is devoted to the Whitham instability theory with applications to water waves.

According to Whitham's nonlinear theory, $\mathcal{L}_a = 0$ gives a dispersion relation that depends on wave amplitude a and has the form

$$\omega = \omega(\boldsymbol{\kappa}, a), \tag{7.7.1}$$

where equations for $\boldsymbol{\kappa}$ and a are no longer uncoupled and constitute a system of partial differential equations. The first important question is whether these equations are hyperbolic or elliptic. This can be answered by a standard and simple method combined with Whitham's conservation equations (7.5.3a) and (7.5.13). For moderately small amplitudes, we use the Stokes expansion of ω in terms of k and a^2 in the form

$$\omega = \omega_0(k) + \omega_2(k)a^2 + \cdots, \tag{7.7.2}$$

We substitute this result in (7.5.3a) and (7.5.13), replace $\omega'(k)$ by its linear value $\omega_0'(k)$, and retain the terms of order a^2 to obtain the equations for k and a^2 in the form

$$\frac{\partial k}{\partial t} + \left[\omega'(k) + \omega_2'(k)a^2\right]\frac{\partial k}{\partial x} + \omega_2(k)\frac{\partial a^2}{\partial x} = 0, \tag{7.7.3}$$

$$\frac{\partial a^2}{\partial t} + \omega_0'(k)\frac{\partial a^2}{\partial x} + \omega_0''(k)a^2\frac{\partial k}{\partial x} = 0. \tag{7.7.4}$$

Neglecting the term $O(a^2)$, these equations can be rewritten as

$$\frac{\partial k}{\partial t} + \omega_0'\frac{\partial k}{\partial x} + \omega_2\frac{\partial a^2}{\partial x} = 0, \tag{7.7.5}$$

$$\frac{\partial a^2}{\partial t} + \omega_0'(k)\frac{\partial a^2}{\partial x} + \omega_0''a^2\frac{\partial k}{\partial x} = 0. \tag{7.7.6}$$

These describe the modulations of a linear dispersive wavetrain and represent a coupled system due to the nonlinear dispersion relation (7.7.2) exhibiting the dependence of ω on both k and a. In matrix form, these equations read

$$\begin{pmatrix} \omega_0' & \omega_2 \\ \omega_0''a^2 & \omega_0' \end{pmatrix}\begin{pmatrix} \frac{\partial k}{\partial x} \\ \frac{\partial a^2}{\partial x} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial k}{\partial t} \\ \frac{\partial a^2}{\partial t} \end{pmatrix} = 0. \tag{7.7.7}$$

Hence, the eigenvalues λ are the roots of the determinant equation

$$|a_{ij} - \lambda b_{ij}| = \begin{vmatrix} \omega_0' - \lambda & \omega_2 \\ \omega_0'' a^2 & \omega_0' - \lambda \end{vmatrix} = 0, \tag{7.7.8}$$

where a_{ij} and b_{ij} are the coefficient matrices of (7.7.7). This determinant equation gives the characteristic velocities

$$\lambda = \left(\frac{dx}{dt}\right) \equiv C(k) = C_0(k) \pm a\left[\omega_2(k)\omega_0''(k)\right]^{1/2} + O\left(a^2\right), \tag{7.7.9ab}$$

where $C_0(k) = \omega_0'(k)$ is the linear group velocity, and, in general, $\omega_0''(k) \neq 0$ for dispersive waves. The equations are hyperbolic or elliptic depending on whether $\omega_2(k)\omega_0''(k) > 0$ or < 0.

In the hyperbolic case, the characteristics are real, and the double characteristic velocity splits into two separate velocities and provides a generalization of the group velocity of nonlinear dispersive waves. In fact, the characteristic velocities (7.7.9ab) are used to define the *nonlinear group velocities*. The splitting of the double characteristic velocity into two different velocities is one of the most significant results of the Whitham theory. This means that any initial disturbance of finite extent will eventually split into two separate disturbances. This prediction is radically different from that of the linearized theory, where an initial disturbance may suffer from a distortion due to dependence of the linear group velocity $C_0(k) = \omega_0'(k)$ on the wavenumber k, but would never split into two. Another significant consequence of nonlinearity in the hyperbolic case is that compressive modulations will suffer from gradual distortion and steepening in the typical hyperbolic manner discussed in Chapter 6. This leads to the multi-valued solutions, and hence, eventually, breaking of waves.

In the elliptic case $(\omega_2, \omega_0'' < 0)$, the characteristics are imaginary. This leads to *ill-posed problems* in the theory of nonlinear wave propagation. Any small sinusoidal disturbances in a and k may be represented by solutions of the form $\exp[ia\{x - C(k)t\}]$, where $C(k)$ is given by (7.7.9ab) for the unperturbed values of a and k. Thus, when $C(k)$ is complex, these disturbances will grow exponentially in time, and hence, the periodic wavetrains become definitely *unstable*.

An application of this analysis to the Stokes waves on deep water reveals that the associated dispersion equation is elliptic in this case. For waves on deep water, the dispersion relation is

$$\omega = \sqrt{gk}\left(1 + \frac{1}{2}a^2k^2\right) + O\left(a^4\right). \tag{7.7.10}$$

This result is compared with the Stokes expansion (7.7.2) to give $\omega_0(k) = \sqrt{gk}$ and $\omega_2(k) = \frac{1}{2}k^2\sqrt{gk}$. Hence, $\omega_0''\omega_2 < 0$, the velocities (7.7.9ab) are complex, and the Stokes waves on deep water are definitely *unstable*. The instability of deep water waves came as a real surprise to researchers in the field in view of the very long and controversial history of the subject. The question of instability went unrecognized for a long period of time, even though special efforts have been made to prove the existence of a permanent shape for periodic water waves for all amplitudes less than the critical value at which the waves assume a sharp-crested form. However, Lighthill's

(1965) theoretical investigation into the elliptic case and Benjamin and Feir's (1967) theoretical and experimental findings have provided conclusive evidence of the instability of Stokes waves on deep water.

For more details of nonlinear dispersive wave phenomena, the reader is referred to Debnath (1994).

7.8 Whitham's Equation: Peaking and Breaking of Waves

In 1865, Rankine conjectured that there exists a wave of extreme height. In a moving frame of reference, the Euler equations are Galilean invariant, and the Bernoulli equation on the free surface takes the form

$$\frac{1}{2}|\nabla\phi|^2 + g\rho\eta = E$$

which expresses the conservation of local energy. The first term represents the kinetic energy of the fluid, where ϕ is the velocity potential, and the second term is the potential energy due to gravity. For the wave of extreme height, $E = g\rho\eta_{\max}$, where $\eta_{\max}$ is the maximum height of the elevation. Thus, at the maximum height of the fluid, the velocity is zero, and there is a *stagnation point* in the flow. Rankine had conjecture that the free surface formed a cusp at the peak, that is, the tangent lines to the free surface are vertical. Stokes (1847) also suggested that such a wave of extreme height exists and the angle subtended at the peak is $120°$. Toland (1978) proved the existence of a wave of extreme height and showed also that if the singularity at the peak is not a cusp, then Stokes' conjecture about the value of the subtended angle is true. Subsequently, Amick et al. (1982) proved rigorously that the singularity at the peak is not a cusp. However, the full Euler equations of motion exhibit singularities, and also there is a limiting amplitude of the periodic waves, or of the solitary wave. This means that the KdV approximation can only be valid for sufficiently small amplitudes. But the KdV equation has smooth solitary and cnoidal waves of arbitrary amplitude.

In his pioneering work on nonlinear water waves, Whitham (1974) observed that the neglect of dispersion in the nonlinear shallow water equations leads to the development of multi-valued solutions with a vertical slope, and hence, eventually breaking occurs. It seems clear that the third derivative dispersive term in the KdV equation produces the periodic and solitary waves which are not found in the shallow water theory. However, the KdV equation cannot describe the observed symmetrical peaking of the crests with a finite angle. On the other hand, the Stokes waves include full effects of dispersion, but they are limited to small amplitude, and describe neither the solitary waves nor the peaking phenomenon.

Although both breaking and peaking are without doubt involved in the governing equations of the exact potential theory, Whitham (1974) developed a mathematical equation that can include all these phenomena. It has been shown earlier that the breaking of shallow water waves is governed by the nonlinear equation

$$\eta_t + c_0\eta_x + d\eta\eta_x = 0, \quad d = 3c_0(2h_0)^{-1}. \tag{7.8.1}$$

On the other hand, the linear equation corresponding to a general linear dispersion relation

$$\frac{\omega}{k} = c(k) \tag{7.8.2}$$

is given by the integrodifferential equation in the form

$$\eta_t + \int_{-\infty}^{\infty} K(x-s)\eta_s(s,t)\,ds = 0, \tag{7.8.3}$$

where the kernel K is given by the inverse Fourier transform of $c(k)$:

$$K(x) = \mathcal{F}^{-1}\{c(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}c(k)\,dk. \tag{7.8.4}$$

Whitham combined the above ideas to formulate a new class of *nonlinear nonlocal* equations

$$\eta_t + d\eta\eta_x + \int_{-\infty}^{\infty} K(x-s)\eta_s(s,t)\,ds = 0. \tag{7.8.5}$$

This is well known as the *Whitham equation* which can, indeed, describe symmetric waves that propagate without change of shape and peak at a critical height, as well as asymmetric waves that invariably break.

Once a wave breaks, it usually continues to travel in the form of a bore as observed in tidal waves. The weak bores have a smooth but oscillatory structure, whereas the strong bores have a structure similar to turbulence with no coherent oscillations. Since the region where waves break is a zone of high energy dissipation, it is natural to include a second derivative dissipative term in the KdV equation to obtain

$$\eta_t + c_0\eta_x + d\eta\eta_x + \mu\eta_{xxx} - \nu\eta_{xx} = 0, \tag{7.8.6}$$

where $\mu = \frac{1}{6}c_0h_0^2$. This is known as the *KdV–Burgers equation* which also arises in nonlinear acoustics for fluids with gas bubbles (Karpman 1975a).

It is convenient to rewrite the Whitham equation (7.8.5) in the form

$$u_t + uu_x + \int_{-\infty}^{\infty} K(x-s)u_s(s,t)\,ds = 0, \tag{7.8.7}$$

and assume that $u(x,t)$ is the classical solution of (7.8.7) with the property $u \to 0$ as $|x| \to \infty$, and the kernel K (or convolution operator) is symmetric.

We next define an integral operator $\tilde{K}$ by

$$(\tilde{K}u)(x,t) = \int_{-\infty}^{\infty} K(x-s)u(s,t)\,ds, \tag{7.8.8}$$

so that it is bounded and self-adjoint in the Hilbert space $L^2(\mathbb{R})$, i.e., $\langle \tilde{K}u, v\rangle = \langle u, \tilde{K}v\rangle$ defined in Debnath and Mikusinski (1999). Also, this operator $\tilde{K}$ commutes with $\partial_x \equiv \frac{\partial}{\partial x}$, i.e., $\tilde{K}\partial_x = \partial_x\tilde{K}$.

We next show that the Whitham equation admits solitary traveling wave solutions in the form $u(x,t) = f(x - Ut)$, where U is the velocity of the wave. Substituting this into (7.8.7) gives

$$-Uf' + ff' + \tilde{K}_0 f' = 0, \tag{7.8.9}$$

where the operator $\tilde{K}_0$ is associated with the kernel $K_0(x) = \frac{\pi}{4}\exp(-\nu|x|)$, $\nu = \frac{\pi}{2}$. It is easy to verify that, for $v \in L^2(\mathbb{R}) \cap C(\mathbb{R})$,

$$\left(\partial_x^2 - \nu^2\right)\tilde{K}_0 v = -\nu^2 v. \tag{7.8.10}$$

Integrating (7.8.9) with respect to x gives

$$-Uf + \frac{1}{2}f^2 + \tilde{K}_0 f - A = 0, \tag{7.8.11}$$

where A is an integrating constant. Applying the operator $(\partial_x^2 - \nu^2)$ to (7.8.11) and using (7.8.10) yields

$$\left(\partial_x^2 - \nu^2\right)\left(Uf - \frac{1}{2}f^2\right) + \nu^2(f - A) = 0. \tag{7.8.12}$$

Multiplying (7.8.12) by $\partial_x(Uf - \frac{1}{2}f^2) = (U - f)f'$ and integrating with respect to x gives

$$\begin{aligned}(U - f)^2 f'^2 - \nu^2\left[\left(Uf - \frac{1}{2}f^2\right)^2 - Uf^2 + \frac{2}{3}f^3\right]& \\ - 2A\nu^2\left(Uf - \frac{1}{2}f^2\right) = B,&\end{aligned} \tag{7.8.13}$$

where B is also a constant of integration.

We consider the special case where $A = B = 0$ so that (7.8.13) can be put into the form

$$(U - f)^2 f'^2 = \nu^2 f^2 Q(f), \tag{7.8.14}$$

where

$$Q(f) \equiv \frac{1}{4}f^2 - \left(U - \frac{2}{3}\right)f + U^2 - U. \tag{7.8.15}$$

Periodic solutions correspond to oscillations of f between two simple zeros of $Q(f)$. If $1 < U < \frac{3}{4}$, there are two simple zeros f_0 and f_1 where $0 < f_0 < f < f_1$. Consequently, (7.8.14) can be rewritten as

$$\left(\frac{dx}{df}\right)^2 = \frac{(U - f)^2}{\nu^2 f^2 Q(f)} \equiv F^2(f) \quad \text{(say)}, \tag{7.8.16}$$

which can be solved in $0 \le f \le f_0$. Integrating (7.8.16) gives

$$(x - x_0)^2 = h^2(f), \qquad \frac{dh}{df} = F(f), \tag{7.8.17}$$

where x_0 is a constant of integration and $f = f_0$ corresponds to $x = x_0$. Equation (7.8.16) shows that the sign of $(\frac{dx}{df})$ is constant because $0 < f_0 < f < f_1$, and tends to $\pm\infty$ as $f \to 0+$ and $f_0 \to f_0 - 0$. It turns out that the solution f represents a solitary wave.

It is well known that there are infinitely many polynomial conservation laws for the shallow water equation, the KdV equation, the Benjamin–Ono equation, and other integrable equations. These equations are special cases of the Whitham equation. Recently, Benguria and Depassier (1989) have shown that these equations are the only representatives of the Whitham equation that possess this property.

We next consider a few conservation laws for the Whitham equation.

The first conservation law is

$$I_1(u) = \int_{-\infty}^{\infty} u(x,t)\,dx = \text{const.}, \tag{7.8.18}$$

that is,

$$\frac{d}{dt}I_1(u) = \frac{d}{dt}I_1(u_0) = 0, \qquad u(x,0) = u_0(x). \tag{7.8.19ab}$$

This is obtained directly from:

$$\begin{aligned}
\frac{d}{dt}I_1(u) &= \int_{-\infty}^{\infty} u_t(x,t)\,dx = -\int_{-\infty}^{\infty}\left(uu_x + \int_{-\infty}^{\infty} K(x-s)u_s(s,t)\,ds\right)dx \\
&= -\frac{1}{2}\int_{-\infty}^{\infty}\frac{d}{dx}(u^2)\,dx - \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}K(x-s)u_s(s,t)\,ds \\
&= \left[\left(-\frac{1}{2}u^2\right)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}u_s(s,t)\,ds\int_{-\infty}^{\infty}K(x-s)\,dx \\
&= -c_0(0)\big[u(s,t)\big]_{-\infty}^{\infty} = 0,
\end{aligned}$$

where

$$c(0) = \int_{-\infty}^{\infty} K(z)\,dz. \tag{7.8.20}$$

Thus it follows that

$$I_1(u) = I_1(u_0) = \text{const.} \tag{7.8.21}$$

This represents the conservation of mass.

The second conservation law $I_2(u) = \text{const.}$ can be derived by multiplying (7.8.7) by u and then integrating the result with respect to x by parts so that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}I_2(u) &= \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}u^2(x,t)\,dx = -\int_{-\infty}^{\infty}\partial_x\left(\frac{u^3}{3}\right)dx - \int_{-\infty}^{\infty}u\partial_x\tilde{K}u\,dx \\
&= -\langle u, \partial_x\tilde{K}u\rangle = \langle \partial_x u, \tilde{K}u\rangle = \langle u, \tilde{K}\partial_x u\rangle = \langle u, \partial_x\tilde{K}u\rangle = 0.
\end{aligned}$$

This means that $\dot{I}_2(u) = 0$, giving $I_2(u) = I_2(u_0) = \text{const.}$

The third conserved quantity $I_3(u)$ is given by

$$I_3(u) = \int_{-\infty}^{\infty} \left(\frac{1}{3}u^3 + u\tilde{K}u\right) dx. \tag{7.8.22}$$

It turns out that

$$\begin{aligned}
\frac{d}{dt} I_3(t) &= \int_{-\infty}^{\infty} u^2 u_t \, dx + \langle u_t, \tilde{K}u \rangle + \langle u, \tilde{K}u_t \rangle \\
&= 2\langle u_t, \tilde{K}u \rangle - \int_{-\infty}^{\infty} (uu_x + \tilde{K}\partial_x u) u^2 \, dx \\
&= -2\left\langle \partial_x \left(\frac{1}{2}u^2 + \tilde{K}u\right), \tilde{K}u \right\rangle - \left(\frac{u^4}{4}\right)_{-\infty}^{\infty} - \langle u^2, \tilde{K}\partial_x u \rangle \\
&= \langle u^2, \partial_x \tilde{K}u \rangle + 2\langle \tilde{K}u, \partial_x \tilde{K}u \rangle - \langle u^2, \tilde{K}\partial_x u \rangle \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial x} (\tilde{K}u)^2 \, dx = 0.
\end{aligned}$$

This quantity $I_3(u)$ represents the energy. Since it is independent of time t, the Whitham equation (7.8.7) can be written in the Hamiltonian form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{H}}{\partial x}\right) = 0 \tag{7.8.23}$$

where the Hamiltonian is $\mathcal{H} = \frac{1}{2} I_3(u)$.

We next make some important comments on the possible extension of a class of nonlinear equations. First, if we eliminate the requirement that the kernel $K(x)$ is even, the Whitham equation (7.8.7) can not only describe the complete dispersion, but also dissipation processes. This extends the class of nonlinear equation under consideration. Second, another natural extension is to generalize (7.8.7) to include nonlocal equations with an arbitrary pseudodifferential operator so that (7.8.7) can be replaced by the following equation:

$$u_t + uu_x + \mathcal{K}(u) = 0, \tag{7.8.24}$$

where the operator $\mathcal{K}(u)$ is given by

$$\mathcal{K}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} K(k) \widehat{u}(k,t) \, dk, \tag{7.8.25}$$

where $\widehat{u}(k,t)$ is the Fourier transform of $u(x,t)$ defined by

$$\widehat{u}(k,t) = \Im\{u(x,t)\} = \int_{-\infty}^{\infty} e^{ikx} u(x,t) \, dx. \tag{7.8.26}$$

The function $K(k)$ introduced to define the operator $\mathcal{K}(u)$ is called the *symbol of the operator*. The dispersion relation $c(k)$ and the symbol $K(k)$ are related by the result

$$K(k) = ikc(k). \tag{7.8.27}$$

This approach reveals a new fact that the Whitham equation represents, as particular cases, many nonlinear equations which are of special interest for physical applications. Examples of such equations include the Kawahara (1972) equation

$$u_t + uu_x + au_{xxx} - u_{xxxxx} = 0, \tag{7.8.28}$$

where $K(k) = -i(ak^3 + k^5)$. This equation describes propagation of signals in electric transmission lines, long waves under ice cover in liquids of finite depth (Ilichev and Marchenko 1989), and water waves with surface tension (Zufira 1987). Another example is the Kuramoto–Sivashinsky equation

$$u_t + \frac{1}{2}u_x^2 + u_{xx} + au_{xxxx} = 0 \tag{7.8.29}$$

which arises in the theory of combustion to model a flame front and also in two-dimensional turbulence (Kuramoto 1984, and Novick-Cohen and Sivashinsky 1986).

Finally, we close this section by adding some specific comments on peaking and breaking of waves described by the Whitham equation (7.8.7) with the initial condition

$$u(x, 0) = u_0(x). \tag{7.8.30}$$

It has been shown by Whitham (1974) that if $K(x) = \alpha \exp(-\beta|x|)$, where α, $\beta > 0$ are constants, the crests of the waves of limiting amplitude have an angle of $110°$, which is very close to the Stokes angle. Naumkin and Shishmarev (1994) proved a theorem which states that the wave remains peaked for all time as long as a solution exists provided the kernel $K(x)$ in (7.8.7) is continuous and absolutely integrable.

On the other hand, waves described by (7.8.7) and (7.8.30) break in a finite time T provided the kernel $K(x)$ in (7.8.7) is *regular* as shown by Seliger (1968).

The first study of the Cauchy problem (7.8.7) and (7.8.30) for the Whitham equation with a singular kernel $K(x)$ was initiated by Naumkin and Shishmarev (1994). They showed that a wave that is sufficiently steep initially at $t = 0$ breaks in a finite time T provided the kernel $K(x)$ is monotonically increasing in the neighborhood of the singular point $x = 0$ and has a singularity of order $|x|^{-\alpha}$, $0 < \alpha < 1$ as $x \to 0$, and also if the kernel $K(x)$ is monotonically decreasing near $x = 0$ and has a singularity of order $|x|^{-\alpha}$, $0 < \alpha < \frac{2}{3}$. For a monotonically decreasing kernel, the integral term in the Whitham equation (7.8.7) represents dissipative effects, whereas the monotonically increasing kernel corresponds to a process of energy pumping.

Making reference to Chapter 7 of Naumkin and Shishmarev (1994) and Amick et al. (1989), we simply state the asymptotic behavior of a solution $u(x, t)$ of the Cauchy problem for the nonlocal nonlinear Whitham equation, as $t \to \infty$,

$$\begin{cases} u_t + u_x^2 + \mathcal{K}(u) = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad x \in \mathbb{R},\ t \geq 0, \tag{7.8.31}$$

where $\mathcal{K}(u)$ is defined by (7.8.25) and (7.8.26).

The asymptotic behavior of $u(x,t)$ as $t \to \infty$ is given by

$$u(x,t) \sim Ae^{-\lambda t}t^{-1/\delta} \int_0^\infty \cos(k\xi)e^{-\omega k^\delta}\, dk + O\big(e^{-\lambda t}t^{-\frac{1}{\delta}-\mu}\big), \tag{7.8.32}$$

where $\xi = xt^{-1/\delta} \geq 0$, the constant A is expressed through the symbol $K(k)$ and the initial function $u_0(x)$, and all other quantities are defined in Naumkin and Shishmarev (1994, p. 180).

7.9 Exercises

1. (a) For the plane wave solutions $\phi(x,t) = A\exp\{i(kx - \omega t)\}$ of the real partial differential equation

$$\frac{\partial \phi}{\partial t} + iP\left(-i\frac{\partial}{\partial x}\right)\phi = 0,$$

where P is an odd function, show that the dispersion relation is $\omega = P(k)$.
(b) For a wave packet described by $\phi(x,t) = \text{Re}[A(x,t)\psi(x,t)]$, where $A(x,t)$ represents a slowly varying complex amplitude function, and where $\psi(x,t) = \exp[i(k_0x - \omega_0 t)]$ denotes the basic carrier wave of frequency $\omega_0 = P(k_0)$ for a given wavenumber $k_0 \neq 0$, show that the amplitude $A(x,t)$ satisfies the equation

$$i\big[A_t + P'(k_0)A_x\big] + \frac{1}{2}P''(k_0)A_{xx} = 0,$$

where $P(k)$ has a Taylor series expansion about $k = k_0$.

2. The dispersion relation (Whitham 1967a) for water waves in an inviscid fluid of arbitrary depth h_0 is

$$\omega = \omega_0(k) + \Omega_2(k)\frac{k^2E}{\rho c_0} + O\big(E^2\big),$$

where $\omega_0(k) = (gk \tanh kh_0)^{\frac{1}{2}}$, $E = \frac{1}{2}g\rho a^2$, and

$$\Omega_2(k) = \frac{(9T_0 - 10T_0^2 + 9)}{8T_0^3} - \frac{1}{kh_0}\left[1 + \frac{1}{4}\frac{(4C_0 - c_0)^2}{(gh_0 - C_0^2)}\right],$$

where $T_0 = \tanh(kh_0)$, $c_0 = \frac{\omega_0(k)}{k_0}$, and $C_0 = \frac{d\omega_0}{dk_0}$. Show that the characteristic velocities given by (7.7.9ab) are

$$C(k) = C_0(k) \pm \left[\frac{\omega_0''(k)\Omega_2(k)Ek^2}{\rho c_0}\right]^{\frac{1}{2}}, \quad \omega_0''(k) < 0.$$

(a) Explain the significance of the solution for $\Omega_2(k) > 0$ or < 0.
(b) If the critical value for instability is determined by the numerical value of (kh_0) for which $\Omega_2(k) = 0$, show that equations are hyperbolic or elliptic according to whether $kh_0 < 1.363$ or > 1.363.

3. If the Whitham averaged variational principle is given by

$$\delta \iint \mathcal{L}(-\theta_t, \theta_x)\, dt\, dx = 0,$$

where $\omega = -\theta_t$ and $k = \theta_x$, show that the Euler equation is

$$\frac{\partial}{\partial t}\mathcal{L}_\omega - \frac{\partial}{\partial x}\mathcal{L}_k = 0.$$

Hence, or otherwise, show that the phase function θ satisfies the second-order, quasi-linear equation

$$\mathcal{L}_{\omega\omega}\theta_{tt} - 2\mathcal{L}_{\omega k}\theta_{tx} + \mathcal{L}_{kk}\theta_{xx} = 0.$$

4. The Whitham equations for the slow modulation of the wave amplitude a and the wavenumber k in the case of two-dimensional deep water waves are

$$\frac{\partial}{\partial t}\left(\frac{a^2}{\omega_0}\right) + \frac{\partial}{\partial x}\left(C\frac{a^2}{\omega_0}\right) = 0 \quad \text{and} \quad \frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0,$$

where $\omega_0 = \sqrt{gk}$ and $C = \frac{g}{2\omega_0}$ is the group velocity. Using Chu and Mei's (1971) dispersion relation

$$\omega = \omega_0\left[1 + \varepsilon^2\left(\frac{1}{2}a^2k^2 + \left\{\left(\frac{a}{\omega_0}\right)_{tt} \operatorname{div} 2\omega_0 a\right\}\right)\right],$$

derive the following equations for the phase function $\phi(x,t)$, where we have used Chu and Mei's notation $W = -2\phi_x$:

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x}\left(a^2\phi_x\right) = 0 \quad \text{and} \quad -2\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial x}\left[-\phi_x^2 + \frac{a^2}{4} + \frac{a_{xx}}{16a}\right] = 0.$$

Show that the second equation can be integrated with respect to x to obtain

$$\phi_t + \frac{1}{2}\phi_x^2 - \frac{1}{8}a^2 - \frac{a_{xx}}{32a} = 0.$$

Using $\Psi = a\exp(4i\phi)$, show that $\Psi(x,t)$ satisfies the nonlinear Schrödinger equation

$$i\Psi_t + \frac{1}{8}\Psi_{xx} + \frac{1}{2}\Psi|\Psi|^2 = 0.$$

5. Show that Whitham's equation (7.8.7) possess the conserved quantity

$$I_4(u) = \int_{-\infty}^{\infty} xu(x,t)\, dx.$$

6. Show that the concentric KdV equation (see Johnson 1997)

$$u_t + \frac{u}{2t} - 6uu_x + u_{xxx} = 0$$

has the following conservation laws:

(a) $\int_{-\infty}^{\infty} \sqrt{t}u\, dx = \text{const.}$,
(b) $\int_{-\infty}^{\infty} tu^2\, dx = \text{const.}$,
(c) $\int_{-\infty}^{\infty} (\sqrt{t}xu + 6t^{3/2}u^2)\, dx = \text{const.}$

7. Using the shallow water equations (2.7.67)–(2.7.70), derive the conservation law

$$\left(\frac{1}{3}m_3 + hm_1\right)_t + \left(\frac{1}{3}m_4 + hm_2 + \frac{1}{2}m_1^2 + \frac{1}{3}h^3\right)_x = 0,$$

where

$$m_n = \int_0^h u^n\, dz,$$

(see Johnson 1997, Benney 1974, and Miura 1974).

8. Consider the KP equation

$$u_t - 6uu_x + u_{xxx} + 3v_y = 0, \quad u_y = v_x.$$

Under suitable decay conditions, show that the second equation produces three conservation laws:
(a) $\int_{-\infty}^{\infty} u\, dx = \text{const.}$,
(b) $\int_{-\infty}^{\infty} u\, dy = \text{const.}$,
(c) $\int_{-\infty}^{\infty} v\, dx = \text{const.}$

9. Show that the Boussinesq equations (see Hirota 1973b)

$$u_t + h_x - 3\left(h^2\right)_x + h_{xxx} = 0, \qquad h_t + u_x = 0$$

have the conserved quantities
(a) $\int_{-\infty}^{\infty} u\, dt$,
(b) $\int_{-\infty}^{\infty} uh\, dx$, and
(c) $\int_{-\infty}^{\infty} (\sqrt{t}xu + 6t^{3/2}u^2)\, dx$.

10. Introducing the averaged Hamiltonian $\mathcal{H}$ by

$$\mathcal{H} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial L}{\partial u_t}\cdot u_t - L\right) d\theta = \omega J - \mathcal{L},$$

where the wave action J is defined by

$$J = \frac{1}{2\pi\omega}\int_0^{2\pi}\left(\frac{\partial L}{\partial u_t}\cdot u_t\right) d\theta = \frac{\partial \mathcal{L}}{\partial \omega},$$

show that the Whitham equations

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial k}\right) = 0$$

can be expressed in the averaged Hamiltonian form

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{H}}{\partial J}\right) = 0 \quad \text{and} \quad \frac{\partial J}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{H}}{\partial k}\right) = 0.$$

8

Nonlinear Diffusion–Reaction Phenomena

The profound study of nature is the most fertile source of mathematical discoveries.

Joseph Fourier

The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. He should take simplicity into consideration in a subordinate way to beauty.... It often happens that the requirements of simplicity and beauty are the same, but where they clash the latter must take precedence.

Paul Dirac

8.1 Introduction

Many physical phenomena are described by the interaction of convection and diffusion and also by the interaction of diffusion and reaction. From a physical point of view, the convection–diffusion process and the diffusion–reaction process are quite fundamental in describing a wide variety of problems in physical, chemical, biological, and engineering sciences. Some nonlinear partial differential equations that model these processes provide many new insights into the question of interaction of nonlinearity and diffusion. It is well known that the Burgers equation is a simple nonlinear model equation representing phenomena described by a balance between convection and diffusion. The Fisher equation is another simple nonlinear model equation which arises in a wide variety of problems involving diffusion and reaction.

To understand these physical processes, this chapter is devoted to the study of both Burgers and Fisher equations and their different kinds of solutions with physical significance. Special attention is given to diffusive wave solutions and traveling wave solutions of the Burgers and Fisher equations. In addition to the standard mathematical methods used for solving the Burgers and Fisher equations, similarity methods

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are developed to find the similarity solutions of both linear and nonlinear diffusion equations with examples of applications.

8.2 Burgers Equation and the Plane Wave Solution

We recall the differential form of the nonlinear conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{8.2.1}$$

To investigate the nature of the discontinuous solution or shock waves, we assume a functional relation $q = Q(\rho)$ and allow a jump discontinuity for ρ and q. In many physical problems of interest, it would be a better approximation to assume that q is a function of the density gradient ρ_x as well as ρ. A simple model is to take

$$q = Q(\rho) - \nu \rho_x, \tag{8.2.2}$$

where ν is a positive constant. Substituting (8.2.2) into (8.2.1), we obtain the *nonlinear diffusion equation*

$$\rho_t + c(\rho)\rho_x = \nu \rho_{xx}, \tag{8.2.3}$$

where $c(\rho) = Q'(\rho)$.

We multiply (8.2.3) by $c'(\rho)$ to obtain

$$\begin{aligned} c_t + cc_x &= \nu c'(\rho)\rho_{xx} \\ &= \nu\{c_{xx} - c''(\rho)\rho_x^2\}. \end{aligned} \tag{8.2.4}$$

If $Q(\rho)$ is a quadratic function in ρ, then $c(\rho)$ is linear in ρ, and $c''(\rho) = 0$. Consequently, (8.2.4) becomes

$$c_t + cc_x = \nu c_{xx}. \tag{8.2.5}$$

As a simple model of turbulence, c is replaced by the fluid velocity field $u(x,t)$ to obtain the well-known *Burgers equation* as

$$u_t + uu_x = \nu u_{xx}, \tag{8.2.6}$$

where ν is the kinematic viscosity. Thus, the Burgers equation is a balance between time evolution, nonlinearity, and diffusion. This is the simplest nonlinear model equation for diffusive waves in fluid dynamics. Burgers (1948) first developed this equation primarily to shed light on the study of turbulence described by the interaction of the two opposite effects of convection and diffusion. However, turbulence is more complex in the sense that it is both three dimensional and statistically random in nature. Equation (8.2.6) arises in many physical problems including one-dimensional turbulence (where this equation had its origin), sound waves in a viscous medium, shock waves in a viscous medium, waves in fluid-filled viscous elastic tubes, and magnetohydrodynamic waves in a medium with finite electrical conductivity. We note that (8.2.6) is *parabolic*, whereas (8.2.6) with $\nu = 0$ is *hyperbolic*. More importantly, the properties of the solution of the parabolic equation are significantly different than those of the hyperbolic equation.

We first solve (8.2.6) for two simple cases: (i) a linearized Burgers equation and (ii) an equation with the linearized convective term cu_x, where c is a constant.

In the first case, the linearized Burgers equation

$$u_t = \nu u_{xx}, \quad -\infty < x < \infty,\ t > 0, \tag{8.2.7}$$

with the initial conditions at $t = 0$

$$u(x,0) = \begin{cases} u_1 & \text{if } x < 0, \\ u_2 & \text{if } x > 0, \end{cases} \tag{8.2.8ab}$$

can readily be solved by applying the Fourier transform with respect to x. The final solution of (8.2.7) with (8.2.8ab) for $t > 0$ is

$$u(x,t) = \frac{1}{2}(u_1 + u_2) - \frac{1}{2}(u_1 - u_2)\,\mathrm{erf}\left(\frac{x}{2\sqrt{\nu t}}\right). \tag{8.2.9}$$

This shows that the presence of the diffusion term νu_{xx} is to smooth out the initial distribution like $(\nu t)^{-\frac{1}{2}}$. The solution (8.2.9) tends to constant values u_1, as $x \to -\infty$, and u_2, as $x \to +\infty$. The absence of the diffusion term in (8.2.6) leads to gradual nonlinear steepening, and eventually breaking. Indeed, equation (8.2.6) combines the two opposite effects of nonlinearity and diffusion. In the absence of the diffusion term in (8.2.6), the resulting equation reduces to the first-order, nonlinear wave equation which admits a progressively distorted wave profile as a solution. Eventually this solution develops a discontinuity as a shock wave.

In the second case, equation (8.2.6) reduces to the linear parabolic equation

$$u_t + cu_x = \nu u_{xx}. \tag{8.2.10}$$

We seek a plane wave solution of (8.2.10) in the form

$$u(x,t) = a\exp\bigl\{i(kx - \omega t)\bigr\}, \tag{8.2.11}$$

where $\operatorname{Im}\omega = -\nu k^2 < 0$, since $\nu > 0$.

Thus, the solution (8.2.11) becomes

$$u(x,t) = ae^{-\nu k^2 t}\exp\bigl[ik(x - ct)\bigr]. \tag{8.2.12}$$

This represents a diffusive wave with wavenumbers k and phase velocity c. The amplitude of the wave decays exponentially with time t, and the decay time $t_0 = (\nu k^2)^{-1}$ becomes smaller as k increases with fixed ν. Thus, the waves of smaller wavelengths decay faster than the waves of longer wavelengths. On the other hand, for a fixed wavenumber k, the decay time decreases as ν increases so that waves of a given wavelength attenuate faster in a medium with a larger ν. This quantity ν may be regarded as a *measure of diffusion*. Finally, after a sufficiently long time $(t \gg t_0)$, only disturbances of long wavelength will survive, whereas all waves of short wavelength will decay very rapidly. For $\nu < 0$, the solution (8.2.12) tends to infinity as $t \to \infty$, and hence, it becomes unstable.

8.3 Traveling Wave Solutions and Shock-Wave Structure

To investigate the effects of nonlinear steepening and diffusion, we seek a traveling wave solution of the Burgers equation (8.2.6) in the form

$$u(x,t) = u(\xi), \quad \xi = x - Ut, \tag{8.3.1ab}$$

where the wave speed U is to be determined and $u(\xi)$ represents the wave form with the property that it tends asymptotically to constant values u_1, as $\xi \to -\infty$, and u_2, as $\xi \to +\infty$. We assume that $u_1 > u_2$. Substituting (8.3.1ab) into the Burgers equation (8.2.6) gives the ordinary differential equation

$$-Uu'(\xi) + uu'(\xi) - \nu u''(\xi) = 0.$$

Integrating this equation yields

$$-Uu(\xi) + \frac{1}{2}u^2 - \nu u'(\xi) = A,$$

where A is a constant of integration, or equivalently,

$$u'(\xi) = \frac{1}{2\nu}\big(u^2 - 2Uu - 2A\big). \tag{8.3.2}$$

Clearly, this suggests that u_1 and u_2 are the roots of the quadratic equation

$$u^2 - 2Uu - 2A = 0. \tag{8.3.3}$$

Hence, U and A are determined from the sum and the product of the roots u_1 and u_2 of (8.3.3), and therefore,

$$U = \frac{1}{2}(u_1 + u_2), \qquad A = -\frac{1}{2}u_1 u_2. \tag{8.3.4ab}$$

Thus, the wave speed is the *average* of the two speeds of asymptotic states at infinity.

Equation (8.3.2) can now be written as

$$2\nu\frac{du}{d\xi} = u^2 - 2Uu - 2A = (u - u_1)(u - u_2).$$

Integrating this equation gives

$$\left(\frac{\xi}{2\nu}\right) = -\int \frac{du}{(u_1 - u)(u - u_2)} = \frac{du}{(u_1 - u_2)} \log\left(\frac{u_1 - u}{(u - u_2)}\right). \tag{8.3.5}$$

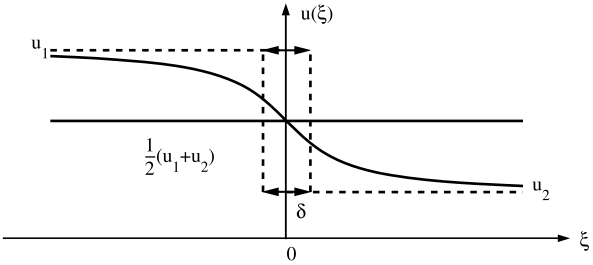

Fig. 8.1 Traveling wave solution $u(\xi)$, $\xi = x - Ut$.

This leads to the solution for $u(\xi)$ in the form

$$u(\xi) = \frac{u_1 + u_2 \exp[(\frac{\xi}{2\nu})(u_1 - u_2)]}{1 + \exp[(\frac{\xi}{2\nu})(u_1 - u_2)]} \tag{8.3.6}$$

$$= u_2 + \frac{(u_1 - u_2)}{1 + \exp[(\frac{\xi}{2\nu})(u_1 - u_2)]}. \tag{8.3.7}$$

Another useful expression for u can be written from (8.3.6) in the form

$$u(\xi) = \frac{1}{2}(u_1 + u_2) + \frac{u_1 + u_2 \exp[(\frac{\xi}{2\nu})(u_1 - u_2)]}{1 + \exp[(\frac{\xi}{2\nu})(u_1 - u_2)]} - \frac{1}{2}(u_1 - u_2)$$

$$= \frac{1}{2}(u_1 + u_2) - \frac{1}{2}(u_1 - u_2)\tanh\left[\left(\frac{\xi}{4\nu}\right)(u_1 - u_2)\right]. \tag{8.3.8}$$

As $u_1 > u_2$, the wave profile $u(\xi)$ decreases monotonically with ξ from the constant value u_1, as $\xi \to -\infty$, to the constant value u_2, as $\xi \to +\infty$, as shown in Figure 8.1. At $\xi = 0$, $u = \frac{1}{2}(u_1 + u_2)$. The shape of the waveform (8.3.8) is significantly affected by the diffusion coefficient ν. This means that the presence of diffusion processes prevents the gradual distortion of the wave profile, and so, it does not break. On the other hand, if the diffusion term is absent $(\nu = 0)$ in (8.2.6), the wave profile will suffer from gradual distortion and steepening, and hence, it would definitely break with development of a shock.

Shock waves are formed as a result of a balance between the steepening effect of the convective (nonlinear) term and the smoothing effect of the linear diffusive terms in the equation of motion. The tendency to steepening has been demonstrated for plane waves by many authors, including Riemann (1858), who first introduced the most general approach.

The upshot of the above analysis is that the convection and diffusion terms in the Burgers equation exhibit opposite effects. The former introduces a sharp discontinuity in the solution profile, whereas the latter tends to diffuse (spread out) the discontinuity into a smooth profile. In view of this property, the solution is called the *diffusive wave*. In the context of fluid flow, ν represents the kinematic viscosity which measures the viscous dissipation.

Multiplying both numerator and denominator of (8.3.6) by $\exp[-\frac{\xi}{2\nu}(u_1-u_2)]$, we write the solution (8.3.6) in the form

$$u(\xi)=\frac{u_2+u_1\exp[-(\frac{\xi}{2\nu})(u_1-u_2)]}{1+\exp[-(\frac{\xi}{2\nu})(u_1-u_2)]}. \tag{8.3.9}$$

The exponential factor in this solution indicates the existence of a thin *transition layer* of thickness δ of the order $\nu/(u_1-u_2)$. This thickness δ can be referred to as the *shock thickness*, which tends to zero as $\nu\to 0$ for fixed u_1 and u_2. Also, δ increases as $u_1\to u_2$ for a fixed ν. If δ is small compared with other typical length scales of the problem, the rapid shock transition can satisfactorily be approximated by a discontinuity. Thus, in the limit as $\nu\to 0$, we might expect that solutions of (8.2.6) tend to solutions of the nonlinear equation

$$u_t+uu_x=0 \tag{8.3.10}$$

together with discontinuous shock waves satisfying the jump condition

$$\frac{1}{2}\left(u_1^2-u_2^2\right)=U(u_1-u_2), \tag{8.3.11}$$

where U is the shock speed given by

$$U=\frac{1}{2}(u_1+u_2). \tag{8.3.12}$$

This is in complete agreement with the above analysis.

8.4 The Exact Solution of the Burgers Equation

We solve the initial-value problem for the Burgers equation

$$u_t+uu_x=\nu u_{xx},\quad x\in\mathbb{R},\ t>0, \tag{8.4.1}$$
$$u(x,0)=F(x),\quad x\in\mathbb{R}. \tag{8.4.2}$$

Special attention will be given to small values of ν or to large values of Reynolds numbers.

Hopf (1950) and Cole (1951) independently discovered a transformation that reduces the Burgers equation to a linear diffusion equation. First, we write (8.4.1) in a form similar to a conservation law,

$$u_t+\left(\frac{1}{2}u^2-\nu u_x\right)_x=0. \tag{8.4.3}$$

This can be regarded as the compatibility condition for a function ψ to exist, such that

$$u = \psi_x \quad \text{and} \tag{8.4.4a}$$

$$\nu u_x - \frac{1}{2}u^2 = \psi_t. \tag{8.4.4b}$$

We substitute the value of u from (8.4.4a) in (8.4.4b) to obtain

$$\nu \psi_{xx} - \frac{1}{2}\psi_x^2 = \psi_t. \tag{8.4.5}$$

Next, we introduce $\psi = -2\nu \log \phi$ so that

$$u = \psi_x = -2\nu \frac{\phi_x}{\phi}. \tag{8.4.6}$$

This is called the *Cole–Hopf transformation*, which, by differentiating, gives

$$\psi_{xx} = 2\nu \left(\frac{\phi_x}{\phi}\right)^2 - \frac{2\nu}{\phi}\phi_{xx} \quad \text{and} \quad \psi_t = -2\nu \frac{\phi_t}{\phi}.$$

Consequently, (8.4.5) reduces to the linear diffusion equation

$$\phi_t = \nu \phi_{xx}. \tag{8.4.7}$$

Many solutions of this equation are well known in the literature. We substitute the given solution for ϕ to find solutions of the Burgers equation.

We now solve equation (8.4.7) subject to the initial condition

$$\phi(x,0) = \Phi(x), \quad x \in \mathbb{R}. \tag{8.4.8a}$$

This can be written in terms of the initial value $u(x,0) = F(x)$ by using (8.4.6):

$$F(x) = u(x,0) = -2\nu \frac{\phi_x(x,0)}{\phi(x,0)}. \tag{8.4.8b}$$

Integrating this result gives

$$\phi(x,0) = \Phi(x) = \exp\left\{-\frac{1}{2\nu}\int_0^x F(\alpha)\,d\alpha\right\}. \tag{8.4.9}$$

The Fourier transform method or the joint Fourier–Laplace transform technique can be used to solve the linear initial-value problem (8.4.7), (8.4.8a), and hence, the standard solution of this problem is

$$\phi(x,t) = \frac{1}{2\sqrt{\pi \nu t}} \int_{-\infty}^{\infty} \Phi(\zeta) \exp\left[-\frac{(x-\zeta)^2}{4\nu t}\right] d\zeta, \tag{8.4.10}$$

where $\Phi(\zeta)$ is given by (8.4.9). We then substitute the value of $\Phi(\zeta)$ to rewrite (8.4.10) in the convenient form

$$\phi(x,t) = \frac{1}{2\sqrt{\pi\nu t}} \int_{-\infty}^{\infty} \exp\left(-\frac{f}{2\nu}\right) d\zeta, \tag{8.4.11}$$

where

$$f(\zeta,x,t) = \int_0^\zeta F(\alpha)\, d\alpha + \frac{(x-\zeta)^2}{2t}. \tag{8.4.12}$$

Thus,

$$\phi_x(x,t) = -\frac{1}{4\nu\sqrt{\pi\nu t}} \int_{-\infty}^{\infty} \left(\frac{x-\zeta}{t}\right) \exp\left(-\frac{f}{2\nu}\right) d\zeta. \tag{8.4.13}$$

Therefore, the exact solution of the Burgers initial-value problem is obtained from (8.4.6) in the form

$$u(x,t) = \frac{\int_{-\infty}^{\infty} (\frac{x-\zeta}{t}) \exp(-\frac{f}{2\nu})\, d\zeta}{\int_{-\infty}^{\infty} \exp(-\frac{f}{2\nu})\, d\zeta}. \tag{8.4.14}$$

This is clearly single-valued and continuous for all values of t. The physical interpretation of this exact solution can hardly be given unless a suitable simple form of $F(x)$ is specified. In many problems, an exact evaluation of the integrals involved in (8.4.14) is almost a formidable task. It is then necessary to resort to asymptotic methods. We next consider the following example to investigate the formation of discontinuities or shock waves.

Example 8.4.1. Find the solution of the Burgers initial-value problem with physical significance for the discontinuous data

$$F(x) = A\delta(x)H(x), \tag{8.4.15}$$

where A is a constant, $\delta(x)$ is the Dirac delta function, and $H(x)$ is the Heaviside unit step function.

To find the solution, first we calculate

$$f(\zeta,x,t) = A\int_{0+}^{\zeta} \delta(\alpha)\, d\alpha + \frac{(x-\zeta)^2}{2t} = \begin{cases} \frac{(x-\zeta)^2}{2t} - A & \text{if } \zeta < 0, \\ \frac{(x-\zeta)^2}{2t} & \text{if } \zeta > 0. \end{cases}$$

Thus, the integral in the numerator of (8.4.14) is

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{x-\zeta}{t}\right) \exp\left(-\frac{f}{2\nu}\right) d\zeta &= \int_{-\infty}^{0} \left(\frac{x-\zeta}{t}\right) \exp\left\{\frac{A}{2\nu} - \frac{(x-\zeta)^2}{4\nu t}\right\} d\zeta \\ &\quad + \int_0^\infty \left(\frac{x-\zeta}{t}\right) \exp\left\{-\frac{(x-\zeta)^2}{4\nu t}\right\} d\zeta \\ &= 2\nu\left(e^R - 1\right) \exp\left(-\frac{x^2}{4\nu t}\right), \end{aligned}$$

which is obtained by substituting $(x-\zeta)/2\sqrt{\nu t} = \alpha$ and writing $(\frac{A}{2\nu}) = R$ which can be interpreted as the Reynolds number (Whitham 1974). Clearly, by small ν, we mean large R ($R \gg 1$), and the large ν corresponds to small R ($R \ll 1$).

Similarly, the integral in the denominator of (8.4.14) gives

$$\int_{-\infty}^{\infty} \exp\left(-\frac{f}{2\nu}\right) d\zeta = 2\sqrt{\nu t}\left[\sqrt{\pi} + \left(e^R - 1\right) \operatorname{erfc}\left(\frac{x}{2\sqrt{\nu t}}\right)\right].$$

So the final solution $u(x,t)$ is obtained from (8.4.14) in the form

$$u(x,t) = \sqrt{\frac{\nu}{t}}\left[\frac{(e^R - 1)\exp(-\frac{x^2}{4\nu t})}{\sqrt{\pi} + (e^R - 1)\frac{\sqrt{\pi}}{2}\operatorname{erfc}(\frac{x}{2\sqrt{\nu t}})}\right]. \tag{8.4.16}$$

Thus, this solution takes the similarity form

$$u(x,t) = \sqrt{\frac{\nu}{t}}\, g(\eta, R), \tag{8.4.17}$$

where $\eta = (x/2\sqrt{\nu t})$ is the *similarity variable*. Two limiting cases (i) $R \to 0$ and (ii) $R \to \infty$ are of interest, and they are considered below.

Case (i) In the limit as $R \to 0(\nu \to \infty)$, the effect of diffusion would be more significant than that of nonlinearity. So, in this case, we can approximate $e^R \sim 1 + R$ and $\operatorname{erfc}(\frac{x}{2\sqrt{\nu t}}) \to 0$ in (8.4.16) to obtain the final solution

$$u(x,t) \sim \frac{A}{2\sqrt{\pi \nu t}} \exp\left(-\frac{x^2}{4\nu t}\right). \tag{8.4.18}$$

This is the *fundamental* (or *source*) solution of the linear diffusion equation. At any time t, the velocity field is Gaussian. The peak height of $u(x,t)$ decreases inversely with $\sqrt{\nu t}$, whereas the width of the peak $(x \sim \sqrt{\nu t})$ increases with $\sqrt{\nu t}$. These are remarkable features of diffusion phenomena.

Case (ii) In this case, $R \to \infty(\nu \to 0)$. So, the nonlinear effect would dominate over diffusion, and hence, discontinuity as a shock is expected to develop. We introduce the similarity variable $\eta = x/\sqrt{2At}$ so that $x/\sqrt{4\nu t} = \eta\sqrt{R}$, and then, rewrite the solution (8.4.16) as

$$u(x,t) = \sqrt{\frac{\nu}{t}}\left[\frac{(e^R - 1)\exp(-R\eta^2)}{\sqrt{\pi} + (e^R - 1)(\frac{\sqrt{\pi}}{2})\operatorname{erfc}(\eta\sqrt{R})}\right], \tag{8.4.19}$$

which has the similarity form

$$u(x,t) = \left(\frac{A}{2Rt}\right)^{\frac{1}{2}} g(\eta, R). \tag{8.4.20}$$

Since $R \to \infty$, $e^R - 1$ can be replaced by e^R, hence, (8.4.19) gives

$$\begin{aligned} u(x,t) &= \sqrt{\frac{\nu}{t}}\left[\frac{\exp\{R(1-\eta^2)\}}{\sqrt{\pi} + (\frac{\sqrt{\pi}}{2})\exp(R)\operatorname{erfc}(\eta\sqrt{R})}\right] \quad \text{for all } \eta, \\ &\sim 0 \quad \text{as } R \to \infty \text{ for } \eta < 0 \text{ and } \eta > 1. \end{aligned} \tag{8.4.21}$$

When $R \to \infty$ and $0 < \eta < 1$, we use the asymptotic result

$$\text{erfc}\big(\eta\sqrt{R}\big) \sim \frac{1}{\eta\sqrt{\pi R}} \exp\big(-\eta^2 R\big)$$

to obtain the solution from (8.4.21) as

$$u(x,t) = \left(\frac{2A}{t}\right)^{\frac{1}{2}} \left[\frac{\eta}{1 + 2\eta\sqrt{\pi R}\exp\{-R(1-\eta^2)\}}\right] \tag{8.4.22}$$

$$\sim \eta\left(\frac{2A}{t}\right)^{\frac{1}{2}} = \frac{x}{t} \quad \text{as } R \to \infty. \tag{8.4.23}$$

It turns out that the final asymptotic solution, as $R \to \infty$, is

$$u(x,t) \sim \begin{cases} \frac{x}{t} & \text{if } 0 < x < \sqrt{2At}, \\ 0 & \text{otherwise.} \end{cases} \tag{8.4.24}$$

This result represents a shock wave at $x = \sqrt{2At}$, and the shock speed is $U = \frac{dx}{dt} = \left(\frac{A}{2t}\right)^{\frac{1}{2}}$. The solution $u(x,t)$ has a jump from zero to $\frac{x}{t} = \left(\frac{2A}{t}\right)^{\frac{1}{2}}$, and hence, the shock condition is satisfied.

Whitham (1974) investigated the structure of the shock wave solution for large values of R. He found two transition layers: one at $\eta = 1$ and the other (weaker) layer at $\eta = 0$. For large but finite R, (8.4.22) indicates a rapid transition from exponentially small values in $\eta > 1$ to $u \sim \sqrt{\frac{2A}{t}}\eta$ in $\eta < 1$. In the transition layer $\eta \approx 1$, solution (8.4.22) can be approximated by

$$u(x,t) \sim \sqrt{\frac{2A}{t}}\big[1 + 2\sqrt{\pi R}\exp\big\{2R(\eta-1)\big\}\big]^{-1}. \tag{8.4.25}$$

This shows that this transition layer has thickness $O(R^{-1})$.

On the other hand, the second layer at $\eta = 0$ does smooth out the discontinuity in the derivative between $u \sim 0$ in $\eta < 0$ to $u \sim \frac{x}{t}$ in $0 < \eta < 1$. It follows from (8.4.21) that this weaker layer occurs for $\eta = O(\frac{1}{\sqrt{R}})$, and hence, (8.4.22) assumes the approximate form

$$u(x,t) \sim 2\sqrt{\frac{\nu}{\pi t}} \cdot \frac{\exp(-R\eta^2)}{\text{erfc}(\eta\sqrt{R})}. \tag{8.4.26}$$

This solution for large R is drawn in Figure 8.2, where $\sqrt{\frac{t}{2\nu}}u$ is plotted against $\eta = \frac{x}{\sqrt{2\nu t}}$. This represents a triangular wave solution of the Burgers equation. As $R \to \infty$, the shock layer at $x = 0$ becomes a discontinuity of u_x.

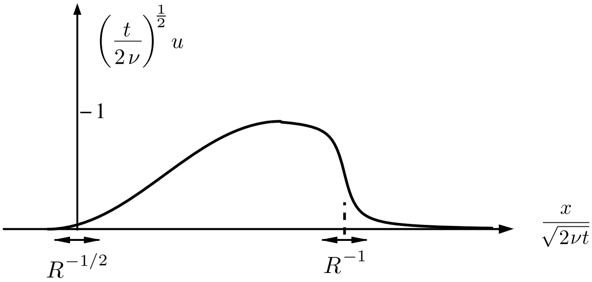

Fig. 8.2 Triangular wave solution of the Burgers equation.

8.5 The Asymptotic Behavior of the Burgers Solution

We use the stationary phase approximation to examine the asymptotic nature of the Burgers solution (8.4.14). We consider any typical integral

$$u(x,t) = \int_a^b F(k) \exp\{it\theta(k)\}\, dk, \tag{8.5.1}$$

where $F(k)$ is a given function and $\theta(k) = (\frac{x}{t})k - \omega(k)$. Integral (8.5.1) can be approximated asymptotically, as $t \to \infty$ for fixed x, by

$$u(x,t) \sim F(k_1)\left[\frac{2\pi}{t|\theta''(k_1)|}\right]^{\frac{1}{2}} \exp\left[i\left\{t\theta(k_1) + \frac{\pi}{4}\operatorname{sgn}\theta''(k_1)\right\}\right], \tag{8.5.2}$$

where k_1 is a stationary point determined by the solution of the equation

$$\theta'(k) = \frac{x}{t} - \omega'(k) = 0, \quad a < k_1 < b \tag{8.5.3}$$

and $\theta''(k_1) \neq 0$.

According to this result, the significant contribution to integrals involved in (8.4.14) comes from stationary points for fixed x and t, that is, from the roots of the equation

$$\frac{\partial f}{\partial \zeta} = F(\zeta) - \left(\frac{x-\zeta}{t}\right) = 0. \tag{8.5.4}$$

Suppose that $\zeta = \xi(x,t)$ is a solution of (8.5.4) representing a stationary point. Invoking the stationary phase formula (8.5.2), the integrals in (8.4.14) have the asymptotic representation as $\nu \to 0$, in the form,

$$\int_{-\infty}^{\infty}\left(\frac{x-\zeta}{t}\right)\exp\left(-\frac{f}{2\nu}\right)d\zeta \sim \left(\frac{x-\zeta}{t}\right)\left[\frac{4\pi\nu}{|f''(\xi)|}\right]^{\frac{1}{2}}\exp\left\{-\frac{f(\xi)}{2\nu}\right\},$$

$$\int_{-\infty}^{\infty}\exp\left(-\frac{f}{2\nu}\right)d\zeta \sim \left[\frac{4\pi\nu}{|f''(\xi)|}\right]^{\frac{1}{2}}\exp\left\{-\frac{f(\xi)}{2\nu}\right\}.$$

Therefore, the final asymptotic solution is

$$u(x,t) \sim \frac{x-\xi}{t}, \tag{8.5.5}$$

where $\xi(x,t)$ satisfies equation (8.5.4).

In other words, the solution assumes the asymptotic form

$$\left.\begin{aligned} u &= F(\xi), \\ \xi &= x - tF(\xi). \end{aligned}\right\} \tag{8.5.6ab}$$

This is identical with the solution of the Burgers equation without the diffusion term $(\nu = 0)$. Here, the stationary point ξ corresponds to the characteristic variable in the context of the first-order, quasi-linear equation. As discussed in Chapter 5, the wave profile described by (8.5.6ab) suffers from gradual distortion and steepening, leads to a multi-valued solution after a sufficiently long time, and, eventually, it breaks with the development of discontinuity as a shock wave. When this state is reached, mathematically, there will be two stationary points of (8.5.4), and then, some modification is required to complete the asymptotic analysis. This leads to *Whitham's geometrical rule of equal area* which is equivalent to the shock condition. We shall not pursue this analysis further, and refer to Whitham (1974) and Burgers (1974) for a detailed discussion.

8.6 The N-Wave Solution

To find an N-wave solution of the Burgers equation, we begin with the source solution of the linear diffusion equation (8.4.7) of the form

$$\phi(x,t) = 1 + \sqrt{\frac{\tau}{t}}\exp\left(-\frac{x^2}{4\nu t}\right), \tag{8.6.1}$$

where τ is a constant.

Substituting (8.6.1) into (8.4.6) gives the solution of the Burgers equation as

$$u(x,t) = \left(\frac{x}{t}\right)\cdot\frac{\sqrt{\frac{\tau}{t}}\exp(-\frac{x^2}{4\nu t})}{1+\sqrt{\frac{\tau}{t}}\exp(-\frac{x^2}{4\nu t})}, \tag{8.6.2}$$

$$= \frac{x}{t}\left[1+\sqrt{\frac{t}{\tau}}\exp\left(\frac{x^2}{4\nu t}\right)\right]^{-1}. \tag{8.6.3}$$

For any time $t > 0$, this solution is shown in Figure 8.3 and has the N-shaped form. Because of this particular shape of the wave profile, it is known as the N-*wave solution* of the Burgers equation. At $t = t_0 > 0$, the profile may be taken as the initial profile. Such N waves are observed in many physical situations including problems governed by the cylindrical or spherical Burgers equation.

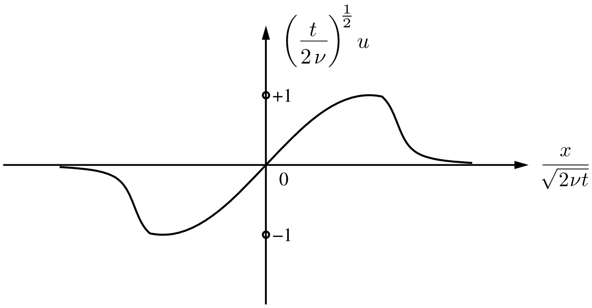

Fig. 8.3 N-wave solution of the Burgers equation.

The area under the negative phase of the wave profile is the same as that over the positive phase of the profile. So, the area under the positive phase of the wave profile is given by

$$\int_0^\infty u(x,t)\,dx = -2\nu\big[\log\phi(x,t)\big]_0^\infty = 2\nu\log\left[1+\sqrt{\frac{\tau}{t}}\right]. \tag{8.6.4}$$

We denote the right-hand side of (8.6.4) in the initial time $t = t_0$ by A and then introduce a Reynolds number R by

$$R = \frac{A}{2\nu} = \log\left[1+\sqrt{\frac{\tau}{t_0}}\right], \tag{8.6.5}$$

so that $\sqrt{\frac{\tau}{t_0}} = (e^R - 1)$, and solution (8.6.3) reduces to

$$u(x,t) = \left(\frac{x}{t}\right)\left[1+\sqrt{\frac{t}{t_0}}\left(e^R-1\right)^{-1}\exp\left(\frac{x^2R}{2At}\right)\right]^{-1}. \tag{8.6.6}$$

When $R \gg 1$, $e^R - 1 \sim e^R$, so that (8.6.6) reduces to the form

$$u(x,t) \sim \left(\frac{x}{t}\right)\left[1+\sqrt{\frac{t}{t_0}}\exp\left\{-R\left(1-\frac{x^2}{2At}\right)\right\}\right]^{-1}. \tag{8.6.7}$$

In the limit, as $R \to \infty$ with fixed t, (8.6.7) gives the shock-wave solution of the Burgers equation in the form

$$u(x,t) \sim \begin{cases} \frac{x}{t} & \text{if} |x| < \sqrt{2At}, \\ 0 & \text{if } |x| > \sqrt{2At}. \end{cases} \tag{8.6.8}$$

In the limit as $t \to \infty$, for fixed ν and τ, solution (8.6.2) takes the form

$$u(x,t) \sim \left(\frac{x}{t}\right)\left(\frac{\tau}{t}\right)^{\frac{1}{2}}\exp\left(-\frac{x^2}{4\nu t}\right). \tag{8.6.9}$$

This corresponds to the dipole solution of the linear diffusion equation.

Finally, in the limit as $R \to 0$, $e^R - 1 \sim R$, and result (8.6.6) gives

$$u(x,t) \sim \left(\frac{xR}{t}\right)\sqrt{\frac{t_0}{t}}\exp\left(-\frac{x^2 R}{2At}\right). \tag{8.6.10}$$

This is identical with (8.6.9), as expected, because $R \to 0$ corresponds to $\nu \to \infty$.

8.7 Burgers Initial- and Boundary-Value Problem

We solve Burgers equation (8.4.1) in $0 < x < l$, $t > 0$ with the following initial and boundary conditions:

$$u(x,0) = u_0 \sin\left(\frac{\pi x}{l}\right), \quad 0 \le x \le l, \tag{8.7.1}$$

$$u(0,t) = u(l,t) = 0, \quad t > 0. \tag{8.7.2}$$

It follows from the Cole–Hopf transformation, that is, from (8.4.9), that

$$\begin{aligned}\phi(x,0) &= \exp\left[-\left(\frac{u_0}{2\nu}\right)\int_0^x \sin\left(\frac{\pi\alpha}{l}\right)d\alpha\right] \\ &= \exp\left[-\left(\frac{u_0 l}{2\pi\nu}\right)\left(1-\cos\frac{\pi x}{l}\right)\right],\end{aligned} \tag{8.7.3}$$

so that boundary conditions (8.7.2) are satisfied.

The standard solution of the linear diffusion equation (8.4.7) is given by

$$\phi(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2\nu t}{l^2}\right)\cos\left(\frac{n\pi x}{l}\right), \tag{8.7.4}$$

where

$$\begin{aligned}a_0 &= \frac{1}{l}\int_0^{\ell} \exp\left[-\left(\frac{u_0 l}{2\pi\nu}\right)\left(1-\cos\frac{\pi x}{l}\right)\right]dx \\ &= \exp\left(-\frac{u_0 l}{2\pi\nu}\right) I_0\left(\frac{l u_0}{2\pi\nu}\right)\end{aligned} \tag{8.7.5}$$

and

$$\begin{aligned}a_n &= \frac{2}{l}\int_0^{\ell} \exp\left[-\left(\frac{l u_0}{2\pi\nu}\right)\left(1-\cos\frac{\pi x}{l}\right)\right]\cos\left(\frac{n\pi x}{l}\right)dx \\ &= 2\exp\left(-\frac{l u_0}{2\pi\nu}\right) I_n\left(\frac{u_0 l}{2\pi\nu}\right),\end{aligned} \tag{8.7.6}$$

where the preceding integrals are evaluated by using standard integrals from Abramowitz and Stegun (1972), and $I_0(x)$ and $I_n(x)$ are the modified Bessel functions of the first kind.

Thus, the solution of the Burgers initial-boundary problem is given by

$$u(x,t)=\left(\frac{4\pi\nu}{l}\right)\frac{\sum_{n=1}^{\infty} nI_n(\frac{u_0 l}{2\pi\nu})\exp(-\frac{n^2\pi^2\nu t}{l^2})\sin(\frac{n\pi x}{l})}{I_0(\frac{u_0 l}{2\pi\nu})+2\sum_{n=1}^{\infty} I_n(\frac{u_0 l}{2\pi\nu})\exp(-\frac{n^2\pi^2\nu t}{l^2})\cos(\frac{n\pi x}{l})}. \quad (8.7.7)$$

At $t=0$, the value of the denominator of (8.7.7) is $\exp[(\frac{u_0 l}{2\pi\nu})\cos(\frac{\pi x}{l})]$. We use this result combined with (8.7.1) and the value of $\phi_x(x,0)$ from (8.7.3) and (8.7.4) to verify that the solution (8.7.7) satisfies the initial condition (8.7.1). We note here that the quantity $(\frac{u_0 l}{\nu})=R$ represents the Reynolds number.

The solution of the linear diffusion equation (8.4.7), subject to the initial and boundary data (8.7.1), (8.7.2), is given by

$$\phi(x,t)=u_0\exp\left(-\frac{\pi^2\nu t}{l^2}\right)\sin\left(\frac{\pi x}{l}\right). \quad (8.7.8)$$

The following conclusions are in order. First, the nonlinear solution (8.7.7) contains an infinite set of higher harmonics with decreasing amplitudes, whereas the linear solution (8.7.8) contains only the fundamental harmonic. Second, the former solution depends on the Reynolds number R rather than the initial amplitude u_0.

We next use the asymptotic expansion of the modified Bessel function

$$I_n\left(\frac{R}{\pi}\right)\sim\frac{1}{\sqrt{2R}}\exp\left(\frac{R}{\pi}\right),\quad \text{as } R\to\infty,$$

to find the asymptotic solution for $u(x,t)$, as $R\to\infty$,

$$u(x,t)\sim\left(\frac{4\pi\nu}{l}\right)\left[\frac{\sum_{n=1}^{\infty}\exp(-\frac{n^2\pi^2\nu t}{l^2})\sin(\frac{n\pi x}{l})}{1+2\sum_{n=1}^{\infty}\exp(-\frac{n^2\pi^2\nu t}{l^2})\cos(\frac{n\pi x}{l})}\right]. \quad (8.7.9)$$

It is interesting to point out that this asymptotic solution is identical with the exact solution. This series solution is extremely rapidly convergent for all values of $(\frac{\nu t}{l^2})$, and hence, it is easy to calculate the wave form. When $\frac{\nu t}{l^2}\gg 1$, the term with $n=1$ in (8.7.9) dominates the series, and hence, we obtain

$$u(x,t)\sim\left(\frac{4\pi\nu}{l}\right)\exp\left(-\frac{\pi^2\nu t}{l^2}\right)\sin\left(\frac{\pi x}{l}\right). \quad (8.7.10)$$

This represents a sinusoidal wave form with an exponentially decaying attenuation. Thus, the ultimate decay of the solution is dominated by diffusion.

Another form of the solution (8.7.9) can be derived by using the Jacobi theta function defined by

$$\vartheta_3(X,T)=1+2\sum_{n=1}^{\infty}\exp\left(-\pi n^2 T\right)\cos(2nX),$$

where $X=\frac{\pi x}{2l}$ and $T=\frac{\nu\pi t}{l^2}$, so that

$$\frac{\partial}{\partial X}\log\vartheta_3(X,T)=2\sum_{n=1}^{\infty}(-1)^n\left\{\sinh\left(\frac{\nu n\pi^2 t}{l^2}\right)\right\}^{-1}\sin(2nX).$$

In view of these results, solution (8.7.9) becomes

$$u(x,t)=\left(\frac{2\pi\nu}{l}\right)\sum_{n=1}^{\infty}(-1)^{n+1}\left\{\sinh\left(\frac{\nu n\pi^2 t}{l^2}\right)\right\}^{-1}\sin\left(\frac{n\pi x}{l}\right). \tag{8.7.11}$$

This is another form of the solution (8.7.9). As suggested by Cole (1951), this solution is approximately equal to

$$u(x,t)\sim\frac{l}{t}\left[\tanh\left(\frac{l-x}{2\nu t}\right)-\left(\frac{l-x}{l}\right)\right]. \tag{8.7.12}$$

This does not depend on the initial amplitude. Physically, the initial sinusoidal profile for large R suffers from a nonlinear steepening effect near $x = l$ because of the higher harmonics generated by convection. However, this wave form steepening may be prevented by the effects of diffusion. As time t becomes large, this effect spreads in the entire wave profile, leading to exponential decay of the harmonics according to (8.7.11).

Finally, for a detailed discussion of the solution of the spatially periodic initial-value problem with general initial conditions, we refer to a paper by Walsh (1969).

8.8 Fisher Equation and Diffusion–Reaction Process

Fisher (1936) first introduced a nonlinear evolution equation to investigate the wave propagation of an advantageous gene in a population. His equation also describes the logistic growth–diffusion process and has the form

$$u_t-\nu u_{xx}=ku\left(1-\frac{u}{\kappa}\right), \tag{8.8.1}$$

where $\nu(>0)$ is a diffusion constant, $k(>0)$ is the linear *growth rate*, and $\kappa(>0)$ is the carrying capacity of the environment. The term $f(u)=ku(1-\frac{u}{\kappa})$ represents a nonlinear growth rate which is proportional to u for small u, but decreases as u increases, and vanishes when $u=\kappa$. It corresponds to the growth of a population u when there is a limit κ on the size of the population that the habitat can support; if $u>\kappa$, then $f(u)<0$, so the population decreases whenever u is greater than the limiting value κ. This interpretation suggests that the habitat can support a certain maximum population so that

$$0\le u(x,0)\le\kappa\quad\text{for } x\in\mathbb{R}. \tag{8.8.2}$$

In recent years, the Fisher equation (8.8.1) has been used as a basis for a wide variety of models for the spatial spread of genes in a population and for chemical wave propagation. It is pertinent to mention recent work on gene-culture waves of advance by Aoki (1987), on the propagation of chemical waves by Arnold et al. (1987), and on the spread of early farming in Europe by Ammerman and Cavalli-Sforva (1971, 1983). It also represents a model equation for the evolution of a neutron population in a nuclear reactor (Canosa 1969, 1973).

The Fisher equation (8.8.1) is a particular case of a general model equation, called the nonlinear *reaction–diffusion* equation, which can be obtained by introducing the net growth rate $f(x,t,u)$ so that it takes the form

$$u_t - \nu u_{xx} = f(x,t,u), \quad x \in \mathbb{R},\ t > 0. \tag{8.8.3}$$

The term f is also referred to as a source or reaction term and it represents the birth–death process in an ecological context. This equation arises in many physical, biological, and chemical problems involving diffusion and nonlinear growth. For example, if a chemically reacting substance is diffusing through a medium, then its concentration $u(x,t)$ satisfies (8.8.3), where f represents the rate of increase of the substance due to the chemical reaction. The temperature distribution $u(x,t)$ satisfies (8.8.3) when a chemical reaction generates heat at a rate depending on the temperature. Other problems described by (8.8.3) include the spread of animal or plant populations and the evolution of neutron populations in a nuclear reactor, where f represents the net growth rate.

We study the Fisher equation as a nonlinear model for a physical system involving linear diffusion and nonlinear growth. It is convenient to introduce the nondimensional quantities x^*, t^*, u^* defined by

$$x^* = \left(\frac{k}{\nu}\right)^{\frac{1}{2}} x, \qquad t^* = kt, \qquad u^* = \kappa^{-1} u, \tag{8.8.4}$$

where $\sqrt{\frac{\nu}{k}}$, k^{-1}, and κ represent the length scale, time scale, and population scale, respectively. Using (8.8.4) and dropping the asterisks, equation (8.8.1) takes the nondimensional form

$$u_t - u_{xx} = u(1-u), \quad x \in \mathbb{R},\ t > 0. \tag{8.8.5}$$

In the spatially homogeneous problem, the stationary states are $u = 0$ and $u = 1$, which represent unstable and stable solutions, respectively. It is then appropriate to look for traveling wave solutions of (8.8.5) for which $0 \le u \le 1$. We seek all solutions of (8.8.5) subject to (8.8.2) with $\kappa = 1$ such that all derivatives of u vanish as $|x| \to \infty$, and

$$\lim_{x\to-\infty} u(x,t) = 1 \quad \text{and} \quad \lim_{x\to\infty} u(x,t) = 0, \quad t \ge 0. \tag{8.8.6}$$

Physically, the first condition implies that the population has its maximum value as $x \to -\infty$, and the second condition represents zero population as $x \to +\infty$.

In their study of a nonlinear system, with applications to combustion and wave propagation in biology and chemistry, Kolmogorov et al. (1937) proved that, for all initial data of the type $0 \le u(x,0) \le 1, x \in \mathbb{R}$, the solution of (8.8.5) is also bounded for all x and t, that is,

$$0 \le u(x,t) \le 1, \quad x \in \mathbb{R},\ t > 0.$$

They also showed that, for the two sets of discontinuous initial data,

$$\text{(i)} \quad u(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases}$$

$$\text{(ii)} \quad u(x,0) = \begin{cases} 1 & \text{if } x < a, \\ f(x) & \text{if } a < x < b, \\ 0 & \text{if } x > b, \end{cases}$$

where $f(x)$ is an arbitrary function, the solution of (8.8.5) evolves, as $t \to \infty$, into a shock-like traveling wave satisfying condition (8.8.6) and propagates to the right with minimum characteristic velocity $c_{\min} = 2$.

It is important to note that equation (8.8.5) is invariant under the transformation $x \to -x$. So, it is sufficient to consider waves propagating to the right only. Kolmogorov et al. (1937) suggested one of the best-known model equations with dissipation in the form

$$u_t - \kappa u_{xx} = F(u), \tag{8.8.7}$$

where $F(u)$ is an arbitrary function of u. This is known as the *KPP equation* and it describes phenomena such as combustion, evolution of genes, and propagation of a nerve pulse in biological systems.

The generalized KPP equation is given by

$$u_t - u^2 + K(u) = 0, \tag{8.8.8}$$

when the linear pseudodifferential operator κ is defined by (7.8.25) and (7.8.26). The equation (8.8.8) corresponds to the symbol $K(k) = 1 + k^2$. The Cauchy condition for (8.8.8) is

$$u(x,0) = u_0(x), \tag{8.8.9}$$

where $u_0(x) \in L(\mathbb{R})$, which is required to ensure the stable solution $(u \to 0)$ of the Cauchy problem for (8.8.8), (8.9.9) as $t \to \infty$. Kolmogorov et al. (1937) proposed a new approach to studying the asymptotic behavior of the solution of the nonlocal KPP equation as $t \to \infty$. This approach is essentially based on application of the explicit form of Green's function for the linear KPP equation. They considered the *step-decaying problem for the KPP equation*, that is, the Cauchy problem with the initial data $u_0(x) \to a_\pm$ as $|x| \to \infty$, where a_+, a_- are constants with $a_+ \ne a_-$. The work of Kolmogorov et al. (1937) led to a large number of studies on the asymptotic solutions as $t \to \infty$ of the Cauchy problem for the KPP equation (see also Bramson 1983). Indeed, the uniform asymptotic behavior of $u(x,t)$ with respect to ξ, as $t \to$

∞, is given by

$$u(x,t) \sim \frac{A}{\sqrt{t}} e^{-t} \int_0^\infty e^{-k^2} \cos(k\xi)\, dk + O\big(e^{-t} t^{-1}\big), \tag{8.8.10}$$

where $\xi = (\frac{|x|}{\sqrt{t}}) \in \mathbb{R}$, and the constant A is expressed in terms of the symbol $K(k)$ and the initial function $u_0(k)$. For details, the reader is referred to Chapter 6 in Naumkin and Shishmarev (1994).

8.9 Traveling Wave Solutions and Stability Analysis

The Fisher equation (8.8.5) is one of the prototype equations which admits traveling wave solutions. We seek a traveling wave solution of (8.8.5) in the form

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{8.9.1}$$

where the wave speed c is to be determined and the waveform $u(\xi)$ satisfies the boundary data (8.8.6), as $\xi \to \mp\infty$. We substitute (8.9.1) into (8.8.5) to obtain the nonlinear, ordinary differential equation

$$u''(\xi) + cu'(\xi) + u(\xi) - u^2(\xi) = 0. \tag{8.9.2}$$

Writing $\frac{du}{d\xi} = v$, equation (8.9.2) gives two first-order equations

$$\frac{du}{d\xi} = 0 \cdot u + v, \qquad \frac{dv}{d\xi} = u(u-1) - cv, \tag{8.9.3ab}$$

or equivalently,

$$\frac{dv}{du} = \frac{u(u-1) - cv}{v}. \tag{8.9.4}$$

This system (8.9.3ab) has a simple interpretation in the Poincaré phase plane. The singular points of this system are the solutions of the equations

$$v = 0, \qquad u(u-1) - cv = 0. \tag{8.9.5ab}$$

Thus, there are two singular points $(u,v) = (0,0)$ and $(1,0)$ which represent the steady states.

We follow the standard phase plane analysis to examine the nature of the nonlinear autonomous system given by

$$\frac{du}{d\xi} = p(u,v), \qquad \frac{dv}{d\xi} = q(u,v).$$

The matrix associated with this system at the critical point (u_0, v_0) is

$$\begin{pmatrix} p_u(u_0,v_0) & p_v(u_0,v_0) \\ q_u(u_0,v_0) & q_v(u_0,v_0) \end{pmatrix}.$$

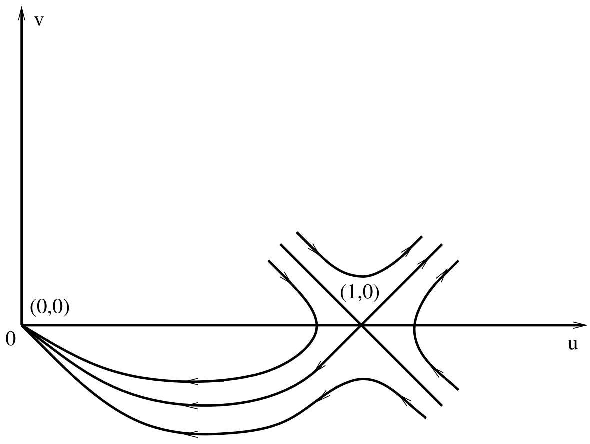

Fig. 8.4 Phase plane trajectories for $c > 2$.

In the present problem, the matrix A at $(0, 0)$ is

$$A(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}.$$

The matrix at $(1, 0)$ is

$$A(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}.$$

The eigenvalues λ of the matrix A at $(0, 0)$ are the roots of the equation

$$\begin{vmatrix} 0-\lambda & 1 \\ -1 & -(c+\lambda) \end{vmatrix} = 0.$$

This equation gives the eigenvalues

$$\lambda = -\frac{1}{2}\left[c \mp \sqrt{c^2-4}\right]. \tag{8.9.6}$$

The eigenvalues are real, distinct, and of the same sign if the discriminant $D = c^2 - 4 > 0$. According to the theory of dynamical systems, the origin is a stable node for $c \geq c_{\min} = 2$, and when $c = c_{\min} = 2$, it is a degenerate node. The phase plane trajectories of equation (8.9.2) for the traveling wave front solution for $c > 2$ are shown in Fig. 8.4. There is a unique separatrix joining the stable node $(0, 0)$ with the saddle point $(1, 0)$. If $c^2 < 4$, the eigenvalues are complex with a negative real part, and hence, the curve is a stable spiral, that is, $u(\xi)$ oscillates in the vicinity of the origin.

On the other hand, the eigenvalues λ of the matrix A at $(1, 0)$ are the roots of the equation

$$\begin{vmatrix} 0-\lambda & 1 \\ 1 & -c-\lambda \end{vmatrix} = 0.$$

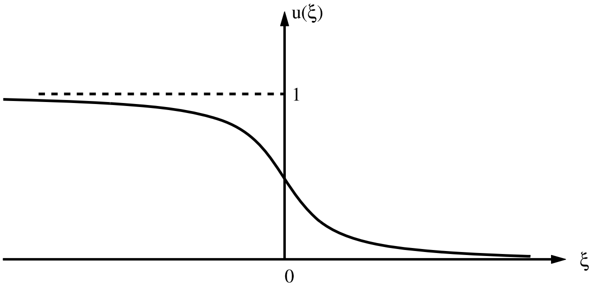

Fig. 8.5 Traveling wave solution.

This equation gives two eigenvalues

$$\lambda = \frac{1}{2}\left[-c \pm \sqrt{c^2+4}\right]. \tag{8.9.7}$$

These eigenvalues are real and of opposite sign, and hence, $(1,0)$ is a saddle point.

Finally, the traveling wave solution $u(\xi)$ of the Fisher equation for $c > 2$, satisfying the boundary conditions (8.8.6), as $\xi \to \mp\infty$, is a monotonically *decreasing function* of ξ when its first derivative vanishes as $\xi \to \mp\infty$. Clearly, this corresponds to the *separatrix* joining the singular points $(1,0)$ and $(0,0)$, as shown in Figure 8.4. A typical traveling wave solution $u(\xi)$ for $c \geq 2$ is shown in Figure 8.5. On the other hand, if $c < 2$, there are traveling wave solutions, but they are physically unrealistic since $u(\xi) < 0$ for some ξ because, in this case, u spirals around the origin. Further, $u \to 0$ at the leading edge with diminishing oscillations about $u = 0$.

We examine the stability of the traveling wave solution of (8.8.5) in the form

$$u(x,t) = U(\xi), \quad \xi = x - ct, \tag{8.9.8}$$

where c is the wave speed.

Such a solution is said to be *asymptotically stable* if a small perturbation imposed on the system at time $t = 0$ decays to zero, as $t \to \infty$. We introduce a coordinate frame moving with the wave speed c, that is, $\xi = x - ct$, $t = t$, so that (8.8.5) reduces to the form

$$u_t - cu_\xi = u_{\xi\xi} + u(1-u). \tag{8.9.9}$$

We look for a perturbed solution of (8.9.9) in the form

$$u(\xi,t) = U(\xi) + \varepsilon v(\xi,t), \tag{8.9.10}$$

where ε $(0 < \varepsilon \ll 1)$ is a small parameter and the second term represents a small perturbation from the given basic solution $U(\xi)$. The perturbed state $v(\xi,t)$ is assumed to have compact support in the moving frame, that is, $v(\xi,t) = 0$ for $|\xi| \geq a$ for some finite $a > 0$. We substitute (8.9.10) into (8.9.9) to find the equation for the perturbed quantity, v, to order ε,

$$v_t - cv_\xi = v_{\xi\xi} + (1 - 2U)v. \tag{8.9.11}$$

We next seek solutions of this equation in the form

$$v(\xi, t) = V(\xi)\exp(-\lambda t). \tag{8.9.12}$$

Substituting this solution into (8.9.11) gives the following linear eigenvalue problem:

$$V''(\xi) + cV'(\xi) + (\lambda + 1 - 2U)V = 0, \tag{8.9.13}$$

where the growth factor λ is considered an eigenvalue. Since the original perturbed state has compact support, we may take the boundary condition on $V(\xi)$ as $V(\pm a) = 0$. According to the general theory of eigenvalue problems, the perturbation will grow in time if the eigenvalues λ are negative, and hence, the system will become unstable. On the other hand, if the eigenvalues λ are positive, then the perturbation will decay to zero, as $t \to \infty$, and hence, the system is asymptotically stable.

To reduce the problem in the standard form, we introduce a new transformation

$$V(\xi) = w(\xi)\exp\left(-\frac{1}{2}c\xi\right) \tag{8.9.14}$$

so that the preceding eigenvalue problem reduces to the form

$$\left.\begin{aligned} w'' + [\lambda - q(\xi)]w = 0, \\ w(-l) = w(l) = 0, \end{aligned}\right\} \tag{8.9.15ab}$$

where

$$q(\xi) = \frac{c^2}{2} - (1 - 2U) \geq 2U(\xi) > 0 \quad \text{for } c \geq 2. \tag{8.9.16}$$

This is a standard eigenvalue problem, and all its eigenvalues are real and positive provided $c \geq 2$. This means that all small perturbations of finite extent decay exponentially to zero, as $t \to \infty$. We conclude that the traveling wave solution of the Fisher equation is asymptotically stable. However, the present perturbation analysis is not completely general. The general problem has been studied by several authors including Hoppensteadt (1975) and Larson (1978).

8.10 Perturbation Solutions of the Fisher Equation

The existence of the traveling wave solution of the Fisher equation was established by using a geometric argument. However, it has not been possible to determine the exact or approximate representation of the solution. We now employ the standard perturbation method to find the asymptotic solution of the boundary-value problem for $u(\xi)$, which satisfies the differential system

$$u'' + cu' + u(1 - u) = 0, \quad -\infty < \xi < \infty, \tag{8.10.1}$$

$$u(-\infty) = 1, \quad u(+\infty) = 0, \tag{8.10.2}$$

where $c \geq 2$.

Since equation (8.10.1) is autonomous, the solution is translation invariant. We choose the value $u(\xi)$ at $\xi = 0$ to be any number in the range of u, and hence, we take $u(0) = \frac{1}{2}$, which is needed to solve the problem. We introduce a new independent variable $z = \frac{\xi}{c} = \sqrt{\varepsilon}\xi$, where $\varepsilon = c^{-2}$ is a small parameter, to transform (8.10.1) and the boundary conditions into the form

$$\varepsilon u''(z) + u'(z) + u - u^2 = 0, \quad -\infty < z < \infty, \tag{8.10.3}$$

$$u(-\infty) = 1, \qquad u(0) = \frac{1}{2}, \qquad u(+\infty) = 0. \tag{8.10.4}$$

We seek a perturbation series expansion for $u(z)$ in powers of ε as

$$u(z, \varepsilon) = u_0(z) + \varepsilon u_1(z) + \varepsilon^2 u_2(z) + \cdots, \tag{8.10.5}$$

where $u_0(z)$, $u_1(z)$, and $u_2(z)$ are to be determined. We substitute (8.10.5) into (8.10.3) and set the coefficients of various powers of ε to be zero to obtain

$$u_0' + u_0 - u_0^2 = 0, \tag{8.10.6}$$

$$u_0(-\infty) = 1, \qquad u_0(0) = \frac{1}{2}, \qquad u_0(+\infty) = 0, \tag{8.10.7}$$

and

$$u_1' + u_1(1 - 2u_0) + u_0'' = 0, \tag{8.10.8}$$

$$u_1(-\infty) = u_1(0) = u_1(+\infty) = 0. \tag{8.10.9}$$

The general solution of (8.10.6) is given by

$$u_0(z) = \left(1 - Ae^z\right)^{-1},$$

where $A = -1$, since $u_0(0) = \frac{1}{2}$.

Thus, the solution reduces to the form

$$u_0(z) = \left(1 + e^z\right)^{-1}. \tag{8.10.10}$$

We then solve (8.10.8), which becomes

$$u_1' + \left(\frac{e^z - 1}{e^z + 1}\right)u_1 - \frac{e^z}{(1 + e^z)^2} = 0. \tag{8.10.11}$$

This can be solved directly by elementary methods, and the solution is given by

$$u_1(z) = \frac{e^z}{(1 + e^z)^2} \log\left\{\frac{4e^z}{(1 + e^z)^2}\right\}. \tag{8.10.12}$$

Thus, the asymptotic solution of the original problem is given by

$$u(z) \sim \left(1 + e^z\right)^{-1} + \frac{\varepsilon^2 e^z}{(1 + e^z)^2} \log\left\{\frac{4e^z}{(1 + e^z)^2}\right\} + O(\varepsilon^2), \tag{8.10.13}$$

where $z = (\frac{\xi}{c}) = \frac{1}{c}(x - ct)$.

This represents an asymptotic traveling wave solution for $c \geq 2$. The present problem is *not* a singular perturbation problem because u and $\frac{du}{dz}$ tend to finite values in the limit, as $\varepsilon \to 0$. Equation (8.10.3) has a uniform limit and the regular perturbation method gives an accurate solution in the whole domain.

8.11 Method of Similarity Solutions of Diffusion Equations

Birkhoff (1950) first recognized that Boltzmann's method of solving the diffusion equation with a concentration-dependent diffusion coefficient is based on the algebraic symmetry of the equation, and special solutions of this equation can be obtained by solving a related ordinary differential equation. Such solutions are called *similarity solutions* because they are geometrically similar. He also suggested that the algebraic symmetry of the partial differential equations can be used to find similarity solutions of other partial differential equations by solving associated ordinary differential equations. Thus, the method of similarity solutions became very successful in dealing with the determination of a group of transformations under which a given partial differential equation is invariant. The simplifying feature of this method is that a similarity transformation of the form $u(x,t) = t^p v(\eta)$, $\eta = xt^{-q}$, can be found which can then be used effectively to reduce the partial differential equation to an ordinary differential equation with η as the independent variable. The resulting ordinary differential equation is relatively easy to solve. In practice, this method is simple and useful in finding solutions of both linear and nonlinear partial differential equations. We illustrate the method of similarity solutions by examples of applications.

Example 8.11.1 (Similarity Solutions of Linear Diffusion Equation). We consider the classical linear diffusion equation with constant diffusion coefficient κ in the form

$$u_t = \kappa u_{xx}, \quad 0 \le x < \infty,\ t > 0, \tag{8.11.1}$$

subject to the following boundary and initial conditions:

$$u(0,t) = 1, \quad u(x,t) \to 0 \quad \text{as } x \to \infty \text{ for } t > 0, \tag{8.11.2ab}$$

$$u(x,0) = 0 \quad \text{for } 0 \le x < \infty. \tag{8.11.3}$$

We introduce a one-parameter set of stretching transformations in the (x,t,u)-space defined by

$$\tilde{x} = a^\alpha x, \qquad \tilde{t} = a^\beta t, \qquad \tilde{u} = a^\gamma u, \tag{8.11.4}$$

under which equation (8.11.1) is invariant, where a is a real parameter which belongs to an open interval I containing $a = 1$, and α, β, and γ are the fixed constants. Usually the set of transformations in the (x,t,u)-space is denoted by T_a, and we write the set explicitly as $T_a : \mathbb{R}^3 \to \mathbb{R}^3$ for each $a \in I$. The set of all such transformations $\{T_a\}$ form a *Lie group* on $\mathbb{R}^3$ with an identity element T_1. This can be seen as follows.

The set $\{T_a\}$ obeys the composition (multiplication) law

$$T_a T_b x = T_{ab} x, \quad \text{for all } a, b \in I. \tag{8.11.5}$$

This law is commutative because $T_{ab} = T_{ba}$. The associative law is also satisfied since

$$T_a(T_b T_c) = T_a(T_{bc}) = T_{abc} = (T_{ab})T_c = (T_a T_b)T_c. \tag{8.11.6}$$

In view of the fact that

$$T_1 T_a = T_{1a} = T_{a1} = T_a \quad \text{for all } a \neq 0, \tag{8.11.7}$$

where T_1 represents the identity transformation.

Finally, we obtain

$$T_a T_{a^{-1}} = T_1 = T_{a^{-1}} T_a. \tag{8.11.8}$$

This shows that the inverse of T_a is $T_{a^{-1}}$.

Clearly,

$$\tilde{u}_{\tilde{t}} = a^{\gamma-\beta} u_t, \qquad \tilde{u}_{\tilde{x}} = a^{\gamma-\alpha} u_x, \qquad \tilde{u}_{\tilde{x}\tilde{x}} = a^{\gamma-2\alpha} u_{xx}, \tag{8.11.9}$$

and hence,

$$\tilde{u}_{\tilde{t}} - \kappa \tilde{u}_{\tilde{x}\tilde{x}} = a^{\gamma-\beta} u_t - \kappa a^{\gamma-2\alpha} u_{xx} = a^{\gamma-\beta}(u_t - \kappa u_{xx}), \tag{8.11.10}$$

provided $\beta = 2\alpha$. Hence, equation (8.11.1) is invariant under the transformation

$$\tilde{x} = a^{\alpha} x, \qquad \tilde{t} = a^{2\alpha} t, \qquad \tilde{u} = a^{\gamma} u, \tag{8.11.11}$$

for any choice of α and γ. The quantities

$$v = t^p u(x,t), \qquad \eta = x t^{-q}, \tag{8.11.12ab}$$

are invariant under T_a provided $p = -(\gamma/\beta)$ and $q = (\alpha/\beta)$. Thus, the invariants of the transformations are given by

$$v(\eta) = u t^{-\gamma/\beta} = u t^{-\gamma/2\alpha}, \qquad \eta = x t^{-\alpha/\beta} = x t^{-\frac{1}{2}}. \tag{8.11.13ab}$$

Substituting (8.11.13ab) into the original equations (8.11.1)–(8.11.3) gives an ordinary differential equation of the form

$$\kappa v''(\eta) + \frac{1}{2}\eta v'(\eta) - \frac{\gamma}{2\alpha} v(\eta) = 0. \tag{8.11.14}$$

The transformed data are then given by

$$u(0,t) = t^{\gamma/2\alpha} v(0) = 1 \quad \text{and} \tag{8.11.15a}$$

$$v(\infty) = 0. \tag{8.11.15b}$$

To make (8.11.15a) independent of t, we require that $\gamma = 0$. Consequently, (8.11.14)–(8.11.15a), (8.11.15b) become

$$v''(\eta) + \left(\frac{\eta}{2\kappa}\right) v'(\eta) = 0, \tag{8.11.16}$$

$$v(0) = 1 \quad \text{and} \quad v(\infty) = 0. \tag{8.11.17ab}$$

Thus, equation (8.11.1) admits the set of transformations (8.11.13ab), which reduces the partial differential equation (8.11.1) to the ordinary differential equation (8.11.14). Result (8.11.13ab) is called a *similarity transformation*, and the new independent variable η is called a *similarity variable*.

Integrating (8.11.16) yields the general solution

$$v(\eta) = A + B \int_0^\eta \exp\left(-\frac{\xi^2}{4\kappa}\right) d\xi, \tag{8.11.18}$$

where A and B are integration constants to be determined by using (8.11.17ab). It turns out that $A = 1$ and $B = -\frac{1}{\sqrt{\pi\kappa}}$. Thus, the solution takes the form

$$\begin{aligned} v(\eta) &= 1 - \frac{1}{\sqrt{\pi\kappa}} \int_0^\eta \exp\left(-\frac{\xi^2}{4\kappa}\right) d\xi \\ &= 1 - \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4\kappa t}} \exp(-\alpha^2)\, d\alpha \\ &= 1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\kappa t}}\right) = \operatorname{erfc}\left(\frac{x}{\sqrt{4\kappa t}}\right), \end{aligned} \tag{8.11.19}$$

where $\operatorname{erf}(z)$ and $\operatorname{erfc}(z)$ are the standard error and complementary error functions, respectively. The solution (8.11.19) is identical with the known solution (1.9.24) which can be obtained by other methods. The present method of solution seems to be simple and powerful, and hence, could be used for other partial differential equations.

Example 8.11.2 (Similarity Solution of the Boltzmann Nonlinear Diffusion Problem). We apply the similarity method to solve the nonlinear diffusion equation

$$u_t = (uu_x)_x, \quad x \in \mathbb{R},\ t > 0, \tag{8.11.20}$$

with the boundary and initial conditions

$$u(\pm\infty, t) = 0, \quad t > 0, \tag{8.11.21}$$

$$u(x, 0) = \delta(x), \quad x \in \mathbb{R}, \tag{8.11.22}$$

and

$$\int_{-\infty}^{\infty} u(x,t)\, dx = 1 \quad \text{for all } t > 0. \tag{8.11.23}$$

We use the set of transformations (8.11.4) in (8.11.20) to obtain

$$\begin{aligned} \tilde{u}_{\tilde{t}} - \tilde{u}_{\tilde{x}}^2 - \tilde{u}\tilde{u}_{\tilde{x}\tilde{x}} &= a^{\gamma-\beta} u_t - a^{2(\gamma-\alpha)} u_x^2 - a^{2(\gamma-\alpha)} u u_{xx} \\ &= a^{\gamma-\beta}\left(u_t - u_x^2 - uu_{xx}\right), \end{aligned} \tag{8.11.24}$$

provided $\gamma = 2\alpha - \beta$. Hence the given equation (8.11.20) is invariant under the set of transformations

$$\tilde{x} = a^{\alpha} x, \qquad \tilde{t} = a^{\beta} t, \qquad \tilde{u} = a^{2\alpha-\beta} u. \tag{8.11.25}$$

The invariants of the set of transformations (8.11.25) are given by

$$v = t^{p} u(x,t), \qquad \eta = x t^{-q}, \tag{8.11.26ab}$$

provided $p = (\beta - 2\alpha)/\beta$ and $q = \alpha/\beta$. Therefore, equation (8.11.20) admits the similarity transformation given by

$$u = t^{-p} v(\eta) = t^{\frac{2\alpha-\beta}{\beta}} v(\eta), \qquad \eta = x t^{-\frac{\alpha}{\beta}}. \tag{8.11.27ab}$$

To determine α and β, we use (8.11.23) to obtain

$$t^{\frac{2\alpha-\beta}{\beta}} \int_{-\infty}^{\infty} v(\eta)\, dx = t^{\frac{3\alpha-\beta}{\beta}} \int_{-\infty}^{\infty} v(\eta)\, d\eta = 1, \tag{8.11.28}$$

which is independent of t provided $(\alpha/\beta) = \frac{1}{3}$. Thus, the similarity transformations (8.11.27ab) assume the form

$$u(x,t) = t^{-\frac{1}{3}} v(\eta), \qquad \eta = x t^{-\frac{1}{3}}. \tag{8.11.29ab}$$

Putting these results into the given equation (8.11.20) gives an ordinary differential equation for $v(\eta)$

$$3(vv')' + \eta v' + v = 0. \tag{8.11.30}$$

Integrating this equation once with respect to η gives

$$3vv' + \eta v = A, \tag{8.11.31}$$

where A is a constant of integration to be determined from the fact that the solution is symmetric about $\eta = 0$, that is, $u_x(0,t) = 0$, and hence, $v'(0) = t^{2/3} u_x(0,t) = 0$. Thus, the constant A must be zero; hence, the resulting equation (8.11.31) can be integrated with the boundary condition (8.11.21), that is, $v(\pm\infty) = 0$, to obtain the solution

$$v(\eta) = \begin{cases} \frac{1}{6}(c^2 - \eta^2) & \text{if } |\eta| < c, \\ 0 & \text{if } |\eta| > c. \end{cases} \tag{8.11.32}$$

Here c is a constant of integration which can be determined from the condition (8.11.28), and hence,

$$1 = \int_{-\infty}^{\infty} v(\eta)\, d\eta = \int_{-c}^{c} v(\eta)\, d\eta = \frac{2}{9} c^3, \tag{8.11.33}$$

giving $c = (\frac{9}{2})^{\frac{1}{3}}$.

The important point about the solution $v(\eta)$ is that $v'(\eta)$ does not tend to zero continuously, as $\eta \to \infty$, as was the case for the linear diffusion problem. The solution $v(\eta)$ represents a parabola, which intersects the η axis at $\eta = \pm c$, and its vertex is at $(0, \frac{1}{6}c^2)$.

Thus, the final solution for $u(x,t)$ becomes

$$u(x,t) = \begin{cases} \frac{1}{6}t^{-1}(c^2 t^{\frac{2}{3}} - x^2) & \text{if } |x| < ct^{\frac{1}{3}}, \\ 0 & \text{if } |x| > ct^{\frac{1}{3}}. \end{cases} \tag{8.11.34}$$

This represents shock wave-like behavior with the propagating wave front at $x = x_f = ct^{\frac{1}{3}}$. The wave front propagates in the medium with the speed $(\frac{dx_f}{dt}) = \frac{1}{3}ct^{-\frac{2}{3}}$ which decreases with time t. This means that the wave slows down as t increases. The solution (8.11.34) shows that u is zero ahead of the wave, and its derivative has a *jump* discontinuity at the front. Recent studies of various nonlinear diffusion models discussed by Murray (1993) and Okubo (1980) for insect and animal dispersal show that grasshoppers exhibit a dispersal behavior similar to that of the above model.

Thus, the major conclusion of this analysis is that the solution of the nonlinear diffusion problem is significantly *different* from the smooth fundamental solution

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right) \tag{8.11.35}$$

of the linear diffusion problem with a point source at $x = 0$ and $t = 0$.

Example 8.11.3. Use the Boltzmann transformation $\eta = \frac{1}{2}(\frac{x}{\sqrt{t}})$ to reduce the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{\kappa(u)\frac{\partial u}{\partial x}\right\}, \quad 0 \le x < \infty,\ t > 0, \tag{8.11.36}$$

with the boundary and initial data

$$u(0,t) = u_0, \quad t > 0; \qquad u(x,0) = u_1, \quad 0 \le x < \infty, \tag{8.11.37}$$

to an ordinary differential equation with the independent variable η. Hence, find the solution of the ordinary differential equation.

Since $\eta = \frac{1}{2}xt^{-\frac{1}{2}}$, we obtain

$$u_t = -\frac{x}{4}t^{-3/2}\left(\frac{du}{d\eta}\right) \quad \text{and} \quad u_x = -\frac{1}{2\sqrt{t}}\left(\frac{du}{d\eta}\right),$$

and hence,

$$\frac{\partial}{\partial x}\left\{\kappa(u)\frac{\partial u}{\partial x}\right\} = \frac{\partial}{\partial x}\left\{\frac{\kappa}{2\sqrt{t}}\frac{du}{d\eta}\right\} = \frac{1}{4t}\frac{d}{d\eta}\left(\kappa\frac{du}{d\eta}\right).$$

Equation (8.11.36) becomes

$$-2\eta\frac{du}{d\eta} = \frac{d}{d\eta}\left(\kappa\frac{du}{d\eta}\right). \tag{8.11.38}$$

The given boundary conditions reduce to

$$u = u_0, \qquad \eta = 0, \quad \text{and} \quad u = u_1, \quad \text{as } \eta \to \infty. \tag{8.11.39}$$

To find the solution, it is convenient to write (8.11.38) with $\tilde{u} = \frac{u}{u_0}$ and $\tilde{u} = 1$ on $\eta = 0$ in the form

$$-\frac{2\eta}{\kappa} \cdot \kappa \cdot \frac{d\tilde{u}}{d\eta} = \frac{d}{d\eta}\left(\kappa \frac{d\tilde{u}}{d\eta}\right). \tag{8.11.40}$$

We solve this equation subject to the condition $\tilde{u} = 0$, as $\eta \to \infty$. Integrating equation (8.11.40) twice, we obtain

$$\tilde{u} = 1 - A\int_0^{\eta} \frac{1}{\kappa} \exp\left[-\int_0^{\eta}\left(\frac{2\xi}{\kappa}\right) d\xi\right] d\xi, \tag{8.11.41}$$

where A is a constant of integration to be determined, so that

$$A\int_0^{\infty} \frac{1}{\kappa} \exp\left[-\int_0^{\eta}\left(\frac{2\xi}{\kappa}\right) d\xi\right] d\xi = 1, \tag{8.11.42}$$

in order that $\tilde{u} = \frac{u_1}{u_0}$, as $\eta \to \infty$, is satisfied and where $\kappa = \kappa(\tilde{u})$ is a known function. The solution (8.11.41) satisfies the condition $\tilde{u} = 1$ on $\eta = 0$.

Considerable attention has been given to the solution of equation (8.11.36) in both semi-infinite $(0 < x < \infty)$ and infinite $(-\infty < x < \infty)$ media for various kinds of concentration-dependent diffusion coefficients. The reader is referred to Chapter 7 of Crank's book (1975).

Example 8.11.4 (Similarity Solution of a More General Nonlinear Diffusion Model). We consider the following nonlinear diffusion equation:

$$\frac{\partial u}{\partial t} = \kappa_0 \frac{\partial}{\partial x}\left[u^n \frac{\partial u}{\partial x}\right], \tag{8.11.43}$$

where κ_0 is a constant.

We follow the analysis of Munier et al. (1981) to obtain the similarity solution of (8.11.43). We first introduce the Kirchhoff transformation

$$v = \int_0^u t^n dt = \frac{u^{n+1}}{n+1}, \quad n+1 \neq 0, \tag{8.11.44}$$

to express (8.11.43) in terms of v as

$$\frac{\partial v}{\partial t} = \kappa_0\big[(n+1)v\big]^{\frac{n}{n+1}} \frac{\partial^2 v}{\partial x^2} = \kappa v^{\frac{n}{n+1}} \frac{\partial^2 v}{\partial x^2}, \tag{8.11.45}$$

where $\kappa = \kappa_0 (n+1)^{\frac{n}{n+1}}$.

We next introduce a one-parameter set of transformations

$$\tilde{x} = a^{\alpha} x, \qquad \tilde{t} = a^{\beta} t, \qquad \tilde{u} = a^{\gamma} v, \tag{8.11.46}$$

to obtain

$$\begin{aligned}\tilde{v}_{\tilde{t}} - \kappa \tilde{v}^{\frac{n}{n+1}} \tilde{v}_{\tilde{x}\tilde{x}} &= a^{\gamma-\beta} v_t - \kappa a^{\frac{n}{n+1}+\gamma-2\alpha} v^{\frac{n}{n+1}} v_{xx} \\ &= a^{\gamma-\beta}\big(v_t - \kappa v^{\frac{n}{n+1}} v_{xx}\big),\end{aligned} \tag{8.11.47}$$

provided that

$$\gamma - \beta = \frac{\gamma n}{n+1} + \gamma - 2\alpha, \quad \text{that is,} \quad \frac{\gamma}{\beta} = \left(\frac{n+1}{n}\right)\left(\frac{2\alpha}{\beta} - 1\right). \tag{8.11.48}$$

The invariants of the transformations are given by

$$\tilde{\eta} = \tilde{x}\tilde{t}^{-\alpha/\beta} = xt^{-\alpha/\beta} = \eta \quad \text{and} \quad \tilde{w} = \tilde{v}\tilde{t}^{-\gamma/\beta} = vt^{-\gamma/\beta} = w. \tag{8.11.49}$$

Thus, equation (8.11.43) admits the similarity transformation in the form

$$v(x,t) = t^{\gamma/\beta} w\big(xt^{-\alpha/\beta}\big). \tag{8.11.50}$$

Substituting (8.11.50) into (8.11.45) gives

$$\left(\frac{n+1}{n}\right)\left(\frac{2\alpha}{\beta} - 1\right) w - \left(\frac{\alpha}{\beta}\right)\eta w'(\eta) = \kappa w^{\frac{n}{n+1}} w''(\eta). \tag{8.11.51}$$

This equation can be integrated by replacing $\frac{\alpha}{\beta} = \frac{1}{n+2}$, and hence, $\frac{\gamma}{\beta} = -\frac{n+1}{n+2}$ so that (8.11.51) reduces to

$$w''(\eta) = -\kappa_0 w^{-\frac{n}{n+1}}\left(w + \frac{1}{n+1} \cdot \eta w'\right), \tag{8.11.52}$$

where $\kappa_0 = (\eta+1)/\kappa(n+2)$. Integrating (8.11.52) once with respect to η gives

$$w'(\eta) = A - \kappa_0 \eta w^{\frac{1}{n+1}}, \tag{8.11.53}$$

where A is an integrating constant. The only case in which equation (8.11.53) admits an explicit analytic solution for w corresponds to $A = 0$. So, in this case, we integrate (8.11.53) to obtain the general solution

$$w^{\frac{n}{n+1}}(\eta) = B - \left(\frac{n\kappa_0}{n+1}\right)\frac{\eta^2}{2}, \tag{8.11.54}$$

where B is a constant of integration. Hence, the solution for $v(x,t)$ follows from (8.11.50) and (8.11.54) as

$$v^{\frac{n}{n+1}}(x,t) = t^{-\frac{n}{n+2}} w^{\frac{n}{n+1}}\big(xt^{-\frac{1}{n+2}}\big) = Bt^{-\frac{n}{n+2}} - \frac{1}{2}\left(\frac{n\kappa_0}{n+1}\right)\left(\frac{x^2}{t}\right). \tag{8.11.55}$$

Finally, result (8.11.44) gives $u(x,t)$ in the form

$$u(x,t) = \big[(n+1)v\big]^{\frac{1}{n+1}} = (n+1)^{\frac{1}{n+1}}\left[Bt^{-\frac{n}{n+2}} - \frac{1}{2}\left(\frac{n\kappa_0}{n+1}\right)\frac{x^2}{t}\right]^{\frac{1}{n}}. \tag{8.11.56}$$

Invoking translation and scaling in t, we obtain the final solution

$$u(x,t) = \left[\tilde{B}(1+\mu t)^{-\frac{n}{n+2}} - \frac{1}{2}\left(\frac{n\kappa_0}{n+2}\right)\frac{\mu x^2}{(1+\mu t)}\right]^{\frac{1}{n}}, \tag{8.11.57}$$

where $\tilde{B}$ and μ are arbitrary constants. It is noted that the similarity solution in terms of $\eta = xt^{-\frac{1}{2}}$ is possible only when $\frac{\alpha}{\beta} = \frac{1}{2}$ with an arbitrary n.

Example 8.11.5 (Heat Transfer in a Superfluid Helium). At a very low temperature, helium reduces to a liquid phase (called *superfluid helium* or *He-II*) with rather unusual properties. One of the most remarkable properties is that heat transport in stationary He-II is described not by the linear Fourier law, but by the nonlinear Gorter–Mellink law

$$q = -\kappa\left(\frac{\partial T}{\partial x}\right)^{\frac{1}{3}}, \tag{8.11.58}$$

where q is the heat flux, κ is a kind of thermal conductivity, and T is the absolute temperature. Combined with the heat balance equation $S(\frac{\partial T}{\partial t}) + (\frac{\partial q}{\partial x}) = 0$ and, in suitable units, (8.11.58) gives the nonlinear heat diffusion equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[\left(\frac{\partial T}{\partial x}\right)^{\frac{1}{3}}\right], \tag{8.11.59}$$

where $\frac{\kappa}{S} = 1$ and S is the heat capacity per unit volume.

This equation is invariant under a one-parameter set of transformations (8.11.4) with $\alpha = 1$ provided $2\gamma - 3\beta = -4$. The invariant solutions for the temperature described by the similarity transformations are given by

$$T(x,t) = t^{\gamma/\beta} v(\eta), \quad \eta = xt^{-\frac{1}{\beta}}, \tag{8.11.60}$$

where $v(\eta)$ satisfies an ordinary differential equation

$$\gamma\frac{d}{d\eta}\left(\frac{dv}{d\eta}\right)^{\frac{1}{3}} + \eta\frac{dv}{d\eta} - \beta v = 0. \tag{8.11.61}$$

This equation is also invariant under the associated transformation group

$$\tilde{\eta} = \mu\eta \quad \text{and} \quad \tilde{v} = \mu^{-2}v, \quad 0 < \mu < \infty. \tag{8.11.62}$$

The clamped-flux case is of special interest because it has been investigated experimentally in connection with the stability of superconducting magnets cooled with superfluid helium. The initial and boundary conditions for this special problem are given by

$$T_x = -\left(\frac{q}{\kappa}\right)^3 \quad \text{at } x = 0, \quad \text{and} \quad T(x,t) \to 0 \quad \text{as } x \to \infty, \text{ for } t > 0, \tag{8.11.63ab}$$

$$T(x,0)=0 \quad \text{for } x>0. \tag{8.11.64}$$

We require that $\gamma=1$ and $\beta=2$, so that these conditions become

$$v'(\eta)=-\left(\frac{q}{\kappa}\right)^3 \quad \text{at } \eta=0, \quad \text{and} \quad v(\eta)\to 0, \quad \text{as } \eta\to\infty. \tag{8.11.65}$$

In this case, the ordinary differential equation is given by (8.11.61) with $\gamma=1$ and $\beta=2$.

8.12 Nonlinear Reaction–Diffusion Equations

The general conservation equation in three space dimensions is

$$\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{F}=f(\mathbf{x},t,u), \tag{8.12.1}$$

where $\mathbf{F}$ is a general flux transport due to diffusion or some other processes and $f(\mathbf{x},t,u)$ is the source or reaction term. For the case of general diffusion processes, we can take $\mathbf{F}=-\kappa\nabla u$, so that equation (8.12.1) becomes

$$\frac{\partial u}{\partial t}=\nabla\cdot(\kappa\nabla u)+f(\mathbf{x},t,u), \tag{8.12.2}$$

where $\kappa=\kappa(\mathbf{x},u)$ is a function of $\mathbf{x}$ and u.

For the case of several chemicals or interacting species, the vector $u_i(\mathbf{x},t)$, $i=1,2,\ldots,n$, represents concentrations or densities each diffusing with its own diffusion coefficient κ_i and interacting according to the vector source term $\mathbf{f}(\mathbf{x},\mathbf{u},t)$. Then, equation (8.12.2) becomes

$$\frac{\partial \mathbf{u}}{\partial t}=\nabla\cdot(\kappa\nabla\mathbf{u})+\mathbf{f}(\mathbf{x},t,\mathbf{u}), \tag{8.12.3}$$

where κ represents a diffusivity matrix with no cross-diffusion among the species, so that κ simply becomes a diagonal matrix. In fact, $\nabla\mathbf{u}$ is a tensor, so that $\nabla\cdot\kappa\nabla\mathbf{u}$ becomes a vector. Equation (8.12.3) represents a system of *nonlinear* (or *interacting population–diffusion*) equations. Turin (1952) used it as an important model for the chemical basis of morphogenesis. Such systems have been widely investigated in the 1970s. One simple model equation as a special case of (8.12.3) is

$$\frac{\partial \mathbf{u}}{\partial t}=\nabla\cdot(\kappa\nabla\mathbf{u})+\mathbf{f}(\mathbf{u}), \tag{8.12.4}$$

where κ is a diagonal matrix.

In particular, when κ is independent of $\mathbf{x}$ and $\mathbf{u}$, equation (8.12.4) reduces to the form

$$\frac{\partial \mathbf{u}}{\partial t}=\kappa\nabla^2\mathbf{u}+f(\mathbf{u}). \tag{8.12.5}$$

Physically, the components of $\mathbf{u}$ represent the concentrations of certain species, which are reacting with each other at each point and also diffusing through the medium. Further, if $\mathbf{f}(\mathbf{u}) = \mathbf{0}$, (8.12.5) becomes a system of diffusion equations

$$\frac{\partial \mathbf{u}}{\partial t} = \kappa \nabla^2 \mathbf{u}. \tag{8.12.6}$$

This is regarded as a mathematical model of the classical kinetics of systems of reaction. For more information, the reader is referred to Kopell and Howard (1973), who studied plane wave solutions of reaction–diffusion equations.

In the one-dimensional case with f as a function of u only, we can write (8.12.2) as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\kappa(u)\frac{\partial u}{\partial x}\right] + f(u), \tag{8.12.7}$$

where usually $\kappa(u) = ku^r$, and k and r are positive constants. This equation with $f \equiv 0$ has been studied much more widely than that of the nonzero f. However, it is of interest to examine equation (8.12.7) with functions $f(u)$ with two simple zeros, one at $u = 0$ and the other at $u = 1$.

We write another version of (8.12.7) with $f(u) = au^m(1-u^n)$, where a, m, and n are positive constants in the form

$$\frac{\partial u}{\partial t} = k\frac{\partial}{\partial x}\left(u^r \frac{\partial u}{\partial x}\right) + au^m(1-u^n). \tag{8.12.8}$$

Pattle (1959) solved this equation without the reaction term $(a = 0)$. On the other hand, Newman (1980) obtained a number of traveling wave solutions $u(\xi)$, $\xi = x - ct$, for various values of r and m with $n = 1$. His study shows that, when $r > 0$, $u(\xi) = 0$ at a prescribed positive value of ξ. Furthermore, when $r = 0$ and $m = 2$, the solution represents an ordinary logistic distribution. According to Newman, this model equation can be applied to problems with growth phenomena, and also to population genetics and combustion processes. Montroll and West (1973) obtained a number of solutions for various problems involving diffusion and growth.

It is convenient to use nondimensional variables $x^* = \sqrt{\frac{a}{k}}x$ and $t^* = at$ to rewrite (8.12.6) in nondimensional form, dropping the asterisks,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^r \frac{\partial u}{\partial x}\right) + u^m(1-u^n). \tag{8.12.9}$$

With $u(x,t) = u(\xi)$, $\xi = x - ct$, we obtain the traveling wave solution of the equation

$$\frac{d}{d\xi}\left(u^r \frac{du}{d\xi}\right) + c\frac{du}{d\xi} + u^m(1-u^n) = 0. \tag{8.12.10}$$

Several authors including Kaliappan (1984) and Murray (1993) obtained an exact traveling wave solution of (8.12.10) for the case where $r = 0$ and $m = 1$ with the boundary conditions

$$u(-\infty) = 1 \quad \text{and} \quad u(+\infty) = 0. \tag{8.12.11}$$

Making reference to Murray's (1993) detailed analysis with $r = 0$ and $m = 1$, we seek an exact solution of (8.12.10) in the form

$$u(\xi) = \frac{1}{(1 + ae^{b\xi})^s}, \tag{8.12.12}$$

where a, b, and s are positive constants to be determined. This solution automatically satisfies the boundary conditions (8.12.11) at $\xi = \pm\infty$. We substitute (8.12.12) in (8.12.10) to obtain the following results:

$$s = \frac{2}{n}, \qquad b = \frac{n}{\sqrt{2(n+2)}}, \qquad c = \frac{n+4}{\sqrt{2(n+2)}}, \tag{8.12.13}$$

$$u(0) = u_0, \qquad a = u_0^{-\frac{n}{2}} - 1. \tag{8.12.14}$$

The second derivative of solution (8.12.12) gives the point of inflection

$$\xi_i = \frac{1}{b} \log\left(\frac{n}{2a}\right), \qquad u_i = \left(1 + \frac{n}{2}\right)^{-\frac{2}{n}}. \tag{8.12.15}$$

The above analysis provides a more general traveling wave solution of the power law logistic equation with diffusion. In particular, when $r = m = 0$ and $n = 1$, (8.12.10) reduces to the Fisher equation which was solved earlier. If we choose $u_0 = \frac{1}{2}$, then $a = \sqrt{2} - 1$, and hence, all quantities b, c, ξ_i, and u_i can be calculated, and solution (8.12.12) is known exactly.

If we set $n = 2$ in (8.12.13)–(8.12.15), then $s = 1$, $b = \frac{1}{\sqrt{2}}$, $c = \frac{3}{\sqrt{2}}$, $a = 1$, $\xi_i = 0$, and $u_i = \frac{1}{2}$. In this case, the solution (8.12.12) reduces to that of the ordinary logistic equation with diffusion.

When $r = 0$ and $m = n + 1$, equation (8.12.9) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^{n+1}\big(1 - u^n\big). \tag{8.12.16}$$

Following Murray (1993), we obtain the traveling wave solution $u(\xi)$ with $\xi = x - ct$ of equation (8.12.16) in the form

$$u(\xi) = \frac{1}{(1 + ae^{b\xi})^s}, \tag{8.12.17}$$

where $s = \frac{1}{n}$, $b = \frac{n}{\sqrt{n+1}}$, and $c = \frac{1}{\sqrt{n+1}}$.

When $n = 1$, equation (8.12.16) is further simplified, and then, $s = 1$, $b = c = \frac{1}{\sqrt{2}}$, and the resulting solution obtained from (8.12.17) is in agreement with Newman's (1980) solution in population genetics. This solution also describes the situation in which the reaction term is modified by the coalition growth instead of the logistic growth.

Another interesting and useful exact solution was obtained by Murray (1993) for the case $r = m = n = 1$. This case describes density-dependent diffusion with logistic population growth, which is represented by the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u(1-u). \tag{8.12.18}$$

We seek a traveling wave solution $u(x,t) = u(\xi)$, $\xi = x - ct$ of (8.12.18) with $u(-\infty) = 1$ and $u(+\infty) = 0$. Thus, equation (8.12.18) reduces to

$$(uu')' + cu' + u(1-u) = 0, \tag{8.12.19}$$

where the prime denotes the derivative with respect to ξ.

Thus, the associated phase plane system is given by

$$\frac{du}{d\xi} = v, \tag{8.12.20a}$$

$$u\frac{dv}{d\xi} = -\big[cv + v^2 + u(1-u)\big]. \tag{8.12.20b}$$

In (8.12.20b), $u = 0$ is a singularity which can be removed by defining a new variable ζ by $\frac{d}{d\zeta} = u\frac{d}{d\xi}$, so that equations (8.12.20a)–(8.12.20b) become

$$\frac{du}{d\zeta} = uv \quad \text{and} \tag{8.12.21a}$$

$$\frac{dv}{d\zeta} = -\big[cv + v^2 + u(1-u)\big]. \tag{8.12.21b}$$

Clearly, the second equation is *not* singular. The singular points in the (u,v)-phase plane are $(u,v) = (0,0)$, $(1,0)$, $(0,-c)$. A phase plane analysis shows that $(1,0)$ and $(0,-c)$ are saddle points, whereas $(0,0)$ is a stable nonlinear node because of the nonlinear term uv in (8.12.21a). Murray (1993) described the phase trajectories of (8.12.21a)–(8.12.21b) for different values of c with graphical representations.

Basically, the model equation (8.12.18) indicates that the population disperses to regions of lower density more rapidly as the population increases. The traveling wave solution $u(\xi)$with boundary conditions $u(-\infty) = 1$, $u(+\infty) = 0$, and $c = \frac{1}{\sqrt{2}}$ is given by

$$u(\xi) = \left\{1 - \exp\left(\frac{\xi - \xi_c}{\sqrt{2}}\right)\right\} H(\xi_c - \xi), \tag{8.12.22}$$

where ξ_c is the wave front and $H(x)$ is the Heaviside unit step function.

Several authors including Aronson (1980), Crank (1975), Fife (1979), Ghez (1988), Gurney and Nisbet (1975), Gurtin and MacCamy (1977), Newman (1980), and Shigesada (1980) have studied various problems of density-dependent diffusion and their extensions.

The above analysis, showing the existence of the traveling wave solutions, can be generalized to more general, nonlinear, diffusion phenomena in which the diffusion coefficient is u^r, for $r \neq 1$, or, even more general, $\kappa(u)$ in (8.12.4) with certain conditions imposed on $\kappa(u)$. In general, nonlinear reaction–diffusion interactions radically change the mathematical and physical features of the solutions of many biological and chemical problems.

Further, we write out the diffusion term in (8.12.4) in full to obtain

$$u_t = f(u) + \kappa'(u)u_x^2 + \kappa(u)u_{xx}. \tag{8.12.23}$$

This shows that the nonlinear diffusion term can be thought of as an equivalent convection with velocity $h'(u) = -\kappa'(u)u_x$. If the convection arises as a natural extension of a conservation law, we can derive an equation with the convection velocity $h'(u)$ in the form

$$\frac{\partial u}{\partial t} + h'(u)u_x = f(u) + \kappa(u)\frac{\partial^2 u}{\partial x^2}. \tag{8.12.24}$$

Such equations arise in a wide variety of problems in physical, chemical, and biological sciences. It is important to point out that the nonlinear convection in reaction–diffusion equations can have a significant effect on the solutions. This is to be expected since there is another major transport process, namely, linear or nonlinear convection. This process may or may not dominate over diffusion effects. However, if the effect of diffusion is negligible compared with that of convection, convection obviously tends to make parts of the wave form grow steeper and steeper, and solutions develop discontinuities in the form of shock waves. Indeed, the development of shock waves and the breaking of a wave form are typical nonlinear convection phenomena.

8.13 Brief Summary of Recent Work

To understand various physical phenomena described by the interaction of diffusion, convection, or relaxation, Lighthill (1956) gave a fully self-contained account of the basic physical features of shock wave development, propagation, and decay, the internal structure of shocks, and their confluence with the derivation of a nonlinear coupled system intermediate between the Navier–Stokes equations and the Burgers equation. He then discussed approximate solutions of several physical problems by using graphical construction of characteristics and the method of steepest descent. He also showed that the Burgers equation is the appropriate one for the investigation of weak planar-wave phenomena. Subsequently, many authors including Blackstock (1964), Walsh (1969), Benton and Platzman (1972), Crighton and Scott (1979), Parker (1980), Rodin (1970), Larson (1978), and Lardner (1986) studied initial boundary-value problems for the Burgers equation and investigated the physical significance of their solutions. It is pertinent to mention that Benton and Platzman (1972) have also compiled an exhaustive list of the possible solutions of the Burgers equation and have illustrated the physically interesting ones by isochronal graphs. At the same time, considerable attention has been given to the determination of self-similar solutions as intermediate asymptotics for nonlinear diffusion equations (see Barenblatt and Zel'dovich 1972; Zel'dovich and Raizer 1966, 1968; Barenblatt 1979, and Newman 1983). Many generalizations of the Burgers equation have been made by several authors including Case and Chiu (1969), Murray (1970a, 1970b, 1973), Penel and Brauner (1974), and Crighton (1979). Kriess and Lorenz (1989) discussed the existence and uniqueness of solutions of boundary-value problems associated with the Burgers equation. Recent developments in the reaction–diffusion equation in general and the Fisher equation in particular are available in papers by Johnson (1970), McKean (1975), Barenblatt (1979), Manoranjan and Mitchell (1983), Hagstrom and Keller (1986), Gazdag and Canosa (1974), Murray (1993), Smoller (1994), Britton (1986), Grindrod (1991), Dunbar (1983), Logan and Dunbar (1992), and Fife (1979). The Fisher equation also represents a nonlinear model equation for the evolution of a neutron population in a nuclear reactor. Canosa (1969, 1973) showed that the Fisher equation describes a balance between linear diffusion and nonlinear local multiplication, and it admits shock-type solutions. Gazdag and Canosa (1974) used the pseudospectral method accurately to discretize spatial derivatives for numerically solving the Fisher equation. Tang and Webber (1991) presented an interesting and precise numerical study of the Fisher equation by a Petrov–Galerkin finite element method. Their analysis shows that any *local* initial disturbance can propagate with a constant limiting speed as time tends to infinity. Both the limiting wave fronts and limiting speed are determined by the system itself and are independent of the initial states. Compared with other numerical methods, Tang and Webber's numerical study is more satisfactory with regard to its accuracy and stability. Larson (1978) studied a more general Fisher equation and determined lower and upper bounds of its solutions. Ablowitz and Zeppetella (1979) have investigated the exact solution and Painlevé transcendents. They have also shown that equation (8.9.2) can be transformed into a simpler nonlinear equation which admits solutions in terms of the Weierstrass $\wp$ function. For the generalized Fisher equation with $f(u)$ in place of $u(1-u)$, Kametaka (1976) proved the existence theorem for the traveling wave solution when certain conditions on $f(u)$ hold, and he investigated the stability of traveling wave and transient solutions. Finally, we refer to the work of Abdelkader (1982), who extended the results of Ablowitz and Zeppetella (1979) for the generalized Fisher equation of the form

$$u'' + cu' + u - u^n = 0, \quad 1 < n < \infty. \tag{8.13.1}$$

For $c = (n+3)/\sqrt{(2n+2)}$, $2 < c < \infty$, the solution of (8.13.1) can be expressed in terms of hyperelliptic integrals. In some special cases, the solution was obtained by Rosenau (1982) in terms of elliptic integrals for a related equation describing nonlinear thermal wave phenomena in a reacting medium.

8.14 Exercises

1. Determine a set of stretching transformations (8.11.4) under which the following equations are invariant:

   (a) $u_{tt} = c^2 u_{xx}$, (b) $uu_t + u_x^2 = 0$,
   (c) $u_t + uu_x = 0$, (d) $u_t = \kappa(u_{rr} + \frac{2}{r}u_r)$,
   (e) $uu_t = u_{xx}$ $(x > 0, t > 0)$, (f) $c^2 u_{tt} + u_{xxxx} = 0$ $(c^2 = \frac{EI}{m})$.

2. Use $\xi = x - ct$ to reduce the reaction–diffusion equation (see Jones and Sleeman 1983)
$$u_t - u_{xx} = f(u)$$
into the form
$$\frac{d^2u}{d\xi^2} + c\frac{du}{d\xi} + f(u) = 0.$$
Solve this ordinary differential equation with the reaction term
$$f(u) = \begin{cases} u & \text{on } D_1 : 0 \le u \le \frac{1}{2}, \\ 1-u & \text{on } D_2 : \frac{1}{2} \le u \le 1 \end{cases}$$
and the boundary conditions
$$u(-\infty) = 1 \quad \text{and} \quad u(+\infty) = 0.$$
Discuss the solution for cases (i) $c \neq 2$ and (ii) $c = 2$.
3. Show that the similarity transformation of the Boltzmann nonlinear equation (8.11.20) with boundary conditions
$$(uu_x)_{x=0} = -a = \text{const.}, \quad u(x,t) \to 0 \quad \text{as } x \to \infty, t > 0,$$
$$u(x,0) = 0 \quad \text{for } x > 0,$$
is
$$u(x,t) = t^{\frac{1}{3}} v(\eta), \quad \eta = xt^{-\frac{2}{3}},$$
where $v(\eta)$ satisfies the nonlinear ordinary differential equation
$$3(vv')' + 2\eta v' - v = 0.$$
4. Show that the Boltzmann equation (8.11.20) is invariant under the one-parameter set of transformations $\tilde{x} = x\exp(\frac{\alpha\tau}{2})$, $\tilde{t} = t + \tau$, $\tilde{u} = u\exp(\alpha\tau)$, $0 < \tau < \infty$. The boundary condition (Barenblatt and Zel'dovich 1972)
$$u(0,t) = u_0 \exp(t), \quad -\infty < t < \infty,$$
is also invariant under the above set of transformations when $\alpha = 1$. Show that the most general invariant solution has the form
$$u(x,t) = \exp(\alpha t) f\left[x\exp\left(-\frac{\alpha t}{2}\right)\right].$$
When $\alpha = 1$, $u(x,t) = \exp(t)v\{x\exp(-\frac{t}{2})\}$. If it is a solution of (8.11.20), then its image under the above set of transformations must be a solution, that is,
$$e^{\alpha\tau}u(x,t) = v(\eta)\exp(t+r), \quad \eta = x\exp\left(\frac{\alpha\tau}{2}\right)\exp\left(-\frac{t+r}{2}\right)$$
is also a solution.

5. Show that the similarity solution of the Burgers equation (8.2.6) with $\upsilon = 1$ is

$$u(x,t) = (2t + a)^{-2} f(\eta), \quad \eta = (2t + a)^{-\frac{1}{2}} x,$$

where $f(\eta)$ satisfies the nonlinear ordinary differential equation

$$f'' - (f - \eta) f' + f = 0.$$

6. Show that the similarity solution of the Burgers equation (8.2.6) with $\nu = 1$ is given by

$$u(x,t) = \frac{1}{\sqrt{t}} f(\eta) - 1, \quad \eta = (x + t) t^{-\frac{1}{2}},$$

where f satisfies the equation

$$f'' + f' \left(\frac{\eta}{2} - f \right) + \frac{1}{2} f = 0.$$

7. The Prandtl–Blasius boundary-layer problem for a flat plate can be described by the equations

$$\begin{aligned} u u_x + v u_y &= \nu u_{yy}, \\ u_x + v_y &= 0, \end{aligned}$$

with the boundary conditions

$$\begin{aligned} (u, v) = (0, 0) & \quad \text{at } y = 0,\ x > 0, \\ u = U & \quad \text{as } y \to \infty,\ x > 0, \\ u = U & \quad \text{at } x = 0,\ y > 0, \end{aligned}$$

where U is the incident velocity. Show that these equations are invariant under the set of transformations

$$\tilde{x} = a^{\alpha} x, \qquad \tilde{y} = a y, \qquad \tilde{u} = a^{\beta} u, \quad \text{and} \quad \tilde{v} = a^{-1} v,$$

where $\alpha - \beta = 2$. Show also that the boundary conditions are all invariant under the above transformations when $\alpha = 2$ and $\beta = 0$. Show that the first two boundary conditions are also invariant even if $\beta \neq 0$ and the last two boundary conditions become $\tilde{u} = a^{\beta} U$.

8. In terms of the stream function ψ defined by $(u, v) = (\psi_y, -\psi_x)$ with $\nu = 1$ and $U = 1$, the Prandtl–Blasius equation in Exercise 7 is given by

$$\psi_{yyy} = \psi_y \psi_{xy} - \psi_x \psi_{yy}.$$

Show that the set of similarity transformations of this equation is

$$\psi(x, y) = x^{\frac{1}{2}} f(\eta), \quad \eta = y x^{-\frac{1}{2}},$$

where $f(\eta)$ satisfies the Blasius equation

$$2f''' + ff'' = 0$$

with boundary conditions

$$f(0) = 0, \qquad f'(0) = 0, \quad \text{and} \quad f'(\infty) = 1.$$

9. The Sparrow et al. (1970) problem of a flat plate with uniform suction or injection is obtained by changing the variables $(\xi, \eta) = (v_\omega \sqrt{\frac{x}{U}}, y\sqrt{\frac{U}{x}})$ in the Prandtl–Blasius equation in Exercise 8, where v_ω is the constant suction velocity. Show that ξ and η are invariant under the set of transformations introduced in Exercise 7. Show that the similarity transformation of this problem with $\nu = 1$ and $U = 1$ is given by $\psi = x^{\frac{1}{2}} f(\xi, \eta)$, where f satisfies the equation

$$2f_{\eta\eta\eta} + ff_{\eta\eta} = \xi(f_\eta f_{\xi\eta} - f_\xi f_{\eta\eta})$$

with boundary conditions

$$f_\eta(\eta = 0) = 0, \quad f_\eta(\eta \to \infty) = 1, \quad \text{and} \quad f + 2\xi = -\xi f_\xi \quad \text{at } \eta = 0.$$

10. In Exercise 1(e), show that

$$v(\eta) = t^{\frac{2\alpha}{\beta} - 1} u(x, t), \quad \eta = x^\beta t^{-\alpha}$$

are invariant under (8.11.4). Using the data $u(x, 0) = 0$, $x > 0$, $u(x, t) \to 0$, as $x \to \infty$, $t > 0$, and $u_x(0, t) = -q$, $t > 0$, where q is a constant, show that $\beta = 1$ and $\alpha = \frac{1}{3}$. Hence, show that $v(\eta)$ satisfies the problem

$$3v'' - v(v - \eta v') = 0, \quad 0 < \eta < \infty,$$
$$v'(0) = -q, \quad \text{and} \quad v(\eta) \to 0 \quad \text{as } \eta \to \infty.$$

11. In Exercise 1(f), show that the quantities

$$v(\eta) = t^{-\frac{\gamma}{\beta}} u \quad \text{and} \quad \eta = xt^{-\alpha/\beta}$$

are invariant under (8.11.4). Hence prove that $v(\eta)$ satisfies the ordinary differential equation

$$v'''' + c^2 \left\{ \left(\frac{\alpha}{\beta}\right)^2 \eta^2 v'' + \frac{\alpha}{\beta}\left(1 + \frac{\alpha}{\beta} - \frac{2\gamma}{\beta}\right)\eta v' + \left(\frac{\gamma}{\beta} - 1\right)\left(\frac{\gamma}{\beta}\right) v \right\} = 0.$$

Derive the equation for $\frac{\alpha}{\beta} = \frac{1}{2} = -\frac{\gamma}{\beta}$.

12. Show that equations

$$\text{(a) } u_t + uu_x + \mu u_{xxx} = 0 \quad \text{and} \quad \text{(b) } u_t + u^p u_x + \mu u_{xxx} = 0$$

are invariant under the transformations

$$\tilde{x} = a^\alpha x, \qquad \tilde{t} = a^\beta t, \quad \text{and} \quad \tilde{u} = au,$$

provided that $\alpha = -\frac{1}{2}$ and $\beta = -\frac{3}{2}$ for equation (a), and $\alpha = -\frac{p}{2}$ and $\beta = -\frac{3p}{2}$ for equation (b).

13. Consider the nonlinear reaction–diffusion model

$$\begin{aligned} u_t - \kappa u_{xx} &= F(u), \quad 0 < x < a,\ t > 0, \\ u_x(0,t) &= 0 = u_x(a,t), \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < a, \end{aligned}$$

where the reaction term F is continuously differentiable and $\sup_{u\in R}|F'(u)| = A < \infty$.

Show that the quantity

$$Q(t) = \int_0^a u_x^2(x,t)\,dx$$

associated with this model satisfies the inequality

$$Q(t) \le Q(0)\exp(\mu t), \quad \mu = 2\bigl(A - \pi^2\bigr).$$

If $A < \pi^2$, show that $Q(t) \to 0$ as $t \to \infty$.

14. Consider the coupled model equations

$$u_t = \kappa\left(u_x - \frac{2u}{c}c_x\right)_x, \qquad c_t + ku = 0,$$

where $u = u(x,t)$ is the population of an organism, $c = c(x,t)$ is the concentration of a nutrient to which the organisms are attracted, and k is a positive constant. Obtain the traveling wave solutions in the form

$$u = u(x - Ut) \quad \text{and} \quad c = c(x - Ut),$$

where

$$U(\pm\infty) = 0, \qquad c(-\infty) = 0, \quad \text{and} \quad c(+\infty) = 1.$$

Draw the graph of the solutions and give a biological interpretation of the solutions.

15. If $av = (1-b)(1-u)$, $a > 0$, $0 < b < 1$, show that the reaction–diffusion system

$$u_t - u_{xx} = u(1 - u - av), \qquad v_t - v_{xx} + buv = 0$$

reduces to the Fisher equation.

16. Consider a simple model of a detonation process governed by the system

$$u_t + uu_x + \lambda_x = \kappa u_{xx}, \qquad \lambda_t = r(u,\lambda),$$

where u is a scaled temperature and λ is the mass fraction of the product species P in a reversible chemical reaction $R \leftrightarrow P$ with the reaction rate $r = 1 - \lambda - \lambda\exp(-u^{-1})$, and where κ is a positive constant. Show that there exist positive traveling wave solutions if the wave speed c exceeds the value of u at $+\infty$ (Logan and Dunbar 1992).

17. Show that the traveling wave solution exists for the wave velocity $c > 2$ for the reaction–convection system (Logan and Shores 1993a, 1993b):

$$u_t + uu_x + v_x = (2-u)(u-1), \qquad v_t + v = 1 - v\exp(u^{-1}).$$

18. (a) Show that the transformation $v(x,t) = \exp(-\alpha t)u(x,t)$ reduces the equation

$$v_t + \alpha v = \kappa u_{xx},$$

into the form

$$u_t = \kappa u_{xx}.$$

(b) Find a change of variable that reduces the diffusion equation

$$u_t = \kappa u_{xx},$$

into the form

$$v_t = v_{xx}.$$

19. Use the transformations

$$a(\tau)\,d\tau \quad \text{and} \quad v(\xi,t) = u\left[\xi + \int_0^t a(\tau)\,d\tau, t\right]$$

to show that the equation

$$u_t + a(t)u_x = \kappa u_{xx},$$

can be transformed into the form

$$v_t = \kappa v_{xx}.$$

20. Using the conservation of density (8.2.1) with $q = \rho u$ and the conservation of momentum, $q_t + (uq)_x = 0$, derive the inviscid Burgers equation

$$u_t + uu_x = 0, \quad \text{or equivalently,} \quad u_t + \left(\frac{1}{2}u^2\right)_x = 0.$$

Hence, or otherwise, show that the solution u_x at any time t is

$$(u_x)(t) = \frac{u_0'}{1+(t-t_0)u_0'},$$

where $u_0' = (u_x)_{t=t_0}$.

21. The Fokker–Planck equation in non-equilibrium statistical mechanics for the probability distribution function $u(x, t)$ is a modified diffusion equation

$$u_t = u_{xx} + (xu)_x, \quad x \in \mathbb{R},\ t > 0.$$

By separation of variables $u = X(x)T(t)$, show that

$$X''(x) + xX' + (1 + n)X = 0 \quad \text{and} \quad \dot{T} + nT = 0.$$

where n is the separation constant.
Assuming $X(x) = f(x)\exp(-\frac{1}{2}x^2)$ and rescaling the independent variables $(x = \sqrt{2}\xi)$, show that $f(\xi) = H_n(\xi)$ satisfies the equation

$$\frac{d^2 f}{d\xi^2} - 2\xi \frac{df}{d\xi} + 2nf = 0.$$

Show that the solution $u(x, t)$ of the Fokker–Planck equation is

$$u(x, t) = \sum_{n=0}^{\infty} a_n \exp\left(-nt - \frac{1}{2}x^2\right) H_n\left(\frac{x}{\sqrt{2}}\right).$$

Find the limiting form of the solution $u(x, t)$ as $t \to \infty$.

9

Solitons and the Inverse Scattering Transform

True Laws of Nature cannot be linear.

Albert Einstein

... the great primary waves of translation cross each other without change of any kind in the same manner as the small oscillations produced on the surface of a pool by a falling stone.

Scott Russell

9.1 Introduction

Dispersion and nonlinearity play a fundamental role in wave motions in nature. The nonlinear shallow water equations that neglect dispersion altogether lead to breaking phenomena of the typical hyperbolic kind with the development of a vertical profile. In particular, the linear dispersive term in the Korteweg–de Vries equation prevents this from ever happening in its solution. In general, breaking can be prevented by including dispersive effects in the shallow water theory. The nonlinear theory provides some insight into the question of how nonlinearity affects dispersive wave motions. Another interesting feature is the instability and subsequent modulation of an initially uniform wave profile.

To understand these features, this chapter is devoted to the Boussinesq and Korteweg–de Vries (KdV) equations and solitons with emphasis on the methods and solutions of these equations that originated from water waves. Special attention is given to the inverse scattering transform, conservation laws and nonlinear transformations, Bäcklund transformations and the nonlinear superposition principle, the Lax formulation and its KdV hierarchy, the ZS (Zakharov and Shabat) scheme, and the AKNS (Ablowitz, Kaup, Newell, and Segur) method. Finally, we consider a new class of solitary waves with compact support which are called *compactons* governed by a two-parameter family of *strongly* dispersive nonlinear equations, $K(m,n)$. Included are the existence of *peakon* (*singular*) solutions of a new strongly nonlinear

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model in shallow water described by the Camassa and Holm equation, and the Harry Dym equation which arises as a generalization of the class of isospectral flows of the Schrödinger operator. As an example of the application of compactons, the solution of nonlinear vibration of an anharmonic mass–spring system is presented. This is followed by a discussion on the existence of new nonlinear intrinsic localized modes in anharmonic crystals. Based on the rotating-wave approximation, Sievers and Takeno (1988) discovered the s-localized modes, while Page (1990) introduced the p-localized modes in a one-dimensional lattice model. These nonlinear localized modes may be treated as compactons.

9.2 The History of the Solitons and Soliton Interactions

Historically, John Scott Russell first experimentally observed the solitary wave, a long water wave without change in shape, on the Edinburgh–Glasgow Canal in 1834. He called it the "*great wave of translation*" and then reported his observations at the British Association in his 1844 paper "Report on Waves." Thus, the solitary wave represents, not a periodic wave, but the propagation of a single isolated symmetrical hump of unchanged form. His discovery of this remarkable phenomenon inspired him further to conduct a series of extensive laboratory experiments on the generation and propagation of such waves. Based on his experimental findings, Russell discovered, empirically, one of the most important relations between the speed U of a solitary wave and its maximum amplitude a above the free surface of liquid of finite depth h in the form

$$U^2 = g(h + a), \tag{9.2.1}$$

where g is the acceleration due to gravity. His experiments stimulated great interest in the subject of water waves and his findings received a strong criticism from Airy (1845) and Stokes (1847). In spite of his remarkable work on the existence of periodic wavetrains representing a typical feature of nonlinear dispersive wave systems, Stokes' conclusion on the existence of the solitary wave was erroneous. However, Stokes (1847) proposed that the free surface elevation of the plane wavetrains on deep water can be expanded in powers of the wave amplitude. His original result for the dispersion relation in deep water is

$$\omega^2 = gk\big(1 + a^2k^2 + \cdots\big). \tag{9.2.2}$$

Stokes' theory predicts that periodic wavetrains are possible in nonlinear dispersive systems, and the dispersion relation involves the wave amplitude, which produces significant qualitative changes in water wave phenomena. In spite of the introduction of the Stokes expansion, definite analytical proof of the existence of a solution representing permanent water waves has been a formidable task. During the first quarter of the twentieth century, there has been a serious controversy on this issue, and, in fact, real doubt was raised about the convergence of the Stokes expansion. The problem was eventually resolved by Levi-Civita (1925), who proved formally

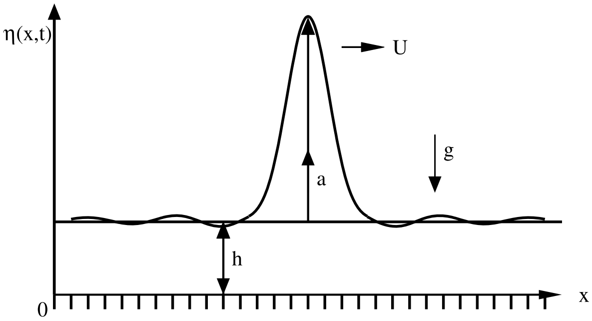

Fig. 9.1 A solitary wave.

that the Stokes expansion for deep water waves converges provided the wave amplitude is very small compared with the wavelength. Almost simultaneously, Struik (1926) extended the proof of convergence to small-amplitude waves on water of finite depth. Subsequently, Krasovskii (1960, 1961) established the existence of permanent periodic waves for all values of amplitude less than the maximum value at which the waves assume a sharp-crested form. Finally, rigorous proofs of the existence of water waves of greatest height have been given by Toland (1978) and Keady and Norbury (1978). Despite these serious attempts to prove the existence of finite-amplitude water waves of permanent form, the independent question of their stability remained unattended until the 1960s except for an isolated study by Korteweg and de Vries in (1895) on long surface waves in water of finite depth. But one of the most remarkable discoveries made in the 1960s was that the periodic Stokes waves on sufficiently deep water are *definitely* unstable! This result seems revolutionary in view of the sustained attempts to prove the existence of Stokes waves of finite amplitude and permanent form. Russell's description of solitary waves contradicted the theories of water waves due to Airy and Stokes; they raised questions on the existence of Russell's solitary waves and conjectured that such waves cannot propagate in a liquid medium without change of form. It was not until the 1870s that Russell's prediction was finally and independently confirmed by both Boussinesq (1871a, 1871b, 1872, 1877) and Rayleigh (1876). From the equations of motion for an inviscid incompressible liquid, they derived formula (9.2.1). In fact, they also showed that the solitary wave profile (see Figure 9.1) $z = \eta(x,t)$ is given by

$$\eta(x,t) = a\,\mathrm{sech}^2\left[\beta(x-Ut)\right], \tag{9.2.3}$$

where $\beta^2 = 3a\,\mathrm{div}\{4h^2(h+a)\}$ for any $a > 0$.

Although these authors found the sech^2 solution, which is valid only if $a \ll h$, they did not write any equation for η that admits (9.2.3) as a solution. However, Boussinesq did a lot more and discovered several new ideas, including a nonlinear evolution equation for such long water waves in the form

$$\eta_{tt} = c^2\left[\eta_{xx} + \frac{3}{2}\left(\frac{\eta^2}{h}\right)_{xx} + \frac{1}{3}h^2\eta_{xxxx}\right], \tag{9.2.4}$$

where $c = \sqrt{gh}$ is the speed of the shallow water waves. This is known as the *Boussinesq* (*bidirectional*) *equation*, which admits the solution

$$\eta(x,t) = a\,\mathrm{sech}^2\left[\left(3a/h^3\right)^{1/2}(x \pm Ut)\right]. \tag{9.2.5}$$

This represents solitary waves traveling in both the positive and negative x-directions.

More than 60 years later, in 1895, two Dutchmen, D.J. Korteweg and G. de Vries, formulated a mathematical model equation to provide an explanation of the phenomenon observed by Scott Russell. They derived the now-famous equation for the propagation of waves in one direction on the surface water of density ρ in the form

$$\eta_t = \frac{c}{h}\left[\left(\varepsilon + \frac{3}{2}\eta\right)\eta_X + \frac{1}{2}\,\sigma\eta_{XXX}\right], \tag{9.2.6}$$

where X is a coordinate chosen to be moving (almost) with the wave, $c = \sqrt{gh}$, ε is a small parameter, and

$$\sigma = h\left(\frac{h^2}{3} - \frac{T}{g\rho}\right) \sim \frac{1}{3}h^3, \tag{9.2.7}$$

when the surface tension $T(\ll \frac{1}{3}g\rho h^2)$ is negligible. Equation (9.2.6) is known as the *Korteweg–de Vries* (*KdV*) *equation*. This is one of the simplest and most useful nonlinear model equations for solitary waves, and it represents the longtime evolution of wave phenomena in which the steepening effect of the nonlinear term is counterbalanced by linear dispersion.

It is convenient to introduce the change of variables $\eta = \eta(X^*, t)$ and $X^* = X + (\varepsilon/h)ct$ which, dropping the asterisks, allows us to rewrite equation (9.2.6) in the form

$$\eta_t = \frac{c}{h}\left(\frac{3}{2}\eta\eta_X + \frac{1}{2}\sigma\eta_{XXX}\right). \tag{9.2.8}$$

Modern developments in the theory and applications of the KdV solitary waves began with the seminal work published as a Los Alamos Scientific Laboratory Report in 1955 by Fermi, Pasta, and Ulam on a numerical model of a discrete nonlinear mass–spring system. In 1914, Debye suggested that the finite thermal conductivity of an anharmonic lattice is due to the nonlinear forces in the springs. This suggestion led Fermi, Pasta, and Ulam to believe that a smooth initial state would eventually relax to an equipartition of energy among all modes because of nonlinearity. But their study led to the striking conclusion that there is no equipartition of energy among the modes. Although all the energy was initially in the lowest modes, after flowing back and forth among various low-order modes, it eventually returns to the lowest mode, and the end state is a series of recurring states. This remarkable fact has become known as the *Fermi–Pasta–Ulam* (FPU) *recurrence phenomenon.* Cercignani (1977) and later on Palais (1997) described the FPU experiment and its relationship to the KdV equation in some detail.

This curious result of the FPU experiment inspired Martin Kruskal and Norman Zabusky to formulate a continuum model for the nonlinear mass–spring system to

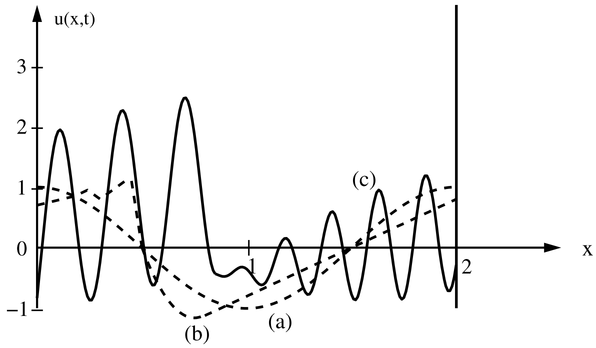

Fig. 9.2 Development of solitary waves: (a) initial profile at $t = 0$, (b) profile at $t = \pi^{-1}$, and (c) wave profile at $t = (3.6)\pi^{-1}$. From Zabusky and Kruskal (1965).

understand why recurrence occurred. In fact, they considered the initial-value problem for the KdV equation,

$$u_t + uu_x + \delta u_{xxx} = 0, \tag{9.2.9}$$

where $\delta = (\frac{h}{\ell})^2$, ℓ is a typical horizontal length scale, with the initial condition

$$u(x, 0) = \cos \pi x, \quad 0 \le x \le 2, \tag{9.2.10}$$

and the periodic boundary conditions with period 2, so that $u(x,t) = u(x+2,t)$ for all t. Their numerical study with $\sqrt{\delta} = 0.022$ produced a lot of new interesting results which are shown in Figure 9.2.

They observed that, initially, the wave steepened in regions where it had a negative slope, a consequence of the dominant effects of nonlinearity over the dispersive term, δu_{xxx}. As the wave steepens, the dispersive effect then becomes significant and balances the nonlinearity. At later times, the solution develops a series of *eight* well-defined waves, each like sech^2 functions with the taller (faster) waves ever catching up and overtaking the shorter (slower) waves. These waves undergo nonlinear interaction according to the KdV equation and then emerge from the interaction without change of form and amplitude, but with only a small change in their phases. So, the most remarkable feature is that these waves retain their identities after the nonlinear interaction. Another surprising fact is that the initial profile reappears, very similarly to the FPU recurrence phenomenon. In view of their preservation of shape and the resemblance to the particle-like character of these waves, Kruskal and Zabusky called these solitary waves, *solitons*, like photon, proton, electron, and other terms for elementary particles.

Historically, the famous 1965 paper of Zabusky and Kruskal marked the birth of the new concept of the soliton, a name intended to signify particle-like quantities. Subsequently, Zabusky (1967) confirmed, numerically, the actual physical interaction of two solitons, and Lax (1968) gave a rigorous analytical proof that the identities of two distinct solitons are preserved through the nonlinear interaction governed by the KdV equation. Physically, when two solitons of different amplitudes (and hence, of different speeds) are placed far apart on the real line, the taller (faster) wave to the left of the shorter (slower) wave, the taller one eventually catches up to the shorter one and then overtakes it. When this happens, they undergo a nonlinear interaction according to the KdV equation and emerge from the interaction completely preserved in form and speed with only a phase shift. Thus, these two remarkable features, (i) steady progressive pulse-like solutions and (ii) the preservation of their shapes and speeds, confirmed the particle-like property of the waves and, hence, the definition of the soliton. Subsequently, Gardner et al. (1967, 1974) and Hirota (1971, 1973a, 1973b) constructed analytical solutions of the KdV equation that provide the description of the interaction among N solitons for any positive integral N. After the discovery of the complete integrability of the KdV equation in 1967, the theory of the KdV equation and its relationship to the Euler equations of motion as an approximate model derived from the theory of asymptotic expansions became of major interest. From a physical point of view, the KdV equation is not only a standard nonlinear model for long water waves in a dispersive medium, it also arises as an approximate model in numerous other fields, including ion-acoustic plasma waves, magnetohydrodynamic waves, and anharmonic lattice vibrations. Experimental confirmation of solitons and their interactions has been provided successfully by Zabusky and Galvin (1971), Hammack and Segur (1974), and Weidman and Maxworthy (1978). Thus, these discoveries have led, in turn, to extensive theoretical, experimental, and computational studies over the last 30 years. Many nonlinear model equations have now been found that possess similar properties, and diverse branches of pure and applied mathematics have been required to explain many of the novel features that have appeared.

Finally, it seems pertinent to mention the following. It is not easy to give a precise definition of a soliton. However, the term can readily be associated with any solution of a nonlinear partial differential equation (or system) that (i) represents a wave of permanent form, (ii) is localized, so that it decays or approaches a constant value at infinity, and (iii) can undergo a strong interaction with other solitons and retain its identity. In the context of the KdV equation and other similar equations, the single-soliton solution is usually referred to as the *solitary wave*, but when more than one of them appear in a solution, they are called *solitons.* In other words, a soliton is a solitary wave when it is infinitely separated from any other soliton. Also, for equations other than the KdV equation, the solitary-wave solution may not be a sech^2 function, but a sech or $\tan^{-1}(e^{\alpha x})$ profile. In fact, some nonlinear equations have solitary-wave solutions but not solitons, whereas others (like the KdV equation) have solitary waves that are solitons. Indeed, the soliton has formed a new paradigm in mathematical physics. In recent years, the concept of the soliton has been used in a very broad sense. For example, the nonlinear Schrödinger equation (NLS), describing waves in plasmas, in superconductors, and in nonlinear optics, etc., yields the *envelope soliton* solution which has the form of a sech profile modulating a monochromatic carrier wave. Unlike the KdV solitons, an NLS soliton does not depend on the amplitude. Physically, the former is called a *low-frequency soliton*, whereas the latter is called a *high-frequency soliton* (Makhankov 1978). Another important model equation, the

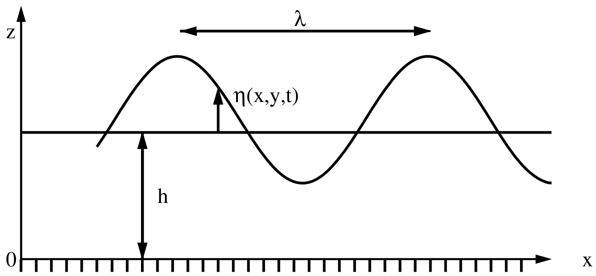

Fig. 9.3 A shallow water wave model.

sine-Gordon (SG) equation, is used to describe nonlinear wave motions in the unified theory of elementary particles, magnetic flows, and dislocations in crystals. This equation admits SG soliton solutions that are called either *kinks* or *antikinks*, whose velocities do not depend on the wave amplitude.

9.3 The Boussinesq and Korteweg–de Vries Equations

We consider an inviscid liquid of constant mean depth h and constant density ρ without surface tension. We assume that the (x, y)-plane is the undisturbed free surface with the z-axis positive upward. The free surface elevation above the undisturbed mean depth h is given by $z = \eta(x, y, t)$, so that the free surface is at $z = H = h + \eta$ and $z = 0$ is the horizontal rigid bottom (see Figure 9.3).

It has already been recognized that the parameters $\varepsilon = a/h$ and $\kappa = ak$, where a is the surface wave amplitude and k is the wavenumber, must both be small for the linearized theory of surface waves to be valid. To develop the nonlinear shallow water theory, it is convenient to introduce the following nondimensional flow variables based on a different length scale h (which could be the fluid depth):

$$\left(x^*, y^*\right) = \frac{1}{l}(x, y), \qquad z^* = \frac{z}{h}, \qquad t^* = \left(\frac{ct}{l}\right), \qquad \eta^* = \frac{\eta}{a}, \qquad \phi^* = \left(\frac{h}{alc}\right)\phi, \tag{9.3.1}$$

where l is a typical horizontal length scale, and $c = \sqrt{gh}$ is the typical horizontal velocity (or shallow water wave speed).

We next introduce two fundamental parameters to characterize the nonlinear shallow water waves:

$$\varepsilon = \frac{a}{h} \quad \text{and} \quad \delta = \frac{h^2}{l^2}, \tag{9.3.2ab}$$

where ε is called the *amplitude* parameter and $\sqrt{\delta}$ is called the *long wavelength* or *shallowness* parameter.

In terms of the preceding nondimensional variables and the parameters, the basic equations for water waves (Debnath 1994, equations (1.6.5)–(1.6.8)) can be written in the nondimensional form, dropping the asterisks:

$$\delta(\phi_{xx}+\phi_{yy})+\phi_{zz}=0, \tag{9.3.3}$$

$$\frac{\partial\phi}{\partial t}+\frac{\varepsilon}{2}\left(\phi_x^2+\phi_y^2\right)+\frac{\varepsilon}{2\delta}\phi_z^2+\eta=0 \quad \text{on } z=1+\varepsilon\eta, \tag{9.3.4}$$

$$\delta\left[\eta_t+\varepsilon(\phi_x\eta_x+\phi_y\eta_y)\right]-\phi_z=0 \quad \text{on } z=1+\varepsilon\eta, \tag{9.3.5}$$

$$\phi_z=0 \quad \text{on } z=0. \tag{9.3.6}$$

It is noted that the parameter $\kappa = ak$ does not enter explicitly in equations (9.3.3)–(9.3.6), but an equivalent parameter $\gamma = a/l$ is associated with ε and δ through $\gamma=(a/h)\cdot(h/l)=\varepsilon\sqrt{\delta}$.

If ε is small, the terms involving ε in (9.3.4), (9.3.5) can be neglected to recover the linearized free surface conditions. However, the assumption that δ is small might be interpreted as the characteristic feature of the shallow water theory. So, we expand ϕ in terms of δ without any assumption about ε, and write

$$\phi=\phi_0+\delta\phi_1+\delta^2\phi_2+\cdots, \tag{9.3.7}$$

and then substitute in (9.3.3)–(9.3.5). The lowest-order term in (9.3.3) is

$$\phi_{0zz}=0, \tag{9.3.8}$$

which, with (9.3.6), yields $\phi_{0z}\equiv 0$, for all z, or $\phi_0=\phi_0(x,y,t)$, which indicates that the horizontal velocity components are independent of the vertical coordinate z in lowest order. Consequently, we use the notation

$$\phi_{0x}=u(x,y,t) \quad \text{and} \quad \phi_{0y}=v(x,y,t). \tag{9.3.9ab}$$

The first- and second-order terms in (9.3.3) are given by

$$\phi_{0xx}+\phi_{0yy}+\phi_{1zz}=0, \tag{9.3.10}$$

$$\phi_{1xx}+\phi_{1yy}+\phi_{2zz}=0. \tag{9.3.11}$$

Integrating (9.3.10) with respect to z and using (9.3.9ab) gives

$$\phi_{1z}=-z(u_x+v_y)+C(x,y,t), \tag{9.3.12}$$

where the arbitrary function $C(x,y,t)$ becomes zero because of the bottom boundary condition (9.3.6). Integrating the resulting equation (9.3.12), again with respect to z and omitting the arbitrary constant, we obtain

$$\phi_1=-\frac{z^2}{2}(u_x+v_y), \tag{9.3.13}$$

so that $\phi_1=0$ at $z=0$ and u and v are then the horizontal velocity components at the bottom boundary.

We next substitute (9.3.13) in (9.3.10), (9.3.11), and then integrate with condition (9.3.6) to determine the arbitrary function. Consequently,

$$\phi_{2z} = \frac{1}{6} z^3 \big[\big(\nabla^2 u\big)_x + \big(\nabla^2 v\big)_y \big], \qquad \phi_2 = \frac{1}{24} z^4 \big[\big(\nabla^2 u\big)_x + \big(\nabla^2 v\big)_y \big], \tag{9.3.14ab}$$

where ∇^2 is the two-dimensional Laplacian.

We next consider the free surface boundary conditions retaining all terms up to order δ, ε in (9.3.4), and δ^2, ε^2, and $\delta\varepsilon$ in (9.3.5). It turns out that

$$\phi_{0t} - \frac{\delta}{2}(u_{tx} + v_{ty}) + \eta + \frac{1}{2}\varepsilon\big(u^2 + v^2\big) = 0, \tag{9.3.15}$$

$$\delta\big[\big\{\eta_t + \varepsilon(u\eta_x + v\eta_y)\big\} + (1 + \varepsilon\eta)(u_x + v_y)\big] = \frac{\delta^2}{6}\big[\big(\nabla^2 u\big)_x + \big(\nabla^2 v\big)_y\big]. \tag{9.3.16}$$

Differentiating (9.3.15) first with respect to x and then with respect to y gives two equations:

$$u_t + \varepsilon(uu_x + vv_x) + \eta_x - \frac{1}{2}\delta(u_{txx} + v_{txy}) = 0, \tag{9.3.17}$$

$$v_t + \varepsilon(uu_y + vv_y) + \eta_y - \frac{1}{2}\delta(u_{txy} + v_{tyy}) = 0. \tag{9.3.18}$$

Simplifying (9.3.16) yields

$$\eta_t + \big[u(1 + \varepsilon\eta)\big]_x + \big[v(1 + \varepsilon\eta)\big]_y = \frac{\delta}{6}\big[\big(\nabla^2 u\big)_x + \big(\nabla^2 v\big)_y\big]. \tag{9.3.19}$$

Equations (9.3.17)–(9.3.19) represent the nondimensional *shallow water equations*.

Using the fact that ϕ_0 is irrotational, that is, $u_y = v_x$ and neglecting terms $O(\delta)$ in (9.3.17)–(9.3.19), we obtain the fundamental shallow water equations

$$u_t + \varepsilon(uu_x + vu_x) + \eta_x = 0, \tag{9.3.20}$$

$$v_t + \varepsilon(uv_x + vv_y) + \eta_y = 0, \tag{9.3.21}$$

$$\eta_t + \big[u(1 + \varepsilon\eta)\big]_x + \big[v(1 + \varepsilon\eta)\big]_y = 0. \tag{9.3.22}$$

This system of three, coupled, nonlinear equations is closed and admits some interesting and useful solutions for u, v, and η. It is equivalent to the boundary-layer equations in fluid mechanics. Finally, it can be linearized when $\varepsilon \ll 1$ to obtain the following dimensional equations:

$$u_t + g\eta_x = 0, \qquad v_t + g\eta_y = 0, \qquad \eta_t + h(u_x + v_y) = 0. \tag{9.3.23abc}$$

Eliminating u and v from these equations gives

$$\eta_{tt} = c^2(\eta_{xx} + \eta_{yy}). \tag{9.3.24}$$

This is a well-known two-dimensional wave equation. It corresponds to the nondispersive shallow water waves that propagate with constant velocity $c = \sqrt{gh}$. This velocity is simply the linearized version of $\sqrt{g(h+\eta)}$. The wave equation has the simple *d'Alembert solution* representing plane progressive waves

$$\eta(x,y,t) = f(k_1 x + l_1 y - \kappa_1 ct) + g(k_2 x + l_2 y - \kappa_2 ct), \tag{9.3.25}$$

where f and g are arbitrary functions and $\kappa_r^2 = (k_r^2 + l_r^2)$, $r = 1, 2$.

We consider the one-dimensional case retaining both ε and δ order terms in (9.3.17)–(9.3.19) so that these equations reduce to the *Boussinesq equations* (1871a, 1871b, 1872)

$$u_t + \varepsilon u u_x + \eta_x - \frac{1}{2}\delta u_{txx} = 0, \tag{9.3.26}$$

$$\eta_t + \big[u(1+\varepsilon\eta)\big]_x - \frac{1}{6}\delta u_{xxx} = 0. \tag{9.3.27}$$

On the other hand, equations (9.3.20)–(9.3.22), expressed in dimensional form, are

$$u_t + u u_x + v u_y + g H_x = 0, \tag{9.3.28}$$

$$v_t + u v_x + v v_y + g H_y = 0, \tag{9.3.29}$$

$$H_t + (uH)_x + (vH)_y = 0, \tag{9.3.30}$$

where $H = (h+\eta)$ is the total depth and $H_x = \eta_x$, since the depth h is constant.

In particular, the one-dimensional version of the shallow water equations follows from (9.3.28)–(9.3.30) and is given by

$$u_t + u u_x + g H_x = 0, \tag{9.3.31}$$

$$H_t + (uH)_x = 0. \tag{9.3.32}$$

This system of approximate shallow water equations is analogous to the exact governing equations of gas dynamics for the case of a compressible flow involving only one space variable (Riabouchinsky 1932).

It is convenient to rewrite these equations in terms of the wave speed $c = \sqrt{gH}$ by using $dc = (g/2c)\,dH$, so that they become

$$u_t + u u_x + 2c c_x = 0, \tag{9.3.33}$$

$$2c_t + c u_x + 2u c_x = 0. \tag{9.3.34}$$

The standard method of characteristics can easily be used to solve (9.3.33) and (9.3.34). Adding and subtracting these equations allows us to rewrite them in the characteristic form

$$\left[\frac{\partial}{\partial t} + (c+u)\frac{\partial}{\partial x}\right](u+2c) = 0, \tag{9.3.35}$$

$$\left[\frac{\partial}{\partial t} + (c-u)\frac{\partial}{\partial x}\right](u-2c) = 0. \tag{9.3.36}$$

Equations (9.3.35), (9.3.36) show that $u + 2c$ propagates in the positive x-direction with velocity $c+u$, and $u-2c$ travels in the negative x-direction with velocity $c-u$, that is, both $u + 2c$ and $u - 2c$ propagate in their respective directions with velocity c relative to the water. In other words,

$$\left.\begin{aligned} u + 2c &= \text{const.} \quad \text{on curves } C_+ \text{ on which } \tfrac{dx}{dt} = u + c, \\ u - 2c &= \text{const.} \quad \text{on curves } C_- \text{ on which } \tfrac{dx}{dt} = u - c, \end{aligned}\right\} \tag{9.3.37ab}$$

where C_+ and C_- are *characteristic curves* of the system of partial differential equations (9.3.31), (9.3.32). A disturbance propagates along these characteristic curves at speed c relative to the flow speed. The quantities $u \pm 2c$ are called the Riemann invariants of the system, and a simple wave is propagating to the right into water of depth h, that is, $u - 2c = c_0 = \sqrt{gh}$. Then, the solution is given by

$$u = f(\xi), \qquad x = \xi + \left(c_0 + \frac{3}{2}u\right)t, \tag{9.3.38}$$

where $u(x,t) = f(x)$ at $t = 0$. However, we note that

$$u_x = \left(1 - \frac{3}{2}u_x t\right) f'(\xi),$$

giving

$$u_x = \frac{2f'(\xi)}{2 + 3tf'(\xi)}. \tag{9.3.39}$$

Thus, if $f'(\xi)$ is anywhere less than zero, u_x tends to infinity as $t \to -2/(3f')$. In terms of the free surface elevation, solution (9.3.38) implies that the wave profile progressively distorts itself, and, in fact, any forward-facing portion of such a wave continually steepens, or the higher parts of the wave tend to catch up with lower parts in front of them. Thus, all of these waves, carrying an increase of elevation, invariably break. The breaking of water waves on beaches is perhaps the most common and the most striking phenomenon in nature.

An alternative system equivalent to the nonlinear evolution equations (9.3.26), (9.3.27) can be derived from the nonlinear shallow water theory, retaining both ε and δ order terms with $\delta < 1$. This system is also known as the *Boussinesq equations* which, in dimensional variables, are given by

$$\eta_t + \big[(h+\eta)u\big]_x = 0, \tag{9.3.40}$$

$$u_t + uu_x + g\eta_x = \frac{1}{3}h^2 u_{xxt}. \tag{9.3.41}$$

They describe the evolution of long water waves that move in both positive and negative x-directions. Eliminating η and neglecting terms smaller than $O(\xi, \delta)$ gives a single *Boussinesq equation* for $u(x,t)$ in the form

$$u_{tt} - c^2 u_{xx} + \frac{1}{2}\left(u^2\right)_{xt} = \frac{1}{3} h^2 u_{xxtt}. \tag{9.3.42}$$

The linearized Boussinesq equation for u and η follows from (9.3.40) and (9.3.41) as

$$\left[\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} - \frac{1}{3} h^2 \frac{\partial^4}{\partial x^2 \partial t^2}\right] \binom{u}{\eta} = 0. \tag{9.3.43}$$

This is in perfect agreement with the infinitesimal wave theory result expanded for small kh. Thus, the third derivative term in 9.3.41) may be identified with the frequency dispersion.

Another equivalent version of the Boussinesq equation is given by

$$\eta_{tt} - c^2 \eta_{xx} = \frac{3}{2}\left(\frac{\eta^2}{h}\right)_{xx} + \frac{1}{3} h^2 \eta_{xxxx}. \tag{9.3.44}$$

There are several features of this equation. It is a nonlinear partial differential equation that incorporates the basic idea of nonlinearity and dispersion. Boussinesq obtained three invariant physical quantities, Q, E, and M, defined by

$$Q = \int_{-\infty}^{\infty} \eta\, dx, \qquad E = \int_{-\infty}^{\infty} \eta^2\, dx, \qquad M = \int_{-\infty}^{\infty} \left[\eta_x^2 - 3\left(\frac{\eta}{h}\right)^3\right] dx, \tag{9.3.45}$$

provided that $\eta \to 0$ as $|x| \to \infty$. Evidently, Q and E represent the volume and the energy of the solitary wave. The third quantity, M, is called the *moment of instability*, and the variational problem, $\delta M = 0$ with E fixed, leads to the unique solitary-wave solution. Boussinesq also derived the results for the amplitude and volume of a solitary wave of given energy in the form

$$a = \frac{3}{4}\left(\frac{E^{3/2}}{h}\right), \qquad Q = 2hE^{1/3}. \tag{9.3.46ab}$$

The former result shows that the amplitude of a solitary wave in a channel varies inversely as the channel depth h.

The Boussinesq equation can then be written in the normalized form

$$u_{tt} - u_{xx} - \frac{3}{2}\left(u^2\right)_{xx} - u_{xxxx} = 0. \tag{9.3.47}$$

This particular form is of special interest because it admits inverse scattering formalism. Equation (9.3.47) has steady progressive wave solutions in the form

$$u(x,t) = 4k^2 f(X), \quad X = kx - \omega t, \tag{9.3.48}$$

where the equation for $f(X)$ can be integrated to obtain

$$f_{XX} = 6A + (4 - 6B) f - 6f^2, \tag{9.3.49}$$

where A is a constant of integration and

$$\omega^2 = k^2 + k^4(4 - 6B). \tag{9.3.50}$$

For the special case $A = B = 0$, a single solitary-wave solution is given by

$$f(X) = \operatorname{sech}^2(X - X_0), \tag{9.3.51}$$

where X_0 is a constant of integration. This result can be used to construct a solution for a series of solitary waves, spaced 2σ apart, in the form

$$f(X) = \sum_{n=-\infty}^{\infty} \operatorname{sech}^2(X - 2n\sigma). \tag{9.3.52}$$

This is a 2σ periodic function that satisfies (9.3.49) for certain values of A and B.

We next assume that ε and δ are comparable, so that all terms $O(\varepsilon, \delta)$ in (9.3.17)–(9.3.19) can be retained. For the case of the two-dimensional wave motion ($v \equiv 0$ and $\partial/\partial y \equiv 0$), these equations become

$$u_t + \eta_x + \varepsilon u u_x - \frac{1}{2}\delta u_{txx} = 0, \tag{9.3.53}$$

$$\eta_t + \big[u(1 + \varepsilon\eta)\big]_x - \frac{1}{6}\delta u_{xxx} = 0. \tag{9.3.54}$$

We now seek steady progressive wave solutions traveling in the positive x-direction only, so that $u = u(x - Ut)$ and $\eta = \eta(x - Ut)$. With the terms of zero order in ε and δ and $U = 1$, we assume a solution of the form

$$u = \eta + \varepsilon P + \delta Q, \tag{9.3.55}$$

where P and Q are unknown functions to be determined. Consequently, equations (9.3.53), (9.3.54) become

$$(\eta + \varepsilon P + \delta Q)_t + \eta_x + \varepsilon\eta\eta_x - \frac{1}{2}\delta\eta_{txx} = 0, \tag{9.3.56}$$

$$\eta_t + \big[(1 + \varepsilon\eta)(\eta + \varepsilon P + \delta Q)\big]_x - \frac{1}{6}\delta\eta_{xxx} = 0. \tag{9.3.57}$$

These equations must be consistent so that we stipulate for the zero order

$$\eta_t = -\eta_x, \qquad P = -\frac{1}{4}\eta^2, \qquad Q = \frac{1}{3}\eta_{xx} = -\frac{1}{3}\eta_{xt}. \tag{9.3.58}$$

We use these results to rewrite both (9.3.56) and (9.3.57) with the assumption that ε and δ are of equal order and small enough for their products and squares to be ignored, so that the ratio $(\varepsilon/\delta) = (al^2/h^3)$ is of order one. Consequently, we obtain a single equation for $\eta(x, t)$ in the form

$$\eta_t + \left(1 + \frac{3}{2}\varepsilon\eta\right)\eta_x + \frac{1}{6}\delta\eta_{xxx} = 0. \tag{9.3.59}$$

This is now universally known as the *Korteweg and de Vries equation* as they discovered it in their 1895 seminal work. We point out that $(\varepsilon/\delta) = al^2/h^3$ is one of the *fundamental* parameters in the theory of nonlinear shallow water waves. Recently, Infeld (1980) considered three-dimensional generalizations of the Boussinesq and Korteweg–de Vries equations.

Example 9.3.1 (Ion-Acoustic Waves and the KdV Equation). A high temperature plasma is a fully ionized gas consisting of electrons and ions that are governed by the equations of continuity and momentum combined with the classical Maxwell equations. Using the subscripts e and i for quantities related to electrons and ions and neglecting dissipation due to collisions, we write the following equations of motion for plasma.

The equation of continuity is

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{u}_j) = 0, \tag{9.3.60}$$

where $n_j (j = e, i)$ is the density and $\mathbf{u}_j$ is the flow velocity.

The equation of motion is

$$m_j n_j \left[\frac{\partial \mathbf{u}_j}{\partial t} + (\mathbf{u}_j \cdot \nabla)\mathbf{u}_j\right] = -\nabla p_j + n_j q_j \left[\mathbf{E} + \frac{1}{c}(\mathbf{u}_j \times \mathbf{B})\right]. \tag{9.3.61}$$

The Maxwell equations are given by

$$\nabla \cdot \mathbf{E} = 4\pi(q_i n_i + q_e n_e), \qquad \nabla \cdot \mathbf{B} = 0, \tag{9.3.62ab}$$

$$\frac{\partial \mathbf{B}}{\partial t} + c(\nabla \times \mathbf{E}) = 0, \qquad -\frac{\partial \mathbf{E}}{\partial t} + c(\nabla \times \mathbf{B}) = 4\pi(q_i n_i \mathbf{u}_i + q_e n_e \mathbf{u}_e). \tag{9.3.63ab}$$

The equation of state is given by

$$p_j = n_j T_j. \tag{9.3.64}$$

In the above equations, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, T is the product of the Boltzmann constant and the temperature, q and m are the charge and mass, respectively, and c is the speed of light.

For an electrostatic wave, that is, for a one-dimensional longitudinal wave ($\nabla \times \mathbf{E} = \mathbf{0}$, $\mathbf{B} = \mathbf{0}$), we set $q_i = -q_e \equiv e$ to obtain

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}(n_j u_j) = 0, \tag{9.3.65}$$

$$\frac{\partial u_j}{\partial t} + u_j \frac{\partial u_j}{\partial x} = \pm \frac{e}{m_j} E - \frac{1}{m_j n_j} \frac{\partial}{\partial x}(T_j n_j), \tag{9.3.66}$$

$$\frac{\partial E}{\partial x} = 4\pi e(n_i - n_e), \qquad \frac{\partial E}{\partial t} = 4\pi e(n_e u_e - n_i u_i). \tag{9.3.67ab}$$

The motion of the cold ions in a hot electron gas with the classical Boltzmann distribution is governed by the simplified form of equations (9.3.60)–(9.3.64). In this case, with $m_e \to 0$ and $T_e = \text{const.}$ with $T_e \gg T_i$, it follows from equations (9.3.65)–(9.3.67ab) that

$$eE = \frac{T_e}{n_e}\frac{\partial n_e}{\partial x}, \tag{9.3.68}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \tag{9.3.69}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = \frac{e}{m_i}E - \frac{1}{m_i n_i}\frac{\partial}{\partial x}(n_i T_i), \tag{9.3.70}$$

$$\frac{\partial E}{\partial x} = 4\pi e(n_i - n_e). \tag{9.3.71}$$

These are the governing equations for quantities n_i, u_i, E, and n_e if T_i is a given constant or, by the adiabatic law, so that $(n_i T_i)$ is proportional to n_i^γ. Introducing the electrostatic potential ϕ, so that $E = -\phi_x$, equation (9.3.69) gives the Boltzmann distribution for electrons $n_e = n_0 \exp(\frac{e\phi}{T_e})$, where n_0 is the number density of electrons (or ions) in the unperturbed state.

To examine the motion of the cold ions in a hot electron gas with the Boltzmann distribution, we use equations (9.3.68) and (9.3.69). It is convenient to nondimensionalize the physical quantities as follows:

$$n_j^* = \frac{n_j}{n_0}, \qquad u_j^* = \frac{u_j}{c_s}, \qquad E^* = \frac{eE}{\sqrt{m_e T_e}}\omega_{pe},$$
$$\phi^* = \frac{e\phi}{T_e}, \qquad x^* = \frac{x}{\lambda_{De}}, \qquad t^* = t\omega_{pi},$$

where $\omega_{pe} = (4\pi n_0/m_e)^{\frac{1}{2}} e$ is the electron plasma frequency,

$\lambda_{De} = \omega_{pe}^{-1}(T_e/m_e)^{\frac{1}{2}}$ is the electron Debye length, $\omega_{pi} = \omega_{pe}(m_e/m_i)^{\frac{1}{2}}$ is the ion plasma frequency, $n_i = n_0 = \text{const.}$, and $c_s = \lambda_{De} \cdot \omega_{pi} = (T_e/m_i)^{\frac{1}{2}}$. Consequently, omitting the asterisks, equations (9.3.69)–(9.3.71) reduce to nondimensional form

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \tag{9.3.72}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{\partial \phi}{\partial x}, \tag{9.3.73}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \exp(\phi) - n_i. \tag{9.3.74}$$

Substituting $n_i = 1 + n_i' \exp[i(kx - \omega t)]$ and $(u_i, \phi) = (u_i', \phi') \exp[i(kx - \omega t)]$ in the above equations gives the linear dispersion relation

$$\omega^2 = k^2\left(1 + k^2\right)^{-1}. \tag{9.3.75}$$

For long waves $(k^2 \ll 1)$, equation (9.3.75) reduces to

$$\omega \approx k\left(1 - \frac{1}{2}k^2\right). \tag{9.3.76}$$

Under the assumption of a quasi-neutral state, $n_i = n_e = \exp(\phi)$ so that $\phi = \log(n_i)$, the equation of motion (9.3.73) becomes

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{1}{n_i}\frac{\partial n_i}{\partial x}. \tag{9.3.77}$$

This is identical to the equation of motion for a gas with unit sound velocity $(c_s = 1)$. Physically, this means that electrons exert a pressure on ions through the electric field $E = -\phi_x$, and hence, waves described by the above dispersion relation are called *ion-acoustic waves*. As the waves steepen by the inertia term, $\phi_{xx} (= n_e - n_i)$ cannot be neglected. Consequently, a soliton is generated by the dispersion involved with the deviation from the quasi-neutral state (the second term of the dispersion relation).

To obtain the soliton solution of the above system, we look for a stationary wave solution with $\xi = x - ct$ and the boundary conditions

$$n_i \to 1, \qquad u_2 \to 0, \qquad \phi \to 0, \qquad \frac{\partial \phi}{\partial \xi} \to 0, \quad \text{as } |\xi| \to \infty. \tag{9.3.78}$$

It turns out that $\phi(\xi, t)$ satisfies the nonlinear equation

$$\frac{1}{2}\left(\frac{\partial \phi}{\partial \xi}\right)^2 = \exp(\phi) + c\left(c^2 - 2\phi\right)^{\frac{1}{2}} - \left(c^2 + 1\right). \tag{9.3.79}$$

This gives the value of c^2 when $\frac{\partial \phi}{\partial \xi} = 0$ at $\phi = \phi_m$, so that

$$c^2 = \frac{\{1 - \exp(\phi_m)\}^2}{2\{\exp(\phi_m) - (1 + \phi_m)\}}. \tag{9.3.80}$$

We assume that $\phi_m \ll 1$, so that $c - 1 = \delta\lambda \ll 1$. Consequently, equation (9.3.79) reduces to the form

$$\left(\frac{\partial \phi}{\partial \xi}\right)^2 = \frac{2}{3}\phi^2 (3\delta c - \phi). \tag{9.3.81}$$

Integrating this equation gives a soliton solution of the form

$$\phi(x,t) = 3(\delta c)\,\text{sech}^2\left[\sqrt{\frac{\delta c}{2}}(x - ct)\right]. \tag{9.3.82}$$

This represents a KdV soliton when the amplitude is finite, but small. As the amplitude of the potential tends to $\phi_m = \frac{1}{2}c^2$, that is, $e\phi_m = \frac{1}{2}m_i c^2$, the ions cannot pass the potential and so will be reflected. As a result, ion orbits will intersect, and hence, the cold ion approximation ceases to be valid. Accordingly, the maximum amplitude of the soliton is given by $\phi_m = \frac{1}{2}c^2$, and, from equation (9.3.80), we obtain $\phi_m \sim 1.3$, and hence, $c \sim 1.6$. This reveals that the maximum Mach number of the soliton is 1.6 as predicted by Sagdeev (1966).

Example 9.3.2 (Derivation of the KdV Equation from the Euler Equations). We discussed this problem at the beginning of this section by using the Laplace equation for the velocity potential under the assumption that $\delta = O(\varepsilon)$ as $\varepsilon \to 0$. Here we follow Johnson (1997) to present another derivation of the KdV equation from the Euler equation in $(1+1)$ dimensions. This approach can be generalized to derive higher dimensional KdV equations.

We consider the problem of surface gravity waves which propagate in the positive x-direction over stationary water of constant depth. The associated Euler equations (2.7.67) and the continuity equation (2.7.68) in $(1+1)$ dimensions are given by

$$u_t + \varepsilon(uu_x + wu_z) = -p_x, \tag{9.3.83}$$

$$\delta\big[w_t + \varepsilon(uw_x + ww_z)\big] = -p_z, \tag{9.3.84}$$

$$u_x + w_z = 0. \tag{9.3.85}$$

The free surface and bottom boundary conditions are obtained from (2.7.69), (2.7.70) in the form

$$w = \eta_t + \varepsilon u\eta_x, \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta, \tag{9.3.86}$$

$$w = 0 \quad \text{on } z = 0. \tag{9.3.87}$$

It can easily be shown that, for any $\sqrt{\delta}$ as $\varepsilon \to 0$, there exists a region in the (x,t)-space where there is a balance between nonlinearity and dispersion which leads to the KdV equation. The region of interest is defined by a scaling of the independent flow variables as

$$x \to \sqrt{\frac{\delta}{\varepsilon}}\,x \quad \text{and} \quad t \to \sqrt{\frac{\delta}{\varepsilon}}\,t, \tag{9.3.88}$$

for any ε and $\sqrt{\delta}$. In order to ensure consistency in the continuity equation, it is necessary to introduce a scaling of w by

$$w \to \sqrt{\frac{\varepsilon}{\delta}}\,w. \tag{9.3.89}$$

Consequently, the net effect of the scalings is to replace δ by ε in equations (9.3.83)–(9.3.87) so that they become

$$u_t + \varepsilon(uu_x + wu_z) = -p_x, \tag{9.3.90}$$

$$\varepsilon\big[w_t + \varepsilon(uw_x + ww_z)\big] = -p_z, \tag{9.3.91}$$

$$u_x + w_z = 0, \tag{9.3.92}$$

$$w = \eta_t + \varepsilon u\eta_x, \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta, \tag{9.3.93}$$

$$w = 0 \quad \text{on } z = 0. \tag{9.3.94}$$

In the limit as $\varepsilon \to 0$, the first-order approximation of equations (9.3.90) and (9.3.93) gives

$$\eta = p, \quad 0 \le z \le 1, \quad \text{and} \quad u_t + \eta_x = 0. \tag{9.3.95}$$

It then follows from (9.3.92) that $w = -zu_x$ which satisfies (9.3.94). The boundary condition (9.3.93) leads to $\eta_t + u_x = 0$ on $z = 1$, which can be combined with (9.3.95) to obtain the linear wave equation

$$\eta_{tt} - \eta_{xx} = 0. \tag{9.3.96}$$

For waves propagating in the positive x-direction, we introduce the far-field variables

$$\xi = x - t \quad \text{and} \quad \tau = \varepsilon t, \tag{9.3.97}$$

so that $\xi = O(1)$ and $\tau = O(1)$ give the far-field region of the problem. This is the region where nonlinearity balances the dispersion to produce the KdV equation.

With the choice of the transformations (9.3.97), equations (9.3.90)–(9.3.94) can be rewritten in the form

$$-u_\xi + \varepsilon(u_\tau + uu_\xi + wu_z) = -p_\xi, \tag{9.3.98}$$
$$\varepsilon\big[-w_\xi + \varepsilon(w_\tau + uw_\xi + ww_z)\big] = -p_z, \tag{9.3.99}$$
$$u_\xi + w_z = 0, \tag{9.3.100}$$
$$w = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi), \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta, \tag{9.3.101}$$
$$w = 0 \quad \text{on } z = 0. \tag{9.3.102}$$

We seek an asymptotic series expansion of the solutions of the system (9.3.98)–(9.3.102) in the form

$$\eta(\xi,\tau,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi,\tau), \qquad q(\xi,\tau,z;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi,\tau,z), \tag{9.3.103}$$

where q (and the corresponding q_n) denotes each of the variables u, w, and p.

Consequently, the leading-order problem is given by

$$u_{0\xi} = p_{0\xi}, \qquad p_{0z} = 0, \qquad u_{0\xi} + w_{0z} = 0, \tag{9.3.104}$$
$$p_0 = \eta_0, \quad w + \eta_{0\xi} = 0 \quad \text{on } z = 1, \tag{9.3.105}$$
$$w = 0 \quad \text{on } z = 0. \tag{9.3.106}$$

These leading-order equations give

$$p_0 = \eta_0, \qquad u_0 = \eta_0, \qquad w_0 + z\eta_{0\xi} = 0, \quad 0 \le z \le 1, \tag{9.3.107}$$

with $u_0 = 0$ whenever $\eta_0 = 0$. The boundary condition on w_0 at $z = 1$ is automatically satisfied.

Using the Taylor series expansion of u, w, and p about $z = 1$, the two surface boundary conditions on $z = 1 + \varepsilon\eta$ are rewritten on $z = 1$, and hence, take the form

$$p_0 + \varepsilon\eta_0 p_{0z} + \varepsilon p_1 = \eta_0 + \varepsilon\eta_1 + O(\varepsilon^2) \quad \text{on } z = 1, \tag{9.3.108}$$
$$w_0 + \varepsilon\eta_0 w_{0z} + \varepsilon w_1 = -\eta_{0\xi} - \varepsilon\eta_{1\xi} + \varepsilon(\eta_{0\tau} + u_0\eta_{0\xi}) + O(\varepsilon^2) \quad \text{on } z = 1. \tag{9.3.109}$$

These conditions are to be used together with (9.3.98), (9.3.99), and (9.3.102).

The equations in the next order are given by

$$-u_{1\xi} + u_{0\tau} + u_0 u_{0\xi} + w_0 u_{0z} = -p_{1\xi}, \qquad p_{1z} = w_{0\xi}, \tag{9.3.110}$$

$$u_{1\xi} + w_{1z} = 0, \tag{9.3.111}$$

$$p_1 + \eta_0 p_0 = \eta_1, \qquad w_1 + \eta_0 + w_{0z} = -\eta_{1\xi} + \eta_{0\tau} + u_0 \eta_{0\xi} \quad \text{on } z = 1, \tag{9.3.112}$$

$$w_1 = 0 \quad \text{on } z = 0. \tag{9.3.113}$$

Noting that

$$u_{0z} = 0, \qquad p_{0z} = 0, \quad \text{and} \quad w_{0z} = -\eta_{0\xi}, \tag{9.3.114}$$

we obtain

$$p_1 = \frac{1}{2}(1 - z^2)\eta_{0\xi\xi} + \eta_1, \tag{9.3.115}$$

and therefore,

$$\begin{aligned} w_{1z} &= -u_{1\xi} = -p_{1\xi} - u_{0\tau} - u_0 u_{0\xi} \\ &= -\left[\eta_{1\xi} + \frac{1}{2}(1 - z^2)\eta_{0\xi\xi\xi} + \eta_{0\tau} + \eta_0 \eta_{0\xi}\right]. \end{aligned} \tag{9.3.116}$$

Finally, we find

$$w_1 = -\left[\eta_{1\xi} + \eta_{0\tau} + \eta_0 + \eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi}\right] z + \frac{1}{6} z^3 \eta_{0\xi\xi\xi}, \tag{9.3.117}$$

which satisfies the bottom boundary condition on $z = 0$.

The free surface boundary condition on $z = 1$ gives

$$\begin{aligned} (w_1)_{z=1} &= -\left(\eta_{1\xi} + \eta_{0\tau} + \eta_0 \eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi}\right) + \frac{1}{6}\eta_{0\xi\xi\xi} \\ &= -\eta_{1\xi} + \eta_{0\tau} + 2\eta_0 \eta_{0\xi}, \end{aligned} \tag{9.3.118}$$

which yields the equation for $\eta_0(\xi, \tau)$ in the form

$$\eta_{0\tau} + \frac{3}{2}\eta_0 \eta_{0\xi} + \frac{1}{6}\eta_{0\xi\xi\xi} = 0. \tag{9.3.119}$$

This is the *Korteweg–de Vries* (*KdV*) *equation*, which describes nonlinear plane gravity waves propagating in the x-direction. It will be described in Section 9.7 that the exact solution of the general initial-value problem for the KdV equation can be obtained, provided the initial data decay sufficiently rapidly as $|\xi| \to \infty$. We may raise the question of whether the asymptotic expansion for η (and hence, for the other flow variables) is uniformly valid as $|\xi| \to \infty$ and as $\tau \to \infty$. For the case of $\tau \to \infty$, this question is difficult to answer because the equations for $\eta_n (n \geq 1)$ are not easy to solve. However, all the available numerical evidence suggests that the asymptotic expansion of η is indeed uniformly valid as $\tau \to \infty$ (at least for solutions which satisfy $\eta \to 0$ as $|\xi| \to \infty$). From a physical point of view, if the waves are allowed to propagate indefinitely, then other physical effects cannot be neglected. In the case of real water waves, the most common physical effects include viscous damping. In practice, the viscous damping seems to be sufficiently weak to allow the dispersive and nonlinear effects to dominate before the waves eventually decay completely.

It is well known that the KdV equation describes nonlinear plane waves which propagate only in the x-direction. However, there are many physical situations in which waves move on a two-dimensional surface. So it is natural to include both x- and y-directions with the appropriate balance of dispersion and nonlinearity. One of the simplest examples is the classical two-dimensional linear wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}), \tag{9.3.120}$$

which describes the propagation of long waves.

This equation has a solution in the form

$$u(x, t) = a \exp\bigl[i(\omega t - \mathbf{k} \cdot \mathbf{x})\bigr], \tag{9.3.121}$$

where a is the wave amplitude, ω is the frequency, $\mathbf{x} = (x, y)$, and the wavenumber vector is $\mathbf{k} = (k, \ell)$.

The dispersion relation is given by

$$\omega^2 = c^2\bigl(k^2 + \ell^2\bigr). \tag{9.3.122}$$

For waves that propagate predominantly in the x-direction, the wavenumber component ℓ becomes small so that the approximate phase velocity is given by

$$c_p = c\left(1 + \frac{1}{2}\frac{\ell^2}{k^2}\right) \quad \text{as } \ell \to 0. \tag{9.3.123}$$

Example 9.3.3 (The Kadomtsev–Petviashvili (KP) Equation or Two-Dimensional KdV Equation). It follows from (9.3.123) that the phase velocity suffers from a small correction provided by the wavenumber component ℓ in the y-direction. In order to ensure that this small correction is the same order as the dispersion and nonlinearity, it it necessary to require $\ell = O(\sqrt{\varepsilon})$ or $\ell^2 = O(\varepsilon)$. This requirement can be incorporated in the governing equations by a scaling of the flow variables as

$$y \to \sqrt{\varepsilon} y \quad \text{and} \quad v \to \sqrt{\varepsilon} v, \tag{9.3.124}$$

and using the same far-field transformations as (9.3.97).

Consequently, equations (2.7.67)–(2.7.71) with the parameters δ replaced by ε reduce to the form

$$-u_\xi + \varepsilon(u_\tau + uu_\xi + \varepsilon vu_y + wu_z) = -p_\xi, \tag{9.3.125}$$

$$-v_\xi + \varepsilon(v_\tau + uv_\xi + \varepsilon vv_y + wv_z) = -p_y, \tag{9.3.126}$$

$$-\varepsilon\bigl[-w_\xi + \varepsilon(w_\tau + uw_\xi + \varepsilon vw_y + ww_z)\bigr] = -p_z, \tag{9.3.127}$$

$$-u_\xi + \varepsilon v_y + w_z = 0, \tag{9.3.128}$$

$$w = -\eta_\xi + \varepsilon(\eta_\tau + u u_\xi + \varepsilon v \eta_y), \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta, \tag{9.3.129}$$

$$w = 0 \quad \text{on } z = 0. \tag{9.3.130}$$

We seek the same asymptotic series solution (9.3.103) valid as $\varepsilon \to 0$ without any change of the leading order problem except that the flow variables involve y so that

$$p_0 = \eta_0, \qquad u_0 = \eta_0, \qquad w_0 = -z\eta_{0\xi}, \quad 0 \le z \le 1, \tag{9.3.131}$$

$$v_{0\xi} = \eta_{0y}. \tag{9.3.132}$$

At the next order, the only difference from Example 9.3.2 is in the continuity equation which becomes

$$w_{1z} = -u_{1\xi} - v_{0y}. \tag{9.3.133}$$

This change leads to the following equation:

$$w_1 = -\left(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \frac{1}{2}\eta_{0\xi\xi\xi} + v_{0y}\right)z + \frac{1}{6}z^3\eta_{0\xi\xi\xi}. \tag{9.3.134}$$

Using equation (9.3.132), the final result is the equation for the leading-order representation of the surface wave in the form

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi} + v_{0y} = 0. \tag{9.3.135}$$

Consequently, differentiating (9.3.135) with respect to ξ and replacing $v_{0\xi}$ by η_{0y} give the evolution equation for $\eta_0(\xi, \tau, y)$ in the form

$$\left(2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi}\right)_\xi + \eta_{0yy} = 0. \tag{9.3.136}$$

This is the *two-dimensional* or, more precisely, the $(1 + 2)$-*dimensional KdV equation*. Obviously, when there is no y-dependence, (9.3.136) reduces to the KdV equation (9.3.119). The two-dimensional KdV equation is also known as the *Kadomtsev–Petviashvili* (*KP*) *equation* (Kadomtsev and Petviashvili 1970). This is another very special completely integrable equation, and it has an exact analytical solution that represents obliquely crossing nonlinear waves. Physically, any number of waves cross obliquely and interact nonlinearly. In particular, the nonlinear interaction of three waves leads to a *resonance phenomenon*. Such an interaction becomes more pronounced, leading to a strongly nonlinear interaction as the wave configuration is more nearly that of parallel waves. This situation can be interpreted as one in which the waves interact nonlinearly over a large distance so that a strong distortion is produced among these waves. The reader is referred to Johnson's (1997) book for more detailed information on oblique interaction of waves.

We now consider the Euler equations (2.7.4)–(2.7.6) and the continuity equation (2.7.2) in cylindrical polar coordinates (r, θ, z). It is convenient to use the nondimensional flow variables, parameters, and scaled variables similar to those defined by (2.7.56), (2.7.57), and (2.7.66) with $r = \ell r^*$ where r^* is a nondimensional variable so that the polar form of Euler equations (2.7.4)–(2.7.6), continuity equation (2.7.2), and the boundary conditions (2.7.64), (2.7.65) in nondimensional form become

$$\frac{Du}{Dt} - \frac{\varepsilon v^2}{r} = -\frac{\partial p}{\partial r}, \qquad \frac{Dv}{Dt} + \frac{\varepsilon uv}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \qquad \delta\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \tag{9.3.137}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \tag{9.3.138}$$

$$w = \eta_t + \varepsilon\left(u\eta_r + \frac{v}{r}\eta_\theta\right), \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta, \tag{9.3.139}$$

$$w = 0 \quad \text{on } z = 0, \tag{9.3.140}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}\right), \tag{9.3.141}$$

and ε and δ are defined by (2.7.58).

For the case of axisymmetric wave motions ($\frac{\partial}{\partial\theta} \equiv 0$), the governing equations become

$$u_t + \varepsilon(uu_r + wu_z) = -p_r, \tag{9.3.142}$$

$$\delta\big[w_t + \varepsilon(uw_r + ww_z)\big] = -p_z, \tag{9.3.143}$$

$$u_r + \frac{u}{r} + w_z = 0, \tag{9.3.144}$$

$$w = \eta_t + \varepsilon u\eta_r, \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta, \tag{9.3.145}$$

$$w = 0 \quad \text{on } z = 0. \tag{9.3.146}$$

In the limit as $\varepsilon \to 0$, the linearized version of equations (9.3.137)–(9.3.141) become

$$u_t = -p_r, \qquad v_t = -\frac{1}{r}p_\theta, \qquad \delta w_t = -p_z, \tag{9.3.147}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0, \tag{9.3.148}$$

$$w = \eta_t, \quad \text{and} \quad p = \eta \quad \text{on } z = 1, \tag{9.3.149}$$

$$w = 0 \quad \text{on } z = 0, \tag{9.3.150}$$

For long waves ($\delta \to 0$), equations (9.3.147)–(9.3.150) lead to the classical wave equation

$$\eta_{tt} = \eta_{rr} + \frac{1}{r}\eta_r + \frac{1}{r^2}\eta_{\theta\theta}. \tag{9.3.151}$$

For axisymmetric surface waves, the wave equation (9.3.151) becomes

$$\eta_{tt} = \eta_{rr} + \frac{1}{r}\eta_r. \tag{9.3.152}$$

This can be solved by using the Hankel transform (see Example 1.10.3).

It is convenient to introduce the characteristic variable $\xi = r - t$ for outgoing waves and $R = \alpha r$ so that $\alpha \to 0$ corresponds to large radius r. In other words, $R = O(1)$, as $\alpha \to 0$, gives $r \to \infty$. The equation (9.3.152) reduces to the following form

$$2\eta_{\xi R} + \frac{1}{R}\eta_\xi = \alpha\left(\eta_{RR} + \frac{1}{R}\eta_R\right) = 0. \tag{9.3.153}$$

In the limit as $\alpha \to 0$, it follows that

$$\sqrt{R}\eta_\xi = g(\xi), \tag{9.3.154}$$

where $g(\xi)$ is an arbitrary function of ξ.

For outgoing waves, the correct solution takes the form for $\alpha \to 0$ $(r \to \infty)$,

$$\eta = \frac{1}{\sqrt{R}}\int g(\xi)\,d\xi = \frac{1}{\sqrt{R}}f(\xi), \tag{9.3.155}$$

where $\eta = 0$ when $f = 0$.

This shows that the amplitude of waves decays as the radius $r \to \infty (R \to \infty)$. This behavior is totally different from the derivation of the KdV equation where the amplitude is uniformly $O(\varepsilon)$. In the present axisymmetric case, the amplitude decreases as the radius r increases so that there is no far-field region where the balance between nonlinearity and dispersion occurs. In other words, the amplitude is so small that nonlinear terms play no role at the leading order. However, there exists a scaling of the flow variables which leads to the *axisymmetric* (*concentric*) *KdV equation* as shown by Johnson (1997).

We recall the axisymmetric Euler equations of motion (9.3.142)–(9.3.146) and introduce scalings in terms of large radial variable R,

$$\xi = \frac{\varepsilon^2}{\delta}(r - t) \quad \text{and} \quad R = \frac{\varepsilon^6}{\delta^2}r. \tag{9.3.156}$$

We next apply the transformations of the flow variables

$$(\eta, p, u, w) = \frac{\varepsilon^3}{\delta}\left(\eta^*, p^*, u^*, \frac{\varepsilon^2}{\delta}w^*\right), \tag{9.3.157}$$

where large distance/time is measured by the scale (δ^2/ε^6) so that

$$\left(\frac{\delta^2}{\varepsilon^6}\right)^{-\frac{1}{2}} = \left(\frac{\varepsilon^3}{\delta}\right),$$

which represents the scale of the amplitude of the waves. It is important to point out that the original wave amplitude parameter ε can now be interpreted based on the amplitude of the wave for $r = O(1)$ and $t = O(1)$. Consequently, the governing equations (9.3.142)–(9.3.146) become, dropping the asterisks in the variables,

$$-u_\xi + \alpha(uu_\xi + wu_z + \alpha uu_R) = -(p_\xi + \alpha p_R), \tag{9.3.158}$$
$$\alpha\left[-w_\xi + \alpha(uw_\xi + ww_z + \alpha uw_R)\right] = -p_z, \tag{9.3.159}$$

$$u_\xi + w_z + \alpha\left(u_R + \frac{1}{R}u\right) = 0, \tag{9.3.160}$$

$$w = -\eta_\xi + \alpha(u\eta_\xi + \alpha u\eta_R), \quad p = \eta \quad \text{on } z = 1 + \alpha\eta, \tag{9.3.161}$$

$$w = 0 \quad \text{on } z = 0, \tag{9.3.162}$$

where $\alpha = (\delta^{-1}\varepsilon^4)$ is a new parameter. These equations are similar in structure to those discussed above with parameter ε, which is now replaced by α in (9.3.158)–(9.3.162) so that the limit as $\alpha \to 0$ is required. This requirement is satisfied (for example, $\varepsilon \to 0$ with δ fixed), and the scaling introduced by (9.3.156) describes the region where the appropriate balance occurs so that the wave amplitude in this region is $O(\alpha)$.

We now seek an asymptotic series solution in the form

$$q(\xi, R, z) = \sum_{n=0}^{\infty} \alpha^n q_n(\xi, R, z), \tag{9.3.163}$$

where q represents each of η, u, w, and p.

In the leading order, we obtain the familiar equations

$$p_0 = \eta_0, \qquad u_0 = \eta_0, \qquad w_0 = -z\eta_{0\xi}, \quad 0 \le z \le 1. \tag{9.3.164}$$

It follows from the continuity equation (9.3.160) that

$$w_{1z} = -u_{1z} - \left(u_{0R} + \frac{1}{R}u_0\right). \tag{9.3.165}$$

Without any more algebraic calculation, it turns out that $\eta_0(\xi, R)$ satisfies the nonlinear evolution equation

$$2\eta_{0R} + \frac{1}{R}\eta_0 + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0. \tag{9.3.166}$$

This is usually referred to as the *axisymmetric* (*concentric*) *KdV equation* which includes a new term $R^{-1}\eta_0$. We may use the large time variable $\tau = (\delta^{-2}\varepsilon^6)t$ so that $R = \tau + \alpha\xi \approx \tau$ (see Johnson 1997 or 1980).

Example 9.3.4 (Derivation of Johnson's Evolution Equation). Johnson (1980) derived a new concentric KdV equation which incorporates weak dependence on the angular coordinate θ. In the derivation of KP equation (9.3.136), $\sqrt{\varepsilon}$ was chosen as the scaling parameter in the y-direction. Similarly, the appropriate scaling on the angular variable θ may be chosen as $\sqrt{\alpha}$. In the derivation of the concentric KdV equation (9.3.166), the parameter α plays the role of ε which is used as the small parameter in the asymptotic solution of the KdV equation.

Following the work of Johnson (1980), we choose the variables ξ and R defined by (9.3.156) and the scaled θ variable as

$$\theta = \sqrt{\alpha}\,\theta^* = \left(\delta^{-1}\varepsilon^2\right), \tag{9.3.167}$$

which introduces a small angular deviation from purely concentric effects. We also use the scaled velocity component in the θ-direction as

$$v = \left(\delta\delta^{-\frac{1}{2}}\varepsilon^5\right)v^*, \tag{9.3.168}$$

so that the scalings on $u = \phi_r$ and $v = \frac{1}{r}\phi_\theta$ are found to be consistent.

Invoking (9.3.156), (9.3.157) combined with (9.3.167), (9.3.168), and dropping the asterisk from all variables the governing equations (9.3.137)–(9.3.140) can be expressed in the following form:

$$-u_\xi + \alpha\left[uu_\xi + wu_z + \alpha\left(uu_R + \frac{v}{R}u_\theta\right)\right] - \frac{\alpha^3}{R}v = -(p_\xi + \alpha p_R), \tag{9.3.169}$$

$$-v_\xi + \alpha\left[uv_\xi + wv_z + \alpha\left(uv_R + \frac{1}{R}vv_\theta\right)\right] + \frac{\alpha^2 uv}{R} = -\frac{1}{R}p_\theta, \tag{9.3.170}$$

$$\alpha\left[-w_\xi + \alpha\left(uw_\xi + ww_z + \alpha uw_R + \frac{\alpha}{R}vw_\theta\right)\right] = -p_z, \tag{9.3.171}$$

$$u_\xi + w_z + \alpha\left[u_R + \frac{1}{R}(u + v_\theta)\right] = 0, \tag{9.3.172}$$

$$w = -\eta_\xi + \alpha\left[u\eta_\xi + \alpha\left(u\eta_R + \frac{1}{R}v\eta_\theta\right)\right], \quad \text{and} \quad p = \eta, \quad \text{on } z = 1 + \alpha\eta, \tag{9.3.173}$$

$$w = 0 \quad \text{on } z = 0. \tag{9.3.174}$$

An asymptotic solution similar to that of (9.3.163) gives the same result (9.3.164) at the leading order with

$$v_{0\xi} = R^{-1}\eta_{0\theta}. \tag{9.3.175}$$

At the next order, the continuity equation (9.3.172) produces a different result in the form

$$w_{1z} = -\left[u_{1\xi} + u_{0R} + \frac{1}{R}(u_0 + v_{0\theta})\right]. \tag{9.3.176}$$

Omitting some algebraic calculations, it turns out that (9.3.176) leads to a KdV type equation

$$2\eta_{0R} + \frac{1}{R}\eta_0 + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} + \frac{1}{R}v_{0\theta} = 0. \tag{9.3.177}$$

Eliminating $v_{0\theta}$ from (9.3.177) with the help of (9.3.175) gives the equation in the form

$$\left(2\eta_{0R} + \frac{1}{R}\eta_0 + 3\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi}\right)_\xi + \frac{1}{R^2}\eta_{0\theta\theta} = 0. \tag{9.3.178}$$

This is known as *Johnson's equation*, as it was first derived by Johnson (1980). In the absence of the θ-dependence, equation (9.3.178) reduces to the concentric KdV equation (9.3.166).

Example 9.3.5 (Transformation of the KP Equation to the Concentric KdV Equation). Replacing τ by t and ξ by x, we rewrite the KP equation (9.3.136) in the form

$$\left(2\eta_t + 3\eta\eta_x + \frac{1}{3}\eta_{xxx}\right)_x + \eta_{yy} = 0. \tag{9.3.179}$$

We apply the transformations

$$\eta = h(x^*, t), \quad x^* = x + \frac{1}{2}\frac{y^2}{t} \tag{9.3.180}$$

to equation (9.3.179) to obtain, dropping the asterisk,

$$\left(2h_t - \left(\frac{y}{t}\right)^2 h_x + 3hh_x + \frac{1}{3}h_{xxx}\right)_x + \frac{1}{t}h_x + \left(\frac{y}{t}\right)^2 h_{xx} = 0. \tag{9.3.181}$$

Integrating this equation with respect to x and using the decay condition for $x \to \infty$ gives the concentric KdV equation for $h(x,t)$

$$2h_t + \frac{1}{t}h + 3hh_x + \frac{1}{3}h_{xxx} = 0. \tag{9.3.182}$$

This equation is similar to (9.3.166) in which R, ξ are replaced by t and x, respectively.

Example 9.3.6 (Transformation of Johnson's Equation to the KdV Equation). We recall Johnson's equation (9.3.178)

$$\left(2\eta_R + \frac{1}{R}\eta + 3\eta\eta_\xi + \frac{1}{3}\eta_{\xi\xi\xi}\right)_\xi + \frac{1}{R^2}\eta_{\theta\theta} = 0. \tag{9.3.183}$$

We use cylindrical polar coordinates $(x = r\cos\theta,\ y = r\sin\theta)$ in the limit as $\theta \to 0$. We write

$$r - t \sim x\left(1 + \frac{1}{2}\frac{y^2}{x^2}\right) - t = x\left(1 + \frac{1}{2}\theta^2\right) - t$$

so that

$$x - t \sim r - t - \frac{1}{2}r\theta^2 = \frac{\delta}{\varepsilon^2}\left(\xi - \frac{1}{2}R\theta^{*2}\right).$$

We next apply the transformations

$$h = \eta(\zeta, R), \quad \zeta = \xi - \frac{1}{2}R\theta^{*2}$$

to equation (9.3.178) and integrate the resulting equation with respect to ξ to obtain the equation

$$2h_R + 3hh_\zeta + \frac{1}{3}\eta_{\zeta\zeta\zeta} = 0. \tag{9.3.184}$$

This is the usual KdV equation where $R = \tau + \alpha\xi$ may be replaced by τ as $\alpha \to 0$ and R may be interpreted as the time variable.

Example 9.3.7 (Derivation of the Boussinesq Equation from the Euler Equations). We consider another example of weakly nonlinear and weakly dispersive waves which propagate in both the positive and negative x-directions. We use the Euler equations (9.3.83)–(9.3.87) with the scaling which replaces δ by ε so that

$$u_t + \varepsilon(uu_x + wu_z) = -p_x, \tag{9.3.185}$$

$$\varepsilon\big[w_t + \varepsilon(uw_x + ww_z)\big] = -p_z, \tag{9.3.186}$$

$$u_x + w_z = 0, \tag{9.3.187}$$

$$w = \eta_t + \varepsilon\eta\eta_x, \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \varepsilon\eta, \tag{9.3.188}$$

$$w = 0 \quad \text{on } z = 0. \tag{9.3.189}$$

We seek as usual an asymptotic expansion of solutions in integral powers of ε as $\varepsilon \to 0$. At the leading order $O(1)$, we obtain

$$p_0 = \eta_0, \quad u_{0t} = -\eta_{0x}, \quad w_0 = -zu_{0x}, \quad u_{0x} = -\eta_{0t}, \quad 0 \le z \le 1. \tag{9.3.190}$$

Consequently, η_0 satisfies the classical wave equation

$$\eta_{0tt} - \eta_{0xx} = 0. \tag{9.3.191}$$

In this problem, we seek solutions that are valid for $x = O(1)$ and $t = O(1)$. At the next order $O(\varepsilon)$, we find

$$p_1 = -\frac{1}{2}\big(1 - z^2\big)u_{0xt} + \eta_1 \tag{9.3.192}$$

so that

$$u_{1t} + u_0u_{0x} = \frac{1}{2}\big(1 - z^2\big)u_{0xxt} - \eta_{1x}. \tag{9.3.193}$$

And the equation

$$w_{1z} = -u_{1x} \tag{9.3.194}$$

leads to the following form:

$$w_{1zt} = -u_{1xt} = \eta_{1xx} - \frac{1}{2}\big(1 - z^2\big)u_{0xxxt} + (u_0u_{0x})_x. \tag{9.3.195}$$

Integrating this equation with respect to z gives

$$w_{1t} = z\left[(u_0u_{0x})_x + \eta_{1xx} - \frac{1}{2}u_{0xxxt}\right] + \frac{1}{6}z^3u_{0xxxt}. \tag{9.3.196}$$

This satisfies $w_{1t} = 0$ or $w_1 = 0$ on $z = 0$. Differentiating with respect to t, the free surface boundary condition on $z = 1$ yields

$$(w_1 + \eta_0w_{0z})_t = (\eta_{1t} + u_0\eta_{0x})_t. \tag{9.3.197}$$

Thus,

$$(u_0 u_{0x})_x + \eta_{1xx} - \frac{1}{3} u_{0xxxt} - (\eta_0 u_{0x})_t = \eta_{1tt} + (u_0 \eta_{0x})_t, \tag{9.3.198}$$

which becomes, by using (9.3.190),

$$\eta_{1tt} - \eta_{1xx} - \left(u_0^2 + \frac{1}{2}\eta_0^2 \right)_{xx} - \frac{1}{3}\eta_{0xxx} = 0. \tag{9.3.199}$$

Combining (9.3.191) and (9.3.199) gives a single equation for $\eta = \eta_0 + \varepsilon\eta_1 + O(\varepsilon^2)$ in the form

$$\eta_{tt} - \eta_{xx} - \varepsilon \left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \eta_t \, dx \right)^2 \right]_{xx} - \frac{\varepsilon}{3}\eta_{xxxx} = O(\varepsilon^2), \tag{9.3.200}$$

where we have used

$$u_0 = -\int_{-\infty}^{x} \eta_0 \, dx$$

under the assumption that $u_0 \to 0$ as $x \to \infty$.

Equation (9.3.200) or equation (9.3.200) with zero on the right-hand side is referred to as the *Boussinesq* (1871a, 1871b) equation. This equation describes weakly *nondispersive* waves which propagate in both the positive and negative x-directions.

Finally, invoking the transformation $\eta^* = \eta - \varepsilon\eta^2$ and the definition

$$x^* = x + \varepsilon \int_{-\infty}^{x} \eta(x, t, \varepsilon) \, dx,$$

the equation for $\eta^*(x^*, t, \varepsilon)$ takes the form, dropping the asterisk,

$$\eta_{tt} - \eta_{xx} - \frac{3}{2}\varepsilon\left(\eta^2\right)_{xx} - \frac{1}{3}\varepsilon\eta_{xxxx} = O\left(\varepsilon^2\right). \tag{9.3.201}$$

This equation or this equation with zero on the right-hand side is the usual form of the *Boussinesq equation*, which is found to be completely integrable for any positive ε. Using the transformations

$$\eta \to -\frac{2}{\varepsilon}\eta, \qquad x \to \sqrt{\frac{\varepsilon}{3}}\, x, \quad \text{and} \quad t \to \sqrt{\frac{\varepsilon}{3}}\, t,$$

equation (9.3.201) reduces to the standard version of the Boussinesq equation as

$$\eta_{tt} - \eta_{xx} + 3\left(\eta^2\right)_{xx} - \eta_{xxxx} = 0. \tag{9.3.202}$$

Example 9.3.8 (Derivation of the KdV Equations for Gravity–Capillary Waves). The two-dimensional exact nonlinear gravity-capillary waves in water of finite depth h are governed by the following equations for the velocity potential $\phi(x, z, t)$ and the free surface elevation $\eta(x, t)$:

$$\nabla^2\phi = \phi_{xx} + \phi_{zz} = 0, \quad 0 < z < h + \eta, \tag{9.3.203}$$

$$\eta_t + \phi_x\eta_x - \phi_z = 0 \quad \text{on } z = h + \eta, \tag{9.3.204}$$

$$\phi_t + g\eta + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) - \frac{T}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} = B \quad \text{on } z = h + \eta, \tag{9.3.205}$$

$$\phi_z = 0 \quad \text{on } z = 0, \tag{9.3.206}$$

where the origin of coordinates is at $z = 0$, T is the surface tension, ρ is the constant density of water, and B is a constant.

In terms of nondimensional flow variables defined by (9.3.1) and two fundamental parameters ε and δ to characterize the nonlinear shallow gravity–capillary waves, equations (9.3.203)–(9.3.206) can be written, dropping the asterisks, in the nondimensional form:

$$\delta\phi_{xx} + \phi_{zz} = 0, \quad 0 < z < 1 + \varepsilon\eta, \tag{9.3.207}$$

$$\delta(\eta_t + \varepsilon\phi_x\eta_x) - \phi_z = 0 \quad \text{on } z = 1 + \varepsilon\eta, \tag{9.3.208}$$

$$\phi_t + \eta + \frac{\varepsilon}{2}\left(\phi_x^2 + \frac{1}{\delta}\phi_z^2\right) - \frac{\tau\delta\eta_{xx}}{(1+\varepsilon^2\delta\eta_x^2)^{3/2}} = 0 \quad \text{on } z = 1 + \varepsilon\eta, \tag{9.3.209}$$

$$\phi_z = 0 \quad \text{on } z = 0, \tag{9.3.210}$$

where $\tau = \left(\frac{T}{g\rho h^2}\right)$ is the Bond number.

Following Vanden-Broeck (2010), we seek series expansion of $\phi(x,z,t)$ in powers of z in the form

$$\phi(x,z,t) = \sum_{n=0}^{\infty} z^n f_n(x,t). \tag{9.3.211}$$

Substituting (9.3.211) into (9.3.207) and (9.3.210) gives

$$f_n = 0, \quad \text{for } n = 1, 3, 5, \ldots, \tag{9.3.212}$$

$$f_n = -\frac{\delta}{n(n-1)}\frac{\partial^2 f_{n-2}}{\partial x^2}, \quad n = 2, 4, 6, \ldots. \tag{9.3.213}$$

Solving (9.3.213) recursively yields

$$\phi = \sum_{n=0}^{\infty}(-1)^n\delta^n\frac{z^n}{(2n)!}\frac{\partial^{2n} f}{\partial x^{2n}}, \quad \text{where } f = f_0. \tag{9.3.214}$$

Substituting (9.3.214) into the free surface conditions (9.3.208)–(9.3.209) yields

$$\eta_t + \left[(1+\varepsilon\eta)f_x\right]_x - \delta\left[\frac{1}{6}(1+\varepsilon\eta)^3 f_{xxxx} + \frac{\varepsilon}{2}(1+\varepsilon\eta)^2\eta_x f_{xxx}\right] + O\left(\delta^2\right) = 0, \tag{9.3.215}$$

$$f_t + \eta + \frac{1}{2}\varepsilon f_x^2 - \frac{\delta}{2}(1+\varepsilon\eta)^2\left(f_{xxt} + \varepsilon f_x f_{xxx} - \varepsilon f_{xx}^2\right) - \frac{\tau\delta\eta_{xx}}{(1+\varepsilon^2\delta\eta_x^2)^{3/2}} + O\left(\delta^2\right) = 0, \tag{9.3.216}$$

Dropping all terms of order δ and then differentiating (9.3.216) with respect to x gives

$$\big[\eta_t + (1+\varepsilon\eta) f_x\big]_x = 0, \tag{9.3.217}$$
$$f_{xt} + \eta_x + \varepsilon f_x f_{xx} - (\tau\delta)\eta_{xxx} = 0. \tag{9.3.218}$$

Equations (9.3.217)–(9.3.218) represent the shallow water equations for the gravity–capillary waves. These define a hyperbolic system of partial differential equations which does not admit traveling wave solutions.

The major result of this analysis is that the KdV equation for gravity–capillary waves can be obtained from (9.3.217)–(9.3.218) for waves moving to the right. To the lowest order in ε and δ, that is, to the order ε^0 and δ^0, these two equations become

$$\eta_t + f_{xx} = 0 \quad \text{and} \quad f_{xt} + \eta_x = 0. \tag{9.3.219}$$

Thus, the solution moving to the right is

$$\eta = f_{xx}, \qquad \eta_t + \eta_x = 0. \tag{9.3.220}$$

We assume solutions to order ε and δ in the form

$$f_x = \eta + \varepsilon P + \delta Q + O\big(\varepsilon^2 + \delta^2\big), \tag{9.3.221}$$

where P and Q are unknown functions of η and its derivatives. Consequently, equations (9.3.115)–(9.3.116) reduce to

$$\eta_t + \eta_x + \varepsilon(P_x + 2\eta\eta_x) + \delta\left(Q_x - \frac{1}{6}\eta_{xxx}\right) = 0, \tag{9.3.222}$$
$$\eta_t + \eta_x + \varepsilon(P_t + \eta\eta_x) + \delta\left(Q_t - \frac{1}{2}\eta_{xxt}\right) - \delta\tau\eta_{xxx} = 0. \tag{9.3.223}$$

Since $\eta_t = -\eta_x + O(\varepsilon, \delta)$, replacing the t-derivatives by minus the x-derivatives in the first order terms in (9.3.223) gives

$$\eta_t + \eta_x + \varepsilon(-P_x + \eta\eta_x) + \delta\left(-Q_x + \frac{1}{2}\eta_{xxx}\right) - \delta\tau\eta_{xxx} = 0. \tag{9.3.224}$$

Equations (9.3.222)–(9.3.224) must be consistent so that the coefficients of ε and δ in these equations are the same. Consequently,

$$2P_x = -\eta\eta_x, \qquad 2Q_x = \left(\frac{2}{3} - \tau\right)\eta_{xxx}. \tag{9.3.225}$$

Integrating (9.3.225) with respect to x gives

$$P = -\frac{1}{4}\eta^2 \quad \text{and} \quad Q = \left(\frac{1}{3} - \frac{1}{2}\tau\right)\eta_{xx}. \tag{9.3.226}$$

Substituting the values for P_x and Q_x into (9.3.224) gives the KdV equation for gravity–capillary waves in the form

$$\eta_t + \left(1 + \frac{3}{2}\varepsilon\eta\right)\eta_x + \frac{1}{6}(1-3\tau)\delta\eta_{xxx} = 0. \tag{9.3.227}$$

In terms of dimensional variables, the KdV equation (9.3.227) reduces to the form

$$\eta_t + c\left(1 + \frac{3}{2h}\eta\right)\eta_x + \frac{ch^2}{6}(1-3\tau)\eta_{xxx} = 0, \tag{9.3.228}$$

where $c = \sqrt{gh}$ is the shallow water wave velocity and the total depth of water is $h + \eta$. It is important to point out that, in the absence of surface tension ($T = 0$, $\tau = 0$), equation (9.3.228) reduces to the celebrated KdV equation (9.3.1).

We next seek a traveling wave solution of (9.3.228) in the form

$$\eta(X) = h\zeta(X), \quad X = x - Ut, \tag{9.3.229}$$

where $\eta(X)$ represents a wave moving to the right with constant velocity U. Substituting (9.3.229) into (9.3.228) gives the ordinary differential equation

$$h(c-U)\zeta' + \left(\frac{3c}{2}\right)h\zeta\zeta' + \frac{1}{6}ch^2(1-3\tau)h\zeta''' = 0, \tag{9.3.230}$$

where the primes denote the derivatives with respect to X.

Integrating (9.3.230) with respect to X yields

$$\left(1 - \frac{U}{c}\right)\zeta + \frac{3}{4}\zeta^2 + \frac{h^2}{6}(1-3\tau)\zeta'' = A, \tag{9.3.231}$$

where A is a constant of integration.

Multiplying (9.3.231) by ζ' and integrating again gives

$$\zeta^3 + 2\left(1 - \frac{U}{c}\right)\zeta^2 + \frac{h^2}{3}(1-3\tau)\zeta'^2 = 4A\zeta + B, \tag{9.3.232}$$

where B is another constant of integration.

We consider a special case when ζ and its derivatives tend to zero at infinity ($X \to \pm\infty$). Hence, $A = B = 0$ so that (9.3.232) gives

$$\left(\frac{d\zeta}{dX}\right)^2 = \frac{3}{h_1^2}\zeta^2(a-\zeta), \quad h_1^2 = h^2(1-3\tau), \tag{9.3.233}$$

where

$$a = 2\left(\frac{U}{c} - 1\right). \tag{9.3.234}$$

The right-hand side of (9.3.233) vanishes at $\zeta = 0$ and $\zeta = a$, and the exact solution of (9.3.233) can be written in the integral form

$$X - X_0 = \left(\frac{h_1^2}{3}\right)^{\frac{1}{2}} \int_0^{\zeta} \frac{d\zeta}{\zeta\sqrt{a-\zeta}}, \tag{9.3.235}$$

which is, by substitution $\zeta = a\,\mathrm{sech}^2\,\theta$,

$$X - X_0 = \left(\frac{h_1^2}{3a}\right)^{\frac{1}{2}}\theta, \tag{9.3.236}$$

where X_0 is an integrating constant, Thus, the solution is

$$\zeta(X) = a\,\mathrm{sech}^2\left[\left(\frac{3a}{4h_1^2}\right)^{\frac{1}{2}}(X - X_0)\right]. \tag{9.3.237}$$

The solution $\zeta(X)$ increases from $\zeta = 0$ as $X \to -\infty$ so that it attains a maximum value $\zeta = \zeta_{\max} = a$ at $X = 0$, and then decreases symmetrically to $\zeta = 0$ as $X \to \infty$. These features imply that $X_0 = 0$ so that

$$\zeta(X) = a\,\mathrm{sech}^2\left[\left(\frac{3a}{4h_1^2}\right)^{\frac{1}{2}}X\right]. \tag{9.3.238}$$

When $\tau < \frac{1}{3}$ and $a > 0$, (9.3.238) represents a wave of elevation traveling with the speed U given by (9.3.234), that is,

$$U = c\left(1 + \frac{a}{2}\right) > c. \tag{9.3.239}$$

When $\tau > \frac{1}{3}$ and $a < 0$, the solution $\zeta(X)$ in(9.3.238) is a wave of depression moving with speed $U < c$. Therefore, the final solution is

$$\eta(x,t) = \eta_0\,\mathrm{sech}^2\left[\left(\frac{3\eta_0}{4h^3(1-3\tau)}\right)^{\frac{1}{2}}(x - Ut)\right], \tag{9.3.240}$$

where the range in η is $\eta_0 = ah_1$. As explained in Section 9.4, $\eta(x,t)$ represents *capillary solitary wave solution* of the KdV equation (9.3.227) for any positive constant η_0. Since $\eta(X) > 0$ for all X, the solution is a wave of elevation which is symmetrical about $X = 0$. It propagates in the medium without change of shape with velocity U which is directly proportional to the amplitude η_0. The width, $[\frac{3\eta_0}{4h^3(1-3\tau)}]^{-\frac{1}{2}}$ of the solution is inversely proportional to $\sqrt{\eta_0}$.

It is important to point out that, when $\tau = \frac{1}{3}$, solution (9.3.238) shows that $(\frac{d\zeta}{dX})$ becomes unbounded. This is due to the fact that the linear dispersive term in (9.3.238) disappears when $\tau = \frac{1}{3}$. Furthermore, equation (9.3.228) is derived under the assumption that ε and δ are both small and of the same order of magnitude, that is, $\varepsilon = \delta = \alpha$, where $0 < \alpha \ll 1$. This assumption has successfully been used in the expansion procedure to derive (9.3.228). However, to derive the evolution equation valid near $\tau = \frac{1}{3}$, it is necessary to include higher order dispersive terms. This can be achieved by making use of $\varepsilon = \alpha^2$ and $\delta = \alpha$ and using the expansion near $\frac{1}{3}$ in the form

$$\tau = \frac{1}{3} + \alpha\tau_1 + \alpha^2\tau_2 + \cdots. \tag{9.3.241}$$

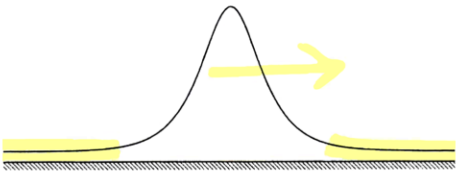

Fig. 9.4 A solitary wave with monotonic decay to a constant level.

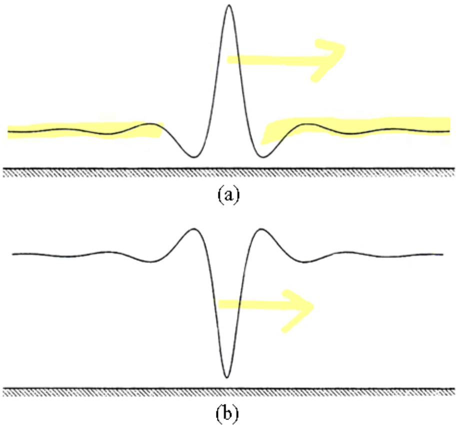

Fig. 9.5 (a) An elevation solitary wave with a decaying oscillatory tail. (b) A depression solitary wave with a decaying oscillatory tail.

Invoking such expansion procedure, the fifth order KdV equation for gravity–capillary waves can be derived in the form

$$\eta_t + c\left(1 + \frac{3}{2h}\eta\right)\eta_x + \frac{ch^2}{6}(1 - 3\tau)\eta_{xxx} + \frac{ch^4}{90}\eta_{xxxxx} = 0. \qquad (9.3.242)$$

We close this section by adding the following comments based on Vanden-Broeck's (2010) computational results of gravity–capillary solitary waves. When $0 < \tau < \frac{1}{3}$, there are no fully nonlinear solutions of small amplitude consistent with (9.3.240). However, there are fully nonlinear solutions characterized by infinite train of ripples in the far field. Such waves are often referred to as *generalized solitary waves* with a series of ripples of constant amplitude in the far field.

The above analysis further reveals that there are three different kinds of capillary solitary waves. The first kind is a solitary wave with a free-surface elevation that tends monotonically to a constant level in the far field as shown in Figure 9.4. The solution (9.3.238) or (9.3.240) is an example of this kind.

The solitary wave of the second kind represents a free-surface elevation (or depression) wave profile which tends to a constant level in the far field with a decaying oscillatory tail as shown in Figures 9.5(a) and 9.5(b).

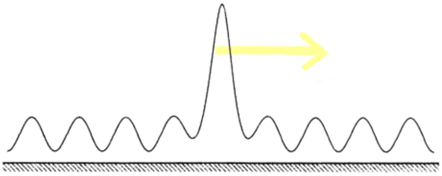

Fig. 9.6 A solitary wave with oscillations of constant amplitude.

The third kind of solitary wave represents a free-surface wave profile that does not tend to a constant level, but is characterized by oscillations of constant amplitude in the far field as shown in Figure 9.6. This is often called the *generalized solitary wave*. Such waves occur when gravity and surface tension are included.

The existence of solitary waves as shown in Figures 9.5(a) and 9.5(b) is related to the minimum in Figure 2.4 in Chapter 2 for $\tau < \frac{1}{3}$. The profiles in Figures 9.5(a) and 9.5(b) look like waves of slowly varying amplitude. However, those waves cannot be expected to be steady unless the phase and group velocities are equal. This is exactly what happens at the minimum in Figure 2.4. This reveals an intuitive explanation why the branches of solitary waves decaying tails bifurcate from the minimum value of the Figure 2.4. Numerical computations of several authors including Vanden-Broeck (2010), Vanden-Broeck and Dias (1992), and others confirmed the existence of such waves.

On the other hand, numerical results of Vanden-Broeck (2010) and Champneys et al. (2002) show that the multiple branches of periodic solutions approach generalized solitary waves (see Figure 9.6) as $\frac{\lambda}{h} = \frac{2\pi}{kh} \to \infty$. In other words, moving from one branch to the next includes two crests or two troughs. In the limit, as $\frac{\lambda}{h} =\to \infty$, this produces a generalized solitary wave with an infinite number of crests and troughs. Thus, these ripples in the tail of the generalized solitary waves seem to be physically unrealistic because they occur on both sides, and hence, they do not satisfy the radiation conditions at infinity. This unrealistic feature has been rigorously confirmed with a negative answer for all τ close to $\frac{1}{3}$ by several authors including Vanden-Broeck (2010) and Champneys et al. (2002).

Further, numerical computations of fully nonlinear solutions predicted by the steady KdV equation (9.3.227) for $\tau < \frac{1}{3}$ show inaccurate answer in the sense that the solutions do not have flat free profiles in the far field as shown in Figure 9.4, but are generalized solitary waves as in Figure 9.6. The fifth order KdV equation also predicts the generalized solitary waves shown in Figure 9.6 when their amplitude is small and τ is close to $\frac{1}{3}$.

9.4 Solutions of the KdV Equation: Solitons and Cnoidal Waves

To find solutions of the KdV equation, it is convenient to rewrite it in terms of dimensional variables as

$$\eta_t + c\left(1 + \frac{3}{2h}\eta\right)\eta_x + \frac{ch^2}{6}\eta_{xxx} = 0, \tag{9.4.1}$$

where $c = \sqrt{gh}$, and the total depth $H = h + \eta$. The first two terms $(\eta_t + c\eta_x)$ describe wave evolution at the shallow water speed c, the third term with coefficient $(3c/2h)$ represents a nonlinear wave steepening, and the last term with coefficient $(ch^2/6)$ describes linear dispersion. Thus, the KdV equation is a balance between time evolution, nonlinearity, and linear dispersion. The dimensional velocity u is obtained from (9.3.55) with (9.3.58) in the form

$$u = \frac{c}{h}\left(\eta - \frac{1}{4h}\eta^2 + \frac{h}{3}\eta_{xx}\right). \tag{9.4.2}$$

We seek a traveling wave solution of (9.4.1) in the frame X so that $\eta = \eta(X)$ and $X = x - Ut$ with $\eta \to 0$, as $|x| \to \infty$, where U is a constant speed. Substituting this solution in (9.4.1) gives

$$(c - U)\eta' + \frac{3c}{2h}\eta\eta' + \frac{ch^2}{6}\eta''' = 0, \tag{9.4.3}$$

where $\eta' = d\eta/dX$. Integrating this equation with respect to X yields

$$(c - U)\eta + \frac{3c}{4h}\eta^2 + \frac{ch^2}{6}\eta'' = A, \tag{9.4.4}$$

where A is an integration constant.

We multiply this equation by $2\eta'$ and integrate again to obtain

$$(c - U)\eta^2 + \left(\frac{c}{2h}\right)\eta^3 + \left(\frac{ch^2}{6}\right)\left(\frac{d\eta}{dX}\right)^2 = 2A\eta + B, \tag{9.4.5}$$

where B is also a constant of integration.

We now consider a special case when η and its derivatives tend to zero at infinity and $A = B = 0$, so that (9.4.5) gives

$$\left(\frac{d\eta}{dX}\right)^2 = \frac{3}{h^3}\eta^2(a - \eta), \tag{9.4.6}$$

where

$$a = 2h\left(\frac{U}{c} - 1\right). \tag{9.4.7}$$

The right-hand side of (9.4.6) vanishes at $\eta = 0$ and $\eta = a$, and the exact solution of (9.4.6) is given by

$$\eta(X) = a\,\mathrm{sech}^2(bX), \quad b = \left(\frac{3a}{4h^3}\right)^{1/2}. \tag{9.4.8ab}$$

Thus,

$$\eta(x,t) = a\,\mathrm{sech}^2\left[\left(\frac{3a}{4h^3}\right)^{1/2}(x - Ut)\right], \tag{9.4.9}$$

where the velocity of the wave is

$$U = c\left(1 + \frac{a}{2h}\right). \tag{9.4.10}$$

This is an *exact* solution of the KdV equation for all a/h; however, the equation is derived with the approximation $(a/h) \ll 1$. The solution (9.4.9) is called a *soliton* (or *solitary wave*) describing a single hump of height a above the undisturbed depth h and tending rapidly to zero away from $X = 0$. The solitary wave propagates to the right with velocity $U(> c)$, which is directly proportional to the amplitude a and has width $b^{-1} = (3a/4h^3)^{-1/2}$, that is, b^{-1} is inversely proportional to the square root of the amplitude a. Another significant feature of the soliton solution is that it travels in the medium without change of shape, which is hardly possible without retaining δ-order terms in the governing equation. A solitary wave profile has already been shown in Figure 9.1.

In the general case, when both A and B are nonzero, (9.4.5) can be rewritten as

$$\frac{h^3}{3}\left(\frac{d\eta}{dX}\right)^2 = -\eta^3 + 2h\left(\frac{U}{c} - 1\right)\eta^2 + \frac{2h}{c}(2A\eta + B) \equiv F(\eta), \tag{9.4.11}$$

where $F(\eta)$ is a cubic with simple zeros.

We seek a real bounded solution for $\eta(X)$, which has a minimum value zero and a maximum value a and oscillates between the two values. For bounded solutions, all three zeros η_1, η_2, η_3 must be real. Without loss of generality, we set $\eta_1 = 0$ and $\eta_2 = a$. Hence, the third zero must be negative so that $\eta_3 = -(b-a)$ with $b > a > 0$. With these choices, $F(\eta) = \eta(a-\eta)(\eta - a + b)$, and equation (9.4.11) assumes the form

$$\frac{h^3}{3}\left(\frac{d\eta}{dX}\right)^2 = \eta(a-\eta)(\eta - a + b), \tag{9.4.12}$$

where

$$U = c\left(1 + \frac{2a-b}{2h}\right), \tag{9.4.13}$$

which is obtained by comparing the coefficients of η^2 in (9.4.11) and (9.4.12).

Writing $a - \eta = p^2$, it follows from equation (9.4.12) that

$$\left(\frac{3}{4h^3}\right)^{1/2} dX = \frac{dp}{[(a-p^2)(b-p^2)]^{1/2}}. \tag{9.4.14}$$

Substituting $p = \sqrt{a}q$ in (9.4.14) gives the standard elliptic integral of the first kind (see Dutta and Debnath 1965)

$$\left(\frac{3b}{4h^3}\right)^{1/2} X = \int_0^q \frac{dq}{[(1-q^2)(1-m^2q^2)]^{1/2}}, \quad m = \left(\frac{a}{b}\right)^{1/2}, \tag{9.4.15}$$

and then, function q can be expressed in terms of the Jacobian *sn function*

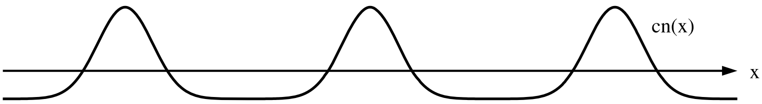

Fig. 9.7 A cnoidal wave.

$$q(X,m) = sn\left[\left(\frac{3b}{4h^3}\right)^{1/2} X, m\right], \tag{9.4.16}$$

where m is the modulus of $sn(z,m)$.

Finally,

$$\eta(X) = a\left[1 - sn^2\left\{\left(\frac{3b}{4h^3}\right)^{1/2} X\right\}\right] = a\,cn^2\left[\left(\frac{3b}{4h^3}\right)^{1/2} X\right], \tag{9.4.17}$$

where $cn(z,m)$ is also the Jacobian elliptic function with a period $2K(m)$, where $K(m)$ is the complete elliptic integral of the first kind defined by

$$K(m) = \int_0^{\pi/2} \left(1 - m^2 \sin^2\theta\right)^{-1/2} d\theta, \tag{9.4.18}$$

and $cn^2(z) + sn^2(z) = 1$.

It is important to note that cnz is periodic, and hence, $\eta(X)$ represents a train of periodic waves in shallow water. Thus, these waves are called *cnoidal waves* with wavelength

$$\lambda = 2\left(\frac{4h^3}{3b}\right)^{1/2} K(m). \tag{9.4.19}$$

The upshot of this analysis is that solution (9.4.17) represents a nonlinear wave whose shape and wavelength (or period) all depend on the amplitude of the wave. A typical cnoidal wave is shown in Figure 9.7. Sometimes, the cnoidal waves with slowly varying amplitude are observed in rivers. More often, wavetrains behind a weak bore (called an *undular bore*) can be regarded as cnoidal waves. Two limiting cases are of special physical interest: (i) $m \to 0$ and (ii) $m \to 1$.

In the first case, $sn\, z \to \sin z$, $cn\, z \to \cos z$ as $m \to 0$ $(a \to 0)$. This corresponds to small-amplitude waves where the linearized KdV equation is appropriate. So, in this limiting case, the solution (9.4.17) becomes

$$\eta(x,t) = \frac{1}{2}a\left[1 + \cos(kx - \omega t)\right], \quad k = \left(\frac{3b}{h^3}\right)^{1/2}, \tag{9.4.20}$$

where the corresponding dispersion relation is

$$\omega = Uk = ck\left(1 - \frac{1}{6}k^2h^2\right). \tag{9.4.21}$$

This corresponds to the first two terms of the series expansion of $(gk \tanh kh)^{1/2}$. Thus, these results are in perfect agreement with the linearized theory.

In the second limiting case, $m \to 1 (a \to b)$, $cnz \longrightarrow \operatorname{sech} z$. Thus, the cnoidal wave solution tends to the classical KdV solitary-wave solution where the wavelength λ, given by (9.4.19), tends to infinity because $K(a) = \infty$ and $K(0) = \pi/2$. The solution identically reduces to (9.4.9) with (9.4.10).

We next report the numerical computation of the KdV equation (9.4.1) due to Berezin and Karpman (1966). In terms of new variables defined by

$$x^* = x - ct, \qquad t^* = t, \qquad \eta^* = \frac{3c}{2h}\eta, \tag{9.4.22}$$

omitting the asterisks, equation (9.4.1) becomes

$$\eta_t + \eta\eta_x + \beta\eta_{xxx} = 0, \tag{9.4.23}$$

where $\beta = (\frac{1}{6})ch^2$.

We examine the numerical solution of (9.4.23) with the initial condition

$$\eta(x, 0) = \eta_0 f\left(\frac{x}{\ell}\right), \tag{9.4.24}$$

where η_0 is constant and $f(\xi)$ is a nondimensional function characterizing the initial wave profile. It is convenient to introduce the dimensionless variables

$$\xi = \frac{x}{\ell}, \qquad \tau = \frac{\eta_0 t}{\ell}, \qquad u = \frac{\eta}{\eta_0} \tag{9.4.25}$$

so that equations (9.4.23) and (9.4.24) reduce to

$$u_\tau + uu_\xi + \sigma^{-2}u_{\xi\xi\xi} = 0, \tag{9.4.26}$$

$$u(\xi, 0) = f(\xi), \tag{9.4.27}$$

where the dimensionless parameter σ is defined by $\sigma = \ell(\frac{\eta_0}{\beta})^{\frac{1}{2}}$.

Berezin and Karpman (1966) obtained the numerical solution of (9.4.26) with the Gaussian initial pulse of the form $u(\xi, 0) = f(\xi) = \exp(-\xi^2)$ and values of the parameter $\sigma = 1.9$ and $\sigma = 16.5$. Their numerical solutions are shown in Figure 9.8.

As shown in Figure 9.8 for case (a), the perturbation splits into a soliton and a wavepacket. In case (b), there are *six* solitons. It is readily seen that the peaks of the solitons lie nearly on a straight line. This is due to the fact that the velocity of the soliton is proportional to its amplitude, so that the distances traversed by the solitons would also be proportional to their amplitudes.

Zabusky's (1967) numerical investigation of the interaction of two solitons reveals that the taller soliton, initially behind, catches up to the shorter one, they undergo a nonlinear interaction and, then, emerge from the interaction without any change in shape and amplitude. The end result is that the taller soliton reappears in front and the shorter one behind. This is essentially strong computational evidence of the stability of solitons.

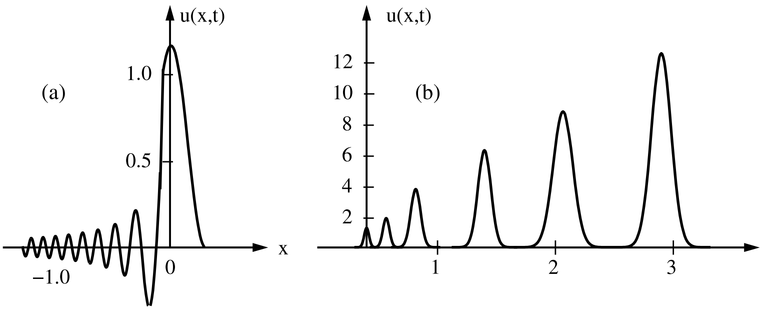

Fig. 9.8 The solutions of the KdV equation $u(x,t)$ for large values of t with the values of the similarity parameter σ: (a) $\sigma = 1.9$, (b) $\sigma = 16.5$. From Berezin and Karpman (1966).

Using the transformation

$$x^* = \varepsilon\beta(x - ct), \qquad t^* = \varepsilon^3 t, \qquad \eta^* = \left(\alpha\varepsilon^2\right)^{-1}\eta$$

with $\alpha\beta(= 3c/2h) = 6$ and $\beta^3(= ch^2/6) = 1$, we write the KdV equation (9.4.1) in the normalized form, dropping the asterisks,

$$\eta_t + 6\eta\eta_x + \eta_{xxx} = 0. \tag{9.4.28}$$

We next seek a steady progressive wave solution of (9.4.28) in the form

$$\eta = 2k^2 f(X), \quad X = kx - \omega t. \tag{9.4.29}$$

Then, the equation for $f(X)$ can be integrated once to obtain

$$f''(X) = 6A + (4 - 6B)f - 6f^2, \tag{9.4.30}$$

where A is a constant of integration and the frequency ω is given by

$$\omega = k^3(4 - 6B). \tag{9.4.31}$$

A single-soliton solution corresponds to the special case $A = B = 0$ and is given by

$$f(X) = \text{sech}^2(X - X_0), \tag{9.4.32}$$

where X_0 is a constant.

Thus, for a series of solitons spaced 2σ apart, we write

$$f(X) = \sum_{n=-\infty}^{\infty} \text{sech}^2(X - 2n\sigma). \tag{9.4.33}$$

This is a 2σ periodic function that satisfies (9.4.30) for some A and B.

The general elliptic function solution of (9.4.28) can be obtained from the integral of (9.4.30), which can be written as

$$f_X^2 = -4C + 12Af + (4 - 6B)f^2 - 4f^3, \tag{9.4.34}$$

where C is a constant of integration. Various asymptotic and numerical results lead to the relations (see Whitham 1984)

$$C = -\frac{1}{2}\frac{dA(\sigma)}{d\sigma} = \frac{1}{4}\frac{d^2B(\sigma)}{d\sigma^2}, \tag{9.4.35ab}$$

and the cubic in (9.4.34) can be factorized as

$$-C + 3Af + \left(1 - \frac{3}{2}B\right)f^2 - f^3 = (f_1 - f)(f - f_2)(f - f_3), \tag{9.4.36}$$

where $f_r(\sigma)$ $(r = 1, 2, 3)$ are determined from $A(\sigma)$, $B(\sigma)$, and $C(\sigma)$. If we set $f_1 > f_2 > f_3$ and, then, the periodic solution oscillates between f_1 at $X = 0$ and f_2 at $X = \sigma$, one particular form of the solution is given by

$$f(X) = f_2 + (f_1 - f_2)cn^2\left(\sqrt{(f_1 - f_3)}X\right), \tag{9.4.37}$$

where the modulus m of $cn(z, m)$ is given by

$$m^2 = \left(\frac{f_1 - f_2}{f_1 - f_3}\right). \tag{9.4.38}$$

Thus, it follows from (9.4.33) and (9.4.37) that the following identity holds:

$$f_2 + (f_1 - f_2)cn^2\left(\sqrt{(f_1 - f_3)}X\right) = \sum_{n=-\infty}^{\infty} \operatorname{sech}^2(X - 2n\sigma), \tag{9.4.39}$$

which can be verified by comparing the periods and poles of the two sides.

It is interesting to point out that there is also a theta function representation of the periodic solution of the KdV equation. The solution may be written in the form

$$f(X) = \frac{\partial^2}{\partial X^2}\log \Theta(X), \tag{9.4.40}$$

where $\Theta(X)$ is a slight modification of the theta function $\vartheta_4(z, q)$ in Dutta and Debnath's (1965) notation, the imaginary period τ in $q = \exp(i\pi\tau)$ has been set equal to $(2i\sigma/\pi)$ so that $q = \exp(-2\sigma)$, and the independent variable z is changed to $(iX + \pi/2)$ and

$$\Theta(X) = \sum_{n=-\infty}^{\infty} \exp\left(-2nX - 2n^2\sigma\right). \tag{9.4.41}$$

A simple direct proof of (9.4.40) is not obvious. However, it is equivalent to (9.4.33), which can be verified by using Jacobi's infinite product formula for the theta functions (Dutta and Debnath 1965, p.108). With the appropriate modification in (9.4.29), the product representation of $\Theta(X)$ is

$$\Theta(X) = G \prod_{r=1}^{\infty} \left(1 + e^{2X-(2r-1)2\sigma}\right) \prod_{r=1}^{\infty} \left(1 + e^{-2X-(r-1)2\sigma}\right). \tag{9.4.42}$$

Thus, the series representation (9.4.41) follows from the preceding analysis by replacing X by $X + \sigma$.

There is a similar situation for the modified KdV equation in the normalized form

$$v_r + 3v^2 v_x + v_{xxx} = 0. \tag{9.4.43}$$

If we seek a steady progressive solution in the form

$$v = \sqrt{2} k f(X), \quad X = kx - \omega t, \tag{9.4.44}$$

then $f(X)$ satisfies the equation

$$F_{XX} = 2A + (2B + 1) f - 2f^3, \tag{9.4.45}$$

where A is a constant of integration and

$$2B = \left(\frac{\omega}{k^3} - 1\right). \tag{9.4.46}$$

A single soliton corresponding to the special case $A = B = 0$ is given by

$$f(X) = \operatorname{sech}(X - X_0), \tag{9.4.47}$$

and hence, a series of solitons spaced 2σ apart is given by

$$f(X) = \sum_{n=-\infty}^{\infty} \exp(X - 2n\sigma). \tag{9.4.48}$$

This is, indeed, an *exact* solution. The required identity between the series of the sech function is found by substituting (9.4.48) into (9.4.45). It follows from Whitham's argument (1974) that

$$A = 0, \qquad B = 6 \sum_{s=1}^{\infty} \frac{\cosh(2s\sigma)}{\sinh^2(2s\sigma)}. \tag{9.4.49}$$

Integrating (9.4.45) once gives

$$f_X^2 = -C + (2B + 1) f^2 - f^4 \equiv \left(f_1^2 - f^2\right)\left(f^2 - f_2^2\right), \tag{9.4.50}$$

where C is a constant of integration and $f_1 \geq f \geq f_2$. This equation admits the periodic solution

$$f(X) = f_1 \, dn(f_1 X, m), \tag{9.4.51}$$

where the modulus m of the elliptic function is

$$m^2 = \left(f_1^2 - f_2^2\right)/f_1^2, \tag{9.4.52}$$

and (9.4.48) gives an identity for the dn function in terms of a sum of the sech functions. The relation between σ and m is given by $\sigma f_1(\sigma) = K(m)$.

Finally, the higher-order modified KdV equation

$$v_t + (p+1)v^p v_x + v_{xxxx} = 0, \quad p > 2, \tag{9.4.53}$$

admits single-soliton solutions in the form

$$v(x,t) = a\,\mathrm{sech}^{2/p}(kx - \omega t). \tag{9.4.54}$$

However, in view of the fractional powers of the sech function, it seems, perhaps, unlikely that there will be any simple superposition formula.

9.5 The Lie Group Method and Similarity Analysis of the KdV Equation

The KdV equation describes the generation and propagation of moderately small-amplitude shallow water waves and many other important phenomena, where a small nonlinearity is combined with a cubic dispersion relation. In many physical situations, its similarity solutions become important.

A simple transformation

$$x \longrightarrow cx, \qquad t \longrightarrow bt, \quad \text{and} \quad 1 + u \longrightarrow au, \tag{9.5.1}$$

can be used to transform the KdV equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{9.5.2}$$

into the general form

$$u_t + \left(\frac{ab}{c}\right)uu_x + \frac{b}{c^3}u_{xxx} = 0, \tag{9.5.3}$$

where a, b, c are nonzero real constants. With suitable choices of the constants, this equation reduces to the standard form (9.4.28), which is invariant under the transformation T_a defined by

$$\tilde{x} = ax, \qquad \tilde{t} = a^3 t, \qquad \tilde{u} = a^{-2}u \tag{9.5.4}$$

for a nonzero real a. The set of all such transformations $\{T_a\}$ forms an infinite *Lie group* with parameter a with the composition (multiplicative) law $T_a T_b = T_{ab}$. This law is commutative. Also, the associative law is satisfied since

$$T_a(T_b T_c) = T_a T_{bc} = T_{abc} = T_{ab} T_c = (T_a T_b) T_c.$$

In view of the fact that

$$T_1 T_a = T_{1a} = T_{a1} = T_a$$

for all $a \neq 0$, T_1 represents the identity transformation.

Finally, we find that

$$T_a T_{a^{-1}} = T_1 = T_{a^{-1}} T_a.$$

This confirms that $T_{a^{-1}}$ is both a left-hand and right-hand inverse of T_a.

The fact that the KdV equation (9.4.28) with 6 replaced by -6 is invariant under a continuous (or Lie) group of transformations T_a suggests that we can seek invariant properties of the solutions. In fact, the quantities

$$u(x,t) = t^p f(\xi), \quad \xi = xt^q \tag{9.5.5ab}$$

are invariant under the transformation T_a provided $p = -\frac{2}{3}$ and $q = -\frac{1}{3}$. It is also easy to check that, if $u(x,t) = -(3t)^{2/3} f(\xi)$ where $\xi = x(3t)^{-1/3}$, then $f(\xi)$ satisfies the ordinary differential equation

$$f''' + (6f - \xi)f' - 2f = 0. \tag{9.5.6}$$

Another substitution of $f = \lambda(dV/d\xi) - V^2$ in (9.5.6), where $V = V(\xi)$ and λ is a constant to be determined, yields a *Painlevé equation of the second kind*

$$V'' - \xi V - 2V^3 = 0. \tag{9.5.7}$$

This equation gives a solution describing a wave profile that decays as $\xi \to +\infty$ and oscillates as $\xi \to -\infty$. The presence of a Painlevé equation, for which each movable singularity is a pole, is not accidental. It is believed that there is a direct correspondence between the appearance of a Painlevé equation for a given partial differential equation and the existence of an inverse scattering transform (and hence, soliton solutions) for that equation.

Zakharov and Shabat (1974) proved an exact reduction of the Boussinesq equation

$$u_{tt} - u_{xx} - \frac{1}{2}\left(u^2\right)_{xx} - \frac{1}{4}u_{xxxx} = 0 \tag{9.5.8}$$

to an ordinary differential equation by looking for a traveling wave solution

$$u(x,t) = f(x - Ut) = f(X), \tag{9.5.9}$$

where $f(X)$ satisfies

$$\left(1 - U^2\right)f'' + \frac{1}{2}\left(f^2\right)'' + \frac{1}{4}f'''' = 0. \tag{9.5.10}$$

This can be integrated twice to obtain the following Painlevé equations, depending on the constants of integration:

$$f'' + 2f^2 + A = 0, \qquad f'' + 2f^2 + X = 0. \tag{9.5.11ab}$$

In general, certain partial differential equations admit similarity solutions in the form

$$u(x,t) = t^{-\alpha} f\left(xt^{-\beta}\right), \tag{9.5.12}$$

where α and β are constants and f satisfies an ordinary differential equation. The equation is invariant under the transformation T_a:

$$\tilde{x} = a^{\beta} x, \qquad \tilde{t} = at, \qquad \tilde{u} = a^{-\alpha} u, \tag{9.5.13}$$

where a is a scalar. The set of all such transformations T_a also forms a continuous group.

It can easily be verified that the normalized KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{9.5.14}$$

has the similarity solution

$$u(x,t) = (3t)^{-2/3} f(\xi), \quad \xi = x(3t)^{-1/3}. \tag{9.5.15ab}$$

Substituting (9.5.15ab) in (9.5.14) gives the following ordinary differential equation for $f(\xi)$ in the form

$$f''' + (6f - \xi) f' - 2f = 0. \tag{9.5.16}$$

In the limit as $\xi \to \infty$, solutions of equation (9.5.16) tend to solutions of the linearized equation

$$f''' - 2f - \xi f' = 0. \tag{9.5.17}$$

This equation admits a solution $f(\xi) = Ai'(\xi)$, where $Ai'(\xi)$ is the Airy function. The other two solutions $Bi'(\xi)$, which is exponentially large, and $Gi'(\xi) \sim -(\pi z^2)^{-1}$ do not decay exponentially (Abramowitz and Stegun 1972, pp. 446–450). Thus we set the boundary condition

$$f(\xi) \sim aAi'(\xi), \quad \text{as } \xi \to \infty, \tag{9.5.18}$$

where a is an amplitude parameter.

The numerical computation of Berezin and Karpman (1966) suggests that, when a is small enough, $f(\xi)$ is oscillatory, as $\xi \to -\infty$, but otherwise, $f(\xi)$ may develop singularities. Rosales (1978) showed that there exists a critical value a_1 of a which separates the oscillatory nature from the singular solutions. For $|a| = a_1$, $f(\xi) \sim \frac{1}{2}\xi$ as $\xi \to -\infty$. For $|a| = a_1$, $f(\xi)$ becomes oscillatory, as $\xi \to -\infty$. On the other hand, when $a > a_1$, $f(\xi)$ develops a singularity.

Some special nonlinear evolution equations admit solutions that are rational functions of the independent variables. For example, $u(x,t) = (x/t)$ is a rational solution of the equation

$$u_t + uu_x = 0. \tag{9.5.19}$$

It turns out that the KdV equation (9.4.28) with the coefficient 6 replaced by -6 also has a simple rational solution. Assuming that $u = u(x)$ and u, u', $u'' \to 0$, as $|x| \to \infty$, the resulting KdV equation $u''' - 6uu' = 0$ can be integrated twice to obtain $u'^2 = 2u^3$, which admits the rational solution

$$u(x,t) = 2x^{-2}. \tag{9.5.20}$$

This is singular at $x = 0$. It can also be shown that equation (9.4.28) with the coefficient 6 replaced by -6 has a rational solution in the form

$$u(x,t) = \frac{6x(x^3 - 24t)}{(x^3 + 12t^2)}. \tag{9.5.21}$$

This is also singular on $x^3 + 12t^2 = 0$. It is conjectured that all rational solutions of the KdV equation are singular!

Zabusky (1967) showed that the modified KdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{9.5.22}$$

has the rational solution

$$u(x,t) = A - (4A)/\left\{4A^2\left(x - 6A^2 t\right)^2 + 1\right\}, \tag{9.5.23}$$

where A is a real constant.

9.6 Conservation Laws and Nonlinear Transformations

An equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0 \tag{9.6.1}$$

is called a *conservation law*, where T and X are known as the *density* and the *flux*, respectively, and neither T nor X involve derivatives with respect to t. If both T and X are integrable on $(-\infty, \infty)$, so that X tends to a constant as $|x| \to \infty$, then equation (9.6.1) can be integrated to obtain

$$\frac{d}{dt}\left[\int_{-\infty}^{\infty} T\, dx\right] = 0, \quad \text{or equivalently,} \tag{9.6.2a}$$

$$\int_{-\infty}^{\infty} T\, dx = \text{const.} \tag{9.6.2b}$$

The integral (9.6.2b) is usually called a *constant of motion*, provided t is interpreted as a time-like variable.

It is convenient to write the KdV equation in the standard form that is much used for the development of the theory of solitons,

$$Ku \equiv u_t - 6uu_x + u_{xxx} = 0. \tag{9.6.3}$$

This is already in conservation form with

$$T = u \quad \text{and} \quad X = u_{xx} - 3u^2. \tag{9.6.4ab}$$

Hence, if T and X are integrable, and u satisfies (9.6.3), then

$$\int_{-\infty}^{\infty} u\,dx = \text{const.}, \tag{9.6.5}$$

which applies to all solutions provided that u and its gradients tend to zero as $|x| \to \infty$. Thus, the first constant of motion is simply the spatial integral of u.

The second conservation law for (9.6.3) can be obtained by multiplying it by u, so that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}u_x^2 - 2u^3\right) = 0. \tag{9.6.6}$$

Thus,

$$\int_{-\infty}^{\infty} \frac{1}{2}u^2\,dx = \text{const.} \tag{9.6.7}$$

for all solutions of the KdV equation which vanish fast enough at infinity. Thus, both u and u^2 are *conserved densities* for the motion associated with the KdV equation.

We know that the KdV equation describes a certain class of nonlinear water waves. In fact, u is associated with the free surface elevation function which, in turn, is also proportional to the velocity in the x-direction. Thus, equation (9.6.5) describes the conservation of mass, and (9.6.7) the conservation of horizontal momentum for water waves within the scope of a shallow water approximation theory. This immediately suggests that there should be a corresponding conserved density that could be associated with the energy of water waves. Indeed, this is true and can be confirmed by adding $3u^2 \times (9.6.3)$ to $u_x \times \{(\partial/\partial x)(9.6.3)\}$ so that the resulting equation can be rewritten as

$$\frac{\partial}{\partial t}\left(u^3 + \frac{1}{2}u_x^2\right) + \frac{\partial}{\partial x}\left(-\frac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 + u_xu_{xxx} - \frac{1}{2}u_{xx}^2\right) = 0. \tag{9.6.8}$$

This gives a third constant of motion,

$$\int_{-\infty}^{\infty}\left(u^3 + \frac{1}{2}u_x^2\right)dx = \text{const.} \tag{9.6.9}$$

Then, by using very laborious methods, Miura et al. (1968) obtained *eight* more conservation laws for the KdV equation. They then conjectured that there exist an infinite number of polynomial conservation laws for the KdV equation. At the same time, a question was raised: Is the existence of soliton solutions of the KdV equation closely related to the existence of an infinite number of conservation laws? Miura et al. (1968) developed an ingenious method of determining an infinite number of conservation laws by introducing the *Miura transformation*,

$$u = v^2 + v_x. \tag{9.6.10}$$

This is similar to the Cole–Hopf transformation used in the Burgers equation which can be reduced to the linear diffusion equation.

A direct substitution of (9.6.10) in (9.6.3) gives

$$2vv_t + v_{xt} - 6\big(v^2 + v_x\big)(2vv_x + v_{xx}) + 6v_xv_{xx} + 2vv_{xxx} + v_{xxxx} = 0,$$

which can then be put into the form

$$\left(2v + \frac{\partial}{\partial x}\right) Mv = 0,$$

where the operator M is defined by

$$Mv \equiv v_t - 6v^2 v_x + v_{xxx} = 0. \tag{9.6.11}$$

This is called the *modified KdV* (*mKdV*) *equation*. Thus, if $Mv = 0$, then u satisfies the KdV equation (9.6.3). In contrast to the Cole–Hopf transformation, the Miura transformation reduces one nonlinear equation to another nonlinear equation, neither of which can easily be solved. However, the Miura transformation establishes a connection between the KdV equation and the Sturm–Liouville problem, and hence, it leads to the inverse scattering method for the exact solution of the initial-value problem for the KdV equation.

Another remarkable fact about the Miura transformation is that there exists an infinite number of conservation laws. It is now convenient to work with w, rather than v, where $v = \frac{1}{2}\varepsilon^{-1} + \varepsilon w$ and ε is an arbitrary real parameter. Consequently, the Miura transformation becomes

$$u = \frac{1}{4}\,\varepsilon^{-2} + w + \varepsilon w_x + \varepsilon^2 w^2. \tag{9.6.12}$$

Since any arbitrary constant can be incorporated in the solution for u, we just write

$$u = w + \varepsilon w_x + \varepsilon^2 w^2. \tag{9.6.13}$$

This is known as the *Gardner transformation*. It is important to note that, since v exists, so does w.

Substitution of (9.6.13) in (9.6.3) gives

$$\begin{aligned}
0 &= u_t - 6uu_x + u_{xxx} \\
&= w_t + \varepsilon w_{xt} + 2\varepsilon^2 w w_t - 6\left(w + \varepsilon w_x + \varepsilon^2 w^2\right)\left(w_x + \varepsilon w_{xx} + 2\varepsilon^2 w w_x\right) \\
&\quad + w_{xxx} + \varepsilon w_{xxxx} + 2\varepsilon^2 (w w_x)_{xx} \\
&= \left(1 + \varepsilon \frac{\partial}{\partial x} + 2\varepsilon^2 w\right)\left\{w_t - 6\left(w + \varepsilon^2 w^2\right) w_x + w_{xxx}\right\}.
\end{aligned}$$

Thus, w satisfies the *Gardner equation*

$$w_t - 6\left(w + \varepsilon^2 w^2\right) w_x + w_{xxx} = 0 \tag{9.6.14}$$

for all ε. We note that, when $\varepsilon = 0$, this equation becomes the KdV equation with $w = u$.

Another interesting fact is that equation (9.6.14) is already in the conservation form

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(w_{xx} - 3w^2 - 2\varepsilon^2 w^3\right) = 0, \tag{9.6.15}$$

with w as the conserved density, that is,

$$\int_{-\infty}^{\infty} w\,dx = \text{const.} \tag{9.6.16}$$

To obtain an infinite number of conservation laws for the KdV equation, we use the arbitrary parameter ε. Since $w \to u$ as $\varepsilon \to 0$, we write an asymptotic expansion of $w(x,t,\varepsilon)$ in ε:

$$w \sim \sum_{n=0}^{\infty} \varepsilon^n w_n(x,t) \quad \text{as } \varepsilon \to 0. \tag{9.6.17}$$

Treating the constant in (9.6.16) similarly, as a power series in ε, and using (9.6.17), we obtain an infinite number of constants of motion. Finally, we substitute (9.6.17) in the Gardner transformation (9.6.13) and, then, equate coefficients of ε^n, for each $n = 0, 1, 2, \ldots$. It is then easy to see that

$$\begin{aligned}
&w_0 = u, \qquad w_1 = -w_{0x} = -u_x, \qquad w_2 = -w_{1x} = -w_0^2 = u_{xx} - u^2,\\
&w_3 = -w_{2x} - 2w_0 w_1 = -\big(u_{xx} - u^2\big)_x + 2uu_x,\\
&w_4 = w_{3x} - 2w_0 w_1 - w_1^2 = -\big\{2uu_x - \big(u_{xx} - u^2\big)_x\big\}_x - 2u\big(u_{xx} - u^2\big) - u_x^2,
\end{aligned}$$

and so on.

In particular, the integrals which are generated by w_0, w_2, and w_4 become the first three integrals (9.6.5), (9.6.7), and (9.6.9), respectively. On the other hand, w_1 and w_3 are exact differentials in x, and hence, the corresponding integrals

$$\int_{-\infty}^{\infty} w_n\,dx = \text{const.} \tag{9.6.18}$$

will not give us any useful result. In other words, if n is even, then (9.6.18) generates an infinite set of conservation laws, and, if n is odd, w_n is an exact differential and, hence, does not lead to any conservation law.

Much of the significant work on the theory of the KdV equation was initiated by the publication of several papers of Gardner et al. (1967, 1974).

We close this section by including a few conservation laws for the Boussinesq equation, the KP equation, and shallow water equations.

It is convenient to introduce $\eta_t = -u_x$ in the Boussinesq equation (9.3.202) so that it becomes

$$-u_{tx} - \eta_{xx} + 3\big(\eta^2\big)_{xx} - \eta_{xxxx} = 0. \tag{9.6.19}$$

Integrating this equation with respect to x and invoking the decay conditions as $|x| \to \infty$, the Boussinesq equation (9.3.202) can be rewritten as the pair of equations

$$\eta_t = -u_x, u_t + \eta_x - \big(3\eta^2\big)_x + \eta_{xxx} = 0. \tag{9.6.20}$$

Consequently, it turns out that

$$\int_{-\infty}^{\infty} \eta_t\,dx = -[u]_{-\infty}^{\infty}, \qquad \int_{-\infty}^{\infty} u_t\,dx = \big[3\eta^2 - \eta - \eta_{xx}\big]_{-\infty}^{\infty}.$$

Thus, we obtain the conservation of mass in the form

$$\int_{-\infty}^{\infty} \eta \, dx = \text{const.}, \tag{9.6.21}$$

and the conservation of momentum as

$$\int_{-\infty}^{\infty} u_t \, dx = \text{const.} \tag{9.6.22}$$

We next recall the KP equation (9.3.135) in the form

$$6u_t + 9uu_x + u_{xx} + 3v_y = 0, \quad v_x = u_y. \tag{9.6.23}$$

Integrating the second equation with respect to x and invoking the decay conditions at infinity gives

$$\frac{\partial}{\partial y}\left(\int_{-\infty}^{\infty} u \, dx\right) = 0, \quad \text{and hence,} \quad \int_{-\infty}^{\infty} u \, dx = c(t). \tag{9.6.24}$$

Physically, the N-soliton solution of the KP equation describes the interaction of waves so that the function $c(t)$ represents a constant and (9.6.24) is also a constant. Consequently,

$$\int_{-\infty}^{\infty} u \, dx = \text{const.} \tag{9.6.25}$$

A similar argument leads to the result

$$\int_{-\infty}^{\infty} v \, dy = \text{const.} \tag{9.6.26}$$

The above results are similar to (9.6.21), (9.6.22).

Integrating the first equation in (9.6.23) yields

$$\frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty} u \, dx\right] + \left[u_{xx} - 3u^2\right]_{-\infty}^{\infty} + 3\frac{\partial}{\partial y}\left[\int_{-\infty}^{\infty} v \, dx\right] = 0. \tag{9.6.27}$$

Using (9.6.25) and suitable decay conditions at infinity gives another conservation law

$$\int_{-\infty}^{\infty} v \, dx = \text{const.} \tag{9.6.28}$$

Clearly, (9.6.26)–(9.6.28) represent *momentum conservation laws*.

Finally, we derive the conservation laws for shallow water waves. We recall shallow water equations (2.7.67)–(2.7.70) in $(1+1)$ dimensions with $\delta \to 0$ and $\varepsilon = 1$ so that $1 + \varepsilon\eta = h(x,t)$. Consequently, these equations become

$$u_t + uu_x + wu_z + h_x = 0, \tag{9.6.29}$$

$$u_x + w_z = 0, \tag{9.6.30}$$

$$w = h_t + uh_x \quad \text{on } z = h, \quad \text{and} \quad w = 0 \quad \text{on } z = 0. \tag{9.6.31}$$

Integrating (9.6.30) with respect to z from 0 to h gives

$$\int_0^h u_x\,dz + [w]_0^h = \frac{\partial}{\partial x}\left[\int_0^h u\,dz\right] + h_t = 0. \tag{9.6.32}$$

Making use of suitable decay conditions, we obtain the *law of mass conservation* as

$$\int_{-\infty}^{\infty} h(x,t)\,dx = \text{const.} \tag{9.6.33}$$

We next multiply (9.6.30) by u and add the result to (9.6.29) to find

$$u_t + 2uu_x + (uw)_z + h_x = 0. \tag{9.6.34}$$

Integrating this equation with respect to z from 0 to h so that

$$\frac{\partial}{\partial t}\left(\int_0^h u\,dz\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}h^2 + \int_0^h u^2\,dz\right) = 0, \tag{9.6.35}$$

leads to the *law of conservation of momentum* as

$$\int_{-\infty}^{\infty}\left(\int_0^h u\,dz\right)dx = \text{const.} \tag{9.6.36}$$

Multiplying (9.6.29) by u, we obtain

$$\left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x + \left(\frac{1}{2}u^2 w\right)_z - \frac{1}{2}u^2 w_z + (uh)_x - hu_x = 0. \tag{9.6.37}$$

We substitute w_z and u_x from (9.6.30) and use the fact that $h = h(x,t)$ to obtain

$$\left(\frac{1}{2}u^2\right)_t + \left(uh + \frac{1}{2}u^3\right)_x + \left(\frac{1}{2}wu^2 + hw\right)_z = 0. \tag{9.6.38}$$

Integrating this result with respect to z and interchanging differentiation under the integral sign gives

$$\left[\frac{1}{2}\left(h^2 + \int_0^h u^2\,dz\right)\right]_t + \left[\int_0^h \left(uh + \frac{1}{3}u^3\right)dz\right]_x = 0. \tag{9.6.39}$$

This result is in the conservation form and yields the *law of conservation of energy*

$$\int_{-\infty}^{\infty}\left(h^2 + \int_0^h u^2\,dz\right)dx = \text{const.} \tag{9.6.40}$$

9.7 The Inverse Scattering Transform (IST) Method

Historically, Gardner et al. (1974) formulated an ingenious method for finding the exact solution of the KdV equation. This novel method has been generalized to solve several other nonlinear equations. In this section, we describe the *inverse scattering transform* for the KdV equation. The exact solution of the KdV equation is obtained by associating its solution with the potential of a time-independent Schrödinger equation. The next step is the solution of the quantum mechanical problem with the initial value for the KdV equation taken as the potential. This involves calculations of the discrete (bound) eigenfunctions, their normalization constants and eigenvalues, and the reflection and transmission coefficients of the continuous (unbounded) states. These results are collectively known as the *scattering data* of the Schrödinger equation. The next step is to determine the evolution of the scattering data for any potential that evolves, according to the KdV equation, from a prescribed initial function. Both discrete and continuous eigenvalues are found to be invariant under such changes, and the normalization constants and the reflection and transmission coefficients evolve according to simple exponential laws. The final step of the method deals with the determination of the potential for any time t from the inversion of the scattering data.

The main problem is to find the exact solution of the general initial-value problem for the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad x \in \mathbb{R},\ t > 0, \tag{9.7.1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{9.7.2}$$

where $u_0(x)$ satisfies certain fairly weak conditions so that the solution $u(x,t)$ exists for all x and t.

We begin with the Miura (1968) transformation defined by

$$u = v^2 + v_x, \tag{9.7.3}$$

where $v = v(x,t)$. Substituting (9.7.3) in (9.7.1) gives

$$2vv_t + v_{xt} - 6\left(v^2 + v_x\right)(2vv_x + v_{xx}) + 6v_x v_{xx} + 2vv_{xxx} + v_{xxxx} = 0.$$

This can be rearranged to obtain equation

$$\left(\frac{\partial}{\partial x} + 2v\right)\left(v_t - 6v^2 v_x + v_{xxx}\right) = 0. \tag{9.7.4}$$

Thus, if v is a solution of the modified KdV equation, then (9.7.3) defines a solution of the KdV equation. We note here that, if u is given, equation (9.7.3) represents the *Riccati equation* for v, so that the transformation

$$v = \frac{\psi_x}{\psi}, \tag{9.7.5}$$

where $\psi(x,t)$ is a nonzero differentiable function, reduces (9.7.3) to the form

$$\psi_{xx} - u\psi = 0. \tag{9.7.6}$$

The function u involved in (9.7.6) is a solution of the KdV equation (9.7.1). We further note that KdV equation (9.7.1) is Galilean invariant, that is, it is invariant under the transformation $\tilde{x} = x + 6\lambda t$, $\tilde{u} = u + \lambda$, where $\lambda \in \mathbb{R}$, so that we can replace u by $u - \lambda$ to obtain a second-order linear equation

$$L\psi = \lambda\psi, \tag{9.7.7}$$

where the operator L is given by

$$L \equiv -\frac{\partial}{\partial x^2} + u. \tag{9.7.8}$$

This is the Schrödinger equation for ψ with the potential $u(x,t)$ and eigenvalue $\lambda = \lambda(t)$ which depends on t because of the parametric dependence on t. However, this Schrödinger equation for ψ is different from the one in quantum mechanics because the potential $u(x,t)$ is *not* known and it is the solution of the KdV equation. So, the method of solution (9.7.7) is not so simple. However, it is well known that equation (9.7.7) admits two distinct kinds of solutions recognized as *bound states*, where $\psi \to 0$ exponentially as $|x| \to \infty$, and *scattering states*, where ψ oscillates with x at infinity. Since both u and λ depend on t, the eigenfunctions $\psi(x,t)$ also depend on t. The quantity t involved in (9.7.7) cannot be regarded as time, but simply as a parameter, so that $u = u(x,t)$ represents a family of potentials. At $t = 0$, the initial condition is given by $u(x,0) = u_0(x)$, and hence, the problem is first to solve the one-dimensional Schrödinger equation for the *known* potential $u_0(x)$,

$$-\left[\frac{\partial^2}{\partial x^2} - u_0(x)\right]\psi(x,t) = \lambda(t)\psi(x,t). \tag{9.7.9}$$

This will determine how the eigenfunctions and eigenvalues evolve as potential changes from the given data according to the KdV equation (9.7.1). The potential $u(x,t)$ for any $t > 0$ can then be determined from the scattering data at time t by the inverse scattering transformation.

To determine the eigenvalues $\lambda(t)$, we first differentiate (9.7.7) with respect to x to obtain

$$\psi_{xxx} - u_x\psi - (u - \lambda)\psi_x = 0,$$

and then with respect to t to derive

$$\psi_{xxt} - (u_t - \lambda_t)\psi - (u - \lambda)\psi_t = 0. \tag{9.7.10}$$

The quantity u_t can be eliminated from (9.7.10) by means of (9.7.1) so that (9.7.9) becomes

$$(-L + \lambda)\psi_t + (u_{xxx} - 6uu_x)\psi + \lambda_t\psi = 0, \tag{9.7.11}$$

where the term $(u_{xxx}\psi)$ can be obtained from the result

$$(u_x\psi)_{xx} = u_{xxx}\psi + u_x\psi_{xx} + 2u_{xx}\psi_x. \tag{9.7.12}$$

We solve this equation for $(u_{xxx}\psi)$ so that the resulting expression can be expressed by the original Schrödinger equation in the form

$$u_{xxx}\psi = (u_x\psi)_{xx} - u_x\psi_{xx} - 2u_{xx}\psi_x = (-L+\lambda)(u_x\psi) - 2u_{xx}\psi_x. \quad (9.7.13)$$

Substituting this result in (9.7.11) yields

$$(-L+\lambda)(\psi_t + u_x\psi) - 2(3uu_x\psi + u_{xx}\psi_x) + \lambda_t\psi = 0. \quad (9.7.14)$$

To replace the quantity $(3uu_x\psi + u_{xx}\psi_x)$, we rewrite the term $u_{xx}\psi_x$ by first using the identity

$$(u\psi_x)_{xx} = u_{xx}\psi_x + 2u_x\psi_{xx} + u\psi_{xxx}. \quad (9.7.15)$$

The quantity ψ_{xx} in the second term on the right-hand side of (9.7.15) can be eliminated by using the Schrödinger equation, and ψ_{xxx} in the third term can be eliminated by the x-derivative of the Schrödinger equation, so that (9.7.15) becomes

$$\begin{aligned}(u\psi_x)_{xx} &= u_{xx}\psi_x + 2u_x(u-\lambda)\psi + u\big[u_x\psi + (u-\lambda)\psi_x\big] \\ &= u_{xx}\psi_x - 2\lambda u_x\psi + 3uu_x\psi + (u-\lambda)u\psi_x. \quad (9.7.16)\end{aligned}$$

We next solve for $(3uu_x\psi + u_{xx}\psi_x)$ by means of the Schrödinger equation and its x-derivatives, so that

$$\begin{aligned}3uu_x\psi + u_{xx}\psi_x &= 3uu_x\psi + (u\psi_x)_{xx} + 2\lambda u_x\psi - 3uu_x\psi - (u-\lambda)u\psi_x \\ &= (-L+\lambda)(u\psi_x) + 2\lambda u_x\psi \\ &= (-L+\lambda)(u\psi_x) + 2\lambda\big[\psi_{xxx} - (u-\lambda)\psi_x\big] \\ &= (-L+\lambda)(u\psi_x + 2\lambda\psi_x). \quad (9.7.17)\end{aligned}$$

Substituting (9.7.17) in (9.7.14) gives

$$(-L+\lambda)\big[\psi_t + u_x\psi - 2(u+2\lambda)\psi_x\big] + \lambda_t\psi = 0. \quad (9.7.18)$$

We next introduce a new quantity Ψ defined by

$$\Psi = \psi_t + u_x\psi - 2(u+2\lambda)\psi_x, \quad (9.7.19)$$

which, by replacing $(u_x\psi)$ with the derivative of (9.7.7) with respect to x, is

$$= \psi_t + \psi_{xxx} - 3(u+\lambda)\psi_x. \quad (9.7.20)$$

In view of (9.7.19), equation (9.7.18) assumes the form

$$(-L+\lambda)\Psi = -\lambda_t\psi. \quad (9.7.21)$$

Equations (9.7.19)–(9.7.21) are the main results for determining the behavior of all quantities involved in the scattering problem as the potential evolves according to the KdV equation.

As stated earlier, the solution of the inverse scattering problem requires a priori knowledge of the scattering data. This includes the discrete eigenvalues, the normalized coefficients for the eigenfunctions, and the reflection and transmission coefficients. We first consider the *discrete spectrum of eigenvalues* (*bound states*).

In view of the fact that $u(x,t)$ decays rapidly as $|x| \to \infty$ for all t, the Schrödinger equation (9.7.7) admits a finite number of discrete eigenstates with negative energy $\lambda_n = -\kappa_n^2$, $n = 1, 2, \ldots, N$, and a continuous spectrum of positive energy ($\lambda = \kappa^2 > 0$). The corresponding discrete eigenfunctions ψ_n belong to the Hilbert space $L^2(\mathbb{R})$. To determine the evolution of the eigenvalues, equation (9.7.21) is multiplied by ψ_n, and we use (9.7.7) to simplify the result to obtain

$$\begin{aligned} -\lambda_{nt}\psi_n^2(x) &= \psi_n(x)(-L+\lambda_n)\varPsi = \psi_n\varPsi_{xx} - \varPsi\psi_{nxx} \\ &= \frac{\partial}{\partial x}\left[\psi_n\frac{\partial \varPsi}{\partial x} - \varPsi\frac{\partial \psi_n}{\partial x}\right]. \end{aligned} \tag{9.7.22}$$

Integrating this equation with respect to x over $(-\infty, \infty)$ gives

$$-\lambda_{nt}\int_{-\infty}^{\infty}\psi_n^2(x)\,dx = \left[\psi_n\frac{\partial \varPsi}{\partial x} - \varPsi\frac{\partial \psi_n}{\partial x}\right]_{-\infty}^{\infty}. \tag{9.7.23}$$

Normalizing the eigenfunctions $\psi_n(x)$ of the bound states by

$$\|\psi_n\| = \left[\int_{-\infty}^{\infty}\psi_n^2(x)\,dx\right]^{\frac{1}{2}} = 1 \tag{9.7.24}$$

and using the fact that $\psi_n(x)$ and its spatial derivatives vanish as $x \to \pm\infty$, we conclude that the right-hand side of (9.7.23) vanishes so that

$$\frac{d\lambda_n}{dt} = 0, \tag{9.7.25}$$

that is, for a potential $u(x,t)$ that evolves with t according to the KdV equation, the eigenvalues of the bound states are invariant with respect to $t : \lambda_n(t) = \lambda_n(0)$ for $n = 1, 2, \ldots, N$. The invariance of the discrete eigenvalues represented by (9.7.25) has several important consequences. First, equation (9.7.21) with $\lambda_{nt}(t) = 0$ implies that $\varPsi_n$ represents the discrete eigenfunctions of the Schrödinger equation (9.7.21) with the eigenvalues $\lambda = \lambda_n$. Second, it follows that (9.7.21) combined with (9.7.19) leads to the equation

$$\psi_{nt} + u_x\psi_n - 2\big(u - 2\kappa_n^2\big)\psi_{nx} = 0. \tag{9.7.26}$$

This can be regarded as a time evolution equation for $\psi_n(x,t)$. The bounded solution for the *n*th eigenfunction can then be represented from its asymptotic behavior in the form

$$\psi_n(x,t) \sim c_n(t)\exp(-\kappa_n x), \quad \text{as } x \to +\infty, \tag{9.7.27}$$

where the real constants c_n are determined by the normalized condition (9.7.24). Substituting this asymptotic form (9.7.27) in (9.7.26) and imposing a major restriction $u(x,t) \to 0$ as $|x| \to \infty$ leads to the equation for $c_n(t)$ in the form

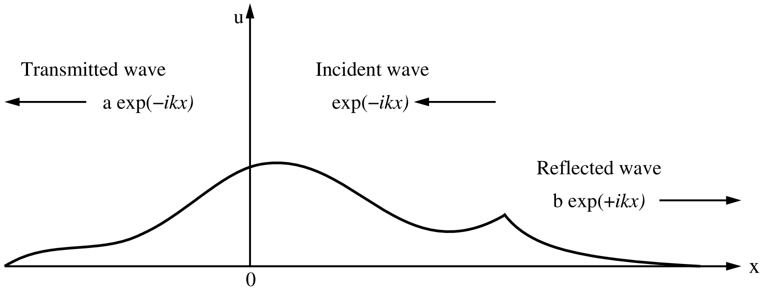

Fig. 9.9 Sketch representing the scattering by a potential u.

$$\frac{d}{dt}c_n(t) - 4\kappa_n^3 c_n(t) = 0. \tag{9.7.28}$$

This equation admits solutions given by

$$c_n(t) = c_n(0)\exp\left(4t\kappa_n^3\right), \tag{9.7.29}$$

where $c_n(0)$, $n = 1, 2, \ldots, N$ are the normalization constants determined at $t = 0$. Thus, the asymptotic representation of $\psi_n(x,t)$ is explicitly given by (9.7.27) with (9.7.29). As a passing remark, we note that the imposed restriction on $u(x,t)$ limits the theory of the KdV soliton and eliminates solutions corresponding to other nonlinear waves.

For a continuous spectrum of eigenvalues (unbounded states), the eigenvalue $\lambda = k^2 > 0$, where k may take any real value. We visualize the wave function ψ as the spatially dependent part of a steady plane wave impinging on the potential u from $x \to +\infty$ for the time-dependent Schrödinger equation. Physically, there will be an interaction of the plane wave with the potential allowing for the existence of *scattering states* that can be treated as the linear combination of a transmitted wave $\exp(-ikx)$ at $x = -\infty$ of amplitude $a(k)$ and a reflected wave $\exp(ikx)$ at $x = +\infty$ of amplitude $b(k)$. In other words, the wave function for the continuous spectrum has asymptotic behavior in the form

$$\psi(x) \sim \exp(-ikx) + b(k)\exp(ikx) \quad \text{as } x \to \infty, \tag{9.7.30a}$$
$$\psi(x) \sim a(k)\exp(-ikx) \quad \text{as } x \to -\infty, \tag{9.7.30b}$$

where $a(k)$ and $b(k)$ are called the *transmission and reflection coefficients*, respectively, and can be determined uniquely from the initial data. The term $\exp(-ikx)$ of unit amplitude in (9.7.30a) is a traveling wave from $x = +\infty$. In general, both $a(k)$ and $b(k)$ are complex functions of a real variable k. A sketch representing the scattering by a potential u is shown in Figure 9.9.

We next consider two distinct eigenfunctions for the same potential u so that

$$\psi_m'' - \left(\kappa_m^2 + u\right)\psi_m = 0 \quad \text{and} \quad \psi_n'' - \left(\kappa_n^2 + u\right)\psi_n = 0, \tag{9.7.31ab}$$

where the primes denote derivatives with respect to x.

Clearly,

$$\left(\kappa_m^2 - \kappa_n^2\right)\psi_m\psi_n = \left(\psi_n\psi_m'' - \psi_m\psi_n''\right) = \frac{d}{dx}W(\psi_n, \psi_m), \tag{9.7.32}$$

where W is the Wronskian of ψ_n and ψ_m.

Integrating (9.7.32) gives, for $m \neq n$,

$$\left(\kappa_m^2 - \kappa_n^2\right)\int_{-\infty}^{\infty}\psi_m\psi_n\,dx = \left[W(\psi_n, \psi_m)\right]_{-\infty}^{\infty} = 0, \tag{9.7.33}$$

because $\psi_n, \psi_m \to 0$ as $|x| \to \infty$. This shows that ψ_m, ψ_n are orthogonal. The continuous eigenfunction ψ is also orthogonal to each discrete eigenfunction ψ_m. Thus, the discrete and continuous eigenfunctions together form a complete orthogonal set, and hence, any function $\psi \in L^2(\mathbb{R})$ can be represented as a linear combination of all the ψ_m's plus an integral of $\psi(x, k)$ over all k.

If ϕ and ψ are two solutions of (9.7.7) with the same value of $\lambda = k^2 (> 0)$, then equation (9.7.7) implies that the derivative of W is zero, that is, $W(\phi, \psi)$ is constant. With an additional restriction that ϕ is proportional to ψ, it turns out that $W(\phi, \psi) = 0$ for all x. Denoting ψ^* as the complex conjugate of the continuous eigenfunction ψ, we can compute $W(\psi, \psi^*)$ at both $x = \pm\infty$ and, then, use (9.7.30a) and (9.7.30b) to obtain

$$W\left(\psi, \psi^*\right) = 2ikaa^* = 2ik\left(1 - bb^*\right). \tag{9.7.34}$$

Or equivalently,

$$|a|^2 + |b|^2 = 1. \tag{9.7.35}$$

This is a statement of the *conservation of energy* in the theory of scattering.

A procedure similar to the case of the discrete spectrum can be employed to determine the scattering coefficients. Integrating (9.7.21) once with respect to x gives

$$\Psi\psi_x - \psi\Psi_x = g(t; k), \tag{9.7.36}$$

where Ψ is defined by (9.7.19) and $g(t; k)$ is an arbitrary function resulting from integration. For the continuous eigenfunction, $\psi(x, k; t)$ is given by (9.7.30b), and hence, the asymptotic result for Ψ follows from (9.7.19):

$$\Psi \sim \left(a_t + 4iak^3\right)e^{-ikx} \quad \text{as } x \to -\infty. \tag{9.7.37}$$

Consequently,

$$(\Psi\psi_x - \psi\Psi_x) \to 0 \quad \text{as } x \to -\infty. \tag{9.7.38}$$

Obviously, $g(t; k) = 0$ for all time t, and (9.7.36) can be integrated to obtain

$$(\Psi/\psi) = h(t; k), \quad \text{or equivalently,} \quad \Psi = \psi h, \tag{9.7.39}$$

where h is another arbitrary function of integration. The use of asymptotic results for ψ and Ψ as $x \to -\infty$ in (9.7.39) yields the equation

$$\frac{da}{dt} + 4iak^3 = ha. \tag{9.7.40}$$

Similarly, we next use (9.7.30a) and the asymptotic form of Ψ, as $x \to -\infty$, in the form

$$\Psi \sim \frac{db}{dt}e^{ikx} + 4ik^3\left(e^{ikx} - be^{-ikx}\right) \tag{9.7.41}$$

in (9.7.39) to obtain

$$\frac{db}{dt}e^{ikx} + 4ik^3\left(e^{-ikx} - be^{ikx}\right) = h\left(e^{-ikx} + be^{ikx}\right). \tag{9.7.42}$$

Thus, it follows from the linear independence of e^{ikx} and e^{-ikx} that

$$\frac{db}{dt} - 4ik^3 b = hb \quad \text{and} \quad h = 4ik^3, \tag{9.7.43ab}$$

and therefore, equation (9.7.40) gives

$$\frac{da}{dt} = 0. \tag{9.7.44}$$

Finally, the solutions for a and b are given by

$$a(k;t) = a(k;0) \quad \text{and} \quad b(k;t) = b(k;0)\exp\left(8ik^3 t\right), \quad t > 0. \tag{9.7.45}$$

This completes the determination of the scattering data $S(t)$, which are summarized as follows:

$$\left.\begin{array}{ll} \kappa_n = \text{const.}; & c_n(t) = c_n(0)\exp(4\kappa_n^3 t), \\ [1ex]a(k;t) = a(k;0); & b(k;t) = b(k;0)\exp(8ik^3 t), \end{array}\right\} \tag{9.7.46}$$

where $c_n(0)$, $a(k;0)$, and $b(k;0)$ are determined from the initial condition for the KdV equation.

The simplest approach to solving the inverse scattering problem involves the integral representation of solutions of the Schrödinger equation (9.7.7) with $\lambda = k^2$,

$$L\psi = k^2\psi. \tag{9.7.47}$$

It is convenient to introduce the following integral representation of the solution of (9.7.47),

$$\phi_k(x) = \exp(ikx) + \int_x^\infty K(x,z)\exp(ikz)\,dz, \tag{9.7.48}$$

$$\phi_{-k}(x) = \exp(-ikx) + \int_x^\infty K(x,z)\exp(-ikz)\,dz. \tag{9.7.49}$$

These are called the *Jost solutions*, which have the following property:

$$\lim_{x\to\infty} \phi_{\pm k}(x) = \exp(\pm ikx). \tag{9.7.50}$$

As already stated earlier, when k is real, the eigenfunctions are continuous (unbounded) states, since $\lambda > 0$. On the other hand, if k is purely imaginary, so that $k = i\kappa$, where κ is real, then $\lambda < 0$, and the eigenfunctions correspond to discrete (bound) states. Henceforth, we adopt k for continuous states and $i\kappa$ for discrete states.

Substituting $\phi_{\pm k}(x)$ in (9.7.47) leads to the inhomogeneous wave equation

$$\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial z^2} = uK, \tag{9.7.51}$$

and $K(x,z)$ is related to the potential $u(x,t)$ by

$$u(x,t) = -2\frac{d}{dx}K(x,x), \tag{9.7.52}$$

where $K(x,z)$ satisfies the following conditions:

$$\lim_{z\to\pm\infty} K(x,z) = 0 \quad \text{and} \quad \lim_{z\to\pm\infty} \frac{\partial K(x,z)}{\partial z} = 0. \tag{9.7.53ab}$$

Because $\phi_k(x)$ and $\phi_{-k}(x)$ are linearly independent, they can be treated as fundamental solutions of the original equation (9.7.47). Hence, the general solution of (9.7.47) corresponding to the eigenvalue k can be written as a linear combination of these two solutions. In particular, we examine the solution $\psi_k(x)$ of (9.7.47) with asymptotic behavior $\psi_k(x) \sim \exp(-ikx)$, as $x \to -\infty$. After some algebraic manipulation, it can be shown that $\psi_k(x)$ is given by

$$\psi_k(x) = \frac{1}{a(k)}\phi_{-k}(x) + \frac{b(k)}{a(k)}\phi_k(x). \tag{9.7.54}$$

Substituting results for $\phi_{\pm k}(x)$ from (9.7.48) and (9.7.49) in (9.7.54) yields

$$\begin{aligned} a(k)\psi_k(x) &= \exp(-ikx) + \int_x^\infty K(x,z)\exp(-ikz)\,dz \\ &\quad + b(k)\left[\exp(ikx) + \int_x^\infty K(x,z)\exp(ikz)\,dz\right]. \end{aligned} \tag{9.7.55}$$

Multiplying this equation by $(2\pi)^{-1}\exp(iky)$, where $y > x$, and integrating with respect to k gives the following result:

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^\infty a(k)\psi_k(x)\exp(iky)\,dk \\ &\quad = \frac{1}{2\pi}\int_{-\infty}^\infty \exp\{ik(y-x)\}\,dk \\ &\qquad + \int_x^\infty K(x,z)\left[\frac{1}{2\pi}\int_{-\infty}^\infty \exp\{ik(y-z)\}\,dk\right]dz \end{aligned}$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\exp\{ik(x+y)\}\,dk$$
$$+\int_x^{\infty} K(x,z)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\exp\{ik(y+z)\}\,dk\right]dz$$
$$= K(x,y)+B(x+y)+\int_x^{\infty} K(x,z)B(y+z)\,dz, \tag{9.7.56}$$

in which the following results are used:

$$B(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)\exp(ikx)\,dk \tag{9.7.57a}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[ik(y-z)\right]dk=\delta(y-x)=0, \quad y>x. \tag{9.7.57b}$$

Clearly, the function $B(x)$ represents the Fourier transform of the reflection coefficient $b(k)$ and is a known quantity. The right-hand side of (9.7.55) has terms involving only an unknown K, which is to be determined. In order to determine K, we first simplify the left-hand side of (9.7.56) by using the theory of residues of analytic functions. Without any further details, it turns out from the residue computation from the simple pole of $a(k)$ at $k=i\kappa$ with $\psi_{i\kappa}(x)=c_\kappa\phi_{i\kappa}(x)$ that the left-hand side of (9.7.56) gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)\psi_k(x)\exp(iky)\,dk$$
$$=-c_\kappa^2\exp\{-\kappa(x+y)\}-\int_{-\infty}^{\infty} c_\kappa^2 K(x,z)\exp\{-\kappa(y+z)\}\,dz. \tag{9.7.58}$$

There is a corresponding contribution to this equation from every bound state of the potential. Therefore, if there are N discrete states, (9.7.58) has to be modified by including a summation over the individual states. Finally, we combine this with result (9.7.56) and incorporate the t-dependence of each quantity to obtain the integral equation for $K(x,y;t)$ in the form

$$K(x,y;t)+B(x+y;t)+\int_x^{\infty} K(x,z;t)B(y+z;t)\,dz=0, \tag{9.7.59}$$

where $B(x,t)$ is now given by

$$B(x,t)=\sum_{n=1}^{N} c_n^2(t)\exp(-\kappa_n x)+\frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,t)\exp(ikx)\,dk. \tag{9.7.60}$$

Equation (9.7.59) with the kernel $B(x,t)$ given by (9.7.60) is known as the *Gelfand–Levitan–Marchenko* (*GLM*) linear integral equation. The solution $K(x,y;t)$ of this integral equation is related to the potential $u(x,t)$ by the following result:

$$u(x,t) = -2\frac{d}{dx}K(x,x;t). \tag{9.7.61}$$

This is the exact solution of the original KdV equation. However, it does not admit any physical interpretation; that can be achieved from the following asymptotic analysis and examples.

For future reference, it is convenient to write the final explicit formula for $B(x,t)$ as

$$B(x,t) = \sum_{n=1}^{N} c_n^2(0)\exp\left(8\kappa_n^3 t - \kappa_n x\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,0)\exp\left(8ik^3 t + ikx\right) dk. \tag{9.7.62}$$

In general, it is not possible to solve the integral equation (9.7.59) except for the reflectionless potentials $(b(k) = 0)$. For the general case, the longtime solution represents N solitons traveling with various speeds to the right, and an oscillatory wavetrain with amplitude decreasing with time can be found to follow the series of solitons.

For a given initial potential energy function, the eigenvalues κ_n are constants for all time. Physically, this means that bound-state energy levels are completely specified and remain unchanged while varying the potential through the KdV equation. Another remarkable fact to observe is that the number of solitons that eventually develops is exactly the number of bound states. Of course, this number depends on the initial state $u(x,0) = u_0(x)$, which is sufficient for determining the number of developing solitons.

This completes the description of the inverse scattering transform method for solving the KdV equation. In summary, two distinct steps are involved in the method: (i) the solution of the Schrödinger equation (the Sturm–Liouville problem) for a given initial condition, $u(x,0) = u_0(x)$, from which we determine the scattering data $S(t)$, and (ii) the solution of the GLM linear integral equation. Even though these two steps may not be technically easy to handle, in principle, the problem is solved. The effectiveness of the method can be best exemplified by many simple but nontrivial examples.

The power and success of the inverse scattering transform method for solving the KdV equation can be attributed to several facts. First, the most remarkable result of the method is the fact that the discrete eigenvalues of the Schrödinger equation do not change as the potential evolves according to the KdV equation. Second, the method has reduced solving a *nonlinear partial differential equation* to solving two *linear* problems: (i) a second-order ordinary differential equation and (ii) an integral equation. Third, the eigenvalues of the ordinary differential equation are constants, and this leads to a major simplification in the evolution equation for ψ. Fourth, the time evolution of the scattering data is explicitly determined from the asymptotic form of ψ, as $|x| \to \infty$. So, this information allows us to solve the inverse scattering problem, and hence, to find the final solution of the KdV equation. The method is presented schematically in Figure 9.10.

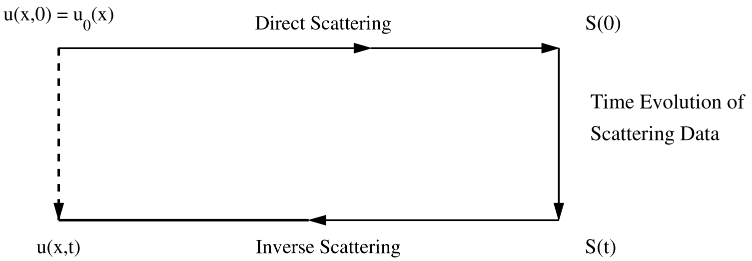

Fig. 9.10 The IST method for the KdV equation.

We now illustrate the method for the reflectionless case $(b(k) = 0)$ with initial profiles given by the following examples.

Example 9.7.1 (Single-Soliton Solution). We consider a particular case where the potential has a single bound state with the eigenvalue $\lambda = -\kappa_1^2$, normalized constant c_0, and with zero reflection coefficient $(b(k,t) \equiv 0)$ for all continuous states k. In this case, B is obtained from (9.7.62) in the form

$$B(x,t) = c_0^2(0)\exp\big(8\kappa_1^3 t - \kappa_1 x\big) = c_0^2 \exp\big(8\kappa_1^3 t - \kappa_1 x\big). \tag{9.7.63}$$

The associated GLM equation (9.7.60) can be solved by assuming a separable kernel

$$K(x,y;t) = \sum_{n=1}^{N} K_n(x,t)\exp(-\kappa_n y), \tag{9.7.64}$$

which, in this case $(N = 1)$, becomes

$$K(x,y;t) = K_1(x,t)\exp(-\kappa_1 y). \tag{9.7.65}$$

The GLM equation (9.7.60) with (9.7.65) is given by

$$\begin{aligned} K(x,y;t) &+ c_0^2 \exp\big\{8\kappa_1^3 t - \kappa_1(x+y)\big\} \\ &+ c_0^2 \exp\big(8\kappa_1^3 t - \kappa_1 y\big)\int_x^{\infty} K(x,z;t)\exp(-\kappa_1 z)\,dz = 0. \end{aligned} \tag{9.7.66}$$

Substituting (9.7.65) in (9.7.66) and performing the integration over z leads to an algebraic equation for $K_1(x,t)$,

$$\begin{aligned} K_1(x,t)\exp(-\kappa_1 y) &+ c_0^2 \exp\big\{8\kappa_1^3 t - \kappa_1(x+y)\big\} \\ &+ \frac{c_0^2}{2\kappa_1}\exp\big(8\kappa_1^3 t - \kappa_1 y - 2\kappa_1 x\big)K_1(x,t) = 0. \end{aligned} \tag{9.7.67}$$

This equation solved for $K_1(x,t)$ can be written in the form

$$K_1(x,t) = -\kappa_1 \exp\big(4\kappa_1^3 t + \kappa_1 x_0\big)\operatorname{sech}\big[\kappa_1\big(x - x_0 - 4\kappa_1^2 t\big)\big], \tag{9.7.68}$$

where x_0 is defined by

$$x_0 = (2\kappa_1)^{-1} \log\left(\frac{c_0^2}{2\kappa_1}\right). \tag{9.7.69}$$

We substitute (9.7.68) in (9.7.65) to obtain the solution for $K(x, y; t)$. Consequently, the solution (9.7.61) of the KdV equation becomes

$$u(x,t) = -2\kappa_1^2 \operatorname{sech}^2\left[\kappa_1(x - Ut) - \kappa_1 x_0\right], \tag{9.7.70}$$

where $U = 4\kappa_1^2$. This represents a single-soliton solution of amplitude $-2\kappa_1^2$ traveling to the right with speed $U = 4\kappa_1^2$ and centered initially at the point x_0.

Example 9.7.2 (N-Soliton Solution for a sech^2 *Potential).* We investigate the scattering problem for the class of sech^2 potentials given by

$$u(x,0) = u_0(x) = -U \operatorname{sech}^2 x, \tag{9.7.71}$$

where U is a constant.

The associated Schrödinger equation is

$$\psi_{xx} + \left(\lambda + U \operatorname{sech}^2 x\right)\psi = 0. \tag{9.7.72}$$

We introduce a change of variable by $y = \tanh x$ so that $-1 < y < 1$ for $-\infty < x < \infty$. In terms of y, the potential (9.7.71) becomes

$$U \operatorname{sech}^2 x = U\left(1 - \tanh^2 x\right) = U\left(1 - y^2\right),$$

and the second derivative in terms of y is given by

$$\frac{d^2}{dx^2} = \frac{dy}{dx}\frac{d}{dy}\left(\frac{dy}{dx}\frac{d}{dy}\right) = \left(1 - y^2\right)\frac{d}{dy}\left[\left(1 - y^2\right)\frac{d}{dy}\right].$$

Consequently, equation (9.7.72) gives the *associated Legendre equation*

$$\frac{d}{dy}\left[\left(1 - y^2\right)\frac{d\psi}{dy}\right] + \left[U + \frac{\lambda}{1 - y^2}\right]\psi = 0. \tag{9.7.73}$$

We set $U = N(N+1)$, where N is a positive integer, and then, consider only the bound states ($b(k,t) = 0$ for all k). Physically, this means that the incident wave is totally reflected, and hence, the associated potential is referred to as *reflectionless*. If $\lambda = -\kappa^2 < 0$, then only bounded solutions for $-1 \le y \le 1$ occur when $\kappa_n = n$, $n = 1, 2, \ldots, N$, and the corresponding discrete eigenfunctions are given by $\psi_n(x) = A_n P_N^n(\tanh x)$, where the associated Legendre functions $P_N^n(y)$ are defined by

$$P_N^n(y) = (-1)^n \left(1 - y^2\right)^{n/2} \frac{d^n}{dy^n} P_N(y) \quad \text{and} \quad P_N(y) = \frac{1}{N!2^N}\frac{d^N}{dy^N}\left(y^2 - 1\right)^N. \tag{9.7.74ab}$$

$P_N(y)$ is the Legendre polynomial of degree N, and the constants of proportionality A_n are determined from the normalization condition (9.7.24). The asymptotic representation of the discrete eigenfunctions has the form

$$\psi_n(x) \sim c_n(t)\exp(-nx) \quad \text{as } x \to +\infty, \tag{9.7.75}$$

where

$$c_n(t) = c_n(0)\exp\left(4n^3 t\right) \tag{9.7.76}$$

and $c_n(0)$ are also determined from the normalization condition.

The function $B(x;t)$ involved in the GLM integral equation (9.7.60), therefore, is given by

$$B(x;t) = \sum_{n=1}^{N} c_n^2(0)\exp\left(8n^3 t - nx\right). \tag{9.7.77}$$

Finally, the GLM equation can easily be solved, and hence, the N-soliton solution assumes the asymptotic form

$$u(x,t) \sim -\sum_{n=1}^{N}\left(2n^2\right)\operatorname{sech}^2\left\{n\left(x-4n^2 t\right) \mp \varepsilon_n\right\} \quad \text{as } t \to \pm\infty, \tag{9.7.78}$$

where the quantities ε_n are known phases.

Clearly, a reflectionless potential with N bound states corresponds to pure N-solitons ordered according to their amplitudes. As $t \to \infty$, the tallest (hence, the fastest) soliton is at the front, followed by a series of progressively shorter (therefore, slower) ones behind. All N solitons interact at $t = 0$ to form the single sech^2 profile, which was prescribed as the initial condition at that instant.

In particular, when $N = 1$, $U = 2$ and $\kappa = \kappa_1 = 1$. The corresponding eigenfunction is given by

$$\psi_1(x) = A_1 P_1^1(\tanh x) = -A_1 \operatorname{sech} x,$$

where the constant A_1 is determined from the condition (9.7.24), which gives $2A^2 = 1$ or $A = \pm(1/\sqrt{2})$. So, the asymptotic nature of the above solution is

$$\psi_1(x) \sim \sqrt{2}e^{-x} \quad \text{as } x \to \infty.$$

Therefore, $c_1(0) = \sqrt{2}$, and $c_1(t) = \sqrt{2}\exp(4t)$, and the function $B(x;t) = 2\exp(8t - x)$.

Finally, the associated GLM equation

$$K(x,z;t) + 2\exp\left[8t-(x+z)\right] + 2\int_x^\infty K(x,y;t)\exp\left[8t-(y+z)\right]dy = 0$$

gives the solution

$$K(x,z;t) = F(x;t)\exp(-z),$$

for some function $F(x;t)$ that satisfies the equation

$$F + 2\exp(8t - x) + 2F\exp(8t)\int_x^\infty e^{-2y}\,dy = 0.$$

This can readily be solved to find the function F expressed as

$$F(x;t) = \frac{-2e^{8t-x}}{1 + e^{8t-2x}}.$$

Thus, the final solution is given by

$$u(x,t) = 2\frac{\partial}{\partial x}\left(\frac{2e^{8t-2x}}{1 + e^{8t-2x}}\right) = -\frac{8e^{2x-8t}}{(1 + e^{2x-8t})} = -2\,\mathrm{sech}^2(x - 4t). \quad (9.7.79)$$

This represents the single-soliton solution of amplitude -2 and speed 4.

Similarly, the two-soliton solution can be obtained from the initial profile (9.7.71) with $N = 2$, so that

$$u(x,0) = -6\,\mathrm{sech}^2 x. \quad (9.7.80)$$

This initial condition is then evolved in time according to the KdV equation, and the solution consists of two solitons where the taller one catches up to the shorter one. They undergo a nonlinear interaction according to the KdV equation and, then, emerge from the interaction unchanged in waveform and amplitude. Eventually, the taller soliton reappears to the right and passes away from the shorter one as t increases. The wave profile $u(x,t)$ with the initial condition (9.7.80) is plotted in Figure 9.11 as a function of x for different values of time: (a) $t = -0.5$, (b) $t = -0.1$, (c) $t = 0.0$, (d) $t = 0.1$, and (e) $t = 0.5$.

Similarly, the three-soliton solution $(N = 3)$ with $u(x,0) = -12\,\mathrm{sech}^2 x$ is shown in Figure 9.12 for different values of t: (a) $t = 0.0$, (b) $t = 0.05$, and (c) $t = 0.2$, where evolving solitons have amplitudes 18, 8, and 2, respectively.

Finally, to obtain the soliton solution, we examine the asymptotic behavior of the exact solution $u(x,t)$ for large x and t by considering only $B(x,t)$ given by (9.7.62). For large t and $x \sim 4\kappa_N^2 t$, where κ_N is the largest eigenvalue $(\kappa_1 < \kappa_2 < \cdots < \kappa_N)$, the Nth term in the series in (9.7.62) dominates over all other terms, and the significant contribution to the integral in (9.7.62) dominates over all other terms. This contribution to the integral in (9.7.62) can be found from the stationary phase approximation (Segur 1973) in the form $t^{-1/3}$ as $t \to \infty$. Thus, it turns out that

$$B(x,t) \sim \alpha\exp(-\kappa_N x), \quad (9.7.81)$$

where

$$\alpha = c_N^2(0)\exp\left(8\kappa_N^3 t\right).$$

With this value of B and $y = x$, the associated GLM equation becomes

$$K(x,x,t) + \alpha\exp(-2\kappa_N x) + \alpha\exp(-\kappa_N x)\int_x^\infty \exp(-\kappa_N z)K(x,z,t)\,dz \sim 0. \quad (9.7.82)$$

This can be easily solved by writing

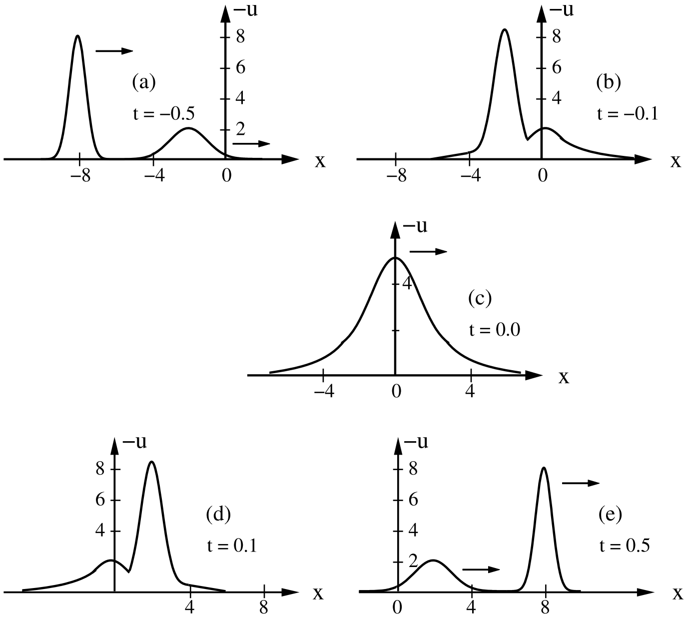

Fig. 9.11 The two-soliton solution. From Drazin and Johnson (1989).

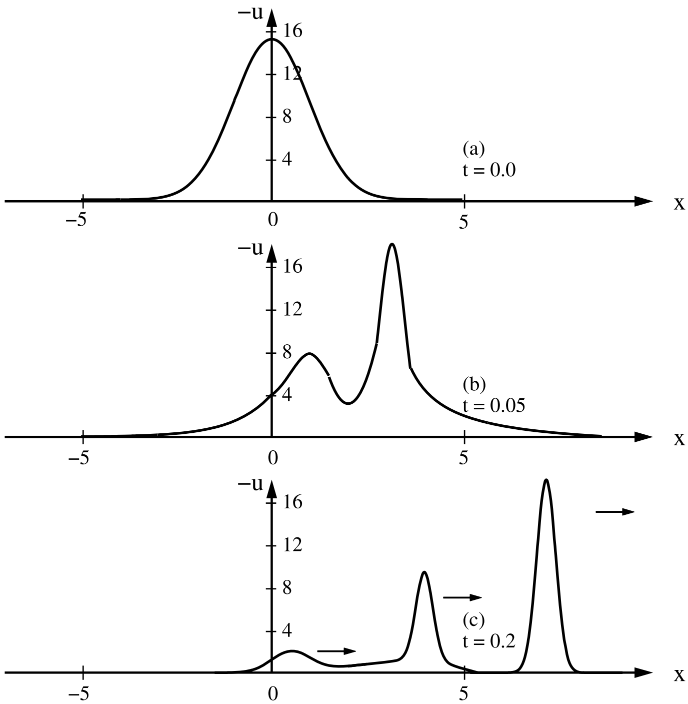

Fig. 9.12 The three-soliton solution. From Drazin and Johnson (1989).

$$K(x,z,t) = F(x,t)\exp(-\kappa_N z), \tag{9.7.83}$$

where F satisfies

$$F + \alpha\left[e^{-\kappa_N x} + F\int_x^\infty e^{-2\kappa_N z}\,dz\right] = 0. \tag{9.7.84}$$

Result (9.7.84) gives

$$F = \left[-\alpha\exp(-\kappa_N x)\right]\Big/\left[1 + \frac{\alpha}{2\kappa_N}\exp(-2\kappa_N x)\right]. \tag{9.7.85}$$

Substituting (9.7.83) in (9.7.61) gives

$$u(x,t) \sim -2\kappa_N^2\,\mathrm{sech}^2\left[\kappa_N(x-x_0) - 4\kappa_N^3 t\right], \tag{9.7.86}$$

where $c_N^2(0)/2\kappa_N = \exp(2\kappa_N x_0)$. Obviously, (9.7.86) asymptotically represents a soliton of amplitude $2\kappa_N^2$ and velocity $4\kappa_N^2$. If, instead of large t, we consider $x \sim 4\kappa_N^2 t$ for any arbitrary n, the significant term in the series involved in (9.7.86) is the nth term, and then, a soliton of amplitude $2\kappa_n^2$ and speed $4\kappa_n^2$ would emerge. Thus, the upshot of this analysis is that the initial (potential well) profile disintegrates into N solitons corresponding to discrete eigenvalues of the associated Schrödinger equation. A further elaborate mathematical analysis of the integral in (9.7.62) reveals a complete solution consisting of a series of N solitons preceded by an oscillatory trail of amplitude decreasing with time. This disintegration process of an initial profile into a series of solitons is usually called *fission*.

Examples 9.7.1 and 9.7.2 illustrate the reflectionless $(b(k) = 0)$ initial profiles. We now give examples for the nonzero reflection coefficient $(b(k) \neq 0)$.

Example 9.7.3 (A Soliton Solution Associated with the Dirac Delta Function Initial Profile). In this case, we use the initial condition as

$$u(x,0) = -u_0\delta(x), \tag{9.7.87}$$

where u_0 is a positive constant. For a discrete eigenstate, $\lambda_1 = -\kappa_1^2$, where $\kappa_1 = \frac{1}{2}u_0$ and the corresponding eigenfunction is given by

$$\psi_1(x) = \begin{cases} \sqrt{\kappa_1}\exp(-\kappa_1 x) & \text{if } x > 0, \\ \sqrt{\kappa_1}\exp(+\kappa_1 x) & \text{if } x < 0. \end{cases} \tag{9.7.88ab}$$

The continuous eigenfunction for $k = \sqrt{\lambda}$ exists and can be written as

$$b(k) = -\frac{u_0}{(u_0 + 2ik)}. \tag{9.7.89}$$

We can then use the time evolution of the scattering data

$$c_1(t) = \sqrt{\kappa_1}\exp\left(4\kappa_1^3 t\right) \quad \text{and} \quad b(k,t) = -\frac{u_0}{(u_0+2ik)}\exp\left(8ik^3 t\right). \tag{9.7.90ab}$$

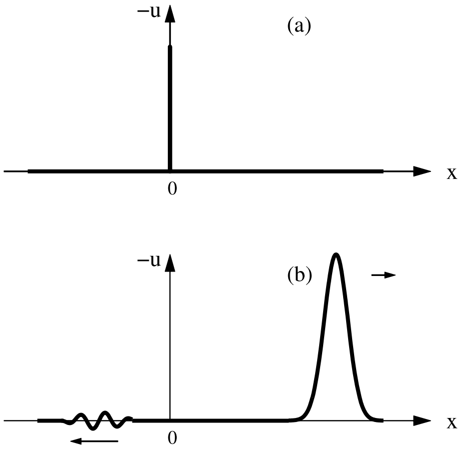

Fig. 9.13 (a) Initial profile and (b) solution at a later time.

The function $B(x,t)$ can be represented as

$$B(x,t) = \kappa_1 \exp\bigl(8\kappa_1^3 t - \kappa_1 x\bigr) - \frac{u_0}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(8ik^3 t + ikx)}{(u_0 + 2ik)}\,dk. \qquad (9.7.91)$$

However, the function $K(x,y;t)$ cannot easily be determined from (9.7.59). But the asymptotic solution for $u(x,t)$ for the single soliton associated with the discrete eigenvalue $\kappa_1 = \frac{1}{2}u_0$ and fixed $(x - u_0^2 t)$ is given by

$$u(x,t) \sim -\frac{1}{2}u_0^2 \operatorname{sech}^2\left\{\frac{1}{2}u_0\bigl(x - u_0^2 t - \varepsilon_1\bigr)\right\}, \quad \text{as } t \to \infty, \qquad (9.7.92)$$

where the phase change is given by $\exp(2\kappa_1\varepsilon_1) = c_0^2(0)/2\kappa_1$.

In fact, the asymptotic solution (9.7.92) is made up of the contribution to $B(x,t)$ from the first term in (9.7.91), and the integral term vanishes where asymptotic solution (9.7.92) is valid. The initial profile represented by the delta function is shown in Figure 9.13(a), and the solution $u(x,t)$ for a subsequent time is plotted in Figure 9.13(b).

Finally, if the amplitude u_0 of the initial profile is negative, there is no discrete eigenvalue, and hence, there is *no* soliton. And only dispersive waves exist in the solution for $t > 0$. As shown above, if $u_0 > 0$, there is only one eigenvalue $\kappa_1 = \frac{u_0}{2}$, and hence, a single soliton with amplitude $2\kappa_1^2 = \frac{1}{2}u_0^2$ is generated.

Example 9.7.4 (Solitons Associated with Negative sech^2 *Initial Profiles).* We use the method described by Crandall (1991) to examine the development of solitons and a dispersive wave associated with a class of potentials given by

$$u(x,0) = u_0(x) = -U \operatorname{sech}^2 x. \qquad (9.7.93)$$

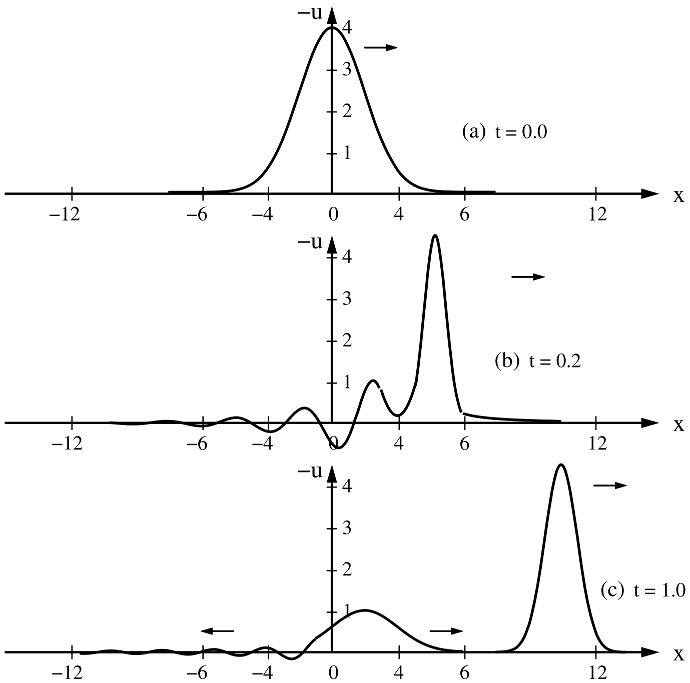

Fig. 9.14 The time evolution of the solution at three different times. From Drazin and Johnson (1989).

According to the general theory of solitons, the total number of discrete eigenvalues for general U is given by

$$\aleph = \left[\left(U + \frac{1}{4}\right)^{\frac{1}{2}} - \frac{1}{2}\right] + 1, \tag{9.7.94}$$

where $[\cdot]$ denotes the integral part and, if $\{(U + \frac{1}{4})^{\frac{1}{2}} - \frac{1}{2}\}$ is an integer, then the $+1$ is omitted.

For $U = 4$, $\aleph = 2$, and hence, there are two discrete eigenvalues, and therefore, a solution with two solitons. Moreover, $U = 4$ cannot be written as $N(N+1)$ for integral N, and the solution also includes a dispersive wave. Finally, the solution of the KdV equation is determined by numerical integration. The end result is that there are contributions from both the discrete and continuous states to the function $B(x+t)$ in equation (9.7.60). The effect of the different contributions is seen in the time evolution of the solution, which reveals the appearance of two solitons moving to the right and a dispersive wavetrain traveling to the left, as shown in Figure 9.14.

We close this section by adding a brief discussion on the number of solitons associated with an initial profile $u(x,0) = u_0(x)$. The solution depends on the discrete eigenvalues of the associated Schrödinger equation. We discuss this point by citing some specific examples.

For the case $u_0(x) = -a\,\mathrm{sech}^2(bx)$, the number of eigenvalues is given by (Landau and Lifshitz 1959, p. 70)

$$\kappa_n = \frac{1}{2}b\left[\left(1+\frac{4a}{b^2}\right)^{1/2} - (2n-1)\right] \geq 0, \tag{9.7.95}$$

where the number of eigenvalues is determined by the parameter P defined by

$$P = \int_{-\infty}^{\infty} \left|u_0(x)\right|^{1/2} dx. \tag{9.7.96}$$

For the sech^2 profile, $P = \pi b\sqrt{a}$. The number of solitons is given by

$$N = \text{largest integer} \leq \frac{1}{2}\left[\left(1+\frac{4P^2}{\pi^2}\right)^{1/2} + 1\right]. \tag{9.7.97}$$

This shows that the number of solitons depends on the parameter P, which, in this case, is proportional to $b\sqrt{a}$. For the delta function case,

$$u_0(x) = -a\delta(x) = -a \lim_{n\to\infty} \sqrt{\frac{n}{\pi}} \exp\left(-nx^2\right), \tag{9.7.98}$$

and $P \to 0$ as $n \to \infty$, and hence, there is only one soliton consistent with (9.7.97). In the other limit as $P \to \infty$, the formula (9.7.97) gives

$$N \sim \frac{P}{\pi}. \tag{9.7.99}$$

Thus, the result (9.7.97) shows that there is always one soliton for small P, and as P increases ($P \to \infty$), the number of solitons increases. Furthermore, when the initial disturbance is large ($P \to \infty$), there are many closely spaced eigenvalues that satisfy the famous Bohr–Sommerfeld rule

$$\oint p\,dx = \oint \left[\lambda - u_0(x)\right]^{1/2} dx = 2\pi\left(n + \frac{1}{2}\right). \tag{9.7.100}$$

Thus, the number of solitons (the largest value of n for $\lambda = 0$) is given by

$$N \sim \frac{1}{\pi}\int_{-\infty}^{\infty} \left|u_0(x)\right|^{1/2} dx = \frac{P}{\pi}. \tag{9.7.101}$$

This ensures the validity of the result obtained in the previous examples. Another approximate formula for the number of solitons with amplitudes in $(a, a+da)$ was first obtained by Karpman (1967) in the form

$$N(a) = \frac{1}{8\pi}\oint\left[\left|u_0(x)\right| - \frac{a}{2}\right]^{-1/2} dx. \tag{9.7.102}$$

This is over the range $0 < a < 2u_{0m} = 2|u_0|_{\max}$, where u_{0m} is the largest value of $|\lambda|$ for the bound states in (9.7.100), since the range of κ is $0 < \kappa < \sqrt{u_{0m}}$. Hence, the total number of solitons is given by

$$N = \int_0^{2u_{0m}} N(a)\, da = \frac{1}{\pi} \int |u_0|^{1/2}\, dx. \tag{9.7.103}$$

This is in agreement with (9.7.101). Thus, the dependence of P on the size (a and b) of a soliton suggests another general result:

$$Q = \int_{-\infty}^{\infty} |u|^{1/2}\, dx. \tag{9.7.104}$$

Physically, this represents an interesting measure of soliton shapes for a single soliton, and (9.7.104) can easily be computed to obtain $Q \sim \sqrt{2}\pi$, which is independent of the amplitude κ_N. Since solitons here are of unit size, $Q = \sqrt{2}\pi N$ for a series of N solitons. This shows that there is a Planck constant for solitons!

It is worth noting here that the parameter P is the value of the integral in the initial disturbance. For large P, $N \sim P/\pi$, and for large time t,

$$Q = \sqrt{2}\pi N \sim \sqrt{2}P. \tag{9.7.105}$$

This describes the close connection between the initial P and the final Q.

9.8 Bäcklund Transformations and the Nonlinear Superposition Principle

Historically, Bäcklund transformations were developed in the 1880s to study the related theories of differential geometry and differential equations. They occurred as an extension of contact transformations, which transform surfaces with a common tangent at a point in one space into surfaces in another space, which also have a common tangent at the corresponding point. One of the earliest Bäcklund transformations was found for the sine-Gordon equation, $u_{xt} = \sin u$. This equation originally arose in differential geometry in connection with the theory of surfaces of constant negative curvature. However, the study of these transformations had been dormant until the recent work on solitons in the 1970s. It has been recognized that partial differential equations admit soliton-like solutions if and only if they admit Bäcklund transformations. Indeed, there is a close relationship between the inverse scattering transform (IST) and a Bäcklund transformation (BT), in the sense that the scattering problem and the associated time dependence that constitute an IST also constitute a BT. In other words, every evolution equation solvable by an IST has a corresponding BT, and, conversely, the existence of BTs always or almost always implies integrability by the IST.

Another approach to deriving conservation laws and the inverse scattering problem is through the use of the Bäcklund transformations. For second-order partial differential equations, the Bäcklund transformation consists of a pair of first-order partial differential equations relating the solutions of the given equation to another solution of the same equation or to a solution of another second-order equation. In general, a Bäcklund transformation for a second-order partial differential equation for a dependent variable $u(x,t)$ is defined by the pair of equations

$$\left.\begin{aligned} w_x &= P(w,u,u_x,u_t,x,t), \\ w_t &= Q(w,u,u_x,u_t,x,t), \end{aligned}\right\} \tag{9.8.1ab}$$

where P and Q are functions of the variables indicated, but not of the derivatives of w. A new equation for w is obtained from the consistency condition.

One of the simplest Bäcklund transformations is the pair of the Cauchy–Riemann equations in complex analysis

$$u_x = v_y \quad \text{and} \quad u_y = -v_x, \tag{9.8.2ab}$$

for the Laplace equations

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0. \tag{9.8.3ab}$$

Thus, if $v(x,y) = xy$ is a simple solution of the Laplace equation, then $u(x,y)$ can be determined from $u_x = x$ and $u_y = -y$. Therefore, $u(x,y) = \frac{1}{2}(x^2 - y^2)$ is another solution of the Laplace equation.

In connection with the Miura transformation (9.6.10), it has been shown that, if v is a solution of the modified KdV equation (9.6.11), then u is a solution of the KdV equation (9.6.3). Since it is possible to eliminate higher derivatives from (9.6.11) by using (9.6.10), we can treat (9.6.10) and (9.6.11), written in the form

$$v_x = u - v^2 \quad \text{and} \quad v_t = 6v^2 v_x - v_{xxx}, \tag{9.8.4ab}$$

as a Bäcklund transformation for the KdV equation (9.6.3).

Wahlquist and Estabrook (1973, 1975, 1976) have developed a more convenient and useful Bäcklund transformation for solutions of the KdV equation. Since the KdV equation is Galilean invariant, we replace v in (9.6.11) by $u - \lambda$, where λ is a real parameter. Using $u = \lambda + v^2 + v_x$, we rewrite the modified KdV equation (9.6.11) in the form

$$v_t - 6\left(v^2 + \lambda\right)v_x + v_{xxx} = 0, \tag{9.8.5}$$

so that u satisfies the KdV equation (9.6.3). Clearly, both v and $-v$ satisfy (9.8.5). This leads us to construct two functions u_1 and u_2, corresponding to v and $-v$, in the form

$$u_1 = \lambda + v^2 + v_x, \qquad u_2 = \lambda + v^2 - v_x \tag{9.8.6ab}$$

for a given λ and v.

Consequently,

$$u_1 - u_2 = 2v_x, \qquad u_1 + u_2 = 2\left(\lambda + v^2\right). \tag{9.8.7ab}$$

It is convenient to define a new potential function w_i such that $u_i = \partial w_i/\partial x$, $i = 1, 2$. Thus, equations (9.8.7ab) reduce to the following pair of equations:

$$w_1 - w_2 = 2v, \tag{9.8.8}$$

$$(w_1 + w_2)_x = 2\lambda + \frac{1}{2}(w_1 - w_2)^2, \tag{9.8.9}$$

where any arbitrary function of t in (9.8.8) has been incorporated in w_i without changing u_i. We next use (9.8.7ab) and (9.8.8) in order to transform (9.8.5) into the form

$$(w_1 - w_2)_t = 3\left(w_{1x}^2 - w_{2x}^2\right) - (w_1 - w_2)_{xxx}. \tag{9.8.10}$$

Thus, the upshot of this analysis is that equations (9.8.9) and (9.8.10) constitute a pair of auto-Bäcklund transformations of the KdV equation, where the former is the x part and the latter is the t part.

We next illustrate the method by solving (9.8.9), (9.8.10) for w_1 with $w_2 = 0$ for all x and t. The resulting equations for w_1 become

$$w_{1x} = 2\lambda + \frac{1}{2}w_1^2 \quad \text{and} \tag{9.8.11a}$$

$$w_{1t} = 3w_{1x}^2 - w_{1xxx}, \tag{9.8.11b}$$

where the former gives

$$w_{1xxx} = \frac{\partial}{\partial x}(w_1 w_{1x}) = w_{1x}^2 + w_1^2 w_{1x}, \tag{9.8.12}$$

and hence, (9.8.11b) can be written, by using (9.8.11a) again and $\lambda = \kappa^2$, as

$$w_{1t} + 4\kappa^2 w_{1x} = 0. \tag{9.8.13}$$

This equation admits the general solution

$$w_1(x,t) = f\left(x - 4\kappa^2 t\right), \tag{9.8.14}$$

where f is an arbitrary function.

On the other hand, (9.8.11a) can readily be integrated to obtain the solution

$$w_1(x,t) = -2\kappa \tanh\left[\kappa x + \alpha(t)\right], \tag{9.8.15}$$

where $\alpha(t)$ is an arbitrary function of t. For consistency of the solutions (9.8.14) and (9.8.15), we require that $\alpha(t) = -4\kappa(\kappa^2 t - x_0)$, where x_0 is an arbitrary constant. Thus, the Bäcklund transformations give the final solution

$$w_1(x,t) = -2\kappa \tanh\left[\kappa\left(x - x_0 - 4\kappa^2 t\right)\right], \tag{9.8.16}$$

and hence, it follows from $u_1 = w_{1x}$ that

$$u_1(x,t) = -2\kappa^2 \operatorname{sech}^2\left[\kappa\left(x - x_0 - 4\kappa^2 t\right)\right]. \tag{9.8.17}$$

This is the soliton solution of the KdV equation (9.6.3).

All of this illustrates how the Bäcklund transformations can be used to obtain the soliton solution of the KdV equation. However, the procedure just demonstrated requires two integrations, one with respect to x and the other with respect to t. Instead of the process of integration involving arbitrary functions, Wahlquist and Estabrook (1973) developed an elegant method, based on the theory of differential forms, for determining solutions of the KdV equation to obtain a one-parameter family of Bäcklund transformations. In this method, two distinct solutions w_1 and w_2 are generated from the Bäcklund transformations by using the same given solution w with two different values λ_1 and λ_2 of λ. Thus, equation (9.8.9) can be written in two different forms,

$$(w_1 + w)_x = 2\lambda_1 + \frac{1}{2}(w_1 - w)^2, \tag{9.8.18a}$$

$$(w_2 + w)_x = 2\lambda_2 + \frac{1}{2}(w_2 - w)^2. \tag{9.8.18b}$$

It is now possible to construct another solution w_{12} from w_1 and λ_2 and, similarly, a solution w_{21} from w_2 and λ_1, so that

$$(w_{12} + w_1)_x = 2\lambda_2 + \frac{1}{2}(w_{12} - w_1)^2, \tag{9.8.19a}$$

$$(w_{21} + w_2)_x = 2\lambda_1 + \frac{1}{2}(w_{21} - w_2)^2. \tag{9.8.19b}$$

We next use Bianchi's theorem of permutability for the Bäcklund transformations in differential geometry, which states that $w_{12} = w_{21}$. Now, we subtract the difference of equations (9.8.18ab) from the difference of equations (9.8.19ab) and use the identity $w_{12} = w_{21}$, so as to produce zero on the left-hand side of the resulting equation:

$$0 = 4(\lambda_2 - \lambda_1) + \frac{1}{2}\left[(w_{12} - w_1)^2 - (w_{21} - w_2)^2 - (w_1 - w)^2 + (w_2 - w)^2\right],$$

whence the solution for $w_{12}(= w_{21})$ is given by

$$w_{12} = w - \frac{4(\lambda_1 - \lambda_2)}{(w_1 - w_2)}. \tag{9.8.20}$$

Thus, it is now possible to find solutions of the KdV equation in a straightforward manner. Equation (9.8.20) is a purely simple algebraic expression, known as the *nonlinear superposition principle*, for constructing solutions. It is possible to generalize superposition formula (9.8.20) and use the procedure to construct multisoliton solutions.

A process similar to the two-soliton solution can be generalized to obtain a three-soliton solution of the KdV equation:

$$\begin{aligned} w_{123} &= w_1 - \frac{4(\lambda_2 - \lambda_3)}{w_{12} - w_{13}} \\ &= \frac{\lambda_1 w_1(w_2 - w_3) + \lambda_2 w_2(w_3 - w_1) + \lambda_3 w_3(w_1 - w_2)}{\lambda_1(w_2 - w_3) + \lambda_2(w_3 - w_1) + \lambda_3(w_1 - w_2)}. \end{aligned} \tag{9.8.21}$$

This represents the *nonlinear superposition principle* for the three-soliton solution.

Thus, the multisoliton can be produced by continuing this process and setting

$$w_s = -4\kappa_s^2 \operatorname{sech}^2\left[\sqrt{2}\kappa_s\left(x - 8\kappa_s^2 t - x_s\right)\right], \tag{9.8.22}$$

where x_s is constant.

In particular, when $w = 0$, w_1 and w_2 represent bounded and unbounded solutions corresponding to bounded and unbounded solitons for u expressed as

$$u_1 = -4\kappa^2 \operatorname{sech}^2\left[\sqrt{2}\kappa\left(x - 8\kappa^2 t - x_0\right)\right] \tag{9.8.23}$$

and

$$u_2 = -4\kappa^2 \operatorname{cosech}^2\left[\sqrt{2}\kappa\left(x - 8\kappa^2 t - x_0\right)\right]. \tag{9.8.24}$$

Although solution (9.8.24) is not of physical interest, it is essential for the construction of a bounded solution for w_{12}, which, in the present case, represents a two-soliton solution.

Example 9.8.1 (Two-Soliton Solution). We apply the nonlinear superposition principle (9.8.20) to derive the two-soliton solution of the KdV equation. We take

$$w_0 = 0, \qquad w_1 = -2\tanh(x - 4t), \qquad w_2 = -4\coth(2x - 32t), \tag{9.8.25}$$

so that $\lambda_1 = 1$ and $\lambda_2 = -4$. Consequently, (9.8.20) becomes

$$w_{12} = -\frac{6}{\{2\coth(2x - 32t) - \tanh(x - 4t)\}}. \tag{9.8.26}$$

The corresponding solution of the KdV equation follows from the result

$$\begin{aligned} u_{12} &= \frac{\partial}{\partial x} w_{12} = -\frac{6\{4\operatorname{cosech}^2(2x - 32t) + \operatorname{sech}^2(x - 4t)\}}{\{2\coth(2x - 32t) - \tanh(x - 4t)\}^2} \\ &= -\frac{6\{4\cosh^2(x - 4t) + \sinh^2(2x - 32t)\}}{\{2\cosh(2x - 32t)\cosh(x - 4t) - \sinh(2x - 32t)\sinh(x - 4t)\}^2} \\ &= -\frac{12\{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)\}}{\{3\cosh(x - 28t) + \cosh(3x - 36t)\}^2}. \end{aligned} \tag{9.8.27}$$

This represents the two-soliton solution associated with the initial profile (9.7.80). The numerical values of (9.8.27) for five different times: (a) $t = -0.5$, (b) $t = -0.1$, (c) $t = 0.0$, (d) $t = 0.1$, and (e) $t = 0.5$ plotted as a function of x give the same wave profile $u(x, t)$ as in Figure 9.11.

We close this section by citing some related work on the use of Bäcklund transformations in finding N-soliton solutions of several nonlinear evolution equations in Wronskian form. Several authors, including Freeman (1984) and Nimmo and Freeman (1983) used this method to obtain N-soliton solutions of KdV, modified KdV, sine-Gordon, KP, Boussinesq, nonlinear Schrödinger equations, and other partial differential difference equations. However, it seems that the scope of this approach is still not yet fully understood.

9.9 The Lax Formulation and the Zakharov and Shabat Scheme

In his 1968 seminal paper, Lax developed an elegant formalism for finding isospectral potentials as solutions of a nonlinear evolution equation with all of its integrals. This work deals with some new and fundamental ideas and deeper results and their application to the KdV model. This work subsequently paved the way to generalizations of the technique as a method for solving other nonlinear partial differential equations. Introducing the Heisenberg picture, Lax developed the method of inverse scattering based upon an abstract formulation of evolution equations and certain properties of operators on a Hilbert space, some of which are familiar in the context of quantum mechanics. His formulation has the feature of associating certain nonlinear evolution equations with linear equations which are analogs of the Schrödinger equation for the KdV equation.

To formulate Lax's method, we consider two linear operators L and M. The eigenvalue equation related to the operator L corresponds to the Schrödinger equation for the KdV equation. The general form of this eigenvalue equation is

$$L\psi = \lambda\psi, \tag{9.9.1}$$

where ψ is the eigenfunction and λ is the corresponding eigenvalue. The operator M describes the change of the eigenvalues with the parameter t, which usually represents time in a nonlinear evolution equation. The general form of this evolution equation is

$$\psi_t = M\psi. \tag{9.9.2}$$

Differentiating (9.9.1) with respect to t gives

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t. \tag{9.9.3}$$

We next eliminate ψ_t from (9.9.3) by using (9.9.2) and obtain

$$L_t\psi + LM\psi = \lambda_t\psi + \lambda M\psi = \lambda_t\psi + M\lambda\psi = \lambda_t\psi + ML\psi, \tag{9.9.4}$$

or equivalently,

$$\frac{\partial L}{\partial t}\psi = \lambda_t\psi + (ML - LM)\psi. \tag{9.9.5}$$

Thus, eigenvalues are constant for nonzero eigenfunctions if and only if

$$\frac{\partial L}{\partial t} = -(LM - ML) = -[L, M], \tag{9.9.6}$$

where $[L, M] = (LM - ML)$ is called the *commutator* of the operators L and M, and the derivative on the left-hand side of (9.9.6) is to be interpreted as the time derivative of the operator *alone*. Equation (9.9.6) is called the *Lax equation* and the operators L and M are called the *Lax pair*. It is the Heisenberg picture of the KdV equation. The problem, of course, is how to determine these operators for a given evolution equation. There is no systematic method of solution of this problem. For a negative integrable hierarchy and in order to find a Lax pair from a given spectral problem, Qiao (1995) and Qiao and Strampp (2002) suggested a general approach to generate integrable equations.

We consider the initial-value problem for $u(x,t)$ which satisfies the nonlinear evolution equation system

$$u_t = N(u), \tag{9.9.7}$$

$$u(x,0) = f(x), \tag{9.9.8}$$

where $u \in Y$ for all t, Y is a suitable function of space, and $N : Y \to Y$ is a nonlinear operator that is independent of t but may involve x or derivatives with respect to x.

We must assume that the evolution equation (9.9.7) can be expressed in the Lax form

$$L_t + (LM - ML) = L_t + [L, M] = 0, \tag{9.9.9}$$

where L and M are linear operators in x on a Hilbert space H and depend on u and $L_t = u_t$ is a scalar operator. We also assume that L is self-adjoint so that $(L\phi, \psi) = (\phi, L\psi)$ for all ϕ and $\psi \in H$ with $(\cdot,\cdot)$ as an inner product.

We now formulate the eigenvalue problem for $\psi \in H$:

$$L\psi = \lambda(t)\psi, \quad t \geq 0,\ x \in \mathbb{R}. \tag{9.9.10}$$

Differentiating with respect to t and making use of (9.9.9), we obtain

$$\lambda_t \psi = (L - \lambda)(\psi_t - M\psi). \tag{9.9.11}$$

The inner product of ψ with this equation yields

$$(\psi, \psi)\lambda_t = \big((L - \lambda)\psi, \lambda_t - M\psi\big), \tag{9.9.12}$$

which is, since $L - \lambda$ is self-adjoint, given by

$$(\psi, \psi)\lambda_t = (0, \psi_t - M\psi) = 0.$$

Hence, $\lambda_t = 0$, confirming that each eigenvalue of L is a constant. Consequently, (9.9.11) becomes

$$L(\psi_t - M\psi) = \lambda(\psi_t - M\psi). \tag{9.9.13}$$

This shows that $(\psi_t - M\psi)$ is an eigenfunction of the operator L with the eigenvalue λ. It is always possible to redefine M by adding the product of the identity operator and a suitable function of t, so that the original equation (9.9.9) remains unchanged. This leads to the time evolution equation for ψ as

$$\psi_t = M\psi, \quad t \geq 0. \tag{9.9.14}$$

Thus, we have the following.

Theorem 9.9.1. *If the evolution equation* (9.9.7) *can be expressed as the Lax equation*

$$L_t + [L, M] = 0, \tag{9.9.15}$$

and if (9.9.10) *holds, then* $\lambda_t = 0$ *and* ψ *satisfies* (9.9.14).

It is not yet clear how to find the operators L and M that satisfy the preceding conditions. To illustrate the Lax method, we choose the Schrödinger operator L in the form

$$L \equiv -\frac{\partial^2}{\partial x^2} + u, \tag{9.9.16}$$

so that $L\psi = \lambda\psi$ becomes the Sturm–Liouville problem for the self-adjoint operator L. With this given L, the problem is to find the operator M. Based on the theory of a linear unitary operator on a Hilbert space H, the linear operator M can be chosen as antisymmetric, so that $(M\phi, \psi) = -(\phi, M\psi)$ for all $\psi, \phi \in H$. So, a suitable linear combination of odd derivatives in x is a natural choice for M. It follows from the inner product that

$$(M\phi, \psi) = \int_{-\infty}^{\infty} \frac{\partial^n \phi}{\partial x^n}\psi\, dx = -\int_{-\infty}^{\infty} \phi \frac{\partial^n \psi}{\partial x^n}\, dx = -(\phi, M\psi), \tag{9.9.17}$$

provided $M = \partial^n \phi / \partial x^n$ for odd n, and ϕ, ψ, and their derivatives with respect to x tend to zero, as $|x| \to \infty$. Moreover, we require that M has sufficient freedom in any unknown constants or functions to make $L_t + [L, M]$ a multiplicative operator, that is, of degree zero. For $n = 1$, the simplest choice for M is $M = c(\partial/\partial x)$, where c is a constant. It then follows that $[L, M] = -cu_x$, which is automatically a multiplicative operator. Thus, the Lax equation is

$$L_t + [L, M] = u_t - cu_x = 0, \tag{9.9.18}$$

and hence, the one-dimensional wave equation

$$u_t - cu_x = 0 \tag{9.9.19}$$

has an associated eigenvalue problem with the eigenvalues that are constants of motion.

The next natural choice is

$$M = -a\frac{\partial^3}{\partial x^3} + A\frac{\partial}{\partial x} + \frac{\partial}{\partial x}A + B, \tag{9.9.20}$$

where a is a constant, $A = A(x, t)$, and $B = B(x, t)$, and the third term on the right-hand side of (9.9.20) can be dropped, but we retain it for convenience. It follows from an algebraic calculation that

$$\begin{aligned}[L, M] = {} & au_{xxx} - A_{xxx} - B_{xx} - 2u_x A \\ & + (3au_{xx} - 4A_{xx} - 2B_x)\frac{\partial}{\partial x} + (3au_x - 4A_x)\frac{\partial^2}{\partial x^2}.\end{aligned}$$

This would be a multiplicative operator if $A = \frac{3}{4}au$ and $B = B(t)$. Consequently, the Lax equation (9.9.15) becomes

$$u_t - \frac{3}{2}auu_x + \frac{a}{4}u_{xxx} = 0. \tag{9.9.21}$$

This is the standard KdV equation if $a = 4$. The operator M defined by (9.9.20) reduces to the form

$$M = -4\frac{\partial^3}{\partial x^3} + 3\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right) + B(t). \tag{9.9.22}$$

Hence, the time evolution equation for ψ can be simplified by using the Sturm–Liouville equation, $\psi_{xx} - (u - \lambda)\psi = 0$, as

$$\begin{aligned}\psi_t &= 4(\lambda\psi - u\psi)_x + 3\psi_x + 3(u\psi)_x + B\psi \\ &= 2(u + 2\lambda)\psi_x - u_x\psi + B\psi.\end{aligned} \tag{9.9.23}$$

We close this section by adding several comments. First, any evolution equations solvable by the IST, like the KdV equation, can be expressed in Lax form. However, the main difficulty is that there is no completely systematic method of finding whether or not a given partial differential equation produces a Lax equation and, if so, how to find the Lax pair L and M. Indeed, Lax proved that there is an infinite number of operators, M, one associated with each odd order of $\partial/\partial x$, and hence, an infinite family of flows u_t under which the spectrum of L is preserved. Second, it is possible to study other spectral equations by choosing alternative forms for L. Third, the restriction that L and M should be limited to the class of scalar operators could be removed. In fact, L and M could be matrix operators. The Lax formulation has already been extended for such operators. Fourth, Zakharov and Shabat (1972, 1974) published a series of notable papers in this field extending the nonlinear Schrödinger (NLS) equation and other evolution equations. For the first time, they have also generalized the Lax formalism for equations with more than one spatial variable. This extension is usually known as the Zakharov and Shabat (ZS) scheme which, essentially, follows the Lax method and recasts it in a matrix form, leading to a matrix Marchenko equation. Finally, we briefly discuss the ZS scheme for non-self-adjoint operators to obtain N-soliton solutions for the NLS equation. Zakharov and Shabat introduced an ingenious method for any nonlinear evolution equation

$$u_t = Nu, \tag{9.9.24}$$

to represent the equation in the form

$$\frac{\partial L}{\partial t} = i[L, M] = i(LM - ML), \tag{9.9.25}$$

where L and M are linear differential operators including the function u in the coefficients and L refers to differentiating u with respect to t in the expression for L.

We consider the eigenvalue problem

$$L\phi = \lambda\phi. \tag{9.9.26}$$

Differentiation of (9.9.26) with respect to t gives

$$i\phi\left(\frac{\partial\lambda}{dt}\right) = (L-\lambda)(i\phi_t - M\phi). \tag{9.9.27}$$

If ϕ satisfies (9.9.26) initially and changes in such a manner that

$$i\phi_t = M\phi, \tag{9.9.28}$$

then ϕ always satisfies (9.9.26). Equations (9.9.26) and (9.9.28) are the pair of equations coupling the function $u(x,t)$ in the coefficients with a scattering problem. Indeed, the nature of ϕ determines the scattering potential in (9.9.26), and the evolution of ϕ in time is given by (9.9.28).

Although this formulation is quite general, the crucial step, still, is to factor L according to (9.9.25). Zakharov and Shabat (1972) introduced 2×2 matrices associated with (9.9.25) as follows:

$$L = i\begin{bmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}\frac{\partial}{\partial x} + \begin{bmatrix} 0 & u^* \\ u & 0 \end{bmatrix}, \tag{9.9.29}$$

$$M = -\alpha\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\partial^2}{\partial x^2} + \begin{bmatrix} \frac{|u|^2}{1+\alpha} & iu_x^* \\ -iu_x & \frac{-|u|^2}{1-\alpha} \end{bmatrix}, \tag{9.9.30}$$

and the NLS equation for complex $u(x,t)$ is given by

$$iu_t + u_{xx} + \gamma|u|^2 u = 0, \tag{9.9.31}$$

where

$$\gamma = 2/\left(1-\alpha^2\right).$$

Thus, the eigenvalue problem (9.9.26) and the evolution equation (9.9.28) complete the inverse scattering problem. The initial-value problem for $u(x,t)$ can be solved for a given initial condition $u(x,0)$. It seems clear that the significant contribution would come from the point spectrum for large times ($t\to\infty$). Physically, the disturbance tends to disintegrate into a series of solitary waves. The mathematical analysis is limited to the asymptotic solutions so that $|u|\to 0$ as $|x|\to\infty$, but a series of solitary waves is expected to be the end result of the instability of wavetrains to modulations.

In general, to date, there is no completely systematic method of determining the linear scattering problem associated with a given nonlinear evolution equation. However, one systematic method is the prolongation structure method, which was introduced by Wahlquist and Estabrook (1975, 1976) using the knowledge of Lie algebra. This method was found to be very useful for several nonlinear evolution equations. Dodd and Gibbon (1997) successfully developed the prolongation structure of a higher-order KdV equation. Subsequently, several authors including Dodd and

Fordy (1983, 1984), Fordy (1990), and Kaup (1980) used the prolongation method for finding the scattering problem of a given nonlinear evolution equation.

Finally, we close this section by including another commutative representation of (9.9.9), so that the theory can be extended to other nonlinear evolution equations. It has already been demonstrated that the Schrödinger equation

$$L\psi = k^2\psi \tag{9.9.32}$$

plays a fundamental role in solving the KdV equation. It has also been seen that if the potential u in this equation varies in time according to the KdV equation, then ψ satisfies a linear equation:

$$\psi_t = -4\psi_{xxx} + 6u\psi_x + 3u_x\psi. \tag{9.9.33}$$

A necessary condition for these equations to be compatible for all λ is that $u(x,t)$ satisfies the KdV equation (9.9.1). This can be seen as follows.

We first transform the Schrödinger equation (9.9.32) into two first-order equations by introducing a new function ϕ so that

$$\psi_x = ik\psi + \phi \quad \text{and} \quad \phi_x = -ik\phi + u\psi. \tag{9.9.34ab}$$

We can use (9.9.34ab) to eliminate all x-derivatives from (9.9.33), which then reduces to the form

$$\psi_t = 4ik^3\psi + 4k^2\phi + 2ik\psi - u_x\psi + 2u\phi. \tag{9.9.35}$$

Using (9.9.34ab), equation (9.9.35) leads to an equation for ϕ in the form

$$\phi_t = -4ik^3\phi + 4k^2u\phi + 2iku_x\psi - 2iku\phi + \left(2u^2 - u_{xx}\right)\psi + u_x\phi. \tag{9.9.36}$$

Introducing a column vector $\varPsi = \binom{\psi}{\phi}$, we can reformulate the system of equations (9.9.34ab)–(9.9.36) in a compact form,

$$\varPsi_x = A(x,t;\lambda)\varPsi \quad \text{and} \tag{9.9.37a}$$
$$\varPsi_t = B(x,t;\lambda)\varPsi, \tag{9.9.37b}$$

where λ is a complex parameter and $A(x,t;\lambda)$ and $B(x,t;\lambda)$ are 2×2 matrices in the form

$$A = \begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}, \tag{9.9.38}$$

$$B = 4i\lambda^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 4\lambda^2 \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} + 2i\lambda \begin{pmatrix} u & 0 \\ u_x & -u \end{pmatrix} + \begin{pmatrix} -u_x & 2u \\ 2u^2 - u_{xx} & u_x \end{pmatrix}, \tag{9.9.39}$$

and A and B depend on x and t through $u(x,t)$.

Differentiating (9.9.37a) with respect to t and (9.9.37b) with respect to x leads to the compatibility equation

$$\frac{\partial A(\lambda)}{\partial t} - \frac{\partial B(\lambda)}{\partial x} + \left[A(\lambda), B(\lambda)\right] = 0. \tag{9.9.40}$$

This is a new commutative representation which can be used to extend the KdV theory to other nonlinear equations. If we consider the case where A and B are polynomial in λ, then the left-hand side of (9.9.40) is also a polynomial in λ. Since equation (9.9.40) is true for all values of λ, all coefficients of this polynomial equation must be identically zero. We next substitute (9.9.38) and (9.9.39) in (9.9.40) to obtain all coefficients which vanish except for the constant term. This means that the left-hand side of (9.9.40) reduces to the matrix

$$\begin{pmatrix} 0 & 0 \\ u_t - 6uu_x - u_{xxx} & 0 \end{pmatrix},$$

which vanishes, so that $u(x,t)$ satisfies the KdV equation.

We illustrate the above method by examples.

Example 9.9.1 (Nonlinear Schrödinger Equation). We consider the form of the 2×2 matrix as

$$A = \begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} + \begin{pmatrix} 0 & iq \\ ir & 0 \end{pmatrix}, \tag{9.9.41}$$

where $q = q(x,t)$ and $r = r(x,t)$ are complex-valued functions of x and t. We chose the matrix $B(\lambda)$ so that (9.9.40) yields certain partial differential equations for q and r. We assume the matrix $B(\lambda)$ as

$$B = 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & q_x \\ -r_x & 0 \end{pmatrix} - i \begin{pmatrix} rq & 0 \\ 0 & -rq \end{pmatrix}. \tag{9.9.42}$$

Substituting these matrices A and B in (9.9.40) gives a system of equations

$$ir_t + r_{xx} + 2qr^2 = 0, \tag{9.9.43a}$$
$$iq_t + q_{xx} + 2rq^2 = 0. \tag{9.9.43b}$$

We next write $r = \overline{q}$ or $r = -\overline{q}$ in the above equations to obtain

$$ir_t + r_{xx} \pm 2|r|^2 r = 0. \tag{9.9.44ab}$$

These are known as the *nonlinear Schrödinger equations*.

Example 9.9.2 (The KdV and Modified KdV Equations). We use the same A as in (9.9.41) and choose B as

$$B = -4i\lambda^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - 4i\lambda^2 \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} + 2\lambda \begin{pmatrix} rq & -iq_x \\ ir_x & -rq \end{pmatrix}$$
$$+ \begin{pmatrix} qr_x - rq_x & iq_{xx} + 2irq^2 \\ ir_{xx} + 2iqr^2 & -qr_x - rq_x \end{pmatrix}, \tag{9.9.45}$$

so that equation (9.9.40) becomes

$$q_t + 6rqq_x + q_{xxx} = 0, \tag{9.9.46a}$$
$$r_t + 6qrr_x + r_{xxx} = 0. \tag{9.9.46b}$$

We assume that q and r are real functions of x and t. When $r = -1$, (9.9.46a) is the famous KdV equation.

When $r = q$ or $r = -q$, equations (9.9.46a) and (9.9.46b) give

$$r_t \pm 6r^2 r_x + r_{xxx} = 0. \tag{9.9.47ab}$$

These are the modified KdV equations.

Example 9.9.3 (The sine-Gordon Equation). We choose $r = q = \frac{1}{2}u_x$, the same matrix A as in (9.9.41), and the matrix B as

$$B(\lambda) = (4i\lambda)^{-1} \begin{pmatrix} \cos u & -i\sin u \\ i\sin u & -\cos u \end{pmatrix},$$

so that equation (9.9.40) leads to the sine-Gordon equation

$$u_{xt} = \sin u. \tag{9.9.48}$$

Example 9.9.4 (The sinh-Gordon Equation). We substitute $r = -q = -\frac{1}{2}u_x$, the same matrix A as in (9.9.41), and

$$B(\lambda) = (4i\lambda)^{-1} \begin{pmatrix} \cosh u & -i\sinh u \\ -i\sinh u & -\cosh u \end{pmatrix},$$

in (9.9.40) to obtain the sinh-Gordon equation

$$u_{xt} = \sinh u. \tag{9.9.49}$$

9.10 The AKNS Method

In 1974, Ablowitz, Kaup, Newell, and Segur (AKNS) generalized the ZS scheme so that their method can be applied to solve many other evolution equations. We briefly outline the AKNS scheme below without all the technical details. We begin this discussion by considering the pair of linear equations

$$\mathbf{u}_x = A\mathbf{u} \quad \text{and} \tag{9.10.1a}$$
$$\mathbf{u}_t = B\mathbf{u}, \tag{9.10.1b}$$

where $\mathbf{u}$ is an n-dimensional vector and A and B are $n \times n$ matrices. Then differentiating (9.10.1a) with respect to t and (9.10.1b) with respect to x and equating the results leads to the equation

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + [A, B] = 0. \tag{9.10.2}$$

This is essentially equivalent to the Lax equation (9.9.15). It turns out that, given A, there is a simple deductive method to find a B so that (9.10.2) contains a nonlinear evolution equation. For (9.10.2) to be effective, the operator A should have a parameter which plays the role of an eigenvalue, say ζ, which satisfies the condition $\zeta_t = 0$. Further, a solution of the related nonlinear evolution equation in an infinite interval can be obtained when the associated scattering problem is such that the inverse scattering method can be carried out. Although there are many nonlinear evolutions which satisfy (9.10.2), at this time, a complete scattering method for many of the associated equations (9.10.1a) has not yet been successfully developed.

As an example, we consider a 2×2 eigenvalue problem for the pair of equations

$$\psi_{1x} = -i\zeta\psi_1 + q\psi_2, \tag{9.10.3a}$$

$$\psi_{2x} = i\zeta\psi_2 + r\psi_1, \tag{9.10.3b}$$

that is,

$$\psi_x = A\psi, \qquad B\psi = -i\zeta\psi, \tag{9.10.4ab}$$

where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $A = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix}$, and $B = \begin{pmatrix} \frac{\partial}{\partial x} & -q \\ r & -\frac{\partial}{\partial x} \end{pmatrix}$. The bounded functions $q(x)$ and $r(x)$, not necessarily real, are *potentials*, and ζ is the eigenvalue. It can easily be shown that there is a direct link between the pair (9.10.3a), (9.10.3b) and the original Schrödinger equation. Differentiating (9.10.3b) with respect to x gives

$$\psi_{2xx} = i\zeta\psi_{2x} + r_x\psi_1 + r\psi_{1x}, \tag{9.10.5}$$

provided that r_x exists. Using (9.10.3ab), equation (9.10.5) can be simplified to obtain

$$\begin{aligned} \psi_{2xx} &= i\zeta\psi_{2x} + r_x\psi_1 + r(-i\zeta\psi_1 + q\psi_2) \\ &= i\zeta\psi_{2x} + \frac{1}{r}(r_x - i\zeta r)(\psi_{2x} - i\zeta\psi_2) + qr\psi_2. \end{aligned}$$

Or equivalently,

$$\psi_{2xx} - \left(\frac{r_x}{r}\right)\psi_{2x} - \left\{qr - i\zeta\left(\frac{r_x}{r}\right) - \zeta^2\right\}\psi = 0. \tag{9.10.6}$$

When $r = -1$, this equation reduces to the Schrödinger equation for ψ_2

$$L\psi_2 + \zeta^2\psi_2 = 0, \tag{9.10.7}$$

with $q = -u$ and $\lambda = \zeta^2$. Thus the system of equations (9.10.4ab) recovers the scattering equation required for solving the KdV equation. The choice $r = -1$ turns out to be a degenerate case since, for all other evolution equations, we assume that both $q(x)$ and $r(x)$ decay sufficiently rapidly as $|x| \to \infty$. This ensures the existence of ψ for $x \in \mathbb{R}$.

We close this section by adding the following comments. Both the ZS and AKNS methods are applicable to KdV and NLS equations and to mKdV and SG equations. Although the two methods overlap, the remarkable difference is that the ZS method is described solely in terms of operators, whereas the AKNS method is expressed in terms of the scattering theory.

9.11 Asymptotic Behavior of the Solution of the KdV–Burgers Equation

The KdV–Burgers (KdVB) equation

$$u_t + 2uu_x + \beta u_{xxx} - \nu u_{xx} = 0, \tag{9.11.1}$$

where β and $\nu (> 0)$ are constants, combines nonlinearity, linear dispersion and dissipation. This equation describes physical phenomena such as weak shock waves in plasmas, propagation of waves in a liquid-filled elastic tube, and the evolution of liquid–gas mixtures.

We consider the asymptotic behavior of the solution of the Cauchy problem for the KdVB equation (9.11.1) with the initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{R}$, as $t \to \infty$. If the initial function $u_0(x)$ decreases sufficiently rapidly at infinity, the uniform asymptotic behavior of this Cauchy problem, as $t \to \infty$, is given by

$$u(x,t) \sim \frac{A}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right) + O\left(\frac{1}{t}\right), \tag{9.11.2}$$

with respect to $\xi = \frac{|x|}{2\sqrt{t}} \geq 0$, where the constant A depends explicitly on $u_0(x)$. The Cauchy problem for (9.11.1) was investigated by several authors. Bona and Schonbek (1985) proved the existence of traveling wave solutions of (9.11.1) and considered their limiting behavior as $a \to 0$ or $b \to 0$. Jeffrey and Xu (1989) found two exact traveling wave solutions of (9.11.1). On the other hand, Avilov et al. (1987) have investigated the step-decaying problem for the KdVB equation by numerical methods. Their study deals with the asymptotic behavior, as $t \to \infty$, of the solution of the Cauchy problem for (9.11.1) with a sufficiently smooth initial condition $u_0(x)$ tending to ± 1 as $|x| \to \infty$.

In their study of ion-acoustic waves with Landau damping, Otto and Sudan (1971) and, in his work on weak shock waves and solitons, Ostrovsky (1976) introduced a new nonlinear dispersive equation of the form

$$u_t + u_x^2 + u_{xxx} + \lambda u + \frac{\mu}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \frac{\text{sgn}(x-\xi)}{\sqrt{|x-\xi|}} u_\xi(\xi,t)\, d\xi = 0, \tag{9.11.3}$$

where λ and μ are nonnegative constants. This equation is now known as the *Otto–Sudan–Ostrovsky* (*OSO*) *equation*, and it describes the nonlinear acoustic effects in the mixture of a gas and liquid bubbles. This equation is a special case of the

Whitham equation (7.8.31), and the symbol $K(k)$ of the operator in (7.8.31) is given by

$$K(k) = \lambda + \mu\sqrt{|k|} - ik^3. \tag{9.11.4}$$

Evidently, when $\lambda = \mu = 0$, the OSO equation (9.11.3) reduces to the KdV equation. On the other hand, the case $\mu = 0$ and $\lambda > 0$ corresponds to the KdV equation with linear dissipation. Several authors including Herman (1990), Kaup and Newell (1978), Karpman and Maslov (1978), and Ko and Kuehl (1980) studied the OSO equation with λ as a small positive parameter and $\mu = 0$. They used the theory of perturbation and the inverse scattering method to solve this equation with only soliton initial conditions and small values of λ. However, the situation with sufficiently general initial perturbations and finite λ seems to be unresolved. In the case $\mu = 0$, the asymptotic behavior, as $t \to \infty$, of the solution of (9.11.3) has an oscillatory character. More detailed asymptotic behavior is available in Chapter 7 of Naumkin and Shishmarev's book (1994).

9.12 Strongly Dispersive Nonlinear Equations and Compactons

Rosenau and Hyman (1993) first discovered a new class of solitary waves with compact support, called *compactons*. This new class of solutions is governed by a two-parameter family of strongly dispersive nonlinear equations, denoted by $K(m,n)$,

$$u_t + a\big(u^m\big)_x + b\big(u^n\big)_{xxx} = 0, \quad m > 0,\ 1 < n \le 3, \tag{9.12.1a}$$

$$u_t - a\big(u^m\big)_x + b\big(u^n\big)_{xxx} = 0, \quad m > 0,\ 1 < n \le 3, \tag{9.12.1b}$$

for certain values of m and n, where a and b are positive real constants. Thus, compactons are defined as solitons with compact support. In other words, they are solitons with finite wavelength or solitons that are free from exponential trails or wings. Unlike the standard KdV soliton, which narrows as the amplitude (speed) increases, the width of a compacton is independent of the amplitude, but its speed depends on its height. Since dispersion increases with amplitude, at high amplitudes, dispersion is more dominant than in the KdV equation, and hence, it can more effectively counterbalance the effect of nonlinear steepening. Numerous numerical experiments of Rosenau and Hyman (1993) confirmed that, when two or more compactons collide, they undergo a nonlinear elastic interaction according to (9.12.1a) and emerge from the interaction with the original form unchanged.

Equation (9.12.1a) with $(+a)$ is called the *focusing branch* and admits traveling solitary-wave solutions. On the other hand, equation (9.12.1b) with $(-a)$ is referred to as the *defocusing branch* and admits solitary-wave solutions with cusps or infinite slopes. Thus, equations (9.12.1a), (9.12.1b) represent two nonlinear models with entirely different physical structures.

We follow Rosenau and Hyman (1993) to find the solution of $K(2,2)$ with $a = b = 1$, that is, the equation

$$u_t + \big(u^2\big)_x + \big(u^2\big)_{xxx} = 0. \tag{9.12.2}$$

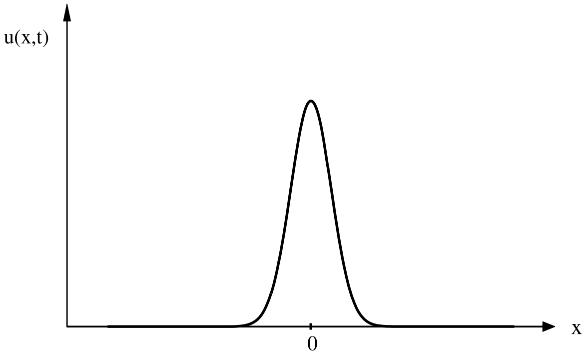

Fig. 9.15 A compacton.

We seek a traveling wave solution $u = u(\xi)$, $\xi = x - ct$ of (9.12.2) and integrate the resulting equation twice to obtain the following nonlinear ordinary differential equation:

$$\left(\frac{\partial u}{\partial \xi}\right)^2 + \frac{1}{4}u^2 - \frac{1}{3}cu + \frac{c_1}{u^2} = c_2, \tag{9.12.3}$$

where c_1 and c_2 are integration constants. Putting $c_1 = c_2 = 0$ leads to the solution

$$u(x,t) = \begin{cases} \left(\frac{4c}{3}\right)\cos^2\left[\frac{1}{4}(x-ct)\right] & \text{if } |x-ct| \le 2\pi, \\ 0 & \text{otherwise.} \end{cases} \tag{9.12.4}$$

This solution is referred to as a *compacton* and is shown in Figure 9.15.

Although the second derivative of the compacton solution is discontinuous at its edges, it represents a solitary wave with compact support because the third derivative acts on u^2 which has smooth derivatives everywhere including the edges. It has already been indicated that dispersion increases with amplitude, and is more dominant at higher amplitude than the KdV soliton. Hence, it can more effectively counterbalance the steepening effects of nonlinearity, so the result is a solitary wave with compact support, or compacton.

In general, there are three distinct traveling wave solutions of (9.12.3). When $c_1 \neq 0$, the solutions represent waves that can be described by elliptic functions. When $c_1 = 0$, there exists a singular trajectory that describes a trigonometric wave solution with period 4π and its amplitude depends on the constant c_2. For $c_2 = 0$, the solution $u(x,t)$ is nonnegative and represents a series of compactons. In view of the degeneracy of $K(2,2)$ at $u = 0$, these compactons do not interact with each other, and therefore, can be separated.

It was also shown by Rosenau and Hyman (1993) that, for a class of general $K(m,n)$ equations, the compacton solution exists only for $1 < n \le 3$, and the singular dispersion at $u = 0$ plays a major role in the compactification. The upper limit $(n \le 3)$ is necessary for the existence of compacton solutions in the classical sense.

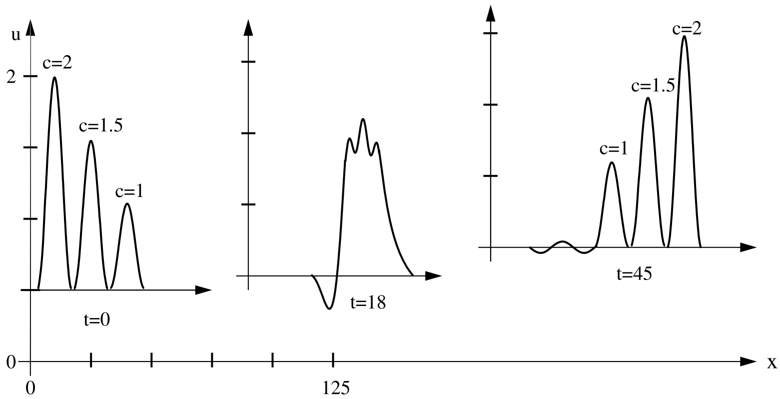

Fig. 9.16 The interaction of three $K(2,2)$ compactons with speeds $c = 2$, 1.5, and 1 starting with centers at $x = 10$, 15, and 40. Rosenau and Hyman (1993).

Based on hundreds of numerical experiments, Rosenau and Hyman (1993) have confirmed that, like solitons, two or more compactons physically interact with each other, and they always remain unchanged after collision except for a slight phase shift. Figure 9.16 exhibits the interaction of three compactons with speeds $c = 2$, 1.5, and 1 and their identities before and after collision.

It was shown by Oron and Rosenau (1989) that $K(m,n)$ type equations arise in the study of nonlinear dispersion in the formation of localized patterns in liquid drops. In their study of a nonlinear model describing new modes of motion of the free surface of a liquid, Ludu and Draayer (1998) demonstrated the existence of localized multiple patterns and nonlinear oscillations which include compactons, solitons, and cnoidal waves as traveling nonaxially symmetric shapes. Subsequently, Ludu et al. (2000) proposed a generalized similarity analysis of nonlinear dispersive equations to find a qualitative description of localized solutions. Their study reveals that compactons fulfill both characteristics of solitons and wavelets with possible new applications to the physics of droplets, bubbles, traveling patterns, fragmentation, fission, and inertial fusions. Dusuel et al. (1998) made an interesting analytical, numerical, and experimental study of physical systems modeled by a nonlinear Klein–Gordon equation with anharmonic coupling, and showed the existence of compactons. In a real physical system, they have also investigated the existence and stability of compactons and kinks consisting of a chain of identical pendulums that are nonlinearly coupled and experience a double-well on-site potential.

In general, compacton solutions of $K(m,n)$ equations for any $m \neq n$ are not yet known. We closely follow the method of solution due to Rosenau and Hyman (1993) and assume the general solution of the $K(n,n)$ equation given by (9.12.1a) in the form

$$u(x,t) = A\left[\sin\left\{k(x-ct)\right\}\right]^{\frac{2}{n-1}}, \tag{9.12.5}$$

or in the form

$$u(x,t) = A\left[\cos\left\{k(x-ct)\right\}\right]^{\frac{2}{n-1}}, \tag{9.12.6}$$

where A and k are constants to be determined.

Substituting these solutions into (9.12.1a) and solving the resulting equations for A and k yields

$$A = \begin{cases} \left(\frac{2nc}{a(n+1)}\right)^{\frac{1}{n-1}} & \text{if } n \text{ is even}, \\ \pm\left(\frac{2nc}{a(n+1)}\right)^{\frac{1}{n-1}} & \text{if } n \text{ is odd}, \end{cases} \tag{9.12.7ab}$$

and

$$k = \pm\frac{(n-1)}{2n}\sqrt{\frac{a}{b}}. \tag{9.12.8}$$

Consequently, the general compacton solutions are:

(i) For even n,

$$u(x,t) = \left[\sqrt{A}\sin\{k(x-ct)\}\right]^{\frac{2}{n-1}} H\big(k|x-ct| - 2n\pi\big) \tag{9.12.9}$$

and

$$u(x,t) = \left[\sqrt{A}\cos\{k(x-ct)\}\right]^{\frac{2}{n-1}} H\big(k|x-ct| - n\pi\big), \tag{9.12.10}$$

where $H(|x|-a) = 1$, for $|x| \le a$, and zero for $|x| > a$.

(ii) For odd n,

$$u(x,t) = \pm\left[\sqrt{A}\sin\{k(x-ct)\}\right]^{\frac{2}{n-1}} H\big(k|x-ct| - 2n\pi\big) \tag{9.12.11}$$

and

$$u(x,t) = \pm\left[\sqrt{A}\cos\{k(x-ct)\}\right]^{\frac{2}{n-1}} H\big(k|x-ct| - n\pi\big). \tag{9.12.12}$$

Similarly, we seek a solution of the one-dimensional *defocusing branch* of $K(n,n)$ equation (9.12.1b) in the form

$$u(x,t) = A\left[\sinh\{k(x-ct)\}\right]^{\frac{2}{n-1}}, \tag{9.12.13}$$

or

$$u(x,t) = A\left[\cosh\{k(x-ct)\}\right]^{\frac{2}{n-1}}, \tag{9.12.14}$$

where A and k are constants to be determined.

Substituting these solutions in (9.12.1b) and solving the resulting equations for A and k gives the solutions for the sinh-profile, where A and k are given by

$$A = \begin{cases} \left(\frac{2nc}{a(n+1)}\right)^{\frac{1}{n-1}} & \text{if } n \text{ is even}, \\ \pm\left(\frac{2nc}{a(n+1)}\right)^{\frac{1}{n-1}} & \text{if } n \text{ is odd}, \end{cases} \tag{9.12.15ab}$$

and

$$k = \pm\frac{(n-1)}{2n}\sqrt{\frac{a}{b}}. \tag{9.12.16}$$

For the cosh-profile, we obtain

$$A = \begin{cases} -\left(\frac{2nc}{a(n+1)}\right)^{\frac{1}{n-1}}, & \text{if } n \text{ is even}, \\ \pm\left(\frac{-2nc}{a(n+1)}\right)^{\frac{1}{n-1}} & \text{if } n \text{ is odd}. \end{cases} \tag{9.12.17ab}$$

Consequently, the general solutions are given as follows:

(i) For even n,

$$u(x,t) = \left[\sqrt{|A|}\sinh\{|k|(x-ct)\}\right]^{\frac{2}{n-1}} \tag{9.12.18}$$

and

$$u(x,t) = -\left[\sqrt{|A|}\cosh\{|k|(x-ct)\}\right]^{\frac{2}{n-1}}. \tag{9.12.19}$$

(ii) For odd n,

$$u(x,t) = \pm\left[\sqrt{|A|}\sinh\{|k|(x-ct)\}\right]^{\frac{2}{n-1}}, \quad c>0, \tag{9.12.20}$$

and

$$u(x,t) = \pm\left[\sqrt{-|A|}\cosh\{|k|(x-ct)\}\right]^{\frac{2}{n-1}}, \quad c<0. \tag{9.12.21}$$

With regard to the compactons, it has been shown by Rosenau and Hyman (1993) and Rosenau (1997) that equations $K(m,n)$ for $m,n=2,3$ admit a finite number of local conservation laws. Extensive numerical experiments for $m=n=2,3$ reveal that many of these compactons have a remarkable particle-like robustness that goes far beyond what could be expected from four local conservation laws. Probably, there exist nonlocal conservation laws which play an important role in compacton dynamics.

As an example of an application of compactons, we consider a vibration of an anharmonic mass–spring system consisting of N initially equally spaced $(h \ll 1)$ mass points m. The potential part of the associated Hamiltonian is

$$H = \sum_{n=1}^{N} \frac{1}{h}(y_{n+1}-y_n)P_n(y), \tag{9.12.22}$$

where $P_n(y) = \frac{1}{N}\alpha_N y^N$, α_N is an anharmonic parameter. For a mixed potential $P(y) = \frac{1}{2}\alpha_2 y^2 + \frac{1}{3}\alpha_3 y^3$, α_2 and α_3 are anharmonic parameters with small α_3. For the purely quartic potential, Rosenau (1994) obtained the nonlinear Boussinesq equation of motion in the continuum limit with $y_x = u$, $\varepsilon = \frac{1}{12}h^2$,

$$u_{tt} = \left(\alpha_3 u + \alpha_3 u^2\right)_{xx} + \varepsilon\alpha_2 u_{xxxx} + 2\varepsilon\alpha_3\left[q\left(\frac{1}{2}\right)\right]_{xx}, \tag{9.12.23}$$

where

$$q(\omega) = u^{1-\omega}\left(u^{\omega}u_x\right)_x. \tag{9.12.24}$$

Rosenau showed that equation (9.12.23) admits both compacton and usual soliton solutions. For the purely quartic potential in normalized units, the equation of motion becomes

$$u_{tt} = \left(u^3\right)_{xx} + \left[u\left(u^2\right)_{xx}\right]_{xx}. \tag{9.12.25}$$

This is clearly a purely cubic nonlinear dispersion equation and fundamentally different from the weakly nonlinear models in that it is nonlinear in the highest-order derivatives, $2u^2 u_{xxxx}$. Among other features, this equation also admits compacton solutions in the form $\sqrt{2}c\cos(x - ct)$. In addition, it also supports *compact breathers* of the form $u = Q(t)Z(x)$, where $Q(t)$ satisfies the nonlinear ordinary differential equation in the form

$$Q''(t) + \kappa^2 Q^3(t) = 0, \tag{9.12.26}$$

where κ is a separation constant. This equation gives the periodic Jacobi elliptic function solution

$$Q(t) = cn\left(\kappa t, \frac{1}{\sqrt{2}}\right). \tag{9.12.27}$$

The function $Z(x)$ satisfies the equation

$$\left[Z\left(Z^2\right)_{xx}\right]_{xx} + \left(Z^3\right)_{xx} + \kappa^2 Z = 0, \tag{9.12.28}$$

which admits the following compacton solution:

$$Z(x) = \begin{cases} \sqrt{8}\kappa\cos\left(\frac{1}{2}x\right) & \text{if } |x| \le \pi, \\ 0 & \text{otherwise.} \end{cases} \tag{9.12.29}$$

Although many properties of compacton solutions are not yet known, extensive numerical studies indicate that a compacton's smoothness at the edge is not informative of its stability. These numerical experiments also show that the low-order dispersion is unable to stabilize the compacton, which decomposes immediately into a series of waves.

The nonlinear model equation

$$u_t + \left[\delta u + \frac{3}{2}\gamma u^2 + q(\omega)\right]_x + \nu u_{txx} = 0, \tag{9.12.30}$$

where δ, γ, ω, and ν are constants, admits compacton solutions, and, for $2\omega = \nu\gamma = 1$, it has a bi-Hamiltonian structure. Rosenau (1994) also proved that the infinite sequence of commuting flows generates an integrable, compacton supporting variant of the Harry Dym equation. In summary, the equation governing the motion of a mass–spring system is a prototype of compacton generating equations. With appropriate scalings, the resulting nonlinear model can be applied to study the motion of ion-acoustic waves and the flow of a two-layer liquid. This model also admits compacton solutions.

We next discuss physical solid models that are inherently discrete where the lattice spacing represents a fundamental physical parameter. Such discrete models admit compacton solutions, that is, soliton solutions with finite wavelength. Soliton-type equations can be derived from such discrete models in which expansions of the wave amplitude and the inverse pulse width normally require a scaling procedure. In other words, the continuum limit approach produces the condition of the slowly varying wave envelope which is consistent with the effect of weak dispersion balanced by a weak nonlinearity. As soon as we deal with compactons instead of typical solitons, the continuum limit approximation can hardly be justified because higher-order derivative terms are numerically small.

We closely follow Kivshar (1993) to consider a one-dimensional lattice model in which each atom interacts with its nearest neighbors by purely *anharmonic forces*. If $u_n(t)$ is the nondimensional displacement function of the nth atom from its equilibrium position, and the atoms interact through quartic anharmonic potentials, the equation of motion for the nth atom is given by

$$\frac{d^2 u_n}{dt^2} = \left[(u_{n+1} - u_n)^3 + (u_{n-1} - u_n)^3\right], \tag{9.12.31}$$

where nondimensional units are employed.

In the continuum limit, the particle number is treated as a continuous variable, and the long wavelength excitation of the nonlinear model equation (9.12.31) can be written as

$$v_{tt} = \left(v^3\right)_{xx} + \cdots, \tag{9.12.32}$$

where $x = av$, $a(=1)$ is the space of the lattice, and $v = (u_{n+1} - u_n)$ is assumed to be a slowly varying function. For short wavelength excitations, the continuum limit approximation can be applied to the wave envelope $\phi_n(x,t)$ defined by the relation $u_n = (-1)^n \phi_n(x,t)$ so that equation (9.12.31) takes the form

$$\phi_{tt} + 16\phi^3 + 6\phi\left(\phi^2\right)_{xx} + \cdots = 0. \tag{9.12.33}$$

Using the method of solution due to Rosenau and Hyman (1993), equations (9.12.32) and (9.12.33) can be solved to describe compacton solution properties. However, these nonlinear evolution equations have higher-order dispersive terms that can be neglected because these terms are numerically small for constant-width solutions. Thus, these nonlinear discrete models seem to be natural models for the description of compacton solutions. We assume that ϕ_n is independent of time t and then seek standing oscillatory solutions of (9.12.31) in the form

$$u_n(t) = (-1)^n \phi_n F(t). \tag{9.12.34}$$

Substituting (9.12.34) into (9.12.31) gives two separable nonlinear equations in the form

$$\frac{d^2 F}{dt^2} + aF^3 = 0, \tag{9.12.35}$$

$$(\phi_{n+1} + \phi_n)^3 + (\phi_{n-1} + \phi_n)^3 = a\phi_n, \tag{9.12.36}$$

where a is a separation constant. Clearly, equation (9.12.35) admits the Jacobi elliptic function solution in the form

$$F(t) = A\,cn(\omega t, k), \tag{9.12.37}$$

where $\omega = \sqrt{a}A$, A is the amplitude, and $k = \frac{1}{\sqrt{2}}$.

Assuming a quasi-linear solution with finite wavelength, the method of Rosenau and Hyman (1993) can be used to seek a solution of (9.12.36) in the form

$$\phi_n = \begin{cases} \cos\{\theta(n-n_0)\} & \text{if } |(n-n_0)\theta| < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{9.12.38}$$

Substituting (9.12.38) into (9.12.37) gives two relations,

$$\tan^2\left(\frac{\theta}{2}\right) = \frac{1}{3}, \quad \text{that is,} \quad \theta = \frac{\pi}{3}, \quad \text{and} \quad a = \frac{27}{4}. \tag{9.12.39}$$

Consequently, the general compacton solution of the lattice equation (9.12.31) is given by

$$u_n(t) = \begin{cases} (-1)^n A\cos\{\theta(n-n_0)\}cn(\omega t, \frac{1}{\sqrt{2}}) & \text{if } |n-n_0| < \frac{3}{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{9.12.40}$$

If the amplitude of the compacton is taken as an independent parameter, the frequency ω of the compacton can be defined in terms of amplitude A by

$$\omega^2 = aA^2. \tag{9.12.41}$$

This is identified as the *nonlinear dispersion relation*.

It is evident that the arbitrary parameter n_0 represents the center of the compacton (9.12.40) so the $n_0 = 0$ corresponds to the compacton center at the particle site $(n = 0)$. The corresponding compacton pattern is shown in Figure 1(a) given by Kivshar (1993). With only three lattice spacings, the compacton mode involves only three neighboring particles oscillating with opposite phases. At $n_0 = 0$, the solution (9.12.40) can be rewritten as

$$u_n(t) = A\left(\ldots, 0, -\frac{1}{2}, 1, -\frac{1}{2}, 0, \ldots\right) cn\left(\omega t, \frac{1}{\sqrt{2}}\right). \tag{9.12.42}$$

This describes the mode pattern through the amplitude of the oscillating particles. On the other hand, when the compacton is centered just between the neighboring particle sites, that is, at $n_0 = \frac{1}{2}$, only two neighboring particles oscillate and the other remain at rest as shown in Figure 1(b) by Kivshar (1993). The mode pattern solution is obtained in the form

$$u_n(t) = \frac{\sqrt{3}}{2}A(\ldots, 0, -1, 1, 0, \ldots)\, cn\left(\omega t, \frac{1}{\sqrt{2}}\right), \tag{9.12.43}$$

where $\frac{\sqrt{3}}{2}A$ is used as a renormalized amplitude of this mode in order to conserve the total energy. Indeed, solution (9.12.40) describes an infinite family of different *localized modes* that are characterized by a particular value of $n_0 \in (0, \frac{1}{2})$. Such a compacton solution has been discovered for a chain of particles with quartic interatomic potentials, and can naturally be used to explain the existence of new intrinsic localized modes in anharmonic crystals. Indeed, in their pioneering work, Sievers and Takeno (1988) and Page (1990) discovered these new modes based on the rotating-wave approximation (RWA) in which only a single frequency component was included in the time dependence. More precisely, the model is described by the equation

$$m\ddot{w}_n = k_2(w_{n+1}+w_{n-1}-2w_n)+k_4\left[(w_{n+1}-w_n)^3+(w_{n-1}-w_n)^3\right], \quad (9.12.44)$$

where k_2 and k_4 are the nearest neighbor harmonic and anharmonic force constants. Using the RWA approximation with only the first harmonic contribution, Sievers and Takeno (1988) obtained what is called the odd-parity s-mode with the displacement function

$$w_n(t) = A\left(\ldots, 0, -\frac{1}{2}, 1, -\frac{1}{2}, 0, \ldots\right)\cos\Omega t, \quad (9.12.45)$$

where A is the amplitude and Ω is the frequency of the mode above the cutoff frequency $\Omega_m^2 = (4k_2/m)$ of the linear spectrum band. The solution (9.12.45) is, indeed, the approximate solution of (9.12.44) in the limit as $(k_4A^2/k_2) \to \infty$. Subsequently, Page (1990) discovered another type of intrinsic localized mode, the even-parity p-*mode* with the displacement function

$$w_n(t) = A(\ldots, 0, -1, 1, 0, \ldots)\cos\Omega t. \quad (9.12.46)$$

In the above limiting case $k_4A^2 \gg k_2$, the contribution of the nonlinear interaction between particles in the model (9.12.44) is much more significant than that of a linear coupling term, so that this model can be treated as model (9.12.31) for the displacement function $u_n = w_n\sqrt{k_4/m}$ which is perturbed by a small linear coupling term. That is why the approximate solutions (9.12.45) and (9.12.46) are very close to the exact solutions (9.12.42) and (9.12.43), respectively. It is pertinent to point out another striking feature of the localized modes in the model (9.12.44) compared to the compacton solution (9.12.40) for the purely anharmonic lattice model described by (9.12.31). Based on a perturbation theory, Sandusky et al. (1992) have demonstrated that the odd-parity s-mode is unstable against certain velocity and displacement perturbations, whereas the even-parity p-mode is absolutely stable against similar perturbations. For positive anharmonicity, both of these modes have amplitude-dependent frequencies above the maximum phonon frequency. Furthermore, the unstable odd-parity mode is observed to evolve into several different kinds of moving localized modes. For certain perturbations the odd-parity mode evolves into a mode which smoothly travels from site to site with a constant speed. These traveling modes exist over a wide range of anharmonicity and can become trapped as the anharmonicity increases. As they travel, these modes have a nonconstant phase difference between adjacent relative displacements. Based on the phenomenon of the *Peierls–Nabarro potential* to the localized mode, Claude et al. (1993) explained this instability of the s-mode. On the other hand, the existence of the exact compacton solution (9.12.40) with arbitrary n_0 suggests that the Peierls–Nabarro potential is absent for the compactons and they, therefore, move freely in the lattice provided the interatomic coupling is purely anharmonic in nature.

As has been demonstrated by Sievers and Takeno (1988), for sufficiently strong anharmonicity, stable *odd-parity* localized excitations are possible at any lattice site with a frequency given by

$$\omega^2 \approx \frac{3}{m}\left(k_2 + \frac{27}{16}k_4 A^2\right), \tag{9.12.47}$$

where m is the mass of the atom and A is the amplitude of oscillations of the central atom in the mode pattern (9.12.45). The above analysis also reveals that anharmonicity is fully responsible for the existence of the new intrinsic localized modes in anharmonic quantum crystals at finite temperature. Furthermore, the general compacton solution can describe well two new intrinsic localized modes obtained in the framework of the RWA approximation. Indeed, the compacton (9.12.40) gives the s-mode pattern when it is centered at the particle site, and it reproduces the p-mode pattern when the compacton (9.12.40) is centered in between the nearest particle sites.

We close this section by stating higher dimensional focusing branches $(+a)$ and defocusing branches $(-a)$ of $K(n,n)$ equations:

$$u_t + a\left(u^n\right)_x + b\left(u^n\right)_{xxx} + c\left(u^n\right)_{yyy} = 0, \tag{9.12.48}$$

$$u_t + a\left(u^n\right)_x + b\left(u^n\right)_{xxx} + c\left(u^n\right)_{yyy} + d\left(u^n\right)_{zzz} = 0, \tag{9.12.49}$$

$$u_t - a\left(u^n\right)_x + b\left(u^n\right)_{xxx} + c\left(u^n\right)_{yyy} = 0, \tag{9.12.50}$$

$$u_t - a\left(u^n\right)_x + b\left(u^n\right)_{xxx} + c\left(u^n\right)_{yyy} + d\left(u^n\right)_{zzz} = 0, \tag{9.12.51}$$

where $n > 1$ and $a > 0$.

For methods of solution of these equations, the reader is referred to Rosenau and Hyman (1993) and Wazwaz (2002).

9.13 The Camassa–Holm (CH) and Degasperis–Procesi (DP) Nonlinear Model Equations

From a physical point of view, it is evident that nonlinearity produces the steepening effects which are counterbalanced by the smoothing effects of dispersion. These effects play a major role in wave peaking and breaking and other physical features of wave phenomena including a variety of weakly singular patterns. In order to understand the major role of these effects, several strongly nonlinear and dispersive models have been developed without a full resolution of the problems, despite over 150 years of progress. Recently, Camassa and Holm (1993) and Camassa et al. (1994) first developed a new, strongly nonlinear, completely integrable model by using asymptotic expansions of the Euler equations for an inviscid incompressible fluid flow in the shallow water regime. Their model is governed by the nonlinear dispersive equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R},\ t > 0, \tag{9.13.1}$$

where $u(x,t)$ is the free surface elevation over a flat rigid bottom. In fact, Fokas and Fuchssteiner (1981) derived (9.13.1) much earlier as an abstract *bi*-Hamiltonian partial differential equation with infinitely many conservation laws (also see McKean

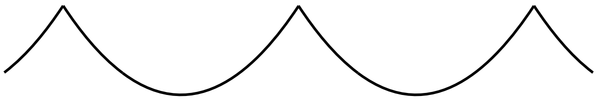

Fig. 9.17 A peakon, or singular solution.

2003). On the other hand, Johnson (2002) gave an alternative derivation of the CH model equation. In fact, Camassa and Holm discovered that their equation is formally integrable in the sense that there exists a Lax pair. Further, the CH model equation has solitary wave solutions which retain their shape and velocity after nonlinear interaction with waves of the same kind, and they are solitons. The most fundamental feature of the CH equation is that it has not only solutions that are global in time but also possesses wave breaking (singular) solutions. In other words, the CH equation has smooth solutions that develop singularities in finite time (or solutions blow-up in finite time). It can be shown that the regular peakon solutions of the CH equation are given by

$$u(x,t) = c\exp\big(-|x-ct|\big), \tag{9.13.2}$$

where c is the velocity of the wave. These are stable solutions (see Constantin and Strauss 2000), and are not classical solutions because they have a peak at their crests and are called *weak solutions*. These solutions are called *peakons*, as shown in Figure 9.17. Indeed, the CH solitary waves lead to breaking, the solution remains bounded but its slope becomes unbounded in finite time (see Constantin and Escher 1998a, 1998b, 1998c). It has recently been shown that the CH equation is locally well-posed in the Sobolev space $H^s(\mathbb{R})$ for $s > \frac{3}{2}$, with solutions depending continuously on initial data, and has global conservative solutions in $H^1(\mathbb{R})$.

The peakons have to be understood as weak solutions, as it is suitable to rewrite the CH equation (9.13.1) in the nonlocal conservation law form

$$(u_t + uu_x) + \partial_x\big(1-\partial_x^2\big)^{-1}\left(u^2 + \frac{1}{2}u_x^2\right) = 0. \tag{9.13.3}$$

Several authors including Dullin et al. (2003) dealt with the traveling wave solutions of a generalized version of the CH equation and found its solutions in an implicit form. Qiao and Zhang (2006) also discussed all possible explicit solutions of the CH equation (9.13.1) with the boundary condition $u \to A = \text{const.}$ as $|x| \to \infty$. When $A = 0$, they found the regular peakon solutions, and, when $A \neq 0$, both new peaked solitons and a new kind of smooth solitons that can be expressed in terms of trigonometric and hyperbolic functions. Using the traveling wave solution as $u(x,t) = \eta(\xi)$, $\xi = x - ct$, with c being the wave velocity, and substituting it in (9.13.1) yields

$$(\eta - c)\big(\eta - \eta''\big)' + 2\eta'\big(\eta - \eta''\big) = 0, \tag{9.13.4}$$

where $\eta' = \frac{d\eta}{d\xi}$.

The CH equation (9.13.1) has peakon solution of the form

$$u(x,t) = \eta(\xi) = c\exp\big(-|x-ct-\xi_0|\big),$$

where $\xi_0 = (x_0 - ct_0)$, $\eta(\xi_0) = c$, $\eta(\pm\infty) = 0$, $\eta'(\xi_0-) = c$, $\eta'(\xi_0+) = -c$, $\eta'(\xi_0-)$ and $\eta'(\xi_0+)$ denote the left-hand derivative and the right-hand derivative of η at ξ_0, respectively.

It can be verified that the CH equation (9.13.1) has the weak traveling wave solution

$$u(x,t) = c\exp\bigl(-|x-ct-\xi_0|\bigr) - \sinh\bigl(|x-ct-\xi_0|\bigr), \tag{9.13.5}$$

where d and ξ_0 are arbitrary real constants.

In particular, when $A = 0$, $d = 0$, and $\xi_0 = 0$, (9.13.5) reduces to the regular peakon solution (9.13.2) that was originally found by Camassa et al. (1994).

When $A \neq 0$, both new peaked solitons and a new kind of smooth solitons which can be expressed in terms of trigonometric and hyperbolic functions emerge.

On the other hand, Degasperis and Procesi (1999) discovered a new nonlinear dispersive equation, known as the *DP equation*, in the form

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R},\ t > 0. \tag{9.13.6}$$

Both CH and DP equations represent models for the propagation of nonlinear shallow water waves which capture the essential features of wave breaking phenomena. The solution of the DP equation is similar to that the CH equation. In particular, their solutions are singular, leading to wave breaking, that is, they are bounded, but their slopes become unbounded in finite time. Escher et al. (2006) examined weak global solutions and the blow-up feature of the DP equation. The asymptotic analysis and numerical simulations confirmed that the DP equation admits smooth solitons and cusp solitons.

Vakhnenko and Parkes (2004) discussed periodic and solitary-wave solutions of the DP equation, and Lenells (2005) investigated the traveling solitary wave solutions of the DP equation. On the other hand, Zhang and Qiao (2007) studied cusp soliton and smooth soliton solutions of the DP equation (9.13.6) with the non-homogeneous boundary conditions $u(x,t) \to A \neq 0$ as $|x| \to \infty$. They also found by direct calculation that the DP equation (9.13.6) admits a new stationary cusp soliton solution in the form

$$u(x,t) = \bigl\{1 - \exp\bigl(-2|x|\bigr)\bigr\}^{\frac{1}{2}} \in W^{1,1}_{\text{loc}}, \tag{9.13.7}$$

for any $x \neq 0$.

If $u \in H^1$, the DP equation (9.13.6) can equivalently be expressed into the following nonlocal conservation law form

$$L(u) = u_t + uu_x + \partial_x\bigl(1 - \partial_x^2\bigr)^{-1}\left(\frac{3}{2}u^2\right) = 0. \tag{9.13.8}$$

However, if $u \notin H^1$, (9.13.8) is no longer equivalent to the DP equation (9.13.6). The cusp soliton solution does not satisfy (9.13.8) because $L(u) = \operatorname{sgn} x \exp(-|x|)$. This means that it is impossible to find the cusp soliton solution of the DP equation (9.13.6) through its nonlocal conservation form (9.13.8).

The DP equation has not only peakon solitons (9.13.2), but also a shock peakon solution in the form

$$u(x,t) = -\frac{1}{(t+k)}\,\mathrm{sgn}(x)e^{-|x|}. \tag{9.13.9}$$

This represents a significantly different solution from that of the CH equation.

The shock-peakon solutions (see Lundmark 2007) can be observed by substituting $(x,t) \to (\varepsilon x, \varepsilon t)$ to (9.13.6) and letting $\varepsilon \to 0$ so that it yields the *inviscid derivative Burgers equation* as

$$(u_t + uu_x)_{xx} = 0, \tag{9.13.10}$$

which produces shock waves.

On the other hand, the celebrated KdV equation is an integrable model for nonlinear shallow water waves and the dispersive effect incorporated into the KdV model prevents breaking. In contrast to the KdV model, the CH and DP model equations capture the essential features of wave breaking. But they do not shed light on the *breaking process* or what happens *after breaking* which are the most fundamental features of water waves for which there appears to be no satisfactory mathematical, physical, or computational theory for a long period of time.

In recent years, Holm and Staley (2003) investigated the following family of $(1+1)$-dimensional nonlinear viscous equation for the fluid velocity $u(x,t)$ in the form

$$m_t + um_x + bu_x m = \nu m_{xx}, \tag{9.13.11}$$

where $u = g * m$ is the *convolution* defined by

$$u(x) = \int_{\mathbb{R}} g(x-\xi)m(\xi)\,d\xi, \tag{9.13.12}$$

which relates velocity u to momentum density m with the kernel $g(x)$ over $\mathbb{R}$. The kernel $g(x)$ is chosen to be the Green's function for the Helmholtz operator $(1-\partial_x^2)$ on the line, that is, $g(x) = \frac{1}{2}\exp(-|x|)$. This means that $m = (u - u_{xx})$. The family of equations (9.13.11) is characterized by the kernel $g(x)$ and the real nondimensional parameter b which is the ratio of the stretching term $(bu_x m)$ to the convective term (um_x) in (9.13.11). The parameter b is also the number of covariant dimensions associated with m, and $\nu\ (>0)$ is the viscosity coefficient associated with m. It is to be noted that the kernel $g(x)$ determines the traveling wave shape and length scale for equation (9.13.11), whereas the constant b represents a balance (or bifurcation) parameter for the nonlinear solution. The quadrative terms in (9.13.11) describe the balance in fluid convection between nonlinear transport and amplification due to b-dimensional stretching term. In a recent work on soliton dynamics, it is shown that equation (9.13.11) for $\nu = 0$ and $b \neq -1$ is included in the family of shallow water equations at quadratic order accuracy that are asymptotically equivalent under Kodama transformations (see Dullin et al. 2003).

In the absence of viscosity $(\nu = 0)$, equation (9.13.11) reduces to $(1+1)$-dimensional b-family equations in the form

$$m_t + um_x + bmu_x = 0, \quad u = g * m. \tag{9.13.13}$$

Using the method of asymptotic integrability, Degasperis and Procesi (1999) showed that equation (9.13.13) cannot be completely integrable unless $b = 2$ or $b = 3$. When $b = 2$, equation (9.13.13) becomes the CH equation (9.13.1). When $b = 3$, equation (9.13.13) reduces to the DP equation (9.13.6). It is shown by Dullin et al. (2003) that the DP equation can be derived from the shallow water elevation equation by an appropriate Kodama transformation. Lundmark and Szmigielski (2003) used the universe scattering method for finding n-peakon solutions of the DP equation, and Holm and Staley (2003) used numerical method to study stability of solitons and peakons. Recently, Gui et al. (2008) investigated the local well-posedness for the peakon b-family of equation (9.13.13) that includes both the CH and the DP equations as special cases and found the precise blow-up feature of strong solutions of (9.13.13) with certain initial conditions.

Among the singular entities, the peakon, a soliton with a finite discontinuity in gradients at its crests, is perhaps the weakest nonanalytic solution ever observed. The peakon solutions have been known for some time (see Fuchssteiner 1981). Camassa and Holm (1993) proved the integrability of equation

$$u_t - u_{xxt} = bu_x + 3uu_x - \left(uu_{xx} + \frac{1}{2}u_x^2\right)_x. \tag{9.13.14}$$

Even if $b = 0$ in (9.13.14), it admits peakon solutions $u(x,t) = c\exp(-|x+ct|)$ which are obtained as a solution of the equation

$$(c-u)\left(u_\xi^2 - u^2\right) = 0, \quad \xi = x + ct. \tag{9.13.15}$$

If $b \neq 0$, equation (9.13.14) yields soliton solutions which are analytic functions.

Another nonlinear model due to Camassa and Holm (1993) is described by the equation

$$(u \pm u_{xx})_t = bu_x + \frac{1}{2}\left[\left(u^2 \pm u_x^2\right)(u \pm u_{xx})\right]_x. \tag{9.13.16}$$

This equation also admits peakon solutions if $b > 0$. When $b < 0$, ordinary solitons emerge, but if $b = 0$, no solitons are possible. For a detailed discussion of this equation (9.13.16) and its peakon solutions, the reader is referred to Rosenau (1997).

It is important to note that the singular solutions of the Camassa–Holm equation look very similar to Stokes' wave of extreme height, but they only appear in the rest frame of reference. This equation is not Galilean invariant.

Another generalized version of the *Camassa–Holm equation* is

$$u_t - u_{xxt} + 3uu_x + 2\kappa u_x = 2u_x u_{xx} + uu_{xxx}, \tag{9.13.17}$$

where κ is a real constant. It has many conservation laws and it has peakon solution when $\kappa = 0$; and its solitary wave solution is stable for $\kappa > 0$. The local well-posedness, global existence, blow-up phenomena, and the well-posedness of global weak solutions of (9.13.17) have been discussed by many authors including Constantin and Escher (1998a, 1998b, 1998c), and Constantin and McKean (1999). Bressan and Constantin (2007) obtained the sharpest results for equation (9.13.17) for the global existence and blow-up solutions. Parker (2004, 2005a, 2005b) and Ohta et al. (2008) constructed soliton solutions of equation (9.13.17).

For $\kappa \neq 0$, the soliton solution of (9.13.17) for the initial value problem is represented graphically by Camassa and Holm (1993). Indeed, the solution for the Gaussian initial condition breaks up into a series of ordered solitons as time progresses. Thus, the soliton train eventually wraps around the periodic domain so that the leading solitons overtake the slower emergent solitons from behind after interaction that causes phase shifts. Hence, the CH equation (9.13.17) represents a model of unidirectional propagation of shallow water waves over a flat bottom. It also arises as a model equation for the axially symmetric waves in a hyperelastic rod. The analysis of the CH equation has provided a challenge and a source of inspiration for many new developments in nonlinear water waves, integrable systems, asymptotics, geometry and Lie groups.

Camassa and Holm (1993) discovered the peakon solitary traveling wave solution of (9.13.17) for nonlinear shallow water waves in the form

$$u(x,t) = c\exp\left[-\left(\frac{|x-ct|}{\alpha}\right)\right], \tag{9.13.18}$$

where the fluid velocity $u(x,t)$ is a function of position x in $\mathbb{R}$ and time t. The peakon solitary wave moves with a velocity equal to its maximum height, at which it has a sharp peak with jump in derivative. So, peakons can be obtained after solving the initial value problem for a partial differential equation derived by an asymptotic expansion of the Euler equations using the small parameters of shallow water wave dynamics. In fact, peakons are nonanalytic solitons that are the superposition of N-soliton solutions as

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{N} p_n(t)\exp\bigl[-\bigl|x-q_n(t)\bigr|/\alpha\bigr], \tag{9.13.19}$$

where the discrete sets $\{p_n(t)\}$ and $\{q_n(t)\}$ of peakon parameters satisfy the canonical Hamiltonian equations. They also arise in the absense of linear dispersion $(\kappa = 0)$ in (9.13.17). Each term in (9.13.19) is a soliton with a sharp peak at its maximum. Expressed using its momentum $m = (1-\alpha^2\partial_x^2)u$, the peakon velocity solution (9.13.19) of dispersionless CH equation becomes a sum of delta functions, supported on a set of points traveling on the real axis. In other words, the peakon velocity (9.13.19) implies

$$m(x,t) = \alpha\sum_{n=1}^{N} p_n(t)\delta\bigl(x-q_n(t)\bigr), \tag{9.13.20}$$

due to the fact that $(1-\alpha^2\partial_x^2)\exp(-\frac{|x|}{\alpha}) = 2\alpha\delta(x)$. These solutions satisfy (9.13.17) for the zero dispersion parameter $(\kappa = 0)$.

Substituting the peakon solution (9.13.19) and (9.13.20) into the dispersionless CH equation

$$m_t + u m_x + 2 m u_x = 0, \quad m = u - \alpha^2 u_{xx}, \tag{9.13.21}$$

gives the celebrated Hamilton's canonical equations for the discrete set of peakon parameters $p_n(t)$ and $q_n(t)$ as

$$\dot{q}_n(t) = \frac{\partial H_N}{\partial p_n} \quad \text{and} \quad \dot{p}_n(t) = -\frac{\partial H_N}{\partial q_n}, \tag{9.13.22}$$

where the Hamiltonian H_N is given by

$$H_N = \frac{1}{4} \sum_{n,m=1}^{N} p_n p_m \exp\left(-\frac{|q_n - q_m|}{\alpha} \right), \tag{9.13.23}$$

$n, m = 1, 2, \ldots, N$.

In their recent work of weakly nonlinear analysis combined with the variational principle for Euler equations, Holm and Staley (2003) made a two-dimensional generalization of the dispersionless CH equation (9.13.21). This generalization is known as the *Euler–Poincaré* (*EP*) *equation* for the Lagrangian consisting of the kinetic energy

$$K = \frac{1}{2} \int \left[|\mathbf{u}|^2 + \alpha^2 (\operatorname{div} \mathbf{u})^2 \right] dx\, dy, \tag{9.13.24}$$

where $\mathbf{u} = (u, v)$ is a two-dimensional fluid velocity vector. They have shown that evolution generated by the kinetic energy in Hamilton's principle results in geodesic motion with respect to the velocity norm $\|\mathbf{u}\|$ which is provided by the Lagrangian of the kinetic energy. The EP equation produced by any choice of kinetic energy norm without imposing incompressibility is called "*EPDiff*" for the *Euler–Poincaré equation for geodesic motion on the diffeomorphisms*. Thus, the EPDiff equation given by Holm and Staley (2003) is

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{m} + \nabla \mathbf{u}^T \cdot \mathbf{m} + \mathbf{m} (\operatorname{div} \mathbf{u}) = 0, \tag{9.13.25}$$

where $\mathbf{m} = \frac{\delta K}{\delta \mathbf{u}}$ is the momentum density, $K = \frac{1}{2} \|\mathbf{u}\|^2$ is the kinetic energy which defines a norm in the fluid velocity $\|\mathbf{u}\|$. Clearly, this equation has no contribution from either the pressure, or the potential energy. However, this equation conserves the velocity norm $\|\mathbf{u}\|$ given by the kinetic energy. In fact, its evolution describes geodesic motion on the diffeomorphisms with respect to this norm. There is an alternative way of expressing the EPDiff equation (9.13.25) in either two or three dimensions as

$$\frac{\partial \mathbf{m}}{\partial t} - \mathbf{u} \times \operatorname{curl} \mathbf{m} + \nabla (\mathbf{u} \cdot \mathbf{m}) + \mathbf{m} (\operatorname{div} \mathbf{u}) = 0. \tag{9.13.26}$$

Remarkably, all three differential operators—gradient, divergence, and curl—are present in this form of EPDiff equation. For the kinetic energy Lagrangian K given by (9.13.24) which is a norm for an *irrotational flow* (curl $\mathbf{u} = \mathbf{0}$), (9.13.25) is the EPDiff equation with the momentum $\mathbf{m} = \delta K/\delta \mathbf{u} = \mathbf{u} - \alpha^2 \nabla(\operatorname{div} \mathbf{u})$.

It is important to point out that the EPDiff equation (9.13.25) has many other interpretations beyond applications in fluid flows. However, the stability problem for EPDiff singular momentum solutions has not yet been solved, even through the stability of the peakon in one dimension was proved by Constantin and Strauss (2000).

On the other hand, Degasperis and Procesi (1999) derived another generalized form of the shallow water model equation

$$u_t - u_{xxt} + 2\kappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{9.13.27}$$

Like (9.13.17), Degasperis et al. (2002) proved that (9.13.27) is also completely integrable and has the Lax pair formulation. It admits an infinite number of conservation laws and its solitary waves interact like solitons (see Matsuno 2005a, 2005b).

Recently, Constantin and Lannes (2009) derived a generalized version of equations (9.13.17) and (9.13.18) in the form

$$u_t - u_x + \frac{3}{2}\varepsilon uu_x + \delta(\alpha u_{xxx} + \beta u_{xxt}) = \varepsilon\delta(au_x u_{xx} + buu_{xxx}), \tag{9.13.28}$$

where ε and δ are defined by (9.3.2ab), α, β, a, and b satisfy certain conditions. They also proved the large time well-posedness on a time scale $O(|\varepsilon|^{-1})$ provided the initial value u_0 belongs to H^s for $s > \frac{5}{2}$. They also discussed the wave breaking phenomena. Using suitable transformations, equation (9.13.28) can be transformed into the form

$$u_t - u_{xxt} + 2\kappa u_x + \alpha uu_x = au_x u_{xx} + buu_{xxx}, \tag{9.13.29}$$

where κ, α, a, and b are constants. Obviously, (9.13.29) is a generalization of both equations (9.13.17) and (9.13.27). The major difference between (9.13.29) for the case $a \neq 2b$ and the CH equation (9.13.17) is that equation (9.13.29) does not have the conservation law

$$I = \int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx \tag{9.13.30}$$

which plays a major in the investigation of the CH equation (9.13.17). Recently, Lai and Wu (2011) investigated local existence and uniqueness of solution for the generalized equation (9.13.29). They also proved the local well-posedness of solutions of (9.13.29) in the Sobolev space $H^s(\mathbb{R})$ for $s > \frac{3}{2}$.

We close this discussion by simply stating two other strongly nonlinear models, the nonlinear shallow water model, and the Harry Dym equation (see Kruskal 1975, Leo et al. 1983),

$$u_t = \left(u^{-\frac{1}{2}}\right)_{xxx}, \tag{9.13.31}$$

which arises as a generalization of the class of isospectral flows of the Schrödinger operator. They lead to essentially new scattering problems, which have not yet been fully explored. There is another version of Harry Dym equation,

$$u_t = \frac{1}{2}\left(D^3 - 4D\right)u^{-\frac{1}{2}}, \quad D \equiv \frac{\partial}{\partial x}, \tag{9.13.32}$$

with an extra transport term.

A recent literature search also suggests that there is a class of hierarchies containing the KdV hierarchy that are obtained from the isospectral problem

$$D^2 + k^2 u(x) - q(x). \tag{9.13.33}$$

These include the KdV flows, the negative KdV flows, the Harry Dym equation, and the Camassa–Holm equation (see Qiao 2003).

Almost all studies of steady and unsteady water waves are based on the assumption that flows are irrotational (that is, $\mathrm{curl}\,\mathbf{u} = \mathbf{0}$). In general, this is a good assumption. However, vorticity is generated at the free surface by wind stress or current, or by friction at the rigid bottom boundary. In fact, a very thin vorticity (or shear) layer is produced at the free surface or at the bottom boundary. In recent years, considerable attention has been given to study water waves by assuming for simplicity that vorticity is constant in the fluid flows. Simmen and Saffman (1985) first investigated numerical solutions for periodic waves in deep water with constant vorticity. Their study showed that the waves have either a limiting configuration with a $120°$ angle at the crests or a trapped bubble at their troughs.

Following Simmen and Saffman (1985), we consider a two-dimensional periodic waves in an inviscid incompressible fluid of infinite depth. The flow is assumed to be rotational and described by a constant vorticity, Ω. We take a frame of reference with the x-axis along the mean water level and in which the flow is steady with gravity is in the negative y-axis. We assume that the steady flow is symmetric with respect to the y-axis. The flow can be described in terms of a stream function $\psi(x,y)$ satisfying

$$\nabla^2 \psi = \Omega \tag{9.13.34}$$

in the flow domain.

We reduce the problem to one governed by the Laplace equation by subtracting a particular solution of (9.13.34). Thus, if we write

$$\psi = \Psi + \frac{\Omega}{2} y^2 - cy, \tag{9.13.35}$$

then

$$\nabla^2 \Psi = 0 \tag{9.13.36}$$

with the requirement that $\Psi \to 0$ as $y \to -\infty$. This defines the quantity c in (9.13.35) uniquely and c is referred to as the wave velocity. In terms of the dimensionless variables by choosing λ as the unit length and c as unit velocity, the dimensionless equation (9.13.35) becomes

$$\psi = \Psi + \frac{\omega}{2} y^2 - y, \tag{9.13.37}$$

where ω is the dimensionless vorticity defined by

$$\omega = \left(\frac{\Omega \lambda}{c}\right). \tag{9.13.38}$$

The function $w(z) = u - iv = \Psi_y + i\Psi_x$ is an analytic function of $z = x + iy$, and the fluid velocity vector is $(u + \omega y - 1, v)$. The function $w(z)$ vanishes at infinity.

Vanden-Broeck's (1996a, 1996b) numerical analysis first confirmed the results of Simmen and Saffman (1985). He then found that there are solution branches that bifurcate from the uniform shear flow $(u, v) = (0, 0)$. As we move along the solution branches, the waves ultimately attain limiting configurations with a $120°$ angle at the crests or a trapped bubble at their troughs. Furthermore, new families of solutions can also be constructed.

Several authors including Teles da Silva and Peregrine (1998), Pullin and Grimshaw (1988), and Vanden-Broeck (1994, 1995) obtained numerical solutions for solitary waves with constant vorticity. They formulated the problem in terms of a stream function ψ such that

$$\nabla^2\psi = -\omega, \tag{9.13.39}$$

where the flow is rotational and characterized by a constant vorticity Ω so that $\omega = (\Omega h/c)$ is the nondimensional vorticity, h is the depth of the fluid, and c is the wave velocity. These authors reduced the problem to the Laplace equation $\nabla^2\Psi = 0$ by subtracting a particular solution of (9.13.39), that is, $\psi = \Psi - \{\frac{\omega}{2}y^2 - (1 + \omega)y\}$. Thus, $w(z) = u - iv = \Psi_y + i\Psi_x$ is an analytic function of $z = x + iy$, where the fluid velocity vector is $\{u - \omega(y - 1) + 1, v\}$. They require the kinematic condition that $v = 0$ on the bottom by reflecting the flow in the bottom. The function $w(z)$ vanishes at infinity.

This problem is then numerically investigated using the flow variables nondimensional by choosing h as the unit length and c as the unit velocity. The flow is then described by the above parameter ω, and two other parameters

$$G = \left(\frac{gh}{c^2}\right) \quad \text{and} \quad a = \frac{\eta}{h}, \tag{9.13.40}$$

where η is the elevation of the wave crests above the level of the free surface. Clearly, G is the nondimensional gravitational parameter, and a is the amplitude parameter. When $\omega = 0$, the flow becomes irrotational. The above authors numerically examined solutions for various values of the parameters ω, G, and a. Their solitary wave solutions are solution branches that bifurcate from the trivial solution ($u = v = 0$ and $a = 0$) at the critical values $G = 1 + \omega$ which is obtained by Benjamin (1962). He studied asymptotic solutions for solitary waves of small amplitude. His analysis showed that, for each value of ω, there is a one-parameter family of solutions that bifurcates from the uniform shear flow at the critical value $G = 1 + \omega$. He also found an asymptotic solution for small values of the parameter a and obtained the following relation between G, ω, and a:

$$G = 1 + \omega - a\left(1 + \omega + \frac{\omega^2}{3}\right). \tag{9.13.41}$$

Vanden-Broeck (1994, 1995) and others generalized Benjamin's results for waves of finite amplitude. They obtained some solution branches which are not associated with Benjamin's solutions and have shown that their numerical values agree with

(9.13.41) as $a \to 0$. As they progress along the solution branches, there exists a critical value $\omega_c \approx -0.32$ of ω so that different limiting configurations occur for $\omega > \omega_c$ and $\omega < \omega_c$. For $\omega < \omega_c$, they found that the solution branches which bifurcate from a uniform shear flow at the critical value $G = 1 + \omega$ do not have limiting configuration with $120°$ angle at the crests. In the limit as $G \to 0$, the solution branches eventually tend to closed regions of constant vorticity in contact with the bottom boundary if G is allowed to become negative. For $\omega > \omega_c$, the solution branches that bifurcate from a uniform shear flow at the critical values $G = 1 + \omega$ have limiting configuration with $120°$ angle at their crests (see Teles Da Silva and Peregrine 1988 and Pullin and Grimshaw 1988).

As a concluding remark, it is necessary to point out that the boundary integral equation method is used by Vanden-Broeck (1996a, 1996b) to compute periodic waves with constant vorticity and to find the solution in the limit as $G \to 0$. This computational results reveal that there are solution branches that tend to configurations with a closed region of fluid in rigid body rotation as $G \to 0$. Further, there are solitary waves with circular closed regions at their crests. Since the solitary waves can be considered as the limit of periodic waves as $(\lambda/h) \to \infty$, Vanden-Broeck's (1996a, 1996b) numerical analysis strongly suggest that configurations with circular regions at the wave crests are a general feature of waves in water of finite depth.

9.14 Exercises

1. Show that $(xu + 3tu^2)$ is a conserved density for the KdV equation (9.7.1).
2. Find three conservation laws for the mKdV equation (Miura et al. 1968).

$$u_t - 6u^2 u_x + u_{xxx} = 0, \quad x \in \mathbb{R},$$

which involve u, u^2, and u^4, respectively.

3. Show that the three conservation laws for the BBM equation

$$u_t - uu_x - u_{xxt} = 0$$

are

$$\text{(i)} \quad u_t - \left(u_{xt} + \frac{1}{2}u^2\right)_x = 0,$$

$$\text{(ii)} \quad \frac{1}{2}\left(u^2 + u_x^2\right)_t - \left(uu_{xt} + \frac{1}{3}u^3\right)_x = 0,$$

and

$$\text{(iii)} \quad \left(\frac{1}{3}u^3\right)_t + \left(u_t^2 - u_{xt}^2 - u^2 u_{xt} - \frac{1}{4}u^4\right)_x = 0.$$

4. Show that the mKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

is invariant under the transformation

$$\tilde{x} = ax, \quad \tilde{t} = a^3 t, \quad \tilde{u} = a^{-1} u \quad (a \neq 0).$$

Hence introduce $u(x,t) = t^{-1/3} f(xt^{-1/3})$ to show that $f(\xi)$ satisfies the non-linear ordinary differential equation

$$f'' - \frac{1}{3}\xi f + 2f^3 = 0, \quad \xi = xt^{-1/3},$$

provided that $f \to 0$ at infinity.

5. Investigate a similarity solution of the cylindrical KdV equation

$$2u_t + \frac{1}{t}u - 3uu_x + \frac{1}{3}u_{xxx} = 0$$

in the form $u(x,t) = -\frac{1}{3}\left(\frac{2}{t^2}\right)^{\frac{1}{3}} f\{x(2t)^{-1/3}\}$. Show that $v(\xi)$ satisfies the equation

$$v'' - \xi v + v^3 = 0,$$

where $f = v^2$ and $\xi = x(2t)^{-1/3}$.

6. Show that the NLS equation

$$iu_t + u_{xx} + \gamma u|u|^2 = 0,$$

where γ is a real constant, is invariant under each of the group transformations

$$\text{(i)} \quad \tilde{x} = ax, \quad \tilde{t} = a^2 t, \quad \tilde{u} = a^{-1} u \quad (a \neq 0),$$
$$\text{(ii)} \quad \tilde{x} = x + a, \quad \tilde{t} = t, \quad \tilde{u} = u.$$

Show that a similarity solution of the form $u(x,t) = t^p f(\xi)$, $\xi = xt^q$ exists for suitable values of p and q. Hence, find the equation for $f(\xi)$.

7. Show that the Kadomtsev–Petviashvili equation (Freeman 1980)

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

has the soliton solution

$$u(x,y,t) = -\frac{1}{2}k^2 \operatorname{sech}^2\left[\frac{1}{2}(kx + \ell y - \omega t)\right],$$

where $\omega = k^3 + (3\ell^2/k)$.

8. Apply the inverse scattering transform method to solve the initial-value problem for the KdV equation (9.7.1) with $u(x,0) = 2\operatorname{sech}^2 x$. Hence, determine the time development of the solution from the above initial data by numerical integration with a program given by Crandall (1991).

9. Use the Bäcklund transformations

$$v_x = u - v^2 \quad \text{and} \quad v_t = -u_{xx} + 2(uv_x + u_x v)$$

to show that u satisfies the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

and that v satisfies the mKdV equation

$$v_t - 6v^2 v_x + v_{xxx} = 0.$$

10. Apply the Bäcklund transformations

$$v_x = -\frac{1}{2\nu}(uv) \quad \text{and} \quad v_t = \frac{v}{4\nu}\left(u^2 - 2\nu u_x\right)$$

to show that u and v satisfy the Burgers and diffusion equations

$$u_t + uu_x = \nu u_{xx} \quad \text{and} \quad v_t = \nu u_{xx}.$$

11. Show that the Kadomtsev and Petviashivili (1970) equation

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

can be derived by selecting the Lax pair

$$L = -\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y} + u \quad \text{and} \quad M = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x + 3\int^x u_y\, dx.$$

12. Show that the operator M for the KdV equation is

$$M = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x}.$$

13. Using *Mathematica*, or otherwise, show that the Lax equation (9.9.6) with

$$L = \frac{\partial^3}{\partial x^3} + u\frac{\partial}{\partial x},$$
$$M = 9\frac{\partial^5}{\partial x^5} + 15u\frac{\partial^3}{\partial x^3} + 15\frac{\partial u}{\partial x}\frac{\partial}{\partial x} + \left(5u^2 + 10\frac{\partial^2 u}{\partial x^2}\right)\frac{\partial}{\partial x}$$

reduces to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^5 u}{\partial x^5} + 5u\frac{\partial^3 u}{\partial x^3} + 5\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x^2} + 5u^2\frac{\partial u}{\partial x}.$$

For $q = ar - 1$, and matrices A and B given by (9.9.41) and (9.9.45), show that (9.9.40) leads to the nonlinear equation

$$r_t + r_{xxx} - 6rr_x + 6ar^2 r_x = 0.$$

14. If the operator M in the Lax equation (9.9.15) is

$$M = -\alpha D^{2n+1} + \sum_{m=1}^{n} \left(U_m D^{2m-1} + D^{2m-1} U_m \right) + A(t),$$

where α is a constant, $u_m = u_m(x,t)$, and $A(t)$ is an arbitrary function, show that the Lax equation reduces to the KdV equation (9.9.21) for $n = 1$, and, for $n = 2$, it becomes the fifth-order KdV equation

$$u_t + 30u^2(Du) - 20(Du)\left(D^2 u\right) - 10u\left(D^3 u\right) + D^5 u = 0.$$

15. If the phase velocity of nonlinear water waves is

$$c(k) = \left[\left(\frac{g}{k} \right) \tanh kh \right]^{\frac{1}{2}} = \sqrt{gh} \left\{ 1 - \frac{1}{6} k^2 h^2 + o\left(k^2 h^2\right) \right\} \quad \text{as } kh \to 0,$$

show that these waves are described by the Whitham equation

$$u_t + u u_x + \int_{-\infty}^{\infty} K(x - \xi) \left(\frac{\partial u}{\partial \xi} \right) d\xi = 0,$$

where the kernel K is determined from the linear theory as the Fourier transform of $c(k)$,

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) \exp(ikx)\, dk.$$

16. Show that the Boussinesq equation

$$u_{tt} - u_{xx} + 3\left(u^2\right)_{xx} - u_{xxxx} = 0$$

has the following conservation laws:
(a) $\int_{-\infty}^{\infty} u\, dx =$ const. (conservation of mass),
(b) $\int_{-\infty}^{\infty} u_t\, dx =$ const. (conservation of momentum).

17. Show that the solitary wave solution of the Boussinesq equation

$$u_{tt} - u_{xx} + 3\left(u^2\right)_{xx} - u_{xxxx} = 0$$

is

$$u(x,t) = a \operatorname{sech}^2 \left[b(x - ct) + d \right],$$

for suitable relations between the constants a, b, c, and d. Verify that the Boussinesq wave propagates in either direction.

18. Show that the KP equation

$$(u_t - 6uu_x + u_{xxxx})_x + 3u_{yy} = 0$$

has the solitary wave solution

$$u(x,y,t) = a \operatorname{sech}^2(kx + ly - \omega t + \alpha)$$

for suitable relations between the constants a, k, l, ω, and α.

19. Show that the KdV equation

$$2u_t + 3uu_x + \frac{1}{3}u_{xxx} = 0$$

has the following conservation laws (see Johnson 1997):
(a) $\int_{-\infty}^{\infty} u\,dx = \text{const.}$ (conservation of mass),
(b) $\int_{-\infty}^{\infty} (u^3 - \frac{1}{3}u_x^2)\,dx = \text{const.}$ (conservation of energy),
(c) $\int_{-\infty}^{\infty} u^2\,dx = \text{const.}$,
(d) $\int_{-\infty}^{\infty} (\frac{45}{4}u^4 - 15uu_x^2 + u_{xx}^2)\,dx = \text{const.}$

20. (a) Show that the CH equation (9.13.1) can be expressed in the nonlocal conservation law form (see Bressan and Constantin 2007)

$$u_t + \left(\frac{u^2}{2}\right)_x + \partial_x\left(1 - \partial_x^2\right)^{-1}\left(u^2 + \frac{1}{2}u_x^2\right) = 0.$$

(b) Verify that the equation in Exercise 20(a) is equivalent to

$$u_t + \left(\frac{u^2}{2}\right)_x + P_x = 0,$$

where P is defined as a convolution

$$P = \frac{1}{2}\exp(-|x|) * \left(u^2 + \frac{1}{2}u_x^2\right).$$

(c) Differentiate the equation in Exercise 20(b) with respect to x to obtain

$$u_{xt} + uu_{xx} + u_x^2 - \left(u^2 + \frac{1}{2}u_x^2\right) + P = 0.$$

(d) Derive two conservation laws with source term

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3} + uP\right)_x = u_x P,$$

$$\left(\frac{u_x^2}{2}\right)_t + \left(\frac{uu_x^2}{2} - \frac{u^3}{3}\right)_x = -u_x P.$$

Hint: Multiply the equation in Exercise 20(b) by u and the equation in Exercise 20(c) by u_x to derive the two conservation law.
(e) Show that the total energy for regular solutions is

$$E(t) = \int_{\mathbb{R}} \left(u^2 + u_x^2\right)\,dx,$$

is constant in time.

21. (a) Consider the Cauchy problem for the dispersive DP equation with $m = u - u_{xx}$ (see Guo 2009):

$$m_t + um_x + 3u_x m + \gamma m_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$m(x,0) = u_0(x) - u_{0xx}(x),$$

where γ is a real constant. If $p(x) = \frac{1}{2}\exp(-|x|)$, $x \in \mathbb{R}$, show that

$$\left(1 - \partial_x^2\right)^{-1} f = p * f \quad \text{for all } f \in L^2(\mathbb{R}),$$

and

$$p * m = u.$$

Using these results, verify that the Cauchy problem can be written in the form

$$u_t + uu_x + \partial_x p * \left(\frac{3}{2}u^2\right) + \gamma u_x = 0, \quad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = u_0(x).$$

(b) Show that the Cauchy problem in Exercise 21(a) with the initial data $u_0(x)$ has a unique global solution $u(x,t)$ which is the sum of a peakon and an antipeakon solution in the form

$$u(x,t) = p_1(t)\exp\left(-\left|x - q_1(t)\right|\right) + p_2(t)\exp\left(-\left|x - q_2(t)\right|\right),$$

for some p_i, $q_i \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$, $i = 1, 2$.

22. Show that $u(x,t) = (c-\gamma)\exp(-|x - ct|)(c \neq \gamma)$ is a global weak solution of the dispersive DP equation given in Exercise 21(a).

10

The Nonlinear Schrödinger Equation and Solitary Waves

. . . Schrödinger and I both had a very strong appreciation of mathematical beauty, and this appreciation of mathematical beauty dominated all our work. It was a sort of act of faith with us that any equations which describe fundamental laws of Nature must have great mathematical beauty in them. It was like a religion with us. It was a very profitable religion to hold, and can be considered the basis of much of our success.

Paul Dirac

. . . the progress of physics will to a large extent depend on the progress of nonlinear mathematics, of methods to solve nonlinear equations . . . and therefore we can learn by comparing different nonlinear problems.

Werner Heisenberg

10.1 Introduction

It has already been indicated in Section 2.3 that the nonlinear Schrödinger (NLS) equation arises in a wide variety of physical problems in fluid mechanics, plasma physics, and nonlinear optics. The most common applications of the NLS equation include self-focusing of beams in nonlinear optics, modeling of propagation of electromagnetic pulses in nonlinear optical fibers which act as wave guides, and stability of Stokes waves in water. Some formal derivations of the NLS equation have been obtained by several methods which include the multiple scales expansions, the asymptotic method, Whitham's (1965a, 1965b) averaged variational equations, and Phillips' (1981) resonant interaction equations. Zakharov and Shabat (1972) developed an ingenious inverse scattering method to show that the NLS equation is completely integrable. The NLS equation is of great importance in adding to our fundamental knowledge of the general theory of nonlinear dispersive waves.

It is now well known that the Korteweg–de Vries equation and the nonlinear Schrödinger equation are the lowest-order nontrivial consequences of a perturbation

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approximation for weakly and strongly nonlinear dispersive wave systems, respectively. This chapter is devoted to the nonlinear Schrödinger equation and its solitary wave solutions. Several properties of the nonlinear Schrödinger equation including envelope solitons and recurrence phenomenon are discussed. Attention is given to conservation laws for the nonlinear Schrödinger equation. Some examples of applications to fluid dynamics, plasma physics, and nonlinear optics are included. Based on an ingenious method of Zakharov and Shabat (1972), the inverse scattering transform for the nonlinear Schrödinger equation is described in some detail.

10.2 The One-Dimensional Linear Schrödinger Equation

We consider the following Fourier integral representation of a quasi-monochromatic plane wave solution:

$$\phi(x,t) = \int_{-\infty}^{\infty} F(k) \exp\big[i\{kx - \omega(k)t\}\big]\, dk, \tag{10.2.1}$$

where the spectrum function $F(k)$ is determined from the given initial or boundary conditions and has the property $F(-k) = F^*(k)$, and $\omega = \omega(k)$ is the dispersion relation. We assume that the initial wave is slowly modulated as it propagates in a dispersive medium. For such a quasi-monochromatic wave, most of the energy is confined in the neighborhood of a specified wavenumber $k = k_0$, so that the spectrum function $F(k)$ has a sharp peak around k_0 with a narrow wavenumber width $(k - k_0) = \delta k = O(\varepsilon)$, and the dispersion relation $\omega(k)$ can be expanded about k_0 as follows:

$$\omega = \omega_0 + (\delta k)\omega_0' + \frac{1}{2}(\delta k)^2 \omega_0'' + \cdots, \tag{10.2.2}$$

where $\omega_0 = \omega(k_0)$, $\omega_0' = \omega'(k_0)$, and $\omega_0'' = \omega''(k_0)$.

Substituting (10.2.2) in (10.2.1) gives

$$\phi(x,t) = \psi(x,t) \exp\big[i\{k_0 x - \omega_0 t\}\big] + c.c., \tag{10.2.3}$$

where $c.c.$ stands for the complex conjugate and $\psi(x,t)$ is the complex amplitude defined by

$$\psi(x,t) = \int_0^{\infty} F(k_0 + \delta k) \exp\left\{ i(x - \omega_0' t)\delta k - \frac{1}{2} i\omega_0''(\delta k)^2 t \right\} d(\delta k), \tag{10.2.4}$$

where it has been assumed that $\omega(-k) = -\omega(k)$. Since (10.2.4) depends on $(x - \omega_0' t)\delta k$ and $(\delta k)^2 t$ where $\delta k = O(\varepsilon)$ is small, the amplitude $\psi(x,t)$ is a slowly varying function of $x^* = (x - \omega_0' t)$ and t. We next introduce slow variables ξ and τ defined by

$$\xi = \varepsilon(x - \omega_0' t) \quad \text{and} \quad \tau = \varepsilon^2 t, \tag{10.2.5ab}$$

so that the wave field assumes the form

$$\phi(x,t) = A(\xi,\tau)\exp\big[i(k_0x - \omega_0 t)\big] + c.c., \tag{10.2.6}$$

where the modulated wave amplitude is given by

$$\begin{aligned} A(\xi,t) &= \int F(k_0+\delta k)\exp\left[i\left\{\delta k(x-\omega_0' t) - \frac{1}{2}\omega_0''(\delta k)^2 t\right\}\right] d(\delta k) \\ &= \int F(k_0+\varepsilon)\exp\left[i\left(\xi - \frac{1}{2}\omega_0''\tau\right)\right] d(\varepsilon). \end{aligned} \tag{10.2.7}$$

In the expression for the wave field (10.2.6), k_0, and hence, $\omega_0 = \omega(k_0)$ are chosen to be constants, and all slow variations of the wavetrain are included in $A(\xi,\tau)$.

A simple computation gives

$$A_\tau = -\frac{1}{2}i\omega_0''A \quad \text{and} \quad A_{\xi\xi} = -A \tag{10.2.8ab}$$

so that the modulated wave amplitude $A(\xi,\tau)$ satisfies the *linear Schrödinger equation*

$$iA_\tau + \frac{1}{2}\omega_0''A_{\xi\xi} = 0. \tag{10.2.9}$$

10.3 The Nonlinear Schrödinger Equation and Solitary Waves

We show below that the nonlinear modulation of a quasi-monochromatic wave is described by the nonlinear Schrödinger equation. To take into account the nonlinearity and the modulation in the far-field approximation, the wavenumber k and frequency ω in the linear dispersion relation are replaced by $k - i\frac{\partial}{\partial x}$ and $\omega + i\frac{\partial}{\partial t}$, respectively. It is convenient to use the nonlinear dispersion relation in the form

$$D\left(k - i\frac{\partial}{\partial x},\ \omega + i\frac{\partial}{\partial t},\ |A|^2\right)A = 0. \tag{10.3.1}$$

We consider the case of a weak nonlinearity and a slow variation of the amplitude, and hence, the amplitude A is assumed to be a slowly varying function of space and time. We next expand (10.3.1) with respect to $|A|^2$, $-i\frac{\partial}{\partial x}$, and $i\frac{\partial}{\partial t}$ to obtain

$$\begin{aligned} D(k,\omega,0) &- i\left(D_k\frac{\partial}{\partial x} - D_\omega\frac{\partial}{\partial t}\right)A \\ &- \frac{1}{2}\left(D_{kk}\frac{\partial^2}{\partial x^2} - 2D_{k\omega}\frac{\partial^2}{\partial x\partial t} + D_{\omega\omega}\frac{\partial^2}{\partial t^2}\right)A \\ &+ \frac{\partial D}{\partial |A|^2}|A|^2A = 0, \end{aligned} \tag{10.3.2}$$

where the first term $D(k,\omega,0) = 0$ is due to the linear dispersion equation.

Introducing the transformation $x^* = x - C_g t$, $t^* = t$, assuming that $A = O(\varepsilon)$, $\frac{\partial}{\partial x^*} = O(\varepsilon)$, and $\frac{\partial}{\partial t^*} = O(\varepsilon^2)$, retaining all terms up to $O(\varepsilon^3)$, and dropping the asterisks, we find that

$$i\frac{\partial A}{\partial t}+p\frac{\partial^2 A}{\partial x^2}+q|A|^2A=0, \tag{10.3.3}$$

where

$$p=\frac{1}{2}\left(\frac{dC_g}{dk}\right), \qquad q=\frac{1}{D_\omega}\left(\frac{\partial D}{\partial|A|^2}\right), \tag{10.3.4}$$

and the following results:

$$C_g=-\frac{D\omega}{Dk}, \qquad \frac{dC_g}{dk}=\left(D_{kk}+2C_gD_{\omega k}+C_g^2D_{\omega\omega}\right)/D_\omega, \tag{10.3.5}$$

have been used.

Equation (10.3.3) is known as the *nonlinear Schrödinger* (*NLS*) *equation*. More explicitly, if the nonlinear dispersion relation is given by

$$\omega=\omega\left(k,a^2\right), \tag{10.3.6}$$

and if we expand ω in a Taylor series about $k=k_0$ and $|a|^2=0$, we obtain

$$\omega\approx\omega_0+(k-k_0)\omega_0'+\frac{1}{2}(k-k_0)^2\omega_0''+\left(\frac{\partial\omega}{\partial|a|^2}\right)_{|a|^2=0}|a|^2. \tag{10.3.7}$$

Replacing $(\omega-\omega_0)$ by $i(\frac{\partial}{\partial t})$, $k-k_0$ by $-i(\frac{\partial}{\partial x})$, and assuming that the resulting operators act on the amplitude function $a(x,t)$, it turns out that

$$i(a_t+\omega_0'a_x)+\frac{1}{2}\omega_0''a_{xx}+\gamma|a|^2a=0, \tag{10.3.8}$$

where

$$\gamma=-\left(\frac{\partial\omega}{\partial|a|^2}\right)_{|a|^2=0}=\text{const.} \tag{10.3.9}$$

Equation (10.3.8) is known as the *nonlinear Schrödinger equation*. If we choose a frame of reference moving with the linear group velocity ω_0', that is, $\xi=x-\omega_0't$ and $\tau=t$, the term involving $a_x$will drop out from (10.3.8), and therefore, the amplitude $a(x,t)$ satisfies the normalized nonlinear Schrödinger equation

$$ia_\tau+\frac{1}{2}\omega_0''a_{\xi\xi}+\gamma|a|^2a=0. \tag{10.3.10}$$

The corresponding dispersion relation is given by

$$\omega=\frac{1}{2}\omega_0''k^2-\gamma a^2. \tag{10.3.11}$$

According to the stability criterion established by Whitham (1974), the wave modulation is stable if $\gamma\omega_0''<0$ or unstable if $\gamma\omega_0''>0$.

To study the solitary wave solution, it is convenient to use the NLS equation in the standard form

$$i\psi_t+\psi_{xx}+\gamma|\psi|^2\psi=0, \quad -\infty<x<\infty,\ t>0. \tag{10.3.12}$$

We then seek waves of permanent form by assuming the solution

$$\psi = f(X)e^{i(mX-nt)}, \quad X = x - Ut, \tag{10.3.13}$$

for some functions f and constant wave speed U to be determined, and where m, n are constants.

Substitution of (10.3.13) in (10.3.12) gives

$$f'' + i(2m - U)f' + \left(n - m^2\right)f + \gamma|f|^2 f = 0. \tag{10.3.14}$$

We eliminate f' by setting $2m - U = 0$, and then write $n = m^2 - \alpha$, so that f can be assumed real. Thus, equation (10.3.14) becomes

$$f'' - \alpha f + \gamma f^3 = 0. \tag{10.3.15}$$

Multiplying this equation by $2f'$ and integrating, we find that

$$f'^2 = A + \alpha f^2 - \frac{\gamma}{2} f^4 \equiv F(f), \tag{10.3.16}$$

where $F(f) \equiv (\alpha_1 - \alpha_2 f^2)(\beta_1 - \beta_2 f^2)$, so that $\alpha = -(\alpha_1\beta_2 + \alpha_2\beta_1)$, $A = \alpha_1\beta_1$, $\gamma = -2(\alpha_2\beta_2)$, and the α's and β's are assumed to be real and distinct.

Evidently, it follows from (10.3.16) that

$$X = \int_0^f \frac{df}{\sqrt{(\alpha_1 - \alpha_2 f^2)(\beta_1 - \beta_2 f^2)}}. \tag{10.3.17}$$

Setting $(\alpha_2/\alpha_1)^{1/2} f = u$ in this integral, we deduce the following elliptic integral of the first kind (Dutta and Debnath 1965):

$$\sigma X = \int_0^f \frac{df}{\sqrt{(1 - u^2)(1 - \kappa^2 u^2)}}, \tag{10.3.18}$$

where $\sigma = (\alpha_2\beta_1)^{1/2}$ and $\kappa = (\alpha_1\beta_2)/(\beta_1\alpha_2)$.

Thus, the final solution can be expressed in terms of the Jacobian elliptic function

$$u = sn(\sigma X, \kappa). \tag{10.3.19a}$$

Thus, the solution for $f(X)$ is given by

$$f(X) = \left(\frac{\alpha_1}{\alpha_2}\right)^{1/2} sn(\sigma X, \kappa). \tag{10.3.19b}$$

In particular, when $A = 0$, $\alpha > 0$, and $\gamma > 0$, we obtain a solitary wave solution. In this case, (10.3.16) can be rewritten

$$\sqrt{\alpha} X = \int_0^f \left\{ f^2\left(1 - \frac{\gamma}{2\alpha} f^2\right)\right\}^{-\frac{1}{2}} df. \tag{10.3.20}$$

Substitution of $(\gamma/2\alpha)^{1/2} f = \operatorname{sech}\theta$ in this integral gives the exact solution

$$f(X) = \left(\frac{2\alpha}{\gamma}\right)^{1/2} \operatorname{sech}\left[\sqrt{\alpha}(x - Ut)\right]. \tag{10.3.21}$$

This represents a solitary wave that propagates without change of shape with constant velocity U. Unlike the solution of the KdV equation, the amplitude and the velocity of the wave are independent parameters. It is noted that the solitary wave exists only for the unstable case $(\gamma > 0)$. This means that small modulations of the unstable wavetrain lead to a series of solitary waves.

The well-known nonlinear dispersion relation for deep water waves is

$$\omega = \sqrt{gk}\left(1 + \frac{1}{2}a^2k^2\right). \tag{10.3.22}$$

Therefore,

$$\omega_0' = \frac{\omega_0}{2k_0}, \qquad \omega_0'' = \frac{\omega_0}{4k_0^2}, \quad \text{and} \quad \gamma = -\frac{1}{2}\omega_0 k_0^2, \tag{10.3.23}$$

and the NLS equation for deep water waves is obtained from (10.3.8) in the form

$$i\left(a_t + \frac{\omega_0}{2k_0}a_x\right) - \frac{\omega_0}{8k_0^2}a_{xx} - \frac{1}{2}\omega_0 k_0^2 |a|^2 a = 0. \tag{10.3.24}$$

The normalized form of this equation in a frame of reference moving with the linear group velocity ω_0' is

$$ia_t - \left(\frac{\omega_0}{8k_0^2}\right)a_{xx} - \frac{1}{2}\omega_0 k_0^2 |a|^2 a = 0. \tag{10.3.25}$$

Since $\gamma\omega_0'' = (\omega_0^2/8) > 0$, this equation confirms the instability of deep water waves. This is one of the most remarkable recent results in the theory of water waves.

We next discuss the uniform solution and the solitary wave solution of the NLS equation (10.3.25). We look for solutions in the form

$$a(x,t) = A(X)\exp\left(i\gamma^2 t\right), \quad X = x - \omega_0' t, \tag{10.3.26}$$

and substitute this in equation (10.3.25) to obtain the following equation:

$$A_{XX} = \frac{8k_0^2}{\omega_0}\left(\gamma^2 A + \frac{1}{2}\omega_0 k_0^2 A^3\right). \tag{10.3.27}$$

We multiply this equation by $2A_X$ and, then, integrate to find

$$A_X^2 = -\left(A_0^4 m'^2 + \frac{8}{\omega_0}\gamma^2 k_0^2 A^2 + 2k_0^4 A^4\right) = \left(A_0^2 - A^2\right)\left(A^2 - m'^2 A_0^3\right), \tag{10.3.28}$$

where $A_0^4 m'^2$ is an integrating constant, $2k_0^4 = 1$, $m'^2 = 1 - m^2$, and $A_0^2 = 4\gamma^2/\omega_0 k_0^2(m^2 - 2)$, which relates A_0, γ, and m.

Finally, we rewrite (10.3.28) as

$$A_0^2\,dX = \frac{dA}{[(1-\frac{A^2}{A_0^2})(\frac{A^2}{A_0^2}-m'^2)]^{1/2}}, \tag{10.3.29a}$$

or equivalently,

$$A_0(X-X_0) = \int^t \frac{ds}{[(1-s^2)(s^2-m'^2)]^{1/2}}, \quad s=(A/A_0). \tag{10.3.29b}$$

This can readily be expressed in terms of the Jacobi *dn* function (Dutta and Debnath 1965):

$$A = A_0\,dn\big[A_0(X-X_0), m\big], \tag{10.3.30}$$

where m is the modulus of the dn function.

In the limit, $m \to 0$, $dnz \to 1$, and $\gamma^2 \to -\frac{1}{2}\omega_0 k_0^2 A_0^2$. Hence, the solution becomes

$$a(x,t) = A(t) = A_0 \exp\left(-\frac{1}{2} i\omega_0 k_0^2 A_0^2 t\right). \tag{10.3.31}$$

On the other hand, when $m \to 1$, $dnz \to \operatorname{sech} z$, and $\gamma^2 \to -\frac{1}{4}\omega_0 k_0^2 A_0^2$. Therefore, the solitary wave solution is

$$a(x,t) = A_0 \exp\left(-\frac{i}{4}\omega_0 k_0^2 A_0^2 t\right) \operatorname{sech}\big[A_0(x-\omega_0' t - X_0)\big]. \tag{10.3.32}$$

10.4 Properties of the Solutions of the Nonlinear Schrödinger Equation

We discuss several important properties of the nonlinear Schrödinger equation in the form

$$iu_\tau + \beta u_{\xi\xi} + \gamma |u|^2 u = 0, \tag{10.4.1}$$

where β and γ are real constants. When $\beta = \frac{1}{2}\omega_0''$, this equation reduces to (10.3.10).

Simple solutions of equation (10.4.1) can always be obtained. However, the nature of the solutions depends on the signs of β and γ. If they are of the same sign, that is, if $\beta\gamma > 0$, the solution of (10.4.1), which tends to zero as $|\xi| \to \infty$, represents a solitary wave solution

$$u(\xi,t) = \sqrt{-\frac{2\nu}{\gamma}}\,\operatorname{sech}\left[\sqrt{\frac{-\nu}{\beta}}\,\xi\right] \exp(-i\nu\tau), \tag{10.4.2}$$

where ν is a constant parameter.

It follows from the Galilean invariance of the Schrödinger equation that equation (10.4.1) is also invariant under the following transformations:

$$\tilde{\xi} = \xi - U\tau, \qquad \tilde{\tau} = \tau, \qquad \tilde{u} = u \exp\left\{-\frac{iU}{2\beta}\left(\xi - \frac{1}{2}U\tau\right)\right\}, \tag{10.4.3}$$

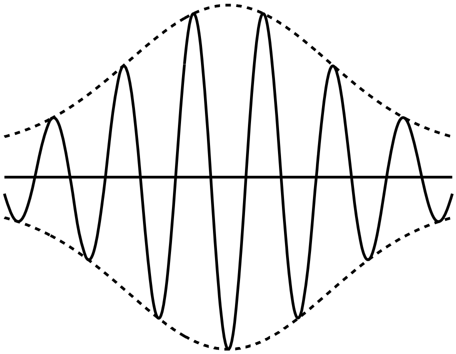

Fig. 10.1 An envelope soliton.

where U is another constant. Thus, equation (10.4.1) admits a solitary wave solution in the form

$$u(\xi,\tau)=\left(-\frac{2\nu}{\gamma}\right)^{\frac{1}{2}}\operatorname{sech}\left[\sqrt{\frac{-\nu}{\beta}}\,\tilde{\xi}\right]\exp\left[i\left(\frac{U}{2\beta}\right)-\frac{i}{2}\left\{\left(\frac{U}{2\beta}\right)^{2}+2\nu\right\}\tau\right], \quad (10.4.4)$$

where ν and U are independent constants. These solutions decrease very rapidly as $|\xi|\to\infty$, so that they may be considered to represent solitary waves which are similar to KdV solitons. However, unlike the KdV equation, the nonlinear Schrödinger equation does not admit a soliton solution corresponding to a steady wave propagating with a constant velocity. In fact, the plane wave part represented by the exponential function and the amplitude of the sech profile propagate with different velocities. In this sense, the solution (10.4.4) is called the *envelope soliton,* as shown by Figure 10.1.

It was first shown numerically by Yajima and Outi (1971) that, like the KdV soliton, the envelope soliton behaves like a particle. This was also proved analytically by Zakharov and Shabat (1972) by the inverse scattering method. An excellent experimental agreement with the *sech*-soliton solution (10.4.4) was found by Hammack as reported by Ablowitz and Segur (1979). Hammack's experimental findings are shown in Figure 7.2 in Debnath (1994, p. 363).

On the other hand, if u tends to a constant value u_0 at infinity, then the plane wave solution exists in the form

$$u(\xi,\tau)=u_0\exp\left[i(\ell\xi-\delta\tau)\right], \quad (10.4.5)$$

where

$$\delta=\left(\beta\ell^2-\gamma u_0^2\right). \quad (10.4.6)$$

However, the plane wave solution is modulationally unstable. This can be shown by introducing two real functions ρ and σ through the expression

$$u=\sqrt{\rho}\exp\left[\left(\frac{i}{2\beta}\right)\int\sigma\,d\xi\right]. \quad (10.4.7)$$

Substituting this expression in (10.4.1) gives

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial \xi}(\rho\sigma) = 0 \tag{10.4.8}$$

and

$$\frac{\partial \sigma}{\partial \tau} + \sigma\frac{\partial \sigma}{\partial \xi} = 2\beta\gamma\frac{\partial \rho}{\partial \xi} + \beta^2\frac{\partial}{\partial \xi}\left[\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial \xi}\left(\frac{1}{\sqrt{\rho}}\frac{\partial \rho}{\partial \xi}\right)\right]. \tag{10.4.9}$$

Since $\beta\gamma > 0$, in the long wavelength approximation, equations (10.4.8), (10.4.9) are equivalent to a hydrodynamic system with negative pressure. In other words, the system becomes elliptic. Thus, we consider that the perturbations modulate a constant amplitude ρ_0 and phase σ_0 so that

$$(\rho, \sigma) = (\rho_0, \sigma_0) + (\delta\rho, \delta\sigma)\exp\{i(\kappa\xi - \Omega\tau)\}. \tag{10.4.10ab}$$

Substituting these perturbations in equations (10.4.8), (10.4.9) leads to an equation for the growth rate Ω in terms of the wavenumber κ as

$$\Omega = -\sigma_0\kappa \pm (-2\beta\gamma\rho_0)^{\frac{1}{2}}\kappa + O(\kappa^3). \tag{10.4.11}$$

This shows that the perturbation grows exponentially as $t \to \infty$ for small wavenumbers κ, provided that ρ_0 is finite.

On the other hand, if $\beta\gamma > 0$, that is, β and γ are of opposite signs, there exists a stable plane wave solution. In this case, the system of equations (10.4.8), (10.4.9) can be reduced to the KdV equation. This follows from the expansion of ρ and σ in terms of the small parameter ε about the constant state, so that we can write

$$\rho = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \cdots \quad \text{and} \quad \sigma = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \cdots. \tag{10.4.12ab}$$

These results combined with the transformation

$$\tilde{\xi} = \sqrt{\varepsilon}\left[\xi - \left\{\sigma_0 + (-2\beta\gamma\rho_0)^{\frac{1}{2}}\right\}\tau\right], \qquad \tilde{\tau} = \varepsilon\tau \tag{10.4.13}$$

reduce the system (10.4.8), (10.4.9) to the KdV equation for ρ_1 and σ_1.

Thus, when $\beta\gamma > 0$, the above solution is often called a *bright soliton* (or *envelope soliton*). The bright pulse arises when $|u|^2$ increases from a finite value at infinity and, subsequently, returns to the same value. When $\beta\gamma < 0$, the solution is of the expansive type in which $|u|^2$ decreases from a finite value at infinity and subsequently returns to the same state. This is often called a *dark soliton* (or *dark pulse*).

So far, the self-modulation of a single mode governed by the dispersion relation has been described. However, when different modes with amplitudes of the same order $O(\varepsilon)$ coexist and undergo mutual interactions and self-modulation, it is possible to split the system of equations into independent nonlinear Schrödinger equations similar to the case of long waves (Oikawa and Yajima 1974a, 1974b). Consequently, envelope solitons associated with various modes, moving separately at the beginning, preserve their identities and, then, propagate as envelope solitons, even after the mutual interactions.

Zakharov and Shabat (1972) discovered that the inverse scattering method gives an exact solution of the initial-value problem for the nonlinear Schrödinger equation. Based on the initial data, which tend to zero rapidly at infinity, their remarkable analysis can be summarized as follows.

(i) An initial wave envelope pulse of arbitrary shape eventually disintegrates into a number of solitons of shorter scales and an oscillatory tail. Each soliton is a permanent, progressive wave. The number and structure of these solitons and the nature of the tail are completely described by the initial data.
(ii) These soliton solutions are definitely stable in the sense that they undergo nonlinear interaction and, then, emerge from the interaction without permanent changes except for a possible change in position and phase.
(iii) The oscillatory tail is relatively small and disperses linearly with amplitude which decays like $t^{-\frac{1}{2}}$, as $t \to \infty$.

Another significant property of the solutions of the nonlinear Schrödinger equation is the *recurrence* phenomenon. In a conservative system, the existence of invariants plays a fundamental role in establishing recurrence phenomena, as shown by Gibbons et al. (1977) and by Thyagaraja (1979, 1981, 1983). As demonstrated by these authors, the recurrence phenomenon is typical only for bounded or periodic domains. Thyagaraja considered the initial-value problem for the nonlinear Schrödinger equation

$$i\psi_t = \psi_{xx} + \gamma|\psi|^2\psi \tag{10.4.14}$$

in the periodic domain $0 \le x \le 1$ with periodic boundary conditions and a real constant γ which can be positive or negative. In many physical problems, γ is positive. We assume the periodic boundary conditions

$$\psi(0,t) = 0 = \psi(1,t) \quad \text{for } t > 0, \tag{10.4.15}$$

or

$$\psi_x(0,t) = 0 = \psi_x(1,t) \quad \text{for } t > 0, \tag{10.4.16}$$

and the initial condition

$$\psi(x,0) = \Psi(x) \quad \text{for all } x \in [0,1]. \tag{10.4.17}$$

Without proof, we assume that this initial-boundary problem for equation (10.4.14) possesses a smooth solution which is uniquely determined by the initial data $\Psi(x)$. To investigate the qualitative properties of the solution, Thyagaraja derived certain a priori bounds involving the integral invariants associated with equation (10.4.14). Two invariants of the problem which are constants of the motion are given by

$$I(t) = \int_0^1 |\psi(x,t)|^2\, dx, \tag{10.4.18}$$

and

$$J(t) = \int_0^1 |\psi_x(x,t)|^2\, dx - \frac{\gamma}{2}\int_0^1 |\psi(x,t)|^4\, dx. \tag{10.4.19}$$

However, Zakharov and Shabat (1972) proved that this problem has an infinite set of integral invariants, provided $\psi(x,t)$ is sufficiently smooth. We consider *any* function $\psi(x,t)$, not necessarily a solution of (10.4.14), which is defined for $|t| > 0$ and sufficiently smooth. We assume that $\psi(x,t)$ satisfies the given boundary conditions and evolves in time, so that functions $I(t)$ and $J(t)$, defined by (10.4.18), (10.4.19), are constants in time. We then find bounds of the Rayleigh quotient $Q(t)$ defined by

$$Q(t)I(t) = \int_0^1 \left|\psi_x(x,t)\right|^2 dx. \tag{10.4.20}$$

Denoting $I(0) = I_0$ and $J(0) = J_0$, we find that

$$Q(t) < \frac{J_0}{I_0}, \tag{10.4.21}$$

provided that $\gamma < 0$.

For $\gamma > 0$, we assume that $|\psi(x,t)|^2$ is minimal at $x = x_0$ for any t, and hence, we obtain

$$\left|\psi(x,t)\right|^2 = \left|\psi(x_0,t)\right|^2 + 2\int_{x_0}^{x} \psi\psi_x \, dx, \tag{10.4.22}$$

which leads to the inequality

$$\left|\psi(x,t)\right|^2 \le \left|\psi(x_0,t)\right|^2 + 2\int_0^1 \psi\psi_x \, dx. \tag{10.4.23}$$

Obviously,

$$\left|\psi(x_0,t)\right|^2 \le \int_0^1 \left|\psi(x_0,t)\right|^2 dx \le \int_0^1 \left|\psi(x,t)\right|^2 dx,$$

and the Schwarz inequality

$$\int_0^1 \left|\psi\psi_0\right| dx \le I(t)\sqrt{Q(t)}.$$

Multiplying (10.4.23) by $|\psi(x,t)|^2$ and integrating gives

$$\int_0^1 \left|\psi(x,t)\right|^4 dx \le I^2 + 2I^2\sqrt{Q(t)}. \tag{10.4.24}$$

Substituting in (10.4.19) yields the quadratic inequality

$$Q(t) \le \left(\frac{J_0}{I_0}\right) + \left(\frac{\gamma}{2}\right) I_0 + \gamma I_0 \sqrt{Q(t)}. \tag{10.4.25}$$

Finally, (10.4.23) can be reorganized to obtain

$$\left|\psi(x,t)\right|^2 \le I_0 + 2I_0\sqrt{Q(t)}. \tag{10.4.26}$$

Since, $Q(t) \geq 0$ by definition, it is easy to solve the inequality (10.4.25) and obtain the following results:

$$Q(t) \leq M^2(I_0, J_0, \gamma), \tag{10.4.27}$$

$$\|\psi\|_\infty^2 \equiv \max_{0\leq x\leq 1} \left|\psi(x,t)\right|^2 \leq I_0(1+2M), \tag{10.4.28}$$

where M is the positive root of the quadratic equation

$$M^2 - \gamma I_0 M - \left\{ \left(\frac{J_0}{I_0}\right) + \left(\frac{\gamma}{2}\right) I_0 \right\} = 0. \tag{10.4.29}$$

To examine the major implications of the a priori bounds in (10.4.27), (10.4.28), we introduce the concept of *Lagrangian stability*. A solution $\psi(x,t)$ of (10.4.14) is said to be *Lagrangian stable* if there exists a constant K independent of t, but possibly dependent on the initial data such that

$$\left\|\psi(x,t)\right\|_\infty^2 = \max_{0\leq x\leq 1} \left|\psi(x,t)\right|^2 \leq K, \tag{10.4.30}$$

for all $|t| \geq 0$.

Clearly, it follows from (10.4.28) with $K = I_0(1+2M)$ that any solution $\psi(x,t)$ is Lagrangian stable.

To give an interpretation of the inequality (10.4.27), we expand $\psi(x,t)$ in a Fourier series in x, so that

$$\psi(x,t) = \sum_{n=-\infty}^{\infty} a_n(t) \exp(2n\pi i x), \tag{10.4.31}$$

where

$$I_0 = \sum_{n=-\infty}^{\infty} \left|a_n(t)\right|^2 \quad \text{and} \quad Q(t) = \left(\frac{4\pi^2}{I_0}\right) \sum_{n=-\infty}^{\infty} n^2 \left|a_n(t)\right|^2. \tag{10.4.32}$$

In analogy with quantum physics, we may interpret $Q(t)/4\pi^2$ as the instantaneous average of n^2, and $(M/2\pi)$ can be interpreted as an upper bound to the rms value of n, the number of modes carrying the "wave energy" I_0. The quantity $(\frac{1}{2\pi})\sqrt{Q}$ is called the number of *effective modes* and is denoted by N_{eff}. The inequality (10.4.27) says that N_{eff} is bounded by $(\frac{M}{2\pi})$, which depends only on the initial values of I_0 and J_0 of the integral invariants and on the *interaction strength* γ. Thus, we have proved that every motion described by equation (10.4.14) with given I_0 and J_0 can be assigned N_{eff} as a measure of the number of effective modes. It is then straightforward to apply Birkhoff's theorem (1927) to conclude that, in general, dynamical systems with a *finite* number of degrees of freedom and bounded motions are "generically" recurrent.

It is interesting to compare the results of this analysis with numerical simulations and experiments. Yuen and Ferguson (1978a, 1978b) investigated spatially periodic solutions of (10.4.14). They numerically evolved initial data for which the linearized form of (10.4.14) is *unstable* to the modulational instability of Benjamin and Feir (1967). The longtime behavior of these solutions reveals that the energy is shared effectively between a *finite number* of wave modes. The number depends on the initial data. Moreover, the solution reconstructed itself after appearing to show a tendency to break up due to linear instability (Yuen and Lake 1980). Both the failure of the energy to thermalize and the tendency to recur have been known since the classic work of Fermi et al. (1955). They observed a remarkable recurrence phenomenon, known as the *Fermi–Pasta–Ulam* (*FPU*) *recurrence*. Initially, if energy is given to the lowest mode with the lowest frequency, then several higher modes are excited by the nonlinear effects, but after a suitable lapse of time during which a flowing back and forth among several low-order modes takes place, the energy eventually returns to the mode comprising the initial state. In other words, the system does not approach thermal equilibrium by the principle of the equipartition of energy due to the nonlinear effects, but the FPU recurrence is observed, and consequently, the nonergodicity of the system can be demonstrated. Subsequently, Zabusky and Kruskal (1965) and Fornberg and Whitham (1978) reported similar observations. From a series of computational and experimental works by Yuen and Lake (1975) and Lake et al. (1977), it has been confirmed that recurrence is a generic phenomenon for any physical problem described by the nonlinear Schrödinger equation. At the same time, the above mathematical analysis is now accepted as conclusive evidence of the FPU recurrence phenomenon.

For a long time it was mistakenly thought that the recurrence property of the NLS equation is attributable to the fact that it admits a multisoliton solution and can be solved *exactly* by the inverse scattering method. However, the above discussion shows, beyond any doubt, that the recurrence of solutions has nothing to do with soliton solutions or exact integrability.

10.5 Conservation Laws for the NLS Equation

Zakharov and Shabat (1972) proved that equation (10.3.12) has an infinite number of polynomial conservation laws. These have the form of an integral, with respect to x, of a polynomial expression in terms of the function $\psi(x,t)$ and its derivatives with respect to x. These laws are somewhat similar to those already proved for the KdV equation. Therefore, the proofs of the conservation laws are based on similar assumptions used in the context of the KdV equation (see Section 9.6).

We prove here three conservation laws for the NLS equation (10.3.12):

$$\int_{-\infty}^{\infty} |\psi|^2\,dx = \text{const.} = C_1, \tag{10.5.1}$$

$$\int_{-\infty}^{\infty} i\big(\psi\overline{\psi}_x - \overline{\psi}\psi_x\big)\,dx = \text{const.} = C_2, \tag{10.5.2}$$

$$\int_{-\infty}^{\infty} \left\{\big(|\psi_x|^2\big) - \frac{1}{2}\gamma|\psi|^4\right\} dx = \text{const.} = C_3. \tag{10.5.3}$$

We multiply (10.3.12) by $\overline{\psi}$ and its complex conjugate by ψ and subtract the latter from the former to obtain

$$i\frac{d}{dt}\left(\psi\overline{\psi}\right) + \frac{d}{dx}\left(\psi_x\overline{\psi} - \overline{\psi}_x\psi\right) = 0, \tag{10.5.4}$$

where the bar denotes the complex conjugate.

Integration with respect to x in $-\infty < x < \infty$ gives

$$i\frac{d}{dt}\int_{-\infty}^{\infty} |\psi|^2\,dx = 0.$$

This proves result (10.5.1).

We multiply (10.3.12) by $\overline{\psi}_x$ and its complex conjugate by ψ_x and, then, add the resulting expressions to obtain

$$i\left(\psi_t\overline{\psi}_x - \overline{\psi}_t\psi_x\right) + \left(\psi_{xx}\overline{\psi}_x + \overline{\psi}_{xx}\psi_x\right) + \gamma|\psi|^2\left(\psi\overline{\psi}_x + \overline{\psi}\psi_x\right) = 0. \tag{10.5.5}$$

We differentiate (10.3.12) and its complex conjugate with respect to x, multiply the former by $\overline{\psi}$ and the latter by ψ, and then, add them together. This leads to the result

$$\begin{aligned} &i\left(\overline{\psi}\psi_{xt} - \psi\overline{\psi}_{xt}\right) + \left(\psi_{xxx}\overline{\psi} + \overline{\psi}_{xxx}\psi\right) \\ &\quad + \gamma\left[\overline{\psi}\left(|\psi|^2\psi\right)_x + \psi\left(|\psi|^2\overline{\psi}\right)_x\right] = 0. \end{aligned} \tag{10.5.6}$$

If we subtract (10.5.6) from (10.5.5) and then simplify, it turns out that

$$\begin{aligned} i\frac{d}{dt}\left(\overline{\psi}\psi_x - \psi\overline{\psi}_x\right) &= \frac{d}{dx}\left(\psi_x\overline{\psi}_x\right) + \frac{d}{dx}\left(\overline{\psi}\psi_{xx} + \psi\overline{\psi}_{xx}\right) \\ &\quad - \frac{d}{dx}\left(\psi_x\overline{\psi}_x\right) + \gamma\frac{d}{dx}\left(\psi\overline{\psi}\right)^2 \\ &= \frac{d}{dx}\left(\overline{\psi}\psi_{xx} + \psi\overline{\psi}_{xx}\right) + \gamma\frac{d}{dx}|\psi|^4. \end{aligned}$$

Integrating this result with respect to x yields

$$\frac{d}{dt}\int_{-\infty}^{\infty} i\left(\overline{\psi}\psi_x - \psi\overline{\psi}_x\right)dx = 0.$$

This proves result (10.5.2).

We multiply (10.3.12) by $\overline{\psi}_t$ and its complex conjugate by ψ_t and add the resulting equations to derive

$$\left(\overline{\psi}_t\psi_{xx} + \overline{\psi}_{xx}\psi_t\right) + \gamma\left(\psi^2\overline{\psi\psi}_t + \overline{\psi}^2\psi\psi_t\right) = 0,$$

or equivalently,

$$\frac{d}{dx}\left(\overline{\psi}_t\psi_x + \psi_t\overline{\psi}_x\right) - \frac{d}{dx}\left(\psi_x\overline{\psi}_x\right) + \frac{\gamma}{2}\frac{d}{dt}\left(\psi^2\overline{\psi}^2\right) = 0.$$

Integrating this result with respect to x gives

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left[|\psi_x|^2 - \frac{\gamma}{2}|\psi|^4\right]dx = \int_{-\infty}^{\infty}\frac{d}{dx}\left(\overline{\psi}_t\psi_x + \psi_t\overline{\psi}_x\right)dx = 0. \quad (10.5.7)$$

This gives (10.5.3).

The preceding three conservation integrals have a simple physical meaning. In fact, the constants of motion C_1, C_2, and C_3 are related to the number of particles, the momentum, and the energy of a system governed by the nonlinear Schrödinger equation.

The analysis of this section reveals several remarkable features of the nonlinear Schrödinger equation. This equation can also be used to investigate instability phenomena in many other physical systems. Like the various forms of the KdV equation, the NLS equation arises in many physical problems, including nonlinear water waves and ocean waves, waves in plasma, propagation of heat pulses in a solid, self-trapping phenomena in nonlinear optics, nonlinear waves in a fluid-filled viscoelastic tube, and various nonlinear instability phenomena in fluids and plasmas.

Whitham's equations for the slow modulation of the wave amplitude a and the wavenumber k in the case of two-dimensional deep water waves are given by

$$\frac{\partial}{\partial t}\left(\frac{a^2}{\omega_0}\right) + \frac{\partial}{\partial x}\left(C\frac{a^2}{\omega_0}\right) = 0, \quad (10.5.8)$$

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0, \quad (10.5.9)$$

where $\omega_0 = \sqrt{gk}$ is the first-order approximation for the wave frequency ω and $C = (g/2\omega_0)$ is the group velocity.

Chu and Mei (1970, 1971) observed that certain terms of the dispersive type, neglected in Whitham's equations to the same order of approximation, must be included to extend the validity of these equations. Whitham's theory is based on the direct use of Stokes' dispersion relation for a uniform wavetrain,

$$\omega = \omega_0\left(1 + \frac{1}{2}\varepsilon^2 a^2 k^2\right), \quad (10.5.10)$$

whereas Chu and Mei added terms of higher derivatives and dispersive type to the expression for ω, so that

$$\omega = \omega_0\left[1 + \varepsilon^2\left(\frac{1}{2}a^2k^2 + \left\{\left(\frac{a}{\omega_0}\right)_{tt} \div 2\omega_0 a\right\}\right)\right]. \quad (10.5.11)$$

They used the expression (10.5.11) to transform (10.5.8), (10.5.9) in a frame of reference moving with the group velocity C and obtained the following nondimensional equations:

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x}\left(a^2\phi_x\right) = 0, \quad (10.5.12)$$

$$-2\frac{\partial^2\phi}{\partial x\partial t}+\frac{\partial}{\partial x}\left[-\phi_x^2+\frac{a^2}{4}+\frac{a_{xx}}{16a}\right]=0, \tag{10.5.13}$$

where we have used Chu and Mei's result $W=-2\phi_x$ and ϕ is a small phase variation. Integrating (10.5.13) with respect to x and setting the constant of integration to be zero gives

$$\phi_t+\frac{1}{2}\phi_x^2-\frac{1}{8}a^2-\frac{a_{xx}}{32a}=0. \tag{10.5.14}$$

A transformation $\Psi=a\exp(4i\phi)$ is used to simplify (10.5.12) and (10.5.14), which reduces to the *nonlinear Schrödinger equation*

$$i\Psi_t+\frac{1}{8}\Psi_{xx}+\frac{1}{2}\Psi|\Psi|^2=0. \tag{10.5.15}$$

This equation has also been derived and exploited by several authors, including Benney and Roskes (1969), Hasimoto and Ono (1972), and Davey and Stewartson (1974) to examine the nonlinear evolution of Stokes' waves on water.

An alternative way to study nonlinear evolution of two- and three-dimensional wavepackets is to use the method of multiple scales in which the small parameter ε is explicitly built into the expansion procedure. The small parameter ε characterizes the wave steepness. This method has been employed by several authors in various fields and has also been used by Hasimoto and Ono (1972) and Davey and Stewartson (1974).

10.6 The Inverse Scattering Method for the Nonlinear Schrödinger Equation

Zakharov and Shabat (1972) developed an ingenious method for solving the nonlinear Schrödinger equation in the form

$$iu_t+u_{xx}+2|u|^2u=0 \tag{10.6.1}$$

by considering the spectral problem

$$v_{1x}=-i\zeta v_1+qv_2, \tag{10.6.2}$$
$$v_{2x}=i\zeta v_2+rv_1. \tag{10.6.3}$$

In matrix notation, they take the form

$$v_x=\begin{pmatrix}-i\zeta & q\\ r & i\zeta\end{pmatrix}v, \quad \text{or} \quad \begin{pmatrix}\frac{\partial}{\partial x} & -q\\ r & -\frac{\partial}{\partial x}\end{pmatrix}v=-i\zeta v, \tag{10.6.4ab}$$

where v is the vector $\binom{v_1}{v_2}$, bounded functions $q(x)$ and $r(x)$ (not necessarily real) are *potentials*, and ζ is the eigenvalue.

The most general linear time dependence is given by

$$v_{1t} = Av_1 + Bv_2, \tag{10.6.5}$$

$$v_{2t} = Cv_1 + Dv_2, \tag{10.6.6}$$

where A, B, C, and D are scalar functions independent of v_1 and v_2 to be determined.

To establish the relation between the pair of equations (10.6.2) and (10.6.3) and the Schrödinger scattering problem, we first differentiate (10.6.3) with respect to x to obtain

$$v_{2xx} = i\zeta v_{2x} + r_x v_1 + r v_{1x}, \tag{10.6.7}$$

provided that r_x exists. By equations (10.6.2) and (10.6.3), equation (10.6.7) becomes

$$\begin{aligned} v_{2xx} &= i\zeta v_{2x} + r_x v_1 + r(-i\zeta v_1 + q v_2) \\ &= i\zeta v_{2x} + \frac{1}{r}(r_x - i\zeta r)(v_{2x} - i\zeta v_2) + rqv_2, \end{aligned}$$

or equivalently,

$$v_{2xx} - \frac{r_x}{r} v_{2x} - \left(qr - i\zeta \frac{r_x}{r} - \zeta^2 \right) v_2 = 0. \tag{10.6.8}$$

The special choice $r = -1$ gives the Schrödinger scattering problem for v_2

$$v_{2xx} + \left(\zeta^2 + q\right) v_2 = 0. \tag{10.6.9}$$

When $\zeta^2 = \lambda$ and $q = -u$, the system (10.6.4ab) reduces to the time-independent Schrödinger equation required for solving the KdV equation which was discussed in Chapter 9.

To determine the coefficients involved in (10.6.5), (10.6.6), we use cross-differentiation of (10.6.2), (10.6.3) and (10.6.5), (10.6.6) combined with the assumption that the eigenvalues are invariant in time ($\zeta_t = 0$), that is, $(v_{jx})_t = (v_{jt})_x$ for $j = 1, 2$, leading to the following equations for A, B, C, and D:

$$\begin{aligned} A_x &= qC - rB, \\ B_x + 2i\zeta B &= q_t - (A - D)q, \\ C_x - 2i\zeta C &= r_t + (A - D)r, \\ -(D)_x &= qC - rB. \end{aligned}$$

Without loss of generality, we can take $-D = A$ so that the preceding system of equations becomes

$$A_x = qC - rB, \tag{10.6.10}$$

$$B_x + 2i\zeta B = q_t - 2Aq, \tag{10.6.11}$$

$$C_x - 2i\zeta C = r_t + 2Ar. \tag{10.6.12}$$

Since the Schrödinger equation is second order in x, we may expect that the series expansions in ζ for A, B, and C include only terms through ζ^2, and so, we write an exact truncated power series

$$A = \sum_{n=0}^{2} A_n \zeta^n, \qquad B = \sum_{n=0}^{2} B_n \zeta^n, \qquad C = \sum_{n=0}^{2} C_n \zeta^n. \tag{10.6.13}$$

Substituting these expressions in (10.6.10)–(10.6.12) and equating coefficients of powers of ζ gives A_n, B_n, and C_n where $n = 0, 1, 2$. The coefficients of ζ^3 in (10.6.11) and (10.6.12) give $B_2 = C_2 = 0$. For coefficients of ζ^2, equation (10.6.10) yields $A_2 = a_2 =$ const. Equations (10.6.11) and (10.6.12) give $(B_1, C_1) = ia_2(q, r)$. For the coefficients ζ, equation (10.6.10) gives $A_1 = a_1 =$ const. For simplicity, we set $a_1 = 0$. However, if $a_1 \neq 0$, a more general evolution equation can be obtained. Then, (10.6.11), (10.6.12) give $(B_0, C_0) = \frac{1}{2}a_2(-q_x, r_x)$. Finally, for the coefficients ζ^0, (10.6.10) gives $A_0 = \frac{1}{2}a_2 qr + a_0$, where a_0 is a constant which is set equal to zero. Consequently, equations (10.6.11) and (10.6.12) for coefficients of ζ^0 give

$$-\frac{1}{2}a_2 q_{xx} = q_t - a_2 q^2 r, \tag{10.6.14}$$

$$\frac{1}{2}a_2 r_{xx} = r_t + a_2 q r^2. \tag{10.6.15}$$

This represents a coupled pair of nonlinear evolution equations. In particular, when $r = \pm q^*$, equations (10.6.14) and (10.6.15) are compatible provided that $a_2 = ia$, where a is real. If we take $a = 2$, we obtain the equation

$$iq_t = q_{xx} \pm 2q^2 q^*. \tag{10.6.16}$$

This equation with the positive sign is called the *nonlinear Schrödinger equation* which admits a special soliton solution, while the equation with the negative sign does *not* give any soliton solution for potentials decaying rapidly as $x \to \infty$ since the spectral operator in (10.6.2) and (10.6.3) is self-adjoint (or Hermitian).

10.7 Examples of Physical Applications in Fluid Dynamics and Plasma Physics

Example 10.7.1 (The Nonlinear Schrödinger Equation for Deep Water and the Benjamin–Feir Instability). One of the simplest solutions of the nonlinear Schrödinger equation (10.3.25) is given by (10.3.31), that is,

$$A(t) = A_0 \exp\left(-\frac{1}{2}i\omega_0 k_0^2 A_0^2 t\right), \tag{10.7.1}$$

where A_0 is a constant. This essentially represents the fundamental component of the Stokes wave. We consider a perturbation of (10.7.1) and express it in the form

$$a(x,t) = A(t)\big[1 + B(x,t)\big], \tag{10.7.2}$$

where $B(x,t)$ is the perturbation function. Substituting this result in (10.3.25) gives

$$\begin{aligned} i(1+B)A_t + iAB_t &- \left(\frac{\omega_0}{8k_0^2}\right)AB_{xx} \\ &= \frac{1}{2}\omega_0 k_0^2 A_0^2\big[(1+B) + BB^*(1+B) \\ &\quad + (B+B^*)B + (B+B^*)\big]A, \end{aligned} \tag{10.7.3}$$

where $B^*(x,t)$ is the complex conjugate of the perturbed function $B(x,t)$. Neglecting the squares of B and simplifying, equation (10.7.3) reduces to

$$iB_t - \left(\frac{\omega_0}{8k_0^2}\right)B_{xx} = \frac{1}{2}\omega_0 k_0^2 A_0^2 (B + B^*). \tag{10.7.4}$$

We look for a solution for the perturbed quantity $B(x,t)$ in the form

$$B(x,t) = B_1 \exp(\Omega t + i\ell x) + B_2 \exp(\Omega^* t - i\ell x), \tag{10.7.5}$$

where B_1 and B_2 are complex constants, ℓ is a real wavenumber, and Ω is a growth rate (possibly a complex quantity) to be determined. Substituting the solution for B in (10.7.4) yields a pair of coupled equations:

$$\left(i\Omega + \frac{\omega_0 \ell^2}{8k_0^2}\right)B_1 - \frac{1}{2}\omega_0 k_0^2 A_0^2 (B_1 + B_2^*) = 0, \tag{10.7.6}$$

$$\left(i\Omega^* + \frac{\omega_0 \ell^2}{8k_0^2}\right)B_2 - \frac{1}{2}\omega_0 k_0^2 A_0^2 (B_1^* + B_2) = 0. \tag{10.7.7}$$

We take the complex conjugate of (10.7.7) to transform it into the form

$$\left(-i\Omega + \frac{\omega_0 \ell^2}{8k_0^2}\right)B_2^* - \frac{1}{2}\omega_0 k_0^2 A_0^2 (B_1 + B_2^*) = 0. \tag{10.7.8}$$

The pair of linear homogeneous equations (10.7.6) and (10.7.8) for B_1 and B_2^* admits a nontrivial eigenvalue for Ω provided

$$\begin{vmatrix} i\Omega + \frac{\omega_0\ell^2}{8k_0^2} - \frac{1}{2}\omega_0 k_0^2 A_0^2 & -\frac{1}{2}\omega_0 k_0^2 A_0^2 \\ -\frac{1}{2}\omega_0 k_0^2 A_0^2 & i\Omega + \frac{\omega_0\ell^2}{8k_0^2} - \frac{1}{2}\omega_0 k_0^2 A_0^2 \end{vmatrix} = 0, \tag{10.7.9}$$

which is equivalent to

$$\Omega^2 = \frac{1}{2}\left(\frac{\omega_0 \ell}{2k_0}\right)^2\left(k_0^2 A_0^2 - \frac{\ell^2}{8k_0^2}\right). \tag{10.7.10}$$

The growth rate Ω is purely imaginary or real (and positive) depending on whether $\ell^2 > 8k_0^4 A_0^2$ or $\ell^2 < 8k_0^4 A_0^2$. The former case represents a wave solution for B, and the latter corresponds to the *Benjamin–Feir* (or *modulational*) *instability* with a criterion in terms of the nondimensional wavenumber $\tilde{\ell} = (\ell/k_0)$ as

$$\tilde{\ell}^2 < 8k_0^2 A_0^2. \tag{10.7.11}$$

Thus, the range of instability is given by

$$0 < \tilde{\ell} < \tilde{\ell}_c = 2\sqrt{2}k_0 A_0. \tag{10.7.12}$$

Since Ω is a function of $\tilde{\ell}$, the maximum instability occurs at $\tilde{\ell} = \tilde{\ell}_{\max} = 2k_0 A_0$, with a maximum growth rate given by

$$(\operatorname{Re}\Omega)_{\max} = \frac{1}{2}\omega_0 k_0^2 A_0^2. \tag{10.7.13}$$

To establish the connection with the Benjamin–Feir instability, we have to find the velocity potential for the fundamental wave mode multiplied by $\exp(kz)$. It turns out that the term proportional to B_1 is the upper sideband, whereas that proportional to B_2 is the lower sideband. The main conclusion of the preceding analysis is that Stokes water waves are definitely *unstable*.

Example 10.7.2 (The Nonlinear Klein–Gordon Equation). We consider the nonlinear Klein–Gordon equation in the form

$$\psi_{tt} - (\psi_{xx} + \psi_{yy}) + m^2\psi + \gamma\psi^3 = 0, \quad x \in \mathbb{R},\ 0 \le y \le a, \tag{10.7.14}$$

where m and γ are constants. We assume the boundary conditions

$$\psi = 0 \quad \text{for } y = 0 \text{ and } y = a. \tag{10.7.15}$$

For the linear case, we assume

$$\psi(x,y,t) = \Psi(y)\exp\big[i(kx - \omega t)\big] \tag{10.7.16}$$

to obtain

$$\Psi(y) = A\sin(\alpha y) + B\cos(\alpha y), \qquad \alpha^2 = \omega^2 - k^2 - m^2. \tag{10.7.17}$$

Since $B = 0$ by (10.7.15), from the boundary conditions, we obtain

$$\alpha^2 = \omega^2 - k^2 - m^2 = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, 3, \ldots, \tag{10.7.18}$$

so that the group velocity $C = \omega'(k) = \frac{\omega}{k}$ and $\omega = \omega(k,n)$. Hence, there exist various modes dependent on n. We consider a wave with $n = 1$, and its amplitude varies slowly due to nonlinearity, so that slow variables ξ and τ defined by (10.2.5ab) can be used. We next assume the following form of the solution:

$$\psi = \sum_{m=1}^{\infty} \varepsilon^m \psi^{(m)}, \qquad \psi^{(m)} = \sum_{\ell=-\infty}^{\infty} \psi_\ell^{(m)}(\xi,\tau,y) \exp\big[i\ell(kx-\omega t)\big]. \tag{10.7.19ab}$$

Retaining only terms of the first order in ε, we obtain

$$\psi_1^{(1)} = \phi(\xi,\tau)\sin(\alpha y), \qquad \psi_\ell^{(1)} = 0, \quad |\ell| \neq 1, \tag{10.7.20}$$

and the second-order terms in ε give

$$2i\ell(k-\omega C)\frac{\partial \psi_\ell^{(1)}}{\partial \xi} + \frac{\partial^2 \psi_\ell^{(2)}}{\partial^2 y} + \big(\ell^2\omega^2 - \ell^2 k^2 - m^2\big)\psi_\ell^{(2)} = 0, \tag{10.7.21}$$

where the first term vanishes because $C = \frac{k}{\omega}$ so that $\psi_\ell^{(2)} = 0$ for $|\ell| \neq 1$. For $\ell = 1$, the terms of the third order in ε yield

$$\frac{\partial^2 \psi_1^{(3)}}{\partial y^2} + \alpha^2 \psi_1^{(3)} + \left[2i\omega\frac{\partial \phi}{\partial \tau} + \big(1-C^2\big)\frac{\partial^2 \phi}{\partial \xi^2}\right]\sin(\alpha y) - 3\gamma|\phi|^2\phi\sin^3(\alpha y) = 0. \tag{10.7.22}$$

Multiplying this equation by $\sin(\alpha y)$ and integrating with respect to y from 0 to a causes the first and second terms to vanish. Consequently, equation (10.7.22) gives

$$i\phi_\tau + \left(\frac{\omega^2-k^2}{2\omega^3}\right)\phi_{\xi\xi} - \frac{9\gamma}{8\omega}|\phi|^2\phi = 0. \tag{10.7.23}$$

This is a *nonlinear Schrödinger equation* for the amplitude $\phi(\xi,\tau)$. Since $\omega^2 - k^2 > 0$, a modulational instability occurs when $\gamma > 0$. Denoting the initial amplitude by A_0, the solitary wave solution is given by

$$\psi_1^{(1)}(\xi,\tau,y) = A_0\sin(\alpha y)\exp\left[\left(\frac{9\gamma A_0^2}{16\omega}\right)(i\tau)\right] \times \operatorname{sech}\left[A\omega\left\{-\frac{9\gamma}{8(\omega^2-k^2)}\right\}^{\frac{1}{2}}\xi\right]. \tag{10.7.24}$$

Example 10.7.3 (Motion of a Vortex Filament and the Nonlinear Schrödinger Equation). The motion of a very thin isolated vortex filament $\mathbf{X} = \mathbf{X}(x,t)$ of radius ε in an incompressible unbounded fluid by its own induction was described asymptotically by Hasimoto (1972) in the form

$$\frac{\partial \mathbf{X}}{\partial t} = G\kappa\mathbf{b}, \tag{10.7.25}$$

where s is the length measured along the filament, t is the time, κ is the curvature, $\mathbf{b}$ is the unit vector along the binormal, and G is the coefficient of local induction

$$G = \left(\frac{\Gamma}{4\pi}\right)\left[\log\left(\varepsilon^{-1}\right) + O(1)\right], \tag{10.7.26}$$

which is proportional to the circulation Γ of the filament and may be treated as constant if the slow variation of the logarithm is compared with that of its argument. The local motion is approximated by that of a thin circular ring with the same curvature. Under this approximation, the tangential motion along the filament due to stretching is neglected, even though it is very important in many cases. Introducing a suitable choice of the units of length and time transforms (10.7.25) into the nondimensional form

$$\dot{\mathbf{X}} = \kappa \mathbf{b}, \tag{10.7.27}$$

where the dot denotes a partial derivative with respect to time. We next include the equations of differential geometry, that is, the Serret–Frenet formulas for the tangent vector $\mathbf{t}$, the principle normal vector $\mathbf{n}$, and the binormal $\mathbf{b}$ ($\mathbf{t} = \mathbf{n} \times \mathbf{b}$ and $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ form a right-handed system),

$$\mathbf{t}' = \kappa \mathbf{n}, \qquad \mathbf{n}' = \tau \mathbf{b} - \kappa \mathbf{t}, \quad \text{and} \quad \mathbf{b}' = -\tau \mathbf{n}, \tag{10.7.28}$$

where a prime denotes a partial derivative with respect to arc length s, κ is the curvature, and τ is the torsion of the curve. When κ and τ are specified at each point of the curve, the shape of the curve, except for its location, is uniquely determined.

We prove that the complex function

$$\psi(s,t) = \kappa(s,t) \exp\left[i \int_0^s \tau(s',t)\, ds'\right],$$

which contains both κ and τ, and hence, completely determines the shape of the filament and satisfies the nonlinear Schrödinger equation. We first combine the last two of the Serret–Frenet formulas (10.7.28) into the complex form

$$(\mathbf{n} + i\mathbf{b})_s + i\tau(\mathbf{n} + i\mathbf{b}) = -\kappa \mathbf{t}. \tag{10.7.29}$$

Introducing

$$\mathbf{N} = (\mathbf{n} + i\mathbf{b}) \exp\left[i \int_0^s \tau\, ds'\right], \qquad \psi = \kappa \exp\left[i \int_0^s \tau\, ds'\right], \tag{10.7.30ab}$$

we obtain

$$\mathbf{N}' = -\psi \mathbf{t}, \tag{10.7.31}$$

and the first equation of (10.7.28) becomes

$$\mathbf{t}' = \frac{1}{2}\left(\psi^* \mathbf{N} + \psi \mathbf{N}^*\right), \tag{10.7.32}$$

where the asterisk denotes the complex conjugate.

This displacement of the position vector along the filament gives a unit tangent vector to the curve, that is,

$$\mathbf{X}_s = \mathbf{t}. \tag{10.7.33}$$

To find the time derivatives of $\mathbf{t}$ and $\mathbf{N}$, we proceed as follows:

$$\frac{\partial \mathbf{t}}{\partial t} = \frac{\partial}{\partial t}(\mathbf{X}_s) = (\kappa \mathbf{b})_s = \kappa_s \mathbf{b} - \kappa\tau\mathbf{n}. \tag{10.7.34}$$

Since

$$\begin{aligned} \kappa_s \mathbf{b} - \kappa\tau\mathbf{n} &= \operatorname{Re}\big[\kappa_s(\mathbf{b} + i\mathbf{n}) + i\kappa\tau(\mathbf{b} + i\mathbf{n})\big] \\ &= \operatorname{Re}\big[i(\kappa_s + i\kappa\tau)(\mathbf{n} - i\mathbf{b})\big] \\ &= \operatorname{Re}\big[i\psi_s \mathbf{N}^*\big] = \frac{i}{2}\big(\psi_s \mathbf{N}^* - \psi_s^* \mathbf{N}\big), \end{aligned} \tag{10.7.35}$$

$$\frac{\partial}{\partial t}\mathbf{t} = \frac{1}{2} i\big(\psi_s \mathbf{N}^* - \psi_s^* \mathbf{N}\big). \tag{10.7.36}$$

In general, the time derivative of $\mathbf{N}$ is a linear combination of $\mathbf{N}$, $\mathbf{N}^*$, and $\mathbf{t}$, and hence, we can write

$$\left(\frac{\partial \mathbf{N}}{\partial t}\right) = \alpha\mathbf{N} + \beta\mathbf{N}^* + \gamma\mathbf{t}, \tag{10.7.37}$$

where α, β, and γ are constants to be determined. It follows from the definition of $\mathbf{N}$ and the fact that $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ are orthogonal that

$$\mathbf{N}\cdot\mathbf{N} = \mathbf{t}\cdot\mathbf{N} = 0 \quad \text{and} \quad \mathbf{N}\cdot\mathbf{N}^* = 2. \tag{10.7.38}$$

We take the dot product of (10.7.37) with each of $\mathbf{N}$, $\mathbf{N}^*$, and $\mathbf{t}$ to derive

$$2\beta = \mathbf{N}\cdot\frac{\partial \mathbf{N}}{\partial t} = \frac{1}{2}(\mathbf{N}\cdot\mathbf{N}) = 0, \qquad 2\alpha = \mathbf{N}^*\cdot\frac{\partial \mathbf{N}}{\partial t}. \tag{10.7.39}$$

Consequently,

$$2(\alpha + \alpha^*) = \frac{\partial}{\partial t}\big(\mathbf{N}\cdot\mathbf{N}^*\big) = 0, \tag{10.7.40}$$

and hence, α must be purely imaginary, so that $\alpha = iR$, where R is real. Finally, since $\frac{\partial}{\partial t}(\mathbf{N}\cdot\mathbf{t}) = 0$,

$$\gamma = \mathbf{t}\cdot\frac{\partial \mathbf{N}}{\partial t} = -\mathbf{N}\cdot\frac{\partial \mathbf{t}}{\partial t} = -\frac{i}{2}\mathbf{N}\cdot\big(\psi_s \mathbf{N}^* - \psi_s^* \mathbf{N}\big), \tag{10.7.41}$$

so that $\gamma = -i\gamma_s$. It turns out that

$$\frac{\partial \mathbf{N}}{\partial t} = i(R\mathbf{N} - \psi_s \mathbf{t}). \tag{10.7.42}$$

The time derivative of (10.7.31) and the s derivatives of (10.7.42) give

$$\mathbf{N}_{st} = \frac{i}{2}\big(\psi\psi_s^* \mathbf{N} - \psi\psi_s \mathbf{N}^*\big) - \mathbf{t}\frac{\partial \psi}{\partial t}, \tag{10.7.43}$$

$$\mathbf{N}_{ts} = \mathbf{N}\left(iR_s - \frac{i}{2}\psi_s\psi^*\right) - \frac{i}{2}\psi\psi_s^* \mathbf{N} - i(R\psi + \psi_{ss})\mathbf{t}. \tag{10.7.44}$$

Equating the components of these two expressions yields

$$\frac{\partial R}{\partial s} = \frac{1}{2}\left(\psi^* \psi_s + \psi \psi_s^*\right) \tag{10.7.45}$$

and

$$\frac{\partial \psi}{\partial t} - i\frac{\partial^2 \psi}{\partial s^2} - iR\psi = 0. \tag{10.7.46}$$

Integrating (10.7.45) gives the value of R as

$$R = \frac{1}{2}\left[\psi\psi^* + A(t)\right], \tag{10.7.47}$$

where $A(t)$ arises due to integration. Substituting the value of R in (10.7.46) leads to the nonlinear Schrödinger equation

$$i\psi_t + \psi_{ss} + \frac{1}{2}\left[|\psi|^2 + A(t)\right]\psi = 0, \tag{10.7.48}$$

where $A(t)$ can be eliminated by the transformation

$$u = \frac{1}{2}\psi \exp\left[-\frac{i}{2}\int_0^t A(t')\, dt'\right]. \tag{10.7.49}$$

This transformation is nothing but a shift of the origin of integration in (10.7.30ab). Therefore, without loss of generality, we may set $A = 0$ in (10.7.48). Consequently, (10.7.48) reduces to the standard NLS equation

$$i\psi_t + \psi_{ss} + \frac{1}{2}|\psi|^2\psi = 0. \tag{10.7.50}$$

We obtain a solitary wave solution which propagates steadily with a constant velocity c along the filament which is a straight line at infinity, that is,

$$\kappa \to 0 \quad \text{as } s \to \infty. \tag{10.7.51}$$

Thus, in the frame of reference, κ and τ are functions of $\xi = s - ct$, that is,

$$\psi = \kappa(\xi) \exp\left[i\int_0^s \tau(\xi)\, ds'\right]. \tag{10.7.52}$$

The real and imaginary parts of (10.7.48) give, respectively,

$$-c\kappa\left[\tau(\xi) - \tau(-ct)\right] = \kappa'' - \kappa\tau^2 + \frac{1}{2}\kappa\left(\kappa^2 + A\right) \tag{10.7.53}$$

and

$$c\kappa' = 2\kappa'\tau + \kappa\tau'. \tag{10.7.54}$$

Integrating (10.7.54) and using (10.7.51) to determine the constant of integration gives $(c - 2\tau)\kappa = 0$, which yields $\tau = \tau_0 = \frac{1}{2}c$ since $\kappa \neq 0$. This means that the torsion is constant along the vortex filament and the velocity of propagation along the filament is twice the torsion. The shape of the vortex filament corresponding to the single solitary wave of the NLS equation, $\tau = 1$, $\kappa = 2\,\mathrm{sech}\, s$, is shown in Figure 10.2. Several other figures have been presented by Hasimoto (1972).

Finally, we use (10.7.51) to integrate (10.7.53) so that the solution becomes

$$\kappa(\xi,t) = 2\nu\,\mathrm{sech}(\nu\xi) = 2\nu\,\mathrm{sech}\big[\nu(s-ct)\big], \tag{10.7.55}$$

provided that A is a constant related to τ_0 and ν by $A = 2(\tau_0^2 - \nu^2)$. The actual shape of the filament is obtained from (10.7.55) by setting $\tau_0 = \frac{1}{2}c$. Physically, the solution represents a helical curve of constant torsion τ_0 with a curvature κ that decreases from a maximum value 2ν at the point $s = ct$ to zero, as $|s| \to \infty$. This single loop of helical motion moves along the vertex line with a velocity $c = 2\tau_0$.

Example 10.7.4 (Langmuir Solitons and Their Instability). We consider one-dimensional Langmuir waves propagating in a plasma of cold ions and hot electrons. We assume that the electrons respond adiabatically to high frequency motions. Using the standard notation for various quantities, which are similar to those involved in Example 9.3.1, the equations of motion for the electrons are given by

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) = 0, \tag{10.7.56}$$

$$\frac{\partial u_e}{\partial t} + u_e\frac{\partial u_e}{\partial x} = -\frac{eE}{m_e} - \left(\frac{3v_{Te}^2}{n_e}\right)\frac{\partial n_e}{\partial x}, \tag{10.7.57}$$

$$\frac{\partial E}{\partial x} = 4\pi e(n_i - n_e). \tag{10.7.58}$$

We decompose the various quantities into a high-frequency part and a low-frequency part characterized by the time scales ω_{pe}^{-1} and ω_{pi}^{-1}, respectively, so that

$$n_e = n_0 + n_\ell + n_{eh}, \qquad n_i = n_0 + n_\ell, \qquad E = E_h + E_\ell, \tag{10.7.59}$$

where n_{eh} and E_h refer to the high-frequency perturbations and n_ℓ and E_ℓ denote the low-frequency perturbations. Assuming charge neutrality in the low-frequency motions, equations (10.7.56)–(10.7.58) reduce to the form

$$\frac{\partial n_{eh}}{\partial t} + \frac{\partial}{\partial x}\big\{(n_0 + n_\ell)u_e\big\} = 0, \tag{10.7.60}$$

$$\frac{\partial u_e}{\partial t} = -\frac{eE_h}{m_e} - \frac{3v_{Te}^2}{(n_0+n_\ell)}\frac{\partial n_{eh}}{\partial x}, \tag{10.7.61}$$

$$\frac{\partial E_h}{\partial x} = -4\pi e n_{eh}. \tag{10.7.62}$$

Eliminating u_e from (10.7.60) and (10.7.61) gives

$$\frac{\partial^2 n_{eh}}{\partial t^2} - (n_0+n_\ell)\left[\left(\frac{e}{m_e}\right)\frac{\partial E_h}{\partial x} + \frac{3v_{Te}^2}{(n_0+n_\ell)}\frac{\partial^2 n_{eh}}{\partial x^2}\right] = 0, \tag{10.7.63}$$

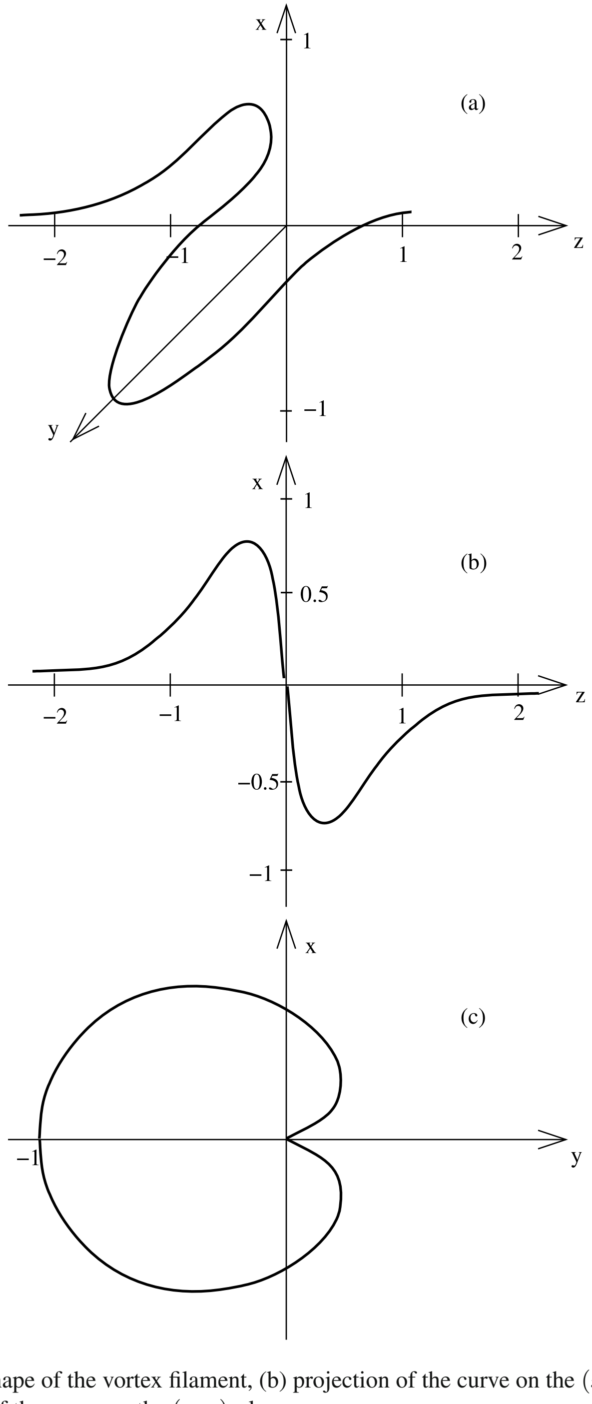

Fig. 10.2 (a) Shape of the vortex filament, (b) projection of the curve on the (x, z)-plane, and (c) projection of the curve on the (x, y)-plane.

which is, by (10.7.62),

$$\frac{\partial^2 E_h}{\partial t^2} + \omega_{pe}^2 \left(1 + \frac{n_\ell}{n_0}\right) E_h - 3v_{Te}^2 \frac{\partial^2 E_h}{\partial x^2} = 0, \tag{10.7.64}$$

where

$$\omega_{pe}^2 = \frac{4\pi e^2 n_0}{m_e}.$$

Equation (10.7.64) describes the trapping of a Langmuir wavepacket in density cavities. We assume that E_h can be decomposed into a fast time-varying component and a slow modulation, that is, we seek a solution in the form

$$E_h(x,t) = \mathcal{E}(x,t) \exp(-i\omega_{pe} t), \tag{10.7.65}$$

where $\mathcal{E}(x,t)$ has a slow modulation in time t. Substituting (10.7.65) in (10.7.64) gives an equation for the modulation function $\mathcal{E}(x,t)$,

$$i\omega_{pe} \frac{\partial \mathcal{E}}{\partial t} + \frac{3}{2} v_{Te}^2 \frac{\partial^2 \mathcal{E}}{\partial x^2} = \frac{1}{2}\left(\omega_{pe}^2 \frac{n_\ell}{n_0}\right) \mathcal{E}. \tag{10.7.66}$$

This equation is based on the hypothesis that the effects of nonlinearity and finite temperature appear as corrections of the same order to the linear dispersion relation of cold plasmas near cutoff values of the wavenumber.

Based on the assumption that electrons respond isothermally to low-frequency motions, the low-frequency motions of plasma are given by

$$\frac{e}{m_e} E_\ell = -\frac{v_{Te}^2}{(n_0 + n_\ell)} \frac{\partial n_\ell}{\partial x} - v_e \frac{\partial u_e}{\partial x}, \tag{10.7.67}$$

$$\frac{\partial n_\ell}{\partial t} + \frac{\partial}{\partial x}\left[(n_0 + n_\ell) u_i\right] = 0, \tag{10.7.68}$$

$$\frac{\partial u_i}{\partial t} - \frac{e}{m_i} E_\ell = 0, \tag{10.7.69}$$

where electron inertia is ignored.

Eliminating u_i from (10.7.68), (10.7.69) yields

$$\frac{\partial^2 n_\ell}{\partial t^2} + \frac{\partial}{\partial x}\left[(n_0 + n_\ell) \frac{e}{m_e} E_\ell\right] = 0. \tag{10.7.70}$$

We next use (10.7.67) to eliminate E_ℓ from (10.7.70), so that equation (10.7.70) becomes

$$\frac{\partial^2 n_\ell}{\partial t^2} - c_s^2 \frac{\partial^2 n_\ell}{\partial x^2} = \left(\frac{n_0 m_e}{m_i}\right) \frac{\partial}{\partial x}\left(u_e \frac{\partial u_e}{\partial x}\right). \tag{10.7.71}$$

Averaging over the time scale ω_{pe}^{-1}, the low-frequency contribution of $(u_e \frac{\partial u_e}{\partial x})$ is given by

$$\left\langle u_e \frac{\partial u_e}{\partial x} \right\rangle = \frac{1}{4}\frac{\partial}{\partial x}\left(\langle |u_e|^2 \rangle\right) = \left(\frac{e}{2\omega_{pe} m_{pe}}\right)^2 \frac{\partial}{\partial x}\left(|E_h|^2\right)$$
$$= (16\pi m_e n_0)^{-1} \frac{\partial}{\partial x}\left(|\mathcal{E}|^2\right). \tag{10.7.72}$$

This represents the fast time-averaged low-frequency ponderomotive force generated by the self-interaction of the high-frequency part of the electric field. This force moves the electrons out of the plasma regions where the electric field has a local maximum. In view of (10.7.72), equation (10.7.71) takes the form

$$\frac{\partial^2 n_\ell}{\partial t^2} - c_s^2 \frac{\partial^2 n_\ell}{\partial x^2} = \frac{1}{16\pi m_i} \frac{\partial^2}{\partial x^2}\left(|\mathcal{E}|^2\right). \tag{10.7.73}$$

This equation describes the formulation of density cavities due to the ponderomotive force related to the electric field.

It is convenient to use the nondimensional quantities defined by

$$x^* = \frac{x}{\lambda_D}, \qquad t^* = \omega_{pe} t, \qquad n_\ell^* = \frac{n_\ell}{n_0}, \quad \text{and} \quad E^* = \frac{E}{4\sqrt{\pi n_0 K T_e}}.$$

In terms of these dimensionless quantities, dropping the asterisks, equations (10.7.66) and (10.7.73) represent *Zakharov's equations*

$$i\varepsilon \frac{\partial \mathcal{E}}{\partial t} + \frac{3}{2}\frac{\partial^2 \mathcal{E}}{\partial x^2} = \frac{1}{2} n_\ell \mathcal{E}, \tag{10.7.74}$$
$$\frac{\partial^2 n_\ell}{\partial t^2} - \frac{\partial^2 n_\ell}{\partial x^2} = \frac{1}{4}\frac{\partial^2}{\partial x^2}\left(|\mathcal{E}|^2\right), \tag{10.7.75}$$

where $\varepsilon = \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}$.

We seek a stationary wave solution of this system as

$$n_\ell, \quad \mathcal{E} \sim \phi(x - Mt) = \phi(\xi), \tag{10.7.76}$$

so that equation (10.7.75) reduces to

$$4 n_\ell \left(1 - M^2\right) = -|\mathcal{E}|^2. \tag{10.7.77}$$

Using (10.7.77), equation (10.7.74) gives the *nonlinear Schrödinger* equation

$$i\varepsilon \mathcal{E}_t + \frac{3}{2}\mathcal{E}_{xx} + \gamma |\mathcal{E}|^2 \mathcal{E} = 0, \tag{10.7.78}$$

where $\gamma = \frac{1}{8}(1 - M^2)^{-1}$.

We write

$$\mathcal{E}(x,t) = \psi(\xi) \exp\left[i\left(\frac{\varepsilon M x}{3} - \omega t\right)\right] \tag{10.7.79}$$

in (10.7.78) so that ψ satisfies the equation

$$\psi'' - \frac{2}{3}\left(\frac{M^2}{6} - \varepsilon\omega\right)\psi + \frac{2}{3}\gamma\psi^3 = 0. \tag{10.7.80}$$

This nonlinear equation admits a solution in the form

$$\psi(\xi) = \frac{1}{8}\left[\frac{1}{2\gamma}\left(\frac{M^2}{6} - \varepsilon\omega\right)\right]^{\frac{1}{2}} \operatorname{sech}\left[\left\{\frac{2}{3}\left(\frac{M^2}{6} - \varepsilon\omega\right)\right\}^{\frac{1}{2}}\xi\right], \tag{10.7.81}$$

so that (10.7.79) gives the explicit representation of $\mathcal{E}(x,t)$. Physically, the solution represents envelope solitons only at subsonic speeds $(M < 1)$. These envelope solitons travel faster, but become smaller and narrower as M increases. Schmidt (1975) investigated the instability and collapse of Langmuir solitons in some detail and found that the envelope solitons of Zakharov's equations are stable to one-dimensional perturbations, but unstable to two-dimensional perturbations. Considerable research on envelope solitons was done by several authors including Karpman (1971, 1975a, 1975b), Gibbons et al. (1977), Rowland et al. (1981), and Infeld and Rowlands (1990).

Example 10.7.5 (Motion of an Electron Fluid). The basic equations of motion for the one-dimensional adiabatic motion of an electron fluid are

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \tag{10.7.82}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{mn}\frac{\partial}{\partial x}(nT) - \frac{e}{m}\frac{\partial \phi}{\partial x} = 0, \tag{10.7.83}$$

$$\frac{d}{dt}\left(n^{-2}T\right) = 0, \tag{10.7.84}$$

$$-\frac{\partial^2 \phi}{\partial x^2} + 4\pi e(n - n_0) = 0, \tag{10.7.85}$$

where n and u are the density and the flow velocity, respectively, ϕ is the electrostatic potential, and T is the electron temperature.

Eliminating T from (10.7.83), (10.7.84) gives

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \left(\frac{3T_0}{mn_0^2}\right)\left(n\frac{\partial n}{\partial x}\right) - \frac{e}{m}\frac{\partial \phi}{\partial x} = 0, \tag{10.7.86}$$

where n_0 and T_0 are the average values of n and T, respectively.

We introduce the transformation of variables

$$\xi = \varepsilon(x - Ct), \qquad \tau = \varepsilon^2 t, \quad \text{and} \quad \theta = kx - \omega t, \tag{10.7.87}$$

where ε is a small parameter and C is a constant to be determined. We expand n, u, and ϕ in powers of ε as

$$n = n_0 + \varepsilon n^{(1)}(\xi,\tau,\theta) + \varepsilon^2 n^{(2)}(\xi,\tau,\theta) + \cdots. \tag{10.7.88}$$

Substituting

$$\phi^{(1)} = \psi(\xi, \tau)\exp(i\theta) + c.c. \tag{10.7.89}$$

from the first-order equation, we obtain the dispersion relation

$$\omega^2 = \omega_p^2 + \left(\frac{3T_0}{m}\right)k^2, \tag{10.7.90}$$

where $\omega_p = (4\pi e^2 n_0/m)^{\frac{1}{2}}$.

The compatibility condition of the second-order equation gives

$$C = \frac{\partial\omega}{\partial k} = \left(\frac{3T_0 k}{m\omega}\right). \tag{10.7.91}$$

Similarly, the compatibility condition of the third-order equation yields

$$i\frac{\partial\psi}{\partial\tau} + \frac{3}{2}\left(\frac{T_0\omega_p^2}{m\omega^3}\right)\frac{\partial^2\psi}{\partial\xi^2} - \left(\frac{e}{m}\right)^2\left(\frac{T_0 k^6}{m\omega\omega_p^6}\right)(16\omega^2 - \omega_p^2)|\psi|^2\psi$$
$$+ \left(\frac{\omega}{n_0}\right)R(\xi, \tau)\psi = 0, \tag{10.7.92}$$

where R is an arbitrary function of ξ and τ. If R is independent of ξ, equation (10.7.92) reduces to the form

$$i\frac{\partial\psi}{\partial\zeta} + \alpha\frac{\partial^2\psi}{\partial\eta^2} + \beta|\psi|^2\psi + i\gamma\psi = 0, \tag{10.7.93}$$

where $\eta = \varepsilon(\xi - Ct) = \varepsilon\xi - C\tau$, $\zeta = \varepsilon^2 t = \varepsilon\tau$, and α, β, and γ are known constants given by Sanuki et al. (1972). In this case, $\alpha\beta < 0$, and γ is a damping coefficient. Equation (10.7.93) is called the *generalized nonlinear Schrödinger equation* since it is modified by including the Landau damping. Sanuki et al. (1972) derived the equation (10.7.93) based on the Vlasov equation to describe the slow, nonlinear modulation of the amplitude of plasma waves including the effects of Landau damping.

Example 10.7.6 (Nonlinear Quasi-Harmonic Waves and Modulational Instability). We derive equations for quasi-harmonic waves in a nonlinear medium from the nonlinear Klein–Gordon equation

$$u_{tt} - c^2 u_{xx} + d^2 u + \sigma u^3 = 0, \tag{10.7.94}$$

where c, d, and σ are constants. We seek a quasi-harmonic wave solution of (10.7.94) with a slowly varying complex amplitude $A(x, t)$ in the form

$$u(x, t) = A(x, t)\exp\left[i(\omega_0 t - k_0 x)\right] + c.c., \tag{10.7.95}$$

with frequency ω_0 and wave number k_0 chosen so as to satisfy the dispersion relation of the linear equation (10.7.94) with $\sigma = 0$

$$\omega^2 = d^2 + c^2 k^2. \tag{10.7.96}$$

We next calculate the second-order derivatives and the cubic term in (10.7.94) so that

$$\begin{aligned}
u_{tt} &= \big(A_{tt} + 2i\omega_0 A_t - \omega_0^2 A\big)e^{i\theta_0} + c.c.,\\
u_{xx} &= \big(A_{xx} - 2ik_0 A_x - k_0^2 A\big)e^{i\theta_0} + c.c.,\\
u^3 &= A^3 e^{3i\theta_0} + A^{*3}e^{-3i\theta_0} + 3A|A|^2 e^{i\theta_0} + 3A^*|A|^2 e^{-i\theta_0},
\end{aligned}$$

where $\theta_0 = \omega_0 t - k_0 x$. Substituting these expressions into (10.7.94) and equating the coefficients of $\exp(i\theta_0)$ yields an equation for the complex amplitude $A(x,t)$

$$i(A_t + c_g A_x) + \frac{1}{2\omega_0}\big(A_{tt} - c^2 A_{xx}\big) + \alpha|A|^2 A = 0, \tag{10.7.97}$$

where $c_g = c^2(k_0/\omega_0) = (\frac{d\omega}{dk})_{k=k_0}$ is the group velocity and $\alpha = (\frac{3\sigma}{2\omega_0})$ is the coefficient of cubic nonlinear term.

We use the variables $\xi = x - c_g t$, $\tau = t$ in (10.7.97) so that the wave envelope varies slowly in the reference frame moving with the group velocity and equation (10.7.97) reduces to the nonlinear Schrödinger equation

$$iA_\tau - \beta A_{\xi\xi} + \alpha|A|^2 A = 0, \tag{10.7.98}$$

where $\beta = c_g^2 - c^2 = \frac{1}{2}(\frac{d^2\omega}{dk^2})$ represents the dispersion parameter.

We next use the transformation $A = a\exp(i\phi)$ to obtain the system of equations for the real variables $a = a(\xi,t)$ and $\phi = \phi(\xi,t)$ in the form

$$\phi_t - \beta\phi_\xi^2 + \beta a^{-1} a_{\xi\xi} = \alpha a^2, \tag{10.7.99}$$

$$a_t - \beta(2a_\xi\phi_\xi + a\phi_{\xi\xi}) = 0. \tag{10.7.100}$$

We next use a new variable $U = -2\beta\phi_\xi$ proportional to the variation of the wave number. This leads to the system of equations

$$U_t + UU_\xi = -q(a^2)_\xi + 2\beta^2\frac{\partial}{\partial\xi}\left(\frac{1}{a}a_{\xi\xi}\right), \tag{10.7.101}$$

$$(a^2)_t + (a^2 U)_\xi = 0, \tag{10.7.102}$$

where $q = 2\alpha\beta = \alpha(\frac{d^2\omega}{dk^2})$ is the product of the dispersion and nonlinear parameters. The instantaneous frequency ω and local wavenumber k of the modulated wave (10.7.95) are

$$\omega = \omega_0 + \phi_t, \qquad k = k_0 - \phi_\xi = k_0 + \frac{1}{2\beta}U(\xi,t). \tag{10.7.103}$$

It follows from (10.7.98)–(10.7.102) that the properties of nonlinear envelope waves are described by the relation between the dispersion parameter β and the nonlinear parameter α. We can use the analysis of nonlinear geometrical optics to study nonlinear envelope waves. We consider only slow modulation so that the term with the third derivative corresponding to the group dispersion may be neglected. Consequently, the modulation equations (10.7.101)–(10.7.102) assume the hydrodynamic form

$$U_t + UU_\xi = -q\left(a^2\right)_\xi, \qquad \left(a^2\right)_t + \left(Ua^2\right)_\xi = 0. \tag{10.7.104}$$

This reveals that, unlike linear theory, the frequency modulation depends on the variation of amplitude even in the geometrical approximation.

We now investigate the modulational instability, that is, the behavior of small perturbations of the amplitude and the frequency of a stationary harmonic wave specified by the constant values $a = a_0$ and $U = U_0 = 0$. We next examine the instability of the monochromatic wave by linearizing the system (10.7.103) in the neighborhood of stationary values and then putting $a = a_0 + a'(\xi,\tau)$ and $U = U'(\xi,\tau)$ so that $a'/a_0 \ll 1$ and $U' \ll 1$. These allow us to obtain a linear system

$$U'_\tau + 2qa_0a'_\xi = 0, \qquad a'_\tau + \frac{1}{2}a_0U'_\xi = 0. \tag{10.7.105}$$

We seek solutions of (10.7.104) in the form of harmonic waves $a', U' \sim \exp[i(\Omega\tau - \kappa\xi)]$ for the real wave number κ. This leads to the dispersion relation of the wave envelope in the form

$$\Omega^2 = qa_0^2\kappa^2, \quad \text{or} \quad \Omega = \pm\sqrt{q}(a_0\kappa). \tag{10.7.106}$$

Thus, if $q > 0$, the frequency Ω has two real values, that is, the system (10.7.103) is hyperbolic and the velocity of wave propagation takes two possible real values $\pm\sqrt{q}a_0$. Consequently, the small perturbation may be represented as a superposition of *slow* and *fast* waves, and the wave is stable in this case.

On the other hand, if $q < 0$, the system (10.7.103) is elliptic and the frequencies are imaginary, and any small initial perturbation will grow in time as $\exp(\sqrt{|q|}a_0\tau) \to \infty$ as $\tau \to \infty$, that is, a small perturbation becomes unstable and *self-modulation* occurs. Physically, the effect of the self-modulation is described by the growth of the sideband components of the wave spectrum into which the energy is transferred from the main component. Thus, the modulational instability occurs provided the nonlinear parameter α and the dispersion parameter β have opposite signs, that is, this product

$$q = 2\alpha\beta = \alpha\frac{d^2\omega}{dk^2} < 0. \tag{10.7.107}$$

This is known as the *Lighthill Criterion* for modulational instability, and it means that the dispersive and nonlinear effects compensate each other so that the spectral components corresponding to modulation interact synchronously with the carrier wave and grow.

There are many physical examples of self-modulation. For example, self-modulation occurs for nonlinear Stokes waves on the surface of deep water, and eventually leads to their decomposition into wave packets. Such phenomena have been observed in a plasma and in electromagnetic waves. The modulational instability also occurs for flexural waves in thin rods and plates.

10.8 Applications to Nonlinear Optics

Nonlinear optics deals with the study of how high intensity light interacts with and propagates through matter. The discovery of laser (light amplification by stimulated emission of radiation) in 1960 led to a remarkably exciting development in the field of nonlinear optics. When a strong laser beam propagates in a medium, it may happen that the refractive index of the medium may change due to various effects produced by the interaction of the beam with the medium. In fact, it is found that the refractive index n is a function of the *intensity* of the light. One of the most interesting topics in nonlinear optics is the self-focusing of beams which can be described by the nonlinear Schrödinger equation. Evidently, this equation and its solitary wave solution play a fundamental role in this new, rapidly growing field of nonlinear optics.

We start with the classical theory for the propagation of light based on the Maxwell equations which, for the first time, combine a unified treatment of electric and magnetic fields. The Maxwell equations for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ and the electric and magnetic induction fields $\mathbf{D}$ and $\mathbf{B}$ are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{10.8.1a}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \tag{10.8.1b}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{10.8.2a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{10.8.2b}$$

where ρ and $\mathbf{j}$ are electric charge and current densities, respectively. The law of conservation of charge is associated with the Maxwell equations. Adding the time derivative of (10.8.2a) with the divergence of (10.8.1b) gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{10.8.3}$$

Integrating this equation over an entire volume V yields

$$\frac{\partial}{\partial t}\int_V \rho\, dV = -\int_V \nabla \cdot \mathbf{j}\, dV = -\int_S \mathbf{j} \cdot \mathbf{n}\, dS. \tag{10.8.4}$$

This expresses the fact that an increase or decrease of charge in a given volume V is affected by the transport of charge, that is, the flow of current, through the boundaries of V. Since we deal with dielectrics in which there are no free charges and in which no current flows, we get $\rho = 0$ and $\mathbf{j} = \mathbf{0}$. Equations (10.8.1a), (10.8.1b), (10.8.2a), (10.8.2b) combined with constitutive relations connecting $\mathbf{B}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{E}$ represent the field equations. The constitutive equations are

$$\mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}, \tag{10.8.5ab}$$

where $\mu = (\varepsilon_0 c^{-2})$ and ε (both of which are constants) are called the *magnetic permeability* and *dielectric constants*, respectively, and c is the speed of light in a vacuum. The vector $\mathbf{P}$ is called the *polarization*, and it is zero in a vacuum. When light propagates in a dielectric medium, the electric field causes distortion in the atomic structure, generating local dipole moments, and thereby induces a polarization field $\mathbf{P}$ which depends on the electric field $\mathbf{E}$. For small to moderate values of the electric field amplitude and if there is no resonance between the electric field and the medium, $\mathbf{P}$ is a linear function of $\mathbf{E}$. In an isotropic medium which responds instantaneously to the electric field, $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$, where χ is called the *electric susceptibility*.

Taking the curl of (10.8.1a) and utilizing (10.8.1b) and (10.8.2a) gives

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{E}) = \left(\varepsilon_0 c^2\right)^{-1} \frac{\partial^2 \mathbf{P}}{\partial t^2}. \tag{10.8.6}$$

This represents the propagation of the electric field, valid for both linear and nonlinear wave propagation. Since $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$, and χ is a constant, (10.8.6) reduces to the wave equation

$$\nabla^2 \mathbf{E} - \left(\frac{n}{c}\right)^2 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \tag{10.8.7}$$

where $n = \sqrt{1+\chi} (> 1)$ is a nondimensional quantity called the *refractive index* of the medium. Equation (10.8.7) shows that the electric field $\mathbf{E}$ propagates with the velocity (c/n).

We seek singly polarized, linear plane wave solutions of equation (10.8.6) in the form

$$\mathbf{E}(\mathbf{x}, t) = \widehat{\mathbf{e}}\left[A \exp(i\theta) + c.c.\right], \tag{10.8.8}$$

where $\widehat{\mathbf{e}}$ is a unit vector in the direction of polarization of $\mathbf{E}$, A is a complex amplitude, $\theta = \mathbf{k} \cdot \mathbf{x} - \omega t$, $\mathbf{k} = (k, \ell, m)$, and $c.c.$ stands for the complex conjugate. These waves are called *monochromatic waves* because they contain only one wavevector $\mathbf{k}$ and one frequency ω. Such solutions (10.8.8) exist for $A \neq 0$, provided that the dispersion relation

$$D(\mathbf{k}, \omega) A = \left[\left(k^2 + \ell^2 + m^2\right) - \left(\frac{n\omega}{c}\right)^2\right] A = 0 \tag{10.8.9}$$

is satisfied.

For nonlinear dispersive media, D also depends on the wave amplitude. In general, $D = 0$ can be solved for any one of the variables, k, ℓ, m, and ω in terms of the other three. In optics, it is customary to solve for the wavenumber in terms of frequency because the refractive index n depends on the frequency, that is, $n = n(\omega)$. For linear or nonlinear waves, the dispersion equation (10.8.9) is algebraic. For wavepackets in a linear dispersive media, (10.8.9) is a linear partial differential equation for A that describes how the wavepacket envelope A moves with the group velocity and is modified by dispersion and diffraction. However, for wavepackets in a nonlinear dispersive medium, (10.8.9) represents a nonlinear partial differential equation for the complex wave envelope A. For the simplest nontrivial case, A satisfies the nonlinear Schrödinger equation, which is a very basic equation in nonlinear optics. Moreover, the dispersion relation is one of the most fundamental concepts in the theory of wave propagation.

The dispersion relation (10.8.9) is also valid when the refractive index of the medium varies slowly, that is, $\nabla n = O(\frac{n}{L}) \ll \frac{n}{\lambda}$, where L is the distance over which n changes. We assume that the electric susceptibility χ, and hence, the refractive index n are slowly varying functions of the space variable x, and we also suppose that $E = \widehat{y}E$ is polarized in the y-direction. This implies that equation (10.8.7) holds for the scalar field $E(x, z, t)$. We seek a solution

$$E(x, z, t) = A \exp(i\theta) + c.c. + \varepsilon E_1, \tag{10.8.10}$$

where $\mathbf{k} = (k, m) = \nabla\theta$ and $\omega = -\theta_t$ denote the wavevector and frequency of the electromagnetic wave, and $\varepsilon = \frac{\lambda}{L} \ll 1$.

Since n varies slowly with x, that is, $n = n(X = \varepsilon x)$, where $\varepsilon = \frac{\lambda}{L} \ll 1$, hence, $\mathbf{k}$ and ω also vary slowly. If ω is assumed to be constant, and since $k = \theta_x$ and $m = \theta_z$, then $k_z = m_x$ holds. Since k is independent of z, we require that m also be constant, and hence, only the wavenumber k can change in the direction of variable refractive index. Substituting the solution (10.8.10) in equation (10.8.7) and assuming that n, A, and k are slowly varying functions of x, that is, functions of X, gives the dispersion relation (10.8.9) to leading order

$$D(k, 0, m, \omega)A = \left[\left(k^2 + m^2\right) - \frac{n^2(X)\omega^2}{c^2}\right]A = 0. \tag{10.8.11}$$

We also find that

$$O(\varepsilon) : \nabla^2 E_1 - \left(\frac{n}{c}\right)^2 \frac{\partial^2 E}{\partial t^2} = \left(-2ik\frac{\partial A}{\partial X} - i\frac{dk}{dX}A\right)e^{i\theta} + c.c. \tag{10.8.12}$$

To eliminate solutions of E_1 that grow like $z \exp(\pm i\theta)$, we must set

$$2k\frac{dA}{dX} + \frac{dk}{dX} \cdot A = 0, \tag{10.8.13}$$

or

$$\left(kA^2\right) = \text{const.} = \left(k_0 A_0^2\right). \tag{10.8.14}$$

Therefore, $k(x)$ and $A(x)$are given by

$$k(x) = 2\left[n^2(X)\frac{\omega^2}{c^2} - m^2\right]^{\frac{1}{2}}, \qquad A(x) = \left(\frac{k_0}{k}\right)^{\frac{1}{2}} A_0, \tag{10.8.15ab}$$

provided that $k \neq 0$.

Equation (10.8.14) is called the *eikonal equation*, and it represents the conservation of wave action. This method of analysis was first proposed by Wentzel, Kramers, Brillouin, and Jeffrey and is known as the *WKBJ method*. Initially, with $(\frac{n\omega}{c}) > m$, the wave cannot propagate, and $k > 0$. However, if $(\frac{n_0\omega}{c}) < m$, there exists a point x, called a *caustic*, at which $n(x_1) \cdot \frac{\omega}{c} = m$ and $k = 0$. At this point, the light rays turn around and the electromagnetic wave is reflected back to the medium with $k = -\sqrt{[(\frac{n\omega}{c})^2 - m^2]}$. The exact nature near $x = x_1$ cannot be determined by the WKBJ method because $k \to 0$, the amplitude A increases like $O(k^{-\frac{1}{2}})$, and the WKBJ method fails. In this case, we can approximate n^2 locally by $\{n^2(x_1) - a(x - x_1)\}$ and seek a solution of the form

$$E(x,t) = B(x)\exp\bigl[i(kz - \omega t)\bigr] + c.c., \quad k = \frac{\omega n_0}{c}. \tag{10.8.16}$$

It turns out that $B(x)$ satisfies the Airy differential equation

$$B_{xx} - \left(\frac{\omega}{c}\right)^2 a(x - x_1)B = 0. \tag{10.8.17}$$

When $x > x_1$, we obtain a solution of (10.8.17) which decays exponentially. On the other hand, when $x < x_1$, the solution of (10.8.17) has two components: one corresponds to an incoming wave with the wavenumber $(-k, 0, m)$, where $k = -\{(\frac{n\omega}{c})^2 - m^2\}^{\frac{1}{2}}$ and the amplitude is given by (10.8.15ab), and the other represents the reflected wave with the wavenumber $(k, 0, m)$. This reveals the fact that light rays turn toward regions of higher refractive index and turn away from regions of lower refractive index. This fact is widely used in the theory of the propagation of light in dielectric materials and has tremendous ramifications. First, the correction to the refractive index is an increasing function of the field intensity, that is,

$$n = n_0 + n_2|A|^2 = n_0\left(1 + \frac{n_2}{n_0}|A|^2\right), \quad n_2 > 0. \tag{10.8.18}$$

If the medium is linear, then $n_2 = 0$. The electromagnetic waves are governed by the standard linear wave equation. So the quantity $\delta = \frac{n_2}{n_0}|A|^2$ measures the strength of nonlinearity, and produces what is called the *Kerr effect*. When light propagates in a Kerr dielectric medium, the light continues to focus in regions of greatest intensity. This instability is called the *self-focusing instability* in optics, and it is widespread in nature. In certain cases, a wavetrain of amplitude which is uniform in space and time becomes unstable, and light intensity concentrates in collapsing filaments. If there is only one transverse direction, a balance between diffraction and nonlinearity leads to the formation of solitons. Hasegawa and Tappert (1973) proved, mathematically, that an optical pulse in a dielectric fiber forms an envelope soliton, that is, the pulses of light either do not change shape or change shape periodically. Mollenauer et al. (1986) demonstrated this effect experimentally. An interesting discussion of these developments has been given by Hasegawa (1990). In two transverse dimensions, the collapse is *no longer* confined, and the light filament continues to intensify.

In reality, light waves are *not* exactly monochromatic. They usually consist of a linear combination of plane waves whose wavevectors and frequencies are confined to a narrow band of order ε about some central values $\mathbf{k}_0$ and $\mathbf{E}_0$, where $0 < \varepsilon \le 1$. Such plane waves are called *wavepackets* represented in the form

$$\mathbf{E} = \widehat{\mathbf{e}}\bigl[A(\mathbf{x},t)\exp\bigl\{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega t) + c.c.\bigr\}\bigr] + O(\varepsilon), \tag{10.8.19}$$

combined with the dispersion relation

$$D(k_0, \ell_0, m_0, \omega_0) = 0, \tag{10.8.20}$$

where the wave amplitude $A(\mathbf{x}, t)$ is a slowly varying function of space and time, that is,

$$\frac{\partial A}{\partial x} = \varepsilon |\mathbf{k}_0|^2 A \ll |\mathbf{k}_0| A. \tag{10.8.21}$$

We introduce the notions of local wavevector $\mathbf{k}$ and local frequency ω, and write the complex amplitude in the polar form $A = a\exp(i\phi)$, where a and ϕ are real. Then the phase of the leading order contribution to the electric field is given by

$$\theta = \mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t + \phi(\mathbf{x}, t), \tag{10.8.22}$$

so that

$$\mathbf{k} = \nabla\theta = \mathbf{k}_0 + \nabla\phi \quad \text{and} \quad \omega = -\theta_t = \omega_0 - \phi_t. \tag{10.8.23ab}$$

Evidently,

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla\omega = 0. \tag{10.8.24}$$

This represents the conservation of wavenumbers.

We consider the evolution of the wavepackets (10.8.19), with $\mathbf{k}_0 = (0, 0, m_0)$ and $\widehat{\mathbf{e}} = (0, 1, 0)$ of an electric field $\mathbf{E}$ that satisfies equation (10.8.7). For simplicity, we first consider the case of constant n. Substituting (10.8.19) in (10.8.7) gives

$$2im_0 \frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial x^2} + 2i\omega_0 \frac{\partial A}{\partial t} - \left(\frac{n}{c}\right)^2 \frac{\partial^2 A}{\partial t^2} = 0, \tag{10.8.25}$$

where $m_0 = \left(\frac{n\omega_0}{c}\right)$. Since each derivative is of order ε, the dominant terms in (10.8.25) are the first and the fourth, so, to the leading order, (10.8.25) yields

$$A_z + \left(\frac{n}{c}\right) A_t = 0. \tag{10.8.26}$$

Thus, to a first-order approximation, A is a function of $\{t - (\frac{n}{c})z\}$ and is constant when $\{t - (\frac{n}{c})z\}$ is constant, that is, in a frame of reference moving with the group velocity

$$\mathbf{C}_g = \nabla_k \omega = \left(0, 0, \frac{c}{n}\right). \tag{10.8.27}$$

In view of (10.8.26), it turns out that $A_{zz} = (\frac{n}{c})^2 A_{tt}$, and so, to order ε^3, equation (10.8.25) becomes

$$\frac{\partial A}{\partial z} + \left(\frac{n}{c}\right) \frac{\partial A}{\partial t} - \left(\frac{i}{2m_0}\right) \frac{\partial^2 A}{\partial x^2} = 0. \tag{10.8.28}$$

This is usually known as the *paraxial* equation because the rays, the normals to the planes of constant phase, of all the plane waves in the wavepacket are nearly parallel. Invoking the change of variables to a frame of reference moving with the group velocity, $z = z$, $\tau = \{t - (\frac{n}{c})z\}$, equation (10.8.28) assumes the form

$$A_z - \left(\frac{i}{2m_0}\right)A_{xx} = 0. \tag{10.8.29}$$

This equation describes the dispersion of a wavepacket in the transverse direction. Such a phenomenon in optics is called *wave diffraction*.

We next recall the dispersion relation (10.8.9) with constant refractive index which was originally derived by using $A\exp\{i(\mathbf{k}\cdot\mathbf{x} - \omega t)\}$ in (10.8.7). If, instead of substituting a plane wave for $\mathbf{E}$, we insert the wavepacket (10.8.19) with $\mathbf{k}_0 = (0, 0, m_0)$, we obtain a modified dispersion relation,

$$D\left(-i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}, m_0 - i\frac{\partial}{\partial z}, \omega_0 + i\frac{\partial}{\partial t}\right)A(x, y, z, t) = 0. \tag{10.8.30}$$

This is an equation for the slowly varying wave envelope A.

We now utilize the frequency dependence of the refractive index, and use $D = (k^2 + \ell^2 + m^2) - \{\frac{n(\omega)\omega}{c}\}^2$, so that equation (10.8.30) becomes

$$\left[\left(-i\frac{\partial}{\partial x}\right)^2 + \left(-i\frac{\partial}{\partial y}\right)^2 + \left(m_0 - i\frac{\partial}{\partial z}\right)^2 \right.$$
$$\left. - \frac{1}{c^2}\left\{n\left(\omega_0 + i\frac{\partial}{\partial t}\right)\left(\omega_0 + i\frac{\partial}{\partial t}\right)\right\}^2\right]A(\mathbf{x}, t) = 0. \tag{10.8.31}$$

To first order, we recover the dispersion relation for the plane waves

$$m_0^2 = n^2(\omega_0)\frac{\omega_0^2}{c^2}. \tag{10.8.32}$$

To the order of ε, we obtain the equation

$$\frac{\partial A}{\partial z} + \left[\frac{\partial}{\partial \omega}\left\{\frac{\omega n(\omega)}{c}\right\}\right]_{\omega=\omega_0}\frac{\partial A}{\partial t} = 0. \tag{10.8.33}$$

This is consistent with the fact that, to the leading order, the wave envelope A of a wavepacket propagates with the group velocity C_g, where $C_g^{-1} = (\frac{\partial m}{\partial \omega})_0$.

Next, we expand (10.8.30) to order ε^2 and retain all terms up to second derivative to obtain the equation

$$(2im_0)A_z + A_{xx} + \left(\frac{2i}{c^2}\right)\omega_0 n(\omega_0)\left\{\frac{\partial}{\partial \omega}(n\omega)\right\}_0 A_t + A_{zz}$$
$$- \frac{1}{c^2}\left[\left\{\frac{\partial}{\partial \omega}(n\omega)\right\}_0^2 + (n_0\omega_0)\left\{\frac{\partial}{\partial \omega^2}(n\omega)\right\}_0\right]A_{tt} = 0. \tag{10.8.34}$$

Using (10.8.33), we can simplify (10.8.34) in the form

$$A_z + \left(\frac{2m}{2\omega}\right)_0 A_t + \frac{1}{2}\left[\left(\frac{\partial^2 m}{\partial k^2}\right)_0 A_{xx} + \left(\frac{\partial^2 m}{\partial \ell^2}\right)_0 A_{yy} + \left(\frac{\partial^2 m}{\partial \omega^2}\right)_0\right] A_{tt} = 0. \tag{10.8.35}$$

We now incorporate the effects of small variations of the refractive index to the order ε^2 in (10.8.35) by replacing n in (10.8.31) by $(n + \delta n)$. Consequently, an extra term $c^{-2}(2n\omega_0^2 \delta n)A$ will be included in equation (10.8.34), and, if we divide across by $(2im_0)$, an additional term $-i\delta n(\frac{\omega_0}{c})A = -i(\frac{\delta n}{n})m_0 A$ is generated in (10.8.35), and this term becomes $-i(\frac{\partial m}{\partial n})_0 \delta n A$. Thus, equation (10.8.35) reduces to the form

$$A_z + \left(\frac{\partial m}{\partial \omega}\right)_0 A_t + \frac{i}{2}\left[\left(\frac{\partial^2 m}{\partial k^2}\right)_0 A_{xx} + \left(\frac{\partial^2 m}{\partial \ell^2}\right)_0 A_{yy} + \left(\frac{\partial^2 m}{\partial \omega^2}\right)_0 A_{tt} - i\left(\frac{\partial m}{\partial n}\right)(\delta n)\right] A = 0. \tag{10.8.36}$$

This equation is widely used in nonlinear optics.

Finally, we derive the nonlinear Schrödinger equation for the evolution of a wavepacket envelope $A(\mathbf{x}, t)$ in a dispersive medium with a nonlinear refractive index n given by (10.8.18). The nonlinear dispersion relation is then given by

$$D\big(k, \ell, m, \omega, |A|^2\big)A = \left\{\big(k^2 + \ell^2 + m^2\big) - \left(\frac{n\omega}{c}\right)^2\right\} A = 0. \tag{10.8.37}$$

For a wave amplitude A of order ε, so that $A_{zz} = O(\varepsilon^3)$, $|A|^2 A = O(\varepsilon^3)$, and hence, the variation of δn in refractive index is of order ε^2, the resulting equation for the wave envelope A is the same as (10.8.36) with $\delta n = n_2 |A|^2$. To include the effects of slow variations of space variables and amplitude, we replace δn in (10.8.36) by $\delta n(x, y) + n_2|A|^2$, so that dropping the subscripts zero on m and the coefficients, equation (10.8.36) becomes

$$A_z + \left(\frac{\partial m}{\partial \omega}\right) A_t + \frac{i}{2}\left[\left(\frac{\partial^2 m}{\partial k^2}\right) A_{xx} + \left(\frac{\partial^2 m}{\partial \ell^2}\right) A_{yy} + \left(\frac{\partial^2 m}{\partial \omega^2}\right) A_{tt}\right] - \left(\frac{im}{n}\right)\big(\delta n + n_2|A|^2\big) A = 0. \tag{10.8.38}$$

Or equivalently,

$$A_z + \left(\frac{\partial m}{\partial \omega}\right) A_t + \frac{i}{2}\big(\alpha A_{xx} + \beta A_{yy} + \gamma A_{tt}\big) - \left(\frac{im}{n}\right)\big(\delta n + n_2|A|^2\big) A = 0, \tag{10.8.39}$$

where the three coefficients of dispersive terms are

$$\alpha = \frac{\partial^2 m}{\partial k^2}, \qquad \beta = \frac{\partial^2 m}{\partial \ell^2}, \quad \text{and} \quad \gamma = \frac{\partial^2 m}{\partial \omega^2}. \tag{10.8.40}$$

If the associated carrier wave is proportional to $\exp\{i(\mathbf{k}\cdot\mathbf{x}-\omega t)\}$, where

$$k=(0,0,m) \quad \text{and} \quad m=\frac{n(\omega)n}{c}, \tag{10.8.41}$$

then

$$\frac{\partial m}{\partial \omega}=\frac{1}{c}\frac{\partial}{\partial \omega}(n\omega) \quad \text{and} \quad \alpha=\beta=-\frac{1}{m}. \tag{10.8.42}$$

Thus, the resulting nonlinear Schrödinger equation becomes

$$A_z+\left\{\frac{1}{c}\frac{\partial}{\partial\omega}(n\omega)\right\}A_t-\frac{i}{2m}(A_{xx}+A_{yy})+\left\{\left(\frac{i}{2c}\right)\frac{\partial^2}{\partial\omega^2}(n\omega)\right\}A_{tt}$$
$$-\left(\frac{im}{n}\right)\left(\delta n+n_2|A|^2\right)A=0. \tag{10.8.43}$$

The above analysis reveals that the NLS equation (10.8.39) or (10.8.43) describing the self-focusing of light beams is the fundamental equation in nonlinear optics.

In particular, if A is independent of x, y, and t, and $\delta n\equiv 0$, the solution of the resulting equation (10.8.39) is given by

$$A(z)=A_0\exp\left[im\left(\frac{n_2}{n}\right)|A_0|^2 z\right]. \tag{10.8.44}$$

This solution can be used to determine the wavenumber of the electric field $E=\widehat{\mathbf{E}}[A(z)\exp\{i(mz-\omega t)\}+c.c.]$. The value of this wavenumber is equal to $m(1+\frac{n_2}{n}|A|^2)$ or $\frac{\omega}{c}(n+n_2|A|^2)$. According to the theory of the NLS equation, it also reveals that, if the product of (mn_2) with any one of three dispersive coefficients α, β, or γ (that is, $\alpha=\beta=-\frac{1}{m}$ and $(mn_2)(-\frac{1}{m})=-n_2$) is negative, then the amplitude-modulated solution (10.8.44) is definitely unstable.

In particular, if the wavepacket envelope A is independent of t with $\alpha=\beta=-\frac{1}{m}$, equation (10.8.39) becomes

$$A_z-\frac{i}{2m}(A_{xx}+A_{yy})-\left(\frac{im}{n}\right)\left(\delta n+n_2|A|^2\right)A=0. \tag{10.8.45}$$

Further, if we assume that $\delta n=0$, this equation reduces to the form

$$iA_z+\frac{1}{2m}(A_{xx}+A_{yy})+\left(\frac{m}{n}\right)n_2|A|^2A=0. \tag{10.8.46}$$

If, initially, the field envelope $A=A_0(z)$ is independent of x and y, then a small perturbation δA confined at some point causes the refractive index $n_2|A|^2$ to increase there. The light rays from the neighboring regions continue to focus toward this region, enhancing the intensity more and more there. As more light focuses there, the refractive index becomes even higher, and even more light converges. Thus, the effect of an increased refractive index leads to *focusing instability* in nonlinear optics. However, this instability is not confined to transverse modulations. We now derive the nonlinear Schrödinger equation for a concentrated wave mode traveling in a waveguide, and, for this model, there is *no* transverse dispersion but *only* dispersion in the direction of the wave propagation. So, if we suppose that $A(z,t)$ is independent of x and y with the fact that $\delta n = 0$, then (10.8.39) becomes

$$A_z + \left(\frac{\partial m}{\partial \omega}\right) A_t + \frac{i}{2}\left(\frac{\partial^2 m}{\partial \omega^2}\right) A_{tt} - im\left(\frac{n_2}{n}\right)|A|^2 A = 0. \tag{10.8.47}$$

Obviously, the focusing instability occurs provided that the product $(\frac{\partial^2 m}{\partial \omega^2}) \times (-\frac{mn_2}{n}) = -\frac{\gamma mn_2}{n}$ is positive. For the diffraction case, the product is $(\frac{n_2}{n})$ whose sign depends on that of n_2. In this case of wave propagation in an optical fiber, it can be shown that $n_2 > 0$ only if $\gamma < 0$. However, in general, γ can be positive or negative in the neighborhood of resonance where the real part of the susceptibility depends on frequency. These situations are referred to as *normal dispersion* and *anomalous dispersion*, respectively. According to the general theory of the NLS equation, there exists a stable soliton solution of (10.8.47) in nonlinear optical fibers. Two solitons of different frequencies have different group velocities and pass through each other without distortion. However, if two solitons have the same frequency, and hence, the same velocity, they undergo a periodic oscillation.

There is a remarkable change in the properties of solutions of (10.8.39) when two transverse dimensions and time are included. In the defocusing case $n_2 < 0$ $(n_2\gamma > 0)$, uniform wavetrain solutions are neutrally stable. In fact, deformations in the uniform wavetrains, in which intensity diminishes locally, neither grow nor decay, but evolve as solitons of a different nature. These solitons are called *dark solitons* because they have intensities lower than those of the ambient wavetrains.

We close this section by adding some brief comments on recent applications of nonlinear optics to modern technology. The use of lasers ranges from high-density data storage on optical disks to greatly improved surgical techniques in ophthalmology, neurosurgery, and dermatology. The discovery of optical fibers led to a revolutionary growth in modern telecommunications. There has been a tremendous impact of optical fibers on the fundamental study of nonlinear optical interactions. Once high-quality, low-loss, single-mode fibers became available in the 1980s, the number of possible new applications has grown in a remarkable fashion. Soliton lasers, modulational instability lasers, SBS lasers, and various kinds of optical devices were built, and the open cavity resonator marked the critical breakthrough in the design of working lasers. It is clear that nonlinear optics is likely to revolutionize future telecommunications and computer technology.

10.9 Exercises

1. (a) Show that the dispersion relation (7.2.11b) for the Boussinesq equation (7.2.11a) in the long-wave approximation $(k \to 0)$ is given by

$$\omega = \omega(k) = c_0 k\left(1 - \frac{1}{2}\beta^2 k^2\right).$$

(b) Hence, show that the Fourier integral representation around $k_0 = 0$ is approximately equal to

$$\phi(x,t) \sim \int_{\delta k} F(\delta k) \exp\left[i\left\{\delta k(x-ct) + \frac{1}{2}c\beta^2(\delta k)^3 t\right\}\right] d(\delta k).$$

(c) Introduce the slow variables $\xi = \varepsilon(x-ct)$ and $\tau = \varepsilon^3 t$, where $\delta k = 0(\varepsilon)$, and then show that the long wave field $\phi(x,t) = A(\xi,\tau)$ satisfies the linearized KdV equation

$$A_\tau + \frac{1}{2}c\beta^2 A_{\xi\xi\xi} = 0.$$

2. Show that the NLS equation

$$iu_t + u_{xx} + |u|^2 u = 0$$

has the combined rational and oscillatory (Peregrine 1983) solution

$$u(x,t) = \left\{1 - 4(1+2it)/\left(1+2x^2+4t^2\right)\right\} \exp(it).$$

3. Show that the NLS equation in Exercise 2 has the Ma solitary wave solution (Ma 1979; Peregrine 1983) of the form

$$u(x,t) = a\exp\left(ia^2 t\right)\left\{1 + 2m(m\cos\theta + in\sin\theta)/f(x,t)\right\}$$

for all real a and m, where $n^2 = (1+m^2)$, $\theta = 2mna^2 t$, and $f(x,t) = n\cosh(ma\sqrt{2}x) + \cos\theta$.

4. Seek a traveling wave solution of the NLS equation in Exercise 2 in the form

$$u(x,t) = r\exp\left\{i(\theta + \omega t)\right\},$$

where $r(x-ct)$ and $\theta(x-ct)$ are real functions and c and ω are real constants. Prove that

$$\theta' = \frac{1}{2}\left(c + \frac{A}{R}\right) \quad \text{and} \quad (R')^2 = -2F(R),$$

where $R = r^2$, $F(R) = R^3 - 2(\omega - \frac{1}{4}c^2)R^2 + BR + A$, and A and B are arbitrary real constants of integration. Hence, show that there exists a solitary wave solution of the form

$$u(x,t) = a\exp\left[i\left\{\frac{1}{2}c(x-ct) + \omega t\right\}\right] \operatorname{sech}\left[\frac{a}{\sqrt{2}}(x-ct)\right],$$

for all a^2 such that

$$a^2 = 2\left(\omega - \frac{1}{4}c^2\right) > 0.$$

5. Following the same method as given in Exercise 4, show that there exists a solitary wave solution (Zakharov and Shabat 1973) of the NLS equation

$$iu_t + u_{xx} - |u|^2 u = 0$$

with $r^2(\xi) = m - 2\kappa^2 \operatorname{sech}^2(\kappa\xi)$ and $c\tan\{\theta(\xi)\} = -2\kappa\tanh(\kappa\xi)$ for all c, where $\xi = x - ct$, $\omega = -m$, and $\kappa = \frac{1}{2}(2m - c^2)^{\frac{1}{2}} > 0$.

6. Show that an auto-Bäcklund transformation (Konno and Wadati 1975) for the NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0$$

is the following pair of equations:

$$\begin{aligned} u_x + v_x &= (u - v)\{4\lambda^2 - |u + v|^2\}^{\frac{1}{2}}, \\ u_t + v_t &= i(u_x - v_x)\{4\lambda^2 - |u + v|^2\}^{\frac{1}{2}} \\ &\quad + \frac{1}{2} i(u + v)\{|u + v|^2 + |u - v|^2\}, \end{aligned}$$

where u and v satisfy the equation. Hence, find the solitary wave solution of this NLS equation by setting $v = 0$ for all x and t.

7. Derive the NLS equation for the local amplitude $A(x, y, t)$ from the resonant interaction equations (Phillips 1981; Debnath 1994) in the form

$$A_t + i\left(\frac{\omega_0}{2k_0}\right) A_x + i\left(\frac{\omega_0}{8k_0^2}\right) A_{xx} - i\left(\frac{\omega_0}{4k_0^2}\right) A_{yy} = \int \dot{b} \exp(i\theta)\, dk.$$

8. If $m = [\{n^2(\omega)(\frac{\omega^2}{c^2})\} - (k^2 + \ell^2)]^{\frac{1}{2}}$, show that

$$\begin{aligned} \frac{\partial m}{\partial k} &= \frac{\partial m}{\partial \ell} = 0, & \frac{\partial^2 m}{\partial k^2} &= \frac{\partial^2 m}{\partial \ell^2} = -\frac{1}{m_0}, \\ \frac{\partial m}{\partial \omega} &= \left[\frac{\partial}{\partial \omega}\left(\frac{n\omega}{c}\right)\right]_0, & \frac{\partial^2 m}{\partial \omega^2} &= \left[\frac{\partial^2}{\partial \omega^2}\left(\frac{n\omega}{c}\right)\right]_0, \end{aligned}$$

where the derivatives are computed at $k = \ell = 0$, $m = m_0 = n(\omega_0)(\frac{\omega_0}{c})$.

9. Show that the solution

$$A(z) = A_0 \exp\left(i\gamma |A_0|^2 z\right)$$

of the nonlinear Schrödinger equation

$$A_z - i\alpha(A_{xx} + A_{yy}) = i\gamma A^2 A^*$$

is unstable. Also show that the maximum growth rate occurs for $k = |A_0|\sqrt{\gamma/\alpha}$.

10. Obtain the solution of the De Broglie wave problem for an electron of mass m in a constant electric field E along the x-axis. The motion of the electron is described by the wave function $\psi(x, t)$ that satisfies the Schrödinger equation

$$i\hbar\psi_t = \left[V(x) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right]\psi,$$

where $\hbar$ is the Planck constant and $V(x) = xE$ is the potential energy of the electron.

11

Nonlinear Klein–Gordon and Sine-Gordon Equations

As long as a branch of science affords an abundance of problems, it is full of life; want of problems means death or cessation of independent development. Just as every human enterprise prosecutes final aims, so mathematical research needs problems. Their solution steels the force of the investigator; thus he discovers new methods and view points and widens his horizon.

David Hilbert

The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and there is no rational explanation for it. It is not at all natural that "laws of nature" exist, much less that man is able to discover them. The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.

Eugene Wigner

11.1 Introduction

This chapter deals with the theory and applications of nonlinear Klein–Gordon (KG) and sine-Gordon (SG) equations. Special emphasis is given to various methods of solutions of these equations. The Green function method combined with integral transforms is employed to solve the linear Klein–Gordon equation. The Whitham averaging procedure and the Whitham averaged Lagrangian principle are used to discuss solutions of the nonlinear Klein–Gordon equation. Included are different ways of finding general and particular solutions of the sine-Gordon equation. Special attention is given to solitons, antisolitons, breather solutions and the energy associated with them, interaction of solitons, Bäcklund transformations, similarity solutions, and the inverse scattering method. Significant features of these methods and solutions are described with other ramifications.

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11.2 The One-Dimensional Linear Klein–Gordon Equation

The one-dimensional, inhomogeneous, Klein–Gordon equation is given by

$$u_{tt} - c^2 u_{xx} + d^2 u = p(x,t), \quad x \in \mathbb{R},\ t > 0 \tag{11.2.1}$$

with the initial boundary conditions

$$u(x,0) = 0 = u_t(x,0) \quad \text{for } x \in \mathbb{R}, \tag{11.2.2ab}$$

$$u(x,t) \to 0 \quad \text{as } |x| \to \infty,\ t > 0, \tag{11.2.3}$$

where c and d are constants.

The Green function $G(x,t)$ associated with this problem satisfies the equation

$$G_{tt} - c^2 G_{xx} + d^2 G = \delta(x)\delta(t), \quad x \in \mathbb{R},\ t > 0, \tag{11.2.4}$$

with the same initial and boundary conditions.

Application of the joint Laplace and Fourier transform (Debnath 1995) gives

$$\bar{\tilde{G}}(k,s) = \left(\frac{1}{\sqrt{2\pi}}\right)\frac{1}{(s^2+\alpha^2)}, \tag{11.2.5}$$

where $\alpha = (c^2k^2 + d^2)^{\frac{1}{2}}$.

The inverse Laplace transform yields

$$\tilde{G}(k,t) = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin\alpha t}{\alpha}\right). \tag{11.2.6}$$

Finally, the inverse Fourier transform leads to the solution

$$\begin{aligned} G(x,t) &= \frac{1}{c}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(k^2+\frac{d^2}{c^2}\right)^{-\frac{1}{2}} \sin\left[ct\sqrt{k^2+\frac{d^2}{c^2}}\right]\exp(ikx)\,dk \\ &= \frac{1}{2c}J_0\left[\frac{d}{c}\sqrt{c^2t^2-x^2}\right]H\left(ct-|x|\right) \\ &= \begin{cases} \frac{1}{2c}J_0[\frac{d}{c}\sqrt{c^2t^2-x^2}] & \text{if } |x| < ct, \\ 0 & \text{if } |x| > ct. \end{cases} \end{aligned} \tag{11.2.7}$$

In the limit as $d \to 0$, this result is in perfect agreement with the solution for the standard wave equation.

If the source is located at (ξ,τ), Green's function assumes the form

$$G(x,t;\xi,\tau) = \begin{cases} \frac{1}{2c}J_0[\frac{d}{c}\{c^2(t-\tau)^2-(x-\xi)^2\}^{\frac{1}{2}}] & \text{if } |x-\xi| < c(t-\tau), \\ 0 & \text{if } |x-\xi| > c(t-\tau). \end{cases} \tag{11.2.8}$$

In terms of the Heaviside functions, Green's function can be expressed as

$$G(x,t;\xi,\tau)=\frac{1}{2c}J_0\left[\frac{d}{c}\left\{c^2(t-\tau)^2-(x-\xi)^2\right\}^{\frac{1}{2}}\right]H\big[x+ct-(\xi+ct)\big]$$
$$\times H\big[\xi+ct-(x+c\tau)\big]. \tag{11.2.9}$$

In the limit as $d\to 0$, this result reduces to that for the wave equation.

The solution of the Klein–Gordon equation (11.2.1) with the initial conditions

$$u(x,0)=f(x),\quad u_t(x,0)=g(x),\quad x\in\mathbb{R}, \tag{11.2.10}$$

can be expressed in terms of Green's function as

$$u(x,t)=\int_0^t\int_{-\infty}^{\infty}p(\xi,\tau)G(x,t;\xi,\tau)\,d\xi\,d\tau$$
$$+\int_{-\infty}^{\infty}\big[g(\xi)G(x,t;\xi,0)-f(\xi)G_\tau(x,t;\xi,0)\big]\,d\xi. \tag{11.2.11}$$

We note that, in the double integral in (11.2.11), $G=0$ for $\tau>1$, and hence, the limit in the τ integral extends only up to t. Also, it follows from (11.2.8) that G is nonzero when $|x-\xi|<c(t-\tau)$, which is equivalent to $x-c(t-\tau)<\xi<x+c(t-\tau)$. Consequently, the double integral in (11.2.11) becomes

$$\int_0^t d\tau\int_{-\infty}^{\infty}p(\xi,\tau)G(x,t;\xi,\tau)\,d\xi$$
$$=\frac{1}{2c}\int_0^t d\tau\int_{x-c(t-\tau)}^{x+c(t-\tau)}p(\xi,\tau)J_0\left[\frac{d}{c}\left\{c^2(t-\tau)^2-(x-\xi)^2\right\}^{\frac{1}{2}}\right]d\xi. \tag{11.2.12}$$

Similarly, since the product of the Heaviside functions vanishes outside the interval $(x-ct,x+ct)$, we find that

$$\int_{-\infty}^{\infty}g(\xi)G(x,t;\xi,0)\,d\xi=\frac{1}{2c}\int_{x-ct}^{x+ct}g(\xi)J_0\left[\frac{d}{c}\left\{c^2t^2-(x-\xi)^2\right\}^{\frac{1}{2}}\right]d\xi. \tag{11.2.13}$$

It follows from direct integration that

$$\big[G_\tau(x,t;\xi,\tau)\big]_{\tau=0}=-\left(\frac{td}{2}\right)\frac{J_0'[\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}]}{\sqrt{c^2t^2-(x-\xi)^2}}$$
$$\times H(x+ct-\xi)H(\xi-x+ct)$$
$$+\frac{1}{2}J_0\left[\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}\right]$$
$$\times\big\{\delta(x+ct-\xi)H(\xi-x+ct)$$
$$+\delta(\xi-x+ct)H(x+ct-\xi)\big\}.$$

In view of this result, the second term within the square bracket of (11.2.11) becomes

$$
\begin{aligned}
&-\int_{-\infty}^{\infty} f(\xi) G_\tau(x,t;\xi,0)\,d\xi \\
&\quad = -\left(\frac{t\,d}{2}\right)\int_{x-ct}^{x+ct} \frac{J_1\left[\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}\right]}{\sqrt{c^2t^2-(x-\xi)^2}} f(\xi)\,d\xi \\
&\qquad + \frac{1}{2}\int_{-\infty}^{\infty} J_0\left(\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}\right)\big\{\delta(x+ct-\xi)H(\xi-x+ct) \\
&\qquad + \delta(\xi-x+ct)H(x+ct-\xi)\big\} f(\xi)\,d\xi \\
&\quad = -\left(\frac{t\,d}{2}\right)\int_{x-ct}^{x+ct} \frac{J_1\left[\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}\right]}{\sqrt{c^2t^2-(x-\xi)^2}} f(\xi)\,d\xi \\
&\qquad + \frac{1}{2}\big[f(x-ct)+f(x+ct)\big],
\end{aligned} \tag{11.2.14}
$$

in which the property of the delta function with $H(2ct) = 1$ is used.

Combining (11.2.12), (11.2.13), and (11.2.14), the final form of solution (11.2.11) is given by

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\big[f(x-ct)+f(x+ct)\big] \\
&\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} J_0\left[\left(\frac{d}{c}\right)\sqrt{c^2t^2-(x-\xi)^2}\right] g(\xi)\,d\xi \\
&\quad - \left(\frac{td}{2}\right)\int_{x-ct}^{x+ct} \frac{J_1\left[\frac{d}{c}\sqrt{c^2t^2-(x-\xi)^2}\right]}{\sqrt{c^2t^2-(x-\xi)^2}} f(\xi)\,d\xi \\
&\quad + \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} J_0\left[\left(\frac{d}{c}\right)\sqrt{c^2(t-\tau)^2-(x-\xi)^2}\right] p(\xi,\tau)\,d\xi.
\end{aligned} \tag{11.2.15}
$$

If $d = 0$, this solution reduces to that for the Cauchy problem for the inhomogeneous wave equation.

11.3 The Two-Dimensional Linear Klein–Gordon Equation

The two-dimensional, linear, inhomogeneous, Klein–Gordon equation is

$$u_{tt} - c^2(u_{xx}+u_{yy}) + d^2u = p(x,y,t), \quad -\infty < x,y < \infty,\ t > 0. \tag{11.3.1}$$

The initial and boundary conditions are

$$u(x,y,0) = 0 = u_t(x,y,0) \quad \text{for all } x \text{ and } y, \tag{11.3.2}$$

$$u(x,y,t) \to 0 \quad \text{as } r = \sqrt{x^2+y^2} \to \infty,\ t > 0. \tag{11.3.3}$$

The Green function $G(x,y,t)$ for the two-dimensional Klein–Gordon equation satisfies the equation

$$G_{tt} - c^2(G_{xx} + G_{yy}) + d^2 G = \delta(x)\delta(y)\delta(t), \quad -\infty < x, y < \infty, t > 0, \tag{11.3.4}$$

with the conditions

$$G(x, y, 0) = 0 = G_t(x, y, 0) \quad \text{for all } x \text{ and } y, \tag{11.3.5}$$

$$G(x, y, t) \to 0 \quad \text{as } \left(x^2 + y^2\right) \to \infty, \; t > 0. \tag{11.3.6}$$

In terms of polar coordinates (r, θ) without θ-dependence, equation (11.3.4) becomes

$$G_{tt} - c^2\left(G_{rr} + \frac{1}{r}G_r\right) + d^2 G = \frac{\delta(r)\delta(t)}{2\pi r}, \quad r > 0, \; t > 0. \tag{11.3.7}$$

Application of the joint Laplace and Hankel transform of order zero gives

$$\bar{\tilde{\tilde{G}}}(\kappa, s) = \frac{1}{2\pi} \cdot \frac{1}{(s^2 + c^2\kappa^2 + d^2)}, \tag{11.3.8}$$

whence, the inverse Laplace transform yields

$$\tilde{G}(\kappa, t) = \frac{1}{2\pi} \cdot \frac{\sin(\alpha t)}{\alpha}, \quad \alpha = \left(c^2\kappa^2 + d^2\right)^{\frac{1}{2}}. \tag{11.3.9}$$

Finally, the inverse Hankel transform leads to the solution

$$G(r, t) = \frac{1}{2\pi c} \int_0^\infty \kappa \, J_0(\kappa r)\left(\kappa^2 + \frac{d^2}{c^2}\right)^{-\frac{1}{2}} \sin\left[ct\left\{\kappa^2 + \frac{d^2}{c^2}\right\}^{\frac{1}{2}}\right] d\kappa, \tag{11.3.10}$$

which, by using the standard table of Hankel transforms, is

$$\begin{aligned} &= \frac{1}{2\pi c} \frac{\cos[(\frac{d}{c})\sqrt{c^2t^2 - r^2}]}{\sqrt{c^2t^2 - r^2}} H(ct - r) \\ &= \frac{1}{2\pi c^2} \cdot \left(t^2 - \frac{r^2}{c^2}\right)^{-\frac{1}{2}} \cos\left[d\sqrt{t^2 - \frac{r^2}{c^2}}\right] H\left(t - \frac{r}{c}\right). \end{aligned} \tag{11.3.11}$$

If the source is located at (ξ, η, τ) instead of the origin, Green's function assumes the form

$$G(x, y, t; \xi, \eta, \tau) = \frac{1}{2\pi c^2} \cdot \left(T^2 - \frac{R^2}{c^2}\right)^{-\frac{1}{2}} \cos\left[d\sqrt{T^2 - \frac{R^2}{c^2}}\right] H\left(T - \frac{R}{c}\right), \tag{11.3.12}$$

where $T = t - \tau$ and $R^2 = (x - \xi)^2 + (y - \eta)^2$.

11.4 The Three-Dimensional Linear Klein–Gordon Equation

The Green function $G(x,t)$ for the three-dimensional Klein–Gordon equation satisfies the equation

$$G_{tt} - c^2\nabla^2 G + d^2 G = \delta(x)\delta(y)\delta(z)\delta(t), \quad -\infty < x, y, z < \infty,\ t > 0, \tag{11.4.1}$$

with the conditions

$$G(\mathbf{x},0) = 0 = G_t(\mathbf{x},0) \quad \text{for all } x, y, z, \tag{11.4.2}$$

$$G(\mathbf{x},0) \to 0 \quad \text{as } |\mathbf{x}| \to \infty, \tag{11.4.3}$$

where $\mathbf{x} = (x,y,z)$.

The application of the joint Laplace and the triple Fourier transform (Debnath 1995) to this problem gives

$$\bar{\tilde{G}}(\kappa, s) = \frac{1}{(2\pi)^{3/2}} \cdot \frac{1}{(s^2 + c^2\kappa^2 + d^2)}, \tag{11.4.4}$$

where $\kappa = (k, l, m)$.

The joint inverse transform gives the solution

$$G(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} \exp\{i(\boldsymbol{\kappa}\cdot\mathbf{x})\} \frac{\sin\alpha t}{\alpha}\, d\boldsymbol{\kappa}, \tag{11.4.5}$$

where $\alpha = (c^2\kappa^2 + d^2)^{\frac{1}{2}}$.

Introducing spherical polar coordinates in the κ-integral and taking the polar axis along the direction of the vector $\mathbf{x}$ so that $\boldsymbol{\kappa}\cdot\mathbf{x} = \kappa r\cos\theta$, $|x| = r$, θ is the polar angle, and $\boldsymbol{\kappa} = \kappa^2 d\kappa \sin\theta\, d\theta\, d\phi$, we obtain

$$\begin{aligned} G(\mathbf{x},t) &= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\infty} \frac{\sin\alpha t}{\alpha} \kappa^2\, d\kappa \int_0^{\pi} \exp(i\kappa r\cos\theta)\sin\theta\, d\theta \\ &= \frac{1}{(2\pi)^2 r} \int_0^{\infty} \sin(\alpha t)\cdot\kappa\sin(\kappa r)\frac{d\kappa}{\alpha} \\ &= -\frac{1}{(2\pi^2 r)} \int_0^{\infty} \frac{\sin\alpha t}{\alpha} \cdot \frac{\partial}{\partial r}\cos(\kappa r)\, d\kappa. \end{aligned} \tag{11.4.6}$$

Because integral (11.4.6) is an even function of κ, and $\frac{\partial}{\partial r}$ can be written outside the integral, we now obtain

$$\begin{aligned} G(\mathbf{x},t) &= -\frac{1}{4\pi^2 rc} \cdot \frac{\partial}{\partial r} \int_{-\infty}^{\infty} \exp(i\kappa r)\left(\kappa^2 + \frac{d^2}{c^2}\right)^{-\frac{1}{2}} \sin\left[ct\sqrt{\kappa^2 + \frac{d^2}{c^2}}\right] d\kappa \\ &= -\frac{1}{4\pi rc} \cdot \frac{\partial}{\partial r} J_0\left(\frac{d}{c}\sqrt{c^2t^2 - r^2}\right) H\left(c^2t^2 - r^2\right) \end{aligned} \tag{11.4.7}$$

in which the standard table of the Fourier transforms is used.

Finally, we carry out the differentiation and use the identity $J_0'(kr) = -kJ_1(kr)$ and $J_0(0) = 1$ to obtain the Green function

$$G(\mathbf{x}, t) = \frac{1}{2\pi}\left[\delta\left(c^2t^2 - r^2\right) - \left(\frac{d}{2c}\right)\frac{J_1\left(\frac{d}{c}\sqrt{c^2t^2 - r^2}\right)}{\sqrt{c^2t^2 - r^2}} H\left(c^2t^2 - r^2\right)\right]. \tag{11.4.8}$$

In the limit as $d \to 0$, this result is in perfect agreement with that for the three-dimensional wave equation.

11.5 The Nonlinear Klein–Gordon Equation and Averaging Techniques

Whitham (1965a, 1965b) has successfully extended the averaging procedure for the study of nonlinear ordinary differential equations associated with oscillation problems to partial differential equations involved in nonlinear dispersive waves. The procedure that he developed was to average over local oscillations and then to obtain partial differential equations describing the slow variations of the basic wave parameters, such as wavenumber, frequency, and amplitude. In all problems where the equations admit locally uniform periodic wavetrains as solutions of the form $u(\mathbf{x}, t) = u(\theta)$, where $\theta = \boldsymbol{\kappa} \cdot \mathbf{x} - \omega t$ is the phase function, $u(\theta)$ is a periodic function of θ, and $\mathbf{k}$ and ω are the wavenumber vector and frequency, respectively. In linear wave problems, $u(\mathbf{x}, t) = a\cos\theta$, which is sinusoidal.

According to the Whitham theory of slowly varying wavetrains, the solution of the form $u = u(\theta, a)$ is maintained, where a is not a constant and θ is not a linear function of x and t. The parameters ω, $\boldsymbol{\kappa}$, and a are assumed to be slowly varying functions of $\mathbf{x}$ and t corresponding to the slow modulation of the wavetrains. In other words, we allow gradual changes in ω, $\boldsymbol{\kappa}$, and a, assuming them to be noticeable only at distances much larger than κ^{-1} and after times much longer than ω^{-1}. This is usually known as the *geometrical optics approximation*. Even though the wave parameters evolve, the general functional form of u is retained in this approach.

We consider the $(1+1)$-*dimensional*, *nonlinear Klein–Gordon* equation

$$u_{tt} - u_{xx} + V'(u) = 0, \tag{11.5.1}$$

where $V(u)$ is a general nonlinear function of u, but not its derivatives, and the prime denotes the derivative with respect to u.

We now multiply (11.5.1) by u_t and u_x and reorganize the terms to derive the following conservation equations:

$$\left[\frac{1}{2}\left(u_t^2 + u_x^2\right) + V\right] + [-u_t u_x]_x = 0, \tag{11.5.2}$$

$$[-u_t u_x]_t + \left[\frac{1}{2}\left(u_t^2 + u_x^2\right) - V\right]_x = 0. \tag{11.5.3}$$

We look for a stationary wave solution of (11.5.1) in the form

$$u = u(\xi), \quad \xi = x - ct \tag{11.5.4ab}$$

so that integration of the resulting equation gives

$$\frac{1}{2}(c^2 - 1)u_\xi^2 + V(u) = A, \tag{11.5.5}$$

where A is an integration constant related to the amplitude of the waves. So, it follows from equation (11.5.5) that (11.5.1) admits stationary wave solutions of the form

$$u_\xi^2 = f(u, c, A). \tag{11.5.6}$$

Based on the phase plane analysis with the phase diagrams of $u_\xi(\xi)$, it can be shown that periodic wave solutions exist for all $V(u)$ with a minimum. So we assume that (11.5.6) admits a solution of the form $u = u(\xi, c, A)$ so that (11.5.6) gives

$$\left(\frac{du}{d\xi}\right) = \left[\frac{2}{(c^2 - 1)}\{A - V(u)\}\right]^{\frac{1}{2}}, \tag{11.5.7}$$

in which we assumed that $c^2 > 1$ and $A > V(u)$. Or equivalently,

$$\xi = \sqrt{\frac{1}{2}(c^2 - 1)} \int \{A - V(u)\}^{-\frac{1}{2}} du. \tag{11.5.8}$$

Clearly, the solution is oscillatory between two consecutive zeros, say $u_1(c, A)$ and $u_2(c, A)$ with $u_2 > u_1$ of the function $f(u, c, A)$ between which it is positive definite. The positive definiteness between u_1 and u_2 is a necessary condition to ensure that $(\frac{du}{d\xi})$ is real. We suppose that ξ_1 and ξ_2 are the values of ξ at which $u(\xi_1, c, A) = u_1(c, A)$ and $u(\xi_2, c, A) = u_2(c, A)$.

Using the analogy of the linear waves, we define the wavelength λ for nonlinear waves by

$$\lambda = \lambda(c, A) = 2\int_{\xi_1}^{\xi_2} d\xi = 2\int_{u_1}^{u_2} \frac{du}{u_\xi} = \sqrt{\frac{1}{2}(c^2 - 1)} \int_{u_1}^{u_2} \frac{du}{\sqrt{A - V(u)}}. \tag{11.5.9}$$

Similarly, the wavenumber $k = k(c, A)$, the frequency $\omega = \omega(c, A)$, and the periodic time $T = T(c, A)$ are defined by

$$k = \frac{2\pi}{\lambda(c, A)}, \qquad \omega = \omega(c, A) = ck(c, A), \qquad T = T(c, A) = \frac{2\pi}{\omega(c, A)}. \tag{11.5.10}$$

In direct analogy with the theory of linear waves, it is possible to find a general solution by considering c and A as slowly varying functions of x and t.

We now describe the Whitham averaging method to eliminate the rapid fluctuations in the field functions taking place at the smaller space scale $x \sim \lambda$ and time scale $t \sim T'$. The significant changes of order $O(1)$ in k, ω, and a take place over the length and time scales of the order of $L = O(x)$ and $T = O(t)$, respectively. The above discussion indicates the function of intermediate scales X and τ, so that $\lambda \ll X \ll L$ and $T' \ll \tau \ll T$. We now define the local average $\overline{f}(x,t)$ of $f(x,t)$ at any point x for a fixed t by the relation

$$\overline{f}(x,t) = \frac{1}{2X}\int_{x-X}^{x+X} f(x',t)\,dx'. \tag{11.5.11}$$

To develop the averaging method, Whitham used the conservation form of the dynamical equation as

$$\frac{\partial \rho}{\partial t} + \nabla\cdot q_i = 0, \quad i = 1,2,\ldots,n. \tag{11.5.12}$$

Obviously,

$$\overline{\left(\frac{\partial \rho}{\partial t}\right)} = \frac{1}{2X}\int_{x-X}^{x+X} \frac{\partial}{\partial t}\rho(x',t)\,dx' = \frac{\partial \overline{\rho}}{\partial t}, \tag{11.5.13}$$

$$\begin{aligned}\overline{\left(\frac{\partial q}{\partial x}\right)} &= \frac{1}{2X}\int_{x-X}^{x+X} \frac{\partial}{\partial x'}q(x',t)\,dx' = \frac{1}{2X}\big[q(x+X,\,t) - q(x-X,\,t)\big]\\ &= \frac{\partial}{\partial x}\left[\frac{1}{2X}\int_{x-X}^{x+X} q(x',t)\,dx'\right] = \frac{\partial \overline{q}}{\partial x}.\end{aligned} \tag{11.5.14}$$

Taking the average of (11.5.12) gives

$$\frac{\partial}{\partial x}\overline{\rho}(x,t,c,A) + \nabla\cdot\overline{q}_i(x,t,c,A) = 0. \tag{11.5.15}$$

The average quantities $\overline{\rho}$ and $\overline{q}_i$ in (11.5.15) still depend explicitly on x and t. To eliminate this explicit dependence, we consider the fact that, since $X \gg \lambda$, the interval $(x-X, x+X)$ contains a large number of waves over which k, ω, etc. are nearly constant. In this case, we can replace $\overline{\rho}$ and $\overline{q}_i$ by $\langle \rho(c,A)\rangle$ and $\langle q_i(c,A)\rangle$, respectively, which are the averages over one wavelength by keeping k, ω, etc. constant on $(x-X,\,x+X)$. When this approximation is made, the errors introduced are of the order of $\frac{\lambda}{X}(\ll 1)$ and $\frac{X}{L}(\ll 1)$, which are very small. We now define the global average $\langle f(c,A)\rangle$ of $f(x,t)$ over a wavelength by

$$\begin{aligned}\big\langle f(c,A)\big\rangle &= \frac{1}{\lambda}\int_x^{x+\lambda} f(x',t)\,dx' = \frac{1}{\lambda}\int_0^{\lambda} f(X,t)\,dX, \quad x' = x+X,\\ &= \frac{1}{\lambda}\int_x^{x+\lambda} f(u,c,A)\,dX,\end{aligned} \tag{11.5.16}$$

where the integrand depends on u.

In view of the average defined by (11.5.16), the averaged conservation equation is obtained from (11.5.12) as

$$\frac{\partial}{\partial t}\big\langle\rho(c,A)\big\rangle+\nabla\cdot\big\langle q_i(c,A)\big\rangle=0, \tag{11.5.17}$$

where c and A are slowly varying functions of x and t. It is noted that the terms in (11.5.17) become of the order of $\frac{\lambda}{L}(\ll 1)$ upon differentiation. The average $\langle f(c,A)\rangle$ defined by (11.5.16) plays an important role in the study of nonlinear waves.

In direct analogy with the adiabatic invariant $I = \oint p\,dq$ in the Hamiltonian mechanics, it is expedient to introduce a new function

$$W(c,A)=\oint u_\xi\,du, \tag{11.5.18}$$

which enables us to express the wavenumber, frequency, and other parameters involved in the wave in terms of the first-order partial derivatives of W with respect to c and A. Furthermore, in many situations, the system of equations involving these partial derivatives of W turns out to be hyperbolic. Consequently, the characteristics of this hyperbolic system define the characteristic speeds associated with the wave. These remarkable features will be discussed further in the context of the Klein–Gordon equation.

In view of the result (11.5.7) for the Klein–Gordon equation, it is convenient to introduce the function $W(c,A)$ given by

$$\begin{aligned}2\pi W(c,A) &= \left(c^2-1\right)\oint u_\xi\cdot du=\sqrt{2\left(c^2-1\right)}\oint\{A-V(u)\}^{\frac{1}{2}}\,du\\ &=\sqrt{\left(c^2-1\right)}G(A),\end{aligned} \tag{11.5.19}$$

where $G(A)$ is another new function defined by

$$G(A)=\oint\big[\{A-V(u)\}\big]^{\frac{1}{2}}\,du, \tag{11.5.20}$$

which is independent of c, but depends only on A. Obviously,

$$G'(A)=\frac{1}{2}\oint\frac{du}{\sqrt{A-V(u)}}=\frac{1}{\sqrt{c^2-1}}\oint d\xi=\frac{\lambda}{\sqrt{c^2-1}}, \tag{11.5.21}$$

in which (11.5.9) is used, and

$$\begin{aligned}G''(A) &= -\frac{1}{2\sqrt{2}}\oint\frac{du}{\{A-V(u)\}^{3/2}}=-\frac{1}{2\sqrt{2}}\oint\frac{1}{\{A-V(u)\}^{3/2}}\left(\frac{du}{d\xi}\right)d\xi\\ &=\frac{1}{(c^2-1)^{3/2}}\int_0^\lambda\left(\frac{du}{d\xi}\right)<0,\end{aligned} \tag{11.5.22}$$

in which (11.5.7) is again utilized.

Further, it follows from (11.5.19) and (11.5.21) that

$$2\pi\left(\frac{\partial W}{\partial A}\right) = \sqrt{(c^2-1)}\,G'(A) = \lambda = \frac{2\pi}{k}. \tag{11.5.23}$$

This shows that the wavenumber k can be defined by the relation

$$k\left(\frac{\partial W}{\partial A}\right) = 1. \tag{11.5.24}$$

We use (11.5.16) to calculate the average values of all quantities involved in equations (11.5.2) and (11.5.3) as follows:

$$\begin{aligned}\left\langle \frac{1}{2}\left(u_t^2+u_x^2\right)\right\rangle &= \left\langle \frac{1}{2}\left(c^2+1\right)u_\xi^2\right\rangle = \frac{1}{2}\left(c^2+1\right)\oint \frac{1}{\lambda}u_\xi^2\,d\xi \\ &= \frac{1}{4\pi}\left(c^2+1\right)\oint ku_\xi\,du = \frac{1}{2}\left(\frac{c^2+1}{c^2-1}\right)kW(c,A),\end{aligned} \tag{11.5.25}$$

in which (11.5.19) is used. Furthermore, we find that

$$\langle -u_t u_x\rangle = \left\langle c u_\xi^2\right\rangle = \frac{c}{(c^2+1)}kW, \tag{11.5.26}$$

$$\langle V(u)\rangle = \left\langle A - \frac{1}{2}\left(c^2+1\right)u_\xi^2\right\rangle = A - \frac{1}{2}kW. \tag{11.5.27}$$

Differentiating (11.5.19) partially with respect to c gives

$$\frac{\partial W}{\partial c} = \frac{c}{(c^2+1)}W. \tag{11.5.28}$$

We substitute the above average values in (11.5.2) and (11.5.3) and then utilize results (11.5.24) and (11.5.28) to derive the following *averaged conservation equation*:

$$\frac{\partial}{\partial t}\big[k(cW_c + AW_A - W)\big] + \frac{\partial}{\partial x}\big[kc(cW_c + AW_A - W) - cA\big] = 0, \tag{11.5.29}$$

$$\frac{\partial}{\partial t}[kW_c] + \frac{\partial}{\partial x}[ckW_c - A] = 0, \tag{11.5.30}$$

where, in writing the second term in (11.5.29), we have added $kc\,AW_A$ and subtracted its equivalent expression cA.

We next calculate partial derivatives of various terms in (11.5.29) and collect the coefficients of c, A, and W to obtain

$$\begin{aligned}&c\left[\frac{\partial}{\partial t}(kW_c) + \frac{\partial}{\partial x}(ckW_c - A)\right] + A\left[\frac{\partial}{\partial t}(kW_A) + \frac{\partial}{\partial x}(ckW_A - c)\right] \\ &\quad - W\left[\frac{\partial k}{\partial t} + \frac{\partial}{\partial x}(ck)\right] = 0.\end{aligned} \tag{11.5.31}$$

The first term in this equation vanishes due to (11.5.30), the second term also vanishes since $kW_A = 1$, and hence, finally, we obtain

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x}(ck) = 0, \tag{11.5.32}$$

which is, introducing the frequency $\omega = ck$,

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{11.5.33}$$

Or equivalently,

$$\frac{\partial k}{\partial t} + \omega'(k)\frac{\partial k}{\partial x} = 0. \tag{11.5.34}$$

This is the fundamental kinematic equation of the wave. Thus, along the characteristics $\frac{dx}{dt} = \omega'(k)$, the wavenumber k (and hence, frequency ω) is conserved.

Substitution of $k = W_A^{-1}$ in (11.5.32) yields

$$\frac{\partial W_A}{\partial t} + c\frac{\partial W_A}{\partial x} - W_A\frac{\partial c}{\partial x} = 0. \tag{11.5.35}$$

Similarly, expanding (11.5.30) combined with (11.5.24) leads to the result

$$\frac{\partial W_c}{\partial t} + c\frac{\partial W_c}{\partial x} - W_A\frac{\partial A}{\partial x} = 0. \tag{11.5.36}$$

Finally, the two independent averaged conservation equations (11.5.35), (11.5.36) can be expressed in the form

$$\frac{DW_A}{Dt} - W_A\frac{\partial c}{\partial x} = 0, \tag{11.5.37}$$

$$\frac{DW_c}{Dt} - W_A\frac{\partial A}{\partial x} = 0, \tag{11.5.38}$$

where the total derivative is defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}. \tag{11.5.39}$$

To determine the characteristics and characteristic speeds, we use (11.5.32) and (11.5.28) and, then, replace W in terms of G to express (11.5.37), (11.5.38) in terms of c and A as

$$G''A_t + cG''A_x + \frac{cG'}{(c^2-1)}c_t + \frac{cG'}{(c^2-1)}c_x = 0, \tag{11.5.40}$$

$$cG'A_t + G'A_x - \frac{G}{(c^2-1)}c_t - \frac{cG}{(c^2-1)}c_x = 0. \tag{11.5.41}$$

These equations admit the following two characteristics $C_\pm$, so that

$$C_+ : \frac{dx}{dt} = \left(\frac{1+c\alpha}{c+\alpha}\right) \quad \text{and} \quad C_- : \frac{dx}{dt} = \left(\frac{1-c\alpha}{c-\alpha}\right), \tag{11.5.42ab}$$

where

$$\alpha = \left(-\frac{GG''}{G'^2}\right)^{\frac{1}{2}}. \tag{11.5.43}$$

The associated compatibility relations are given by

$$\frac{dc}{(c^2-1)} - \left(-\frac{G''}{G}\right)^{\frac{1}{2}} dA = 0 \quad \text{along } C_+, \tag{11.5.44}$$

$$\frac{dc}{(c^2-1)} + \left(-\frac{G''}{G}\right)^{\frac{1}{2}} dA = 0 \quad \text{along } C_-. \tag{11.5.45}$$

These relations define two Riemann invariants

$$r = \int_{c_0}^{c} \frac{dc}{(c^2-1)} - \int_{A_0}^{A} \left(-\frac{G''}{G}\right)^{\frac{1}{2}} dA, \tag{11.5.46}$$

$$s = \int_{c_0}^{c} \frac{dc}{(c^2-1)} + \int_{A_0}^{A} \left(-\frac{G''}{G}\right)^{\frac{1}{2}} dA. \tag{11.5.47}$$

Equations (11.5.42ab) reveal that there are two characteristic speeds $(1 \pm c\alpha)/(c \pm \alpha)$. If we consider a wavetrain which is initially uniform with $c = c_0$ and $A = A_0$ outside some finite region, then, after some interaction period, the disturbance splits into two simple waves separated by a domain of constant values of c and A. In one simple wave, the characteristics C_+ are straight lines which carry constant values of r, and the other Riemann invariant is constant everywhere. On the other hand, in the second simple wave, r has the same value everywhere, and s is constant along the characteristics C_-, which are straight lines. Between these two simple waves, c and A assume constant values. Since the wavenumber and amplitude are expressible in terms of c and A, the qualitative features similar to those as stated above are also applicable to them.

The energy propagation speed is then obtained from (11.5.29) as the ratio of energy flux to energy density and has the form

$$\begin{aligned}
&\frac{kc(cW_c + AW_A - W) - cA}{k(cW_c + AW_A - W)} \\
&\quad = \frac{kc(cW_c - W) + cA - cA}{k(cW_c - W) + A} \quad \big(\text{by } (11.5.24)\big) \\
&\quad = \frac{c(cW_c - W)}{(cW_c - W) + \frac{A}{k}} = \frac{cW}{W + (c^2-1)(\frac{A}{k})} \quad \big(\text{by } (11.5.28)\big),
\end{aligned}$$

which, by results (11.5.19) and (11.5.21), is

$$= \frac{cG}{G + (c^2-1)AG'}. \tag{11.5.48}$$

This is another important new speed which is definitely not one of the two characteristic speeds. This new speed does not seem to have a simple physical interpretation. Thus, in the present fully nonlinear case, there exist three important speeds—two characteristic speeds and one energy propagation speed. Clearly, one of them should be interpreted as the group velocity.

11.6 The Klein–Gordon Equation and the Whitham Averaged Variational Principle

The Lagrangian density of the Klein–Gordon equation (11.5.1) is given by

$$L = \frac{1}{2}\left(u_t^2 - u_x^2\right) - V(u). \tag{11.6.1}$$

We seek a progressive, periodic wave solution of the nonlinear Klein–Gordon equation (11.5.1) in the form

$$u = u(\theta), \quad \theta = kx - \omega t, \tag{11.6.2}$$

where the local wavenumber and local frequency are given by $k = \theta_x$ and $\omega = -\theta_t$, respectively. Substituting (11.6.2) in equation (11.5.1) gives

$$\left(\omega^2 - k^2\right)u_{\theta\theta} + V'(u) = 0, \tag{11.6.3}$$

which gives the integral

$$\frac{1}{2}\left(\omega^2 - k^2\right)u_\theta^2 + V(u) = A, \tag{11.6.4}$$

where A is a constant of integration. This equation can be solved by a quadrature for a suitable form of $V(u)$ and has a solution, periodic in θ, given by

$$\theta = \left\{\frac{1}{2}\left(\omega^2 - k^2\right)\right\}^{\frac{1}{2}} \int \frac{du}{\sqrt{A - V(u)}}. \tag{11.6.5}$$

Since u is a periodic function with period 2π, we require that

$$2\pi = \left\{\frac{1}{2}\left(\omega^2 - k^2\right)\right\}^{\frac{1}{2}} \oint \frac{du}{\sqrt{A - V(u)}}, \tag{11.6.6}$$

where $\oint$ denotes integration over a complete period. Result (11.6.6) represents a relation between ω, k, and A, and hence, it represents the dispersion relation.

According to the Whitham theory, the wave parameters k, ω, and A are slowly varying functions of x and t corresponding to the slow modulation of the wave. These parameters can then be determined from the averaged Lagrangian defined by Whitham (1965a, 1965b) as

$$\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L\, d\theta, \tag{11.6.7}$$

where L is given by (11.6.1) which is simplified by using (11.6.2) so that

$$L = \frac{1}{2}\left(\omega^2 - k^2\right)u_\theta^2 - V(u) = \left(\omega^2 - k^2\right)u_\theta^2 - A. \tag{11.6.8}$$

Substituting (11.6.8) in (11.6.7) gives

$$\begin{aligned}
\mathcal{L}(\omega, k, A) &= \frac{1}{2\pi}\left(\omega^2 - k^2\right)\int_0^{2\pi}\left(\frac{\partial u}{\partial \theta}\right)^2 d\theta - \frac{A}{2\pi}\int_0^{2\pi} d\theta \\
&= \frac{1}{2\pi}\left(\omega^2 - k^2\right)\int_0^{2\pi}\left(\frac{\partial u}{\partial \theta}\right) du - A,
\end{aligned}$$

which is, by (11.6.4),

$$\begin{aligned}
&= \frac{1}{2\pi}\left\{\left(\omega^2 - k^2\right)\right\}^{\frac{1}{2}}\oint \sqrt{A - V(u)}\, du - A \\
&= \frac{1}{2\pi}\left(\omega^2 - k^2\right)^{\frac{1}{2}} G(A) - A,
\end{aligned} \tag{11.6.9}$$

where $G(A)$ is defined by (11.5.20).

According to the Whitham theory, wave parameters ω, k, and A are determined from the averaged variational principle

$$\delta \iint \mathcal{L}(\omega, k, A)\, dx\, dt = 0, \tag{11.6.10}$$

where ω and k are related by $\omega = -\theta_t$ and $k = \theta_x$, and hence, they cannot be varied independently. The variational equations from variations δA and $\delta\theta$ in (11.6.10) are given by

$$\delta A : \frac{\partial \mathcal{L}}{\partial A} = 0, \tag{11.6.11}$$

$$\delta\theta : \frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \theta_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \theta_x}\right) = -\frac{\partial}{\partial t}(\mathcal{L}_\omega) + \frac{\partial}{\partial x}(\mathcal{L}_k) = 0. \tag{11.6.12}$$

Thus, by (11.6.9), equation (11.6.11) becomes

$$\frac{1}{2\pi}\left(\omega^2 - k^2\right)^{\frac{1}{2}} G'(A) = 1, \tag{11.6.13}$$

or equivalently,

$$k = \frac{2\pi}{\sqrt{(c^2 - 1)}G'(A)}, \qquad \omega = ck = \frac{2\pi c}{\sqrt{(c^2 - 1)}G'(A)}. \tag{11.6.14ab}$$

In terms of $G(A)$, we can express $k_t + \omega_x = 0$ in the form

$$\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{c^2-1}G'(A)}\right]+\frac{\partial}{\partial x}\left[\frac{c}{\sqrt{c^2-1}G'(A)}\right]=0. \tag{11.6.15}$$

Similarly, the amplitude equation (11.6.12) can be expressed as

$$\frac{\partial}{\partial t}\left[\frac{cG(A)}{\sqrt{c^2-1}}\right]+\frac{\partial}{\partial x}\left[\frac{G(A)}{\sqrt{c^2-1}}\right]=0. \tag{11.6.16}$$

The Whitham Lagrangian formulation is relatively more simple than the averaging procedure discussed in the previous section. Indeed, the averaged Lagrangian is a useful quantity in nonlinear theory whenever it is available. It is also important to point out that the present analysis is applied to slowly varying nonlinear waves. However, the variational method is applicable to *weak* nonlinear waves and also to *all* nonlinear wavetrains.

In particular, for the linear Klein–Gordon equation, $V'(u)=u$ so that $V(u)=\frac{1}{2}u^2$. Hence the function $G(A)$ defined by (11.5.20) can easily be computed by using the linear solution for $u=-a\cos\theta$ where $2A=a^2$. It turns out that $G(A)=2\pi A$ so that $G'(A)=2\pi$. Consequently, the averaged Lagrangian $\mathcal{L}$ can be obtained explicitly from (11.6.9) as

$$\mathcal{L}=\left[\left(\omega^2-k^2\right)^{\frac{1}{2}}-1\right]A, \quad A=\frac{1}{2}a^2, \tag{11.6.17}$$

so that the linear dispersion relation is $\mathcal{L}_A=0$, that is,

$$\omega^2-k^2=1. \tag{11.6.18}$$

This is independent of the amplitude A, as expected in the linear theory. This also confirms the fact that the Whitham variational method is applicable to both linear and nonlinear wave propagation problems.

11.7 The Sine-Gordon Equation: Soliton and Antisoliton Solutions

The sine-Gordon equation has a long history that begins in the latter part of the nineteenth century when this equation was discovered in differential geometry in connection with the theory of surfaces of constant negative curvature. Various methods for finding particular solutions of this equation were developed at that time. One of the methods is known as the Bäcklund transformations, which will be considered in a subsequent section. Other methods include traveling wave solutions, the similarity method, the inverse scattering method, and the method of separation of variables, which deals with the representation of solutions as functions of independent variables. All of these methods will be discussed in subsequent sections.

It has already been indicated in Chapter 2 that the sine-Gordon equation is one of the basic nonlinear evolution equations and has recently been employed to describe various important nonlinear physical phenomena. The standard form of this equation is

$$\frac{\partial^2 u}{\partial X^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial T^2} = \sin u. \tag{11.7.1}$$

In terms of the characteristic coordinate transformations

$$x = \pm\frac{1}{2}(X - cT), \qquad t = \frac{1}{2}(X + cT), \tag{11.7.2ab}$$

equation (11.7.1) takes the following form:

$$u_{xt} = \sin u. \tag{11.7.3}$$

Thus, the two forms (11.7.1) and (11.7.3) of the sine-Gordon equation are interchangeable. The equation is invariant under $u \to u + 2n\pi$, where $n = \pm 1, \pm 2, \pm 3, \ldots$. The transformation $u \to u + (2n+1)\pi$ leads to the replacement of $\sin u$ by $-\sin u$ in the equation.

The linearized form of equation (11.7.1) is given by

$$u_{XX} - c^{-2} u_{TT} = u. \tag{11.7.4}$$

A plane wave solution of the form $u(X,T) = A\exp[i(kX - \omega T)]$ exists provided the dispersion relation

$$\omega^2 = c^2\left(1 + k^2\right) \tag{11.7.5}$$

is satisfied. This shows that ω is real for all real k, and the equilibrium solution $u = 0$ of (11.7.1) is stable. On the other hand, if $u = \tilde{u} + \pi$ for a small perturbation $\tilde{u}$, (11.7.1) can be linearized to obtain

$$\tilde{u}_{XX} - c^{-2}\tilde{u}_{TT} = -\tilde{u}. \tag{11.7.6}$$

A plane wave solution of the form $\tilde{u}(X,T) = A\exp[i(kX - \omega T)]$ also exists, provided the dispersion relation

$$\omega^2 = c^2\left(k^2 - 1\right) \tag{11.7.7}$$

is satisfied. Hence $\omega^2 < 0$, if $0 \le k^2 < 1$, and $\tilde{u}(X,T)$ will grow exponentially in time T, that is, the equilibrium solution $u = \pi$ is definitely unstable. Physically, this is quite natural because, when the term u_{XX} is absent (that is, when $k = 0$), the sine-Gordon equation (11.7.1) reduces to the equation of a simple pendulum representing finite oscillations. Thus, the question of stability or instability is an important feature of the equilibrium solutions of the sine-Gordon equation.

We next discuss the single-soliton solutions of the sine-Gordon equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \sin u. \tag{11.7.8}$$

We seek a solution of the form $u(x,t) = \phi(x - Ut) = \phi(\xi)$ which corresponds to a wave traveling with a velocity U and, then, substitute in (11.7.8) to obtain an ordinary differential equation for $\phi(\xi)$ in the form

$$\frac{d^2\phi}{d\xi^2} - V^2\frac{d^2u}{d\xi^2} = \sin\phi, \tag{11.7.9}$$

where $V^2 = \frac{U^2}{c^2}$. Dividing both sides by $(1 - V^2)$ and multiplying by $\phi'(\xi)$ gives

$$\phi'(\xi)\phi''(\xi) = \left(\frac{\sin u}{1 - V^2}\right)\phi'(\xi). \tag{11.7.10}$$

Or equivalently,

$$\frac{d}{d\xi}\left[\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + \frac{\cos u}{1 - V^2}\right] = 0, \tag{11.7.11}$$

that is,

$$\frac{1}{2}\left(\frac{d\phi}{d\xi}\right)^2 + \frac{\cos u}{(1 - V^2)} = B = \text{const.} \tag{11.7.12}$$

Solving for $\phi'(\xi)$ gives a first-order, ordinary differential equation for $\phi(\xi)$

$$\frac{d\phi}{d\xi} = \left(2A - \frac{2\cos\phi}{1 - V^2}\right)^{\frac{1}{2}}. \tag{11.7.13}$$

We can separate the variables and, then, integrate to obtain

$$\int_{\phi_0}^{\phi}\frac{d\psi}{\sqrt{A - \cos\psi}} = \left(\frac{2}{1 - V^2}\right)^{\frac{1}{2}}\int_{\xi_0}^{\xi} d\eta, \tag{11.7.14}$$

where $A = B\sqrt{1 - V^2}$. This result depends on the two parameters V (or $U = cV$), the velocity of the soliton, and B, an integrating constant. Thus, the stable solutions represent solitary waves, periodic waves, or a monotonically increasing function of ξ (Scott 1969) depending on the sign and magnitudes of U and A. When $A = 1$, a solitary wave solution exists for *any* velocity, so that $0 < |V| < 1$ (or $0 < |U| < c$). In this case, we use the trigonometric identities $1 - \cos\phi = 2\sin^2(\frac{1}{2}\phi)$ and $\frac{d}{d\phi}[\log\tan(\frac{1}{4}\phi)] = (2\sin\frac{1}{2}\phi)^{-1}$ to simplify both sides of (11.7.14), so that

$$\left(\frac{2}{1 - V^2}\right)^{\frac{1}{2}}(\xi - \xi_0) = \sqrt{2}\int_{\phi_0}^{\phi}\frac{d\psi}{2\sin\frac{1}{2}\psi} = \sqrt{2}\log\left[\frac{\tan(\frac{1}{4}\phi)}{\tan(\frac{1}{4}\phi_0)}\right]. \tag{11.7.15}$$

This leads to the solution for $\phi(\xi)$ as

$$\phi(\xi) = 4\tan^{-1}\left\{\alpha\exp\left(\frac{\xi - \xi_0}{\sqrt{1 - V^2}}\right)\right\}, \tag{11.7.16}$$

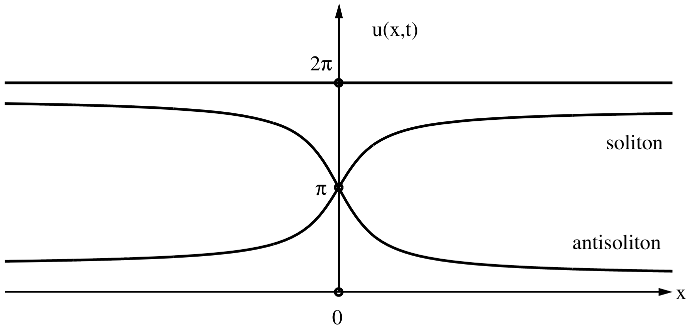

Fig. 11.1 The soliton and antisoliton solutions.

where $\alpha = \tan(\frac{1}{4}\phi_0)$ and ξ_0 are constants of integration which can be incorporated in several ways. We choose $\alpha = 1$ and $\xi_0 = 0$ to obtain one simple solitary wave solution for $u(x,t)$ in the form

$$u(x,t) = 4\tan^{-1}\left\{\exp\left(\frac{x - Vt}{\sqrt{1-V^2}}\right)\right\}. \tag{11.7.17}$$

This is called the *soliton* (or *kink*) solution of the sine-Gordon equation and represents a continuous profile with $u \to 0$ as $x \to -\infty$, and $u \to 2\pi$ as $x \to +\infty$, shown in Figure 11.1. The soliton propagates in the positive x-direction with velocity V.

Another solution can be obtained from (11.7.15) in the form

$$u(x,t) = 4\tan^{-1}\left[\exp\left(-\frac{x - Vt}{\sqrt{1-V^2}}\right)\right] = 4\cot^{-1}\left[\exp\left(\frac{x - Vt}{\sqrt{1-V^2}}\right)\right]. \tag{11.7.18}$$

This is called the *antisoliton* (or *antikink*). This solution profile travels in the negative x-direction with velocity V. It represents a continuous profile with $u \to 2\pi$ as $x \to -\infty$, and $u \to 0$ as $x \to +\infty$, also shown in Figure 11.1.

The partial derivatives of $u(x,t)$, that is, u_x and u_t, also represent solitary waves which are given by

$$u_x(x,t) = \phi'(x - Ut) \quad \text{and} \quad u_t(x,t) = -U\phi'(x - Ut), \tag{11.7.19ab}$$

where $\phi'(\xi)$ is obtained from (11.7.16) as

$$\phi'(\xi) = \frac{4\tan(\frac{1}{4}\phi)}{\sqrt{1-V^2}\{1+\tan^2(\frac{1}{4}\phi)\}} = \frac{2\text{sech}\{(x-Vt)/\sqrt{1-V^2}\}}{\sqrt{1-V^2}}$$

in which $\tan(\frac{1}{4}\phi)$ is replaced by $\exp(\frac{x-Vt}{\sqrt{1-V^2}})$.

If $A \neq 1$, other types of solutions of the sine-Gordon equation can be obtained. In particular, when $V > 1$ and $|A| < 1$, the integral in (11.7.14) with $\phi_0 = 0$ and $\xi_0 = 0$ can be written as

$$\xi = \left(\frac{1-V^2}{2}\right)^{\frac{1}{2}} \int_0^{\phi} \frac{d\psi}{\sqrt{(1-\cos\psi)-(1-A)}}$$
$$= \left(\frac{1-V^2}{2}\right)^{\frac{1}{2}} \int_0^{\phi} \frac{d\psi}{\sqrt{2}\sqrt{\sin^2\frac{\psi}{2}-m^2}}, \quad m^2 = \frac{1}{2}(1-A),$$

which, substituting $s = \frac{1}{m}\sin(\frac{\psi}{2})$, is

$$= \left(V^2-1\right)^{\frac{1}{2}} \int_0^{s} \frac{ds}{\sqrt{1-s^2}\sqrt{1-s^2m^2}}. \tag{11.7.20}$$

This can be expressed in terms of Jacobi's elliptic function (Dutta and Debnath 1965) as

$$s = \frac{1}{m}\sin\left(\frac{\phi}{2}\right) = sn\left(\frac{\xi}{\sqrt{V^2-1}}, m\right), \tag{11.7.21}$$

where m is the modulus of Jacobi's elliptic function $sn(z,m)$. Thus, the final solution is given by

$$u(x,r) = \phi(\xi) = 2\sin^{-1}\left[m sn\left(\frac{\xi}{\sqrt{V^2-1}}, m\right)\right]. \tag{11.7.22}$$

11.8 The Solution of the Sine-Gordon Equation by Separation of Variables

The form of the solution (11.7.17) suggests that the transformation

$$v(x,t) = \tan\left(\frac{1}{4}u\right) \tag{11.8.1}$$

can be used to transform the sine-Gordon equation

$$u_{xx} - u_{tt} = \sin u \tag{11.8.2}$$

into the form

$$\left(1+v^2\right)\left(v_{xx}-v_{tt}-v\right) - 2v\left(v_x^2-v_t^2-v^2\right) = 0, \tag{11.8.3}$$

in which the trigonometric identity $\sin u = 4v(1-v^2)/(1+v^2)^2$ has been utilized. We seek solutions by the separation of variables in the form

$$v(x,t) = \tan\left(\frac{1}{4}u\right) = \frac{\phi(x)}{\psi(t)}, \tag{11.8.4}$$

for some functions $\phi(x)$ and $\psi(t)$ to be determined. Substituting (11.8.4) in equation (11.8.3) and using the identity $\sin 4\theta = 4\tan\theta(1-\tan^2\theta)/(1+\tan^2\theta)^2$, we obtain

$$\left(\phi^2+\psi^2\right)\left(\frac{\phi_{xx}}{\phi}+\frac{\psi_{tt}}{\psi}\right)-2\left(\phi_x^2+\psi_t^2\right)=\left(\phi^2-\psi^2\right). \tag{11.8.5}$$

Differentiating this equation with respect to x and t enables us to separate the variables so that

$$\frac{1}{(\phi\phi_x)}\left(\frac{\phi_{xx}}{\phi}\right)_x=-\frac{1}{(\psi\psi_t)}\left(\frac{\psi_{tt}}{\psi}\right)_t=-4\kappa^2, \tag{11.8.6ab}$$

where $-4\kappa^2$ is a separation constant. Each of these ordinary differential equations (11.8.6ab) can be integrated twice to find

$$\phi'^2=-\kappa^2\phi^4+a\phi^2+b \quad \text{and} \quad \psi^2=\kappa^2\psi^4+c\psi^2+d. \tag{11.8.7ab}$$

Substituting these equations in (11.8.5) reveals that $a-c=1$ and $b+d=0$. Setting $a=m^2$ and $b=n^2$, we obtain

$$\phi_x^2=-\kappa^2\phi^4+m^2\phi^2+n^2 \quad \text{and} \quad \psi_t^2=\kappa^2\psi^4+\left(m^2-1\right)\psi^2-n^2, \tag{11.8.8ab}$$

where m and n are integration constants. In general, these equations can be solved in terms of elliptic functions. However, we solve these equations for the following special cases of interest:

Case 1. $\kappa=n=0$ and $m>1$.

In this case, equations (11.8.8ab) take the form

$$\phi_x=\pm m\phi \quad \text{and} \quad \psi_t=\pm\sqrt{m^2-1}\,\psi, \tag{11.8.9ab}$$

which give exponential solutions

$$\phi(x)=a_1\exp(\pm mx) \quad \text{and} \quad \psi(x)=a_2\exp\left(\pm\sqrt{m^2-1}\,t\right), \tag{11.8.10ab}$$

where a_1 and a_2 are integration constants. Thus, the solution (11.8.4) becomes

$$u(x,t)=4\tan^{-1}\left[\alpha\exp\left(\pm\frac{x\pm Ut}{\sqrt{1-U^2}}\right)\right], \tag{11.8.11}$$

where $\alpha=\frac{a_1}{a_2}$ and $U=\frac{\sqrt{m^2-1}}{m}$ (or $m=(1-U^2)^{-\frac{1}{2}}$) are constants.

Evidently, one of these solutions (11.8.11)

$$u(x,t)=4\tan^{-1}\left[\alpha\exp\{m(x-Ut)\}\right] \tag{11.8.12}$$

is identical with (11.7.17) for $\alpha=1$, and hence, it represents the *soliton* (or *kink*) solution of the sine-Gordon equation. On the other hand, from (11.8.11), another solution can be found in the form

$$u(x,t)=4\tan^{-1}\left[\alpha\exp\{-m(x-Ut)\}\right]=4\cot^{-1}\left[\frac{1}{\alpha}\exp\{m(x-Ut)\}\right]. \tag{11.8.13}$$

This is also identical with (11.7.18) for $\alpha = 1$ and represents the *antisoliton* (or *antikink*) solution of (11.8.2).

Furthermore, u_x and u_t also represent solitary wave solutions given by

$$u_x(x,t) = \pm 2m \operatorname{sech}\big[m(x \pm Ut) + \log \alpha\big] \tag{11.8.14}$$

and

$$u_t(x,t) = \pm 2\sqrt{m^2-1}\operatorname{sech}\big[m(x \pm Ut) + \log \alpha\big]. \tag{11.8.15}$$

Two other choices in sign in (11.8.11) represent *antisoliton* (or *antikink*) solutions of the sine-Gordon equation.

Case 2. $\kappa = 0$, $m^2 > 1$, and $n \neq 0$.

In this case, solutions of (11.8.8ab) are obtained by integration in terms of hyperbolic functions as

$$\begin{aligned} \phi(x) &= \pm\left(\frac{n}{m}\right)\operatorname{sech}(mx + a_1), \\ \psi(t) &= \frac{n}{\sqrt{m^2-1}}\cosh\left(\sqrt{m^2-1}\,t + a_2\right), \end{aligned} \tag{11.8.16}$$

where a_1 and a_2 are integrating constants. Thus, the solution is given by

$$u(x,t) = \pm\tan^{-1}\left[\frac{U\sinh(mx + a_1)}{\cosh(\sqrt{m^2-1}t + a_2)}\right]. \tag{11.8.17}$$

Since this result is made up of the ratio (ϕ/ψ), it does not depend on n. In particular, when $a_1 = a_2 = 0$, solution (11.8.17) can be expressed in the form

$$u(x,t) = \pm 4\tan^{-1}\left[\frac{U\sinh(mx)}{\cosh(mUt)}\right], \quad 0 < U^2 < 1. \tag{11.8.18}$$

This solution was first discovered by Perring and Skyrme (1962) based on numerical calculations for two interacting solitons. They verified analytically that (11.8.18) is, indeed, the exact solution, and it describes the interaction of two solitons similar to those of the KdV solitons, where solitons emerge from the interaction only with a slight change of phase. To show that the solution describes the interaction of two equal solitons, we investigate it asymptotically as $t \to \pm\infty$. Expressing $\sinh(mx)$ in terms of a sum of two exponentials, we first note that

$$\cosh(mUt) \sim \frac{1}{2}\exp\big(mU|t|\big) \quad \text{as } t \to \pm\infty.$$

In view of these results, it follows at once that

$$v(x,t) \sim U\exp\big\{m(x+Ut)\big\} - U\exp\big\{-m(x-Ut)\big\} \quad \text{as } t \to -\infty. \tag{11.8.19}$$

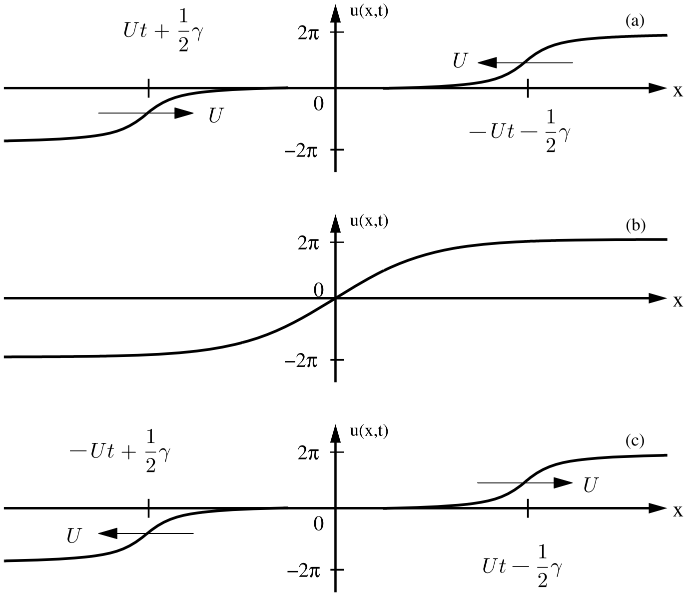

Fig. 11.2 Interaction of two equal solitons (11.8.18) for $U > 0$. (a) $t \to -\infty$, (b) $t = 0$, and (c) $t \to +\infty$. From Drazin (1983).

Solution (11.8.19) is uniformly valid for all x and represents two distinct solitons approaching one another at equal but opposite speed U, as shown in Figure 11.2. More precisely, in the limit as $x \to -\infty$ and $t \to -\infty$, the solution becomes

$$u(x,t) \sim -4\tan^{-1}\left[U\exp\{-m(x-Ut)\}\right]. \tag{11.8.20}$$

Solution (11.8.20) represents a pulse moving in the positive x-direction, as $u(x,t)$ increases from -2π to 0, as x passes through the value Ut.

As $x \to +\infty$ and $t \to -\infty$, the solution reduces to the form

$$u(x,t) \sim 4\tan^{-1}\left[U\exp\{-m(x+Ut)\}\right]. \tag{11.8.21}$$

This also represents a pulse traveling in the negative x-direction, as $u(x,t)$ increases from 0 to 2π, as x passes through the value $-Ut$. At $t = 0$, two pulses suffer from an interaction. All of these results are shown in Figure 11.2.

Similarly, as $t \to \infty$, the asymptotic solution is given by

$$v(x,t) \sim -U\exp\left(-\frac{x+Ut}{\sqrt{1-U^2}}\right) + U\exp\left(-\frac{x-Ut}{\sqrt{1-U^2}}\right). \tag{11.8.22}$$

In other words, as $x \to -\infty$ and $t \to +\infty$, this asymptotic solution is

$$u(x,t) \sim -4\tan^{-1}\left[U\exp\left(-\frac{x+Ut}{\sqrt{1-U^2}}\right)\right]. \tag{11.8.23}$$

As $x \to +\infty$ and $t \to +\infty$, this asymptotic solution for $u(x,t)$ is given by

$$u(x,t) \sim 4\tan^{-1}\left[U\exp\left(\frac{x-Ut}{\sqrt{1-U^2}}\right)\right]. \tag{11.8.24}$$

Since u varies from -2π to $+2\pi$ as x changes from $-\infty$ to $+\infty$, the corresponding solution is called a 4π *pulse*.

Finally, the only visible evidence of the interaction remaining, as $t \to +\infty$, is a longitudinal displacement of each soliton. To show this, we observe the exponential form of the solution

$$\pm U\exp\big[\pm m(x+Ut)\big] = \pm\exp\left[\pm m\left(x+Ut\pm\frac{1}{2}\gamma\right)\right], \tag{11.8.25}$$

where $\gamma = 2\sqrt{1-U^2}\log(U^{-1})$ represents the displacement of each soliton which is retarded by the interaction.

The above soliton solutions produce u_x and u_t. Clearly, for $U > 0$, we find

$$u_x(x,t) = \frac{4U\cosh(mx)\cosh(mUt)}{\sqrt{1-U^2}[\sinh^2(mx)+\cosh^2(mUt)]}. \tag{11.8.26}$$

This represents the interaction of two hump-shaped solitons as shown in Figure 11.3. This also shows, more clearly, the interaction of two solitons for $t \to \pm\infty$.

Case 3. $\kappa \neq 0$, $n = 0$, and $m^2 > 1$.

In this case, the solution $u(x,t)$ is given by

$$u(x,t) = -4\tan^{-1}\left[\frac{m}{\sqrt{m^2-1}}\cdot\frac{\sinh(\sqrt{m^2-1}\,t+a_2)}{\cosh(mx+a_1)}\right]. \tag{11.8.27}$$

This result is somewhat similar to that of (11.8.17), and can be interpreted physically with a given boundary condition $u_x(0,t) = 0$ for all time t. It represents a soliton moving toward the boundary and reflected back as an antisoliton, and hence, solution (11.8.27) represents an interaction of a soliton with an antisoliton.

In the limit as $m \to 1$, solution (11.8.27) reduces to the form

$$u(x,t) \sim -4\tan^{-1}\big[(t+a_2)\text{sech}\,(x+a_1)\big]. \tag{11.8.28}$$

Case 4. $\kappa \neq 0$, $n = 0$, and $m^2 < 1$.

This case gives a *new* solution which can be obtained from

$$u(x,t) \sim -4\tan^{-1}\left[\frac{m}{\sqrt{1-m^2}}\cdot\frac{\sinh(\omega t+a_2)}{\cosh(mx+a_1)}\right], \tag{11.8.29}$$

where $\omega = \sqrt{1-m^2} = mU$. This is known as the *breather solution* of the sine-Gordon equation and represents a pulse-type structure of a soliton. For fixed x, the

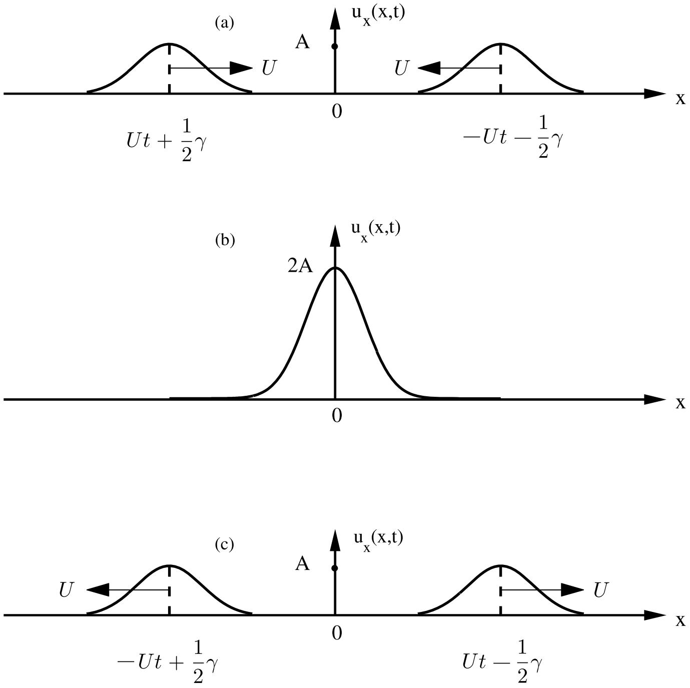

Fig. 11.3 Interaction of two hump-shaped solitons for (a) $t \to -\infty$, (b) $t = 0$, and (c) $t \to +\infty$ with amplitude $A = 2U/\sqrt{1-U^2}$. From Drazin (1983).

solution is a periodic function of time t with frequency $\omega = \sqrt{1-m^2}$. For $m = 0.8$, the breather solution as a function of x is shown in Figure 11.4.

The case for $m \ll 1$ corresponds to a *small-amplitude breather* solution. This can be derived from (11.8.29) by expanding the inverse tangent function for small m and retaining only the first term. This leads to a limiting solution in the form

$$u(x,t) \sim 2im\,\text{sech}(mx)\exp\left[i\left(1-\frac{m^2}{2}\right)t\right] + c.c., \tag{11.8.30}$$

$$= A(x,t)\exp(it), \tag{11.8.31}$$

where $A(x,t) = 2im\,\text{sech}(mx)\exp(\frac{itm^2}{2})$. It can easily be verified that $A(x,t)$ satisfies the $(1+1)$-dimensional nonlinear Schrödinger equation

$$2iA_t - A_{xx} - |A|^2 A = 0, \tag{11.8.32}$$

where the term $A_{tt} = O(m^4)$ has been neglected.

Finally, more general solutions of equations (11.8.8ab) can be found in terms of elliptic functions. It is also of interest to investigate the solutions of the sine-Gordon equation confined to a finite region of space by boundaries. It turns out that the oscillations of the soliton's position closely resemble those of a particle confined by

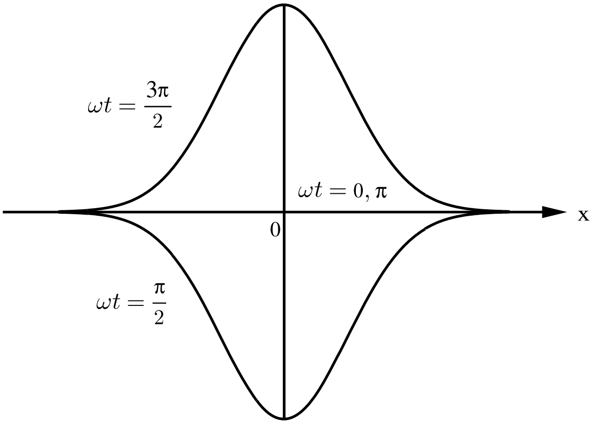

Fig. 11.4 Breather solution of (11.8.2) for $m = 0.8$.

an external potential. This confirms the fact that solitons behave like *elastic particles*. For details, the reader is referred to De Leonardis and Trullinger (1980) and Newell and Kaup (1978).

It may be noted that a variety of solutions of the sine-Gordon equation have been studied by Lamb (1973, 1980).

Example 11.8.1 (Energy of a Soliton, an Antisoliton, and a Breather). We consider the energy associated with the soliton and the antisoliton of the sine-Gordon equation

$$u_{xx} - u_{tt} = \sin u. \tag{11.8.33}$$

The Lagrangian density of this equation is given by

$$\mathcal{L} = \frac{1}{2}\left(u_t^2 - u_x^2\right) - (1 - \cos u). \tag{11.8.34}$$

The Hamiltonian density has the form

$$\mathcal{H} = u_t\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) - \mathcal{L}, \tag{11.8.35}$$

which, due to (11.8.34), is

$$= \frac{1}{2}\left(u_t^2 + u_x^2\right) + (1 - \cos u). \tag{11.8.36}$$

For the single-soliton solution

$$u(x,t) = 4\tan^{-1}\left[\exp\left\{m(x - Ut)\right\}\right], \tag{11.8.37}$$

where $m = (1 - U^2)^{-\frac{1}{2}}$, we obtain the Hamiltonian density $\mathcal{H}$ and energy E as

$$\mathcal{H} = 4m^2 \operatorname{sech}^2\left\{m(x - Ut)\right\}, \tag{11.8.38}$$

$$E = \int_{-\infty}^{\infty} \mathcal{H}\, dx = 8m. \tag{11.8.39}$$

For the antisoliton solution

$$u(x,t) = 4\tan^{-1}\left[\exp\left\{-m(x-Ut)\right\}\right], \tag{11.8.40}$$

the energy E is also equal to $8m$. This shows that the solution represents the elastic interaction of a soliton and an antisoliton, each of which has energy $8m$.

For the soliton–antisoliton solution (11.8.27), we find that

$$\mathcal{H} = \left(\frac{8m^2}{\kappa}\right) \cdot \frac{\text{sech}^2\{mx(p + q\tanh^2 mx)\}}{(1 + r\text{sech}^2 mx)^2}, \tag{11.8.41}$$

where $\kappa^2 = m^2 - 1 > 0$, $p = \kappa^2 + (1+\kappa^2)\text{sech}^2(\kappa t)$, $q = m^2 \sinh^2(\kappa t)$, and $r = (\frac{m}{\kappa})^2 \sinh^2(\kappa t)$. In this case, it is easy to check that the energy $E = 16m$, where $m > 1$.

Finally, the breather solution is given by (11.8.29). The associated energy is $E = 16m$, where $m > 1$. The energy is, thus, equal to the sum of energies of two free solitons (or antisolitons) and leads to the physical interpretation of the breather as a *bound state* composed of a pair of a soliton and an antisoliton.

11.9 Bäcklund Transformations for the Sine-Gordon Equation

We consider the characteristic form of the sine-Gordon equation

$$u_{xt} = \sin u. \tag{11.9.1}$$

In 1880, Bäcklund discovered the transformation as the pair of equations

$$\frac{1}{2}(u-v)_x = a\sin\frac{1}{2}(u+v), \tag{11.9.2a}$$

$$\frac{1}{2}(u+v)_t = \frac{1}{a}\sin\frac{1}{2}(u-v), \tag{11.9.2b}$$

where a is a nonzero arbitrary real constant. This pair of equations is called the *Bäcklund transformations* of (11.9.1). The cross-differentiation of (11.9.2a), (11.9.2b) gives

$$\frac{1}{2}(u-v)_{tx} = \frac{a}{2}(u+v)_t \cos\frac{1}{2}(u+v) = \sin\frac{1}{2}(u-v)\cos\frac{1}{2}(u+v), \tag{11.9.3}$$

$$\frac{1}{2}(u+v)_{xt} = \frac{1}{2a}(u-v)_x \cos\frac{1}{2}(u-v) = \sin\frac{1}{2}(u+v)\cos\frac{1}{2}(u-v). \tag{11.9.4}$$

Adding and subtracting these equations, we obtain

$$u_{xt} = \sin u \quad \text{and} \quad v_{xt} = \sin v. \tag{11.9.5ab}$$

Evidently, the Bäcklund transformations (11.9.2a), (11.9.2b) relate two solutions u and v that satisfy the same equation (11.9.1). Since both u and v satisfy the same sine-Gordon equation, the pair (11.9.2a), (11.9.2b) is referred to as an *auto-Bäcklund transformation* for (11.9.1). The auto-Bäcklund transformation can be used to construct a sequence of solutions of the sine-Gordon equation beginning with any given solution.

We next use (11.9.2a), (11.9.2b) to solve the sine-Gordon equation (11.9.1). Obviously, $u(x,t)=0$ is a trivial solution of (11.9.1) for all x and t. We use this solution to construct a nontrivial solution. We set $v=0$, so that the Bäcklund transformation (11.9.2a), (11.9.2b) becomes

$$u_x = 2a\sin\left(\frac{1}{2}u\right), \qquad u_t = \frac{2}{a}\sin\left(\frac{1}{2}u\right). \tag{11.9.6ab}$$

Integrating these equations gives

$$2ax = \int^u \operatorname{cosec}\left(\frac{1}{2}u\right)du = 2\log\left|\tan\left(\frac{1}{4}u\right)\right| + A(t), \tag{11.9.7}$$

$$\frac{2t}{a} = \int^u \operatorname{cosec}\left(\frac{1}{2}u\right)du = 2\log\left|\tan\left(\frac{1}{4}u\right)\right| + B(x), \tag{11.9.8}$$

where $A(t)$ and $B(x)$ are arbitrary functions involved as a result of integration. It follows from (11.9.7) and (11.9.8) that

$$\tan\left(\frac{u}{4}\right) = \alpha\exp\left(ax + \frac{t}{a}\right), \tag{11.9.9}$$

where α is a constant. Or equivalently,

$$u(x,t) = 4\tan^{-1}\left[\alpha\exp\left(ax+\frac{t}{a}\right)\right]. \tag{11.9.10}$$

This represents a *new solution* of the sine-Gordon equation (11.9.1) and it describes a *soliton* (or *kink*) solution.

Returning to the original sine-Gordon equation (11.7.1) with $x=\frac{1}{2}(X-cT)$ and $t=\frac{1}{2}(X+cT)$, we obtain

$$\begin{aligned} u(X,T) &= 4\tan^{-1}\left[\alpha\exp\left\{\frac{a}{2}(X-cT)+\frac{1}{2a}(X+cT)\right\}\right] \\ &= 4\tan^{-1}\left[\alpha\exp\{\pm m(X-UT)\}\right], \end{aligned} \tag{11.9.11}$$

where $\frac{1}{2}(a+\frac{1}{a}) = m = (1-U^2)^{-\frac{1}{2}}$, $|U|<1$, and $\frac{c}{2}(a-\frac{1}{a}) = Um$ so that

$$a = m\left(1+\frac{U}{c}\right) \quad \text{and} \quad U = c\left(\frac{a^2-1}{a^2+1}\right).$$

Thus the solution (11.9.11) represents a soliton (or kink) and an antisoliton (or antikink) corresponding to the positive or negative sign in (11.9.11). These solutions are of width m, and they propagate with the constant velocity U.

We next construct a new soliton solution representing two interacting solitons by means of a purely algebraic method from a known solution u_0. We first replace u and v by u_1 and u_2, respectively, in (11.9.2a), (11.9.2b), and, then, obtain two solutions corresponding to two arbitrary constants $a = a_1$ and $a = a_2$ with $a_1 \neq a_2$. Using these solutions, we seek two more solutions u_3 and u_4 corresponding to parameters $a = a_2$ and u_2 with $a = a_1$ and u_2, respectively. According to the above procedure, we obtain four relations from (11.9.2a), (11.9.2b):

$$\frac{\partial u_1}{\partial x} = \frac{\partial u_0}{\partial x} + 2a_1 \sin \frac{1}{2}(u_1 + u_0), \tag{11.9.12}$$

$$\frac{\partial u_2}{\partial x} = \frac{\partial u_0}{\partial x} + 2a_2 \sin \frac{1}{2}(u_2 + u_0), \tag{11.9.13}$$

$$\frac{\partial u_3}{\partial x} = \frac{\partial u_1}{\partial x} + 2a_2 \sin \frac{1}{2}(u_3 + u_1), \tag{11.9.14}$$

$$\frac{\partial u_4}{\partial x} = \frac{\partial u_2}{\partial x} + 2a_1 \sin \frac{1}{2}(u_4 + u_2). \tag{11.9.15}$$

We set $u_3 = u_4 = v_2$ in (11.9.14) and (11.9.15) and choose suitable values for integrating constants. Then, subtracting the resulting expressions (11.9.15) from (11.9.12), and (11.9.14) from (11.9.13), we obtain

$$a_1 \left[\sin \frac{1}{2}(v_2 + u_2) - \sin \frac{1}{2}(u_1 + u_0) \right] = a_2 \left[\sin \frac{1}{2}(v_2 + u_1) - \sin \frac{1}{2}(u_2 + u_0) \right].$$

This result can be simplified by using the standard formula for the difference of two sine functions to obtain

$$a_1 \sin \left[\frac{1}{4} \{ (v_2 - u_0) - (u_1 - u_2) \} \right] = a_2 \sin \left[\frac{1}{4} \{ (v_2 - u_0) + (u_1 - u_2) \} \right].$$

Further simplification of this result gives

$$\tan \left\{ \frac{1}{4}(v_2 - u_0) \right\} = \left(\frac{a_1 + a_2}{a_1 - a_2} \right) \tan \left\{ \frac{1}{4}(u_1 - u_2) \right\}, \tag{11.9.16}$$

or equivalently,

$$v_2 = 4 \tan^{-1} \left[\left(\frac{a_1 + a_2}{a_1 - a_2} \right) \tan \left\{ \frac{1}{4}(u_1 - u_2) \right\} \right] + u_0. \tag{11.9.17}$$

This gives a *new solution* in terms of given solutions u_0, u_1, and u_2 and it can be regarded as the nonlinear superposition principle for the sine-Gordon equation. It is important to point out that the same solution (11.9.17) can be derived from (11.9.2b). It is also possible to continue this procedure many times, and hence, this leads to an n interacting soliton solution v_n from an $(n-1)$-soliton solution v_{n-1}. In particular, the solution for two interacting solitons can be obtained in the form

$$\tan \left(\frac{1}{4} v_2 \right) = \left(\frac{a_1 + a_2}{a_1 - a_2} \right) \frac{\sinh[\frac{1}{2}(\xi_1 - \xi_2)]}{\cosh[\frac{1}{2}(\xi_1 + \xi_2)]}, \tag{11.9.18}$$

where

$$\xi_r = \pm\left(1+U_r^2\right)^{-\frac{1}{2}}(X-UT), \qquad a_r = \pm\left[(1-U_r)/(1+U_r)\right]^{\frac{1}{2}}, \quad r=1,2,$$

and solutions (11.9.11) and (11.9.16) have been used.

We next set $U_1 = -U_2 = U$ to reduce the solution (11.9.18) to the form

$$\tan\left(\frac{1}{4}v_2\right) = \frac{U\sinh(mX)}{\cosh(mUt)}. \tag{11.9.19}$$

This is exactly the same result (11.8.18) obtained by Perring and Skyrme (1962). It was already investigated as the asymptotic behavior of (11.9.19), as $t \to \pm\infty$, in Section 11.8. The asymptotic solution describes the interaction of two-soliton solutions.

11.10 The Solution of the Sine-Gordon Equation by the Inverse Scattering Method

We use the characteristic form of the sine-Gordon equation (11.9.1). Following Lamb (1980), we describe the method of solution of (11.9.1) by the inverse scattering approach. The essence of this approach is to relate equation (11.9.1) to a pair of linear scattering equations for v_1 and v_2,

$$\left.\begin{aligned} \frac{\partial v_1}{\partial x} + i\zeta v_1 &= -\tfrac{1}{2}u_x v_2, \\ \frac{\partial v_2}{\partial x} - i\zeta v_2 &= \tfrac{1}{2}u_x v_1. \end{aligned}\right\} \tag{11.10.1ab}$$

These equations, together with the conditions that v_1 and v_2 are bounded as $x \to \pm\infty$, may be considered as a problem of determining the eigenvalue ζ for a given function $q = -\frac{1}{2}u_x$ called the *potential* corresponding to a solution $u(x,t)$ of (11.9.1).

We note that, if v_1 and v_2 are bounded and $q \to 0$ sufficiently rapidly as $x \to \pm\infty$, then $v_1 \sim a_1\exp(-i\zeta x)$ and $v_2 \sim a_2\exp(i\zeta x)$, as $x \to \pm\infty$. So, there exists an unbounded state if and only if ζ is real. If ζ is taken as a fixed, real number, then $\zeta_t = 0$. Differentiating (11.10.1ab) with respect to t, integrating with respect to x, and then, combining with equation (11.9.1) gives the evolution equations for the vector eigenfunctions (v_1, v_2) in the form

$$\left.\begin{aligned} \frac{\partial v_1}{\partial t} &= \frac{i}{4\zeta}(v_1\cos u - v_2\sin u), \\ \frac{\partial v_2}{\partial t} &= \frac{i}{4\zeta}(v_1\sin u + v_2\cos u). \end{aligned}\right\} \tag{11.10.2ab}$$

Since the solution u is related to q through $q = -\frac{1}{2}u_x$ so that

$$u(x,t) = -2\int_{-\infty}^{x} q(\xi,t)\,d\xi, \tag{11.10.3}$$

it follows that $u \to 0$ as $x \to -\infty$. Hence, equations (11.10.2ab) give

$$v_{1t} \sim \frac{i}{4\zeta} v_1, \tag{11.10.4a}$$

$$v_{2t} \sim -\frac{i}{4\zeta} v_2 \quad \text{as } x \to -\infty. \tag{11.10.4b}$$

The form of the equations (11.10.2ab) as $x \to +\infty$ depends on the value of $u(x,t)$ as $x \to +\infty$. The only cases that can be solved without any problem are those in which $u(x,t) \to 2n\pi$ as $x \to +\infty$.

From (11.10.4a) v_1 is either zero or tends to $\exp(-\frac{it}{4\zeta})$ as $x \to -\infty$. Similarly, (11.10.4b) shows that v_2 is either zero or tends to $\exp(-\frac{it}{4\zeta})$ as $x \to -\infty$. We then consider the solution that is proportional to the fundamental solutions given by

$$\phi_1(x,k) = \exp(-ikx) + \int_{-\infty}^{x} A_1(x,\xi) \exp(-ik\xi)\, d\xi \tag{11.10.5}$$

and

$$\phi_2(x,k) = \int_{-\infty}^{x} A_2(x,\xi) \exp(-ik\xi)\, d\xi, \tag{11.10.6}$$

where A_1 and A_2 represent the scattering (or wake) and the reflected particles. Then,

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \sim \exp\left(\frac{it}{4\zeta}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\zeta x) \quad \text{as } x \to -\infty. \tag{11.10.7}$$

The form of the solution can be represented as a linear combination of two linearly independent solutions, that is,

$$v \sim \exp\left(\frac{it}{4\zeta}\right) \left[c_{11} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(i\zeta x) + c_{12} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\zeta x) \right] \quad \text{as } x \to +\infty. \tag{11.10.8}$$

Consequently, in the limit as $x \to +\infty$,

$$\left.\begin{aligned} v_1 &\sim c_{11}(\zeta,t) \exp[i(\tfrac{t}{4\zeta} - \zeta x)], \\ v_2 &\sim c_{12}(\zeta,t) \exp[i(\tfrac{t}{4\zeta} + \zeta x)], \end{aligned}\right\} \tag{11.10.9ab}$$

where the time dependence of c_{11} and c_{12} is found from (11.10.2ab) in the limit as $x \to +\infty$. Since we consider the case where $u \to 2n\pi$ as $x \to +\infty$,

$$v_{1t} \sim \frac{i}{4\zeta} v_1, \quad v_{2t} \sim -\frac{i}{4\zeta} v_2 \quad \text{as } x \to +\infty. \tag{11.10.10ab}$$

Following the method used for the modified KdV equation, it turns out that

$$c_{11}(\zeta,t) = c_{11}(\zeta,0) \exp\left(-\frac{it}{2\zeta}\right), \qquad c_{12}(\zeta,t) = c_{12}(\zeta,0).$$

Using the notation

$$R_R(\zeta,t) = c_{11}(\zeta,t)/c_{12}(\zeta,t) \quad \text{and} \quad m_{L\ell}(k_\ell,t) = -ic_{11}(k_\ell,t)/c_{12}(k_\ell,t),$$

we find the following result:

$$\begin{aligned}\Omega_R(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_R(\zeta,0)\exp\left[-i\left(\frac{t}{2\zeta}-\zeta z\right)\right]d\zeta \\ -\sum_{\ell=1}^{N} m_{R\ell}(k_\ell,0)\exp\left[-i\left(\frac{t}{2k_\ell}-k_\ell z\right)\right].\end{aligned} \tag{11.10.11}$$

Similarly, we can derive

$$\begin{aligned}\Omega_L(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_L(\zeta,0)\exp\left[-i\left(\frac{t}{2\zeta}-\zeta z\right)\right]d\zeta \\ -\sum_{\ell=1}^{N} m_{L\ell}(k_\ell,0)\exp\left[-i\left(\frac{t}{2k_\ell}-k_\ell z\right)\right].\end{aligned} \tag{11.10.12}$$

A procedure similar to that used for the modified KdV equation can be employed to determine the pure soliton solutions without any difficulty. Making reference to Lamb (1980), the multisoliton solutions are then given by

$$u(x,t) = -4\tan^{-1}\left[\frac{\text{Im}|I-iM|}{\text{Re}|I-iM|}\right], \tag{11.10.13}$$

where I is a unit matrix of order N, N is the number of zeros of $c_{12}(\zeta,0)$ in the upper half-plane, and M is an $N\times N$ matrix with time-dependent elements

$$M_{Lj}(k_j,t) = m_{Lj}(k_j,0)\exp\left(\frac{it}{2k_j}\right). \tag{11.10.14}$$

Here, we consider the simplest case which deals with a single pole on the imaginary axis at $k_1 = \frac{1}{2}ia_1$ so that

$$m_{L1}(k_1,t) = m_{L1}(k_1,0)\exp\left(\frac{t}{a_1}\right) \tag{11.10.15a}$$

and

$$M_{L1} = \frac{1}{a}m_{L1}(k_1,0)\exp\left(a_1x+\frac{t}{a_1}\right). \tag{11.10.15b}$$

Consequently, the single-soliton solution is derived from (11.10.13) as

$$u(x,t) = 4\tan^{-1}\left[\exp\left(a_1x+\frac{t}{a_1}+\gamma\right)\right], \tag{11.10.16}$$

where $\gamma = \log[\frac{1}{a}\cdot m_{L1}(k_1,0)]$. This solution is identical to (11.8.11) which was found earlier by a different, but simpler approach.

Finally, the solution of the sine-Gordon equation representing the interaction of two solitons can also be obtained from two poles on the imaginary axis. This case is similar to that of the multisoliton solution of the KdV equation. Here, we set $k_r = \frac{1}{2} i a_r$, $r = 1, 2$. Then the expression $|I - iM_L|$ takes the form

$$\mathrm{Re}|I - iM_L| = 1 - \frac{m_{L1} m_{L2}}{a_1 a_2} \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 \exp\bigl\{(a_1 + a_2)x\bigr\}, \quad (11.10.17)$$

$$\mathrm{Im}|I - iM_L| = -\left[\frac{m_{L1}}{a_1} \exp(a_1 x) + \frac{m_{L2}}{a_2} \exp(a_2 x)\right]. \quad (11.10.18)$$

Consequently, the two-soliton solution takes the form

$$u(x,t) = -4 \tan^{-1}\left[\left(\frac{a_1 + a_2}{a_1 - a_2}\right) \frac{\cosh \frac{1}{2}(u_1 - u_2)}{\sinh \frac{1}{2}(u_1 + u_2)}\right], \quad a_1 > a_2, \quad (11.10.19)$$

where

$$u_r = \left(a_r x + \frac{t}{a_r} + \gamma_r\right), \qquad \exp(\gamma_r) = \left(\frac{a_1 - a_2}{a_1 + a_2}\right)\left[\frac{m_{Lr}(k, 0)}{a_r}\right]. \quad (11.10.20)$$

The above solutions are those special solutions which evolve into pure multisoliton solutions.

11.11 The Similarity Method for the Sine-Gordon Equation

To obtain a similarity solution for the characteristic form of the sine-Gordon equation (11.9.1), we assume that this equation is invariant under a group of transformations

$$\tilde{x} = a^{\alpha} x, \qquad \tilde{t} = a^{\beta} t, \quad \text{and} \quad \tilde{u} = a^{\gamma} u, \quad (11.11.1)$$

for suitable values of constants α, β, and γ. It follows that

$$\tilde{u}_{\tilde{x}\tilde{t}} - \sin \tilde{u} = a^{\gamma - \alpha - \beta} u_{xt} - \sin\bigl(a^{\gamma} u\bigr) = u_{xt} - \sin u, \quad (11.11.2)$$

provided that $\gamma = 0$ and $\alpha = -\beta$. This shows that equation (11.9.1) admits similarity transformations given by

$$u(x,t) = v(\eta), \quad \eta = x t^{-\alpha/\beta} = xt. \quad (11.11.3)$$

Using $\frac{\partial}{\partial x} = t \frac{d}{d\eta}$, $\frac{\partial}{\partial t} = x \frac{d}{d\eta}$, equation (11.9.1) reduces to the ordinary differential equation

$$\eta v''(\eta) + v'(\eta) = \sin v, \quad (11.11.4)$$

where the prime denotes differentiation with respect to η.

We next introduce a new dependent variable $w = \exp(iv)$ in (11.11.4) to derive an equation for w in the form

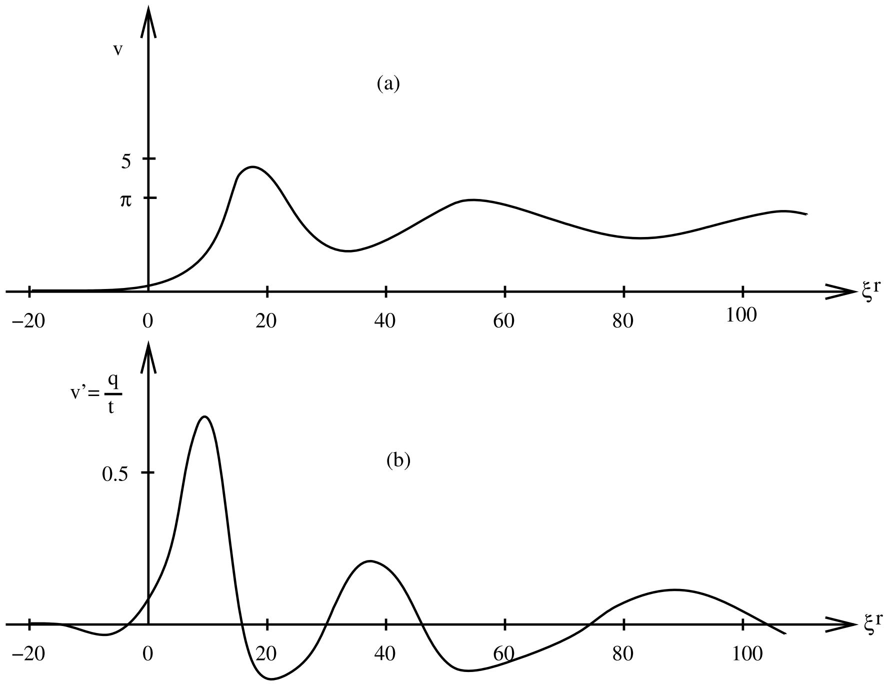

Fig. 11.5 Similarity solution of (11.11.4). From Lamb (1980).

$$w'' - \frac{w'^2}{w} + \frac{1}{2\eta}\left(2w' - w^2 + 1\right) = 0. \tag{11.11.5}$$

Equation (11.11.5) is a special case of the equation that defines the Painlevé equation of the third kind (Davis 1962, p. 185). However, it is normally not easy to find a solution of (11.11.5); hence, it is necessary to use numerical integration to do so. An example of such a solution is given by Lamb (1980) with a phase plane analysis and graphical representations of functions v and $v' = -\frac{2q}{t} = \frac{u_x}{t}$ by (11.11.3). The numerical solutions for v and v' shown in Figure 11.5 satisfy the conditions $v(0) = 0.1$ and $v'(0) = \sin v(0)$, giving a solution that is finite at $\eta = 0$.

11.12 Nonlinear Optics and the Sine-Gordon Equation

McCall and Hahn (1967, 1969) discovered a remarkable new wave mode which propagates in a two-level atomic system without attenuation. When a sufficiently intense electromagnetic wave is incident on the system as a short pulse, then the number of atoms in the excited state in the leading part of the pulse exceeds that in the ground state due to the absorption of the electromagnetic wave. However, the electromagnetic wave in the rear part of the pulse is again emitted due to induced emission so that the system transmits a pulse without absorption. This is called *self-induced transparency* (SIT).

In SIT, a medium transmits a light pulse at the resonant frequency without resonance absorption. When an electromagnetic wave is incident on a two-level atomic system, an ensemble of atoms has a ground state of energy E_1 and an excited state of energy E_2. If the frequency of the wave is equal to the transition frequency $\omega_0 = (E_2 - E_1)/\hbar$, the transition from the ground state to the excited state occurs due to the resonance absorption of the electromagnetic wave. If the present system is composed of a large number of atoms, the atoms excited to the higher level lose energy by collision with atoms and the slow irreversible decay of the number of excited atoms to lower levels. However, the remarkable fact is that the pulse propagates as a soliton without attenuation of energy.

To describe the self-induced transparency phenomena, we may regard the electromagnetic wave as a modulated, circularly polarized plane wave traveling along the x-axis so that the electric field $\mathbf{E}(x,t)$ is represented by

$$\mathbf{E}(x,t) = \mathcal{E}(x,t)\big(\hat{\mathbf{j}}\cos\theta + \hat{\mathbf{k}}\sin\theta\big), \tag{11.12.1}$$

where $\theta = (kx - \omega t) + \psi(x)$, $k = \frac{n\omega}{c}$, is the refractive index determined by the linear dispersion relation of the medium, the amplitude $\mathcal{E}(x,t)$ and the phase $\psi(x)$ vary slowly compared with that of the carrier wave, that is,

$$\left|\frac{\partial \mathcal{E}}{\partial x}\right| \ll |k\mathcal{E}|, \qquad \left|\frac{\partial \mathcal{E}}{\partial t}\right| \ll |\omega\mathcal{E}|, \tag{11.12.2}$$

and similar results hold for $\psi(x)$.

The electric field $\mathbf{E}(x,t)$ can be obtained from the Maxwell equations and satisfies the wave equation

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} - \left(\frac{n}{c}\right)^2 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \left(\frac{4\pi}{c^2}\right)\frac{\partial^2 \mathbf{P}}{\partial t^2}, \tag{11.12.3}$$

where $\mathbf{P}(x,t)$ is the polarization of the medium caused by the electromagnetic wave. If the equation for $\mathbf{P}(x,t)$ is given, this equation combined with (11.12.1) forms a closed system for two unknowns $\mathcal{E}$ and ψ.

We assume that the medium is made up of N two-level atoms and the transition frequencies ω_0 of individual atoms are not all the same but are distributed about the frequency ω_0 of the incident wave field in accordance with the spectral density function $f(\Delta\omega)$, which is normalized by the conditions

$$\Delta\omega = (\omega_0 - \omega) \quad \text{and} \quad \int_{-\infty}^{\infty} f(\Delta\omega)\, d(\Delta\omega) = 1. \tag{11.12.4}$$

Consequently, the polarization of the medium reduces to

$$\mathbf{P}(x,t) = N\int_{-\infty}^{\infty} f(\Delta\omega)\mathbf{p}(\Delta\omega, x, t)\, d(\Delta\omega), \tag{11.12.5}$$

where $\mathbf{p}(\Delta\omega, x, t)$ is the polarization of a two-level atom induced by the electromagnetic wave and $\mathbf{p}(\Delta\omega, x, t)$ satisfies the slow-variation conditions of (11.12.2).

Thus, when the polarization $\mathbf{p}$ consists of two components, one is in phase with the carrier wave of the electric field $\mathbf{E}(x,t)$ and the other is out of phase, so that it can be expressed as

$$\begin{aligned}\mathbf{p}(\varDelta\omega,x,t) &= u(\varDelta\omega,x,t)\big(\hat{\mathbf{j}}\cos\theta+\hat{\mathbf{k}}\sin\theta\big)\\ &\quad+v(\varDelta\omega,x,t)\big(\hat{\mathbf{j}}\sin\theta-\hat{\mathbf{k}}\cos\theta\big),\end{aligned} \tag{11.12.6}$$

where u and v correspond to the modulations of the electric field and become slowly varying functions of x and t.

Therefore, the problem reduces to determining the equations for u and v, which can be obtained from the Schrödinger equation for a two-level atom in the incident electric field. Invoking the electric dipole approximation and neglecting relaxation effects of the excited states, McCall and Hahn (1969) derived the following equations for u and v:

$$\frac{\partial u}{\partial t} = v\varDelta\omega, \tag{11.12.7}$$

$$\frac{\partial v}{\partial t} = -u\varDelta\omega - \left(\frac{\kappa^2}{\omega}\right)\mathcal{E}W, \tag{11.12.8}$$

$$\frac{\partial W}{\partial t} = v\mathcal{E}\omega, \tag{11.12.9}$$

where u and v are usually called the *dispersive* and *absorptive* components of the electric dipole, respectively, $\kappa = (2p/\hbar)$, p is the magnitude of the electric dipole moment, and W is the energy of the two-level atom, that is, $W = -\frac{1}{2}\hbar\omega_0$ in the ground state and $W = \frac{1}{2}\hbar\omega$ in the excited state. Moreover, we assume that all two-level atoms are in the ground state until the dipole transitions occur.

Thus, equations (11.12.3) and (11.12.7)–(11.12.9) constitute a closed system for u, v, W, and $\mathbf{E}$. Since u, v, and $\mathbf{E}$ are slowly varying functions corresponding to the modulation of $\mathbf{p}$, we derive equations for the modulational quantities $\mathcal{E}$ and ψ from (11.12.3). Substituting (11.12.1), (11.12.5), and (11.12.6) in equation (11.12.3) and using the slow-variation conditions (11.12.2), we derive the following equations for $\mathcal{E}$ and ψ:

$$\frac{\partial\mathcal{E}}{\partial x} + \frac{n}{c}\frac{\partial\mathcal{E}}{\partial t} = -\frac{2\pi\omega N}{nc}\int_{-\infty}^{\infty} vf(\varDelta\omega)\,d(\varDelta\omega), \tag{11.12.10}$$

$$\mathcal{E}\frac{\partial\psi}{\partial x} = \frac{2\pi\omega N}{nc}\int_{-\infty}^{\infty} uf(\varDelta\omega)\,d(\varDelta\omega). \tag{11.12.11}$$

Thus, we conclude that five equations (11.12.7)–(11.12.11) for u, v, W, $\mathcal{E}$, and ψ describe the self-induced transparency phenomena. In particular, if all the transition frequencies of N two-level atoms are identical, so that $f(\varDelta\omega) = \delta(\varDelta\omega)$ and the frequency ω of the electromagnetic wave is equal to that frequency ($\varDelta\omega = 0$), equations (11.12.7)–(11.12.11) can be reduced to the sine-Gordon equation. To show this, we can set $u \equiv 0$ and $\psi \equiv 0$ because of (11.12.7) and (11.12.11). Consequently, the system of equations (11.12.8)–(11.12.10) assumes the form

$$\frac{\partial v}{\partial t} = -\left(\frac{\kappa^2}{\omega_0}\right)\mathcal{E}W, \tag{11.12.12}$$

$$\frac{\partial W}{\partial t} = \omega_0 v\mathcal{E}, \tag{11.12.13}$$

$$\frac{\partial \mathcal{E}}{\partial x} + \frac{n}{c}\frac{\partial \mathcal{E}}{\partial t} = -\frac{2\pi\omega_0 N}{nc}v. \tag{11.12.14}$$

Introducing a new function $\phi(x,t)$ by

$$\phi(x,t) = \kappa\int_{-\infty}^{t}\mathcal{E}(x,\tau)\,d\tau \tag{11.12.15}$$

from (11.12.12), (11.12.13), we can write $v(x,t)$ and $W(x,t)$ in the form

$$v(x,t) = p\sin\{\phi(x,t)\}, \tag{11.12.16a}$$

$$W(x,t) = W_0\cos\{\phi(x,t)\}, \tag{11.12.16b}$$

where $W_0 = -\frac{\hbar\omega_0}{2}$, and we have used the boundary condition that $W(x,t) = -\frac{1}{2}\hbar\omega_0$, as $t \to -\infty$, that is, all the two-level atoms are in the ground state. Using equations (11.12.15) and (11.12.16a), equation (11.12.14) reduces to the sine-Gordon equation

$$\phi_{xx} + \frac{1}{c'}\phi_{tt} = -\gamma^2\sin\phi, \tag{11.12.17}$$

where $c' = \frac{c}{n}$ and $\gamma^2 = (\pi N\omega_0\hbar)/nc$.

Using a suitable transformation of variables, equation (11.12.17) can be reduced to the canonical form of the sine-Gordon equation,

$$\phi_{xx} - \phi_{tt} = \sin\phi. \tag{11.12.18}$$

11.13 Nonlinear Lattices and the Toda-Lattice Soliton

We follow Toda (1967) and write the Toda lattice equation (2.3.36) with unit mass $(m = 1)$ as the first-order system

$$\frac{ds_n}{dt} = ae^{-br_n} - a = f(r), \tag{11.13.1}$$

$$\frac{dr_n}{dt} = 2s_n - s_{n+1} - s_{n-2}. \tag{11.13.2}$$

We note that

$$\ddot{s}_n = -abe^{-br_n}\frac{dr_n}{dt} = -b\left(a + \frac{ds_n}{dt}\right)\left(\frac{dr_n}{dt}\right)$$

so that

$$\frac{\ddot{s}_n}{b(a+\dot{s}_n)} = s_{n+1} - 2s_n - 2s_{n-1}, \tag{11.13.3}$$

where the dot denotes the derivative with respect to time t.

We next seek a traveling wave solution

$$s_n(t) = f(\theta), \quad \theta = \omega t - \alpha n \tag{11.13.4}$$

so that $f(\theta)$ satisfies the equation

$$\frac{\omega^2 f''}{b(a + \omega f')} = f(\theta + \alpha) - 2f(\theta) + f(\theta - \alpha). \tag{11.13.5}$$

In order to find $f(\theta)$, Toda suggested the identity involving elliptic functions

$$dn^2(\theta + \alpha) - dn^2(\theta - \alpha) = -2k^2 \frac{d}{d\alpha}\left[\frac{sn\,\theta\; cn\,\theta\; dn\,\theta sn^2\,\alpha}{1 - k^2 sn^2\,\theta\; sn^2\,\alpha}\right], \tag{11.13.6}$$

where k is the modulus of the Jacobi elliptic functions. Using the Jacobi Epsilon function defined by

$$E(\theta) = \int_0^\theta dn^2 x\, dx, \tag{11.13.7}$$

equation (11.13.6) is integrated to obtain

$$E(\theta + \alpha) - 2E(\theta) + E(\theta - \alpha) = -2k^2 \frac{sn\,\theta\; cn\,\theta\; dn\,\theta\; sn^2\alpha}{(1 - k^2\, sn^2\,\theta\; sn^2\,\alpha)}. \tag{11.13.8}$$

This is close to equation (11.13.5). We further note that $E'(\theta) = dn^2\,\theta = 1 - k^2\, sn^2\,\theta$ and $E''(\theta) = -2k^2\, sn\theta\; cn\,\theta\; dn\,\theta$ so that (11.13.8) can be written as

$$\frac{E''(\theta)}{\beta + E'(\theta)} = E(\theta + \alpha) - 2E(\theta) + E(\theta - \alpha), \tag{11.13.9}$$

where $\beta = \frac{1}{sn^2\alpha} - 1$. The function $E(\theta)$ is not a periodic function, but it is associated with the periodic Jacobi zeta function

$$Z(\theta) \equiv E(\theta) - \frac{E(k)}{K(k)}\theta, \tag{11.13.10}$$

where $K(k) = K$ and $E(k) = E$ are complete elliptic integrals of the first and second kind, respectively, and $2K$ is the period of $Z(\theta)$.

We next normalize the phase so a period corresponds to a unit increase in θ and write $\dot{s}_n(t)$ in the form

$$\dot{s}_n(t) = \frac{(2K\omega)^2}{b}\left[dn^2\{2K(\omega t - \alpha n); k\} - \frac{E}{K}\right]. \tag{11.13.11}$$

Thus, it follows from (11.13.1)–(11.13.2) that

$$r_n(t) = -\frac{1}{b}\log\left(1 + \frac{\dot{s}_n}{a}\right). \tag{11.13.12}$$

In the limit as $k \to 1$, $E(k) \to 1$ and $K(k) \to \infty$ and $dn(z,k) \to \text{sech}(z)$ so that the solitary wave solution is

$$r_n(t) = -\frac{1}{b} \log\left[1 + \sinh^2 \kappa \,\text{sech}^2\left(\kappa n \pm t\sqrt{ab} \sinh \kappa\right)\right], \tag{11.13.13}$$

where κ is defined in such a way that

$$2K\alpha \to \kappa, \qquad 2K\omega \to \sqrt{ab} \sinh \kappa. \tag{11.13.14}$$

It follows from (11.13.13) that the velocity of the localized lattice wave can be expressed in terms of the amplitude parameter κ as

$$v = \sqrt{ab}\frac{\sinh \kappa}{\kappa}. \tag{11.13.15}$$

The above analysis of the Toda lattice problem leads to the following conclusions:

1. The solution (11.13.13) represents the Toda lattice soliton. The minus sign in (11.13.13) indicates that the Toda lattice soliton (TLS) is a compression wave.
2. The amplitude of the TLS tends to zero as $\sinh \kappa \to 0$ so that it reduces to a solution of the linear equation (2.3.37) moving with the velocity $v = \sqrt{ab}$.
3. The velocity of a TLS is greater than that of small amplitude waves.
4. All these results of this analysis hold for all forms of the Toda potential from the linear limit of (2.3.37) to the hard sphere limit as $b \to \infty$, while ab remains finite.
5. It also follows from (2.3.33)–(2.3.34) that Toda's infinite chain of unit masses linked by the potential (2.3.35) is governed by the differential equation (2.3.34). The transformation

$$r_n(t) = -\frac{1}{b} \log\left(1 + V_n(t)\right) \tag{11.13.16}$$

reveals than V_n satisfies the equation

$$\frac{d^2}{dt^2} \log(1 + V_n) = ab(V_{n+1} - 2V_n + V_{n-1}). \tag{11.13.17}$$

This represents that propagation of voltage through a nonlinear lattice filter. Introducing the time scale $\tau = t\sqrt{ab}$, Hirota (1973a, 1973b) showed that

$$V_n(t) = \frac{d^2}{d\tau^2} f_n(\tau), \tag{11.13.18}$$

where

$$\begin{aligned}
f_n(\tau) &= \sum_{\mu=0,1} \exp\left[\sum_{i<j}^{(N)} B_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i X_i\right], \\
X_i &= \omega_i \tau - \kappa_i n + \gamma_i, \\
\omega_i &= \pm 2 \sinh\left(\frac{\kappa_i}{2}\right), \kappa_i, \gamma_i \text{ are constants}, \\
\exp(B_{ij}) &= -\frac{(\omega_i - \omega_j)^2 - 4\sinh^2[(\kappa_i - \kappa_j)/2]}{(\omega_i + \omega_j)^2 - 4\sinh^2[(\kappa_i + \kappa_j)/2]}.
\end{aligned} \tag{11.13.19}$$

The results show that collisions of Toda solitons can be described by the N-soliton solutions given by (11.13.18) and (11.13.19).

In 1970s, it has been realized that a mass–spring lattice is completely integrable for the special form of the interaction potential (2.3.35) discovered by Toda (1967a, 1967b). However, even though exact integrability is lost for any other potential, it is still possible to obtain solitary wave solution of equation (2.3.34) in the form

$$r_n(t) = R(n - vt) = R(z), \tag{11.13.20}$$

with $R(z) \to 0$ as $|z| \to \infty$ so that equation (2.3.34) reduces to the form

$$v^2 R''(z) = V'\big[R(z+1)\big] - 2V'\big[R(z)\big] + V'\big[R(z-1)\big]. \tag{11.13.21}$$

This is a general equation for long lattice solitary waves, where $R(z) = r_n(t)$ is the separation of adjacent lattice sites and $V[R(z)]$ is the associated spring potential. Several authors have suggested methods for finding numerical or analytical approximation for lattice solitary waves (LSW). Eilbeck and Flesch (1990) have shown how to obtain fairly accurate numerical solutions of such equations (11.13.21) based on pseudospectral methods. They found $R(z)$ as a finite Fourier cosine series of n terms over a finite interval. On the other hand, Duncan et al. (1993) have developed a rather general approach based on Fourier transform analysis. They defined the Fourier transform of $F(z) = V'[R(z)]$ by

$$\tilde{F}(k) = \mathcal{F}\big\{F(z)\big\} = \int_{-\infty}^{\infty} e^{-ikz} F(z)\, dz, \tag{11.13.22}$$

and similarly, $\tilde{R}(k) = \mathcal{F}\{R(z)\}$. Thus, the Fourier transform of (11.13.21) is

$$v^2 \tilde{R}(k) = \left(\frac{4}{k^2}\sin^2\frac{k}{2}\right)\tilde{F}(k) \approx \left(1 + \frac{k^2}{12}\right)^{-1}\tilde{F}(k). \tag{11.13.23}$$

This leads to the differential equation

$$v^2\left(1 - \frac{1}{12}\frac{d^2}{dz^2}\right)R(z) = V'\big[R(z)\big]. \tag{11.13.24}$$

Or equivalently,

$$R''(z) = \frac{12}{v^2}\big(v^2 R(z) - V'\big[R(z)\big]\big). \tag{11.13.25}$$

This equation can be integrated in terms of elliptic or hyperbolic functions for different forms of the function $V'[R(z)]$.

For a potential of the form

$$V(r) = \frac{1}{2}r^2 + \frac{ar^3}{3}, \tag{11.13.26}$$

Collins (1981) has obtained an approximate inversion of the different operator on the right-hand side of equation (11.13.21). We follow Duncan et al. (1993) with potential (11.13.26) to find equation (11.13.25) in the form

$$R''(z) = \frac{12}{v^2}\left(v^2-1\right)R(z) - \left(\frac{12a}{v^2}\right)R^2. \tag{11.13.27}$$

We use a method similar to that of the KdV equation in Section 9.4 to find the solitary wave solution of (11.13.27) in the form

$$R(z) = r_n(t) = \left(\frac{3}{2a}\right)\left(v^2-1\right)\operatorname{sech}^2\left[(n-vt)\sqrt{3\left(v^2-1\right)/v^2}\right]. \tag{11.13.28}$$

Thus, this LSW can be either compressive for $a<0$ or expansive for $a>0$. In both cases, the velocity of the LSW is greater than unity, the wave velocity in the low-amplitude limit. In the limit as $v^2 \to 1+$ with $a<0$, the solution (11.13.28) tends to the Toda lattice soliton (11.13.13). On the other hand, as $v^2 \to 1+$, the amplitude of the LSW (11.13.28) tends to zero so that its width increases without bound. This means that (11.13.28) represents a large solitary waves that extend over many lattice spacings.

For more information about the existence of solitary waves on mass–spring lattices for a wider classes of potential functions and about the two-dimensional Toda lattice, the reader is referred to Scott (2003), Biondini and Wang (2010).

11.14 Exercises

1. Use the Fourier transform method to show that the Fourier transform solution of the $(n+1)$-dimensional, inhomogeneous, Klein–Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)u + d^2 u = q(\mathbf{x},t), \quad \mathbf{x}\in\mathbb{R}^n,\ t>0,$$

with the Cauchy data

$$u(\mathbf{x},0) = f(\mathbf{x}) \quad \text{and} \quad u_t(\mathbf{x},0) = g(\mathbf{x}) \quad \text{for all } \mathbf{x}\in\mathbb{R}^n$$

is

$$\begin{aligned} U(\mathbf{k},t) &= F(\mathbf{k})\cos\left(t\sqrt{c^2|\mathbf{k}|^2+d^2}\,\right) \\ &\quad + \frac{G(\mathbf{k})}{\sqrt{c^2|\mathbf{k}|^2+d^2}}\sin\left(t\sqrt{c^2|\mathbf{k}|^2+d^2}\,\right) \\ &\quad + \int_0^t \frac{\sin(\tau\sqrt{c^2|\mathbf{k}|^2+d^2})}{\sqrt{c^2|\mathbf{k}|^2+d^2}} Q(\mathbf{k},t-\tau)\,d\tau, \end{aligned}$$

where $U(\mathbf{k},t) = \mathcal{F}\{u(\mathbf{x},t)\}$, $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, and $\mathbf{k}=(k_1,k_2,\ldots,k_n)$.

2. Show that the energy integral associated with the Klein–Gordon equation in Exercise 1 is given by

$$E(t) = \int \left(|u_t|^2 + c^2|\nabla u|^2 + d^2|u|^2\right) d\mathbf{x}.$$

Show that $E(t)$ is independent of t.

3. Seek a similarity solution of the Klein–Gordon equation

$$u_{tt} - u_{xx} = u^3$$

in the form $u(x,t) = t^m f(xt^n)$ for suitable values of m and n. Show that $f(z)$ satisfies the equation

$$\left(z^2 - 1\right) f'' + 4zf' + 2f = f^3, \quad \text{where } z = xt^{-1}.$$

4. If $V(u)$ involved in the Klein–Gordon equation (11.5.1) is given (Whitham 1974) by

$$V(u) = \frac{1}{2}u^2 + bu^4 + \cdots,$$

derive the result

$$u(\theta) = a\cos\theta + \frac{1}{8}a^3 b\cos 3\theta + \cdots \quad \text{as } a \to 0,$$

where the nonlinear dispersion relation is given by

$$\omega^2 = 1 + k^2 + 3a^2 b + \cdots \quad \text{as } a \to 0$$

and the nonlinear amplitude is

$$A = \frac{1}{2}a^2 + \frac{9}{8}a^4 b + \cdots \quad \text{as } a \to 0.$$

5. Verify that

$$u(x,t) = 4\tan^{-1}\left[\alpha \exp\left\{\frac{x - Ut}{\sqrt{1-U^2}}\right\}\right]$$

is a solution of the sine-Gordon equation (11.8.2), where α and U are constants.

6. Show that

$$u(x,t) = 4\tan^{-1}\left[\frac{U\cosh\{x/\sqrt{1-U^2}\}}{\sinh\{Ut/\sqrt{1-U^2}\}}\right]$$

is an exact antisoliton solution of the sine-Gordon equation, where $0 < |U| < 1$. Examine the asymptotic nature of $u(x,t)$, as $t \to \pm\infty$, with physical significance.

7. Show that the transformations

$$\xi = \frac{1}{2}\left(x - \frac{t}{c}\right), \qquad \eta = \frac{1}{2}\left(x + \frac{t}{c}\right)$$

transform the equation

$$u_{xx} - c^2 u_{tt} = \sin u$$

into the form

$$u_{\xi\eta} = \sin u.$$

8. (a) Show that the sine-Gordon equation

$$u_{xx} - u_{tt} = \sin u$$

is invariant under the transformations $\tilde{x} = m(x - Ut)$ and $\tilde{t} = m(x - Ut)$, where $m = (1 - U^2)^{-\frac{1}{2}}$ and $-1 < U < 1$.
(b) Hence or otherwise, derive that

$$u(x,t) = 4\tan^{-1}\left[\frac{(1-\beta^2)^{\frac{1}{2}}}{\beta}\,\frac{\sin\{\alpha\beta(t - Ux - t_0)\}}{\cosh\{\alpha(1-\beta^2)^{\frac{1}{2}}(x - Ut - x_0)\}}\right]$$

is a solution of the sine-Gordon equation, where $\beta \neq 0$, x_0, t_0, and U are real constants, $-1 < \beta < 1$, and $-1 < U < 1$.

9. (a) Use the Bäcklund transformation (11.9.2a), (11.9.2b) to show that four solutions u_r $(r = 1, 2, 3, 4)$ satisfy the relation

$$\tan\left\{\frac{1}{4}(u_4 - u_1)\right\} = \left(\frac{a_1 + a_2}{a_1 - a_2}\right)\tan\left\{\frac{1}{4}(u_2 - u_3)\right\},$$

where a_1 and a_2 are nonzero constants.
(b) Apply the relation in (a) to obtain a soliton solution of the sine-Gordon equation by setting $u_1 = 0$, and u_2 and u_3 as different soliton solutions.

10. Examine the asymptotic nature of the two-soliton solution (11.10.19), as $t \to \pm\infty$.

11. Derive the solitary wave solution of the sinh-Gordon equation

$$u_{xt} = \sinh u.$$

12. (a) Use the similarity solution of the sine-Gordon equation (11.9.1) of the form $u(x,t) = f(xt^n)$ for suitable n to show that $w(z) = \exp\{if(z)\}$ yields a Painlevé equation for $w(z)$.
(b) Use the same method as in (a) to discuss the sinh-Gordon equation in Exercise 11.

13. Verify the following conservation laws (Lamb 1971) for equation (11.9.1):

$$\left(\frac{1}{2}u_t^2\right)_x - (1 - \cos u)_t = 0, \qquad (1 - \cos u)_x - \left(\frac{1}{2}u_x^2\right)_t = 0,$$

$$\left(\frac{1}{6}u_t^6 - \frac{2}{3}u_t^2 u_{tt}^2 + \frac{8}{9}u_t^3 u_{ttt} + \frac{4}{3}u_{ttt}^2\right)_x + \left\{\left(\frac{1}{9}u_t^4 - \frac{4}{3}u_{tt}^2\right)\cos u\right\}_t = 0,$$

$$\left(\frac{1}{4}u_t^4 - u_{tt}^2\right)_x + \left(u_t^2\cos u\right)_t = 0.$$

14. Using the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(u_t^2 - u_x^2) - V(u),$$

where $V'(u) = \sin u$, derive the sine-Gordon equation

$$u_{xx} - u_{tt} = \sin u.$$

15. Show that the bilinear form of the sine-Gordon equation in Exercise 14 is the pair of equations

$$\big(D_x^2 - D_t^2 - 1\big)(\phi \cdot \psi) = 0 \quad \text{and} \quad \big(D_x^2 - D_t^2\big)(\phi \cdot \phi - \psi \cdot \psi) = 0,$$

where $u = 4\tan^{-1}(\frac{\psi}{\phi})$.
16. Consider the dissipative wave equation

$$u_{tt} - c^2\nabla^2 u + \nu u_t = 0,$$

where $\nu(> 0)$ is the coefficient of dissipation. Show that the energy is decreasing:

$$\frac{dE}{dt} = -\nu \iiint u_t^2 \, d\mathbf{x} \le 0.$$

17. Use $u(x,t) = \exp(-at)v(x,t)$ to transform the one-dimensional dissipative wave equation in Exercise 16. For $\nu = a^2$, show that the transformed equation is the Klein–Gordon equation with imaginary mass (ia).
18. Solve the fractional Klein–Gordon equation (Debnath and Bhatta 2004)

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} - c^2 u_{xx} + d^2 u &= p(x,t), \quad x \in \mathbb{R},\ t > 0\ (1 < \alpha \le 2), \\ u(x,0) &= 0 = u_t(x,0), \quad x \in \mathbb{R}, \\ u(x,t) &\to 0 \quad \text{as } |x| \to \infty,\ t > 0. \end{aligned}$$

19. Solve the initial-value problem for the KG equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} + d^2 u = 0, &\quad x \in \mathbb{R},\ t > 0, \\ u(x,0) = 0, &\quad u_t(x,0) = \delta(x), \quad x \in \mathbb{R}. \end{aligned}$$

12

Asymptotic Methods and Nonlinear Evolution Equations

The organic unity of mathematics is inherent in the nature of this science, for mathematics is the foundation of all exact knowledge of natural phenomena.

David Hilbert

It seems to be one of the fundamental features of nature that fundamental physics laws are described in terms of great beauty and power.
As time goes on, it becomes increasingly evident that the rules that the mathematician finds interesting are the same as those that nature has chosen.

Paul Dirac

12.1 Introduction

Many physical systems involving nonlinear wave propagation include the effects of dispersion, dissipation, and/or the inhomogeneous property of the medium. The governing equations are usually derived from conservation laws. In simple cases, these equations are hyperbolic. However, in general, the physical processes involved are so complex that the governing equations are very complicated, and hence, are not integrable by analytic methods. So, special attention is given to seeking mathematical methods which lead to a less complicated problem, yet retain all of the important physical features. In recent years, several asymptotic methods have been developed for the derivation of the *evolution equations* which describe how some dynamical variables evolve in time and space. So, we begin this chapter with one simple method of construction of the linear evolution equation from a given frequency–wavenumber dispersion relation of the form

$$\omega = f(k). \tag{12.1.1}$$

This relation is multiplied by $-iU(k)\exp[i(kx-\omega t)]$ and integrated with respect to the wavenumber k from $-\infty$ to ∞ to obtain the equation

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$$\frac{\partial u}{\partial t} = L(u) = -i \int_{-\infty}^{\infty} f(k) U(k) \exp\bigl[i(kx - \omega t)\bigr]\, dk, \tag{12.1.2}$$

where $L(u)$ represents an operator and $U(k)$ is an arbitrary function which is related to the function $u(x,t)$ by the inverse Fourier transform

$$u(x,t) = \int_{-\infty}^{\infty} U(k) \exp\bigl[i(kx - \omega t)\bigr]\, dk. \tag{12.1.3}$$

Equation (12.1.2) represents the fundamental result that determines the structure of the evolution equation. In particular, if $f(k)$ is a polynomial in k, then $L(u)$ is a differential operator. More explicitly, if

$$f(k) = ck - i\gamma k^2 - \alpha_3 k^3 - \cdots - \alpha_n k^n, \tag{12.1.4}$$

where c, γ, $\alpha_3, \ldots, \alpha_n$ are arbitrary constants, then (12.1.2) gives the explicit evolution equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - \gamma \frac{\partial^2 u}{\partial x^2} + \alpha_3 \frac{\partial^3 u}{\partial x^3} + \cdots + \alpha_n \frac{\partial^n u}{\partial x^n} = 0. \tag{12.1.5}$$

If $\alpha_3 = \alpha_4 = \cdots = \alpha_n = 0$, (12.1.5) reduces to the linear Burgers equation. When $\alpha_n \equiv 0$ for $n \le 4$, equation (12.1.5) represents the linear KdV–Burgers equation. On the other hand, if $\gamma = -i\nu$ for real ν and $\alpha_n = 0$ for $n \le 3$, then (12.1.5) reduces to the linear Schrödinger equation.

The above analysis reveals that the polynomial dispersion relation corresponds to systems governed by evolution equations of the differential type. On the other hand, if the dispersion relation has a transcendental form, the corresponding evolution is of integro-differential type. However, there is *no* such simple method of construction of evolution equations from a given nonlinear dispersion relation. For nonlinear problems, asymptotic methods are employed to construct evolution equations. The linear part of the associated evolution equation can be recovered from (12.1.2). So, the rest of this chapter is devoted to deriving evolution equations for various nonlinear systems by asymptotic methods.

The idea of a far-field asymptotic behavior of a physical system originated from the properties inherent in a given evolution equation which do not depend, in any sensitive manner, on the details of the initial conditions. Thus, such properties are generally observed after a sufficiently long period of time for a large class of initial conditions. In this sense, it is *not* truly useful to study the initial-value problems in detail. Rather, it is more useful to investigate the asymptotic behavior of a physical system in the limit as time goes to infinity. In many situations, these far-field solutions merely represent the simple waves described in Chapter 6.

Included here are reductive perturbation methods for describing weakly dispersive systems, strongly dispersive systems, and quasi-linear dissipative systems. The perturbation method due to Ostrovsky and Pelinovsky (1971) is developed in Section 12.6 for a general system of first-order nonlinear partial differential equations that can be applied to both conservative and nonconservative systems. The Gardener–Morikawa transformation is introduced in Section 12.4 to derive a fairly general nonlinear evolution equation. This is followed by several examples of applications, including nonlinear oscillations of an elastic string with dispersive effects, the Burgers equation in gas dynamics, the KdV equation for ion-acoustic waves, and nonlinear shallow water waves on an uneven bottom. In Section 12.5, the reductive perturbation method is also applied to describe strongly nonlinear dispersive systems and the generalized nonlinear Schrödinger equation. The method of multiple scales and asymptotic analysis are introduced in Sections 12.7 and 12.8 and used to derive both the nonlinear Schrödinger and the KdV equations from nonlinear dispersive systems. Special attention is given to the asymptotic method that has been utilized for the derivation of the nonlinear Schrödinger equation and the Davey–Stewartson equations with direct applications to the instability of Stokes waves in water. Several conservation laws for the Davey–Stewartson equations are also derived in Section 12.9.

12.2 The Reductive Perturbation Method and Quasi-linear Hyperbolic Systems

Historically, this method was first formulated by Taniuti and his collaborators (Taniuti and Wei 1968; Taniuti and Washimi 1968; Kakutani et al. 1968; Taniuti and Yajima 1969; Asano and Ono 1971; Taniuti 1974) in a more general form applicable to both weakly dispersive and weakly dissipative systems.

We first consider a first-order, quasi-linear totally hyperbolic system of equations

$$U_t + A(U)U_x = 0, \tag{12.2.1}$$

where U is a column vector with n components $u_1, u_2, \ldots, u_n$ and A is an $n \times n$ matrix. Such a system of equations has already been investigated in connection with simple waves.

We expand the vector U about a constant state solution $U^{(0)}$ of (12.2.1) in terms of a small parameter ε which represents the smallness of the amplitude of U in the form

$$U = U^{(0)} + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \cdots . \tag{12.2.2}$$

Since the amplitude is small, the solution of (12.2.1) can be approximated by the linearized form of the system in which $A(U)$ is replaced by $A(U^{(0)})$. As the system is totally hyperbolic, there are n real distinct eigenvalues $\lambda^{(r)}$ of $A(U)$, and hence, there are n *constant* eigenvalues $\lambda_0^{(r)}$ of $A(U^{(0)})$, where $r = 1, 2, \ldots, n$. In the linear case, the characteristics are straight lines. On the other hand, for a nonlinear system, as time evolves, characteristics will no longer be parallel lines, and hence, after a sufficiently long time, characteristics intersect and lead to the breakdown of differentiability. To investigate the process more explicitly, we expand the jth family of characteristics of (12.2.1) in powers of ε,

$$\frac{dx}{dt} = \lambda^{(j)}, \tag{12.2.3}$$

and, then, retain only terms of order ε to obtain

$$\frac{dx'}{dt} = \varepsilon \sum_{r=1}^{n} u_r^{(1)} \left[\frac{\partial \lambda^{(j)}}{\partial u_r^{(j)}}\right]_{U=U^{(0)}} = \varepsilon U^{(1)} \nabla_u \lambda_0^{(j)}, \tag{12.2.4}$$

where $x' = x - t\lambda_0^{(j)}$, so that the coordinate system moves with speed $\lambda_0^{(j)}$, $\lambda_0^{(j)} = \lambda^{(j)}(U^{(0)})$, $u_r^{(1)}$ is the rth element of the vector $U^{(1)}$, and ∇_u is the gradient operator with respect to $u_1^{(1)}, u_2^{(1)}, \ldots, u_n^{(1)}$.

It follows from result (12.2.4) that the characteristics suffer from deviation from parallel straight lines by an amount of the order ε. Using the time scale εt, the phase velocity $\frac{dx'}{d(\varepsilon t)}$, moving with the speed $\lambda_0^{(j)}$ in the reference frame, reduces to the order of unity. Hence, for small $t \ll \varepsilon^{-1}$, the phase velocity can be well approximated by the phase velocity for the linear case. However, for large time scales, $t \sim \varepsilon^{-1}$, the characteristics deviate significantly from those found in the linear case, even if the amplitude is very small ($\varepsilon \ll 1$). Consequently, the far-field solutions can be described by using the stretched time scale $t \sim \varepsilon^{-1}$.

It also follows from (12.2.4) that $\nabla_u \lambda_0^{(j)}$ is constant, and hence, U is a function of x' and εt. Consequently, the time derivative of $U^{(1)}$ is of the order ε and becomes small. This means that x' is constant and $U^{(1)}$ varies slowly with time along the linear parallel equiphase straight lines. All of these lead to the use of the following transformations for the far-field solutions of (12.2.1):

$$x' = x - \lambda_0^{(j)} t, \qquad t' = \varepsilon t. \tag{12.2.5ab}$$

Clearly, $\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}$ and $\frac{\partial}{\partial t} = \varepsilon \frac{\partial}{\partial t'} - \lambda_0^{(j)} \frac{\partial}{\partial x'}$. Hence, we use (12.2.1), (12.2.2) together with the expansion of the matrix A as

$$A = A_0 + \varepsilon U^{(0)} \nabla_u A_0 + \cdots, \tag{12.2.6}$$

where $A_0 = A(U^{(0)})$. Thus, we obtain the following system of perturbation equations:

$$O(\varepsilon): \left[\lambda_0^{(j)} I - A_0\right] U_{x'}^{(1)} = 0, \tag{12.2.7}$$

$$O(\varepsilon^2): \left[-\lambda_0^{(j)} I + A_0\right] U_{x'}^{(2)} + U_{t'}^{(1)} + \left(U^{(1)} \nabla_u A_0\right) U_{x'}^{(1)} = 0. \tag{12.2.8}$$

Equation (12.2.7) shows that $U_{x'}^{(1)}$ must be proportional to a right-eigenvector $R_0^{(j)}$ of A_0 corresponding to the eigenvalue $\lambda_0^{(j)}$. Integrating (12.2.7) with respect to x' gives the solution

$$U^{(1)} = \phi_j^{(1)} R_0^{(j)} + V_j^{(1)}(t'), \tag{12.2.9}$$

where $\phi_j^{(1)} = \phi_j^{(1)}(x', t')$ is a function of x' and t' and $V_j^{(1)}(t')$ are vector functions of t' only that arise from integration and so can be determined by the boundary conditions. However, functions $\phi_j^{(1)}(x', t')$ cannot be determined from (12.2.9), and we have to consider (12.2.8) to find them. Equation (12.2.8) is an algebraic equation for $U_{x'}^{(2)}$, with a multiplying factor $(-\lambda_0^{(j)} I + A_0)$. If $\lambda_0^{(j)}$ is an eigenvalue of A_0, the eigenvalue equation is given by

$$\left|A_0 - \lambda_0^{(j)} I\right| = 0. \tag{12.2.10}$$

If $L_0^{(j)}$ is a left-eigenvector of A_0 corresponding to the eigenvalue $\lambda_0^{(j)}$, then

$$L_0^{(j)}\left(-\lambda_0^{(j)} I + A_0\right) = 0. \tag{12.2.11}$$

Thus, a necessary and sufficient condition for the existence of the solution of (12.2.8) is

$$L_0^{(j)}\left[U_{t'}^{(1)} + (U^{(1)}\nabla_u A_0)U_{x'}^{(1)}\right] = 0. \tag{12.2.12}$$

Without loss of generality, we set $V_j^{(1)}(t') \equiv 0$ in (12.2.9) so that the resulting equation (12.2.9) can be substituted in (12.2.12) to obtain the *nonlinear* partial differential equation for $\phi_j^{(1)}(x', t')$ in the form

$$\frac{\partial}{\partial t'}\phi_j^{(1)} + \alpha_j \frac{\partial}{\partial x'}\phi_j^{(1)} = 0, \tag{12.2.13}$$

where α_j is given by

$$\alpha_j = L_0^{(j)}\left[R_0^{(j)}\nabla_u A_0\right]R_0^{(j)}/L_0^{(j)}R_0^{(j)}, \tag{12.2.14}$$

which, after some simplification, reduces to

$$\alpha_j = \left[\nabla_u \lambda_0^{(j)}\right]R_0^{(j)}. \tag{12.2.15}$$

Thus, the present asymptotic analysis allows us to derive (12.2.13), which represents the basic *scalar evolution equation* describing the far-field behavior of the jth (column vector) mode of wave propagation of the hyperbolic system (12.2.1). Further, to the order of ε, the solution of (12.2.1) can be written as

$$U = U^{(j)} + \varepsilon\phi_j^{(1)}R_0^{(j)}, \tag{12.2.16}$$

where $\phi_j^{(1)}$ is a solution of (12.2.13).

Evidently, the characteristic equation of (12.2.13) is

$$\frac{dx'}{dt'} = \alpha_j \phi_j^{(1)}, \tag{12.2.17}$$

which ensures the existence of the far-field behavior, provided that $\alpha_j \neq 0$. The solution (12.2.16) represents simple waves already discussed in Chapter 6 that describe the far-field behavior of (12.2.1) until a discontinuity is developed in the form of shock waves.

The present method of analysis is called the *reductive perturbation method* because it reduces the determination of the far-field behavior of a system to the solution of a scalar, nonlinear evolution equation, such as (12.2.13).

Finally, we close this section by adding a remark on the nonuniqueness of the far-field solution of hyperbolic equations. Equation (12.2.13) is invariant under the transformation

$$\tilde{x} = x', \qquad \tilde{t} = \varepsilon^{-1} t', \qquad \widetilde{\phi^{(1)}} = \varepsilon \phi^{(1)}. \tag{12.2.18}$$

However, this transformation is not unique because equation (12.2.13) is also invariant under the following transformation:

$$\tilde{x} = \varepsilon^{-a} x, \qquad \tilde{t} = \varepsilon^{-(a+1)} t', \qquad \widetilde{\phi^{(1)}} = \varepsilon \phi^{(1)}, \tag{12.2.19}$$

where a is an arbitrary constant. This transformation can be found from (12.2.5ab), and the corresponding transformation

$$\xi = \varepsilon^{a}(x - \lambda_0 t), \qquad \tau = \varepsilon^{(a+1)} t, \tag{12.2.20}$$

leads to equation (12.2.13). This transformation was first introduced by Gardner and Morikawa (1960) for a solitary wave in a plasma propagating normally across a magnetic field and is known as the *Gardner–Morikawa transformation*. Since a is arbitrary, solutions vary more slowly over space and time for large values of a than they do for smaller values. Thus, the far-field solutions are not determined uniquely for hyperbolic equations. The breaking time t_B after which smooth solutions cease to exist is determined by the initial conditions, so that, in the case of hyperbolic systems, the time scale for the far-field solution must be less than t_B. Thus, the concept of a far-field solution for the hyperbolic case is not meaningful unless t_B is sufficiently large. It is possible that a discontinuity (or shock) is developed before an initial disturbance breaks up into simple waves.

The above asymptotic method can be applied to physical systems with dissipative terms which prevent breaking. So, smooth solutions exist for all time for dissipating systems, so that $t_B \to \infty$, unlike the case of hyperbolic equations. This leads us to consider an arbitrarily long time evolution of a solution. For example, in the case of a perfect gas, the viscosity begins to act effectively as the sound wave steepens, and these two effects, the nonlinearity leading to breaking and the dissipation that causes smoothing, balance one another to produce smooth solutions. As a result, shock waves are developed and propagated as already described by the Burgers equation in Chapter 8.

12.3 Quasi-linear Dissipative Systems

We now apply the reductive perturbation method to solve a general dissipative system which becomes totally hyperbolic when the dissipative terms are neglected. Using the same notation as in (12.2.1), we consider the system of dissipative equations described by

$$U_t + A(U)U_x + K_1(K_2 U_x)_x = 0, \tag{12.3.1}$$

where K_1 and K_2 are $n \times n$ matrices which are functions of U and the eigenvalues of $A(U)$ are all real and distinct because of the first-order hyperbolic system embedded in the system of equations (12.3.1).

We now adopt a frame of reference that moves with a constant velocity λ_0 which is equal to one of the eigenvalues of the matrix $A_0 = A(U^{(0)})$ associated with the constant state $U^{(0)}$. We assume that the dispersion relation for small wavenumbers k of the linearized system about $U^{(0)}$ is given by

$$\omega = \lambda_0 k + i\mu k^2 + \cdots, \tag{12.3.2}$$

where $\mu < 0$. It can be anticipated that equation (12.3.1) can be reduced to the Burgers equation by the Gardner–Morikawa transformation (12.2.20) with $a = 1$. In view of this transformation, equation (12.3.1) admits approximate solutions which vary slowly over space and more slowly with time due to weak nonlinearity and weak dissipating effects. With $\xi = \varepsilon(x - \lambda_0 t)$ and $\tau = \varepsilon^2 t$, we apply the same perturbation solution (12.2.2) and use the following results:

$$\frac{\partial}{\partial x} = \varepsilon \frac{\partial}{\partial \xi}, \qquad \frac{\partial^2}{\partial x^2} = \varepsilon^2 \frac{\partial^2}{\partial \xi^2}, \qquad \frac{\partial}{\partial t} = -\varepsilon\lambda_0 \frac{\partial}{\partial \xi} + \varepsilon \frac{\partial}{\partial \tau}, \tag{12.3.3}$$

in the original equation (12.3.1). Then, equating the lowest-order terms $O(\varepsilon^2)$ gives

$$(\lambda_0 I - A_0) U_\xi^{(1)} = 0, \tag{12.3.4}$$

so that integration of this result gives

$$U^{(1)} = \phi^{(1)}(\xi, \tau) R_0, \tag{12.3.5}$$

where the scalar function $\phi^{(1)}$ is to be determined, R_0 is a right-eigenvector of A_0 corresponding to the eigenvalue λ_0, and the boundary condition $U^{(1)} \to 0$, as $\xi \to \infty$, has been utilized. It is noted that (12.3.5) is not affected by the dissipative term to this order of approximation.

Using the same argument as before, we obtain the following equation to the order $O(\varepsilon^3)$:

$$(-\lambda_0 I + A_0) U_\xi^{(2)} + U_\tau^{(1)} + U^{(1)} \nabla_u A_0 U_\xi^{(1)} + K_0 U_{\xi\xi}^{(1)} = 0, \tag{12.3.6}$$

where $K_0 = (K_1 K_2)_{U=U^{(0)}}$.

Multiplying from the left by a left-eigenvector L_0 corresponding to the eigenvalue λ_0 and using the same argument as in Section 12.2 yields the necessary and sufficient condition for the existence of solutions for $U_\xi^{(2)}$ as

$$L_0 \left[U_\tau^{(1)} + U^{(1)} \nabla_u A_0 U_\xi^{(1)} + K_0 U_{\xi\xi}^{(1)} \right] = 0. \tag{12.3.7}$$

Substituting $U^{(1)}$ from (12.3.5) in (12.3.7) gives the following quasi-linear, second-order equation for $\phi^{(1)}(\xi, \tau) \equiv \phi(\xi, \tau)$:

$$\phi_\tau + \alpha \phi \phi_\xi + \mu \phi_{\xi\xi} = 0, \tag{12.3.8}$$

where

$$\alpha = (\nabla_u \lambda_0) R_0 \quad \text{and} \tag{12.3.9a}$$

$$\mu = \frac{L_0 K_0 R_0}{L_0 R_0}. \tag{12.3.9b}$$

Thus, the scalar function $\phi(\xi, \tau)$ satisfies the Burgers equation (12.3.8), provided that $\mu < 0$. This equation governs the far-field behavior of the system (12.3.1), corresponding to the characteristic field λ_0 of the hyperbolic part of the system. Finally, the solution of system (12.3.1) to the order $O(\varepsilon)$ is given by

$$U = U^{(0)} + \varepsilon \phi^{(1)}(\xi, \tau) R_0. \tag{12.3.10}$$

Thus, $\phi^{(1)}(\xi, \tau) \equiv \phi(\xi, \tau)$ satisfies a nonlinear evolution equation (12.3.8) which determines the far-field behavior corresponding to the characteristic field λ_0.

12.4 Weakly Nonlinear Dispersive Systems and the Korteweg–de Vries Equation

The dispersion relation for nonlinear waves represents an algebraic equation for the frequency ω, wavenumber k, and amplitude $a^2 : \omega = \omega(k, a^2)$. If the amplitude is small, the dispersion relation is usually determined by the linearized equation, so that both the phase and group velocities are functions of the wavenumber k only. The dispersion is said to be *weak* if the group velocity of the wave is almost constant in a certain range of frequencies, and such that $|\frac{\partial \omega}{\partial k}| \gg |\frac{\partial^2 \omega}{\partial k^2}|$. The Korteweg–de Vries (KdV) equation is a classic example of weak dispersion. However, if this inequality is not valid, the dispersion is said to be *strong*. The nonlinear Schrödinger (NLS) equation is a famous example of strong dispersion.

We develop the reductive perturbation method to solve a general weakly nonlinear dispersive system in a homogeneous medium given by

$$U_t + AU_x + \left[\sum_{\beta=1}^{s} \prod_{\alpha=1}^{p} \left(H_\alpha^\beta \frac{\partial}{\partial t} + K_\alpha^\beta \frac{\partial}{\partial x}\right)\right] U = 0, \quad p \geq 2, \tag{12.4.1}$$

where U is a column vector of n components $u_1, u_2, \ldots, u_n (n \geq 2)$, and A, H_α^β, and K_α^β are $n \times n$ matrices, which are functions of U. For suitable H_α^β and K_α^β, the nonlinear system can be dispersive because the highest spatial derivative is of order three, and it is weakly dispersive because it does not contain an $(n \times 1)$ column vector of the form $B(U)$ on the left-hand side of (12.4.1). Further, we can take into account the nonhomogeneity of the medium by adding an extra term BV_x on the left-hand side of equation (12.4.1).

We first consider a solution of (12.4.1) representing small derivations around a steady equilibrium state U_0 in the form

$$U = U_0 + U_1 \exp\left[i(kx - \omega t)\right]. \tag{12.4.2}$$

Substituting (12.4.2) in (12.4.1) and retaining only the first power of U_1 gives

$$\left[\left(-\frac{\omega}{k}\right)I + A_0 + (ik)^{p-1}\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\left(K_{\alpha 0}^{\beta} - \frac{\omega}{k}H_{\alpha 0}^{\beta}\right)\right]U_1 = 0, \tag{12.4.3}$$

where the zero subscript denotes the value when $U = U_0$. Thus, equation (12.4.3) leads to the dispersion relation

$$\left|\left(-\frac{\omega}{k}\right)I + A_0 + (ik)^{p-1}\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\left(K_{\alpha 0}^{\beta} - \frac{\omega}{k}H_{\alpha 0}^{\beta}\right)\right| = 0. \tag{12.4.4}$$

For the case of long waves, that is, for small k, equation (12.4.3) can be solved by the method of successive approximations. The zeroth-order approximation is given by

$$\left(-\frac{\omega}{k}I + A_0\right)U_{10} = 0. \tag{12.4.5}$$

Obviously, the zeroth-order dispersion relation is of order n in $(\frac{\omega}{k})$. We assume that λ_0 is a nondegenerate eigenvalue of A_0 with L_0 and R_0 as corresponding left- and right- eigenvectors, respectively. To this approximation, we find

$$\frac{\omega}{k} = \lambda_0, \qquad U_{10} = R_0 \text{ or some scalar multiple of } R_0. \tag{12.4.6}$$

Then we obtain the next approximation by substituting λ_0 for $\frac{\omega}{k}$ and R_0 for U_1 in the neglected terms. Consequently,

$$\left[\left(-\frac{\omega}{k}\right)I + A_0 + (ik)^{p-1}\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\left(K_{\alpha 0}^{\beta} - \lambda_0 H_{\alpha 0}^{\beta}\right)\right]R_0 = 0, \tag{12.4.7}$$

so that the multiplication of this equation by L_0 yields

$$\frac{\omega}{k} = \lambda_0 + \frac{(ik)^{p-1}L_0}{L_0 R_0}\left[\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\left(K_{\alpha 0}^{\beta} - \lambda_0 H_{\alpha 0}^{\beta}\right)\right]R_0. \tag{12.4.8}$$

Proceeding in this manner gives a higher-order approximation to $(\frac{\omega}{k})$ as

$$\frac{\omega}{k} = \lambda_0 + C_1 k^{p-1} + C_2 k^{2(p-1)} + \cdots, \tag{12.4.9}$$

where the coefficient C_1 is given by

$$C_1 = (L_0 R_0)^{-1}\left[i^{p-1}L_0\left\{\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\left(K_{\alpha 0}^{\beta} - \lambda_0 H_{\alpha 0}^{\beta}\right)\right\}R_0\right], \tag{12.4.10}$$

and similarly for C_2. In this system, it is necessary to assume that $C_1 \neq 0$.

Neglecting the third term in the original equation (12.4.1), the characteristic curves of the reduced equation can be expressed in the form

$$\frac{dx}{dt} = \lambda_0 + \varepsilon\lambda_1 + O(\varepsilon^2), \tag{12.4.11}$$

where ε is a small *nonzero* parameter which determines the effects of *nonlinearity*.

A simple comparison of (12.4.9) with (12.4.11) leads to the fact that there can be a coupling between nonlinearity and dispersion (or dissipation) at the order of ε, provided that

$$k \sim \varepsilon^a, \quad a = (p-1)^{-1}. \tag{12.4.12}$$

This means that ε^a times the wavelength is of order unity, so that we can define the moving coordinate ξ by

$$\xi = \varepsilon^a (x - \lambda_0 t), \tag{12.4.13}$$

in which the wave can be described when nonlinearity is weak and the wavelength is long. Differentiating (12.4.13) with respect to x gives

$$\begin{aligned} \frac{d\xi}{dx} &= \varepsilon^a \left[1 - \lambda_0 \left(\frac{dx}{dt}\right)^{-1}\right] \\ &= \varepsilon^a \left[1 - \frac{\lambda_0}{\lambda_0 + \varepsilon\lambda_1 + O(\varepsilon^2)}\right] = \varepsilon^{a+1}\left[\frac{\lambda_1}{\lambda_0} + O(\varepsilon)\right], \end{aligned} \tag{12.4.14}$$

or equivalently,

$$\frac{d\xi}{d\eta} = \frac{\lambda_1}{\lambda_0} + O(\varepsilon), \tag{12.4.15a}$$

$$\eta = \varepsilon^{a+1} x. \tag{12.4.15b}$$

This allows us to introduce a *stretched variable* η by (12.4.15b) so that $\frac{d\xi}{d\eta} = O(1)$.

Similarly, differentiating (12.4.13) with respect to t gives

$$\frac{d\xi}{dt} = \varepsilon^{a+1}\left[\lambda_1 + O(\varepsilon)\right]. \tag{12.4.16}$$

This enables us to define another *stretched variable* τ by

$$\tau = \varepsilon^{a+1} t \tag{12.4.17}$$

such that $\frac{d\xi}{d\tau} = O(1)$.

We thus obtain two sets of stretched variables: (i) ξ, η and (ii) ξ, τ. Using the first set defined by (12.4.13) and (12.4.15b) gives the characteristics in terms of these stretched variables in the form

$$\frac{dx}{dt} = \lambda_0 + \varepsilon\lambda_0 \frac{d\xi}{d\eta}. \tag{12.4.18}$$

The first set of stretched variables can be used to deal with an initial-value problem.

Similarly, the characteristics associated with the second set ξ, τ are given by

$$\frac{dx}{dt} = \lambda_0 + \frac{d\xi}{d\tau}. \tag{12.4.19}$$

The second set of stretched variables introduces the *Gardner–Morikawa transformation* and can usually be applied to deal with boundary-value problems.

It is clear from the above analysis that the scale of coordinate stretching is uniquely determined by the given governing equation (12.4.1). The reduction perturbation analysis has been developed by Taniuti and Wei (1968) for the case of a homogeneous medium and, subsequently, modified by Asano and Ono (1971) to account for moderate nonhomogeneity. When the medium is not homogeneous, as is the case with most physical systems, the interaction of the wave with the nonhomogeneity also becomes important. To state the stretched variable transformations associated with a nonhomogeneous medium, we briefly outline the basic ideas of Asano and Ono (1971) as follows:

The steady state of the system of (12.4.1) with the term BV_x on the left-hand side, denoted by the zero subscript, is described by

$$A_0 U_{0x} + \sum_{\beta=1}^{s} \prod_{\alpha=1}^{p} \left(K_{\alpha 0}^{\beta} \frac{\partial}{\partial x} \right) U_0 + B_0 V_x = 0. \tag{12.4.20}$$

This arises from a more general system of equations for weakly nonlinear waves in a homogeneous medium.

With the assumption that U_0 and V are slowly varying functions of x, the stretched variable η can be defined by (12.4.15b) with $a = (p-1)^{-1}$. We can also determine the order of ε from the slow variations of U_0 and V with x. To account for the nonhomogeneity of the medium, we introduce the stretched variable ξ and η redefined by the following relations:

$$\xi = \varepsilon^a \left[\int \lambda_0^{-1} dx - t \right] \quad \text{and} \quad \eta = \varepsilon^{a+1} x, \tag{12.4.21ab}$$

where λ_0 is a nondegenerate eigenvalue of the matrix A_0, equal to the velocity of the linear wave. In terms of stretched coordinates, equation (12.4.20) becomes

$$A_0 U_{0\eta} + B_0 V_\eta = 0, \tag{12.4.22}$$

where the terms of the order ε^p are ignored.

The reader is referred to Asano and Ono (1971) or Bhatnagar (1979) for a complete analysis. However, we simply state the final evolution equation for $\phi(\xi, \eta)$ in the form

$$\phi_\eta + (\alpha_1 \phi + \alpha_2)\phi_\xi + \beta_1 \phi_{\xi\xi\ldots\xi} + \gamma_1 \phi + \gamma_2 = 0, \tag{12.4.23}$$

where the partial derivatives in the third term of (12.4.23) occur p times,

$$
\begin{aligned}
U_1 &= R_0\phi(\xi,\eta) + W(\eta),\\
\alpha_1 &= (\lambda_0^2 L_0 R_0)^{-1} L_0\{(\nabla_U \cdot A)_0 \cdot R_0\} R_0,\\
\alpha_2 &= (\lambda_0^2 L_0 R_0)^{-1} L_0\{(\nabla_U \cdot A)_0 \cdot W\} R_0,\\
\beta_1 &= (\lambda_0 L_0 R_0)^{-1} L_0 \left[\sum_{\beta=1}^{s} \prod_{\alpha=1}^{p} \left(-H_{\alpha 0}^{\beta} + \frac{1}{\lambda_0} K_{\alpha 0}^{\beta}\right)\right] R_0,\\
\gamma_1 &= (\lambda_0 L_0 R_0)^{-1} \big[\lambda_0 L_0 R_0 + L_0\big\{(\nabla_U A)_0 R_0\big\} U_{0\eta} + L_0\big\{(\nabla_U B)_0 R_0\big\} V_\eta\big],\\
\gamma_2 &= (\lambda_0 L_0 R_0)^{-1} \big[\lambda_0 L_0 W_\eta + L_0\big\{(\nabla_U A)_0 W\big\} U_{0\eta} + L_0\big\{(\nabla_U B)_0 W\big\} V_\eta\big],
\end{aligned}
$$

and W is an arbitrary function of η alone.

Following Taniuti and Wei (1968), we outline the fundamental ideas of the reductive perturbation method for a systematic reduction of a fairly general nonlinear system of equations (12.4.1) to a single, tractable, nonlinear evolution equation describing the far-field behavior.

We introduce the Gardner–Morikawa transformation

$$
\xi = \varepsilon^a (x - \lambda t), \quad \tau = \varepsilon^{a+1} t, \quad a = (p-1)^{-1}, \tag{12.4.24}
$$

and assume expansions about a constant solution $U^{(0)}$ in terms of the small parameter ε of the form

$$
U = U^{(0)} + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \cdots, \tag{12.4.25}
$$

$$
A = \sum_{j=0}^{\infty} \varepsilon^j A_j, \qquad H_\alpha^\beta = \sum_{j=0}^{\infty} \varepsilon^j H_{\alpha j}^\beta, \qquad K_\alpha^\beta = \sum_{j=0}^{\infty} \varepsilon^j K_{\alpha j}^\beta. \tag{12.4.26}
$$

Substituting (12.4.25), (12.4.26) in (12.4.1) and using (12.4.24) enables us to rewrite (12.4.1) in terms of derivatives with respect to ξ and τ. Equating the coefficients of like powers in ε to zero yields

$$
O\big(\varepsilon^{a+1}\big) : (-\lambda I + A_0)\frac{\partial U^{(1)}}{\partial \xi} = 0, \tag{12.4.27}
$$

$$
\begin{aligned}
O\big(\varepsilon^{a+2}\big) : (-\lambda I + A_0)\frac{\partial U^{(2)}}{\partial \xi} + \frac{\partial U^{(1)}}{\partial \tau} + \big[U^{(1)} \cdot (\nabla_U A)_0\big]\frac{\partial U^{(1)}}{\partial \xi}&\\
+ \sum_{\beta=1}^{s} \prod_{\alpha=1}^{p} \big(-\lambda H_{\alpha 0}^{\beta} + K_{\alpha 0}^{\beta}\big)\frac{\partial^p U^{(1)}}{\partial \xi^p} = 0,&
\end{aligned} \tag{12.4.28}
$$

where, again, ∇_U denotes the gradient operator with respect to U, $(U_0 \cdot \nabla_U)$ represents the operator $\sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i}$, and A_1 is written as

$$
U^{(1)} \cdot (\nabla_U A)_{U=U^{(0)}} \equiv U^{(1)} \cdot (\nabla_U A)_0.
$$

Introducing a right-eigenvector R_0 of A_0 corresponding to the eigenvalue λ, so that $(A_0 - \lambda I)R_0 = 0$ and integrating (12.4.27) with the boundary conditions $U^{(1)} \to 0$, as $\xi \to \infty$, gives

$$U^{(1)} = \phi^{(1)}(\xi,\tau)R_0 + V_1(\tau), \tag{12.4.29}$$

where $\phi^{(1)}$ is one of the components of $U^{(1)}$, and $V_1(\tau)$ is an arbitrary vector-valued function of τ to be determined from the given boundary condition for $U^{(1)}$. Since $U^{(1)} \to 0$ as $\xi \to \infty$, $V_1(\tau) = 0$, and the solution (12.4.29) becomes

$$U^{(1)} = \phi^{(1)}(\xi,\tau)R_0. \tag{12.4.30}$$

To solve for $\frac{\partial}{\partial\xi}U^{(2)}$, a compatibility condition is required. To find this condition, we multiply (12.4.28) on the left by a left eigenvector L_0 and use the fact that $L_0(-\lambda I + A_0) = 0$ to obtain

$$L_0\frac{\partial U^{(1)}}{\partial\tau} + L_0\cdot\big[U^{(1)}\cdot(\nabla_U A)_0\big]\frac{\partial U^{(1)}}{\partial\xi} + L_0\cdot\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\big(-\lambda H_{\alpha 0}^{\beta} + K_{\alpha 0}^{\beta}\big)\frac{\partial^p U^{(1)}}{\partial\xi^p} = 0. \tag{12.4.31}$$

Substituting (12.4.30) in this compatibility condition and dropping the superscript (1) leads to the nonlinear evolution equation for $\phi^{(1)}(\xi,\tau)$ of the form

$$\frac{\partial\phi}{\partial\tau} + \alpha_1\phi\frac{\partial\phi}{\partial\xi} + \beta_1\frac{\partial^p\phi}{\partial\xi^p} = 0, \tag{12.4.32}$$

where α_1 and β_1 are constants defined by

$$\alpha_1 = L_0\cdot\big[R_0(\nabla_U A)_0\big]/(L_0R_0), \tag{12.4.33}$$

$$\beta_1 = L_0\cdot\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\big(-\lambda H_{\alpha 0}^{\beta} + K_{\alpha 0}^{\beta}\big)R_0/(L_0R_0). \tag{12.4.34}$$

When $p = 3$, equation (12.4.32) reduces to the KdV equation, whereas for $p = 2$, (12.4.32) becomes the Burgers equation. Thus, the far field of the general nonlinear system (12.4.1) associated with the λ characteristics field satisfies the nonlinear equation (12.4.32), which admits the solution to the order $O(\varepsilon)$ in the form

$$U = U^{(0)} + \varepsilon\phi(\xi,\tau)R_0. \tag{12.4.35}$$

Taniuti and Wei (1968) have shown that the reduction perturbation method can be extended to the exceptional case in which the eigenspace of A_0 comprises invariant subspaces. Su and Gardner (1969) derived both the KdV equation and the Burgers equation by using a similar perturbation technique.

Finally, we describe far-field behavior by the modified KdV (mKdV) equation. The above analysis reveals that if the characteristic field associated with the hyperbolic part of system (12.4.1) is *exceptional (linearly degenerate)*, then the nonlinear term in the KdV equation (12.4.32) vanishes $(\alpha_1 = 0)$, and hence, (12.4.32) becomes linear. Such a linear equation cannot describe inherently nonlinear, far-field behavior, and hence, it is necessary to introduce a different scaling to retain the effects of weak nonlinearity on the far-field behavior.

We consider the situation in which the KdV equation can be replaced by the modified KdV equation in the form

$$u_t + u^s u_x + \mu u_{xxx} = 0 \tag{12.4.36}$$

as the appropriate far-field equation. This equation is invariant under the transformations

$$\tilde{x} = \varepsilon^{\alpha} x, \qquad \tilde{t} = \varepsilon^{\beta} t, \qquad \tilde{u} = \varepsilon u, \tag{12.4.37}$$

provided that $\alpha = -\frac{s}{2}$ and $\beta = -\frac{3s}{2}$.

The solution of (12.4.36) is exceptional with respect to the characteristics field related to its hyperbolic part if the solution is expanded about $u = 0$. This follows from the fact that $\lambda = u^s$, so $(\nabla_u \lambda)_{u=0} = (s u^{s-1})_{u=0} = 0$. Using the standard argument, it can be shown that the solution of (12.4.32) is also invariant under the transformation (12.4.37). So, if the characteristic field corresponding to the eigenvalue λ of $A(U)$ in (12.4.1) is exceptional, we can expand about a constant state $U^{(0)}$ after the change of variables $\xi = \varepsilon^{s/2}(x - \lambda_0 t)$ and $\tau = \varepsilon^{3s/2} t$ and, then, write

$$U = U^{(0)} + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \cdots . \tag{12.4.38}$$

Following the same argument used before, it can be shown that far-field behavior can be described by the modified KdV equation

$$\phi_\tau + \alpha_1 \phi^s \phi_\xi + \mu \phi_{\xi\xi\xi} = 0, \tag{12.4.39}$$

where $\alpha_1 \neq 0$.

A wide variety of physical problems in plasma physics, nonlinear optics, and solid and fluid dynamics have been investigated by the reductive perturbation method or its extensions. The reader is referred to extensive references cited in several books and review articles by Jeffrey and Kakutani (1972), Taniuti (1974), Jeffrey and Kawahara (1982), Jeffrey and Engelbrecht (1994), Asano (1970), Asano et al. (1969, 1970), Asano and Ono (1971), and Kakutani (1971a, 1971b) who have investigated nonlinear wave propagation problems in inhomogeneous media. Asano (1974) examined the effects of weak dissipation or instability on nonlinear waves. The perturbation method has been generalized to allow for wave modulation in a system of nonlinear integro-partial differential equations by Taniuti and Yajima (1973). On the other hand, Nozaki and Taniuti (1973) and Oikawa and Yajima (1973, 1974a, 1974b) have considered the nonlinear interaction of three monochromatic waves and the interactions of solitary waves and nonlinear modulating waves.

In addition, we may mention that a similar reduction perturbation theory has been developed in other contexts, for example, in studying the behavior of the partial differential equations in the neighborhood of a critical point and in describing the complete history of a pulse with a curved wave front. Relevant references include Kulikovskii and Slobodkina (1967), Bhatnagar and Prasad (1971), and Prasad (1973, 1975). We next illustrate the method by examples.

Example 12.4.1 (Oscillations of Nonlinear Elastic String with Dispersive Effects). This example is taken from Taniuti and Nishihara (1983). The transverse displacement $y(x,t)$ of such a string of infinite length is governed by the nonlinear equation

$$y_{tt} - c^2 y_{xx} - \sigma^2 y_{xxxx} = 0, \tag{12.4.40}$$

where $c^2 = (1 + y_x^2)^{-3/2}$ and σ is a real constant. Equation (12.4.40) can be linearized when $c \sim 1$, and hence, it admits a plane wave solution $y(x,t) = A\exp[i(kx - \omega t)]$, provided that the dispersion relation

$$\omega^2 = k^2 - \sigma^2 k^4 \tag{12.4.41}$$

is satisfied. For appropriate long waves (large wavelength or small wavenumber), $\omega^2 > 0$, and so $\omega(k)$ is real, which confirms that the waves are dispersive.

We now write $y_t = u$ and $y_x = v$, so that equation (12.4.40) can be replaced by a system of equations like (12.4.1) which has the form

$$U_t + AU_x + KU_{xxx} = 0, \tag{12.4.42}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad K = \begin{pmatrix} 0 & -\sigma^2 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of the matrix A are $\lambda^{(\pm)} = \pm c$, and hence, it follows that $\lambda_0^{\pm} = \pm 1$, for oscillations about the equilibrium position in which $u = 0$, whereas $L_0^{\pm} = (\pm 1, 1)$ and $R_0^{(\pm)} = (1, \pm 1)^T = \binom{1}{\mp 1}$, so that $\nabla_u \lambda^{(\pm)} = (0,0)$. Consequently, $[\nabla_u \lambda^{(\pm)}]_0 R_0^{(\pm)} = 0$, which confirms that each of the two characteristic fields associated with (12.4.42) is exceptional. The appropriate far-field equation in the present case satisfies the modified KdV equation

$$\phi_\tau - \frac{3}{4}\phi^2 \phi_\xi + \frac{\sigma^2}{2}\phi_{\xi\xi\xi} = 0. \tag{12.4.43}$$

When both dissipation and weak dispersion are present in a physical system, the far-field behavior is governed by the KdV–Burgers equation. This equation *cannot* be derived by the standard reductive perturbation method. However, the necessary generalization of the method was given by Cramer and Sen (1992), including the case in which α in (12.3.9a) can change sign in a continuous manner, as the equilibrium solution $U^{(0)}$ is varied. This has led to the systematic study of nonlinear systems with *positive* or *negative* nonlinear terms depending on $\alpha > 0$ or $\alpha < 0$.

Example 12.4.2 (Gas Dynamics and the Burgers Equation). For an ideal gas ($p = R\rho T$), the conservation laws for mass, momentum, and energy are given by the following equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{12.4.44}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\nu}{\rho}\frac{\partial^2 u}{\partial x^2}=0, \tag{12.4.45}$$

$$\frac{\partial p}{\partial t}+\gamma p\frac{\partial u}{\partial x}+u\frac{\partial p}{\partial x}-\frac{\kappa(\gamma-1)}{R}\frac{\partial}{\partial x}\left[\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)-\frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial x}\right)\right]$$
$$+\nu(\gamma-1)\left[u\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial x}(uu_x)\right]=0, \tag{12.4.46}$$

where ρ is the density, p is the pressure, u is the flow velocity, ν is the kinematic viscosity, κ is the thermal conductivity, and R and γ are gas constants.

This system of equations can be expressed in the matrix form equation (12.4.1) with $s=p=2$ if U, A, and K are given as follows:

$$U=\begin{pmatrix}\rho\\ u\\ p\end{pmatrix},\qquad A=\begin{pmatrix}u & \rho & 0\\ 0 & u & \rho^{-1}\\ 0 & \gamma p & u\end{pmatrix},$$

$$K_1^1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \kappa(\gamma-1)R^{-1} & -\nu(\gamma-1) & -\kappa(\gamma-1)R^{-1}\end{pmatrix},$$

$$K_2^1=\begin{pmatrix}p/\rho^2 & 0 & 0\\ 0 & u & 0\\ 0 & 0 & \rho^{-1}\end{pmatrix},\qquad K_1^2=\begin{pmatrix}0 & 0 & 0\\ 0 & -\frac{\nu}{\rho} & 0\\ 0 & \nu(\gamma-1)u & 0\end{pmatrix},$$

$$K_2^2=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The eigenvalues of matrix A are u, $u\pm(\frac{\gamma p}{\rho})^{\frac{1}{2}}$ so that they reduce to 0 and ± 1 for the constant state $U^{(0)}=(1,0,\gamma^{-1})$. The eigenvectors L_0 and R_0 for a wave moving with velocity $\lambda_0=1$ are given by $L_0=(0,1,1)$ and $R_0=(1,1,1)$. This leads to the Burgers equation

$$\frac{\partial \rho^{(1)}}{\partial \tau}+\alpha_1\rho^{(1)}\frac{\partial \rho^{(1)}}{\partial \xi}+\beta_1\frac{\partial \rho^{(1)}}{\partial \xi^2}=0, \tag{12.4.47}$$

where $\rho^{(1)}$ is the first-order term of the expansion $\rho=1+\varepsilon\rho^{(1)}+\cdots$, the boundary condition $U\to U^{(0)}$ as $x\to\infty$ has been assumed, and

$$\alpha_1=\frac{1}{2}(\gamma+1)\quad\text{and}\quad\beta_1=\frac{1}{2}\left[(\gamma R)^{-1}\kappa(\gamma-1)^2+\nu\right].$$

Example 12.4.3 (Ion-Acoustic Waves and the KdV Equation). The motion of cold ions in a hot electron gas with the Boltzmann distribution law is governed by the equations

$$\frac{\partial n_i}{\partial t}+\frac{\partial}{\partial x}(n_i u_i)=0, \tag{12.4.48}$$

$$\left(\frac{T_e}{n_e}\right)\left(\frac{\partial n_e}{\partial x}\right) = eE, \tag{12.4.49}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = \left(\frac{e}{m_i}\right)E - \frac{1}{m_i n_i}\frac{\partial}{\partial x}(n_i T_i), \tag{12.4.50}$$

$$\frac{\partial E}{\partial x} = 4\pi e(n_i - n_e), \tag{12.4.51}$$

where the subscripts i and e refer to quantities related to ions and electrons, respectively, n_j $(j = i, e)$ is the density, u_j $(j = i, e)$ is the flow velocity, T_j is the product of the Boltzmann constant and the temperature, e is the charge of the ions, $-e$ is the charge of the electrons, and E is the electric field.

To apply the reductive perturbation method to this system of ion-acoustic waves, we eliminate n_i and E from equations (12.4.48)–(12.4.51) to obtain

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + \frac{1}{n_e}\frac{\partial n_e}{\partial x} = 0, \tag{12.4.52}$$

$$\frac{\partial u_e}{\partial t} + \frac{\partial}{\partial x}(n_e n_i) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + u_i \frac{\partial}{\partial x}\right)\left[n_e^{-1}\left(\frac{\partial n_e}{\partial x}\right)\right] = 0. \tag{12.4.53}$$

This system of equations corresponds to system (12.4.1) with $p = 3$ and $s = 1$. The associated matrices U, A, H, and K are given by

$$U = \begin{pmatrix} n_e \\ n_i \end{pmatrix}, \qquad A = \begin{pmatrix} u_i & n_e \\ n_e^{-1} & u_i \end{pmatrix},$$

$$H_1 = 0, \qquad K_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} u_i & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_3 = 0, \quad \text{and} \quad K_3 = \begin{pmatrix} n_e^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

For $U^{(0)} = \binom{1}{0}$, the eigenvalues of A_0 are $\lambda_0 = \pm 1$. For the case of $\lambda_0 = 1$, we obtain

$$L_0 = (1, 1) \qquad R_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \nabla_u \lambda_0 = (0, 1).$$

Thus, from (12.4.33) and (12.4.34), it turns out that $\alpha_1 = 1$, and $\beta_1 = \frac{1}{2}$, and the density of electrons $n_e^{(1)}$ to the first-order approximation satisfies the KdV equation

$$\frac{\partial n_e^{(1)}}{\partial \tau} + n_e^{(1)} \frac{\partial n_e^{(1)}}{\partial \xi} + \frac{1}{2}\frac{\partial^3 n_e^{(1)}}{\partial \xi^3} = 0. \tag{12.4.54}$$

In thermodynamic equilibrium at a constant temperature, electrons must satisfy the Boltzmann distribution law $n_e = n_0 \exp(\frac{e\phi}{T_e})$, where n_0 is the number density of electrons in the undisturbed state and ϕ is the electrostatic potential associated with the wave motions. Expanding $\exp(\frac{e\phi}{T_e})$, we can obtain $n_e^{(1)} = \phi^{(1)}$, and hence, the soliton solution of the KdV equation agrees with the solution for $\phi^{(1)}$.

Example 12.4.4 (Nonlinear Shallow Water Waves on an Uneven Bottom). For long waves on a beach, Peregrine (1967) derived the following system of coupled equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla h + \frac{1}{6}H^2\frac{\partial}{\partial t}\nabla(\nabla\cdot\mathbf{u}) - \frac{1}{2}H\frac{\partial}{\partial t}\nabla\big[\nabla\cdot(H\mathbf{u})\big] = 0, \tag{12.4.55}$$

$$\frac{\partial h}{\partial t} + \nabla\cdot\big[(H+h)\mathbf{u}\big] = 0, \tag{12.4.56}$$

where $\mathbf{u}$ is the horizontal velocity averaged over the vertical direction, h is the wave amplitude, and H is the depth of still water, which depends only on the horizontal coordinates.

For the motion of water in the x-direction only, Peregrine's equations become

$$h_t + uh_x + (H+h)u_x + uH_x = 0, \tag{12.4.57}$$

$$u_t + uu_x + h_x - \frac{1}{2}H(Hu_t)_{xx} + \frac{1}{6}H^2u_{txx} = 0. \tag{12.4.58}$$

It is noted here that only the first-order x- and t-derivatives are involved in equation (12.4.57), whereas equation (12.4.58) contains the third-order derivatives in which the x-derivative is repeated twice, but the t-derivative occurs only once. These comments are useful for choosing the matrices H_α^β and K_α^β in the original equation (12.4.1). In this problem, $p = 3$ and equations (12.4.57), (12.4.58) can be expressed in the form (12.4.1) with an extra term BV_x on its left side by choosing $s = 1$ and

$$U = \begin{pmatrix} h \\ u \end{pmatrix}, \qquad A = \begin{pmatrix} u & H+h \\ 1 & u \end{pmatrix}, \qquad V = H, \qquad B = \begin{pmatrix} u \\ 0 \end{pmatrix},$$

$$H_1^1 = H_2^1 = 0, \qquad H_3^1 = \begin{pmatrix} 0 & H \\ 0 & 1 \end{pmatrix},$$

$$K_1^1 = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2}H & \frac{1}{6}H^2 \end{pmatrix}, \qquad K_2^1 = I,\ K_3^1 = 0.$$

For the equilibrium state $h = u = 0$, we have the matrix

$$A_0 = \begin{pmatrix} 0 & H \\ 1 & 0 \end{pmatrix}, \quad \text{and hence,} \quad \lambda_0 = \pm\sqrt{H}.$$

A left-eigenvector L_0 and a right-eigenvector R_0 are given by

$$L_0 = (1, \lambda_0) \quad \text{and} \quad R_0 = \begin{pmatrix} H \\ \lambda_0 \end{pmatrix}.$$

To write the nonlinear evolution equation (12.4.23), we calculate its coefficients α_1, β_1, and γ_1 for $\lambda_0 = \sqrt{H}$ as

$$\begin{aligned} \alpha_1 &= \frac{3}{2}H^{-\frac{1}{2}}, \qquad \beta_1 = \frac{1}{6}\sqrt{H}, \\ \gamma_1 &= \frac{5}{4}\frac{1}{H}\left(\frac{dH}{d\eta}\right), \quad \text{and} \quad \gamma_2 = 0. \end{aligned} \tag{12.4.59}$$

Consequently, the KdV equation for $\psi = H^{5/4}\phi$ assumes the form

$$\psi_\eta + \left(\frac{3}{2}H^{-7/4}\right)\psi\psi_\xi + \left(\frac{1}{6}\sqrt{H}\right)\psi_{\xi\xi\xi} = 0, \tag{12.4.60}$$

where the coefficients of this equation are functions of η only.

12.5 Strongly Nonlinear Dispersive Systems and the NLS Equation

We consider a general, strongly nonlinear, dispersive system that can be described by the equation

$$U_t + AU_x + B + \left[\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}\left(H_\alpha^\beta \frac{\partial}{\partial t} + K_\alpha^\beta \frac{\partial}{\partial x}\right)\right]U = 0, \tag{12.5.1}$$

where U is a column vector with the n components $u_1, u_2, \ldots, u_n$ $(n \geq 2)$, A, H_α^β, and K_α^β are $n \times n$ matrices, and B is a column vector of n components, all of which are functions of U. We make an important assumption that the linear dispersion relation of (12.5.1) admits strong dispersion because of the term B. For simplicity, we consider the problem of nonlinear wave modulation based on a system of equations without the last term in (12.5.1), so that (12.5.1) gives the first-order equation

$$U_t + AU_x + B = 0. \tag{12.5.2}$$

Many physical problems exhibiting strong dispersion are described by this equation. A constant equilibrium solution $U = U_0$ of this equation is required to satisfy the algebraic equation

$$B(U_0) = 0. \tag{12.5.3}$$

We assume here that a constant solution U_0 exists and that a plane wave solution of infinitesimal amplitude is superimposed on the constant state, so that

$$U = U_0 + (\delta U_k)\exp\big[i(kx - \omega t)\big] + c.c., \tag{12.5.4}$$

where $c.c.$ stands for the complex conjugate of the preceding expression. Substituting (12.5.4) in (12.5.2) and linearizing with respect to δU_k gives the linear dispersion relation

$$|\mp i\omega I \pm ikA_0 + \nabla_u B_0| = 0, \tag{12.5.5}$$

where I is the unit matrix, A_0 and B_0 are matrices evaluated at $U = U_0$, and ∇_U is the gradient in U space. Relation (12.5.5) determines ω as a function of k, and hence, we assume that ω is given by a single real root of (12.5.5). We then introduce a matrix W_m defined by

$$W_m = \mp im\omega I \pm imkA_0 + \nabla_u B_0. \tag{12.5.6}$$

To consider the modulation of a linear plane wave, we introduce the slow variables ξ and τ as

$$\xi = \varepsilon(x - \lambda t) \quad \text{and} \quad \tau = \varepsilon^2 t, \tag{12.5.7ab}$$

where $\lambda = (\frac{d\omega}{dk})$ is the group velocity. The main objective here is to investigate how such a plane wave is modulated by nonlinear effects. We assume that the solution of (12.5.2) is of the form

$$U = U_0 + \sum_{n=1}^{\infty} \varepsilon^n U_n, \qquad U_n = \sum_{m=-\infty}^{\infty} U_n^{(m)}(\xi,\tau)\exp\bigl[im(kx-\omega t)\bigr], \tag{12.5.8ab}$$

where U_n is assumed to be real, so that $U_n^{(m)}$ is equal to its complex conjugates $U_n^{(m)*}$.

We consider modulation of the plane carrier wave with wavenumber k and frequency ω. We may set $U_1^{(m)} = 0$ except for $m = \pm 1$. Substituting (12.5.8ab) into (12.5.2) and using (12.5.7ab) leads to the following perturbation equations:

$$O(\varepsilon) : W_m U_1^{(m)} = 0, \tag{12.5.9}$$

$$\begin{aligned} O\bigl(\varepsilon^2\bigr) : W_m U_2^{(m)} &+ (-\lambda I + A_0)\left(\frac{\partial U_1^{(m)}}{\partial \xi}\right) \\ &+ \left\langle U_1 \cdot \nabla_U A_0 \sum_{r=-\infty}^{\infty} irkU_1^{(r)} \exp\bigl[ir(kx-\omega t)\bigr]\right\rangle^{(m)} \\ &+ \frac{1}{2}\nabla_U\nabla_U : \langle U_1U_1\rangle^{(m)} = 0, \end{aligned} \tag{12.5.10}$$

$$\begin{aligned} O\bigl(\varepsilon^3\bigr) : W_m U_3^{(m)} &+ (-\lambda I + A_0)\frac{\partial U_2^{(m)}}{\partial \xi} + \frac{\partial U_1^{(m)}}{\partial \tau} \\ &+ \left\langle U_1 \cdot \nabla_U A_0 \frac{\partial U_1}{\partial \xi}\right\rangle^{(m)} + \left\langle U_1 \cdot \nabla_U A_0 \sum_r irkU_2^{(r)} Z^{(r)}\right\rangle^{(m)} \\ &+ \frac{1}{2}\left\langle U_1U_1 : \nabla_U\nabla_U A_0 \sum_r irkU_1^{(r)} Z^{(r)}\right\rangle^{(m)} \\ &+ \left\langle U_2 \cdot \nabla_U A_0 \sum_r irkU_1^{(r)} Z^{(r)}\right\rangle^{(m)} + \nabla_U\nabla_U B_0 : \langle U_1U_2\rangle^{(m)} \\ &+ \frac{1}{6}\nabla_U\nabla_U\nabla_U B_0 : \langle U_1U_1U_1\rangle^{(m)} = 0, \end{aligned} \tag{12.5.11}$$

where $\langle\cdot\rangle^{(m)}$represents the coefficient of the mth harmonic and

$$Z^{(r)} \equiv \exp\big[ir(kx - \omega t)\big].$$

Thus, from (12.5.9), we obtain

$$U_1^{(m)} = \phi_1(\xi, \tau)R \quad \text{for } |m| = 1, \tag{12.5.12}$$

where ϕ_1 is a complex function of slow variables ξ and τ to be determined and R is a right-eigenvector of W_1 corresponding to the eigenvalue λ so that

$$W_1 R = 0, \tag{12.5.13}$$

whereas, for $|m| \neq 1$,

$$U_1^{(m)} = 0. \tag{12.5.14}$$

The third and fourth terms in equation (12.5.10) result from self-interaction of the fundamental harmonic $U_1^{(1)}\exp[i(kx-\omega t)]$, and they do not vanish only for $|m| = 2$ and for the slow mode $m = 0$. This implies that the second harmonic and the slow mode exist for the second order of ε. Thus, for $m = 1$, we simplify (12.5.10) by using (12.5.12) and (12.5.14) to obtain the equation

$$W_1 U_2^{(1)} + (-\lambda I + A_0)R\left(\frac{\partial \phi_1}{\partial \xi}\right) = 0. \tag{12.5.15}$$

Since $\det W_1 = 0$, equation (12.5.15) admits a solution for $U_2^{(1)}$ provided that the compatibility condition

$$L(-\lambda I + A_0)R = 0 \tag{12.5.16}$$

is satisfied, where L is a left-eigenvector corresponding to λ, so that

$$LW_1 = 0. \tag{12.5.17}$$

Differentiating (12.5.13) with respect to k and using (12.5.6) gives

$$i(-\lambda I + A_0)R + W_1\left(\frac{\partial R}{\partial k}\right) = 0. \tag{12.5.18}$$

Multiplying this equation by L from the left and using (12.5.17) yields the compatibility condition (12.5.16). Then, it follows from (12.5.15) combined with (12.5.18) that

$$U_1^{(2)} = \phi^{(2)}(\xi, \tau)R - i\left(\frac{\partial \phi^{(1)}}{\partial \xi}\right)\left(\frac{\partial R}{\partial k}\right), \tag{12.5.19}$$

where $\phi^{(2)}$ is a complex function of ξ and τ.

Solving equation (12.5.10) for $U_2^{(0)}$ and $U_2^{(\pm 2)}$, when $m = 0$ and $m = \pm 2$, we obtain solutions in the forms

$$
\begin{aligned}
U_2^{(0)} &= -W_0^{-1}\Big[ik(\nabla_U A_0 \cdot R^* - c.c.) \\
&\qquad + \frac{1}{2}(\nabla_U \nabla_U B_0 : R^* R + c.c.)\Big]\big|\phi^{(1)}\big|^2, \qquad (12.5.20)\\
U_2^{(2)} &= -W_2^{-1}\Big[ik(\nabla_U A_0 \cdot R)R + \frac{1}{2}\nabla_U \nabla_U B_0 : RR\Big]\big\{\phi^{(1)}\big\}^2. \qquad (12.5.21)
\end{aligned}
$$

The nonzero terms in the summation over r in the third-order equation (12.5.11) are only $U_1^{(\pm 1)}$, $U_2^{(\pm 1)}$, $U_2^{(0)}$, and $U_2^{(\pm 2)}$. However, for $m = 1$, there is no contribution from the term $U_2^{(\pm 1)}$ due to the fact that

$$
\big\langle Z^{(1)} Z^{(r)} \big\rangle^{(1)} = \delta(r) \quad \text{and} \quad \big\langle Z^{(-1)} Z^{(r)} \big\rangle^{(1)} = \delta(r-2),
$$

and hence, we note that, for $|m| = 1$, the nonlinear terms in (12.5.11) do not contain $\phi^{(2)}$. If we multiply equation (12.5.11) for $|m| = 1$ by L from the left and use equations (12.5.12)–(12.5.14) and (12.5.19)–(12.5.21), we can eliminate all terms containing $\phi^{(2)}$ because the first and the second terms vanish by equations (12.5.17) and (12.5.16). Consequently, all terms involving $\phi^{(2)}$ vanish, and hence, we obtain the equation for $\phi^{(1)}$ in the form

$$
i\frac{\partial \phi^{(1)}}{\partial \tau} + p\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} + q\big|\phi^{(1)}\big|^2 \phi^{(1)} = 0, \qquad (12.5.22)
$$

where the coefficients p and q are given by

$$
\begin{aligned}
p &= \frac{1}{(L \cdot R)} \cdot L(-\lambda I + A_0) \cdot \frac{\partial R}{\partial k}, \qquad (12.5.23)\\
q &= -\frac{iL}{(L \cdot R)}\Big[ik\Big\{2(\nabla_u A_0 \cdot R^*)R_2^{(2)} - \big(\nabla_u A_0 \cdot R_2^{(2)}\big)R^* \\
&\quad + \big(\nabla_u A_0 \cdot R_0^{(2)}\big)R + (\nabla_u \nabla_u A_0 : RR^*)R - \frac{1}{2}(\nabla_u \nabla_u A_0 : RR)R^*\Big\} \\
&\quad + \nabla_u \nabla_u B_0 : \big(RR_0^{(2)} + R^* R_2^{(2)}\big) + \frac{1}{2}\nabla_u \nabla_u \nabla_u B_0 : RR^* R\Big], \qquad (12.5.24)
\end{aligned}
$$

where $R_2^{(2)}$ and $R_0^{(2)}$ are constant column vectors defined by

$$
U_2^{(2)} = R_2^{(2)}\big\{\phi^{(1)}\big\}^2 \quad \text{and} \quad U_0^{(2)} = R_2^{(2)}\big|\phi^{(1)}\big|^2 \qquad (12.5.25ab)
$$

and are given by

$$
\begin{aligned}
R_2^{(2)} &= -W_2^{-1}\Big[ik(\nabla_u A_0 \cdot R)R + \frac{1}{2}\nabla_u \nabla_u B_0 : RR\Big], \qquad (12.5.26)\\
R_0^{(2)} &= -\,W_0^{-1}\Big[ik\big\{(\nabla_u A_0 \cdot R^*)R - c.c.\big\} \\
&\qquad + \frac{1}{2}(\nabla_u \nabla_u B_0 : R^* R) + c.c.\Big]. \qquad (12.5.27)
\end{aligned}
$$

We next show that $p = \frac{1}{2}\omega''$ as follows. Differentiating (12.5.18) with respect to k gives

$$-i\omega'' R + 2i(-\lambda I + A_0)\frac{\partial R}{\partial k} + W_1 \frac{\partial^2 R}{\partial k^2} = 0. \tag{12.5.28}$$

This is multiplied by L from the left to obtain, by (12.5.17),

$$-i\omega''(L \cdot R) + 2iL(-\lambda I + A_0) \cdot \left(\frac{\partial R}{\partial k}\right) = 0, \tag{12.5.29}$$

which gives $p = \frac{1}{2}\omega''$ from (12.5.23). We then derive the *generalized nonlinear Schrödinger equation*

$$-i\phi_\tau + \frac{1}{2}\omega''\phi_{\xi\xi} + q|\phi|^2\phi = 0, \tag{12.5.30}$$

where ϕ is a function of $\xi = \varepsilon(x - \omega' t)$ and $\tau = \varepsilon^2 t$, and q is not necessarily real. In particular, when q is real, (12.5.30) is called the *nonlinear Schrödinger equation*. Asano (1974) obtained the most general form of this equation for the case of three-dimensional modulation of nonlinear waves as

$$i\phi_t + \frac{1}{2}\sum_{i,j}\left(\frac{\partial^2\omega}{\partial k_i \partial k_j}\right)\left(\frac{\partial^2\phi}{\partial \xi_i \partial \xi_j}\right) + q|\phi|^2\phi + i\gamma\phi = 0, \tag{12.5.31}$$

where $k_i (i = 1, 2, 3)$ and ω denote the components of the wavevector and the frequency of the plane carrier wave, respectively, and $\xi_i = \varepsilon\{x_i - (\frac{\partial\omega}{\partial k_i})t\}$. The last term $i\gamma\phi$ is involved to incorporate weak dissipation and/or unstable effects.

Equation (12.5.30) can be solved exactly. The transformations

$$\xi = X, \qquad \frac{1}{2}\omega''\tau = T, \quad \text{and} \quad \psi = \left|\frac{q}{\omega''}\right|^{\frac{1}{2}}\phi$$

are used to put it into the canonical form

$$i\frac{\partial\psi}{\partial T} + \psi_{XX} + 2\sigma|\psi|^2\psi = 0, \tag{12.5.32}$$

where $\sigma = \operatorname{sgn}(\frac{q}{\omega''})$. This canonical equation has been solved in Chapter 10 by the inverse scattering method. For $\sigma = +1$, the asymptotic solution of the initial-value problem for (12.5.32) represents a sequence of envelope solitons

$$\begin{aligned}\psi(X,T) = 2\eta \operatorname{sech}\bigl\{2\eta(X + 4\nu T - X_0)\bigr\} \\ \times \exp\bigl[-2i\nu X - 4i\bigl(\nu^2 - \eta^2\bigr) - i\phi_0\bigr]\end{aligned} \tag{12.5.33}$$

and radiation modes, where η and ν are parameters to be determined from the initial conditions.

12.6 The Perturbation Method of Ostrovsky and Pelinovsky

Whitham's theory of averaged Lagrangian principles has been a major contribution to the field of nonlinear dispersive waves. This theory is of great interest from the viewpoint of its mathematical generality and physical clarity. Based on his averaged Lagrangian principles, Whitham derived his fundamental equations for describing slowly varying nonlinear dispersive waves. This method is directly applicable to a conservative system for which the Lagrangian is known. However, the determination of the Lagrangian is *not* always an easy problem. Furthermore, Whitham's theory cannot be applied readily to nonconservative dynamical systems.

To extend Whitham's theory, Ostrovsky and Pelinovsky (1971, 1972) have developed an asymptotic perturbation method for a general system of first-order nonlinear partial differential equations that can be applied to both conservative and nonconservative systems. They also considered a dynamical system which can be described by nonlinear partial differential equations of the Lagrangian type that include dissipation. Ostrovsky and Pelinovsky (1972) also showed that the equations of the first approximation can be deduced from the generalized Hamilton variational principle. This section deals with the perturbation method due to Ostrovsky and Pelinovsky.

We consider a system of first-order nonlinear partial differential equations

$$M(\xi,\tau,u,u_x,u_t)=\varepsilon N(\xi,\tau,u,u_x,u_t), \tag{12.6.1}$$

where $u(\mathbf{x},t)$ is an n-dimensional column vector function, operators M and N denote sets of nonlinear functions $\boldsymbol{\xi}=\varepsilon\mathbf{x}$ and $\tau=\varepsilon t$, $\boldsymbol{\xi}$ and $\mathbf{x}$ are three-dimensional vectors, and the parameter ε characterizes the deviation from the stationary and uniform state of the medium. We also assume that operators M and N are sufficiently smooth in terms of their arguments.

For $\varepsilon=0$ and for constants $\boldsymbol{\xi}$ and τ, the system of equations (12.6.1) is assumed to have stationary plane wave solutions in the form

$$u=U(\theta) \quad \text{and} \quad \theta=\boldsymbol{\kappa}\cdot\mathbf{x}-\omega t. \tag{12.6.2ab}$$

These solutions can be determined from the ordinary differential equations

$$M\big\{\boldsymbol{\xi},\tau,U(\theta),-\omega U_\theta,\ \boldsymbol{\kappa}U_\theta\big\}=0, \tag{12.6.3}$$

where $\boldsymbol{\xi}$ and τ are assumed to be constants. It is noted that solutions of (12.6.3) depend on $m(\leq n)$ arbitrary constants of integration θ_0, $\mathbf{A}=(A_2,A_3,\ldots,A_m)$ and on the parameters $\boldsymbol{\xi}$, τ, $\boldsymbol{\kappa}$, and ω. We also assume that the frequency ω, the wavenumber vector $\boldsymbol{\kappa}$, and constant $\mathbf{A}$ satisfy the dispersion relation

$$\omega=\omega(\boldsymbol{\kappa},\mathbf{A}), \tag{12.6.4}$$

and that they will be chosen so that $u=U(\theta)$ is a periodic function of θ with period 2π. The nonlinear wave profile is then determined from the basic equation (12.6.3) and may, in general, be different from a sinusoidal shape.

We now look for a solution in the form of the asymptotic series

$$u = U(\theta, \mathbf{A}, \boldsymbol{\xi}, \tau) + \sum_{n=1}^{\infty} \varepsilon^n u^{(n)}(\mathbf{x}, t), \tag{12.6.5}$$

$$\theta = \theta^{(0)}(\mathbf{x}, t, \boldsymbol{\xi}, \tau) + \sum_{n=1}^{\infty} \varepsilon^n \theta^{(n)}(\boldsymbol{\xi}, \tau), \tag{12.6.6}$$

$$A(\boldsymbol{\xi}, \tau) = \sum_{n=0}^{\infty} \varepsilon^n A^{(n)}(\boldsymbol{\xi}, \tau), \tag{12.6.7}$$

$$\boldsymbol{\kappa}(\boldsymbol{\xi}, \tau) = \theta_{\mathbf{x}}, \quad \text{and} \quad \omega(\boldsymbol{\xi}, \tau) = -\theta_t \tag{12.6.8ab}$$

so that

$$\frac{\partial \boldsymbol{\kappa}}{\partial t} + \frac{\partial \omega}{\partial \mathbf{x}} = 0. \tag{12.6.9}$$

We also expand operators M and N in an asymptotic series to obtain

$$\begin{aligned} M &= M^{(0)}(U, -\omega U_\theta, \kappa U_\theta) \\ &\quad + \varepsilon \left[\frac{\partial M^{(0)}}{\partial U} u^{(1)} + \frac{\partial M^{(0)}}{\partial U_{\mathbf{x}}} \left(U_\xi + u_{\mathbf{x}}^{(1)}\right) + \frac{\partial M^{(0)}}{\partial U_t} \left(U_\tau + u_t^{(1)}\right) \right] \\ &\quad + O\left(\varepsilon^2\right), \end{aligned} \tag{12.6.10}$$

$$\varepsilon N = \varepsilon N^{(0)}(U, -\omega U_\theta, \boldsymbol{\kappa} U_\theta) + O\left(\varepsilon^2\right), \tag{12.6.11}$$

where each term of the form $(\frac{\partial M^{(0)}}{\partial U})U$ represents a product of a square matrix and a column vector.

We substitute (12.6.5)–(12.6.11) in (12.6.1) and equate coefficients of similar powers of ε to obtain

$$P\left(\theta, \boldsymbol{\xi}, \tau, \frac{\partial}{\partial t}, \frac{\partial}{\partial \mathbf{x}}\right) u^{(r)} = Q^{(n)}, \tag{12.6.12}$$

where $n = 1, 2, 3, \ldots,$

$$P \equiv \frac{\partial M^{(0)}}{\partial U} + \frac{\partial M^{(0)}}{\partial U_t} \cdot \frac{\partial}{\partial t} + \frac{\partial \mathbf{M}^{(0)}}{\partial U_{\mathbf{x}}} \cdot \frac{\partial}{\partial \mathbf{x}}, \tag{12.6.13}$$

$$Q^{(1)} \equiv N^{(0)} - \frac{\partial M^{(0)}}{\partial U_t} \cdot \frac{\partial U}{\partial \tau} - \frac{\partial M^{(0)}}{\partial U_{\mathbf{x}}} \cdot \frac{\partial U}{\partial \boldsymbol{\xi}}, \tag{12.6.14}$$

and similar expressions for $Q^{(2)}, Q^{(3)}, \ldots$.

To determine functions θ and $\mathbf{A}$ to any desired approximation, we have to solve a system of nonhomogeneous, linear, partial differential equations with periodic coefficients and nonhomogeneous terms, which are also periodic in θ. Using the following results:

$$\frac{\partial}{\partial \mathbf{x}} \equiv \boldsymbol{\kappa} \frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial}{\partial t} \equiv -\omega \frac{\partial}{\partial \theta} \tag{12.6.15ab}$$

in equation (12.6.12), the solution $u^{(n)}$ can be represented in the form

$$u^{(n)} = R \int_0^{\theta} R^{-1} Q^{(n)} \, d\theta', \tag{12.6.16}$$

where R is a matrix consisting of vectors of the system of solutions of the variational equations

$$P\left(\theta, \boldsymbol{\xi}, \tau, \omega, \boldsymbol{\kappa}, \frac{\partial}{\partial \theta}\right)\phi = 0, \tag{12.6.17}$$

and R^{-1} is the inverse of the matrix R. It should be pointed out that the set of m particular vectors is found in terms of U as follows:

$$R_1 = \frac{\partial U}{\partial \theta}, \quad \text{and} \quad R_s = \frac{\partial U}{\partial A_s}, \quad s = 2, 3, \ldots, m. \tag{12.6.18}$$

Clearly, R_1 is a periodic function of θ, and R_s can be expressed in the form

$$R_s = \frac{\partial U}{\partial A_s} + B_s \theta \frac{\partial U}{\partial \theta}, \tag{12.6.19}$$

where B_s is an arbitrary constant to be determined from the dependence of $\boldsymbol{\kappa}$ and ω on the amplitude $\mathbf{A}$.

The remaining $(n - m)$ vectors can be expressed as

$$R_\ell = \exp(\lambda_\ell \theta) f_\ell(\theta), \tag{12.6.20}$$

where $f_\ell(\theta)$ is a periodic function with period 2π.

We assume that the characteristic exponent λ_ℓ has no positive real part and, if $\mathrm{Re}\lambda_\ell = 0$, then, $\mathrm{Im}\lambda_\ell \neq 0, \pm 1, \ldots$. Substituting R and R^{-1} in (12.6.16) yields

$$\begin{aligned} u^{(n)} &= U_\theta \int_0^{\theta} \left[U_{\theta'} Q^{(n)} + \sum_{r=2}^{m} \int_0^{\theta'} U_{A_r} Q^{(n)} \, d\theta'' \right] d\theta' \\ &\quad + \sum_{r=2}^{m} U_{A_r} \int_0^{\theta} U_{A_r} Q^{(n)} \, d\theta' \\ &\quad + \sum_{s=m+1}^{n} f_s \exp(\lambda_s \theta) \int_0^{\theta} \exp(-\lambda_s \theta') \big(f_s^* Q^{(n)}\big) \, d\theta'. \end{aligned} \tag{12.6.21}$$

The last term of (12.6.21) is bounded for all θ due to the nature of λ. In order for $u^{(n)}$ to remain bounded as a function of θ, it must satisfy the orthogonality conditions

$$\int_0^{2\pi} U_{A_r} Q^{(n)} \, d\theta = 0 \quad \text{and} \tag{12.6.22a}$$

$$\int_0^{2\pi} U_\theta Q^{(n)} \, d\theta = 0, \tag{12.6.22b}$$

for $n = 1, 2, \ldots$ and $r = 2, 3, \ldots, m$.

The equations of the first-order approximation can be obtained in a different form. We multiply equation (12.6.1) by U_A, integrate over θ, and then, subtract equation (12.6.22a) from the resulting integral expression to obtain

$$\int_0^{2\pi} M(\xi,\tau,U,U_{\mathbf{x}},U_t)U_{A_r}\,d\theta = \varepsilon\int_0^{2\pi} N(\xi,\tau,U,-\omega U_\theta,\boldsymbol{\kappa}U_\theta)U_{A_r}\,d\theta. \tag{12.6.23}$$

Similarly, multiplying (12.6.1) by U_θ, integrating over θ, and subtracting equation (12.6.22b) from the integrated result gives

$$\int_0^{2\pi} M(\xi,\tau,U,U_{\mathbf{x}},U_t)U_\theta\,d\theta = \varepsilon\int_0^{2\pi} N(\xi,\tau,U,-\omega U_\theta,\boldsymbol{\kappa}U_\theta)U_\theta\,d\theta. \tag{12.6.24}$$

The three equations (12.6.9), (12.6.23), and (12.6.24) form a closed set for the unknown functions $\boldsymbol{\kappa}$, ω, and $\mathbf{A}$. It is necessary to solve the second-order approximation equations to determine the phase function θ of order $O(\varepsilon)$.

In their paper, Ostrovsky and Pelinovsky (1972) have shown that equations of the first-order approximation can be derived from the generalized Hamilton variational principle in the averaged form. They examined the system of equations

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial u_t}\right)+\frac{\partial}{\partial\mathbf{x}}\left(\frac{\partial\mathcal{L}}{\partial u_{\mathbf{x}}}\right)-\frac{\partial\mathcal{L}}{\partial u}=\phi, \tag{12.6.25}$$

$$\phi=\phi^{(0)}+\varepsilon\phi^{(1)}+O\big(\varepsilon^2\big), \tag{12.6.26}$$

where $\mathcal{L}$ is the Lagrangian density, ϕ is the density of the nonconservative term, $u(\mathbf{x},t)$ is an n-dimensional vector function, and ε is a small parameter. We assume that $\mathcal{L}$ and ϕ are sufficiently smooth functions of u, u_t, $u_{\mathbf{x}}$ and also of slow variables $\boldsymbol{\xi}\equiv\varepsilon\mathbf{x}$ and $\tau=\varepsilon t$.

Equation (12.6.25) can be derived from the generalized Hamilton variational principle

$$\iint(\delta\mathcal{L}+\delta W)\,d\mathbf{x}\,dt=0, \tag{12.6.27}$$

where $\delta W=\phi\delta u$.

We conclude the perturbation analysis of Ostrovsky and Pelinovsky by adding a couple of comments. First, this analysis is valid for nonlinear systems with arbitrary, but *not necessarily, weak nonlinearity*. Second, the reader is referred to the work of Ostrovsky and Pelinovsky (1971, 1972) and Gorschkov et al. (1974) for further information.

12.7 The Method of Multiple Scales

Multiple scales or, more precisely, scales of different orders arise in many physical problems because different physical effects usually manifest themselves over different length and time scales. Naturally, the concept of multiple scales is involved, explicitly or implicitly, in almost all asymptotic perturbation methods. Hence, it seems to have wide applicability to many problems that involve physical phenomena which occur in relation to various scales. Although the method of multiple scales covers a wide variety of perturbation procedures, the method will be formulated here in the context of nonlinear wave propagation.

Historically, Sturrock (1957) first introduced the method of multiple scales or, more precisely, the *derivative expansion method* for an investigation of nonlinear effects in electron plasmas. Nonlinear effects are analyzed by the above perturbation procedure to study the incoherent interaction which is responsible for spectral decay, that is, for the redistribution of energy in wavenumber space—the breakdown of organized large-scale motion into disorganized small-scale motion. Subsequently, Sandri (1963, 1965, 1967) used the method of multiple scales that completely separates the different time scales exhibited by the evolution of a gas to provide the foundation of nonequilibrium statistical mechanics. Several other authors including Stuart (1960), Watson (1960), Hasimoto and Ono (1972), Frieman (1963), Nayfeh (1965a, 1965b, 1971, 1973), Nayfeh and Hassan (1971), and Kawahara (1973, 1975a, 1975b) successfully applied the derivative expansion method to solve different problems in fluid mechanics and plasma physics. These authors showed that a systematic application of the derivative expansion method can effectively take into account the full effect of amplitude modulation of a quasi-monochromatic wave, and thereby can lead to the nonlinear Schrödinger equation. It has also been shown that the derivative expansion can be applied successfully to problems with long-wave approximation, which leads to the Korteweg–de Vries equation.

We consider the nonlinear partial differential equation

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \lambda\right) u(x,t) = N\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \mu\right) u^2(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{12.7.1}$$

where L and N are differential operators and λ, μ are fixed parameters. We introduce the sets of independent variables $x_0, x_1, x_2, \ldots, x_m$ and $t_0, t_1, t_2, \ldots, t_m$ defined by

$$x_n = \varepsilon^n x \quad \text{and} \quad t_n = \varepsilon^n t, \tag{12.7.2ab}$$

where ε is a small parameter characterizing the smallness of the associated terms. Consequently, the dependent variable $u(x,t)$ can be regarded as a function of these new variables so that $u(x,t) = u(x_0, x_1, \ldots, x_m, t_0, t_1, \ldots, t_m)$.

Since the method is called the derivative expansion method, it is appropriate to introduce expansions of the derivative operators:

$$\frac{\partial}{\partial x} = \sum_{n=0}^{m} \varepsilon^n \frac{\partial}{\partial x_n} \quad \text{and} \quad \frac{\partial}{\partial t} = \sum_{n=0}^{m} \varepsilon^n \frac{\partial}{\partial t_n}. \tag{12.7.3ab}$$

Using these expansions in (12.7.1) leads to the following results:

$$
\begin{aligned}
L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\lambda\right) &= \sum_{n=0}^{m}\varepsilon^n L_n\left(\frac{\partial}{\partial t_0},\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_m},\frac{\partial}{\partial x_0},\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_m},\lambda\right) \\
&\quad + O\big(\varepsilon^{m+1}\big), \\
N\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\mu\right) &= \sum_{n=0}^{m}\varepsilon^n N_n\left(\frac{\partial}{\partial t_0},\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_m},\frac{\partial}{\partial x_0},\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_m},\mu\right) \\
&\quad + O\big(\varepsilon^{m+1}\big).
\end{aligned}
\tag{12.7.4ab}
$$

We also assume that $u(x,t)$ has the asymptotic representation

$$
\begin{aligned}
&u(x_0,x_1,\ldots,x_m,t_0,t_1,\ldots,t_m) \\
&\quad = \sum_{n=1}^{m}\varepsilon^n u_n(x_0,x_1,\ldots,x_m,t_0,t_1,\ldots,t_m) + O\big(\varepsilon^{m+1}\big).
\end{aligned}
\tag{12.7.5}
$$

In general, $u(x,t)$ can be expanded in terms of another small parameter δ which measures the degree of nonlinearity of the wave field, and it is assumed that the new parameter is related to ε. For simplicity, however, $u(x,t)$ has been expanded in powers of ε.

We next substitute (12.7.4ab) and (12.7.5) in equation (12.7.1) and then equate coefficients of like powers of ε to obtain a system of perturbation equations from which it is possible to determine the functions u_n successively. The underlying assumption is that each perturbed quantity u_n must be nonsecular (bounded) at each stage of the perturbation process. Thus, the method of multiple scales can be applied effectively to a general dispersive wave system with or without small dissipation. In the rest of this section, we follow Jeffrey and Kawahara (1982) to illustrate the general method of multiple scales by applying it to the nonlinear Boussinesq equation

$$
u_{tt} - c^2 u_{xx} - \lambda u_{xxtt} = \frac{1}{2}\big(u^2\big)_{xx}, \tag{12.7.6}
$$

where c and λ are fixed constants. The dispersion relation for the linearized Boussinesq equation is given by

$$
D(\omega,k) \equiv c^2k^2 - \omega^2 - \lambda k^2\omega^2 = 0. \tag{12.7.7}
$$

Comparing the Boussinesq equation (12.7.6) with (12.7.1) yields

$$
L \equiv \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} - \lambda\frac{\partial^4}{\partial x^2\partial t^2}, \qquad N \equiv \frac{1}{2}\frac{\partial^2}{\partial x^2}, \tag{12.7.8ab}
$$

so that equation (12.7.6) can be written symbolically as

$$
L[u] = N\big[u^2\big]. \tag{12.7.9}
$$

In view of the power series expansions (12.7.4ab), the first few operators are given by

$$L_0 \equiv \frac{\partial^2}{\partial t_0^2} - c^2 \frac{\partial^2}{\partial x_0^2} - \lambda \frac{\partial^4}{\partial x_0^2 \partial t_0^2},$$

$$L_1 \equiv 2\frac{\partial^2}{\partial t_0 \partial t_1} - 2c^2 \frac{\partial^2}{\partial x_0 \partial x_1} - 2\lambda \left(\frac{\partial^4}{\partial t_0 \partial t_1 \partial x_0^2} + 2\frac{\partial^4}{\partial t_0^2 \partial x_0 \partial x_1} \right),$$

$$L_2 \equiv \frac{\partial^2}{\partial t_1^2} + 2\frac{\partial^2}{\partial t_0 \partial t_2} - c^2 \left(\frac{\partial^2}{\partial x_1^2} + 2\frac{\partial^2}{\partial x_0 \partial x_2} \right) - \lambda \left(\frac{\partial^4}{\partial t_0^2 \partial x_1^2} + 2\frac{\partial^4}{\partial t_0^2 \partial x_0 \partial x_2} + 4\frac{\partial^4}{\partial t_0 \partial t_1 \partial x_0 \partial x_1} + \frac{\partial^4}{\partial t_1^2 \partial x_0^2} + 2\frac{\partial^4}{\partial t_0 \partial t_2 \partial x_0^2} \right),$$

and

$$N_0 \equiv \frac{1}{2}\frac{\partial^2}{\partial x_0^2}, \qquad N_1 \equiv \frac{\partial^2}{\partial x_0 \partial x_2}, \quad \text{and} \quad N_2 \equiv \frac{1}{2}\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_0 \partial x_2}.$$

Utilizing these operators, equation (12.7.9) reduces to a system of perturbation equations in the form:

$$O(\varepsilon) : Lu_1 = 0, \tag{12.7.10}$$

$$O(\varepsilon^2) : L_0 u_2 + L_1 u_1 = N_0 u_1^2, \tag{12.7.11}$$

$$O(\varepsilon^3) : L_0 u_3 + L_1 u_2 + L_2 u_1 = N_0[2u_1 u_2] + N_1 u_1^2, \tag{12.7.12}$$

$$O(\varepsilon^4) : L_0 u_4 + L_1 u_3 + L_2 u_2 + L_3 u_1 = N_0\left[u_2^2 + 2u_1 u_3\right] + N_1[2u_1 u_2] + N_2 u_1^2. \tag{12.7.13}$$

We consider a strongly dispersive wave system, and it is necessary to take into account nonlinear modulation of the wavetrain; that is, amplitude and phase suffer from a slow variation over a localized region in space. Consequently, it is reasonable to assume that the first-order perturbation solution of the first equation (12.7.10) has the form

$$u_1 = A(x_1, x_2, \ldots, x_m, t_1, t_2, \ldots, t_m) \exp(i\theta) + c.c., \tag{12.7.14}$$

where $\theta = (kx_0 - \omega t_0)$, the amplitude A is a complex function of the slow variables, and $c.c.$ represents the complex conjugate of the preceding expression. Moreover, k and ω satisfy the dispersion relation (12.7.7).

Invoking (12.7.14) in the second equation (12.7.11) of the perturbation system yields

$$L_0 u_2 = -i\left(\frac{\partial D}{\partial \omega}\frac{\partial A}{\partial t_1} - \frac{\partial D}{\partial k}\frac{\partial A}{\partial x_1} \right) \exp(i\theta) - 2k^2 A^2 \exp(2i\theta) + c.c., \tag{12.7.15}$$

where

$$\frac{\partial D}{\partial \omega} = -2\left(1 + \lambda k^2\right)\omega, \qquad \frac{\partial D}{\partial k} = 2\left(c^2 - \lambda \omega^2\right)k. \tag{12.7.16ab}$$

To determine u_2 from (12.7.15), it is necessary to integrate it, so that the result produces what is called a *secular* (*unbounded*) *term*, that is, the term grows without bound, as time $t \to \infty$, due to the form of L_0 and $\exp(i\theta)$ in the first term on the right-hand side of (12.7.15). This means that such a solution for u_2 will not be uniformly valid as $t \to \infty$, no matter how small ε may be. To eliminate the nonuniformity in the perturbation expansion, the coefficient of $\exp(i\theta)$ in (12.7.15) must vanish identically. This leads to the following condition for nonsecularity:

$$\frac{\partial A}{\partial t_1} + C_g(k)\frac{\partial A}{\partial x_1} = 0, \tag{12.7.17}$$

where

$$C_g(k) = \frac{\partial \omega}{\partial k} = -\frac{D_k}{D_\omega} \tag{12.7.18}$$

represents the group velocity of the linearized problem.

Consequently, the uniformly valid solution for u_2 can be obtained from (12.7.15) in the form

$$\begin{aligned} u_2 = &-\frac{2(kA)^2}{D(2\omega, 2k)}\exp(2i\theta) + F(x_1, x_2, \ldots, x_m, t_1, t_2, \ldots, t_m)\exp(i\theta) + c.c. \\ &+ G(x_1, x_2, \ldots, x_m, t_1, t_2, \ldots, t_m), \end{aligned} \tag{12.7.19}$$

where F and G are complex and real functions, respectively, of higher-order scales that can be found from higher-order perturbations.

Substituting u_1 and u_2 in equation (12.7.12) leads to another nonsecular (bounded) condition

$$\begin{aligned} & i\left(\frac{\partial D}{\partial \omega}\cdot\frac{\partial A}{\partial t_2} - \frac{\partial D}{\partial k}\cdot\frac{\partial A}{\partial x_2}\right) \\ &\quad - \frac{1}{2}\left(\frac{\partial^2 D}{\partial \omega^2}\cdot\frac{\partial^2 A}{\partial t_1^2} - 2\frac{\partial^2 D}{\partial k\partial\omega}\cdot\frac{\partial^2 A}{\partial x_1\partial t_1} + \frac{\partial^2 D}{\partial k^2}\cdot\frac{\partial^2 A}{\partial x_1^2}\right) \\ &\quad + k^2\left\{-\frac{2k^2}{D(2\omega, 2k)}A^2A^* + GA\right\} \\ &\quad + i\left(\frac{\partial D}{\partial \omega}\frac{\partial F}{\partial t_1} - \frac{\partial D}{\partial k}\frac{\partial F}{\partial x_1}\right) = 0, \end{aligned} \tag{12.7.20}$$

and its complex conjugate relation.

The variation of G follows from the nonsecular condition obtained from the fourth equation (12.7.13) of the perturbation system, and hence, it turns out that

$$\frac{\partial^2 G}{\partial t_1^2} - c^2\frac{\partial^2 G}{\partial x_1^2} = \frac{\partial^2 (AA^*)}{\partial x_1^2}. \tag{12.7.21}$$

If A and G depend only on x_1 and t_1 through $\xi = x_1 - C_g t_1$, that is, if they are considered in a coordinate system moving with the group velocity C_g, equation (12.7.21) can be integrated with respect to ξ and then solved to obtain

$$G(\xi, x_2, \ldots, x_m, t_2, t_3, \ldots, t_m) = \left(C_g^2 - c^2\right)^{-1}AA^* + \alpha, \tag{12.7.22}$$

provided that $C_g \neq c$ and $\alpha = \alpha(x_2, x_3, \ldots, x_m, t_2, t_3, \ldots, t_m)$ is a slowly varying function of higher-order slow variables to be determined from the initial and boundary conditions. However, if $C_g = c$, the resonant interaction between short and long waves occurs, and this case can be handled separately.

To obtain an equation for the amplitude A, it is necessary to find F involved in (12.7.20). Since F is a coefficient of a secular term involved in u_2 and thus is an unbounded term proportional to $\exp(i\theta)$, it follows that F is associated with the unbounded terms of the higher-order solutions u_n. It is possible to show that these unbounded (resonant) terms can be transferred to the lower-order solution u_1 and the transferred quantity then can be considered as a new amplitude A. Therefore, the last term in (12.7.20) may be dropped. Finally, using (12.7.17) and (12.7.22) in the nonsecular condition (12.7.20) yields

$$\begin{aligned} i\left(\frac{\partial A}{\partial t_2} + C_g \frac{\partial A}{\partial x_2}\right) + \frac{1}{2} C_g'(k) \frac{\partial^2 A}{\partial \xi^2} \\ + \left(\frac{k^2}{D_\omega}\right)\left[\left(\frac{1}{C_g^2 - c^2} + \frac{1}{6\lambda\omega^2}\right)|A|^2 A + \alpha A\right] = 0, \end{aligned} \tag{12.7.23}$$

where $A = A(\xi, x_2, \ldots, x_m, t_2, t_3, \ldots, t_m)$, α is independent of ξ, and hence, (12.7.22) indicates that α can be treated as a constant with respect to the ξ coordinate.

In a frame of reference moving with group velocity C_g and the coordinate transformations defined by

$$\begin{aligned} \xi &= \frac{1}{\varepsilon}(x_2 - C_g t_2) = (x_1 - C_g t_1) = \varepsilon(x_2 - C_g t) \quad \text{and} \\ \tau &= t_2 = \varepsilon t_1 = \varepsilon^2 t, \end{aligned} \tag{12.7.24ab}$$

we can express equation (12.7.23) in the following canonical form of the *nonlinear Schrödinger equation*:

$$i\frac{\partial A}{\partial \tau} + \frac{1}{2} C_g'(k) \frac{\partial^2 A}{\partial \xi^2} + \left(\frac{k^2}{D_\omega}\right)\left[\left(\frac{1}{C_g^2 - c^2} + \frac{1}{6\lambda\omega^2}\right)|A|^2 A + \alpha A\right] = 0. \tag{12.7.25}$$

Thus, the upshot of this perturbation analysis is that the amplitude modulation $A(\xi, \tau)$ for a strongly dispersive system satisfies the nonlinear Schrödinger equation.

We close this section by adding some relevant comments on the asymptotic expansion in terms of another small parameter δ. First, the coordinate transformations defined by (12.7.24ab) are nothing but the Gardner–Morikawa transformations that have been used in the reductive perturbation analysis. Second, $u(x, t)$ has a more general asymptotic expansion in the form

$$u(x, t) = \sum_{n=1}^{\infty} \delta^n u_n(x_0, x_1, \ldots, x_m, t_0, t_1, \ldots, t_m), \tag{12.7.26}$$

where δ is a new small parameter characterizing the smallness of terms that measure the degree of nonlinearity. Several cases arise depending on the relative importance of δ and ε. The case $\delta = \varepsilon$ has already been discussed above. The case $\delta = \varepsilon^2$ corresponds to the dispersive wave systems dominated by linear propagation of envelopes. This case can be analyzed similarly to the case of $\delta = \varepsilon$, and hence, the wave amplitude A satisfies the *linear equation* similar to (12.7.25) *without* the nonlinear term $|A|^2 A$. In the case $\delta = \sqrt{\varepsilon}$, the nonlinear interaction dominates, and hence, the nonlinear term occurs in the first nonsecular condition. A similar analysis leads to the following equation:

$$i\left(\frac{\partial A}{\partial t_1} + C_g \frac{\partial A}{\partial x_1}\right) + \left(\frac{k^2}{D_\omega}\right)\left[\frac{1}{6\lambda\omega^2}|A|^2 A + \alpha A\right] = 0, \tag{12.7.27}$$

where A and α are functions of slow scales $x_1, x_2, \ldots, x_m, t_1, t_2, \ldots, t_m$. The independent variables in (12.7.27) are the first-order slow variables x_1 and t_1. The slowly varying function α can be found with the aid of the same equation (12.7.21) that was obtained from the higher-order perturbation analysis. Thus, α can be treated as a function of slow variables $x_2, x_3, \ldots, x_m, t_2, t_3, \ldots, t_m$ to be determined from the prescribed boundary and/or initial conditions. Hence, the amplitude equation (12.7.27) in the nonsecular condition remains valid for the first-order slow variables. Additional slower variables $x_2, t_2, x_3, t_3, \ldots$ in (12.7.27) can be treated as slow parameters for this order of approximation.

Inoue and Matsumoto (1974) applied the method of multiple scales to a general system of equations of the form

$$A(U)\frac{\partial U}{\partial t} + B(U)\frac{\partial U}{\partial x} + C(U) = 0, \tag{12.7.28}$$

where U is a column vector with n components $u_1, u_2, \ldots, u_n$, A and B are $n \times n$ matrices, and C is a column vector.

According to the method of multiple scales, we introduce a new set of independent variables as

$$x_n = \varepsilon^n x \quad \text{and} \quad t_n = \varepsilon^n t, \tag{12.7.29ab}$$

where $n = 0, 1, 2, \ldots, m$ and, then, the dependent variable U is expanded in the form

$$U = U_0 + \sum_{n=1}^{m+1} \varepsilon^n U_n(x_0, x_1, \ldots, x_m, t_0, t_1, \ldots, t_m) + O\left(\varepsilon^{m+2}\right), \tag{12.7.30}$$

where U_0 is a constant solution vector which satisfies the compatibility condition

$$C(U_0) = 0. \tag{12.7.31}$$

Invoking the asymptotic expansion of the derivatives into multiple scale independent variables, as in (12.7.3ab), the perturbation analysis for the general system of equations can be carried out without any difficulty. However, we will not pursue the analysis further. For more details of perturbation analysis, the reader is referred to Inoue and Matsumoto (1974) and Jeffrey and Kawahara (1982).

In passing we may mention that other investigators (Leibovich and Seebass 1972) have also applied the method of multiple scales in deriving the KdV and Burgers equations. Prasad and Ravindran (1977) have developed a more general method to derive similar model equations for the propagation of nonlinear curved waves in several dimensions.

12.8 Asymptotic Expansions and Method of Multiple Scales

As we have seen in this chapter, the method of asymptotic expansion is fairly direct and requires no deep knowledge of the theory of the subject. Despite the long history of the subject, the method has not been free from controversy, mainly over the interpretation of divergent series which may arise in applied mathematics. The method is very successful and is often used based on the assumption that a solution of the given equation exists as a suitable asymptotic expansion with respect to the relevant small parameter.

This section deals with examples of applications of asymptotic methods to the solution of partial differential equations in the context of problems in wave propagation.

We again use the Boussinesq equation (12.7.6) to illustrate this method. The dispersion relation for this equation shows that $\omega \to 0$ in the long-wavelength limit as $k \to 0$. It is then evident from the expansions (12.7.4ab) that both operators $\frac{\partial}{\partial x_0}$ and $\frac{\partial}{\partial t_0}$ can be removed so that $L_0 = L_1 = N_0 = N_1 = 0$. Consequently, the system of operators (12.7.4ab) reduces to

$$L_2 \equiv \frac{\partial^2}{\partial t_1^2} - c^2 \frac{\partial^2}{\partial x_1^2}, \tag{12.8.1}$$

$$L_3 \equiv 2\frac{\partial^2}{\partial t_1 \partial t_1} - 2c^2 \frac{\partial^2}{\partial x_1 \partial x_2}, \tag{12.8.2}$$

$$L_4 \equiv \frac{\partial^2}{\partial t_2^2} + 2\frac{\partial^2}{\partial t_1 \partial t_3} - c^2\left(\frac{\partial^2}{\partial x_2^2} + 2\frac{\partial^2}{\partial x_1 \partial x_3} - \lambda \frac{\partial^4}{\partial t_1^2 \partial x_1^2}\right), \tag{12.8.3}$$

and

$$N_2 \equiv \frac{1}{2}\frac{\partial^2}{\partial x_1^2} \quad \text{and} \quad N_3 \equiv \frac{\partial^2}{\partial x_1 \partial x_2}. \tag{12.8.4ab}$$

We write the asymptotic expansion for $u(x,t)$ in the form

$$u(x,t) = \sum_{n=1}^{M} \delta^n u_n(x_1, x_2, \ldots, x_m, t_1, t_2, \ldots, t_m) + O\big(\delta^{M+1}\big). \tag{12.8.5}$$

Several possible cases arise depending on the relative importance of dispersion and nonlinearity in this problem.

Case (i): $\delta = \varepsilon^2$.

In this case, the Boussinesq equation reduces to the following coupled system:

$$O(\varepsilon^4) : L_2 u_1 = 0, \tag{12.8.6}$$

$$O(\varepsilon^5) : L_3 u_1 = 0, \tag{12.8.7}$$

$$O(\varepsilon^6) : L_2 u_2 + L_4 u_1 = N_2 u_1^2. \tag{12.8.8}$$

Equation (12.8.6) yields the linear wave equation

$$\left(\frac{\partial^2}{\partial t_1^2} - c^2 \frac{\partial^2}{\partial x_1^2}\right) u_1 = 0. \tag{12.8.9}$$

This admits a solution for $u_1$which is a function of $\xi_1 = x_1 - ct_1$ for waves propagating to the right, or of $\xi_2 = x_1 + ct_1$ for waves traveling to the left with constant velocity c. Using these results in (12.8.7) combined with terms of order $O(\varepsilon^5)$ yields

$$\left(\frac{\partial}{\partial t_2} + c\frac{\partial}{\partial x_2}\right)\frac{\partial}{\partial \xi_1}\{u_1(\xi_1, x_2, \ldots, x_m, t_2, t_3, \ldots, t_m)\} = 0. \tag{12.8.10}$$

This equation is satisfied if u_1 depends on x_2 and t_2 through $\xi_2 = x_2 - ct_2$, and hence, equation (12.8.8) reduces to the form

$$\left(\frac{\partial^2}{\partial t_1^2} - c^2\frac{\partial^2}{\partial x_1^2}\right) u_2 - \left(2c\frac{\partial^2}{\partial \xi_1 \partial t_3} + 2c^2 \frac{\partial^2}{\partial \xi_1 \partial x_3} + \lambda c^2 \frac{\partial^4}{\partial \xi_1^4}\right) u_1 = \frac{1}{2}\frac{\partial^2 u_1^2}{\partial \xi_1^2}. \tag{12.8.11}$$

Finally, we assume that u_2 depends on x_1 and t_1 through $\xi_1 = x_1 - ct_1$ so that equation (12.8.11) with $u_1 = \phi$ takes the form

$$\frac{\partial \phi}{\partial t_3} + c\frac{\partial \phi}{\partial x_3} + \frac{1}{2}(\lambda c)\frac{\partial^3 \phi}{\partial \xi_1^3} + \frac{1}{2c}\left(\phi \frac{\partial \phi}{\partial \xi_1}\right) = 0, \tag{12.8.12}$$

in which integration is carried out with respect to ξ_1.

In a frame of reference moving with the phase velocity c of long waves, that is, by setting $\xi_3 = x_3 - ct_3$, $\tau = t_3$ and, then, replacing ξ_1 by ξ, equation (12.8.12) reduces to the canonical form of the KdV equation in terms of the independent variables ξ and τ

$$\frac{\partial \phi}{\partial \tau} + \frac{1}{2c}\left(\phi\frac{\partial \phi}{\partial \xi}\right) + \frac{1}{2}(\lambda c)\frac{\partial^3 \phi}{\partial \xi^3} = 0. \tag{12.8.13}$$

This reveals the fact that, when deriving the KdV equation, the frame of reference moves with the phase velocity but *not* with the group velocity of the wave.

Case (ii): $\delta = \varepsilon^3$.

In this case, the asymptotic expansion enables us to derive an equation similar to (12.8.12) *without* the nonlinear term.

Case (iii): $\delta = \varepsilon$.

In this case, we obtain a nonlinear equation *without* a dispersive term in the form

$$\frac{\partial \phi}{\partial t_2} + c\frac{\partial \phi}{\partial x_2} + \frac{1}{2c}\left(\phi \frac{\partial \phi}{\partial \xi}\right) = 0. \tag{12.8.14}$$

Obviously, the independent variables are the lower-order scales x_2 and t_2 in equation (12.8.14).

Example 12.8.1 (The Korteweg–de Vries (KdV) Equation). We consider the nonlinear partial differential equation (Johnson 1997)

$$u_{tt} - u_{xx} = \varepsilon(u^2 + u_{xx})_{xx}, \quad x \in \mathbb{R},\ t \geq 0, \tag{12.8.15}$$

where ε is a small parameter that represents the characteristics of both small amplitude and long waves.

We seek a solution of (12.8.15) in terms of the small parameter ε as

$$u(x, t; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n u_n(x, t) \quad \text{as } \varepsilon \to 0, \tag{12.8.16}$$

where both x and t are $O(1)$. We assume that equation (12.8.15) is to be solved with the appropriate initial condition at $t = 0$.

We substitute (12.8.16) in (12.8.15), collect together like powers of ε, and then set equal to zero for each coefficient of ε^n. Consequently, (12.8.16) is a solution of (12.8.15) provided

$$u_{0tt} - u_{0xx} = 0, \tag{12.8.17}$$

$$u_{1tt} - u_{1xx} = \left(u_0^2 + u_{0xx}\right)_{xx}. \tag{12.8.18}$$

Clearly, the d'Alembert solution of (12.8.17) is given by

$$u_0(x, t) = f(x - t) + g(x + t), \tag{12.8.19}$$

where f and g are arbitrary functions of their arguments.

For simplicity, we assume that the initial data is such as to generate only the wave traveling to the positive x-direction so that

$$u(x, 0; \varepsilon) = f(x), \qquad u_t(x, 0; \varepsilon) = -f'(x). \tag{12.8.20}$$

With $u_0 = f(x - t)$ as the solution, equation (12.8.18) becomes

$$u_{1tt} - u_{1xx} = \left(f^2 + f''\right)'', \tag{12.8.21}$$

where the prime denotes the derivative with respect to $(x - t)$. We next introduce the characteristic variables ξ, η for (12.8.21)

$$\xi = x - t, \qquad \eta = x + t, \tag{12.8.22}$$

so that

$$-4u_{1\xi\eta} = \left(f^2 + f''\right)'', \tag{12.8.23}$$

and hence,

$$u_1(\xi,\eta) = -\frac{1}{4}\eta\left(f^2 + f''\right)' + A(\xi) + B(\eta), \tag{12.8.24}$$

where $f = f(\xi)$, and the arbitrary functions A and B are determined from the initial data (12.8.20) with the following requirements for $u_1(x,t)$:

$$u_1(x,0) = 0 = u_{1t}(x,0). \tag{12.8.25}$$

Consequently, we have

$$u_1(\xi,\eta) = \frac{1}{4}\left[(\xi-\eta)\left\{f^2(\xi) + f''(\xi)\right\}' - f^2(\xi) - f''(\xi) + f^2(\eta) + f''(\eta)\right],$$

and therefore,

$$u_1(x,t) = \frac{1}{4}\left[F(x+t) - F(x-t)\right] - \frac{1}{2}tF'(x-t), \tag{12.8.26}$$

where $F = f^2 + f''$.

Thus, the asymptotic expansion is

$$u(x,t;\varepsilon) = f(x-t) - \frac{\varepsilon}{4}\left[2tF'(x-t) + F(x-t) - F(x+t)\right]. \tag{12.8.27}$$

If f has compact support, or at least for $f(x) \to 0$ sufficiently rapidly as $x \to \pm\infty$, the asymptotic result (12.8.27) is not uniformly valid for $\varepsilon t = O(1)$. We now investigate the solution of (12.8.15) using the transformations

$$\xi = x - t = O(1), \qquad \tau = \varepsilon t = O(1) \quad \text{as } \varepsilon \to 0. \tag{12.8.28}$$

It is noted that the asymptotic expansion (12.8.27) would be nonuniform (breaking down) for any values of ξ where the first, second, or third derivative of $f(\xi)$ is not defined. The solution (12.8.27) is also nonuniform for any t, no matter how well behaved $f(\xi)$ may be. Furthermore, the solution is of no physical interest for a constant value of f.

In wave propagation problems, the region where a large time (or distance) variable τ is involved is called the *far-field region*, and the region where $t = O(1)$ is then referred to as the *near-field region*. We also observe that, for $\xi = x - t = O(1)$, $t = O(\varepsilon^{-1})$ implies that $x = O(\varepsilon^{-1})$.

We next apply the transformation (12.8.28) to equation (12.8.15) and use the fact that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \tau}\cdot\frac{\partial \tau}{\partial x} = \frac{\partial}{\partial \xi},$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \tau}\cdot\frac{\partial \tau}{\partial t} = \varepsilon\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \xi},$$

so that the equation for $u(x,t;\varepsilon) = U(\xi,\tau;\varepsilon)$ reduces to the form

$$\varepsilon U_{\tau\tau} - 2U_{\tau\xi} = \left(U^2 + U_{\xi\xi}\right)_{\xi\xi}. \tag{12.8.29}$$

We seek an asymptotic solution of this equation in the form

$$U(\xi,\tau;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n U_n(\xi,\tau) \quad \text{as } \varepsilon \to 0, \tag{12.8.30}$$

where $\xi = O(1)$ and $\tau = O(1)$.

Thus, $U_0(\xi,\tau)$ satisfies the equation

$$2U_{0\tau\xi} + \left(U_0^2 + U_{0\xi\xi}\right)_{\xi\xi} = 0, \tag{12.8.31}$$

or equivalently,

$$2(U_{0\tau} + U_0 U_{0\xi}) + U_{0\xi\xi\xi} = 0, \tag{12.8.32}$$

where decay conditions are imposed as $\xi \to \pm\infty$.

This is one form of the *Korteweg–de Vries equation*. The solution of this equation satisfying the matching condition

$$U_0 \to f(\xi) \quad \text{as } \tau \to 0$$

corresponds to the initial-value problem for equation (12.8.32). Such a solution exists, provided $f(\xi)$ decays sufficiently rapidly as $\xi \to \pm\infty$. The inverse scattering method was used to obtain the solution in Chapter 9. The solution for $U_0(\xi,\tau)$ represents a one-term uniformly valid asymptotic expansion for $\tau \geq 0$ and $\tau = O(1)$ as $\varepsilon \to 0$.

Finally, the next term in the expansion (12.8.30) satisfies the equation

$$2U_{1\tau\xi} + 2(U_0U_1)_{\xi\xi} + U_{1\xi\xi\xi\xi} = U_{0\tau\tau},$$

or equivalently,

$$2U_{1\tau} + 2(U_0U_1)_{\xi} + U_{1\xi\xi\xi} = -\left(U_0^2 + U_{0\xi\xi}\right)_{\tau}, \tag{12.8.33}$$

where equation (12.8.32) was used for $U_{0\tau}$ and the decay conditions have been imposed as $\xi \to \pm\infty$. The mathematical analysis hereafter is not particularly simple. However, the solution for U_1 is found by writing $U_1 = U_{0\xi} V(\xi,\tau)$, which can be investigated to check whether the asymptotic expansion (12.8.30) is valid as $\tau \to \infty$. This involves a lengthy and somewhat complicated discussion if the general term U_n is to be included. We will not pursue this problem further. We conclude by making a final statement that the far-field asymptotic expansion of the solution for problems of this type is usually uniformly valid, provided $f(x)$ is sufficiently smooth and decays to zero sufficiently rapidly at infinity.

Finally, we describe the method of multiple scales to solve another example which is typical of some problems in water waves.

Example 12.8.2 (The Nonlinear Schrödinger equation). We consider the nonlinear partial differential equation (Johnson 1997)

$$u_{tt} - c^2 u_{xx} - u + \varepsilon (u u_x)_x = 0, \tag{12.8.34}$$

where ε is a small parameter. For $\varepsilon = 0$ this equation admits traveling wave solutions in the form

$$u(x,t) = A \exp\left[i(\omega t - kx)\right] + c.c., \tag{12.8.35}$$

where $c.c.$ represents the complex conjugate, and A is the complex amplitude of the wave.

Substituting (12.8.35) in (12.8.34) with $\varepsilon = 0$ gives the dispersion relation

$$\omega^2 = c^2 k^2 - 1, \tag{12.8.36}$$

which admits a real solution for ω if $|ck| > 1$. We assume $ck > 1$ so that there are two possible phase velocities of the wave given by

$$c_p = \frac{\omega}{k} = \pm\sqrt{c^2 - k^{-2}}. \tag{12.8.37}$$

For a given k and a phase velocity c_p, there exists a harmonic wave solution of equation (12.8.34) which evolves slowly on suitable scales. It is convenient to introduce slow variables

$$\zeta = \varepsilon(x - c_g t), \qquad \tau = \varepsilon^2 t, \tag{12.8.38}$$

where the group velocity c_g is, in general, not equal to the phase velocity c_p and is known at this point. In addition, introducing a new variable

$$\xi = x - c_p t,$$

and using the following identities:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \varepsilon \frac{\partial}{\partial \zeta}, \qquad \frac{\partial}{\partial t} = -c_p \frac{\partial}{\partial \xi} - \varepsilon c_g \frac{\partial}{\partial \zeta} + \varepsilon^2 \frac{\partial}{\partial \tau},$$

in the original equation (12.8.34) with $c = 1$ for $u(x,t;\varepsilon) = U(\xi,\zeta,\tau;\varepsilon)$ reduces to the form

$$\begin{aligned} &\left(c_p^2 - 1\right) U_{\xi\xi} - U + 2\varepsilon(c_p c_g - 1) U_{\xi\zeta} + \varepsilon^2\left[\left(c_g^2 - 1\right) U_{\zeta\zeta} - 2c_p U_{\xi\tau}\right] \\ &\quad + \varepsilon (U U_\xi)_\xi + \varepsilon^2\left[(U U_\xi)_\zeta + (U U_\zeta)_\xi\right] = O\left(\varepsilon^3\right), \end{aligned} \tag{12.8.39}$$

where only terms $O(\varepsilon^2)$ are retained in the above equation. Thus, the function $U(x,t;\varepsilon)$ is now treated as a function of the new variables ξ, ζ, τ. In view of the scales $O(1)$, $O(\varepsilon^{-1})$, and $O(\varepsilon^{-2})$, the method adopted here is called the *method of multiple scales*.

We now seek a solution in the form of the asymptotic expansion

$$U(\xi,\zeta,\tau;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n U_n(\xi,\zeta,\tau) \quad \text{as } \varepsilon \to 0, \tag{12.8.40}$$

where ξ, ζ, and τ are all $O(1)$.

Consequently, U_0 satisfies the linear equation

$$\left(c_p^2 - 1\right)U_{0\xi\xi} - U_0 = 0. \tag{12.8.41}$$

We seek a solution in the form

$$U_0(\xi, \zeta, \tau) = A_{01}(\zeta, \tau)e^{ik\xi} + c.c. \tag{12.8.42}$$

with

$$c_p^2 = \left(1 - \frac{1}{k^2}\right), \quad k > 1, \tag{12.8.43}$$

where the first subscript in A_{01} represents the term ε^0, and the second term is related to the choice $E^m = \exp(imk\xi)$, $m = 1, 2, \ldots$.

At the next order ε^1, we find the nonlinear equation

$$\begin{aligned}\left(c_p^2 - 1\right)U_{1\xi\xi} - U_1 &= 2(1 - c_p c_g)U_{0\xi\zeta} + (U_0 U_{0\xi})_\xi \\ &= (1 - c_p c_g)(ikA_{01}E + c.c.) - \left(2k^2 A_{01}^2 E^2 + c.c.\right),\end{aligned} \tag{12.8.44}$$

and U_1 is a harmonic function only if $c_p c_g - 1 = 0$, which determines c_g. If we do not make this choice for c_g, then U_1 includes a particular integral proportional to (ξE) which would lead to the nonuniform asymptotic expansion (12.8.40) as $|\xi| \to \infty$. Terms like (ξE) are called *secular*, while a uniform solution in ξ is possible only if harmonic terms in ξ are allowed in the solution for U_1. The amplitude A_{01} of the wave moves with the group velocity c_g which satisfies the condition $c_p c_g = 1$, which can also be verified from the definition of the group velocity

$$c_g = \frac{d\omega}{dk} = \frac{d}{dk}(kc_p) \quad \text{with (12.8.43).}$$

Thus, the solution for $U_1(\xi, \zeta, \tau)$ can be written as

$$\begin{aligned}U_0(\xi, \zeta, \tau) &= A_{11}(\zeta, \tau)E + \frac{2k^2 A_{01}^2}{4k^2(1 - c_p^2) - 1}E^2 + c.c. \\ &= A_{11}(\zeta, \tau)E + \frac{2}{3}k^2 A_{01}^2 E^2 + c.c.,\end{aligned} \tag{12.8.45}$$

where A_{11} is the amplitude of the fundamental wave $E = \exp(ik\xi)$. It is noted that U_1 includes a higher harmonic E^2 and its complex conjugate E^{-2}.

With the condition $c_p c_g = 1$, we obtain the equation for $U_2(\xi, \zeta, \tau)$ as

$$\begin{aligned}\left(c_p^2 - 1\right)U_{2\xi\xi} - U_2 = {}& \left(1 - c_g^2\right)U_{0\zeta\zeta} + 2c_p U_{0\xi\tau} \\ & - \left(U_0^2\right)_{\xi\zeta} - (U_0 U_1)_{\xi\xi}.\end{aligned} \tag{12.8.46}$$

We impose the condition that the solution for U_2 must contain terms harmonic in ξ so that any terms in E^1 involved in the forcing terms in (12.8.46) must vanish. Such terms can arise only from the equation

$$
\begin{aligned}
&(1-c_g^2)U_{0\zeta\zeta}+2c_pU_{0\xi\tau}-(U_0U_1)_{\xi\xi}\\
&\quad=(1-c_g^2)[A_{01\zeta\zeta}E+c.c.]+2c_p\big[(ikE)A_{01\tau}+c.c.\big]\\
&\qquad-\frac{\partial^2}{\partial\xi^2}\bigg[\big(A_{01}E+\overline{A}_{01}E^{-1}\big)\\
&\qquad+\bigg\{\frac{2}{3}k^2\big(A_{01}^2E^2+\overline{A}_{01}^2E^{-2}\big)+A_{11}E+\overline{A}_{11}E^{-1}\bigg\}\bigg], \qquad (12.8.47)
\end{aligned}
$$

where the bar denotes the complex conjugate. The coefficient of E involved in this expression must vanish (and, of course, its conjugate for E^{-1}) to satisfy the condition imposed on U_2 so that

$$
(2ikc_p)A_{01\tau}+\big(1-c_g^2\big)A_{01\zeta\zeta}+\frac{2}{3}k^4A_{01}|A_{01}|^2=0, \qquad (12.8.48)
$$

where all the other terms generate higher harmonics in U_2, which is acceptable for uniformity in the asymptotic solution as $|\xi|\to\infty$. This equation (12.8.48) is well known as the *nonlinear Schrödinger* (*NLS*) *equation*, which describes the nonlinear evolution of the wave amplitude.

12.9 Derivation of the NLS Equation and Davey–Stewartson Evolution Equations

This section deals with two derivations that lead to a description of the evolution of wavepackets for surface gravity waves on water of finite depth. We use an asymptotic method to derive first the $(1+1)$-dimensional nonlinear Schrödinger equation and then $(2+1)$-dimensional Davey–Stewartson equations.

Example 12.9.1 (Derivation of the NLS Equation from the Laplace Equation for the Velocity Potential). We recall the governing equation (9.3.3)–(9.3.6) for nonlinear water waves with no y-dependence in the form

$$
\begin{aligned}
&\phi_{zz}+\delta\phi_{xx}=0, &&(12.9.1)\\
&\phi_t+\eta+\frac{1}{2}\varepsilon\big(\phi_x^2+\delta^{-1}\phi_z^2\big)=0 \quad \text{on } z=1+\varepsilon\eta, &&(12.9.2)\\
&\phi_z=\delta(\eta_t+\varepsilon\phi_x\eta_x) \quad \text{on } z=1+\varepsilon\eta, &&(12.9.3)\\
&\phi_z=0 \quad \text{on } z=0, &&(12.9.4)
\end{aligned}
$$

where parameters δ and ε are given by (9.3.2ab).

We next introduce the new variables

$$
\xi=x-c_pt, \qquad \zeta=\varepsilon(x-c_gt), \qquad \tau=\varepsilon^2t, \qquad (12.9.5)
$$

where c_p and c_g are to be determined.

In view of the transformations (12.9.5), equations (12.9.1)–(12.9.4) reduce to

$$\phi_{zz} + \delta\big(\phi_{\xi\xi} + 2\varepsilon\phi_{\xi\zeta} + \varepsilon^2\phi_{\zeta\zeta}\big) = 0, \tag{12.9.6}$$

$$\varepsilon^2\phi_\tau + \eta - \big(\varepsilon c_g\phi_\zeta + c_p\phi_\xi\big) + \frac{\varepsilon}{2}\big[\delta^{-1}\phi_z^2 + (\phi_\xi + \varepsilon\phi_\zeta)^2\big] = 0 \quad \text{on } z = 1 + \varepsilon\eta, \tag{12.9.7}$$

$$\phi_z = \delta\big[\varepsilon^2\eta_\tau - (c_p\eta_\xi + \varepsilon c_g\eta_\zeta) + \varepsilon(\phi_\xi + \varepsilon\phi_\zeta)(\eta_\xi + \varepsilon\eta_\zeta)\big] \quad \text{on } z = 1 + \varepsilon\eta, \tag{12.9.8}$$

$$\phi_z = 0 \quad \text{on } z = 0. \tag{12.9.9}$$

We seek an asymptotic solution of this system in the form

$$\phi = \sum_{n=0}^{\infty} \varepsilon^n \phi_n(\xi, \zeta, \tau, z), \qquad \eta = \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi, \zeta, \tau) \quad \text{as } \varepsilon \to 0, \tag{12.9.10}$$

which are periodic functions in ξ.

In the leading order $O(1)$, the problem is given by

$$\phi_{0zz} + \delta\phi_{0\xi\xi} = 0, \tag{12.9.11}$$

$$\eta_0 - c_p\phi_{0\xi} = 0, \qquad \phi_{0z} = -\delta c_p\eta_{0\xi} \quad \text{on } z = 1, \tag{12.9.12}$$

$$\phi_{0z} = 0 \quad \text{on } z = 0. \tag{12.9.13}$$

We seek the solution of this leading order system as

$$\phi_0 = f_0(\zeta, \tau) + \Phi_0(z, \zeta, \tau)E + c.c., \qquad \eta_0 = A_0(\zeta, \tau)E + c.c., \tag{12.9.14}$$

where $E = \exp(ik\xi)$ and $c.c.$ denotes the complex conjugate of the terms in E. Clearly, the above solution describes a harmonic wave of wavenumber k propagating with the velocity c_p.

Evidently, the Laplace equation (12.9.11) with (12.9.14) becomes

$$\Phi_{0zz} = \delta k^2\Phi_0. \tag{12.9.15}$$

The solution of (12.9.15) with the bottom boundary condition (12.9.13) is

$$\Phi_0 = F_0(\zeta, \tau)\cosh\big(\sqrt{\delta}kz\big), \tag{12.9.16}$$

where the function $F_0 = F_0(\zeta, \tau)$ is to be determined.

The two surface boundary conditions (12.9.12) yield

$$\sqrt{\delta}kF_0 \sinh\sqrt{\delta}k = -i\delta kc_pA_0, \qquad ikc_pF_0\cosh\sqrt{\delta}k = A_0, \tag{12.9.17}$$

which give

$$c_p^2 = \frac{\tanh\sqrt{\delta}k}{\sqrt{\delta}k}, \qquad F_0 = -\left(\frac{iA_0}{kc_p}\right)\operatorname{sech}\big(\sqrt{\delta}k\big). \tag{12.9.18ab}$$

Thus, solution (12.9.16) becomes

$$\Phi_0 = -i\sqrt{\delta}c_p A_0 \frac{\cosh(\sqrt{\delta}kz)}{\sinh(\sqrt{\delta}k)}, \tag{12.9.19}$$

where the wave amplitude $A_0 = A_0(\zeta, \tau)$ is to be determined.

At the next order $O(\varepsilon)$, we proceed to collect all terms and use the surface boundary conditions on $z = 1$ to obtain

$$\phi_{1zz} + \delta(\phi_{1\xi\xi} + 2\phi_{0\xi\zeta}) = 0, \tag{12.9.20}$$

$$\eta_1 - c_g\phi_{0\zeta} - c_p(\phi_{1\xi} + \phi_{0\xi z}) + \frac{1}{2}\left(\delta^{-1}\phi_{0z}^2 + \phi_{0\xi}^2\right) = 0 \quad \text{on } z = 1, \tag{12.9.21}$$

$$\phi_{1z} + \eta_0\phi_{0zz} = \delta(\phi_{0\xi}\eta_{0\xi} - c_g\eta_{0\zeta} - c_p\eta_{1\xi}) \quad \text{on } z = 1, \tag{12.9.22}$$

$$\phi_{1z} = 0 \quad \text{on } z = 0. \tag{12.9.23}$$

It is evident that the nonlinearity of the surface boundary conditions generates terms E^0, E^2, and E^{-2} from equations (12.9.20)–(12.9.23). The nonlinear interactions produce higher harmonics E^2, E^{-2} and their complex conjugates in addition to the fundamental harmonics E^1 with E^{-1} introduced by (12.9.14). We seek a periodic solution in ξ and remove all secular terms that contribute to the nonperiodic terms to the solution. In order to carry out this program further, we employ the asymptotic procedure similar to that used in Examples 12.8.1 and 12.8.2. Thus, we write

$$\phi_m = \sum_{n=0}^{m+1} \Phi_{mn}E^n + \text{c.c.}, \qquad \eta_m = \sum_{n=0}^{m+1} A_{mn}E^n + \text{c.c.}, \tag{12.9.24}$$

where $m = 1, 2, 3, \ldots, \Phi_{mn}(z, \zeta, \tau)$ and $A_{mn}(\zeta, \tau)$ are yet to be determined, and $c.c.$ refers only to terms E^n, $n = 1, 2, 3, \ldots$. The terms corresponding to $n = 0$ are not harmonic (or oscillatory) solutions, but they are periodic. At each higher order in ε, higher-order harmonics are produced so that E^2 appears first at $O(\varepsilon)$, E^3 first at $O(\varepsilon^2)$, and so on.

The Laplace equation (12.9.20) for ϕ_1 yields

$$\Phi_{10zz} = 0, \qquad \Phi_{12zz} - 4\delta k^2\Phi_{12} = 0 \tag{12.9.25}$$

and

$$\Phi_{11zz} - \delta k^2\Phi_{11} + 2i\delta k\Phi_{0\zeta} = 0. \tag{12.9.26}$$

These equations have solutions satisfying the bottom boundary condition (12.9.23) as

$$\Phi_{10} = F_{10}(\zeta, \tau), \qquad \Phi_{12} = F_{12}(\zeta, \tau)\cosh\left(2\sqrt{\delta}kz\right), \tag{12.9.27}$$

$$\Phi_{11} = F_{11}(\zeta, \tau)\cosh\left(\sqrt{\delta}kz\right) - i\sqrt{\delta}F_{0\zeta}z\sinh\left(\sqrt{\delta}kz\right), \tag{12.9.28}$$

where $F_{1n}(\zeta, \tau)$ are arbitrary functions of ζ and τ. These solutions are then utilized in the two free surface boundary conditions (12.9.21), (12.9.22) to obtain six equations arising from coefficients of E^0, E^1 and E^2. Equation (12.9.21) gives

$$E^0 : -c_g f_{0\zeta} + i\sqrt{\delta}k^2 c_p(A_0\overline{F}_0 + \overline{A}_0 F_0)\sinh\sqrt{\delta}k + A_{10} + k^2 F_0\overline{F}_0\left(\sinh^2\sqrt{\delta}k + \cosh^2\sqrt{\delta}k\right) = 0, \tag{12.9.29}$$

$$E^1 : -c_g F_{0\zeta}\cosh\sqrt{\delta}k - ikc_p\left(F_{11}\cosh\sqrt{\delta}k - i\sqrt{\delta}F_{0\zeta}\sinh\sqrt{\delta}k\right) + A_{11} = 0, \tag{12.9.30}$$

$$E^2 : -ikc_p\left(2F_{12}\cosh 2\sqrt{\delta}k + \sqrt{\delta}kA_0F_0\sinh\sqrt{\delta}k\right) + A_{12} - \frac{1}{2}k^2F_0^2 = 0, \tag{12.9.31}$$

and similarly, equation (12.9.22) leads to

$$E^0 : (A_0\overline{F}_0 + \overline{A}_0F_0)\delta k^2\cosh\sqrt{\delta}k = \delta k^2(A_0\overline{F}_0 + \overline{A}_0F_0)\cosh\sqrt{\delta}k, \tag{12.9.32}$$

$$E^1 : \sqrt{\delta}kF_{11}\sinh\sqrt{\delta}k - i\sqrt{\delta}F_{0\zeta}\left(\sinh\sqrt{\delta}k + \sqrt{\delta}k\cosh\sqrt{\delta}k\right) = -\delta(c_gA_{0\zeta} + ikc_pA_{11}), \tag{12.9.33}$$

$$E^2 : 2\sqrt{\delta}kF_{12}\sinh 2\sqrt{\delta}k + \delta k^2A_0F_0\cosh\sqrt{\delta}k = -\delta\left(2ikc_pA_{12} + k^2A_0F_0\cosh\sqrt{\delta}k\right), \tag{12.9.34}$$

where the bar denotes the complex conjugate.

It is noted that (12.9.32) is identically satisfied and that with result (12.9.18ab) and for Φ_0 used in equation (12.9.29), we find

$$A_{10} = -\frac{2\sqrt{\delta}k}{\sinh 2\sqrt{\delta}k}A_0\overline{A}_0 + c_g f_{0\zeta}. \tag{12.9.35}$$

Equation (12.9.30) leads us directly to obtain

$$A_{11} = c_gF_{0\zeta}\cosh\sqrt{\delta}k + ikc_p\left(F_{11}\cosh\sqrt{\delta}k - i\sqrt{\delta}F_{0\zeta}\sinh\sqrt{\delta}k\right). \tag{12.9.36}$$

If this result (12.9.36) is utilized in equation (12.9.33), we see that F_{11} cancels identically. If we next use the result for c_p in (12.9.18a) first and then F_0 from (12.9.18b) in (12.9.33), it turns out that $A_{0\zeta}$ also cancels and the result becomes

$$c_g = \frac{1}{2}c_p\left(1 + 2\sqrt{\delta}k\operatorname{cosech}2\sqrt{\delta}k\right). \tag{12.9.37}$$

This is the group velocity for water waves.

Finally, we use (12.9.18ab) to solve (12.9.31) and (12.9.34) for F_{12} and A_{12} so that

$$F_{12} = -\left(\frac{3i}{4}\right)\frac{\delta kc_pA_0^2}{\sinh^4(\sqrt{\delta}k)}, \quad A_{12} = \frac{\sqrt{\delta}k\cosh(\sqrt{\delta}k)}{2\sinh^3(\sqrt{\delta}k)}\left(2\cosh^2\sqrt{\delta}k + 1\right)A_0^2, \tag{12.9.38}$$

where $A_0 = A_0(\zeta, \tau)$ is to be determined.

In the next order $O(\varepsilon^2)$, equation (12.9.6) and the expansion of the free surface boundary conditions (12.9.7), (12.9.8) give

$$\phi_{2zz} + \delta(\phi_{2\xi\xi} + 2\phi_{1\xi\zeta} + \phi_{2\zeta\zeta}) = 0, \tag{12.9.39}$$

$$\begin{aligned} &\phi_{0\tau} - c_g\eta_0\phi_{0\zeta z} - c_p\left(\phi_{1\zeta} + \phi_{2\xi} + \eta_0\phi_{1\xi z} + \eta_1\phi_{0\xi z} + \frac{1}{2}\eta_0^2\phi_{0\xi zz}\right) + \eta_2 \\ &\quad + \delta^{-1}(\eta_0\phi_{0zz} + \phi_{1z})\phi_{0z} + (\eta_0\phi_{0\xi z} + \phi_{1\xi} + \phi_{0\zeta})\phi_{0\xi} = 0 \\ &\quad \text{on } z = 1, \end{aligned} \tag{12.9.40}$$

$$\begin{aligned} &\phi_{2z} + \eta_0\phi_{1zz} + \frac{1}{2}\eta_0^2\phi_{0zzz} + \eta_1\phi_{0zz} - \delta^{-1}\big[\eta_{0\tau} - (c_g\eta_{1\zeta} + c_p\eta_{2\xi}) \\ &\quad + (\eta_{1\xi} + \eta_{0\zeta})\phi_{0\xi} + (\eta_0\phi_{0\xi z} + \phi_{1\xi} + \phi_{0\zeta})\eta_{0\xi}\big] = 0 \quad \text{on } z = 1, \end{aligned} \tag{12.9.41}$$

and

$$\phi_{2z} = 0 \quad \text{on } z = 0. \tag{12.9.42}$$

It is important to point out that the periodic solution adopted in (12.9.24) is now utilized for $m = 2$ so that the higher harmonics E^3 now appears for the first time in this analysis. Consequently, equation (12.9.39) then yields

$$\Phi_{20zz} + \delta f_{0\zeta\zeta} = 0, \qquad \Phi_{21zz} - \delta k^2\Phi_{21} + \delta(2ik\Phi_{11\zeta} + \Phi_{0\zeta\zeta}) = 0, \tag{12.9.43}$$

and so on. The solution for $\Phi_{21}(z,\zeta,\tau)$ satisfying the bottom boundary condition (12.9.42) then becomes

$$\begin{aligned} \Phi_{21} &= F_{21}\cosh\sqrt{\delta}kz - \sqrt{\delta}\left(iF_{11\zeta} + \frac{1}{2k}F_{0\zeta\zeta}\right)z\sinh\sqrt{\delta}kz \\ &\quad + \frac{1}{2}\delta F_{0\zeta\zeta}\left(\frac{z}{k\sqrt{\delta}}\sinh\sqrt{\delta}kz - z^2\cosh\sqrt{\delta}kz\right). \end{aligned} \tag{12.9.44}$$

The free surface boundary condition (12.9.41) reduces to, for terms E^1,

$$\begin{aligned} &\Phi_{21z} + \overline{A}_0\Phi_{12z} + \frac{1}{2}\big(A_0^2\overline{\Phi}_{0zzz} + 2A_0\overline{A}_0\Phi_{0zzz}\big) + A_{10}\Phi_{0zz} + A_{12}\overline{\Phi}_{0zz} \\ &\quad = \delta\big[A_{0\tau} - (c_gA_{11\zeta} + ikc_pA_{21}) + 2k^2A_{12}\overline{\Phi}_{0zz} - k^2A_0\big(\overline{A}_0\Phi_{0z} - A_0\overline{\Phi}_{0z}\big) \\ &\qquad + k^2\overline{A}_0(A_0\Phi_{0z} + 2\Phi_{12})\big] \quad \text{on } z = 1. \end{aligned} \tag{12.9.45}$$

The free surface boundary condition (12.9.40) on $z = 1$ gives, for terms E^1,

$$\begin{aligned} &\Phi_{0\tau} - c_g\Phi_{11\zeta} - ik\big\{\big(c_p\Phi_{21} + 2c_p\overline{A}_0\Phi_{12z}\big) + c_p(A_{10}\Phi_{0z} - A_{12}\Phi_{1z})\big\} \\ &\quad - \frac{1}{2}ikc_p\big(2A_0\overline{A}_0\Phi_{0zz} - A_0^2\,\overline{\Phi}_{0zz}\big) + A_{21} \\ &\quad + \delta^{-1}\big[\big(A_0\overline{\Phi}_{0zz} + \overline{A}_0\Phi_{0zz}\big)\Phi_{0z} + (A_0\Phi_{0zz} + \Phi_{12z})\overline{\Phi}_{0z}\big] \\ &\quad - k^2\big(\overline{A}_0\Phi_{0z} - A_0\overline{\Phi}_{0z}\big)\Phi_0 + k^2(A_0\Phi_{0z} + 2\Phi_{12})\overline{\Phi}_0 = 0. \end{aligned} \tag{12.9.46}$$

Clearly, the equation for A_0 arises from the terms which occur after elimination of secular terms at E^1. The procedure is relatively simple to describe, but rather tedious to discuss. However, we can eliminate A_{12} from (12.9.45) and (12.9.46), and then introduce the functions obtained before including Φ_{21} from (12.9.44). Simplifying the result so that F_{21} cancels identically due to the definition of c_p, and A_{11} (or F_{11}) also cancels after substitution of c_g from (12.9.37), the final equation for $A_0(\zeta, \tau)$ can be obtained in the form

$$-2ikc_p A_{0\tau} + \alpha A_{0\zeta\zeta} + \beta|A_0|^2 A_0 = 0, \tag{12.9.47}$$

where

$$\alpha = c_g^2 - (1 - s\tanh s)\text{sech}^2 s = -(kc_p)\frac{d^2}{dk^2}(kc_p), \qquad \omega(k) = kc_p, \tag{12.9.48}$$

and

$$\beta = \left(k^2 c_p^{-2}\right)\left[\frac{1}{2}\left(1 + 9\coth^2 s - 13\,\text{sech}^2 s - 2\tanh^4 s\right) - \left(1 - c_g^2\right)^{-1}\left(2c_p + c_g\,\text{sech}^2 s\right)^2\right], \tag{12.9.49}$$

with $s = \sqrt{\delta}k$.

Equation (12.9.47) is well known as the *nonlinear Schrödinger* (*NLS*) *equation* for the evolution of nonlinear water waves, and it is one of the completely integrable types of equations. It is easy to check that $\alpha > 0$ for all $s = k\sqrt{\delta}$, but β changes its sign from positive to negative as s decreases its value across $s = k\sqrt{\delta} \approx 1.363$. It is important to point out that the nature of the solutions of (12.9.47) depend on the signs of α and β. Indeed, the condition $\beta\alpha^{-1} > 0$ $(\alpha\beta > 0)$ ensures the existence of a solitary wave solution of the NLS equation (12.9.47), and the soliton decays to zero at infinity. There exists some experimental and numerical evidence in support of the solitary wave solution. On the other hand, when

$$\frac{\beta}{\alpha} < 0 \quad (\alpha\beta < 0),$$

no such solitary wave solution exists. In other words, the solution grows exponentially as $\tau \to \infty$, that is, the solution becomes unstable.

Example 12.9.2 (Derivation of the Davey–Stewartson (DS) Equations). Example 12.9.1 deals with the NLS equation which describes the modulation of the amplitude of the nonlinear water waves propagating only in one direction. We closely follow Davey and Stewartson (1974) to consider the two-dimensional problem of the propagation of a plane wave. This two-dimensional problem incorporates slow (or weak) dependence on both x- and y-coordinates with the fast oscillations only in the x-direction. We present the evolution of wavepackets that will propagate in the x-direction with slowly varying structure in both x- and y-directions. However, the group velocity is still associated with the wave propagation in the x-direction.

We recall governing equations (9.3.3)–(9.3.6) and then introduce the variables

$$\xi = x - c_p t, \qquad \zeta = \varepsilon(x - c_g t), \qquad y^* = \varepsilon y, \qquad \tau = \varepsilon^2 t. \tag{12.9.50}$$

Consequently, equations (9.3.3)–(9.3.6) become, dropping the asterisk in y^*,

$$\phi_{zz} + \delta\big[\phi_{\xi\xi} + 2\varepsilon\phi_{\xi\zeta} + \varepsilon(\phi_{\zeta\zeta} + \phi_{yy})\big] = 0, \tag{12.9.51}$$

$$\varepsilon^2\phi_\tau - \varepsilon c_g\phi_\zeta - c_p\phi_\xi + \eta + \frac{1}{2}\varepsilon\left[\frac{1}{\delta}\phi_z^2 + (\phi_\xi + \varepsilon\phi_\zeta)^2 + \varepsilon\phi_y^2\right] = 0$$
$$\text{on } z = 1 + \varepsilon\eta, \tag{12.9.52}$$

$$\phi_z = \delta\big[\varepsilon^2\eta_\tau - (\varepsilon c_g\eta_\zeta + c_p\eta_\xi) + \varepsilon(\phi_\xi + \varepsilon\phi_\zeta)(\eta_\xi + \varepsilon\eta_\zeta) + \varepsilon^3\phi_y\eta_y\big]$$
$$\text{on } z = 1 + \varepsilon\eta, \tag{12.9.53}$$

and

$$\phi_z = 0 \quad \text{on } z = 0. \tag{12.9.54}$$

Proceeding to collect all terms not larger than $O(\varepsilon^2)$, the only contribution from the y-dependence will come from the term ϕ_{yy} in the Laplace equation (12.9.51). All other terms involving derivatives of y produce new nonlinear interactions that will arise first at $O(\varepsilon^3)$. We can use the calculation similar to that already presented in Example 12.9.1 for the nonlinear Schrödinger equation. Thus, we avoid the details here and present only major steps to derive a pair of Davey–Stewartson equations.

We seek asymptotic solutions in the form

$$\phi = f_0(\zeta, y, \tau) + \sum_{m=0}^{\infty}\varepsilon^m\left[\sum_{n=0}^{m+1}\Phi_{mn}(z, \zeta, y, \tau)E^n + c.c.\right], \tag{12.9.55}$$

$$\eta = \sum_{m=0}^{\infty}\varepsilon^m\left[\sum_{n=0}^{m+1}A_{mn}(\zeta, y, \tau)E^n + c.c.\right], \tag{12.9.56}$$

where $E = \exp(ik\xi)$ and $A_{00} = 0$ so that the first-order approximation to the surface gravity waves is purely harmonic. We expect results already obtained for all the terms at $O(\varepsilon^0)$ and $O(\varepsilon)$. The main differences will appear at $O(\varepsilon^2)$, and the problem at ε^2E^0 yields an equation for f_0,

$$\big(1 - c_g^2\big)f_{0\zeta\zeta} + f_{0yy} = -\frac{1}{c_p^2}\big(2c_p + c_g\operatorname{sech}^2\sqrt{\delta}k\big)\big(|A_0|^2\big)_\zeta, \tag{12.9.57}$$

where $A_0 = A_{01}$ is given.

The free surface boundary conditions for the terms ε^2E give

$$\begin{aligned}&-2ikc_pA_{0\tau} + \alpha A_{0\zeta\zeta} - c_pc_gA_{0yy}\\&\quad+\frac{1}{2}\big(k^2c_p^{-2}\big)\big(1 + 9\coth^2 s - 13\operatorname{sech}^2 s - 2\tanh^4 s\big)A_0|A_0|^2\\&\quad+k^2\big(2c_p + c_g\operatorname{sech}^2 s\big)A_0f_{0\zeta} = 0,\end{aligned} \tag{12.9.58}$$

where $s = \sqrt{\delta}k$.

These two equations (12.9.57), (12.9.58) are called the *Davey–Stewartson* (*DS*) *equations* for the modulation of two-dimensional harmonic waves.

Using α, β given by (12.9.48) and (12.9.49) and introducing a new parameter γ by

$$\gamma = 2c_p + c_g \operatorname{sech}^2 s, \tag{12.9.59}$$

the Davey–Stewartson equations can be written in a compact form

$$(1 - c_g^2) f_{0\zeta\zeta} + f_{0yy} = -\left(\frac{\gamma}{c_p^2}\right)\left(|A_0|^2\right)_\zeta, \tag{12.9.60}$$

$$- 2ikc_p A_{0\tau} + \alpha A_{0\zeta\zeta} - c_p c_g A_{0yy} + \left[\beta + \frac{\gamma^2 k^2}{c_p^2(1 - c_g^2)}\right] A_0|A_0|^2 + \gamma k^2 A_0 f_{0\zeta} = 0. \tag{12.9.61}$$

It is noted here that $\gamma > 0$ and $c_p c_g > 0$. For no y-dependence, equation (12.9.60) becomes after integration

$$(1 - c_g^2) f_{0\zeta} = -\left(\frac{\gamma}{c_p^2}\right)|A_0|^2. \tag{12.9.62}$$

This equation (12.9.62) describes the mean drift generated by the nonlinear interaction of the wave motion. This is usually called the *Stokes drift*.

In the absence of y-dependence with the assumption that $f_{0\zeta} = 0$, and hence, $A_0 \equiv 0$, equation (12.9.61) reduces to the NLS equation

$$-2ikc_p A_{0\tau} + \alpha A_{0\zeta\zeta} + \beta|A_0|^2 A_0 = 0. \tag{12.9.63}$$

This is identical with the NLS equation (12.9.47).

It has been an interesting exercise to present several conservation laws for the nonlinear Schrödinger equation in Section 10.5. However, the derivation of the conservation laws for the Davey–Stewartson equations is not easy because these equation are coupled and depend on the three variables τ, ζ, and y. In order to derive a few conservation laws, it is convenient to rewrite the DS equations (12.9.60), (12.9.61) by replacing τ by t, ζ by x, f_0 by f, and A_0 by A so that they read as

$$a f_{xx} + f_{yy} - b\left(|A|^2\right)_x = 0, \tag{12.9.64}$$

$$-icA_t + dA_{xx} - eA_{yy} + hA|A|^2 + Af_x = 0, \tag{12.9.65}$$

where a, b, c, d, e, and h are real constants and f is a real function.

Equation (12.9.64) is already in the conservation form

$$\frac{\partial}{\partial x}\left(af_x - b|A|^2\right) + \frac{\partial}{\partial y}(f_y) = 0, \tag{12.9.66}$$

which gives

$$\frac{\partial}{\partial x}\left[\int_{-\infty}^{\infty}\left(af_x - b|A|^2\right) dy\right] + [f_y]_{-\infty}^{\infty} = 0. \tag{12.9.67}$$

Under the decay conditions as $|x| \to \infty$, this equation takes the form

$$a\frac{\partial}{\partial x}\left(\int_{-\infty}^{\infty} f\,dy\right) = b\int_{-\infty}^{\infty} |A|^2\,dy = g(t), \tag{12.9.68}$$

where $g(t)$ is an arbitrary function of t. If, for some appropriate value of x, the left-hand side of (12.9.68) vanishes, then $g(t) = 0$ for all t. Consequently, (12.9.68) gives

$$a\left(\int_{-\infty}^{\infty} f dy\right) = b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |A|^2\,dx\,dy = \text{const.} \tag{12.9.69}$$

This leads to another conservation law

$$\int_{-\infty}^{\infty} f\,dy = \text{const.} \tag{12.9.70}$$

We next take the complex conjugate of (12.9.65) to obtain

$$ic\overline{A}_t + d\overline{A}_{xx} - e\overline{A}_{yy} + h\overline{A}|A|^2 + \overline{A}f_x = 0. \tag{12.9.71}$$

We next multiply (12.9.71) by A and (12.9.65) by $\overline{A}$ and then subtract the latter result from the former result to find

$$ic\frac{\partial}{\partial x}(A\overline{A}) + d\frac{\partial}{\partial x}(A\overline{A}_x - \overline{A}A_x) + e\frac{\partial}{\partial y}(\overline{A}A_y - A\overline{A}_y) = 0. \tag{12.9.72}$$

This leads to the following results in the conservation form:

$$ic\frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty} |A|^2\,dx\right] + e\frac{\partial}{\partial y}\left[\int_{-\infty}^{\infty} (\overline{A}A_y - A\overline{A}_y)\,dx\right] = 0 \tag{12.9.73}$$

and

$$ic\frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty} |A|^2\,dy\right] + d\frac{d}{dx}\left[\int_{-\infty}^{\infty} (A\overline{A}_x - \overline{A}A_x)\,dy\right] = 0, \tag{12.9.74}$$

provided decay conditions are satisfied as $|x| \to \infty$ for fixed y, and as $|y| \to \infty$ for fixed x. This means that waves at infinity are not parallel to either the x- or y-directions. Moreover, if decay conditions hold for $|x| \to \infty$ and $|y| \to \infty$, that is, the solution decays sufficiently rapidly as $(x^2 + y^2) \to \infty$, it turns out that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |A|^2\,dx\,dy = \text{const.} \tag{12.9.75}$$

This represents a conserved quantity that arises for a certain class of solutions. This is usually referred to as the law of *conservation of mass*.

Finally, we close this section by representing one of the most direct applications of the NLS and DS equations. This application deals with the stability of the Stokes waves. We recall the NLS equation (12.9.47) and rewrite it in the form

$$-2ikc_p A_{0\tau} + \alpha A_{0\zeta\zeta} + \beta A_0 |A_0|^2 = 0, \tag{12.9.76}$$

where α and β are given by (12.9.48) and (12.9.49).

We seek a nonlinear plane wave solution of (12.9.76) in the form

$$A_0(\zeta, \tau) = A \exp\big[i(m\zeta - \sigma\tau)\big], \tag{12.9.77}$$

where the amplitude A is a complex constant and the wavenumber m is real. This solution exists provided σ satisfies the dispersion relation

$$2kc_p\sigma = \beta|A|^2 - \alpha m^2, \tag{12.9.78}$$

where c_p, α, and β are functions of the wavenumber $k(>0)$ of the primary wave described by (12.9.14) and $m(>0)$ is the wavenumber of the modulation. Thus, the primary wave solution becomes

$$\eta_0 = A \exp\big\{i(kx - \omega t + m\zeta - \sigma\tau)\big\} + c.c., \tag{12.9.79}$$

where $\tau = \varepsilon^2 t$ and σ is given by (12.9.78) with $m = 0$.

The solution (12.9.79) represents the Stokes wave of constant amplitude A and wavenumber k with the dispersion relation

$$2kc_p\sigma = \beta|A|^2. \tag{12.9.80}$$

Thus, the Stokes wave assumes the form

$$\eta_0(x,t) = A \exp\big[i\big\{kx - \big(\omega + \varepsilon^2\sigma\big)t\big\}\big]. \tag{12.9.81}$$

We next employ the NLS equation to examine the stability of the Stokes wave. The NLS equation describes the modulation of the amplitude of the harmonic wave represented by $\exp(ikx)$. We seek a solution which is a small perturbation of the nonlinear plane wave solution (12.9.81) so that

$$A_0 = A(1 + \mu a) \exp\big[i(-\sigma\tau + \mu\theta)\big], \tag{12.9.82}$$

where $a = a(\zeta, \tau)$ and $\theta = \theta(\zeta, \tau)$ are real perturbation functions and μ is a parameter with the dispersion relation (12.9.80).

Substituting (12.9.82) into the original NLS equation (12.9.76) gives, dropping the exponential term in (12.9.82),

$$\begin{aligned}
&-2ikc_p\big[\mu a_\tau + i(1+\mu a)(\mu\theta_\tau - \sigma)\big] \\
&\quad + \alpha\big[\mu a_{\zeta\zeta} + 2i\mu^2 a_\zeta\theta_\zeta + \mu(1+\mu a)\big(i\theta_{\zeta\zeta} - \mu\theta_\zeta^2\big)\big] \\
&\quad + \beta(1+\mu a)^3|A|^2 = 0.
\end{aligned} \tag{12.9.83}$$

The leading terms $O(1)$ as $\mu \to 0$ cancel due to (12.9.80), and hence, the main perturbation terms $O(\mu)$ are given by

$$-2ikc_p\big[a_\tau + i(\theta_\tau - \sigma)\big] + \alpha(a_{\zeta\zeta} + i\theta_{\zeta\zeta}) + 3\beta|A|^2 a = 0. \tag{12.9.84}$$

In view of the fact that a and θ are real functions, we use (12.9.80) to eliminate σ from (12.9.84) so that

$$2kc_p\theta_\tau + \alpha a_{\zeta\zeta} + 2\beta|A|^2 a = 0, \tag{12.9.85}$$

$$-2kc_p a_\tau + \alpha\theta_{\zeta\zeta} = 0. \tag{12.9.86}$$

This is a pair of linear partial differential equations with constant coefficients. We seek a solution of the form

$$\begin{pmatrix} a \\ \theta \end{pmatrix} = \begin{pmatrix} a_0 \\ \theta_0 \end{pmatrix} \exp\bigl[i(\kappa\zeta - \Omega\tau)\bigr] + c.c., \tag{12.9.87}$$

where a_0, θ_0, $\kappa(> 0)$, and Ω are constants. This solution exists, provided the dispersion relation for Ω

$$(2kc_p\Omega)^2 = \alpha^2\kappa^2\left[\kappa^2 - 2\left(\frac{\beta}{\alpha}\right)|A|^2\right] \tag{12.9.88}$$

is satisfied. Or equivalently,

$$(2kc_p\Omega) = (\alpha\kappa)\left[\kappa^2 - 2\left(\frac{\beta}{\alpha}\right)|A|^2\right]^{\frac{1}{2}}. \tag{12.9.89}$$

Some new and striking conclusions can be drawn from (12.9.89). If $\beta\alpha^{-1} > 0$ $(\alpha\beta > 0)$, then Ω is imaginary for some wavenumber κ, provided

$$0 < \kappa < \sqrt{2}|A|\sqrt{\frac{\beta}{\alpha}}. \tag{12.9.90}$$

In this case, the amplitude modulation grows exponentially as $\tau \to \infty$. This means that the Stokes wave is definitely unstable for a range of wavenumbers κ given by (12.9.90). On the other hand, if $\beta\alpha^{-1} < 0$ $(\alpha\beta < 0)$, Ω is real for all values of the wavenumber κ. This implies that the amplitude modulation persists, but does not grow. In other words, the Stokes wave is neutrally stable for all values of κ.

Another most simple and direct interpretation of the instability phenomenon of the Stokes wave can be given by writing the leading order fundamental wave mode at $t = 0$ as

$$\eta_0 = A(1 + \mu a)\exp\bigl[i(kx + \mu\theta)\bigr] + c.c., \tag{12.9.91}$$

which is, by using (12.9.87) with $\zeta = \varepsilon(x - c_g t)$ at $t = 0$,

$$= A\exp\bigl[i(kx + \mu\theta)\bigr] + A\mu a_0 \exp\bigl[i(k + \varepsilon\kappa)x + i\mu\theta\bigr] + c.c. \tag{12.9.92}$$

as $\mu \to 0$ for fixed x and ε.

Evidently, the perturbation to the fundamental mode has the term μa_0 which has a wavelength $k + \varepsilon\kappa$ that is close to k. This means that a perturbation with a wavenumber close to that of the fundamental mode will produce an unstable solution whenever $\beta\alpha^{-1} > 0$. Thus, it is impossible to generate stable waves with a precisely fixed wavenumber both in nature and in the laboratory. In other words, a wave with a small deviation from a fixed wavenumber k would always occur and lead to the instability phenomenon. This kind of instability is associated with a small change in the wavenumber of the fundamental mode and is now well known as the *Benjamin–Feir* (1967) *side-band instability*. It is often observed in nature that a set of plane waves gradually breaks down along the wave fronts into a series of wave groups.

12.10 Exercises

1. The simplest equation for compressible fluid flows is a 2×2 matrix of the continuity and momentum equations in two independent variables x and t, that is,

$$U_t + F_x = S, \quad \text{or equivalently,} \quad U_t + AU_x = S,$$

where

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 \\ S_2 \end{bmatrix}.$$

(a) Find the matrix A and its eigenvalues for the inviscid Burgers equation. Show that the conservation laws are not hyperbolic.
(b) For compressible flows of velocity u, define the density u_1 and the momentum $u_2 = uu_1$, $F_1 = uu_1 = u_2$, $F_2 = uu_2 + P = u_2^2/u_1 + P$, where $P = P(u_1)$ is the pressure, write the conservation equation in the matrix form

$$U_t + AU_x = S,$$

and find the matrix A, its eigenvalues, eigenvectors, and the Riemann invariants.
(c) Write the conservation matrix equation for the $n \times n$ matrix A.

13

Tables of Integral Transforms

In this chapter, we provide a set of *short* tables of integral transforms of the functions that are either cited in the text or are in most common use in mathematical, physical, and engineering applications. For exhaustive lists of integral transforms, the reader is referred to Erdélyi et al. (1954), Campbell and Foster (1948), Ditkin and Prudnikov (1965), Doetsch (1970), Marichev (1983), Debnath (1995), Debnath and Bhatta (2007), and Oberhettinger (1972).

13.1 Fourier Transforms

	$f(x)$	$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ikx)f(x)\,dx$
1	$f(ax+b)$	$\frac{1}{\lvert a\rvert}\exp(\frac{ibk}{a})F(\frac{k}{a})$
2	$e^{ibx}f(ax)$	$\frac{1}{\lvert a\rvert}F(\frac{k-b}{a})$
3	$f^{(n)}(x)$	$(ik)^n F(k)$
4	$x^n f(x)$	$i^n F^{(n)}(k)$
5	$\overline{f(-x)}$	$\overline{F(k)}$
6	$\overline{f(x)}$	$\overline{F(-k)}$
7	$f(x)*g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-\xi)g(\xi)\,d\xi$	$F(k)G(k)$

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	$f(x)$	$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ikx) f(x)\, dx$
8	If $f_t(x) = \frac{1}{\sqrt{4\pi\kappa t}} \exp(-\frac{x^2}{4\kappa t})$, $f_t(x) * f_s(x)$	$F_{s+t}(k)$
9	$f(x)g(x)$	$F(k) * G(k)$
10	$f(x) \cos ax$	$\frac{1}{2}[F(k-a) + F(k+a)]$
11	$f(x) \sin ax$	$\frac{1}{2i}[F(k-a) - F(k+a)]$
12	$F(x)$	$f(-k)$
13	$x^n\, e^{iax}$	$\frac{1}{\sqrt{2\pi}} i^n \delta^{(n)}(k-a)$
14	$\chi_{[-a,a]}(x) = H(a - \|x\|)$	$\sqrt{\frac{2}{\pi}}(\frac{\sin ak}{k})$
15	$(1 - \frac{\|x\|}{a})H(1 - \frac{\|x\|}{a})$	$\frac{a}{\sqrt{2\pi}}(\frac{ak}{2})^{-2} \sin^2(\frac{ak}{2})$
16	$\exp(-a\|x\|), \quad a > 0$	$(\sqrt{\frac{2}{\pi}})a(a^2 + k^2)^{-1}$
17	$x \exp(-a\|x\|), \quad a > 0$	$(\sqrt{\frac{2}{\pi}})(-2aik)(a^2 + k^2)^{-2}$
18	$\exp(-ax^2), \quad a > 0$	$\frac{1}{\sqrt{2a}} \exp(-\frac{k^2}{4a})$
19	$(x^2 + a^2)^{-1}, \quad a > 0$	$\sqrt{\frac{\pi}{2}} \frac{\exp(-a\|k\|)}{a}$
20	$x(x^2 + a^2)^{-1}$	$\sqrt{\frac{\pi}{2}}(\frac{-ik}{2a}) \exp(-a\|k\|)$
21	$\begin{cases} c & \text{if } a \le x \le b, \\ 0 & \text{otherwise} \end{cases}$	$\frac{ic}{\sqrt{2\pi}} \frac{1}{k}(e^{-ibk} - e^{-iak})$
22	$\|x\| \exp(-a\|x\|), \quad a > 0$	$\sqrt{\frac{2}{\pi}}(a^2 - k^2)(a^2 + k^2)^{-2}$
23	$\frac{\sin ax}{x}$	$\sqrt{\frac{\pi}{2}} H(a - \|k\|)$
24	$\exp\{-x(a - i\omega)\}H(x)$	$\frac{1}{\sqrt{2\pi}} \frac{i}{(\omega - k + ia)}$
25	$(a^2 - x^2)^{-\frac{1}{2}} H(a - \|x\|)$	$\sqrt{\frac{\pi}{2}} J_0(ak)$
26	$\frac{\sin[b(x^2+a^2)^{\frac{1}{2}}]}{(x^2+a^2)^{\frac{1}{2}}}$	$\sqrt{\frac{\pi}{2}} J_0(a\sqrt{b^2 - k^2}) H(b - \|k\|)$

	$f(x)$	$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ikx) f(x)\, dx$
27	$\frac{\cos(b\sqrt{a^2-x^2})}{(a^2-x^2)^{\frac{1}{2}}} H(a-\lvert x\rvert)$	$\sqrt{\frac{\pi}{2}} J_0(a\sqrt{b^2+k^2})$
28	$e^{-ax} H(x), \quad a > 0$	$\frac{1}{\sqrt{2\pi}} (a - ik)(a^2+k^2)^{-1}$
29	$\frac{1}{\sqrt{\lvert x\rvert}} \exp(-a\lvert x\rvert), \quad a > 0$	$(a^2+k^2)^{-\frac{1}{2}} [a + (a^2+k^2)^{\frac{1}{2}}]^{\frac{1}{2}}$
30	$\delta^{(n)}(x-a), \quad n = 0, 1, 2, \ldots$	$\frac{1}{\sqrt{2\pi}} (ik)^n \exp(-iak)$
31	$\exp(iax)$	$\sqrt{2\pi} \quad \delta(k-a)$
32	$x^n \operatorname{sgn} x$	$\sqrt{\frac{2}{\pi}} \frac{n!}{(ik)^{n+1}}$
33	1	$\sqrt{2\pi}\delta(k)$
34	x	$\sqrt{2\pi} i \delta'(k)$
35	x^n	$\sqrt{2\pi} i^n \delta^{(n)}(k)$
36	x^{-1}	$-i\sqrt{\frac{\pi}{2}} \operatorname{sgn} k$
37	x^{-n}	$-i\sqrt{\frac{\pi}{2}} [\frac{(-ik)^{n-1}}{(n-1)!} \operatorname{sgn} k]$
38	$\cos(ax^2), \quad a > 0$	$\frac{1}{2\sqrt{a}} [\cos(\frac{k^2}{4a}) + \sin(\frac{k^2}{4a})]$
39	$\sin(ax^2), \quad a > 0$	$\frac{1}{2\sqrt{a}} [\cos(\frac{k^2}{4a}) - \sin(\frac{k^2}{4a})]$

13.2 Fourier Sine Transforms

	$f(x)$	$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin(kx) f(x)\, dx$
1	$\exp(-ax), \quad a > 0$	$\sqrt{\frac{2}{\pi}} k(a^2+k^2)^{-1}$
2	$x \exp(-ax), \quad a > 0$	$\sqrt{\frac{2}{\pi}} (2ak)(a^2+k^2)^{-2}$
3	$x^{\alpha-1}, \quad 0 < \alpha < 1$	$\sqrt{\frac{2}{\pi}} k^{-\alpha} \Gamma(\alpha) \sin(\frac{\pi\alpha}{2})$
4	$\frac{1}{\sqrt{x}}$	$\frac{1}{\sqrt{k}}, \quad k > 0$

	$f(x)$	$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(kx) f(x)\, dx$
5	$x^{\alpha-1} e^{-ax}, \quad \alpha > -1, a > 0$	$\sqrt{\frac{2}{\pi}} \Gamma(\alpha) r^{-\alpha} \sin(\alpha\theta)$, where $r = (a^2 + k^2)^{\frac{1}{2}}, \ \theta = \tan^{-1}(\frac{k}{a})$
6	$x^{-1} e^{-ax}, \quad a > 0$	$\sqrt{\frac{2}{\pi}} \tan^{-1}(\frac{k}{a}), \quad k > 0$
7	$x \exp(-a^2 x^2)$	$2^{-3/2}(\frac{k}{a^3}) \exp(-\frac{k^2}{4a^2})$
8	$\text{erfc}(ax)$	$\sqrt{\frac{2}{\pi}} \frac{1}{k}[1 - \exp(-\frac{k^2}{4a^2})]$
9	$x(a^2 + x^2)^{-1}$	$\sqrt{\frac{\pi}{2}} \exp(-ak), \quad a > 0$
10	$x(a^2 + x^2)^{-2}$	$\sqrt{\frac{\pi}{2}} (\frac{k}{2a}) \exp(-ak), \quad (a > 0)$
11	$H(a - x), \quad a > 0$	$\sqrt{\frac{2}{\pi}} \frac{1}{k}(1 - \cos ak)$
12	$x^{-1} J_0(ax)$	$\begin{cases} \sqrt{\frac{2}{\pi}} \sin^{-1}(\frac{k}{a}) & \text{if } 0 < k < a, \\ \sqrt{\frac{\pi}{2}} & \text{if } a < k < \infty \end{cases}$
13	$x(a^2 + x^2)^{-1} J_0(bx),$ $a > 0, \ b > 0$	$\sqrt{\frac{\pi}{2}}\, e^{-ak} I_0(ab), \quad a < k < \infty$
14	$J_0(a\sqrt{x}), \quad a > 0$	$\sqrt{\frac{2}{\pi}} \frac{1}{k} \cos(\frac{a^2}{4k})$
15	$(x^2 - a^2)^{\nu - \frac{1}{2}} H(x - a),$ $\lvert\nu\rvert < \frac{1}{2}$	$2^{\nu - \frac{1}{2}} (\frac{a}{k})^\nu \Gamma(\nu + \frac{1}{2}) J_{-\nu}(ak)$
16	$x^{1-\nu}(x^2 + a^2)^{-1} J_\nu(ax),$ $\nu - \frac{3}{2}, \ a, b > 0$	$\sqrt{\frac{\pi}{2}} a^{-\nu} \exp(-ak) I_\nu(ab),$ $a < k < \infty$
17	$x^{-\nu} J_{\nu+1}(ax), \quad \nu > -\frac{1}{2}$	$\frac{k(a^2 - k^2)^{\nu - \frac{1}{2}}}{2^{\nu - \frac{1}{2}} a^{\nu+1} \Gamma(\nu + \frac{1}{2})} H(a - k)$
18	$\text{erfc}(ax)$	$\sqrt{\frac{2}{\pi}} \frac{1}{k}[1 - \exp(-\frac{k^2}{4a^2})]$
19	$x^{-\alpha}, \quad 0 < \alpha < 2$	$\Gamma(1 - \alpha) k^{\alpha-1} \cos(\frac{\alpha\pi}{2})$
20	$(ax - x^2)^{\alpha - \frac{1}{2}} H(a - x),$ $\alpha > -\frac{1}{2}$	$\sqrt{2}\Gamma(\alpha + \frac{1}{2})(\frac{a}{k})^\alpha \sin(\frac{ak}{2}) J_\alpha(\frac{ak}{2})$

13.3 Fourier Cosine Transforms

	$f(x)$	$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(kx) f(x)\, dx$
1	$\exp(-ax), \quad a > 0$	$(\sqrt{\frac{2}{\pi}}) a (a^2 + k^2)^{-1}$
2	$x \exp(-ax), \quad a > 0$	$(\sqrt{\frac{2}{\pi}})(a^2 - k^2)(a^2 + k^2)^{-2}$
3	$\exp(-a^2 x^2)$	$\frac{1}{a\sqrt{2}} \exp(-\frac{k^2}{4a^2})$
4	$H(a - x)$	$\sqrt{\frac{2}{\pi}} (\frac{\sin ak}{k})$
5	$x^{a-1}, \quad 0 < a < 1$	$\sqrt{\frac{2}{\pi}} \Gamma(a) k^{-a} \cos(\frac{a\pi}{2})$
6	$\cos(ax^2)$	$\frac{1}{2\sqrt{a}} [\cos(\frac{k^2}{4a}) + \sin(\frac{k^2}{4a})]$
7	$\sin(ax^2), \quad a > 0$	$\frac{1}{2\sqrt{a}} [\cos(\frac{k^2}{4a}) - \sin(\frac{k^2}{4a})]$
8	$(a^2 - x^2)^{\nu - \frac{1}{2}} H(a - x), \quad \nu > -\frac{1}{2}$	$2^{\nu - \frac{1}{2}} \Gamma(\nu + \frac{1}{2}) (\frac{a}{k})^\nu J_\nu(ak)$
9	$(a^2 + x^2)^{-1} J_0(bx), \quad a, b > 0$	$\sqrt{\frac{\pi}{2}} a^{-1} \exp(-ak) I_0(ab),$ $b < k < \infty$
10	$x^{-\nu} J_\nu(ax), \quad \nu > -\frac{1}{2}$	$\frac{(a^2 - k^2)^{\nu - \frac{1}{2}} H(a - k)}{2^{\nu - \frac{1}{2}} a^\nu \Gamma(\nu + \frac{1}{2})}$
11	$(x^2 + a^2)^{-\frac{1}{2}} \exp[-b(x^2 + a^2)^{\frac{1}{2}}]$	$K_0[a(k^2 + b^2)^{\frac{1}{2}}], \quad a > 0,\ b > 0$
12	$x^{\nu - 1} e^{-ax}, \quad \nu > 0,\ a > 0$	$\sqrt{\frac{2}{\pi}} \Gamma(\nu) r^{-\nu} \cos n\theta,$ where $r = (a^2 + k^2)^{\frac{1}{2}}, \theta = \tan^{-1}(\frac{k}{a})$
13	$\frac{2}{x} e^{-x} \sin x$	$\sqrt{\frac{2}{\pi}} \tan^{-1}(\frac{2}{k^2})$
14	$\sin[a(b^2 - x^2)^{\frac{1}{2}} H(b - x)]$	$\sqrt{\frac{\pi}{2}} (ab)(a^2 + k^2)^{-\frac{1}{2}}$ $\times J_1[b(a^2 + k^2)^{\frac{1}{2}}]$
15	$\frac{(1 - x^2)}{(1 + x^2)^2}$	$\sqrt{\frac{\pi}{2}} k \exp(-k)$
16	$x^{-\alpha}, \quad 0 < \alpha < 1$	$\sqrt{\frac{\pi}{2}} \frac{k^{\alpha - 1}}{\Gamma(\alpha)} \sec(\frac{\pi\alpha}{2})$
17	$(\frac{1}{a} + x) e^{-ax}$	$\sqrt{\frac{\pi}{2}} \frac{2a^2}{(a^2 + k^2)^2}$

	$f(x)$	$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(kx) f(x)\, dx$
18	$\log(1 + \frac{a^2}{k^2}), \quad a > 0$	$\sqrt{2\pi} \frac{(1-e^{-ak})}{k}$
19	$\log(\frac{a^2+x^2}{b^2+x^2}), \quad a, b > 0$	$\sqrt{2\pi} \frac{(e^{-bk}-e^{-ak})}{k}$
20	$a(x^2 + a^2)^{-1}, \quad a > 0$	$\sqrt{\frac{\pi}{2}} \exp(-ak), \quad k > 0$

13.4 Laplace Transforms

	$f(t)$	$\bar{f}(s) = \int_0^\infty \exp(-st) f(t)\, dt$
1	$e^{-at} f(t)$	$\bar{f}(s + a)$
2	$f(at)$	$\frac{1}{\lvert a \rvert} \bar{f}(\frac{s}{a})$
3	$f^{(n)}(t)$	$s^n \bar{f}(s) - \sum_{r=1}^{n} s^{n-r} f^{(r-1)}(0)$
4	$\int_0^t f(\tau)\, d\tau$	$\frac{1}{s} \bar{f}(s)$
5	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} \bar{f}(s)$
6	$f(t - a) H(t - a)$	$\exp(-as) \bar{f}(s)$
7	$(f * g)(t) = \int_0^t f(t - \tau) g(\tau)\, d\tau$	$\bar{f}(s) \bar{g}(s)$
8	$\frac{t^{n-1}}{(n-1)!} * f(t) = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)\, d\tau$	$\frac{\bar{f}(s)}{s^n}$
9	$[t]$ = greatest integer = largest integer that, is less than or equal to t	$\frac{1}{s}(e^s - 1)^{-1}$
10	$f(t)$ is periodic with period a	$(1 - e^{-as})^{-1} \int_0^a e^{-st} f(t)\, dt$
11	$f(t) = H(t) - 2H(t - a) + 2H(t - 2a) - 2H(t - 3a) + \cdots$	$\frac{1}{s} \tanh(\frac{as}{2})$
12	$t^n, \quad n = 0, 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$\overline{f}(s) = \int_0^\infty \exp(-st) f(t)\, dt$
13	e^{at}	$\frac{1}{s-a}$
14	$t^n e^{-at}$	$\frac{\Gamma(n+1)}{(s+a)^{n+1}}$
15	$t^a, \quad a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$
16	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
17	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
18	$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
19	$2\sqrt{t}$	$\frac{1}{s}\sqrt{\frac{\pi}{s}}$
20	$t^{-1/2} \exp(-\frac{a}{t})$	$\sqrt{\frac{\pi}{s}} \exp(-2\sqrt{as})$
21	$t^{-3/2} \exp(-\frac{a}{t})$	$\sqrt{\frac{\pi}{a}} \exp(-2\sqrt{as})$
22	$\frac{1}{\sqrt{\pi t}}(1+2at)e^{at}$	$\frac{s}{(s-a)\sqrt{s-a}}$
23	$\frac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$	$\sqrt{s-a} - \sqrt{s-b}$
24	$\exp(a^2 t)\, \mathrm{erf}(a\sqrt{t})$	$\frac{a}{\sqrt{s}(s-a^2)}$
25	$\exp(a^2 t)\, \mathrm{erfc}(a\sqrt{t})$	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$
26	$\frac{1}{\sqrt{\pi t}} + a \exp(a^2 t)\, \mathrm{erf}(a\sqrt{t})$	$\frac{\sqrt{s}}{(s-a^2)}$
27	$\frac{1}{\sqrt{\pi t}} - a \exp(a^2 t)\, \mathrm{erfc}(a\sqrt{t})$	$\frac{1}{\sqrt{s}+a}$
28	$\frac{\exp(-at)}{\sqrt{b-a}}\, \mathrm{erf}(\sqrt{(b-a)t})$	$\frac{1}{(s+a)\sqrt{s+b}}$
29	$\frac{1}{2} e^{i\omega t}[\exp(-\lambda z)\, \mathrm{erfc}(\zeta - \sqrt{i\omega t})$ $+ \exp(\lambda z)\, \mathrm{erfc}(\zeta + \sqrt{i\omega t})]$ where $\zeta = z/2\sqrt{\nu t},\ \lambda = \sqrt{\frac{i\omega}{\nu}}$	$(s - i\omega)^{-1} \exp(-z\sqrt{\frac{s}{\nu}})$
30	$\frac{1}{2}[\exp(-ab)\, \mathrm{erfc}(\frac{b-2at}{2\sqrt{t}})$ $+ \exp(ab)\, \mathrm{erfc}(\frac{b+2at}{2\sqrt{t}})]$	$\exp[-b(s+a^2)^{\frac{1}{2}}]$
31	$J_0(at)$	$(s^2+a^2)^{-\frac{1}{2}}$

	$f(t)$	$\bar{f}(s) = \int_0^\infty \exp(-st) f(t)\, dt$
32	$I_0(at)$	$(s^2 - a^2)^{-\frac{1}{2}}$
33	$t^{\alpha-1} \exp(-at), \quad \alpha > 0$	$\Gamma(\alpha)(s+a)^{-\alpha}$
34	$t^{-1} J_\nu(at)$	$\nu^{-1} a^\nu (\sqrt{s^2+a^2} + s)^{-\nu}$, $\mathrm{Re}\,\nu > -\frac{1}{2}$
35	$J_0(a\sqrt{t})$	$\frac{1}{s} \exp(-\frac{a^2}{4s})$
36	$(\frac{2}{a})^\nu t^{\nu/2} J_\nu(a\sqrt{t})$	$s^{-(\nu+1)} \exp(-\frac{a^2}{4s}), \quad \mathrm{Re}\,\nu > -\frac{1}{2}$
37	$\frac{a}{2t\sqrt{\pi t}} \exp(-\frac{a^2}{4t})$	$\exp(-a\sqrt{s}), \quad a > 0$
38	$\frac{1}{\sqrt{\pi t}} \exp(-\frac{a^2}{4t})$	$\frac{1}{\sqrt{s}} \exp(-a\sqrt{s}), \quad a \geq 0$
39	$\exp(-\frac{a^2 t^2}{4})$	$\frac{\sqrt{\pi}}{a} \exp(\frac{s^2}{a^2})\, \mathrm{erfc}(\frac{s}{a}), \quad a \geq 0$
40	$(t^2 - a^2)^{-\frac{1}{2}} H(t-a)$	$K_0(as), \quad a > 0$
41	$\delta^{(n)}(t-a), \quad n = 0, 1, \ldots$	$s^n \exp(-as)$
42	$t^{m\alpha+\beta-1} E_{\alpha,\beta}^{(m)}(\pm at)$, $m = 0, 1, 2, \ldots$	$\frac{m!\, s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}$
43	$\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})} (\frac{t}{2a})^\nu J_\nu(at)$	$(s^2+a^2)^{-(\nu+\frac{1}{2})}, \quad \mathrm{Re}\,\nu > -\frac{1}{2}$
44	$\frac{1}{2} e^{-ct} [\exp(-a\sqrt{b-c})$ $\times \mathrm{erfc}\{\frac{a}{\sqrt{4t}} - \sqrt{(b-c)t}\}$ $- \exp(a\sqrt{b-c})$ $\times \mathrm{erfc}\{\frac{a}{\sqrt{4t}} + \sqrt{(b-c)t}\}]$	$\frac{\exp(-a\sqrt{s+b})}{(s+c)\sqrt{(s+b)}}$
45	$\frac{1}{2} e^{-ct} [\exp(-a\sqrt{b-c})$ $\times \mathrm{erfc}\{\frac{a}{\sqrt{4t}} - t\sqrt{b-c}\}$ $- \exp(a\sqrt{b-c})$ $\times \mathrm{erfc}\{\frac{a}{\sqrt{4t}} + t\sqrt{b-c}\}]$	$\frac{\exp(-a\sqrt{s+b})}{(s+c)}$
46	$e^{-bt} [\sqrt{\frac{4t}{\pi}} \exp(-\frac{a^2}{4t})$ $- a\, \mathrm{erfc}(\frac{a}{\sqrt{4t}})]$	$\frac{\exp(-a\sqrt{s+b})}{(s+b)^{3/2}}$

	$f(t)$	$\overline{f}(s)=\int_0^\infty \exp(-st)f(t)\,dt$
47	$e^{-bt}[(t+\frac{1}{2}a^2)\operatorname{erfc}(\frac{a}{\sqrt{4t}}) - \sqrt{\frac{ta^2}{\pi}}\exp(-\frac{a^2}{4t})]$	$\frac{\exp(-a\sqrt{s+b})}{(s+b)^2}$
48	$\exp(ab+a^2t)\operatorname{erfc}(a\sqrt{t}+\frac{b}{2\sqrt{t}})$	$\frac{e^{-b\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})}, \quad b\ge 0$
49	$L_n(t)=\frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t})$	$\frac{1}{s}(1-\frac{1}{s})^n$
50	$\frac{n!}{(2n)!\sqrt{\pi t}}H_{2n}(\sqrt{t})$	$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$
51	$f(t)g(t)$	$\frac{1}{2\pi i}[\bar{f}(s)*\bar{g}(s)]$

13.5 Hankel Transforms

	$f(r)$	Order n	$\tilde{f}_n(\kappa)=\int_0^\infty rJ_n(\kappa r)f(r)\,dr$
1	$f(ar)$	0	$\frac{1}{a^2}\tilde{f}_n(\frac{\kappa}{a}), \quad a>0$
2	$f'(r)$	1	$-\kappa\tilde{f}_0(\kappa)$
3	$\nabla^2 f=\frac{1}{r}\frac{d}{dr}(\frac{df}{dr})$	0	$-\kappa^2\tilde{f}_0(\kappa)$
4	$(\nabla^2-\frac{n^2}{r^2})f$	>-1	$-\kappa^2\tilde{f}_n(\kappa)$
5	$H(a-r)$	0	$\frac{a}{\kappa}J_1(a\kappa)$
6	$\exp(-ar)$	0	$a(a^2+\kappa^2)^{-\frac{3}{2}}$
7	$\frac{1}{r}\exp(-ar), \quad a>0$	0	$(a^2+\kappa^2)^{-\frac{1}{2}}$
8	$(a^2-r^2)H(a-r)$	0	$\frac{4a}{\kappa^3}J_1(a\kappa)-\frac{2a^2}{\kappa^2}J_0(a\kappa)$
9	$a(a^2+r^2)^{-\frac{3}{2}}$	0	$\exp(-a\kappa)$
10	$\frac{1}{r}\cos(ar)$	0	$(\kappa^2-a^2)^{-\frac{1}{2}}H(\kappa-a)$

	$f(r)$	Order n	$\tilde{f}_n(\kappa) = \int_0^\infty r J_n(\kappa r) f(r)\, dr$
11	$\frac{1}{r}\sin(ar)$	0	$(a^2 - \kappa^2)^{-\frac{1}{2}} H(a-\kappa)$
12	$\frac{1}{r^2}(1-\cos ar)$	0	$\cosh^{-1}(\frac{a}{\kappa}) H(a-\kappa)$
13	$\frac{1}{r} J_1(ar)$	0	$\frac{1}{a} H(a-\kappa), \quad a > 0$
14	$Y_0(ar)$	0	$(\frac{2}{\pi})(a^2 - \kappa^2)^{-1}$
15	$K_0(ar)$	0	$(a^2 + \kappa^2)^{-1}$
16	$\frac{\delta(r)}{r}$	0	1
17	$\delta(r-a)$	>-1	$a J_n(a\kappa)$
18	$(r^2+b^2)^{-\frac{1}{2}}$ $\times \exp\{-a(r^2+b^2)^{\frac{1}{2}}\}$	0	$(\kappa^2 + a^2)^{-\frac{1}{2}} \exp\{-b(\kappa^2+a^2)^{\frac{1}{2}}\}$
19	$(r^2+a^2)^{-\frac{1}{2}}$	0	$\frac{1}{\kappa}\exp(-a\kappa)$
20	$\exp(-ar)$	1	$\kappa(a^2+\kappa^2)^{-3/2}$
21	$\frac{\sin ar}{r}$	1	$\frac{a H(\kappa - a)}{\kappa(\kappa^2 - a^2)^{\frac{1}{2}}}$
22	$\frac{1}{r}\exp(-ar)$	1	$\frac{1}{\kappa}[1 - \frac{a}{(\kappa^2+a^2)^{\frac{1}{2}}}]$
23	$\frac{1}{r^2}\exp(-ar)$	1	$\frac{1}{\kappa}[(\kappa^2+a^2)^{\frac{1}{2}} - a]$
24	$r^n H(a-r)$	>-1	$\frac{1}{\kappa} a^{n+1} J_{n+1}(a\kappa)$
25	$r^n \exp(-ar), \quad \mathrm{Re}\, a > 0$	>-1	$\frac{1}{\sqrt{\pi}} \frac{2^{n+1}\Gamma(n+\frac{3}{2}) a \kappa^n}{(a^2+\kappa^2)^{n+\frac{3}{2}}}$
26	$r^n \exp(-ar^2)$	>-1	$\frac{\kappa^n}{(2a)^{n+1}} \exp(-\frac{\kappa^2}{4a})$
27	r^{a-1}	>-1	$\frac{2a\Gamma[\frac{1}{2}(a+n+1)]}{\kappa^{a+1}\Gamma[\frac{1}{2}(1-a+n)]}$
28	$r^n(a^2-r^2)^{m-n-1}$ $\times H(a-r)$	>-1	$2^{m-n-1}\Gamma(m-n) a^m \kappa^{n-m} J_m(a\kappa)$
29	$r^m \exp(-r^2/a^2)$	>-1	$\frac{\kappa^n a^{m+n+2}}{2^{n+1}\Gamma(n+1)} \Gamma(1+\frac{m}{2}+\frac{n}{2})$ $\times {}_1F_1(1+\frac{m}{2}+\frac{n}{2};\ n+1;\ -\frac{1}{4}a^2\kappa^2)$

	$f(r)$	Order n	$\tilde{f}_n(\kappa) = \int_0^\infty rJ_n(\kappa r)f(r)\,dr$
30	$\frac{1}{r}J_{n+1}(ar)$	>-1	$\kappa^n a^{-(n+1)}H(a-\kappa), \quad a>0$
31	$r^n(a^2-r^2)^m H(a-r),$ $m>-1$	>-1	$2^m a^n \Gamma(m+1)(\frac{a}{\kappa})^{m+1}$ $\times J_{n+m+1}(a\kappa)$
32	$\frac{1}{r^2}J_n(ar)$	$>\frac{1}{2}$	$\begin{cases} \frac{1}{2n}(\frac{\kappa}{a})^n & \text{if } 0<\kappa\le a, \\ \frac{1}{2n}(\frac{a}{\kappa})^n & \text{if } a<\kappa<\infty \end{cases}$
33	$\frac{r^n}{(a^2+r^2)^{m+1}}, \quad a>0$	>-1	$(\frac{\kappa}{2})^m \frac{a^{n-m}}{\Gamma(m+1)}K_{n-m}(a\kappa)$
34	$\exp(-p^2r^2)J_n(ar)$	>-1	$(2p^2)^{-1}\exp(-\frac{a^2+\kappa^2}{4p^2})I_n(\frac{a\kappa}{2p^2})$
35	$\frac{1}{r}\exp(-ar)$	>-1	$\frac{((\kappa^2+a^2)^{\frac{1}{2}}-a)^n}{\kappa^n(\kappa^2+a^2)^{\frac{1}{2}}}$
36	$\frac{r^n}{(r^2+a^2)^{n+1}}$	>-1	$(\frac{\kappa}{2})^n \frac{K_0(a\kappa)}{\Gamma(n+1)}$
37	$\frac{r^n}{(a^2-r^2)^{n+\frac{1}{2}}}H(a-r)$	<1	$\frac{1}{\sqrt{\pi}}(\frac{\kappa}{2})^n\Gamma(\frac{1}{2}-n)(\frac{\sin a\kappa}{\kappa})$
38	$r^{-1}\exp(-ar^2)$	1	$\frac{1}{\kappa}[1-\exp(-\frac{\kappa^2}{4a})]$
39	$r^{-1}\sin(ar^2), \quad a>0$	1	$\frac{1}{\kappa}\sin(\frac{\kappa^2}{4a})$
40	$r^{-1}\cos(ar^2), \quad a>0$	1	$1-\cos(\frac{\kappa^2}{4a})$
41	$\exp(-ar), \quad a>0$	>-1	$\frac{(a+n\sqrt{\kappa^2+a^2})}{(\kappa^2+a^2)^{3/2}}(\frac{\kappa}{a+\sqrt{a^2+\kappa^2}})^n$
42	$\exp(-ar^2)J_0(br)$	0	$\frac{a}{2}\exp(-\frac{\kappa^2-b^2}{4a})I_0(\frac{b\kappa}{2a})$
43	$\frac{H(a-r)}{\sqrt{a^2-r^2}}$	0	$\sqrt{\frac{a\pi}{2\kappa}}J_{\frac{1}{2}}(a\kappa), \quad a>0$
44	$\frac{r^n H(a-r)}{\sqrt{a^2-r^2}}$	>-1	$\sqrt{\frac{\pi}{2\kappa}}a^{n+\frac{1}{2}}J_{n+1}(a\kappa), \quad a>0$
45	$r^{-2}\sin r$	0	$\sin^{-1}(\frac{1}{\kappa}), \quad \kappa>1$
46	$\exp(-ar^2)$	0	$\frac{1}{2a}\exp(-\frac{\kappa^2}{4a})$
47	$r\exp(-ar^2)$	0	$\frac{\kappa}{4a^2}\exp(-\frac{\kappa^2}{4a})$

	$f(r)$	Order n	$\tilde{f}_n(\kappa)=\int_0^\infty rJ_n(\kappa r)f(r)\,dr$
48	If $a=\sigma+i\omega$ in 46, $f(r)=e^{-\sigma r^2}\cos(\omega r^2)$	0	$2\Omega\exp(-\sigma\Omega\kappa^2)$ $\times\{\sigma\cos(\omega\Omega\kappa^2)+\omega\sin(\omega\Omega\kappa^2)\}$, where $\Omega=[4(\sigma^2+\omega^2)]^{-1}$
49	$f(r)=e^{-\sigma r^2}\sin(\omega r^2)$	0	$2\Omega\exp(-\sigma\Omega\kappa^2)$ $\times\{-\omega\cos(\omega\Omega\kappa^2)$ $+\sigma\sin(\omega\Omega\kappa^2)\}$, where $\Omega=[4(\sigma^2+\omega^2)]^{-1}$
50	$\delta(ar-b)=\frac{1}{a}\delta(r-\frac{b}{a})$, $a>0$	>-1	$\frac{b}{a^2}J_n(\frac{b\kappa}{a})$, $a>0$
51	$\delta^{(m)}(ar-b)$, $a>0$	>-1	$\frac{(-1)^m\kappa^{m-1}}{a^2}$ $\times[\frac{b}{a}J_n^{(m)}(\frac{b\kappa}{a})+mJ_n^{(m-1)}(\frac{b\kappa}{a})]$
52	$\frac{r^n}{(r^2+a^2)^{m+1}}$	>-1	$(\frac{\kappa}{2})^m\frac{a^{n-m}}{\Gamma(m+1)}K_{n-m}(a\kappa)$
53	$r^{m-n}J_m(ar)$	>-1	$\frac{a^m2^{m-n+1}(\kappa^2-a^2)^{n-m-1}}{\kappa^n\Gamma(n-m)}$, $\kappa>a$

13.6 Finite Hankel Transforms

	$f(r)$	Order n	$\tilde{f}_n(k_i)=\int_0^a rJ_n(rk_i)f(r)\,dr$
1	c, where c is a constant	0	$(\frac{ac}{k_i})J_1(ak_i)$
2	(a^2-r^2)	0	$\frac{4a}{k_i^3}J_1(ak_i)$
3	$(a^2-r^2)^{-\frac{1}{2}}$	0	$k_i^{-1}\sin(ak_i)$
4	$\frac{J_0(\alpha r)}{J_0(\alpha a)}$	0	$-\frac{ak_i}{(\alpha^2-k_i^2)}J_1(ak_i)$
5	$\frac{1}{r}$	1	$k_i^{-1}\{1-J_0(ak_i)\}$
6	$r^{-1}(a^2-r^2)^{-\frac{1}{2}}$	1	$\frac{(1-\cos ak_i)}{(ak_i)}$
7	r^n	>-1	$\frac{a^{n+1}}{k_i}J_{n+1}(ak_i)$
8	$\frac{J_\nu(\alpha r)}{J_\nu(\alpha a)}$	>-1	$\frac{ak_i}{(\alpha^2-k_i^2)}J'_\nu(ak_i)$

	$f(r)$	Order n	$\tilde{f}_n(k_i) = \int_0^a rJ_n(rk_i)f(r)\,dr$
9	$r^{-n}(a^2 - r^2)^{-\frac{1}{2}}$	>-1	$\frac{\pi}{2}\{J_{\frac{n}{2}}(\frac{ak_i}{2})\}^2$
10	$r^n(a^2 - r^2)^{-(n+\frac{1}{2})}$	$<\frac{1}{2}$	$\frac{\Gamma(\frac{1}{2}-n)}{\sqrt{\pi}2^n} k_i^{n-1} \sin(ak_i)$
11	$r^{n-1}(a^2 - r^2)^{n-\frac{1}{2}}$	$>-\frac{1}{2}$	$\frac{\sqrt{\pi}}{2}\Gamma(n+\frac{1}{2})(\frac{2}{k_i})^n a^{2n} J_n^2(\frac{ak_i}{2})$

A

Some Special Functions and Their Properties

The main purpose of this appendix is to introduce several special functions and to state their basic properties that are most frequently used in the theory and applications of ordinary and partial differential equations. The subject is, of course, too vast to be treated adequately in so short a space, so that only the more important results will be stated. For a fuller discussion of these topics and of further properties of these functions the reader is referred to the standard treatises on the subject.

A-1 Gamma, Beta, and Error Functions

The *gamma function* (also called the *factorial function*) is defined by a definite integral in which a variable appears as a parameter

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \quad x > 0. \tag{A-1.1}$$

The integral (A-1.1) is uniformly convergent for all x in $[a, b]$ where $0 < a \leq b < \infty$, and hence, $\Gamma(x)$ is a continuous function for all $x > 0$.

Integrating (A-1.1) by parts, we obtain the fundamental property of $\Gamma(x)$

$$\begin{aligned}\Gamma(x) &= \left[-e^{-t} t^{x-1}\right]_0^\infty + (x-1)\int_0^\infty e^{-t} t^{x-2}\, dt \\ &= (x-1)\Gamma(x-1) \quad \text{for } x-1>0.\end{aligned}$$

Then we replace x by $x+1$ to obtain the fundamental result

$$\Gamma(x+1) = x\,\Gamma(x). \tag{A-1.2}$$

In particular, when $x = n$ is a positive integer, we make repeated use of (A-1.2) to obtain

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots \\ &= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\Gamma(1) = n!,\end{aligned} \tag{A-1.3}$$

where $\Gamma(1) = 1$.

L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers*,
DOI 10.1007/978-0-8176-8265-1,

We put $t = u^2$ in (A-1.1) to obtain

$$\Gamma(x) = 2\int_0^\infty \exp(-u^2)u^{2x-1}\,du, \quad x > 0. \tag{A-1.4}$$

Letting $x = \frac{1}{2}$, we find

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty \exp(-u^2)\,du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}. \tag{A-1.5}$$

Using (A-1.2), we deduce

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}. \tag{A-1.6}$$

Similarly, we can obtain the values of $\Gamma(\frac{5}{2}), \Gamma(\frac{7}{2}), \ldots, \Gamma(\frac{2n+1}{2})$.

The gamma function can also be defined for negative values of x by the rewritten form of (A-1.2) as

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \quad x \neq 0, -1, -2, \ldots. \tag{A-1.7}$$

For example,

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\,\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}, \tag{A-1.8}$$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{4}{3}\sqrt{\pi}. \tag{A-1.9}$$

We differentiate (A-1.1) with respect to x to obtain

$$\begin{aligned}\frac{d}{dx}\Gamma(x) = \Gamma'(x) &= \int_0^\infty \frac{d}{dx}\left(t^x\right)\frac{e^{-t}}{t}\,dt \\ &= \int_0^\infty \frac{d}{dx}\left[\exp(x\log t)\right]\frac{e^{-t}}{t}\,dt = \int_0^\infty t^{x-1}(\log t)e^{-t}\,dt.\end{aligned} \tag{A-1.10}$$

At $x = 1$, this gives

$$\Gamma'(1) = \int_0^\infty e^{-t}\log t\,dt = -\gamma, \tag{A-1.11}$$

where γ is called the *Euler constant* and has the value 0.5772.

The graph of the gamma function is shown in Figure A.1.

The volume, V_n, and the surface area, S_n, of a sphere of radius r in n-dimensional space $\mathbb{R}^n$ are given by

$$V_n = \frac{\{\Gamma(\frac{1}{2})\}^n r^n}{\Gamma(\frac{n}{2}+1)}, \qquad S_n = \frac{2\{\Gamma(\frac{1}{2})\}^n r^{n-1}}{\Gamma(\frac{n}{2})}.$$

Thus, $\frac{dV_n}{dr} = S_n$.

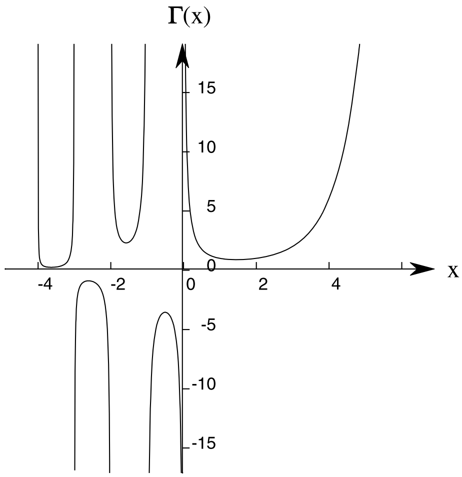

Fig. A.1 The gamma function.

In particular, when $n = 2, 3, \ldots$, we get $V_2 = \pi r^2$, $S_2 = 2\pi r$; $V_3 = \frac{4}{3}\pi r^3$, $S_3 = 4\pi r^2$; etc.

Using (A-1.2) and (A-1.5), we obtain the following results:

$$V_{2m} = \frac{\pi^m r^{2m}}{m!}, \qquad S_{2m} = \frac{2\pi^m r^{2m-1}}{(m-1)!},$$
$$V_{2m+1} = \frac{2(2\pi)^m r^{2m+1}}{1.3.5\cdots(2m+1)}, \qquad S_{2m+1} = \frac{2^{2m+1} m! \pi^m r^{2m}}{(2m)!}.$$

Legendre Duplication Formula

Several useful properties of the gamma function are recorded below for reference without proof. We begin with

$$2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2x). \tag{A-1.12}$$

In particular, when $x = n$ $(n = 0, 1, 2, \ldots)$,

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,(2n)!}{2^{2n}\,n!}. \tag{A-1.13}$$

The following properties also hold for $\Gamma(x)$:

$$\Gamma(x)\Gamma(1-x) = \pi \operatorname{cosec} \pi x, \quad x \text{ is a noninteger}, \tag{A-1.14}$$
$$\Gamma(x) = p^x \int_0^\infty \exp(-pt)\, t^{x-1}\, dt, \tag{A-1.15}$$
$$\Gamma(x) = \int_{-\infty}^{\infty} \exp\left(xt - e^t\right) dt. \tag{A-1.16}$$

$$\Gamma(x+1) \sim \sqrt{2\pi}\exp(-x)x^{x+\frac{1}{2}} \quad \text{for large } x, \tag{A-1.17}$$

$$n! \sim \sqrt{2\pi}\exp(-n)x^{n+\frac{1}{2}} \quad \text{for large } n. \tag{A-1.18}$$

The latter formulas are known as *Stirling approximation* of $\Gamma(x+1)$ for large x and of $n!$ for large n.

The *incomplete gamma function*, $\gamma(x,a)$, is defined by the integral

$$\gamma(a,x) = \int_0^x e^{-t}t^{a-1}\,dt, \quad a > 0. \tag{A-1.19}$$

The *complementary incomplete gamma* function, $\Gamma(a,x)$, is defined by the integral

$$\Gamma(a,x) = \int_x^\infty e^{-t}\,t^{a-1}\,dt, \quad a > 0. \tag{A-1.20}$$

Thus, it follows that

$$\gamma(a,x) + \Gamma(a,x) = \Gamma(a). \tag{A-1.21}$$

The *beta function*, denoted by $B(x,y)$, is defined by the integral

$$B(x,y) = \int_0^t t^{x-1}(1-t)^{y-1}\,dt, \quad x > 0,\ y > 0. \tag{A-1.22}$$

The beta function $B(x,y)$ is *symmetric* with respect to its arguments x and y, that is,

$$B(x,y) = B(y,x). \tag{A-1.23}$$

This follows from (A-1.22) by the change of variable $1-t=u$, that is,

$$B(x,y) = \int_0^1 u^{y-1}(1-u)^{x-1}\,du = B(y,x).$$

If we make the change of variable $t = u/(1+u)$ in (A-1.22), we obtain another integral representation of the beta function

$$B(x,y) = \int_0^\infty u^{x-1}(1+u)^{-(x+y)}\,du = \int_0^\infty u^{y-1}(1+u)^{-(x+y)}\,du. \tag{A-1.24}$$

Putting $t = \cos^2\theta$ in (A-1.22), we derive

$$B(x,y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta\,\sin^{2y-1}\theta\,d\theta. \tag{A-1.25}$$

Several important results are recorded below for ready reference without proof:

$$B(1,1) = 1, \qquad B\left(\frac{1}{2},\frac{1}{2}\right) = \pi, \tag{A-1.26}$$

$$B(x,y) = \left(\frac{x-1}{x+y-1}\right)B(x-1,y), \tag{A-1.27}$$

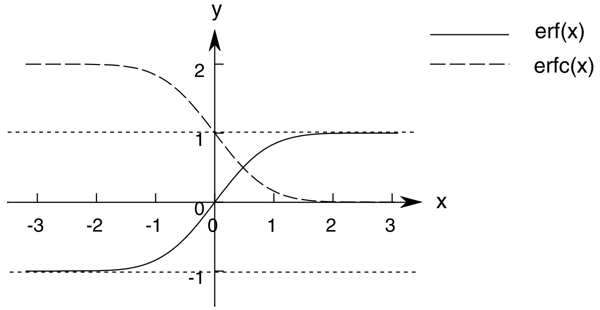

Fig. A.2 The error function and the complementary error function.

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{A-1.28}$$

$$B\left(\frac{1+x}{2}, \frac{1-x}{2}\right) = \pi \sec\left(\frac{\pi x}{2}\right), \quad 0 < x < 1. \tag{A-1.29}$$

The *error function*, $\text{erf}(x)$, is defined by the integral

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt, \quad -\infty < x < \infty. \tag{A-1.30}$$

Clearly, it follows from (A-1.30) that

$$\text{erf}(-x) = -\text{erf}(x), \tag{A-1.31}$$

$$\frac{d}{dx}\left[\text{erf}(x)\right] = \frac{2}{\sqrt{\pi}} \exp\left(-x^2\right), \tag{A-1.32}$$

$$\text{erf}(0) = 0, \qquad \text{erf}(\infty) = 1. \tag{A-1.33}$$

The *complementary error function*, $\text{erfc}(x)$, is defined by the integral

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt. \tag{A-1.34}$$

Clearly, it follows that

$$\text{erfc}(x) = 1 - \text{erf}(x), \tag{A-1.35}$$

$$\text{erfc}(0) = 1, \qquad \text{erfc}(\infty) = 0. \tag{A-1.36}$$

The graphs of $\text{erf}(x)$ and $\text{erfc}(x)$ are shown in Figure A.2.

$$\text{erfc}(x) \sim \frac{1}{x\sqrt{\pi}} \exp\left(-x^2\right) \quad \text{for large } x. \tag{A-1.37}$$

Closely associated with the error function are the Fresnel integrals, which are defined by

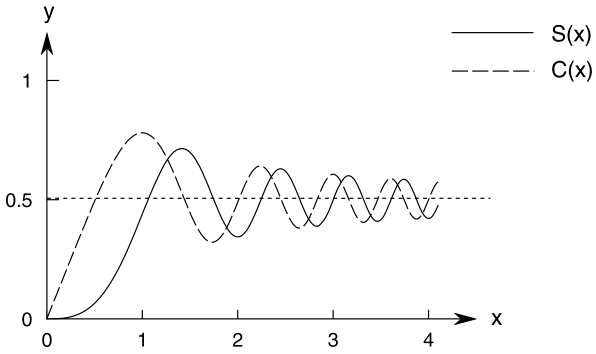

Fig. A.3 The Fresnel integrals $C(x)$ and $S(x)$.

$$C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt. \tag{A-1.38}$$

These integrals arise in diffraction problems in optics, in water waves, in elasticity, and elsewhere.

Clearly, it follows from (A-1.38) that

$$C(0) = 0 = S(0), \tag{A-1.39}$$

$$C(\infty) = S(\infty) = \frac{\pi}{2}, \tag{A-1.40}$$

$$\frac{d}{dx}C(x) = \cos\left(\frac{\pi x^2}{2}\right), \qquad \frac{d}{dx}S(x) = \sin\left(\frac{\pi x^2}{2}\right). \tag{A-1.41}$$

It also follows from (A-1.38) that $C(x)$ has extrema at the points where $x^2 = (2n+1)$, $n = 0, 1, 2, 3, \ldots$, and $S(x)$ has extrema at the points where $x^2 = 2n$, $n = 1, 2, 3, \ldots$. The largest maxima occur first and are found to be $C(1) = 0.7799$ and $S(\sqrt{2}) = 0.7139$. We also infer that both $C(x)$ and $S(x)$ are oscillatory about the line $y = 0.5$. The graphs of $C(x)$ and $S(x)$ for non-negative real x are shown in Figure A.3.

We prove further properties of the Gamma and the Beta functions. We first prove that

$$\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} x \, dx = \frac{1}{2} B(p,q). \tag{A-1.42}$$

We put $\sin^2 x = t$ so that the left hand side of the above integral becomes

$$\frac{1}{2}\int_0^{\pi/2} \sin^{2p-2} x \cos^{2q-2} x \cdot 2\cos x \sin x \, dx$$
$$= \frac{1}{2}\int_0^1 t^{p-1}(1-q)^{q-1}\, dt = \frac{1}{2}B(p,q). \tag{A-1.43}$$

We next prove that

$$\Gamma(2p)\Gamma\left(\frac{1}{2}\right) = 2^{2p-1}\Gamma(p)\,\Gamma\left(p+\frac{1}{2}\right). \tag{A-1.44}$$

We have

$$\int_0^{\pi/2} \sin^{2p} 2x\,dx = \frac{1}{2}\int_0^{\pi} \sin^{2p}\theta\,d\theta = 2\int_0^{\pi/2} \frac{1}{2}\sin^{2p}\theta\,d\theta, \quad (2x=\theta).$$

Putting $q=\frac{1}{2}$ in (A-1.42) with $x = 2\theta$ gives

$$\begin{aligned} B\left(p+\frac{1}{2},\frac{1}{2}\right) &= 2\int_0^{\pi/2} \sin^{2p} x\,dx = 2\int_0^{\pi/2} \sin^{2p} 2\theta\,d\theta \\ &= 2^{2p+1}\int_0^{\pi/2} \sin^{2p}\theta\cos^{2p}\theta\,d\theta \\ &= 2^{2p}B\left(p+\frac{1}{2},p+\frac{1}{2}\right), \end{aligned}$$

which is, using (A-1.28) and (A-1.2),

$$\frac{\Gamma(p+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(p+1)} = 2^{2p}\frac{\Gamma(p+\frac{1}{2})\Gamma(p+\frac{1}{2})}{\Gamma(2p+1)} = 2^{2p}\frac{\Gamma(p+\frac{1}{2})\Gamma(p+\frac{1}{2})}{2p\,\Gamma(2p)},$$

or equivalently,

$$\Gamma(2p)\,\Gamma\left(\frac{1}{2}\right) = 2^{2p-1}\Gamma(p)\Gamma\left(p+\frac{1}{2}\right).$$

We next define

$$f(n,t) = \begin{cases} (1-\frac{t}{n})^n t^{x-1} & \text{if } 0 \le t \le n, \\ 0 & \text{if } t \ge n. \end{cases} \tag{A-1.45}$$

Using

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x},$$

we obtain, for fixed t,

$$\lim_{n\to\infty} f(n,t) = \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n t^{x-1} = e^{-t}\,t^{x-1}. \tag{A-1.46}$$

Hence, for $x > 0$,

$$\begin{aligned} \Gamma(x) &= \int_0^\infty e^{-t}t^{x-1}\,dt = \int_0^\infty \lim_{n\to\infty} f(n,t)\,dt \\ &= \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt, \end{aligned}$$

which is, putting $\frac{t}{n} = z$,

$$\begin{aligned}
&= \lim_{n\to\infty} n^x \int_0^1 (1-z)^n z^{x-1}\,dz \\
&= \lim_{n\to\infty} n^x \left\{ \left[(1-z)^n \cdot \frac{z^x}{x} \right]_0^1 + n \int_0^1 (1-z)^{n-1} \frac{z}{x}\,dz \right\} \\
&= \lim_{n\to\infty} n^x \cdot \left(\frac{n}{x}\right) \int_0^1 (1-z)^{n-1} z^x\,dz \\
&= \lim_{n\to\infty} n^x \frac{n\cdot(n-1)\cdots 1}{x\cdot(x+1)\cdots(n+x-1)} \int_0^1 z^{n+x-1}\,dz \\
&= \lim_{n\to\infty} \frac{n^x\, n!}{x(x+1)\cdots(x+n)}.
\end{aligned} \tag{A-1.47}$$

This is the celebrated *Gauss formula*.

We next prove that, for $0 < x < 1$,

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{A-1.48}$$

Since x and $x-1$ are positive and not integers, we use the Gauss formula (A-1.47) so that

$$\begin{aligned}
\Gamma(x)\Gamma(1-x) &= \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n)} \frac{n^{1-x} n!}{(1-x)(1-x+1)\cdots(1-x+n)} \\
&= \lim_{n\to\infty} \frac{1}{x} \cdot \frac{(n!)^2 n}{(1+x)(1-x)\cdots(n+x)(n-x)(n+1-x)} \\
&= \lim_{n\to\infty} \frac{1}{x} \frac{(n!)^2}{(1-x^2)(4-x^2)\cdots(n^2-x^2)} \cdot \frac{1}{\{1+\frac{1-x}{n}\}} \\
&= \frac{1}{x} \lim_{n\to\infty} \frac{1}{(1-x^2)(1-\frac{x^2}{2^2})\cdots(1-\frac{x^2}{n^2})} \\
&= \frac{1}{x} \lim_{n\to\infty} \left\{ \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) \right\}^{-1},
\end{aligned}$$

which is, by the product formula for the sine function,

$$= \frac{1}{x} \cdot \frac{x\pi}{\sin \pi x} = \frac{\pi}{\sin \pi x}.$$

Finally, we show that the $(2n)$th order moment of the standard normal probability density function

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{A-1.49}$$

is

$$E\left(X^{2n}\right) = \frac{2^n}{\sqrt{\pi}}\,\Gamma\left(n+\frac{1}{2}\right). \tag{A-1.50}$$

We have

$$\begin{aligned}
E\left(X^{2n}\right) &= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} x^{2n}\exp\left(-\frac{x^2}{2}\right)dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty} x^{2n}\exp\left(-\frac{x^2}{2}\right)dx, \quad \left(\frac{x^2}{2}=t\right) \\
&= \sqrt{\frac{2}{\pi}}\int_0^{\infty}(2t)^n\exp(-t)(2t)^{-\frac{1}{2}}\,dt \\
&= \frac{1}{\sqrt{\pi}}2^n\int_0^{\infty}t^{n-\frac{1}{2}}e^{-t}\,dt = \frac{1}{\sqrt{\pi}}2^n\Gamma\left(n+\frac{1}{2}\right).
\end{aligned}$$

Using $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we obtain

$$E\left(X^{2n}\right) = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1. \tag{A-1.51}$$

A random variable X with values in $(0,1)$ has the *Beta distribution* if its density function is, for some p, $q > 0$,

$$f(x) = \frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}, \quad 0<x<1; \tag{A-1.52}$$

$$E\left(X^n\right) = \frac{1}{B(p,q)}\int_0^1 x^n x^{p-1}(1-x)^{q-1}\,dx = \frac{B(n+p,q)}{B(p,q)}. \tag{A-1.53}$$

When $n=1$,

$$E(X) = \frac{B(p+1,q)}{B(p,q)} = \frac{p}{p+q}. \tag{A-1.54}$$

A random variable with values in $(0,\infty)$ has the *Gamma distribution* if, for some $p>0$ and $q>0$,

$$f(x) = \frac{q^p}{\Gamma(p)}x^{p-1}e^{-qx}. \tag{A-1.55}$$

A-2 Bessel and Airy Functions

The Bessel function of the first kind of order v (non-negative real number) is denoted by $J_v(x)$ and defined by

$$J_v(x) = x^v\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{2^{2r+v}r!\,\Gamma(r+v+1)}. \tag{A-2.1}$$

This series is convergent for all x.

The Bessel function $y = J_v(x)$ satisfies the *Bessel equation*

$$x^2y'' + xy' + \left(x^2 - v^2\right)y = 0. \quad \text{(A-2.2)}$$

When v is *not* a positive integer or zero, $J_v(x)$ and $J_{-v}(x)$ are two linearly independent solutions so that

$$y = AJ_v(x) + BJ_{-v}(x) \quad \text{(A-2.3)}$$

is the general solution of (A-2.2), where A and B are arbitrary constants.

However, when $v = n$, where n is a *positive integer* or *zero*, $J_n(x)$ and $J_{-n}(x)$ are no longer independent, but are related by the equation

$$J_{-n}(x) = (-1)^n J_n(x). \quad \text{(A-2.4)}$$

Thus, when n is a positive integer or zero, equation (A-2.2) has only *one* solution given by

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r}. \quad \text{(A-2.5)}$$

A second solution, known as *Neumann's or Webber's solution*, $Y_n(x)$ is given by

$$Y_n(x) = \lim_{v\to n} Y_v(x), \quad \text{(A-2.6)}$$

where

$$Y_v(x) = \frac{(\cos v\pi)J_v(x) - J_{-v}(x)}{\sin v\pi}. \quad \text{(A-2.7)}$$

Thus, the general solution of (A-2.2) is

$$y(x) = A\,J_n(x) + B\,Y_n(x), \quad \text{(A-2.8)}$$

where A and B are arbitrary constants.

In particular, from (A-2.5),

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r}, \quad \text{(A-2.9)}$$

$$J_1(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(r+1)!}\left(\frac{x}{2}\right)^{2r+1}. \quad \text{(A-2.10)}$$

Clearly, it follows from (A-2.9) and (A-2.10) that

$$J_0'(x) = -J_1(x). \quad \text{(A-2.11)}$$

Bessel's equation may not always arise in the standard form given in (A-2.2), but more frequently as

$$x^2y'' + xy' + \left(k^2x^2 - v^2\right)y = 0 \quad \text{(A-2.12)}$$

with the general solution

$$y(x) = AJ_v(kx) + BY_v(kx). \tag{A-2.13}$$

The *recurrence relations* are recorded below for easy reference without proof:

$$J_{v+1}(x) = \left(\frac{v}{x}\right) J_v(x) - J_v'(x), \tag{A-2.14}$$

$$J_{v-1}(x) = \left(\frac{v}{x}\right) J_v(x) + J_v'(x), \tag{A-2.15}$$

$$J_{v-1}(x) + J_{v+1}(x) = \left(\frac{2v}{x}\right) J_v(x), \tag{A-2.16}$$

$$J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x). \tag{A-2.17}$$

We have, from (A-2.5),

$$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r 2^{-(n+2r)}}{r!(n+r)!} x^{2n+2r}.$$

Differentiating both sides of this result with respect to x and using the fact that $2(n+r)/(n+r)! = 2/(n+r-1)!$, it turns out that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(-1)^r 2^{-(n+2r+1)}}{r!(n+r-1)!} x^{2n+2r-1} = x^n J_{n-1}(x). \tag{A-2.18}$$

Similarly, we can show

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x). \tag{A-2.19}$$

The generating function for the Bessel function is

$$\exp\left[\frac{1}{2}x\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x). \tag{A-2.20}$$

The integral representation of $J_n(x)$ is

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta. \tag{A-2.21}$$

The following are known as the *Lommel integrals*:

$$\int_0^a x\, J_n(px) J_n(qx)\, dx = \frac{a}{(q^2 - p^2)} \left[p\, J_n(qa) J_n'(pa) - q\, J_n(pa)\, J_n'(qa)\right], \quad p \neq q, \tag{A-2.22}$$

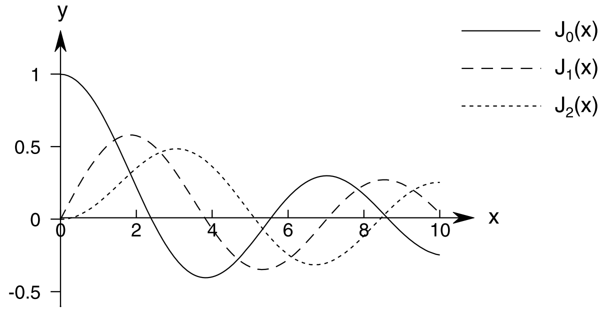

Fig. A.4 Graphs of $y = J_0(x)$, $J_1(x)$, and $J_2(x)$.

and

$$\int_0^a x\,J_n^2(px)\,dx = \frac{a^2}{2}\left[J_n'^2(pa) + \left(1 - \frac{n^2}{p^2a^2}\right)J_n^2(pa)\right]. \tag{A-2.23}$$

When $n = \pm\frac{1}{2}$,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x. \tag{A-2.24}$$

A rough idea of the shape of the Bessel functions when x is large may be obtained from equation (A-2.2). Substitution of $y = x^{-\frac{1}{2}}u(x)$ eliminates the first derivative, and hence, gives the equation

$$u'' + \left(1 - \frac{4n^2 - 1}{4x^2}\right)u = 0. \tag{A-2.25}$$

For large x, this equation approximately becomes

$$u'' + u = 0. \tag{A-2.26}$$

This equation admits the solution $u(x) = A\cos(x + \varepsilon)$, that is,

$$y = \frac{A}{\sqrt{x}}\cos(x + \varepsilon). \tag{A-2.27}$$

This suggests that $J_n(x)$ is oscillatory and has an infinite number of zeros. It also tends to zero as $x \to \infty$. The graphs of $J_n(x)$ for $n = 0, 1, 2$ and for $n = \pm\frac{1}{2}$ are shown in Figures A.4 and A.5, respectively.

An important special case arises in particular physical problems when $k^2 = -1$ in equation (A-2.12). we then have the *modified Bessel equation*

$$x^2y'' + xy' - \left(x^2 + v^2\right)y = 0, \tag{A-2.28}$$

with the general solution

$$y = AJ_v(ix) + BY_v(ix). \tag{A-2.29}$$

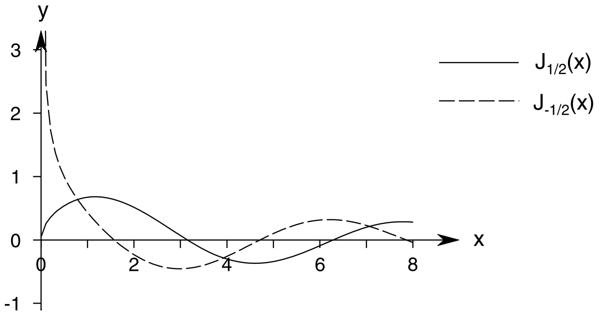

Fig. A.5 Graphs of $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$.

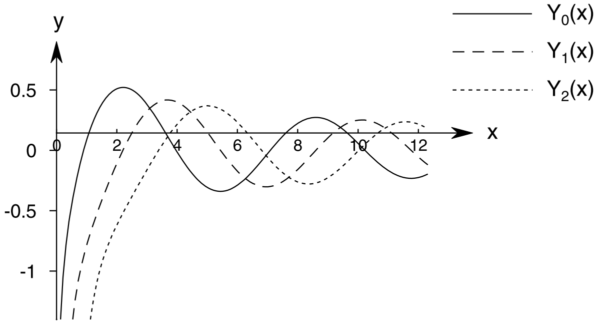

Fig. A.6 Graphs of $y = Y_0(x)$, $Y_1(x)$, and $Y_2(x)$.

We now define a new function

$$I_v(x) = i^{-v} J_v(ix), \tag{A-2.30}$$

and then use the series (A-2.1) for $J_v(x)$ so that

$$I_v(x) = i^{-v} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(r+v+1)} \left(\frac{ix}{2}\right)^{v+2r} = \sum_{r=0}^{\infty} \frac{1}{r!\Gamma(r+v+1)} \left(\frac{x}{2}\right)^{v+2r}. \tag{A-2.31}$$

Similarly, we can find the second solution, $K_v(x)$, of the modified Bessel equation (A-2.28). Usually, $I_v(x)$ and $K_v(x)$ are called *modified Bessel functions* and their properties can be obtained in a similar way to those of $J_v(x)$ and $Y_v(x)$. The graphs of $Y_0(x)$, $Y_1(x)$, and $Y_2(x)$ are shown in Figure A.6.

We state a few important infinite integrals involving Bessel functions which arise frequently in the application of Hankel transforms.

$$\int_0^{\infty} \exp(-at) J_v(bt) t^v \, dt = \frac{(2b)^v \Gamma(v+\frac{1}{2})}{\sqrt{\pi}(a^2+b^2)^{v+\frac{1}{2}}}, \quad v > -\frac{1}{2}, \tag{A-2.32}$$

$$\int_0^{\infty} \exp(-at) J_v(bt) t^{v+1} \, dt = \frac{2a(2b)^v \Gamma(v+\frac{3}{2})}{\sqrt{\pi}(a^2+b^2)^{v+\frac{3}{2}}}, \quad v > -1, \tag{A-2.33}$$

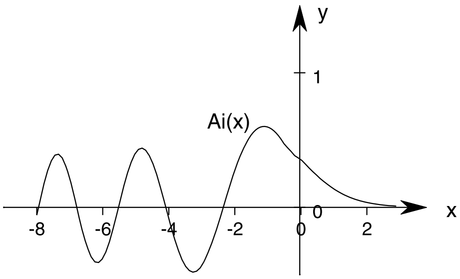

Fig. A.7 The Airy function.

$$\int_0^\infty \exp(-a^2t^2)\, J_v(bt)t^{v+1}\, dt = \frac{b^v}{(2a^2)^{v+1}} \exp\left(-\frac{b^2}{4a^2}\right), \quad v > -1, \tag{A-2.34}$$

$$\int_0^\infty \exp(-a^2t^2) J_v(bt) J_v(ct) t\, dt = \frac{1}{2a^2} \exp\left(-\frac{b^2+c^2}{4a^2}\right) I_v\left(\frac{bc}{2a^2}\right), \quad v > -1, \tag{A-2.35}$$

$$\int_0^\infty t^{2\mu-v-1}\, J_v(t)\, dt = \frac{2^{2\mu-v-1}\Gamma(\mu)}{\Gamma(v-\mu+1)}, \quad 0 < \mu < \frac{1}{2},\ v > -\frac{1}{2}. \tag{A-2.36}$$

The *Airy function*, $y = Ai(x)$, is the first solution of the differential equation

$$y'' - xy = 0. \tag{A-2.37}$$

The second solution is denoted by $Bi(x)$. Then these functions are expressed in terms of the Bessel and modified Bessel functions in the form

$$Ai(x) = \sqrt{\frac{x}{3}}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) - I_{\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right)\right] = \frac{1}{\pi}\sqrt{\frac{x}{3}} K_{\frac{1}{3}}(\xi), \tag{A-2.38}$$

$$Bi(x) = \sqrt{\frac{x}{3}}\left[I_{-\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right) + I_{\frac{1}{3}}\left(\frac{2}{3}x^{3/2}\right)\right] = \sqrt{\frac{x}{3}}\,\mathrm{Re}\left[e^{\frac{i\pi}{6}} H_{\frac{1}{3}}(-i\xi)\right], \tag{A-2.39}$$

where $\xi = \frac{2}{3}x^{3/2}$. The integral representation of $Ai(x)$ is

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + x\,t\right) dt. \tag{A-2.40}$$

The graph of $y = Ai(x)$ is shown in Figure A.7 using the values of $Ai(x)$ at $x = 0$ and $x \to \infty$:

$$Ai(0) = \frac{1}{\sqrt{3}} Bi(0) = \frac{1}{3^{3/2}\Gamma(\frac{2}{3})} = 0.355028,$$

$$\left[Ai(x), Bi(x)\right] \to [0, \infty] \quad \text{as } x \to \infty.$$

Similarly, the graph of $y = Bi(x)$ can be drawn.

The integral representation of $Bi(x)$ is

$$Bi(x) = \frac{1}{\pi} \int_0^{\infty} \left[\exp\left(xt - \frac{t^3}{3} \right) + \sin\left(xt + \frac{t^3}{3} \right) \right] dt. \tag{A-2.41}$$

A slightly more general integral representation of $Ai(ax)$ is

$$Ai(ax) = \frac{1}{\pi a} \int_0^{\infty} \cos\left(xt + \frac{t^3}{3a^3} \right) dt. \tag{A-2.42}$$

When $a = 1$, this reduces to (A-2.40).

The general power series solution of the Airy equation (A-2.37) is given by

$$\begin{aligned} y(x) = a_0 &\left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)} + \cdots \right] \\ &+ a_1 \left[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)} + \cdots \right] \end{aligned} \tag{A-2.43}$$

$$\begin{aligned} = a_0 &\left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{3 \cdot 4 \cdots (3n-4)(3n-3)(3n-1)(3n)} \right] \\ &+ a_1 \left[x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n-3)(3n-2)(3n)(3n+1)} \right], \end{aligned} \tag{A-2.44}$$

where a_0 and a_1 are arbitrary constants of integration.

Finally, the asymptotic representations of $Ai(x)$ and $Bi(x)$ are given by

$$Ai(x) \approx \frac{1}{2\sqrt{\pi} x^{1/4}} \exp\left(-\frac{2}{3} x^{3/2} \right) \quad \text{as } x \to +\infty, \tag{A-2.45}$$

$$Bi(x) \approx \frac{1}{\sqrt{\pi} x^{1/4}} \exp\left(\frac{3}{2} x^{3/2} \right) \quad \text{as } x \to \infty. \tag{A-2.46}$$

A-3 Legendre and Associated Legendre Functions

The Legendre polynomials, $P_n(x)$, are defined by the *Rodrigues formula*

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1 \right)^n. \tag{A-3.1}$$

The first seven Legendre polynomials are

$$P_0(x) = 1,$$
$$P_1(x) = x,$$

$$
\begin{aligned}
P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right),\\
P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right),\\
P_4(x) &= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right),\\
P_5(x) &= \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right),\\
P_6(x) &= \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right).
\end{aligned}
$$

The generating function for the Legendre polynomials is

$$\left(1 - 2xt + t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x). \tag{A-3.2}$$

This function provides more information about the Legendre polynomials. For example,

$$P_n(1) = 1, \qquad P_n(-1) = (-1)^n, \tag{A-3.3}$$

$$P_{2n}(0) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n n!} = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \tag{A-3.4}$$

$$P_{2n+1}(0) = 0, \quad n = 0, 1, 2, \ldots, \tag{A-3.5}$$

$$P_n(-x) = (-1)^n P_n(x), \qquad \frac{d^n}{dx^n} P_n(x) = \frac{(2n)!}{2^n n!}, \tag{A-3.6}$$

where the double factorial is defined by

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1) \quad \text{and} \quad (2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n).$$

The graphs of the first four Legendre polynomials are shown in Figure A.8.

The recurrence relations for the Legendre polynomials are

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \tag{A-3.7}$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \tag{A-3.8}$$

$$\left(1 - x^2\right)P'_n(x) = nP_{n-1}(x) - nxP_n(x), \tag{A-3.9}$$

$$\left(1 - x^2\right)P'_n(x) = (n+1)x\, P_n(x) - (n+1)\, P_{n+1}(x). \tag{A-3.10}$$

The Legendre polynomials, $y = P_n(x)$, satisfy the Legendre differential equation

$$\left(1 - x^2\right)y'' - 2x\, y' + n(n+1)y = 0. \tag{A-3.11}$$

If n is *not* an integer, both solutions of (A-3.11) diverge at $x = \pm 1$.

The orthogonal relation is

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = \frac{2}{(2n+1)}\delta_{nm}. \tag{A-3.12}$$

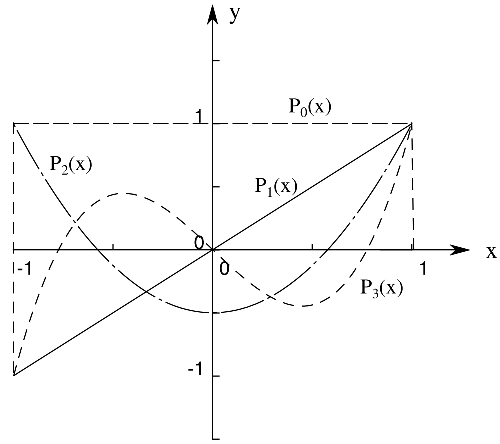

Fig. A.8 Graphs of $y = P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$.

The *associated Legendre functions* are defined by

$$P_n^m(x) = \left(1-x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x) = \frac{1}{2^n n!}\left(1-x^2\right)^{\frac{m}{2}} \frac{d^{m+n}}{dx^{m+n}}\left(x^2-1\right)^n, \quad \text{(A-3.13)}$$

where $0 \le m \le n$.

Clearly, it follows that

$$P_n^0(x) = P_n(x), \quad \text{(A-3.14)}$$

$$P_n^m(-x) = (-1)^{n+m} P_n^m(x), \qquad P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x). \quad \text{(A-3.15)}$$

The generating function for $P_n^m(x)$ is

$$\frac{(2m)!(1-x^2)^{\frac{m}{2}}}{2^m m!(1-2tx+t^2)^{m+\frac{1}{2}}} = \sum_{r=0}^{\infty} P_{r+m}^m(x) t^r. \quad \text{(A-3.16)}$$

The recurrence relations are

$$(2n+1)xP_n^m(x) = (n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x), \quad \text{(A-3.17)}$$

$$2\left(1-x^2\right)^{\frac{1}{2}} \frac{d}{dx} P_n^m(x) = P_n^{m+1}(x) - (n+m)(n-m+1)P_n^{m-1}(x). \quad \text{(A-3.18)}$$

The associated Legendre functions $P_n^m(x)$ are solutions of the differential equation

$$\left(1-x^2\right)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{(1-x^2)}\right] y = 0. \quad \text{(A-3.19)}$$

This reduces to the Legendre equation when $m = 0$.

Listed below are a few associated Legendre functions with $x = \cos\theta$:

$$
\begin{aligned}
P_1^1(x) &= \left(1-x^2\right)^{\frac{1}{2}} = \sin\theta, \\
P_2^1(x) &= 3x\left(1-x^2\right)^{\frac{1}{2}} = 3\cos\theta\sin\theta, \\
P_2^2(x) &= 3\left(1-x^2\right) = 3\sin^2\theta, \\
P_3^1(x) &= \frac{3}{2}\left(5x^2-1\right)\left(1-x^2\right)^{\frac{1}{2}} = \frac{3}{2}\left(5\cos^2\theta-1\right)\sin\theta, \\
P_3^2(x) &= 15x\left(1-x^2\right) = 15\cos\theta\sin^2\theta, \\
P_3^3(x) &= 15\left(1-x^2\right)^{3/2} = 15\sin^3\theta.
\end{aligned}
$$

The orthogonal relations are

$$\int_{-1}^{1} P_n^m(x) P_t^m(x)\, dx = \frac{2}{(2\ell+1)} \cdot \frac{(\ell+m)!}{(\ell-m)!}\delta_{n\ell}, \tag{A-3.20}$$

$$\int_{-1}^{1} \left(1-x^2\right)^{-1} P_n^m(x) P_n^\ell(x)\, dx = \frac{(n+m)!}{m(n-m)!}\delta_{m\ell}. \tag{A-3.21}$$

A-4 Jacobi and Gegenbauer Polynomials

The *Jacobi polynomials*, $P_n^{(\alpha,\beta)}(x)$, of degree n are defined by the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right], \tag{A-4.1}$$

where $\alpha > -1$ and $\beta > -1$.

When $\alpha = \beta = 0$, the Jacobi polynomials become Legendre polynomials, that is,

$$P_n(x) = P_n^{(0,0)}(x), \quad n = 0, 1, 2, \ldots. \tag{A-4.2}$$

On the other hand, the associated Laguerre functions arise as the limit

$$L_n^\alpha(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right). \tag{A-4.3}$$

The recurrence relations for $P_n^{(\alpha,\beta)}(x)$ are

$$
\begin{aligned}
&2(n+1)(\alpha+\beta+n+1)(\alpha+\beta+2n)P_{n+1}^{(\alpha,\beta)}(x) \\
&\quad = (\alpha+\beta+2n+1)\left[\left(\alpha^2-\beta^2\right) + x(\alpha+\beta+2n+2)(\alpha+\beta+2n)\right]P_n^{(\alpha,\beta)}(x) \\
&\qquad - 2(\alpha+n)(\beta+n)(\alpha+\beta+2n+2)P_{n-1}^{(\alpha,\beta)}(x),
\end{aligned} \tag{A-4.4}
$$

where $n = 1, 2, 3, \ldots$, and

$$P_n^{(\alpha,\beta-1)}(x) - P_n^{(\alpha-1,\beta)}(x) = P_{n-1}^{(\alpha,\beta)}(x). \tag{A-4.5}$$

The generating function for Jacobi polynomials is

$$2^{(\alpha+\beta)} R^{-1}(1-t+R)^{-\alpha}(1+t+R)^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) t^n, \tag{A-4.6}$$

where $R = (1-2xt+t^2)^{\frac{1}{2}}$.

The Jacobi polynomials, $y = P_n^{(\alpha,\beta)}(x)$, satisfy the differential equation

$$(1-x^2)y'' + \big[(\beta-\alpha) - (\alpha+\beta+2)x\big]y' + n(n+\alpha+\beta+1)y = 0. \tag{A-4.7}$$

The orthogonal relation is

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\, dx = \begin{cases} 0 & \text{if } n \neq m, \\ \delta_n & \text{if } n = m, \end{cases} \tag{A-4.8}$$

where

$$\delta_n = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}. \tag{A-4.9}$$

When $\alpha = \beta = v - \frac{1}{2}$, the Jacobi polynomials reduce to the *Gegenbauer polynomials*, $C_n^v(x)$, which are defined by the Rodrigues formula

$$C_n^v(x) = \frac{(-1)^n}{2^n n!}\left(1-x^2\right)^{v-\frac{1}{2}} \frac{d^n}{dx^n}\left[\left(1-x^2\right)^{v+n-\frac{1}{2}}\right]. \tag{A-4.10}$$

The generating function for $C_n^v(x)$ of degree n is

$$\left(1-2x\,t+t^2\right)^{-v} = \sum_{n=0}^{\infty} C_n^v(x)\, t^n, \quad |t|<1,\ |x| \leq 1,\ v > -\frac{1}{2}. \tag{A-4.11}$$

The recurrence relations are

$$(n+1)C_{n+1}^v(x) - 2(v+n)xC_n^v(x) + (2v+n-1)C_{n-1}^v(x) = 0, \tag{A-4.12}$$

$$(n+1)C_{n+1}^v(x) - 2vC_n^{v+1}(x) + 2vC_{n-1}^{v+1}(x) = 0, \tag{A-4.13}$$

$$\frac{d}{dx}\left[C_n^v(x)\right] = 2vC_{n+1}^{v+1}(x). \tag{A-4.14}$$

The differential equation satisfied by $y = C_n^v(x)$ is

$$(1-x^2)y'' - (2v+1)xy' + n(n+2v)y = 0. \tag{A-4.15}$$

The orthogonal property is

$$\int_{-1}^{1} \left(1-x^2\right)^{v-\frac{1}{2}} C_n^v(x) C_m^v(x)\, dx = \delta_n \delta_{nm}, \tag{A-4.16}$$

where

$$\delta_n = \frac{2^{1-2v} n \Gamma(n+2v)}{n!(n+v)[\Gamma(v)]^2}. \tag{A-4.17}$$

When $v = \frac{1}{2}$, the Gegenbauer polynomials reduce to Legendre polynomials, that is,

$$C_n^{\frac{1}{2}}(x) = P_n(x). \tag{A-4.18}$$

The Hermite polynomials can also be obtained from the Gegenbauer polynomials as the limit

$$H_n(x) = n! \lim_{v\to\infty} v^{-n/2} C_n^v \left(\frac{x}{\sqrt{v}}\right). \tag{A-4.19}$$

Finally, when $\alpha = \beta = \frac{1}{2}$, the Gegenbauer polynomials reduce to the well-known *Chebyshev polynomials*, $T_n(x)$, which are defined by a solution of the second order difference equation

$$u_{n+2} - 2x\, u_{n+1} + u_n = 0, \quad |x| \le 1, \tag{A-4.20}$$

$$u(0) = u_0 \quad \text{and} \quad u(1) = u_1. \tag{A-4.21}$$

The generating function for $T_n(x)$ is

$$\frac{(1-t^2)}{(1-2x\,t+t^2)} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x) t^n, \quad |x| \le 1,\ t < 1. \tag{A-4.22}$$

The first seven Chebyshev polynomials of degree n of the first kind are

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x,\\
T_2(x) &= 2x^2 - 1,\\
T_3(x) &= 4x^3 - 3x,\\
T_4(x) &= 8x^4 - 8x^2 + 1,\\
T_5(x) &= 16x^5 - 20x^3 + 5x,\\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1.
\end{aligned}$$

The graphs of the first four Chebyshev polynomials are shown in Figure A.9.

The Chebyshev polynomials $y = T_n(x)$ satisfy the differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0. \tag{A-4.23}$$

It follows from (A-4.22) that $T_n(x)$ satisfies the recurrence relations

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, \tag{A-4.24}$$

$$T_{n+m}(x) - 2T_n(x)T_m(x) + T_{n-m}(x) = 0, \tag{A-4.25}$$

$$(1-x^2)T_n'(x) + nxT_n(x) - nT_{n-1}(x) = 0. \tag{A-4.26}$$

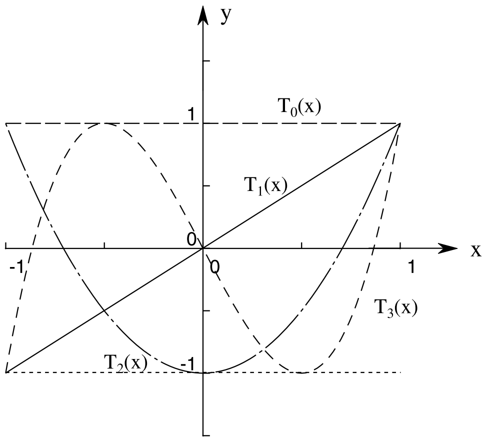

Fig. A.9 Chebyshev polynomials $y = T_n(x)$.

The parity relation for $T_n(x)$ is

$$T_n(-x) = (-1)^n T_n(x). \tag{A-4.27}$$

The Rodrigues formula is

$$T_n(x) = \frac{\sqrt{\pi}(-1)^n(1-x^2)^{\frac{1}{2}}}{2^n(n-\frac{1}{2})!} \cdot \frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n-\frac{1}{2}}\right]. \tag{A-4.28}$$

The orthogonal relation for $T_n(x)$ is

$$\int_{-1}^{1} \left(1-x^2\right)^{-\frac{1}{2}} T_m(x)\, T_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{\pi}{2} & \text{if } m = n, \\ \pi & \text{if } m = n = 0. \end{cases} \tag{A-4.29}$$

The *Chebyshev polynomials of the second kind*, $U_n(x)$, are defined by

$$U_n(x) = \left(1-x^2\right)^{-\frac{1}{2}} \sin\left[(n+1)\cos^{-1} x\right], \quad -1 \leq x \leq 1. \tag{A-4.30}$$

The generating function for $U_n(x)$ is

$$\left(1-2x\,t+t^2\right)^{-1} = \sum_{n=0}^{\infty} U_n(x) t^n, \quad |x| < 1, \ |t| < 1. \tag{A-4.31}$$

The first seven Chebyshev polynomials $U_n(x)$ are given by

$$\begin{aligned} U_0(x) &= 1, \\ U_1(x) &= 2x, \\ U_2(x) &= 4x^2 - 1, \end{aligned}$$

$$U_3(x) = 8x^3 - 4x,$$
$$U_4(x) = 16x^4 - 12x^2 + 1,$$
$$U_5(x) = 32x^5 - 32x^3 + 6x,$$
$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1.$$

The differential equation for $y = U_n(x)$ is

$$(1 - x^2)y'' - 3xy' + n(n + 2)y = 0. \tag{A-4.32}$$

The recurrence relations are

$$U_{n+1}(x) - 2x\,U_n(x) + U_{n-1}(x) = 0, \tag{A-4.33}$$
$$(1 - x^2)U_n'(x) + nxU_n(x) - (n + 1)U_{n-1}(x) = 0. \tag{A-4.34}$$

The parity relation is

$$U_n(-x) = (-1)^n U_n(x). \tag{A-4.35}$$

The Rodrigues formula is

$$U_n(x) = \frac{\sqrt{\pi}(-1)^n(n + 1)}{2^{n+1}(n + \frac{1}{2})!(1 - x^2)^{\frac{1}{2}}}\frac{d^n}{dx^n}\left[(1 - x^2)^{n+\frac{1}{2}}\right]. \tag{A-4.36}$$

The orthogonal relation for $U_n(x)$ is

$$\int_{-1}^{1}(1 - x^2)^{\frac{1}{2}}\,U_m(x)\,U_n(x)\,dx = \frac{\pi}{2}\delta_{mn}. \tag{A-4.37}$$

A-5 Laguerre and Associated Laguerre Functions

The Laguerre polynomials $L_n(x)$ are defined by the *Rodrigues formula*

$$L_n(x) = e^x\frac{d^n}{dx^n}\left(x^n\,e^{-x}\right), \tag{A-5.1}$$

where $n = 0, 1, 2, 3, \ldots$.

The first seven Laguerre polynomials are

$$L_0(x) = 1,$$
$$L_1(x) = 1 - x,$$
$$L_2(x) = 2 - 4x + x^2,$$
$$L_3(x) = 6 - 18x + 9x^2 - x^3,$$
$$L_4(x) = 24 - 96x + 72x^2 - 16x^3 + x^4,$$
$$L_5(x) = 120 - 600x + 600x^2 - 200x^3 + 25x^4 - x^5,$$
$$L_6(x) = 720 - 4320x + 5400x^2 - 2400x^3 + 450x^4 - 36x^5 + x^6.$$

The generating function is

$$(1-t)^{-1}\exp\left(\frac{x\,t}{1-t}\right)=\sum_{n=0}^{\infty}t^n\,L_n(x). \tag{A-5.2}$$

In particular,

$$L_n(0)=1. \tag{A-5.3}$$

The orthogonal relation for the Laguerre polynomial is

$$\int_0^{\infty}e^{-x}L_m(x)L_n(x)\,dx=(n!)^2\delta_{nm}. \tag{A-5.4}$$

The recurrence relations are

$$(n+1)L_{n+1}(x)=(2n+1-x)L_n(x)-nL_{n-1}(x), \tag{A-5.5}$$

$$xL_n'(x)=nL_n(x)-nL_{n-1}(x), \tag{A-5.6}$$

$$L_n'(x)=L_{n-1}'(x)-L_{n-1}(x). \tag{A-5.7}$$

The Laguerre polynomials, $y=L_n(x)$, satisfy the *Laguerre differential equation*

$$xy''+(1-x)y'+ny=0. \tag{A-5.8}$$

The *associated Laguerre polynomials* are defined by

$$L_n^m(x)=\frac{d^m}{dx^m}L_n(x)\quad \text{for } n\geq m. \tag{A-5.9}$$

The generating function for $L_n^m(x)$ is

$$(1-z)^{-(m+1)}\exp\left(-\frac{x\,z}{1-z}\right)=\sum_{n=0}^{\infty}L_n^m(x)z^n,\qquad |z|<1. \tag{A-5.10}$$

It follows from this that

$$L_n^m(0)=\frac{(n+m)!}{n!m!}. \tag{A-5.11}$$

The associated Laguerre function satisfies the *recurrence relation*

$$(n+1)L_{n+1}^m(x)=(2n+m+1-x)L_n^m(x)-(n+m)L_{n-1}^m(x), \tag{A-5.12}$$

$$x\frac{d}{dx}L_n^m(x)=n\,L_n^m(x)-(n+m)L_{n-1}^m(x). \tag{A-5.13}$$

The associated Laguerre function, $y=L_n^m(x)$, satisfies the associated Laguerre differential equation

$$xy''+(m+1-x)y'+ny=0. \tag{A-5.14}$$

The Rodrigues formula for $L_n^m(x)$ is

$$L_n^m(x)=\frac{e^x x^{-m}}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+m}\right). \tag{A-5.15}$$

The *orthogonal relation* for $L_n^m(x)$ is

$$\int_0^{\infty}e^{-x}x^mL_n^m(x)L_l^m(x)\,dx=\frac{(n+m)!}{n!}\delta_{n\,l}. \tag{A-5.16}$$

A-6 Hermite Polynomials and Weber–Hermite Functions

The Hermite polynomials $H_n(x)$ are defined by the Rodrigues formula

$$H_n(x) = (-1)^n \exp\left(x^2\right) \frac{d^n}{dx^n}\left[\exp\left(-x^2\right)\right], \tag{A-6.1}$$

where $n = 0, 1, 2, 3, \ldots$.

The first seven Hermite polynomials are

$$\begin{aligned}
H_0(x) &= 1, \\
H_1(x) &= 2x, \\
H_2(x) &= 4x^2 - 2, \\
H_3(x) &= 8x^3 - 12x, \\
H_4(x) &= 16x^4 - 48x^2 + 12, \\
H_5(x) &= 32x^5 - 16x^3 + 120x, \\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120.
\end{aligned}$$

The *generating function* is

$$\exp\left(2xt - t^2\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x). \tag{A-6.2}$$

It follows from (A-6.2) that $H_n(x)$ satisfies the *parity relation*

$$H_n(-x) = (-1)^n H_n(x). \tag{A-6.3}$$

Also, it follows from (A-6.2) that

$$H_{2n+1}(0) = 0, \qquad H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}. \tag{A-6.4}$$

The *recurrence relations* for Hermite polynomials are

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \tag{A-6.5}$$

$$H_n'(x) = 2xH_{n-1}(x). \tag{A-6.6}$$

The Hermite polynomials, $y = H_n(x)$, are solutions of the *Hermite differential equation*

$$y'' - 2xy' + 2ny = 0. \tag{A-6.7}$$

The orthogonal property of Hermite polynomials is

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) H_n(x) H_m(x)\, dx = 2^n n! \sqrt{\pi}\, \delta_{mn}. \tag{A-6.8}$$

With repeated use of integration by parts, it follows from (A-6.1) that

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) H_n(x) x^m dx = 0, \quad m = 0, 1, \ldots, (n-1), \tag{A-6.9}$$

$$\int_{-\infty}^{\infty} \exp\left(-x^2\right) H_n(x) x^n \, dx = \sqrt{\pi} n!. \tag{A-6.10}$$

The *Weber–Hermite function*, or simply *Hermite function*,

$$y = h_n(x) = \exp\left(-\frac{x^2}{2}\right) H_n(x) \tag{A-6.11}$$

satisfies the Hermite differential equation

$$y'' + \left(\lambda - x^2\right) y = 0, \quad x \in \mathbb{R}, \tag{A-6.12}$$

where $\lambda = 2n + 1$. If $\lambda \neq 2n + 1$, then y is not finite as $|x| \to \infty$.

The Hermite functions $\{h_n(x)\}_0^\infty$ form an orthogonal basis for the Hilbert space $L^2(\mathbb{R})$ with weight function 1. They satisfy the following fundamental properties:

$$\begin{aligned} &h_n'(x) + x h_n(x) - 2_n h_{n-1}(x) = 0, \\ &h_n'(x) - x h_n(x) + h_{n+1}(x) = 0, \\ &h_n''(x) - x^2 h_n(x) + (2n+1) h_x = 0, \\ &\mathcal{F}\left\{h_n(x)\right\} = \tilde{h}_n(k) = (-i)^n h_n(k). \end{aligned}$$

The normalized Weber–Hermite functions are given by

$$\psi_n(x) = 2^{-n/2} \pi^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2}\right) H_n(x). \tag{A-6.13}$$

Physically, they represent quantum-mechanical oscillator wave functions. The graphs of these functions are shown in Figure A.10.

A-7 Mittag-Leffler Function

Another important function that has widespread use in fractional calculus and fractional differential equation is the *Mittag-Leffler function*. The Mittag-Leffler function is an entire function defined by the series

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0. \tag{A-7.1}$$

The graph of the Mittag-Leffler function is shown in Figure A.11.

The generalized Mittag-Leffler function, $E_{\alpha,\beta}(z)$, is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0. \tag{A-7.2}$$

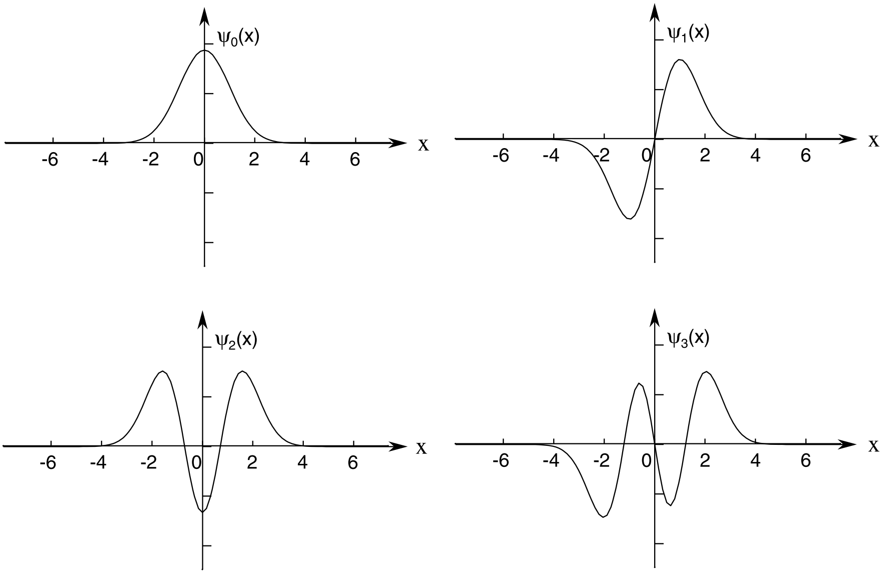

Fig. A.10 The normalized Weber–Hermite functions.

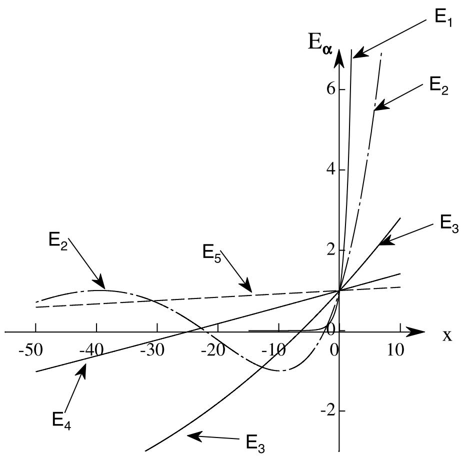

Fig. A.11 Graph of the Mittag-Leffler function $E_\alpha(x)$.

Also the inverse Laplace transform yields

$$\mathcal{L}^{-1}\left\{\frac{m!s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}\right\} = t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}\left(\pm a t^\alpha\right), \tag{A-7.3}$$

where

$$E^{(m)}_{\alpha,\beta}(z) = \frac{d^m}{dz^m}E_{\alpha,\beta}(z). \tag{A-7.4}$$

Obviously,

$$E_{\alpha,1}(z) = E_\alpha(z), \qquad E_{1,1}(z) = E_1(z) = e^z. \tag{A-7.5}$$

A-8 The Jacobi Elliptic Integrals and Elliptic Functions

The parametric equation of an ellipse is given by

$$x = a\sin\theta, \quad y = b\cos\theta, \quad (a > b),\ 0 \le \theta \le \phi. \tag{A-8.1}$$

Using the arclength formula from calculus, the length of the elliptic arc (A-8.1) is

$$\begin{aligned} ds^2 &= dx^2 + dy^2 = \left(a^2\cos^2\theta + b^2\sin^2\theta\right)d\theta^2 \\ &= a^2\left(1 - \frac{a^2 - b^2}{a^2}\sin^2\theta\right)d\theta^2 = a^2\left(1 - m^2\sin^2\theta\right)d\theta^2, \end{aligned} \tag{A-8.2}$$

where $e = m = (\frac{a^2-b^2}{a^2})^{\frac{1}{2}} < 1$ is the eccentricity of the ellipse.

Consequently, (A-8.2) gives the length of the elliptic arc

$$s = \int_0^s ds = a\int_0^\phi \sqrt{1 - m^2\sin^2\theta}\, d\theta. \tag{A-8.3}$$

This integral cannot be evaluated in terms of elementary functions. Because of its origin, it is called an *elliptic integral*. In general, there are three classes of elliptic integrals, *called elliptic integrals of the first, second and third* kinds, and they defined by

$$F(\phi, m) = \int_0^\phi \frac{d\theta}{\sqrt{1 - m^2\sin^2\theta}}, \tag{A-8.4}$$

$$E(\phi, m) = \int_0^\phi \sqrt{1 - m^2\sin^2\theta}\, d\theta, \tag{A-8.5}$$

$$\Pi(\phi, m, n) = \int_0^\phi \frac{d\theta}{(1 + n^2\sin^2\theta)\sqrt{1 - m^2\sin^2\theta}}, \quad (m \ne n), \tag{A-8.6}$$

where the parameter ϕ is called the *amplitude*, $\phi = am(F, m)$ so that $am(0, m) = 0$ and m $(0 < m < 1)$ is called the *modulus*. When $\phi = \frac{\pi}{2}$, (A-8.4)–(A-8.6) are referred to as *complete elliptic integrals of the first*, *second and third kinds*, and they are denoted by special symbols: $K(m) = F(\frac{\pi}{2}, m)$, $E(m) = E(\frac{\pi}{2}, m)$ and $\Pi(m, n) = \Pi(\frac{\pi}{2}, m, n)$.

Putting $x = \sin\theta$, $0 \le \theta \le \phi$, the first, second and third elliptic integrals can be written in equivalent forms as

$$F(\phi, m) = \int_0^{\sin\phi} \frac{dx}{\sqrt{(1 - x^2)(1 - m^2x^2)}}, \tag{A-8.7}$$

$$E(\phi,m)=\int_0^{\sin\phi}\frac{\sqrt{1-m^2x^2}}{\sqrt{1-x^2}}\,dx, \tag{A-8.8}$$

$$\Pi(\phi,m,n)=\int_0^{\sin\phi}\frac{dx}{(1+n^2x^2)\sqrt{(1-x^2)(1-m^2x^2)}}. \tag{A-8.9}$$

Using (A-8.7)–(A-8.9), the Jacobi elliptic functions, $sn(u,m)$, $cn(u,m)$ and $dn(u,m)$, are defined by

$$sn(u,m)=\sin\phi, \tag{A-8.10}$$

$$cn(u,m)=\cos\phi=\left(1-sn^2\,u\right)^{\frac{1}{2}}, \tag{A-8.11}$$

$$dn(u,m)=\left(1-m^2\sin^2\phi\right)^{\frac{1}{2}}=\left(1-m^2\,sn^2\,u\right)^{\frac{1}{2}} \tag{A-8.12}$$

so that

$$sn(-u)=-\,sn\,u,\qquad cn(-u)=cn\,u,\qquad dn(-u)=dn\,u, \tag{A-8.13}$$

$$sn(0)=0,\quad\text{and}\quad cn(0)=dn(0)=1. \tag{A-8.14}$$

The following limiting results also hold

$$\lim_{m\to 0} sn(u,m)=\sin u,\qquad \lim_{m\to 0} cn(z,m)=\cos u,\qquad \lim_{m\to 0} dn(z,m)=1, \tag{A-8.15}$$

$$\lim_{m\to 1} sn(u,m)=\tanh u,\qquad \lim_{m\to 1} cn(u,m)=\operatorname{sech} u,\qquad \lim_{m\to 1} dn(u,m)=\operatorname{sech} u. \tag{A-8.16}$$

Making reference to Dutta and Debnath (1965) without proof, we state the following basic properties of the Jacobi elliptic functions:

$$sn^2\,u+cn^2\,u=1,\qquad dn^2\,u+m^2\,sn^2\,u=1,\qquad dn^2\,u-m^2\,cn^2\,u=1-m^2, \tag{A-8.17}$$

$$\frac{d}{du}\,sn\,u=cn\,u\,dn\,u,\qquad \frac{d}{du}\,cn\,u=-\,sn\,u\,dn\,u, \tag{A-8.18}$$

and

$$\frac{d}{du}\,dn\,u=-m^2\,sn\,u\,cn\,u. \tag{A-8.19}$$

Putting $sn(u,m)=x$ in the first result in (A-8.18) gives the differential equation

$$\frac{dx}{du}=\sqrt{\left(1-x^2\right)\left(1-m^2x^2\right)}, \tag{A-8.20}$$

so that it leads to the *Legendre normal form* for the sn-function

$$u=\int_0^{sn(u,m)}\frac{dx}{\sqrt{(1-x^2)(1-m^2x^2)}}. \tag{A-8.21}$$

Similarly, it follows from (A-8.18)–(A-8.19) that the *Legendre normal forms* for cn and dn functions are

$$u = \int_{cn(u,m)}^{1} \frac{dx}{\sqrt{(1-x^2)(m'^2+m^2x^2)}}, \tag{A-8.22}$$

$$u = \int_{dn(u,m)}^{1} \frac{dx}{\sqrt{(1-x^2)(x^2-m'^2)}}, \tag{A-8.23}$$

where $m' = \sqrt{1-m^2}$ is called the *complementary modulus* of the elliptic integral.

The complete elliptic integrals are then given by

$$K(m) = F\left(\frac{\pi}{2}, m\right) = K'(m'), \qquad E(m) = E\left(\frac{\pi}{2}, m\right) = E'(m'). \tag{A-8.24}$$

The limiting values of $K(m)$ and $E(m)$ are given as follows:

$$\lim_{m\to 0} K(m) = K(0) = \frac{\pi}{2}, \qquad \lim_{m\to 0} E(m) = E(0) = \frac{\pi}{2}, \tag{A-8.25}$$

$$\lim_{m\to 1} K(m) = K(1) = \infty, \qquad \lim_{m\to 1} E(m) = E(1) = 1. \tag{A-8.26}$$

Finally, the addition theorems for sn, cn, and dn functions are

$$sn(u+v) = \frac{sn\,u\;cn\,v\;dn\,v + sn\,v\;cn\,u\;dn\,u}{(1-m^2\,sn^2\,u\;sn^2\,v)}, \tag{A-8.27}$$

$$cn(u+v) = \frac{cn\,u\;cn\,v - sn\,u\;dn\,u\;sn\,v\;dn\,v}{(1-m^2\,sn^2\,u\;sn^2\,v)}, \tag{A-8.28}$$

$$dn(u+v) = \frac{dn\,u\;dn\,v - m^2\,sn\,u\;cn\,u\;sn\,v\;cn\,v}{(1-m^2\,sn^2\,u\;sn^2\,v)}, \tag{A-8.29}$$

where $sn(u+4K) = sn\,u$, $cn(u+4K) = cn\,u$, and $dn(u+2K) = dn\,u$ so that sn and cn are periodic functions of period $4K$, and dn is a periodic function of period $2K$.

B

Fourier Series, Generalized Functions, and Fourier and Laplace Transforms

The main purpose of this appendix is to discuss Fourier series and generalized functions, and to state their basic properties that are most frequently used in the theory and applications of ordinary and partial differential equations. The subject is, of course, too vast to be treated adequately in so short a space, so that only the more important results will be stated. Included are basic properties of Fourier and Laplace transforms which are used in finding solutions of ordinary and partial differential equations. For a fuller discussion of these topics and of further properties of these functions the reader is referred to the standard treatises on the subjects including Debnath and Bhatta (2007).

B-1 Fourier Series and Its Basic Properties

If $f(x)$ is a periodic function of period 2π defined in $(-\pi,\pi)$, then $f(x)$ can be represented as an infinite series in terms of trigonometric functions in the form

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{B-1.1}$$

This is known as the *Fourier Series*. If we assume that the infinite series is term-by-term integrable on $(-\pi,\pi)$, then

$$\int_{-\pi}^{\pi}f(x)\,dx=\int_{-\pi}^{\pi}\left[\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\right]dx=\pi a_0$$

so that

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx. \tag{B-1.2}$$

Multiplying both sides of (B-1.1) by $\cos mx$ and integrating the resulting series from $-\pi$ to π gives

L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers*,
DOI 10.1007/978-0-8176-8265-1,

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$
$$= \int_{-\pi}^{\pi}\left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx\,dx = \pi a_n, \quad m = n.$$

Thus,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx. \tag{B-1.3}$$

Similarly, multiplying (B-1.1) by $\sin mx$ and integrating over $(-\pi, \pi)$ gives

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \tag{B-1.4}$$

The Fourier coefficients a_n and b_n given by (B-1.2)–(B-1.4) are known as the *Euler–Fourier formulas*.

If $f(x)$ is an even function of x defined on $[-\pi, \pi]$, then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx, \quad n = 0, 1, 2, \ldots, \tag{B-1.5}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0, \quad n = 1, 2, 3, \ldots. \tag{B-1.6}$$

Hence, the Fourier series of an even function $f(x)$ can be written as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \tag{B-1.7}$$

where a_n are given by (B-1.5).

Similarly, if $f(x)$ is an odd function of x on $(-\pi, \pi)$, the Fourier series of an odd function is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \tag{B-1.8}$$

where $a_n = 0$ for all n, and b_n are given by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$
$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx, \quad n = 1, 2, 3, \ldots. \tag{B-1.9}$$

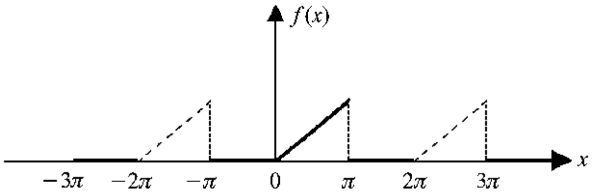

Fig. B.1 The function $f(x)$ and its extension.

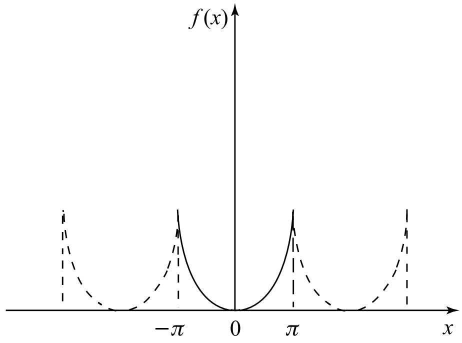

Fig. B.2 Graph of $f(x) = x^2$ and its extension.

Example B-1.1. The Fourier series of $f(x)$ (see Figure B.1)

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0, \\ x & \text{if } 0 \le x < \pi \end{cases}$$

is given by (B-1.1), where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{1}{\pi n^2}\big[(-1)^n - 1\big], \\ b_n &= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = -\frac{1}{n}(-1)^n. \end{aligned}$$

Hence, the Fourier series of $f(x)$ is

$$f(x) = \frac{\pi}{4} - \sum_{n=1}^{\infty}\left[\frac{2}{\pi(2n-1)^2}\cos(2n-1)x + \frac{(-1)^n}{n}\sin nx\right]. \quad \text{(B-1.10)}$$

Example B-1.2. The Fourier series of the (see Figure B.2) function

$$f(x) = x^2, \quad -\pi \le x \le \pi \text{ with } f(x \pm 2n\pi) = f(x),\ n = 1, 2, 3, \ldots,$$

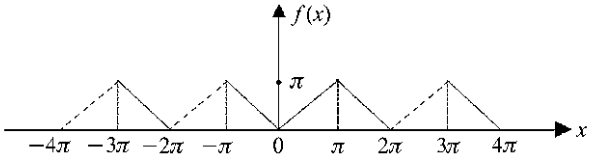

Fig. B.3 The triangular wave function and its extension.

is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \tag{B-1.11}$$

Since $f(x)$ is even, $b_n = 0$ for all $n \geq 1$, it turns out that

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \frac{2\pi^2}{3}, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx \\ &= \frac{4}{n^2}(-1)^n, \quad n \geq 1. \end{aligned}$$

Example B-1.3. The triangular wave function (see Figure B.3)

$$f(x) = |x| = \begin{cases} -x & \text{if } -\pi \leq x < 0, \\ x & \text{if } 0 \leq x < \pi, \end{cases}$$

with $f(x \pm 2\pi n) = f(x)$ has the Fourier cosine series representation

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x. \tag{B-1.12}$$

In this case, $f(x)$ is even, thus, $b_n = 0$ for all $n \geq 1$, and

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \, dx = \frac{2}{\pi} \int_0^{\pi} x \, dx = \pi, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx \\ &= \frac{2}{\pi n^2} \left[(-1)^n - 1\right], \end{aligned}$$

and so

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even}, \\ -\frac{4}{\pi n^2} & \text{if } n \text{ is odd}. \end{cases}$$

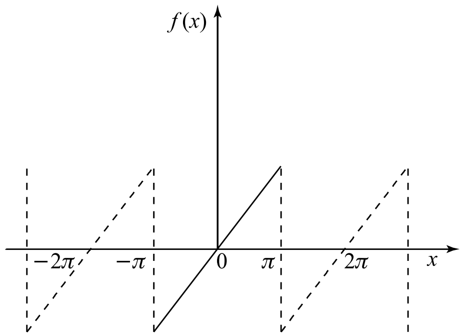

Fig. B.4 The sawtooth wave function.

Example B-1.4. The sawtooth wave function (see Figure B.4)

$$f(x) = x, \quad -\pi < x < \pi,$$

with $f(x) = f(x \pm 2n\pi)$, $n = 1, 2, 3, \ldots$, has the Fourier sine series expansion

$$f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}. \tag{B-1.13}$$

In this case, $f(x)$ is odd and hence, $a_n = 0$ for all $n \geq 0$, and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx = \frac{2}{n}(-1)^{n+1}.$$

Example B-1.5. The Fourier series representation of the square wave function (see Figure B.5) defined by

$$f(x) = \operatorname{sgn}(x) = \begin{cases} 1 & \text{if } 0 < x \leq \pi, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } -\pi \leq x < 0 \end{cases}$$

is given by

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)x. \tag{B-1.14}$$

Obviously, $f(x)$ is odd, and hence, $a_n = 0$ for all $n \geq 0$, and b_n is given by

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn} x \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} \sin nx \, dx \\ &= \frac{2}{\pi}\left[\frac{1-(-1)^n}{n}\right] = \begin{cases} 0 & \text{if } n \text{ is even}, \\ \frac{4}{n\pi} & \text{if } n \text{ is odd}. \end{cases} \end{aligned}$$

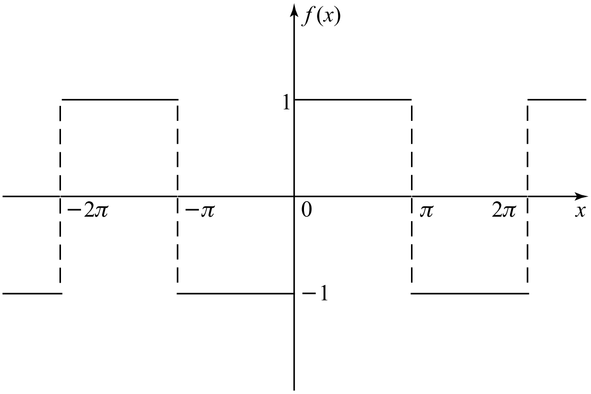

Fig. B.5 The square wave function and its extension.

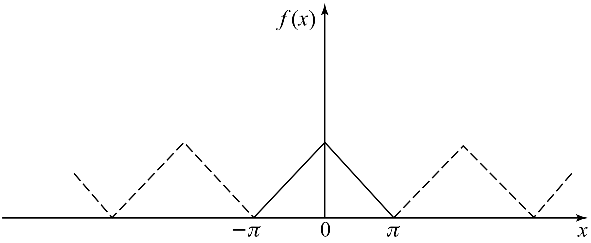

Fig. B.6 The triangular wave function and its extension.

Example B-1.6. The triangular wave function (see Figure B.6) $f(x)$ on $[-\pi, \pi]$ is given by

$$f(x) = \begin{cases} \pi + x & \text{if } -\pi \leq x \leq 0, \\ \pi - x & \text{if } 0 \leq x \leq \pi. \end{cases}$$

Since $f(x)$ is even, $b_n = 0$ for $n \geq 1$, and

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx = \pi, \\ a_n &= \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx = \frac{2}{\pi}\left[\frac{1}{n^2}\big(1 - (-1)^n\big)\right] \\ &= \frac{2}{\pi n^2} \cdot \begin{cases} 0 & \text{if } n \text{ is even}, \\ 2 & \text{if } n \text{ is odd}. \end{cases} \end{aligned}$$

Thus, the Fourier series for $f(x)$ is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x. \end{aligned} \tag{B-1.15}$$

If $f(x)$ is a periodic function of period $2l$ and is defined on $[-l, l]$, then the Fourier representation of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right], \tag{B-1.16}$$

where a_0, a_n, and b_n are given by the Euler–Fourier formulas:

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx, \tag{B-1.17}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \tag{B-1.18}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx. \tag{B-1.19}$$

If $f(x)$ is an even function of period $2l$ defined on $[-l, l]$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \tag{B-1.20}$$

where $b_n = 0$, $n \geq 1$, and

$$a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \quad n = 0, 1, 2, \ldots. \tag{B-1.21}$$

If $f(x)$ is an odd function of period $2l$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \tag{B-1.22}$$

where $a_n = 0$, $n \geq 0$, and

$$b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx. \tag{B-1.23}$$

The functions $f(x)$ in all Examples B-1.1–B-1.6 can be defined on $[-l, l]$ and the corresponding Fourier series can be obtained directly by calculating Fourier coefficients a_n and b_n, or by using the transformation $x = \frac{\pi t}{l}$. In Example B-1.1, the Fourier series on $[-l, l]$ is

$$f(x) = \frac{l}{4} - \frac{l}{\pi} \sum_{n=1}^{\infty} \left[\frac{2}{\pi(2n-1)^2} \cos\frac{(2n-1)\pi x}{l} + \frac{(-1)^n}{n} \sin\left(\frac{n\pi x}{l}\right) \right]. \tag{B-1.24}$$

In Example B-1.2, the Fourier series on $[-l, l]$ is given by

$$f(x) = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{l}\right). \tag{B-1.25}$$

In Example B-1.3, the Fourier series on $[-l, l]$ is

$$f(x) = \frac{l}{2} - \left(\frac{4l}{\pi^2}\right) \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left\{(2n-1)\frac{x\pi}{l}\right\}. \tag{B-1.26}$$

When $l = \pi$, this reduces to (B-1.12).

In Example B-1.4, the Fourier series on $[-l, l]$ is

$$f(x) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right). \tag{B-1.27}$$

In Example B-1.5, the Fourier series of $\text{sgn}(x)$ on $[-l, l]$ is

$$\text{sgn}(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\left\{(2n-1)\frac{x\pi}{l}\right\}. \tag{B-1.28}$$

In Example B-1.6, the Fourier series on $[-l, l]$ is

$$f(x) = \frac{l}{2} + \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)\frac{\pi x}{l}. \tag{B-1.29}$$

It is sometimes convenient to represent a function $f(x)$ by a Fourier series in complex form. This expansion can easily be derived from the Fourier series (B-1.1), that is,

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2} a_n \left(e^{inx} + e^{-inx}\right) + \frac{b_n}{2i}\left(e^{inx} - e^{-inx}\right)\right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx}\right] \\ &= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{inx} + c_{-n} e^{-inx}\right) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \end{aligned}$$

where

$$\begin{aligned} c_0 &= \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx, \\ c_n &= \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(\cos nx - i \sin nx)\,dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\,dx, \end{aligned}$$

$$c_{-n} = \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx + i\sin nx)\,dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx = \overline{c_n}.$$

Thus, we obtain the Fourier series of $f(x)$ in complex form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad -\pi < x < \pi, \tag{B-1.30}$$

where the Fourier coefficients c_n are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx. \tag{B-1.31}$$

Multiplying (B-1.30) by $\frac{1}{2\pi}f(x)$ and integrating from $-\pi$ to π gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \sum_{n=-\infty}^{\infty} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}\,dx$$
$$= \sum_{n=-\infty}^{\infty} c_n \cdot c_{-n}$$
$$= \sum_{n=-\infty}^{\infty} c_n\overline{c_n} = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{B-1.32}$$

Thus, (B-1.32) is known as the *Parseval formula* for a complex Fourier series.

We next consider the nth partial sum

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx), \tag{B-1.33}$$

of the Fourier series for $f(x)$ in (B-1.1) defined on $[-\pi,\pi]$. If $\int_{-\pi}^{\pi} f^2(x)\,dx$ exists and is finite, then

$$0 \le \int_{-\pi}^{\pi}\left[f(x) - s_n(x)\right]^2 dx$$
$$= \int_{-\pi}^{\pi} f^2(x)\,dx - 2\int_{-\pi}^{\pi} f(x)s_n(x)\,dx + \int_{-\pi}^{\pi} s_n^2(x)\,dx. \tag{B-1.34}$$

It follows from the definition of the Fourier coefficients (B-1.2)–(B-1.4) and the orthogonality of the cosine and sine functions that

$$\int_{-\pi}^{\pi} f(x)s_n(x)\,dx = \int_{-\pi}^{\pi} f(x)\left[\frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)\right]dx$$
$$= \frac{\pi a_0^2}{2} + \pi\sum_{k=1}^{n}(a_k^2 + b_k^2).$$

Similarly, it turns out that

$$\begin{aligned}\int_{-\pi}^{\pi} s_n^2(x)\,dx &= \int_{-\pi}^{\pi}\left[\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)\right]^2 dx\\ &= \int_{-\pi}^{\pi}\frac{a_0^2}{4}\,dx+\sum_{k=1}^{n}\left[a_k^2\int_{-\pi}^{\pi}\cos^2 kx\,dx+b_k^2\int_{-\pi}^{\pi}\sin^2 kx\,dx\right]\\ &= \frac{\pi a_0^2}{2}+\pi\sum_{k=1}^{n}(a_k^2+b_k^2).\end{aligned}$$

Consequently, (B-1.34) reduces to

$$\begin{aligned}0 &\le \int_{-\pi}^{\pi}[f(x)-s_n(x)]^2\,dx\\ &= \int_{-\pi}^{\pi} f^2(x)\,dx-\left[\frac{\pi a_0^2}{2}+\pi\sum_{k=1}^{n}(a_k^2+b_k^2)\right].\end{aligned} \tag{B-1.35}$$

This leads to the inequality

$$\frac{a_0^2}{2}+\sum_{k=1}^{n}(a_k^2+b_k^2)\le\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx, \tag{B-1.36}$$

and since the right-hand side of (B-1.36) is independent of n, it follows in the limit as $n\to\infty$ that

$$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)\le\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx. \tag{B-1.37}$$

This is known as the *Bessel inequality* for a Fourier series.

Since the left-hand side of (B-1.36) is non-decreasing and is bounded above, the series

$$\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2) \tag{B-1.38}$$

converges. Thus, the necessary condition for the convergence of the series (B-1.38) is that

$$\lim_{k\to\infty} a_k = 0 \quad\text{and}\quad \lim_{k\to\infty} b_k = 0. \tag{B-1.39}$$

That is,

$$\lim_{k\to\infty}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = 0, \qquad \lim_{k\to\infty}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = 0. \tag{B-1.40}$$

These results are known as the *Riemann–Lebesgue Lemma*.

The Fourier series is said to converge in the mean to $f(x)$ when

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\left[f(x)-s_n(x)\right]^2 dx = 0. \tag{B-1.41}$$

If the Fourier series converges in the mean to $f(x)$, then

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,dx. \tag{B-1.42}$$

This is called *Parseval's relation*, and it is one of the fundamental results in the theory of Fourier series. This relation can formally be derived from the convergence of the Fourier series to $f(x)$ on $[-\pi,\pi]$. In other words, if

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx), \tag{B-1.43}$$

where a_0, a_n, and b_n are given by (B-1.2)–(B-1.4), we multiply by (B-1.43) by $\frac{1}{\pi}f(x)$ and integrate the resulting expression from $-\pi$ to π to obtain

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)\,dx\\ &= \frac{a_0}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx\\ &\quad+\sum_{n=1}^{\infty}\left[\frac{1}{\pi}a_n\int_{-\pi}^{\pi}f(x)\cos nx\,dx+\frac{1}{\pi}b_n\int_{-\pi}^{\pi}f(x)\sin nx\,dx\right].\end{aligned} \tag{B-1.44}$$

Replacing all integrals on the right hand side of (B-1.44) by the Fourier coefficients gives the Parseval relation (B-1.42).

If two (2π)-periodic integrable functions $f(x)$ and $g(x)$ defined on $[-\pi,\pi]$ have the Fourier series expansions

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx), \tag{B-1.45}$$

$$g(x)=\frac{1}{2}\alpha_0+\sum_{n=1}^{\infty}(\alpha_n\cos nx+\beta_n\sin nx), \tag{B-1.46}$$

where a_0, a_n, and b_n are given by (B-1.1)–(B-1.3), and α_0, α_n, and β_n are given by results similar to those of (B-1.1)–(B-1.3), then the following Parseval's relations hold

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)+g(x)\right]^2 dx=\frac{1}{2}(a_0+\alpha_0)^2+\sum_{n=1}^{\infty}\left[(a_n+\alpha_n)^2+(b_n+\beta_n)^2\right], \tag{B-1.47}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(x)-g(x)\right]^2 dx=\frac{1}{2}(a_0-\alpha_0)^2+\sum_{n=1}^{\infty}\left[(a_n-\alpha_n)^2+(b_n-\beta_n)^2\right]. \tag{B-1.48}$$

Subtracting (B-1.48) from (B-1.47) yields

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx=\frac{1}{2}a_0\alpha_0+\sum_{n=1}^{\infty}(a_n\alpha_n+b_n\beta_n). \tag{B-1.49}$$

This is a general Parseval relation for the product function $f(x)g(x)$. When $f=g$, (B-1.49) reduces to (B-1.42).

In the context of the complex Fourier series expansion (B-1.30) of a (2π)-periodic function $f(x)$, we can replace the Fourier coefficient c_n by $\widehat{f}(n)$ so that

$$c_n=\widehat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}f(x)\,dx. \tag{B-1.50}$$

The concept of *convolution* of two (2π)-periodic integrable functions f and g on $\mathbb{R}$ arises naturally, so that we define their convolution $(f*g)(x)$ on $[-\pi,\pi]$ by

$$(f*g)(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)\,dt. \tag{B-1.51}$$

Physically, the convolution $(f*g)(x)$ represents an integral output of the two functions f and g in contrast to the ordinary pointwise product (output) $f(x)g(x)$. Clearly, the convolution is commutative, that is,

$$(f*g)(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)g(x-\xi)\,d\xi=(g*f)(x). \tag{B-1.52}$$

Another interpretation of a convolution is that it represents an *average* (or *mean*) *value*. In particular, if $g=1$ in (B-1.52), then $f*g$ is constant and is equal to

$$(f*1)(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,d\xi. \tag{B-1.53}$$

This means that $(f*1)(x)$ represents the average value of $f(x)$ on $[-\pi,\pi]$.

In addition to commutativity, the convolution satisfies the following algebraic and analytic properties for any constant α and β:

$$f*(\alpha g+\beta h)=\alpha(f*g)+\beta(f*h)\quad \text{(Distributive)}, \tag{B-1.54}$$

$$(f*g)*h=f*(g*h)\quad \text{(Associative)}, \tag{B-1.55}$$

$$(f*g)(x)\text{ is continuous}\quad \text{(Continuity)}, \tag{B-1.56}$$

$$\widehat{f*g}(n)=\widehat{f}(n)\widehat{g}(n)\quad \text{(Convolution)}. \tag{B-1.57}$$

To prove (B-1.57), we use the definition

$$\begin{aligned}
\widehat{f*g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-inx}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)g(x-t)\,dt\right]dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-int}f(t)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-in(x-t)}g(x-t)\,dx\right]dt, \quad x-t=s \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-int}f(t)\,dt\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}g(s)e^{-ins}\,ds\right] \\
&= \widehat{f}(n)\widehat{g}(n).
\end{aligned}$$

The nth partial sum of a complex Fourier series (B-1.30) is

$$\begin{aligned}
s_n(x) &= \sum_{k=-n}^{n}c_k e^{ikx} = \sum_{k=-n}^{n}e^{ikx}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)e^{-ikt}\,dt\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\left[\sum_{k=-n}^{n}e^{ik(x-t)}\right]dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)D_n(x-t)\,dt, \qquad \text{(B-1.58)} \\
&= D_n(x)*f(x), \qquad \text{(B-1.59)}
\end{aligned}$$

where $D_n(x)$ is called the *Dirichlet kernel* defined by

$$\begin{aligned}
D_n(x) &= \sum_{k=-n}^{n}e^{ikx} = 1+\sum_{k=1}^{n}\left(e^{ikx}+e^{-ikx}\right) \\
&= 1+\sum_{k=1}^{n}2\cos kx. \qquad \text{(B-1.60)}
\end{aligned}$$

We next make the following observations regarding the genesis of the convolution in the theory of Fourier. The convolution $(D_n*f)(x)$ arises in the nth partial sum $s_n(x)$ of the Fourier series of $f(x)$. Thus, the problem of understanding $s_n(x)$ reduces to that of $(D_n*f)(x)$.

It follows from (B-1.60) that

$$\begin{aligned}
\frac{1}{2}D_n(x)\sin\frac{x}{2} &= \frac{1}{2}\sin\frac{x}{2}+\sin\frac{x}{2}(\cos x+\cos 2x+\cdots+\cos nx) \\
&= \frac{1}{2}\sin\frac{x}{2}+\frac{1}{2}\left[\left(\sin\frac{3x}{2}-\sin\frac{x}{2}\right)+\left(\sin\frac{5x}{2}-\sin\frac{3x}{2}\right)\right. \\
&\quad \left.+\cdots+\left(\sin\left(n+\frac{1}{2}\right)x-\sin\left(n-\frac{1}{2}\right)x\right)\right] \\
&= \frac{1}{2}\sin\left(n+\frac{1}{2}\right)x.
\end{aligned}$$

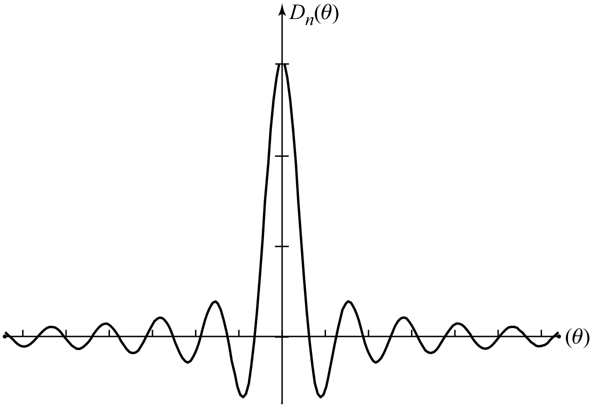

Fig. B.7 Graph of $D_n(x = \theta)$ against $(x = \theta)$.

Thus, the exact form of the Dirichlet kernel is

$$D_n(x) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}}. \tag{B-1.61}$$

This reveals that the denominator of the Dirichlet kernel $D_n(x)$ vanishes at the points $x = 2\pi m$ which are removable points of discontinuity. Furthermore, it follows from (B-1.60) that $D_n(x)$ is an even function with period 2π and satisfies

$$\int_{-\pi}^{\pi} D_n(x)\,dx = 2\pi. \tag{B-1.62}$$

The graph of $D_n(x = \theta)$ is shown in Figure B.7. It looks very similar to that of the diffusion kernel as shown in Figure 1.6 in Chapter 1 except for its symmetric oscillatory trail in both positive and negative θ-axes.

It follows from (B-1.58), by putting $t - x = \xi$ and noting that $D_n(\xi)$ is even, that

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi}\int_{-\pi-x}^{\pi-x} f(x+\xi)D_n(\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\xi)D_n(\xi)\,d\xi. \end{aligned} \tag{B-1.63}$$

We close this section by stating the *Pointwise Convergence Theorem*: If $f(x)$ is a piecewise smooth and periodic function with period 2π on $[-\pi, \pi]$, then, for any x in $(-\pi, \pi)$,

$$\begin{aligned} \lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left[\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\right] \\ &= \frac{1}{2}\big[f(x+0) + f(x-0)\big], \end{aligned} \tag{B-1.64}$$

or equivalently,

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \frac{1}{2}\left[f(x+) + f(x-)\right], \tag{B-1.65}$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx, \quad k = 0, 1, 2, 3, \ldots, \tag{B-1.66}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx, \quad k = 1, 2, 3, \ldots. \tag{B-1.67}$$

We refera to Myint-U and Debnath (2007) for its proof.

Another remarkable feature of Fourier series of a function at its ordinary points of discontinuity deals with the behavior of its nth partial sums $s_n(x)$ as $n \to \infty$. At points where $f(x)$ is continuous, the nth partial sums $s_n(x)$ approach smoothly the value $f(x)$ as $n \to \infty$. However, for the functions $f(x) = x$ in Example B-1.4 or $f(x) = sgn x$ in Example B-1.5, the graphs of their partial sums exhibit a large error in the neighborhood of points of discontinuity at $x = 0$ and $x = \pm\pi$ independent of the number of terms in their partial sums. In other words, the partial sums do not converge smoothly to the mean value. Instead, they overshoot the mark at each end of the jumps of the function. The explanation of this phenomenon was first provided by J.W. Gibbs (1839–1903) in 1899, who showed that overshooting was not the result of computational errors. This feature is typical for the Fourier series of a function at the points of discontinuity, and is now universally known as the *Gibbs phenomenon*.

One of the most effective and useful applications of Fourier series deals with the summation of infinite series in closed form which is one of the major problems in mathematics. We shall use Examples B-1.1–B-1.6 to derive the sums of many important numerical series.

Substituting $x = 0$ in (B-1.10) gives the numerical series

$$0 = f(0) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Hence,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8},$$

or equivalently,

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}. \tag{B-1.68}$$

This can be used to obtain another numerical series

$$S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}. \tag{B-1.69}$$

In fact,

$$\begin{aligned} S &= \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) \\ &= \frac{\pi^2}{8} + \frac{1}{4} S. \end{aligned}$$

Thus, $S(1 - \frac{1}{4}) = \frac{\pi^2}{8}$, which gives (B-1.69).

Putting $x = 0$ in (B-1.11) leads to the numerical series

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

or equivalently,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Thus,

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}. \tag{B-1.70}$$

Substituting $x = \frac{\pi}{2}$ in the series (B-1.13) gives

$$\begin{aligned} \frac{\pi}{2} &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} \\ &= 2\left(\frac{\sin \frac{\pi}{2}}{1} - \frac{\sin \frac{2\pi}{2}}{2} + \frac{\sin \frac{3\pi}{2}}{3} - \frac{\sin \frac{4\pi}{2}}{4} + \cdots\right) \\ &= 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right). \end{aligned}$$

Therefore,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}. \tag{B-1.71}$$

This is celebrated *Leibniz series* for π discovered by Leibniz in 1673. It is also known as the *Gregory series* independently discovered by James Gregory (1638–1675) in around 1670.

Putting $x = \frac{\pi}{4}$ in (B-1.13) gives another numerical series

$$\frac{\pi}{8} = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots\right) - \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

In view of (B-1.71), this leads to the following series

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \frac{\pi}{2\sqrt{2}}. \tag{B-1.72}$$

Subtracting (B-1.70) from (B-1.69) yields

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \cdots = \frac{\pi^2}{24}. \tag{B-1.73}$$

In Example B-1.2, the Fourier series for $f(x) = x^2$ is given by (B-1.11). It follows from the Parseval relation (B-1.42) that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} a_n^2 = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^4},$$

or equivalently,

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty}\frac{16}{n^4}.$$

Thus,

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}. \tag{B-1.74}$$

If we apply the Parseval formula (B-1.42) to Example B-1.3, we can derive the following numerical series

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}. \tag{B-1.75}$$

A similar calculation can be used for the Fourier series of $f(x) = x(\pi - x)$, $0 < x < \pi$, in the form

$$f(x) = \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right). \tag{B-1.76}$$

Therefore, we can show that

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}, \tag{B-1.77}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}. \tag{B-1.78}$$

In 1.15 Exercises, Problem 10, the initial conditions are $f(x)$ and $g(x) = 0$ defined on $0 \le x \le l$ by

$$f(x) = \begin{cases} \frac{2hx}{l} & \text{if } 0 \le x \le \frac{l}{2}, \\ \frac{2h(l-x)}{l} & \text{if } \frac{l}{2} \le x \le l, \end{cases} \tag{B-1.79}$$

where the midpoint of the string of length l is held at a vertical distance h from the equilibrium position for $t < 0$ and released at time $t = 0$.

We expand $f(x)$ in a Fourier sine series $(a_n \equiv 0)$ so that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \tag{B-1.80}$$

where

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \\ &= \frac{2}{l}\left[\int_0^{l/2}\left(\frac{2hx}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l}\frac{2h(l-x)}{l}\sin\left(\frac{n\pi x}{l}\right)dx\right], \end{aligned}$$

which is, integrating by parts,

$$\begin{aligned} &= \frac{4h}{l^2}\left[\left(-\frac{l}{n\pi}\right)x\cos\left(\frac{n\pi x}{l}\right)\bigg|_0^{l/2} - \int_0^{l/2}\cos\left(\frac{n\pi x}{l}\right)dx \right. \\ &\quad \left. + (l-x)\cos\frac{n\pi x}{l}\bigg|_{l/2}^{l} + \int_{l/2}^{l}\cos\frac{n\pi x}{l}\,dx\right] \\ &= \left(\frac{8h}{n^2\pi^2}\right)\sin\left(\frac{n\pi}{2}\right). \end{aligned} \tag{B-1.81}$$

Thus, when $h = 1$ and $l = \pi$, the Fourier sine series on the interval $0 < x < \pi$ for the function $f(x)$ is given by

$$f(x) = \frac{8}{\pi^2}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots\right). \tag{B-1.82}$$

Thus, the Fourier sine series for $f(x) = x$ on $0 < x < \frac{\pi}{2}$ is

$$x = \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \cdots\right). \tag{B-1.83}$$

Putting $x = \frac{\pi}{2}$ in (B-1.83) gives the numerical series

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots. \tag{B-1.84}$$

The Fourier sine series (B-1.83) for x can be integrated from x to $\frac{\pi}{2}$ term-by-term so that

$$\frac{1}{2}\left(\frac{\pi^2}{4} - x^2\right) = \frac{4}{\pi}\left(\cos x - \frac{1}{3^3}\cos 3x + \frac{1}{5^3}\cos 5x - \cdots\right). \tag{B-1.85}$$

Substituting $x = 0$ into (B-1.85) gives the numerical series

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}. \tag{B-1.86}$$

Integrating (B-1.85) with respect to x from 0 to x leads to

$$\frac{1}{2}\left(\frac{\pi^2}{4}x - \frac{x^3}{3}\right) = \frac{4}{\pi}\left(\sin x - \frac{1}{3^4}\sin 3x + \frac{1}{5^4}\sin 5x - \cdots\right). \tag{B-1.87}$$

Putting $x = \frac{\pi}{2}$ in (B-1.87) yields the numerical series

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi}{8} \cdot \frac{\pi^3}{12} = \frac{\pi^4}{96}. \tag{B-1.88}$$

Integrating (B-1.87) again from x to $\frac{\pi}{2}$ gives

$$\begin{aligned} &\frac{\pi^2}{16}\left[\left(\frac{\pi}{2}\right)^2 - x^2\right] - \frac{1}{24}\left[\left(\frac{\pi}{2}\right)^4 - x^4\right] \\ &\quad = \frac{4}{\pi}\left(\cos x - \frac{1}{3^5}\cos 3x + \frac{1}{5^5}\cos 5x - \cdots\right). \end{aligned} \tag{B-1.89}$$

Substituting $x = 0$ in (B-1.89) gives the numerical series

$$\frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \cdots = \frac{5\pi^5}{1536}. \tag{B-1.90}$$

Integrating (B-1.89) again from 0 to x yields a new numerical series

$$\begin{aligned} &\left(\frac{\pi^4}{64}x - \frac{\pi^2}{48}x^3 - \frac{\pi^4}{384}x + \frac{1}{120}x^5\right) \\ &\quad = \frac{4}{\pi}\left(\sin x - \frac{1}{3^6}\sin 3x + \frac{1}{5^6}\sin 5x - \cdots\right). \end{aligned} \tag{B-1.91}$$

Putting $x = \frac{\pi}{2}$ in (B-1.91) leads to another numerical series

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots = \frac{\pi^6}{960}. \tag{B-1.92}$$

We consider applications of Fourier series to differential equations. As a simple application, we discuss the periodic solution of a non-homogeneous simple harmonic oscillator

$$\ddot{x} + \omega^2 x = f(t), \tag{B-1.93}$$

where the forcing function $f(t)$ has a Fourier series expansion

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt), \tag{B-1.94}$$

the coefficients A_0, A_n, and B_n are known.

We seek a uniformly convergent Fourier series solution of (B-1.93) in the form

$$x(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt). \tag{B-1.95}$$

Differentiating (B-1.95) term by term and assuming the series for $\dot{x}(t)$ and $\ddot{x}(t)$ converge uniformly, substituting in (B-1.93) gives

$$\begin{aligned}\sum_{n=1}^{\infty}(-n^2+\omega^2)a_n \cos nt + \frac{a_0}{2}\omega_0^2 + \sum_{n=1}^{\infty}(-n^2+\omega^2)b_n \sin nt \\ = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt).\end{aligned} \tag{B-1.96}$$

Consequently, equating the coefficients, we obtain

$$a_0 = \frac{A_0}{\omega^2}, \qquad a_n = \frac{A_n}{\omega^2 - n^2}, \qquad b_n = \frac{B_n}{\omega^2 - n^2}.$$

Thus, the solution of (B-1.93) is given by

$$x(t) = \frac{A_0}{2\omega^2} + \sum_{n=1}^{\infty}\frac{(A_n \cos nt + B_n \sin nt)}{\omega^2 - n^2} \tag{B-1.97}$$

provided $\omega \neq n$. However, if $\omega^2 = n^2$ for some integer n, the solution becomes unbounded. This phenomenon is known as *resonance*. If damping is included in equation (B-1.93), the solution will remain bounded.

However, when $\omega^2 = n^2$, we can write

$$g(t) = f(t) - A_\omega \cos \omega t + B_\omega \sin \omega t,$$

and then we can solve separately $\ddot{x} + \omega^2 x = g(t)$ and $\ddot{x} + \omega^2 x = A_\omega \cos \omega t + B_\omega \sin \omega t$. The second equation gives rise to the *non-periodic solutions* $(At+B)\cos \omega t$ and $(A't+B')\sin \omega t$. The sum of the solutions of the two equations gives a solution of the original equation. Equation (B-1.93) can be solved by the use of integrating factor in the form

$$x(t) = \exp(-i\omega t)\int_0^t \exp(2i\omega\tau)\left[\int_0^\tau e^{i\omega x} f(x)\,dx\right]d\tau. \tag{B-1.98}$$

We next apply the method of Fourier series to solve the damped harmonic oscillator governed by

$$\ddot{x} + k\dot{x} + \omega^2 x = f(t), \tag{B-1.99}$$

where $k\dot{x}$ $(k > 0)$ is the damping term.

If $f(t)$ is periodic with period 2π and has the same Fourier series expansion (B-1.94), then we can find a Fourier series solution of (B-1.95) by differentiating and substituting in (B-1.99) so that equation (B-1.99) takes the form

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(-n^2 a_n + nkb_n + \omega^2 a_n\right)\cos nt + \left(\frac{\omega^2 a_0}{2}\right) \\
&\quad + \sum_{n=1}^{\infty}\left(-n^2 b_n - nka_n + \omega^2 b_n\right)\sin nt \\
&= \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt). \qquad \text{(B-1.100)}
\end{aligned}
$$

Equating the coefficients from both sides gives $a_0 = \frac{A_0}{\omega^2}$ and

$$
\begin{aligned}
\left(\omega^2 - n^2\right)a_n + nkb_n &= A_n, \\
-nka_n + \left(\omega^2 - n^2\right)b_n &= B_n.
\end{aligned}
$$

Solving for a_n and b_n gives

$$
a_n = \frac{\begin{vmatrix} A_n & nk \\ B_n & \omega^2 - n^2 \end{vmatrix}}{\begin{vmatrix} \omega^2 - n^2 & nk \\ -nk & \omega^2 - n^2 \end{vmatrix}} = \frac{1}{D}\left[\left(\omega^2 - n^2\right)A_n - nkB_n\right], \qquad \text{(B-1.101)}
$$

$$
b_n = \frac{\begin{vmatrix} \omega^2 - n^2 & A_n \\ -nk & B_n \end{vmatrix}}{\begin{vmatrix} \omega^2 - n^2 & nk \\ -nk & \omega^2 - n^2 \end{vmatrix}} = \frac{1}{D}\left[\left(\omega^2 - n^2\right)B_n + nkA_n\right], \qquad \text{(B-1.102)}
$$

where

$$
D = \left(\omega^2 - n^2\right)^2 + n^2 k^2. \qquad \text{(B-1.103)}
$$

Thus, the coefficients a_0, a_n, and b_n of the Fourier series solution (B-1.95) of the damped simple harmonic equation (B-1.99) are given by (B-1.101)–(B-1.102), and the solution represents the periodic response. There may be a transient response depending on the initial conditions, and the transient solution will eventually decay as $t \to \infty$. Because of the presence of the damping term, there will be *no resonance* when $n = \omega$.

We close this section by adding an example of an application of Fourier series to a simple boundary value problem

$$
\frac{d^2 u}{dx^2} = \lambda u = f(x), \quad 0 < x < l, \qquad \text{(B-1.104)}
$$

$$
u(0) = 0 = u(l), \qquad \text{(B-1.105)}
$$

where λ is a given parameter.

For simplicity, we assume that $f(x)$ has a Fourier sine series expansion

$$
f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l, \qquad \text{(B-1.106)}
$$

where b_n are known. We seek the Fourier sine series solution of the boundary value problem (B-1.104)–(B-1.105) in the form

$$u(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l. \tag{B-1.107}$$

We assume that this series may be differentiated twice to give

$$\frac{d^2u}{dx^2} = \sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{l^2}\right) B_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l.$$

We substitute $u(x)$ and $u''(x)$ into (B-1.104) to obtain

$$\sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{l^2} + \lambda\right) B_n \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l.$$

Consequently, the coefficients B_n are given by

$$B_n = \left(\lambda - \frac{n^2\pi^2}{l^2}\right)^{-1} b_n, \tag{B-1.108}$$

provided $\lambda \neq (\frac{n\pi}{l})^2$.

If $\lambda = (\frac{n\pi}{l})^2$ for all or some values of positive integer n, then B_n cannot be determined, and hence no solution exists unless $B_n \equiv 0$.

Thus, the Fourier series solution of the boundary value problem is

$$u(x) = \sum_{n=1}^{\infty}\left(\frac{l^2 b_n}{\lambda l^2 - n^2\pi^2}\right) \sin\left(\frac{n\pi x}{l}\right), \tag{B-1.109}$$

where the zero denominator must be handled separately.

B-2 Generalized Functions (Distributions)

The most widely known example of a generalized function is the *Dirac delta function* $\delta(x)$ which was first introduced by P.M.M. Dirac in 1920s as a mathematical device in the formulation of quantum mechanics. The Dirac delta function $\delta(x)$ is defined by

$$\delta(x) = 0 \quad \text{for all } x \neq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{B-2.1}$$

In the ordinary sense, $\delta(x)$ cannot be considered as a function because if a function vanishes everywhere except at a single point $x = 0$, then its integral over any interval must be zero, so that integrating it over an interval including the origin cannot be equal to one. However, from the physical point of view, the delta function is the natural mathematical quantity which can be used to describe many of the abstractions which arise in physical sciences. For example, the mass-density function

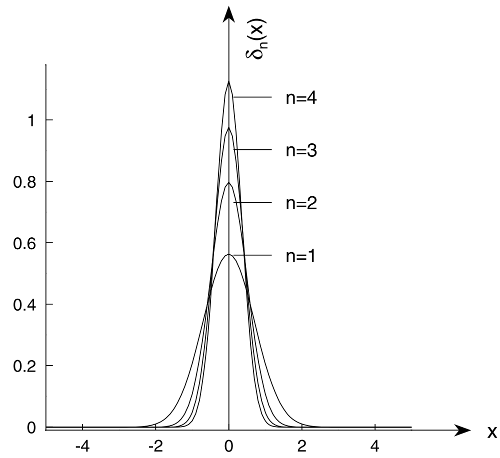

Fig. B.8 The sequence of delta functions, $\delta_n(x)$.

is zero everywhere except at $x = 0$, where it is infinite because a finite mass is concentrated in zero length, and it is so infinitely large here that the integral is non-zero even through the integrand is positive over an infinitesimally small region only. This makes sense physically, though it is mathematically absurd. So the delta function $\delta(x)$ can be used as the mass-density function $\rho(x)$ describing the mass distribution per unit length of a rod at a point x. Similarly, the point charge, the impulsive force, the point dipole, and the frequency response of an undamped harmonic oscillator are all aptly represented by the delta function or other generalized functions.

In general, the generalized functions play a major role in Fourier analysis and in the theory of partial differential equations. The function $f(x) = 1$ has no Fourier transform in the ordinary Fourier transform theory, but it has Fourier transform $\sqrt{2\pi}\delta(k)$ in the generalized function theory. Thus, the generalized functions remove difficulties which existed in the classical Fourier analysis. There is a beautiful analogy with the way in which the use of complex numbers helps in the solution of quadratic equations as within the realm of real numbers only certain quadratic equations have no solutions.

Sometimes, the Dirac delta function $\delta(x)$ is defined by its fundamental property

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a), \tag{B-2.2}$$

where $f(x)$ is any continuous function in any interval containing the point $x = a$.

Although there are no ordinary functions which have the properties required of the delta function, we may approximate $\delta(x)$ in one dimension by a sequence of ordinary functions

$$\delta_n(x) = \sqrt{\frac{n}{\pi}}\exp\left(-nx^2\right), \quad n = 1, 2, 3, \ldots. \tag{B-2.3}$$

Clearly, $\delta_n(x) \to 0$ as $n \to \infty$ for any $x \neq 0$ and $\delta_n(0) \to \infty$ as $n \to \infty$ as shown in Figure B.8. Also, for all $n = 1, 2, 3, \ldots,$

$$\int_{-\infty}^{\infty} \delta_n(x)\, dx = 1$$

and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x)\, dx = \int_{-\infty}^{\infty} \delta(x)\, dx = 1,$$

as expected. Thus, the delta function can be considered as the limit of a sequence of ordinary Gaussian functions, and we write

$$\delta(x) = \lim_{n\to\infty} \sqrt{\frac{n}{\pi}} \exp\left(-nx^2\right). \tag{B-2.4}$$

As mentioned in Chapter 1, the major problem of finding the solution of inhomogeneous partial differential equations deals with the construction of the Green's function in each case. This problem becomes easier by the use of a generalized function together with the methods of Fourier and Laplace transforms. In spherical polar coordinates (r, θ, ϕ), the Laplacian is

$$\nabla^2 \psi(r) = \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right), \tag{B-2.5}$$

which is independent of θ and ϕ. If $\psi(r) = \frac{1}{r}$, then $\nabla^2\psi(r) = 0$ for all $r \neq 0$. At $r = 0$, $\nabla^2\psi(r)$ does not exist. By the divergence theorem (or Gauss' theorem)

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV = \int_V \operatorname{div} \nabla\left(\frac{1}{r}\right) dV = \int_S \nabla\left(\frac{1}{r}\right) \cdot \mathbf{n}\, \mathbf{dS}, \tag{B-2.6}$$

where V is the volume of a sphere of radius a and center at $r = 0$ so that

$$\left[\nabla\left(\frac{1}{r}\right)\right]_{r=a} = \hat{e}_r \left[\frac{d}{dr}\left(\frac{1}{r}\right)\right]_{r=a} = -\frac{\hat{e}_r}{a^2}, \tag{B-2.7}$$

where $\hat{e}_r$ is the direction of the outward normal $\mathbf{n}$ to the surface of the sphere. It follows from (B-2.6) that

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV = -\frac{1}{a^2} \int_S \hat{e}_r \cdot \mathbf{n}\, \mathbf{ds} = -4\pi. \tag{B-2.8}$$

Thus, the Laplacian $\nabla^2(\frac{1}{r})$ is a function of r which has the following fundamental properties:

(i) It vanishes for $r \neq 0$.
(ii) It is not defined at $r = 0$.
(iii) And its integral over any sphere with center at $r = 0$ is -4π.

All of these lead to the result

$$\int_V \nabla^2\left(\frac{1}{r}\right) dV = \begin{cases} -4\pi & \text{if } V \text{ contains } r = 0, \\ 0 & \text{if } V \text{ does not contain } r = 0. \end{cases} \tag{B-2.9}$$

This may be written in the compact form

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r), \tag{B-2.10}$$

where $\delta(r)$ is the Dirac delta function with the property

$$\int_V \delta(r)\, dV = \begin{cases} 1 & \text{if } V \text{ contains } r = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{B-2.11}$$

This result is a special case of the general definition of the vector form of the delta function

$$\int_V f(\mathbf{r})\delta(\mathbf{r}-\mathbf{a})\, d\mathbf{r} = \begin{cases} f(\mathbf{a}) & \text{if } V \text{ contains the point } \mathbf{a}, \\ 0 & \text{otherwise,} \end{cases} \tag{B-2.12}$$

obtained by setting $f(\mathbf{r}) = 1$ and $\mathbf{a} = \mathbf{0}$ in (B-2.12).

For the results (B-2.11) and (B-2.12) to be valid, it is certainly sufficient that the functions be continuous and infinitely differentiable everywhere. Result (B-2.12) reduces to (B-2.2) in one dimension.

We would not go into great detail, but refer to the famous books of Lighthill (1958) and Jones (1982) for the introduction to the subject of generalized functions.

A *good function*, $g(x)$, is a function in $C^\infty(\mathbb{R})$ that decays sufficiently rapidly so that $g(x)$ and all of its derivatives decay to zero faster than $|x|^{-N}$ as $|x| \to \infty$ for all $N > 0$. In other words, suppose that for each positive integer N and n,

$$\lim_{|x|\to\infty} x^N g^{(n)}(x) = 0, \tag{B-2.13}$$

then $g(x)$ is called a *good function*.

Usually, the class of good functions is represented by $\mathbb{S}$. The good functions play an important role in Fourier analysis because the inversion, convolution, and differentiation theorems as well as many others take simple forms with no problem of convergence. The rapid decay and infinite differentiability properties of good functions lead to the fact that the Fourier transform of a good function is also a good function.

Good functions also play an important role in the theory of generalized functions. A good function of bounded support is a special type of good function that also plays an important part in the theory of generalized functions. Good functions also have the following important properties. The sum (or difference) of two good functions is also a good function. The product and convolution of two good functions are good functions. The derivative of a good function is a good function; $x^n\, g(x)$ is a good function for all non-negative integers n whenever $g(x)$ is a good function. A good function belongs to L^p (a class of pth power Lebesgue integrable functions) for every p in $1 \le p \le \infty$. The integral of a good function is not necessarily good. However, if $\phi(x)$ is a good function, then the function g defined for all x by

$$g(x) = \int_{-\infty}^{x} \phi(t)\, dt$$

is a good function if and only if $\int_{-\infty}^{\infty} \phi(t)\, dt$ exists.

Good functions are not only continuous, but are also uniformly continuous on $\mathbb{R}$ and absolutely continuous on $\mathbb{R}$. However, a good function cannot necessarily be represented by a Taylor series expansion on every interval. As an example, consider a good function of bounded support

$$g(x) = \begin{cases} \exp[-(1-x^2)^{-1}] & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

The function g is infinitely differentiable at $x = \pm 1$, as it must be in order to be good. It does not have a Taylor series expansion in every interval because a Taylor expansion based on the various derivatives of g at any point x satisfying $|x| > 1$ would lead to zero value for all x.

For example, $\exp(-x^2)$, $x\exp(-x^2)$, $(1+x^2)^{-1}\exp(-x^2)$, and $\operatorname{sech}^2 x$ are good functions, while $\exp(-|x|)$ is not differentiable at $x = 0$, and the function $(1+x^2)^{-1}$ is not a good function as it decays too slowly as $|x| \to \infty$.

A sequence of good functions, $\{f_n(x)\}$, is called *regular* if, for any good function $g(x)$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x)\, dx \tag{B-2.14}$$

exists. For example, $f_n(x) = \frac{1}{n}\phi(x)$ is a regular sequence for any good function $\phi(x)$, since

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x)\, dx = \lim_{n\to\infty} \frac{1}{n} \int_{-\infty}^{\infty} \phi(x) g(x)\, dx = 0.$$

Two regular sequences of good functions are equivalent if, for any good function $g(x)$, the limit (B-2.14) exists and is the same for each sequence.

A *generalized function*, $f(x)$, is a regular sequence of good functions, and two generalized functions are equal if their defining sequences are equivalent. Generalized functions are, therefore, only defined in terms of their action on integrals of good functions if

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x) g(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x)\, dx = \lim_{n\to\infty} \langle f_n,\, g\rangle \tag{B-2.15}$$

for any good function $g(x)$, where the symbol $\langle f,\, g\rangle$ is used to denote the action of the generalized function $f(x)$ on the good function $g(x)$, or $\langle f,\, g\rangle$ represents the number that f associates with g. If $f(x)$ is an ordinary function such that

$(1+x^2)^{-N} f(x)$ is integrable on $(-\infty, \infty)$ for some N, then the generalized function $f(x)$ equivalent to the ordinary function is defined as any sequence of good functions $\{f_n(x)\}$ such that, for any good function $g(x)$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x) g(x)\, dx = \int_{-\infty}^{\infty} f(x) g(x)\, dx. \tag{B-2.16}$$

For example, the generalized function equivalent to zero can be represented by either of the sequences $\{\frac{\phi(x)}{n}\}$ and $\{\frac{\phi(x)}{n^2}\}$.

The unit function, $I(x)$, is defined by

$$\int_{-\infty}^{\infty} I(x) g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx \tag{B-2.17}$$

for any good function $g(x)$. A very important and useful good function that defines the unit function is $\{\exp(-\frac{x^2}{4n})\}$. Thus, the unit function is the generalized function that is equivalent to the ordinary function $f(x) = 1$.

The *Heaviside function*, $H(x)$, is defined by

$$\int_{-\infty}^{\infty} H(x) g(x)\, dx = \int_{0}^{\infty} g(x)\, dx. \tag{B-2.18}$$

The generalized function $H(x)$ is equivalent to the ordinary unit function

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{B-2.19}$$

and since generalized functions are defined through the action on integrals of good functions, the value of $H(x)$ at $x = 0$ does not have significance here.

The *sign function*, $\operatorname{sgn}(x)$, is defined by

$$\int_{-\infty}^{\infty} \operatorname{sgn}(x) g(x)\, dx = \int_{0}^{\infty} g(x)\, dx - \int_{-\infty}^{0} g(x)\, dx \tag{B-2.20}$$

for any good function $g(x)$. Thus, $\operatorname{sgn}(x)$ can be identified with the ordinary function

$$\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ +1 & \text{if } x > 0. \end{cases} \tag{B-2.21}$$

In fact, $\operatorname{sgn}(x) = 2H(x) - I(x)$, which can be seen as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} \operatorname{sgn}(x) g(x)\, dx &= \int_{-\infty}^{\infty} \big[2H(x) - I(x)\big] g(x)\, dx \\ &= 2\int_{-\infty}^{\infty} H(x) g(x)\, dx - \int_{-\infty}^{\infty} I(x) g(x)\, dx \\ &= 2\int_{0}^{\infty} g(x)\, dx - \int_{-\infty}^{\infty} g(x)\, dx \end{aligned}$$

$$= \int_0^\infty g(x)\,dx - \int_{-\infty}^0 g(x)\,dx.$$

The following results are also true

$$x\delta(x) = 0, \tag{B-2.22}$$

$$\delta(x-a) = \delta(a-x). \tag{B-2.23}$$

Result (B-2.23) shows that $\delta(x)$ is an even function.

Clearly, the result

$$\int_{-\infty}^{x} \delta(y)\,dy = \begin{Bmatrix} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0 \end{Bmatrix} = H(x)$$

shows that

$$\frac{d}{dx}H(x) = \delta(x). \tag{B-2.24}$$

The Fourier transform of the Dirac delta function is

$$\mathcal{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\delta(x)\,dx = \frac{1}{\sqrt{2\pi}}. \tag{B-2.25}$$

Hence,

$$\delta(x) = \mathcal{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk. \tag{B-2.26}$$

This is an integral representation of the *delta function* extensively used in quantum mechanics. Also, (B-2.26) can be rewritten as

$$\delta(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dx. \tag{B-2.27}$$

The Dirac delta function, $\delta(x)$, is defined so that for any good function $g(x)$,

$$\langle \delta, g \rangle = \int_{-\infty}^{\infty} \delta(x)g(x)\,dx = g(0). \tag{B-2.28}$$

Derivatives of generalized functions are defined by the derivatives of any equivalent sequences of good functions. We can integrate by parts using any member of the sequences and assuming $g(x)$ vanishes at infinity. We can obtain this definition as follows:

$$\langle f', g \rangle = \int_{-\infty}^{\infty} f'(x)g(x)\,dx$$

$$= \left[f(x)g(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)g'(x)\,dx = -\langle f, g' \rangle. \tag{B-2.29}$$

The derivative of a generalized function f is the generalized function f' defined by

$$\langle f', g\rangle = -\langle f, g'\rangle \tag{B-2.30}$$

for any good function g.

The differential calculus of generalized functions can easily be developed with locally integrable functions. To every locally integrable function f, there corresponds a *generalized function* (or *distribution*) defined by

$$\langle f, \phi\rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx, \tag{B-2.31}$$

where ϕ is a test function on $\mathbb{R} \to \mathbb{C}$ with bounded support (ϕ is infinitely differentiable and such that its derivatives of all orders exist and are continuous).

The derivative of a generalized function f is the generalized function f' defined by

$$\langle f', \phi\rangle = -\langle f, \phi'\rangle \tag{B-2.32}$$

for all test functions ϕ. This definition follows from the fact that

$$\begin{aligned}\langle f', \phi\rangle &= \int_{-\infty}^{\infty} f'(x)\phi(x)\,dx \\ &= \left[f(x)\phi(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)\,dx, = -\langle f, \phi'\rangle\end{aligned}$$

which was obtained from integration by parts and using the fact that ϕ vanishes at infinity.

It is easy to check that $H'(x) = \delta(x)$, for

$$\begin{aligned}\langle H', \phi\rangle &= \int_{-\infty}^{\infty} H'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} H(x)\phi'(x)\,dx \\ &= -\int_{0}^{\infty} \phi'(x)\,dx = -\left[\phi(x)\right]_{0}^{\infty} = \phi(0) = \langle \delta, \phi\rangle.\end{aligned}$$

Another result is

$$\langle \delta', \phi\rangle = -\int_{-\infty}^{\infty} \delta(x)\phi'(x)\,dx = -\phi'(0).$$

It is easy to verify that

$$f(x)\delta(x) = f(0)\delta(x).$$

We next define $|x| = x\operatorname{sgn}(x)$ and calculate its derivative as follows. We have

$$\begin{aligned}\frac{d}{dx}|x| = \frac{d}{dx}\{x\operatorname{sgn}(x)\} &= x\frac{d}{dx}\{\operatorname{sgn}(x)\} + \operatorname{sgn}(x)\frac{dx}{dx} \\ &= x\frac{d}{dx}\{2H(x) - I(x)\} + \operatorname{sgn}(x) \\ &= 2x\delta(x) + \operatorname{sgn}(x) = \operatorname{sgn}(x),\end{aligned} \tag{B-2.33}$$

which is true since $\operatorname{sgn}(x) = 2H(x) - I(x)$ and $x\delta(x) = 0$.

Similarly, we can show that

$$\frac{d}{dx}\{\operatorname{sgn}(x)\} = 2H'(x) = 2\delta(x). \tag{B-2.34}$$

Theorem B-2.1. *The Fourier transform of a good function is a good function.*

Proof. The Fourier transform of a good function $f(x)$ exists and is defined by

$$\mathcal{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx. \tag{B-2.35}$$

Differentiating $F(k)$ n times and integrating N times by parts, we get

$$\begin{aligned} \left|F^{(n)}(k)\right| &\le \left| \frac{(-1)^N}{(-ik)^N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{d^N}{dx^N}\{(-ix)^n f(x)\}\, dx \right| \\ &\le \frac{1}{|k|^N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left| \frac{d^N}{dx^N}\{x^n f(x)\} \right| dx. \end{aligned}$$

Evidently, all derivatives tend to zero as fast as $|k|^{-N}$ as $|k| \to \infty$ for any $N > 0$, and hence $F(k)$ is a good function.

Theorem B-2.2. *If $f(x)$ is a good function with the Fourier transform (B-2.35), then the inverse Fourier transform is given by*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k)\, dk. \tag{B-2.36}$$

Proof. For any $\epsilon > 0$, we have

$$\mathcal{F}\{e^{-\epsilon x^2} F(-x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx-\epsilon x^2} \left\{ \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt \right\} dx.$$

Since f is a good function, the order of integration can be interchanged to obtain

$$\mathcal{F}\{e^{-\epsilon x^2} F(-x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, dt \int_{-\infty}^{\infty} e^{-i(k-t)x-\epsilon x^2}\, dx,$$

which is, by a similar calculation as that used in Example 1.7.1 in Chapter 1,

$$= \frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} \exp\left[-\frac{(k-t)^2}{4\epsilon}\right] f(t)\, dt.$$

Using the fact that

$$\frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} \exp\left[-\frac{(k-t)^2}{4\epsilon}\right] dt = 1,$$

we can write

$$\mathcal{F}\{e^{-\epsilon x^2}F(-x)\} - f(k)\cdot 1$$
$$= \frac{1}{\sqrt{4\pi\epsilon}}\int_{-\infty}^{\infty}\left[f(t)-f(k)\right]\exp\left[-\frac{(k-t)^2}{4\epsilon}\right]dt. \qquad \text{(B-2.37)}$$

Since f is a good function, we have

$$\left|\frac{f(t)-f(k)}{t-k}\right| \le \max_{x\in\mathbb{R}}\left|f'(x)\right|.$$

It follows from (B-2.37) that

$$\left|\mathcal{F}\{e^{-\epsilon x^2}F(-x)\} - f(k)\right|$$
$$\le \frac{1}{\sqrt{4\pi\epsilon}}\max_{x\in\mathbb{R}}\left|f'(x)\right|\int_{-\infty}^{\infty}|t-k|\;\exp\left[-\frac{(t-k)^2}{4\epsilon}\right]dt$$
$$= \frac{1}{\sqrt{4\pi\epsilon}}\max_{x\in\mathbb{R}}\left|f'(x)\right|4\epsilon\int_{-\infty}^{\infty}|\alpha|e^{-\alpha^2}\,d\alpha \to 0$$

as $\epsilon \to 0$, where $\alpha = \frac{t-k}{2\sqrt{\epsilon}}$.

Consequently,

$$f(k) = \mathcal{F}\{F(-x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}F(-x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ikx}F(x)\,dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\,dx\int_{-\infty}^{\infty}e^{-i\xi x}f(\xi)\,d\xi.$$

Interchanging k with x, this reduces to the celebrated *Fourier integral formula*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\left[\int_{-\infty}^{\infty}e^{-ik\xi}f(\xi)\,d\xi\right]dk. \qquad \text{(B-2.38)}$$

Hence, the theorem is proved.

Example B-2.1. The Fourier transform of a constant function c is

$$\mathcal{F}\{c\} = \sqrt{2\pi}c\delta(k). \qquad \text{(B-2.39)}$$

In the ordinary sense,

$$\mathcal{F}\{c\} = \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ikx}\,dx$$

is not a well defined integral; it diverges. However, it can be treated as a generalized function, namely taking $c = cI(x)$ and considering $\{\exp(-\frac{x^2}{4n})\}$ as an equivalent sequence to the unit function $I(x)$ gives

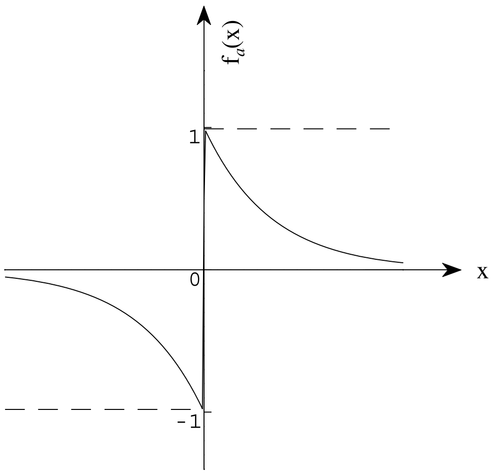

Fig. B.9 Graph of the function $f_a(x)$.

$$\mathcal{F}\left\{c\exp\left(-\frac{x^2}{4n}\right)\right\} = \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-ikx - \frac{x^2}{4n}\right)dx,$$

which, by Example 1.7.1, is

$$\begin{aligned} &= c\sqrt{2n}\exp\left(-nk^2\right) = \sqrt{2\pi}c\sqrt{\frac{n}{\pi}}\exp\left(-nk^2\right), \\ &= \sqrt{2\pi}c\delta_n(k) \to \sqrt{2\pi}c\delta(k) \quad \text{as } n \to \infty, \end{aligned}$$

since $\{\delta_n(k)\} = \{\sqrt{\frac{n}{\pi}}\exp(-nk^2)\}$ is a sequence equivalent to the delta function defined by (B-2.4).

Example B-2.2. Show that

$$\mathcal{F}\{e^{-ax}H(x)\} = \frac{1}{\sqrt{2\pi}(ik+a)}, \quad a > 0. \tag{B-2.40}$$

We have, by definition,

$$\mathcal{F}\{e^{-ax}H(x)\} = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\{-x(ik+a)\}\,dx = \frac{1}{\sqrt{2\pi}(ik+a)}.$$

Example B-2.3. By considering the function (see Figure B.9)

$$f_a(x) = e^{-ax}H(x) - e^{ax}H(-x), \quad a > 0, \tag{B-2.41}$$

find the Fourier transform of $\operatorname{sgn}(x)$. In Figure B.9, the vertical axis (y-axis) represents $f_a(x)$ and the horizontal axis represents the x-axis.

We have, by definition,

$$\mathcal{F}\{f_a(x)\} = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \exp\{(a-ik)x\}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} \exp\{-(a+ik)x\}\,dx$$
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+ik} - \frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{(-ik)}{a^2+k^2}.$$

In the limit as $a \to 0$, $f_a(x) \to \operatorname{sgn}(x)$ and then

$$\mathcal{F}\{\operatorname{sgn}(x)\} = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{ik}. \tag{B-2.42}$$

B-3 Basic Properties of the Fourier Transforms

The Fourier transform of $f(x)$, $\mathcal{F}\{f(x)\} = F(k)$, is defined by (1.7.1) and its inverse $\mathcal{F}^{-1}\{F(k)\} = f(x)$ is defined by (1.7.2).

We state the following properties of the Fourier transform:

(a) (Shifting)

$$\mathcal{F}\{f(x-a)\} = e^{-iak}\mathcal{F}\{f(x)\}, \tag{B-3.1}$$

(b) (Scaling)

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|}F\left(\frac{k}{a}\right), \tag{B-3.2}$$

(c) (Conjugate)

$$\mathcal{F}\{\overline{f(-x)}\} = \overline{\mathcal{F}\{f(x)\}}, \tag{B-3.3}$$

(d) (Translation)

$$\mathcal{F}\{e^{iax}f(x)\} = F(k-a), \tag{B-3.4}$$

(e) (Duality)

$$\mathcal{F}\{f(x)\} = F(k), \quad \text{and} \quad \mathcal{F}\{F(x)\} = f(-k), \tag{B-3.5}$$

(f) (Composition)

$$\int_{-\infty}^{\infty} F(k)g(k)e^{ikx}\,dk = \int_{-\infty}^{\infty} f(\xi)G(\xi-x)\,d\xi, \tag{B-3.6}$$

where

$$F(k) = \mathcal{F}\{f(\xi)\} \quad \text{and} \quad G(\xi) = \mathcal{F}\{g(k)\}.$$

(g) (Riemann–Lebesgue Lemma) If $\mathcal{F}\{f(x)\} = F(k)$, then

$$\lim_{|k|\to\infty} F(k) = 0. \tag{B-3.7}$$

(h) (Poisson Summation Formula)

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \left(\frac{\sqrt{2\pi}}{2a}\right) \sum_{n=-\infty}^{\infty} F\left(\frac{n\pi}{a}\right). \tag{B-3.8}$$

When $a = \pi$,

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} F(n). \tag{B-3.9}$$

When $2a = 1$, formula (B-3.8) becomes

$$\sum_{n=-\infty}^{\infty} f(n) = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} F(n). \tag{B-3.10}$$

The convolution $(f * g)(x)$ of two integrable function $f(x)$ and $g(x)$ is defined by (1.7.13) and the convolution Theorem 1.7.1 states that

$$\mathcal{F}\big\{(f * g)(x)\big\} = F(k)G(k), \tag{B-3.11}$$

or equivalently,

$$\mathcal{F}^{-1}\big\{F(k)G(k)\big\} = (f * g)(x). \tag{B-3.12}$$

Proof. We have

$$\mathcal{F}\big\{(f * g)(x)\big\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \big[(f * g)(x)\big]\, dx,$$

which is, by definition of the convolution,

$$\begin{aligned}
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \left[\int_{-\infty}^{\infty} f(x-\xi) g(\xi)\, d\xi\right] dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik\xi} g(\xi)\, d\xi \int_{-\infty}^{\infty} e^{-ik(x-\xi)} f(x-\xi)\, d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik\xi} g(\xi)\, d\xi \int_{-\infty}^{\infty} e^{-ik\eta} f(\eta)\, d\eta, \quad (x - \xi = \eta) \\
&= F(k)G(k).
\end{aligned}$$

This completes the proof.

It is often convenient to delete the factor $\frac{1}{\sqrt{2\pi}}$ in the definition of convolution (1.7.13) as this factor does not have any effect on the properties of the convolution. The convolution has the following algebraic properties:

$$f * g = g * f \quad \text{(Commutative)}, \tag{B-3.13}$$

$$f * (g * h) = (f * g) * h \quad \text{(Associative)}, \tag{B-3.14}$$

$$(\alpha f + \beta g) * h = \alpha(f * h) + \beta(g * h) \quad \text{(Distributive)}, \tag{B-3.15}$$

$$f * \delta = f = \delta * f \quad \text{(Identity)}, \tag{B-3.16}$$

where α and β are any two constants, and $\delta(x)$ is the Dirac delta function.

$$\overline{(f * g)}(x) = (\overline{f} * \overline{g})(x), \tag{B-3.17}$$

$$x\big[(f * g)(x)\big] = \big[xf(x)\big] * g(x) + \big[f(x) * xg(x)\big], \tag{B-3.18}$$

$$f(ax + b) * g(ax + c) = \frac{1}{|a|} h(ax + b + c), \tag{B-3.19}$$

where $h(x) = (f * g)(x)$.

In particular,

$$f(x + b) * g(x + c) = h(x + b + c). \tag{B-3.20}$$

If a_1, a_2, b_1, b_2 are any constants and $f_1 * g_1$, $f_1 * g_2$, $f_2 * g_1$, and $f_2 * g_2$ exist, then

$$(a_1 f_1 + a_2 f_2) * (b_1 g_1 + b_2 g_2) = a_1 b_1 (f_1 * g_1) + a_1 b_2 (f_1 * g_2) + a_2 b_1 (f_2 * g_1) + a_2 b_2 (f_2 * g_2). \tag{B-3.21}$$

To prove (B-3.14), we have

$$\begin{aligned}
\big[f * (g * h)\big](x) &= \int_{-\infty}^{\infty} f(x - \xi)(g * h)(\xi)\, d\xi \\
&= \int_{-\infty}^{\infty} f(x - \xi)\, d\xi \int_{-\infty}^{\infty} g(\xi - t) h(t)\, dt \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x - \xi) g(\xi - t)\, d\xi\right] h(t)\, dt, \quad (\xi - t = \eta) \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x - t - \eta) g(\eta)\, d\eta\right] h(t)\, dt, \\
&= \int_{-\infty}^{\infty} (f * g)(x - t) h(t)\, dt = \big[(f * g) * h\big](x).
\end{aligned}$$

To prove (B-3.18), we apply the Fourier transform to the left hand side so that

$$\begin{aligned}
\mathcal{F}\big\{x(f * g)(x)\big\} &= \int_{-\infty}^{\infty} e^{-ikx} x (f * g)(x)\, dx \\
&= \int_{-\infty}^{\infty} x e^{-ikx}\, dx \int_{-\infty}^{\infty} f(x - \xi) g(\xi)\, d\xi \\
&= \int_{-\infty}^{\infty} (x - \xi + \xi) e^{-ikx}\, dx \int_{-\infty}^{\infty} f(x - \xi) g(\xi)\, d\xi \\
&= \int_{-\infty}^{\infty} (x - \xi) e^{-ikx}\, dx \int_{-\infty}^{\infty} f(x - \xi) g(\xi)\, d\xi
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{\infty} e^{-ikx}\,dx \int_{-\infty}^{\infty} f(x-\xi)\xi g(\xi)\,d\xi \\
&= \int_{-\infty}^{\infty} e^{-ikx}\,dx \int_{-\infty}^{\infty} (x-\xi) f(x-\xi) g(\xi)\,d\xi \\
&\quad + \int_{-\infty}^{\infty} e^{-ikx}\,dx \int_{-\infty}^{\infty} f(x-\xi)\xi g(\xi)\,d\xi \\
&= \int_{-\infty}^{\infty} e^{-ikx}\big[xf(x) * g(x)\big]\,dx \\
&\quad + \int_{-\infty}^{\infty} e^{-ikx}\big[f(x) * xg(x)\big]\,dx \\
&= \mathcal{F}\big[xf(x) * g(x)\big] + \mathcal{F}\big[f(x) * xg(x)\big].
\end{aligned}
$$

We next apply $\mathcal{F}^{-1}$ to obtain result (B-3.18).

To prove (B-3.19), we apply the Fourier transform to its left-hand side and use the convolution theorem so that

$$
\begin{aligned}
&\mathcal{F}\big\{f(ax+b) * g(ax+c)\big\} \\
&\quad = \mathcal{F}\big\{f(ax+b)\big\}\mathcal{F}\big\{g(ax+c)\big\} \\
&\quad = \frac{1}{|a|}\exp\left(\frac{ikb}{a}\right)F\left(\frac{k}{a}\right)\cdot\frac{1}{|a|}\exp\left(\frac{ikc}{a}\right)G\left(\frac{k}{a}\right) \\
&\quad = \frac{1}{|a|^2}\cdot\exp\left[\frac{ik(b+c)}{a}\right]H\left(\frac{k}{a}\right), \quad \text{since } H(k) = F(k)G(k) \\
&\quad = \frac{1}{|a|}\mathcal{F}\big\{h(ax+b+c)\big\}.
\end{aligned}
$$

We next apply $\mathcal{F}^{-1}$ to both sides to obtain (B-3.19).

Theorem B-3.1 (General Parseval's Relation). *If* $\mathcal{F}\{f(x)\} = F(k)$ *and* $\mathcal{F}\{g(x)\} = G(k)$, *then*

$$
\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} F(k)\overline{G(k)}\,dk. \tag{B-3.22}
$$

Proof. We proceed formally to obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} e^{ikx}F(k)\,dk \int_{-\infty}^{\infty} e^{-ilx}\overline{G(l)}\,dl \\
&= \int_{-\infty}^{\infty} F(k)\,dk \int_{-\infty}^{\infty} \overline{G(l)}\,dl \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-l)x}\,dx \\
&= \int_{-\infty}^{\infty} F(k)\,dk \int_{-\infty}^{\infty} \overline{G(l)}\delta(k-l)\,dl \\
&= \int_{-\infty}^{\infty} F(k)\overline{G(k)}\,dk.
\end{aligned}
$$

Thus, the proof is complete.

In particular, if $f(x) = g(x)$, then

$$\int_{-\infty}^{\infty} f(x)\overline{f(x)}\,dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}\,dk,$$

or equivalently,

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(k)|^2\,dk. \tag{B-3.23}$$

This is well known as the *Parseval relation*.

The function space $L^2(\mathbb{R})$ of all complex-valued Lebesgue square integrable functions with the *inner product* (f, g) defined by

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx \tag{B-3.24}$$

is a complex Hilbert space with the *norm* $\|f\|_2$ defined by

$$\|f\|_2 = \sqrt{(f, f)} = \left(\int_{-\infty}^{\infty} \left|f(x)\right|^2\,dx\right)^{\frac{1}{2}}. \tag{B-3.25}$$

In terms of this norm, the Parseval relation (B-3.23) takes the form

$$\|f\|_2 = \|F\|_2 = \|\mathcal{F}f\|_2. \tag{B-3.26}$$

This means that the Fourier transform is a *unitary transformation* on the Schwartz space $S(\mathbb{R})$ which consists of the set of all infinitely differentiable functions f so that f and all its derivatives are *rapidly decreasing* in the sense that

$$\sup_{x\in\mathbb{R}} |x|^m \left|f^{(n)}(x)\right| < \infty \quad \text{for every } m, n \geq 0.$$

Physically, the quantity $\|f\|_2$ is a measure of energy and $\|F\|_2$ represents the *power spectrum* of f.

The following examples of the Fourier transforms are useful in applied mathematics (see Debnath and Bhatta 2007).

If $\chi_{[-a,a]}(x)$ is the characteristic function of $[-a, a]$ defined by

$$\chi_{[-a,a]}(x) = H\big(a - |x|\big) = \begin{cases} 1 & \text{if } |x| < a, \\ 0 & \text{if } |x| > a, \end{cases} \tag{B-3.27}$$

then

$$\mathcal{F}\big\{\chi_{[-a,a]}(x)\big\} = F_a(k) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right). \tag{B-3.28}$$

If

$$f(x) = \left(1 - \frac{|x|}{a}\right)H\left(1 - \frac{|x|}{a}\right), \tag{B-3.29}$$

then

$$\mathcal{F}\{f(x)\} = \frac{a}{\sqrt{2\pi}}\left(\frac{ak}{2}\right)^{-2}\sin^2\left(\frac{ak}{2}\right). \tag{B-3.30}$$

The following analytic properties of the convolution also hold:

$$\frac{d}{dx}\big[(f*g)(x)\big] = \big(f'*g\big)(x) = \big(f*g'\big)(x), \tag{B-3.31}$$

$$\frac{d^2}{dx^2}\big[(f*g)(x)\big] = \big(f'*g'\big)(x) = \big(f''*g\big)(x), \tag{B-3.32}$$

$$(f*g)^{(m+n)}(x) = \big(f^{(m)}*g^{(n)}\big)(x), \tag{B-3.33}$$

$$\int_{-\infty}^{\infty}(f*g)(x)\,dx = \int_{-\infty}^{\infty}f(\xi)\,d\xi\int_{-\infty}^{\infty}g(\eta)\,d\eta, \tag{B-3.34}$$

$$(f*\chi_{[a,b]})(x) = \int_a^b f(x-\xi)\,d\xi = \int_{x-b}^{x-a}f(\eta)\,d\eta, \tag{B-3.35}$$

$$\mathcal{F}\{f(x)g(x)\} = (F*G)(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(k-\xi)G(\xi)\,d\xi, \tag{B-3.36}$$

$$f(x) = (f*\delta)(x) = \mathcal{F}^{-1}\{F(k)\}. \tag{B-3.37}$$

To prove (B-3.31), we have

$$\begin{aligned}(f*g)'(x) &= \frac{d}{dx}\int_{-\infty}^{\infty}f(x-\xi)g(\xi)\,d\xi = \int_{-\infty}^{\infty}f'(x-\xi)g(\xi)\,d\xi \\ &= \big(f'*g\big)(x).\end{aligned}$$

To prove (B-3.33), we first apply the Fourier transform to the left-hand side and then use the convolution Theorem 1.7.1 so that

$$\begin{aligned}\mathcal{F}\{(f*g)^{m+n}(x)\} &= (ik)^{m+n}\mathcal{F}\{(f*g)(x)\} = (ik)^{m+n}F(k)G(k) \\ &= \big[(ik)^m F(k)\big]\big[(ik)^n G(k)\big] = \mathcal{F}\{f^{(m)}(x)\}\mathcal{F}\{g^{(n)}(x)\} \\ &= \mathcal{F}\{\big(f^{(m)}*g^{(n)}\big)(x)\}.\end{aligned}$$

The use of the inverse Fourier transform proves the result (B-3.33).

To prove that the integral of the convolution satisfies (B-3.34), we have

$$\begin{aligned}\int_{-\infty}^{\infty}(f*g)(x)\,dx &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(x-\xi)g(\xi)\,d\xi\right]dx \\ &= \int_{-\infty}^{\infty}g(\xi)\left[\int_{-\infty}^{\infty}f(x-\xi)\,d\xi\right]d\xi, \quad (x-\xi=\eta) \\ &= \int_{-\infty}^{\infty}g(\xi)\,d\xi\int_{-\infty}^{\infty}f(\eta)\,d\eta.\end{aligned}$$

We next prove (B-3.36) which is a dual result of (B-3.11) as follows. We have

$$\mathcal{F}\{f(x)g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)g(x)\,dx,$$

which is, by replacing $g(x)$ by its inverse Fourier transform formula,

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik'x} G(k')\,dk' \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k')\,dk' \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix(k-k')} f(x)\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k-k')G(k')\,dk' \\
&= (F * G)(k).
\end{aligned}$$

Thus, the results (B-3.11) and (B-3.36) represent a *duality* since the Fourier transform of the convolution product of two functions is equal to the ordinary product of their Fourier transforms, and the Fourier transform of the ordinary product of two functions is equal to the convolution product of their Fourier transforms.

If the diffusion kernel function $G_t(x)$ is

$$G_t(x) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right), \tag{B-3.38}$$

then

$$(G_t * G_s)(x) = G_{t+s}(x). \tag{B-3.39}$$

To prove (B-3.39), we apply the Fourier transform without the factor $\frac{1}{\sqrt{2\pi}}$. This means that $\mathcal{F}\{\exp(-ax^2)\} = \sqrt{\frac{\pi}{a}} \exp(-\frac{k^2}{4a})$. Consequently,

$$\begin{aligned}
\mathcal{F}\{(G_t * G_s)(x)\} &= \mathcal{F}\{G_t(x)\}\mathcal{F}\{G_s(x)\} \\
&= \exp(-\kappa k^2 t) \exp(-\kappa k^2 s) \\
&= \exp[-k^2\kappa(t+s)] = \mathcal{F}\{G_{t+s}(x)\}.
\end{aligned}$$

The inverse Fourier transform gives the result.

If $g(x) = \frac{1}{2a}H(a - |x|)$, then $(f * g)(x)$ is the average value of $f(x)$ on $[x-a, x+a]$.

Using the value of $g(x)$ gives

$$\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi = \frac{1}{2a} \int_{-a}^{a} f(x-\xi)\,d\xi \\
&= \frac{1}{2a} \int_{x-a}^{x+a} f(\eta)\,d\eta.
\end{aligned}$$

We close this section by adding an application of the convolution to the solution of the wave equation in Example 1.7.2. The solution of the Fourier transform $U(k,t) = \mathcal{F}\{u(x,t)\}$ of the transformed wave equation (1.7.17) is given by

$$U(k,t) = A\cos ckt + B\sin ckt, \tag{B-3.40}$$

where

$$U(k,0) = F(k) \quad \text{and} \quad \left(\frac{dU}{dk}\right)_{t=0} = G(k). \tag{B-3.41}$$

Consequently, the use of (B-3.41) gives

$$U(k,t) = F(k)\cos ckt + \frac{G(k)}{ck}\sin ckt. \tag{B-3.42}$$

Using the results

$$\mathcal{F}\{\delta(x-ct)+\delta(x+ct)\} = \sqrt{\frac{2}{\pi}}\cos(ckt),$$

$$\mathcal{F}\{\chi_{[-ct,ct]}(x)\} = \mathcal{F}\{H(ct-|x|)\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ckt}{k}\right),$$

applying the inverse Fourier transform to (B-3.42) gives the solution

$$u(x,t) = \mathcal{F}^{-1}\{F(k)\cos ckt\} + \frac{1}{c}\mathcal{F}^{-1}\left\{G(k)\frac{\sin ckt}{k}\right\}.$$

Application of the convolution Theorem 1.7.1 gives the solution

$$\begin{aligned} u(x,t) &= f(x) * \sqrt{\frac{\pi}{2}}\left[\delta(x-ct)+\delta(x+ct)\right] \\ &\quad + \frac{1}{c}\left[g(x) * \sqrt{\frac{\pi}{2}}\chi_{[-ct,ct]}(x)\right] \qquad \text{(B-3.43)} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} f(x-\xi)\{\delta(\xi-ct)+\delta(\xi+ct)\}\,d\xi \\ &\quad + \frac{1}{2c}\int_{-\infty}^{\infty} g(x-\xi)\chi_{[-ct,ct]}(\xi)\,d\xi \\ &= \frac{1}{2}\left[f(x-ct)+f(x+ct)\right] + \frac{1}{2c}\int_{-ct}^{ct} g(x-\xi)\,d\xi, \quad (x-\xi=\alpha) \\ &= \frac{1}{2}\left[f(x-ct)+f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\,d\alpha. \qquad \text{(B-3.44)} \end{aligned}$$

This is identical with the d'Alembert solution of the wave equation.

B-4 Basic Properties of Laplace Transforms

The Laplace transform of a continuous or piecewise continuous function $f(t)$ for $t > 0$ is denoted by $\mathcal{L}\{f(t)\} = \overline{f}(s)$, and it defined by (1.9.1) and the inverse

Laplace transform, $\mathcal{L}^{-1}\{\overline{f}(s)\} = f(t)$, is defined by (1.9.2). In this section, some basic properties of the Laplace transform are presented. For more information, the reader is referred to Debnath and Bhatta (2007):

$$\mathcal{L}\{e^{at}f(t)\} = \overline{f}(s-a) \quad \text{(shifting)}, \tag{B-4.1}$$

$$\mathcal{L}\{f(at+b)\} = \frac{1}{|a|}\exp\left(\frac{b}{a}s\right)\overline{f}\left(\frac{s}{a}\right) \quad \text{(scaling)}, \tag{B-4.2}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\overline{f}(s), \quad n \geq 1, \tag{B-4.3}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(s)\,ds, \tag{B-4.4}$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{\overline{f}(s)}{s}. \tag{B-4.5}$$

If $\mathcal{L}\{f(t)\} = \overline{f}(s)$, then

$$\mathcal{L}^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau, \tag{B-4.6}$$

$$\mathcal{L}^{-1}\left\{\frac{\overline{f}(s)}{s^2}\right\} = \int_0^t \left\{\int_0^{t_1} f(\tau)\,d\tau\right\} = \int_0^t (t-\tau)f(\tau)\,d\tau. \tag{B-4.7}$$

In general,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{\overline{f}(s)}{s^n}\right\} &= \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} f(\tau)\,d\tau\,dt_1 \cdots dt_{n-1} \\ &= \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)\,d\tau.\end{aligned} \tag{B-4.8}$$

If $f(t)$ is a periodic function with period T and $\mathcal{L}\{f(t)\}$ exists, then

$$\mathcal{L}\{f(t)\} = \left(1 - e^{-sT}\right)^{-1} \int_0^T e^{-st} f(t)\,dt. \tag{B-4.9}$$

For example, if $f(t)$ is a square wave function with period $2a$ defined by

$$f(t) = H(t) - 2H(t-a) + 2H(t-2a) - 2H(t-3a) + \cdots, \tag{B-4.10}$$

then the Laplace transform of $f(t)$ is

$$\overline{f}(s) = \frac{1}{s}\tanh\left(\frac{as}{2}\right). \tag{B-4.11}$$

Clearly, the graph of $f(t)$ shows that $f(t) = 1$ if $0 < t < a$ and $f(t) = -1$ if $0 < a < t < 2a$. Thus, result (B-4.9) can be used to find

$$
\begin{aligned}
\overline{f}(s) &= \left(1-e^{-2as}\right)^{-1} \int_0^{2a} e^{-st} f(t)\, dt \\
&= \left(1-e^{-2as}\right)^{-1} \left[\int_0^{a} e^{-st} dt - \int_a^{2a} e^{-st}\, dt\right] \\
&= \left(1-e^{-2as}\right)^{-1} \cdot \frac{1}{s}\left(1-2e^{-sa}+e^{-2as}\right) \\
&= \frac{1}{s}\frac{(1-e^{-as})^2}{(1-e^{-2as})} = \frac{1}{s}\frac{(1-e^{-sa})}{(1+e^{-sa})} = \frac{1}{s}\tanh\left(\frac{as}{2}\right).
\end{aligned}
$$

Theorem B-4.1 (Convolution Theorem). *If $f(t) * g(t)$ is the* Laplace Convolution *of $f(t)$ and $g(t)$ defined by*

$$f(t) * g(t) = (f * g)(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau, \tag{B-4.12}$$

and if $\mathcal{L}\{f(t)\} = \overline{f}(s)$ and $\mathcal{L}\{g(t)\} = \overline{g}(s)$, then

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \overline{f}(s)\overline{g}(s), \tag{B-4.13}$$

or equivalently,

$$\mathcal{L}^{-1}\{\overline{f}(s)\overline{g}(s)\} = f(t) * g(t). \tag{B-4.14}$$

To prove the convolution theorem, we have

$$
\begin{aligned}
\mathcal{L}\{f(t) * g(t)\} &= \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)\, d\tau\right] dt, \\
&= \int_0^\infty \int_0^t \left[e^{-s(t-\tau)} f(t-\tau) \cdot e^{-s\tau} g(\tau)\, d\tau\right] dt,
\end{aligned}
$$

which is, reversing the order of integration,

$$
\begin{aligned}
&= \int_0^\infty \left[\int_\tau^\infty e^{-s(t-\tau)} f(t-\tau)\, dt\right] e^{-s\tau} g(\tau)\, d\tau, \\
&= \int_0^\infty e^{-sx} f(x)\, dx \int_0^\infty e^{-s\tau} g(\tau)\, d\tau, \quad (t-\tau = x) \\
&= \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \overline{f}(s)\overline{g}(s).
\end{aligned}
$$

In case of the Laplace transform, we proved the convolution theorem (B-4.12) and now we prove the dual result

$$\mathcal{L}\{f(t)g(t)\} = \frac{1}{2\pi i}\bar{f}(s) * \bar{g}(s), \tag{B-4.15}$$

where

$$f(t) = \frac{1}{\sqrt{2\pi i}}\int_{c_1 - i\infty}^{c_1 + i\infty} e^{st}\bar{f}(s)\, ds, \quad g(t) = \frac{1}{\sqrt{2\pi i}}\int_{c_2 - i\infty}^{c_2 + i\infty} e^{st}\bar{g}(s)\, ds,$$
$$c_1, c_2 > 0.$$

We have, by definition and replacing $g(t)$ by its inverse Laplace transform,

$$\begin{aligned}
\mathcal{L}\{f(t)g(t)\} &= \int_0^{\infty} e^{-st} f(t)g(t)\,dt \\
&= \int_0^{\infty} e^{-st} f(t)\,dt \cdot \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} e^{zt}\bar{g}(z)\,dz \\
&= \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \bar{g}(z)\,dz \int_0^{\infty} e^{-t(s-z)} f(t)\,dt \\
&= \frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty} \bar{f}(s-z)\bar{g}(z)\,dz, \quad \mathrm{Re}(s-z) \geq c_1 \\
&= \frac{1}{2\pi i}\bar{f}(s) * \bar{g}(s), \quad \mathrm{Re}\, s \geq c_1 + c_2.
\end{aligned}$$

Like the above duality property for the Fourier transform, results (B-4.12) and (B-4.16) represent the *duality property for the Laplace transform*.

The following properties of the Laplace convolution (B-4.12) also hold:

$$\mathcal{L}\{(f*g)'(t)\} = \mathcal{L}\{(f'*g)(t)\} + f(0)\bar{g}(s) = s\bar{f}(s)\bar{g}(s), \tag{B-4.16}$$

where the prime denotes the derivative with respect to t.

Similarly,

$$\mathcal{L}\{(f*g)'(t)\} = \mathcal{L}\{(f*g')(t)\} + g(0)\bar{f}(s) = s\bar{f}(s)\bar{g}(s). \tag{B-4.17}$$

Results similar to (B-4.16)–(B-4.17) can be proved for the second and the higher derivatives of $(f*g)(t)$.

The Duhamel formulas follow from (B-4.16) and (B-4.17) as

$$\mathcal{L}^{-1}\{s\bar{f}(s)\bar{g}(s)\} = f(0)g(t) + \int_0^t g(t-\tau)f'(\tau)\,d\tau, \tag{B-4.18}$$

or equivalently,

$$\mathcal{L}^{-1}\{s\bar{f}(s)\bar{g}(s)\} = g(0)f(t) + \int_0^t f(t-\tau)g'(\tau)\,d\tau. \tag{B-4.19}$$

If $f_p(t) = t^{p-1}e^{-t}$, $t > 0$, then direct differentiation gives

$$f_p'(t) = (p-1)f_{p-1}(t) - f_p(t). \tag{B-4.20}$$

It also follows that $f_p(t) * f_q(t)$ exists for all p, $q > 0$ and satisfies the following identities

$$(f_p * f_q)(t) = B(p,q)f_{p+q}(t) \tag{B-4.21}$$

$$(f_p * f_q)'(t) = B(p,q)\big[(p+q-1)f_{p+q-1}(t) - f_{p+q}(t)\big] \tag{B-4.22}$$

$$(f_p * f_q)'(t) = f_p' * f_q(t) = \big[(p-1)f_{p-1}(t) - f_p(t)\big] * f_q(t), \tag{B-4.23}$$

$$= (p-1)B(p-1,q)f_{p+q-1}(t) - B(p,q)f_{p+q}(t), \tag{B-4.24}$$

where $B(p,q)$ are the Beta function of p and q.

To prove (B-4.21), we have

$$\begin{aligned}(f_p * f_q)(t) &= \int_0^t f_p(t-\tau) f_q(\tau)\, d\tau \\ &= \int_0^t (t-\tau)^{p-1} e^{-(t-\tau)} \tau^{q-1} e^{-\tau}\, d\tau, \quad (t-\tau = tu) \\ &= e^{-t} t^{p+q-1} \int_0^1 u^{p-1}(1-u)^{q-1}\, du, \\ &= B(p,q) e^{-t} t^{p+q-1} = B(p,q) f_{p+q}(t).\end{aligned}$$

Then

$$(f_p * f_q)'(t) = B(p,q) f'_{p+q}(t) = B(p,q)\big[(p+q-1) f_{p+q-1}(t) - f_{p+q}(t)\big].$$

The Laplace convolution (B-4.12) also satisfies the properties similar to (B-3.13). In particular,

$$\mathcal{L}\big\{(f_1 * f_2 * \cdots * f_n)(t)\big\} = \overline{f}_1(s)\overline{f}_2(s)\cdots\overline{f}_n(s). \tag{B-4.25}$$

$$\mathcal{L}\big\{f^{*n}(t)\big\} = \big\{\overline{f}(s)\big\}^n, \tag{B-4.26}$$

where f^{*n} is the nth convolution product defined by

$$f^{*n}(t) = (f * f * \cdots * f)(t). \tag{B-4.27}$$

In general, mathematical operations such as addition, $f(x)+g(x)$, multiplication, $f(x)g(x)$, and composition, $f(g(x))$, for two functions $f(x)$ and $g(x)$ form a new function or an *ordinary output*. On the other hand, the convolution $f(x) * g(x)$ represents the *integral output* of two functions f and g, and it plays a central role in the subjects, such as Fourier series, Fourier transforms, number theory, harmonic analysis, probability theory, and almost any integral transform. We have already stated the convolution Theorem 1.7.1 for the Fourier transform, and the convolution Theorem B-4.1 for the Laplace transform of two functions. Obviously, these theorems can be generalized to obtain a relation between the n-fold convolution of n functions and the product of the transforms of these functions.

We next establish a nice connection between the Weierstrass transform and the two-sided Laplace transform of $f(\xi)$ on $-\infty < \xi < \infty$ defined by

$$\mathcal{L}\big\{f(\xi)\big\} = \overline{f}(s) = \int_{-\infty}^{\infty} e^{-s\xi} f(\xi)\, d\xi \tag{B-4.28}$$

which is the Fourier transform of $f(\xi)$ for $s = ik$ so that

$$\mathcal{L}\big\{f(\xi)\big\} = \overline{f}(s) = \sqrt{2\pi}\mathcal{F}\big\{f(\xi)\big\}(ik) = \sqrt{2\pi}F(ik). \tag{B-4.29}$$

Thus, the two-sided Laplace transform of $f(\xi) = \exp(-\frac{\xi^2}{4a})$ is given by

$$\mathcal{L}\left\{\exp\left(-\frac{\xi^2}{4a}\right)\right\} = \sqrt{2\pi}\mathcal{F}\left\{e^{-\frac{\xi^2}{4a}}\right\}(ik = s)$$
$$= \sqrt{4\pi a}\exp\left(as^2\right). \tag{B-4.30}$$

In Example 1.7.6 in Chapter 1, the solution of the Cauchy problem for the diffusion equation is given by

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)G(x-\xi)\,d\xi = f(x) * G(x), \tag{B-4.31}$$

where the diffusion kernel $G(x)$ is given by

$$G(x) = \frac{1}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right). \tag{B-4.32}$$

The above results (B-4.31)–(B-4.32) are used to introduce the *Weierstrass transform* of $f(\xi)$ with positive parameter t in the form

$$\mathcal{W}\{f(\xi)\}(x) = F(x) = \int_{-\infty}^{\infty} f(\xi)G(x-\xi)\,d\xi, \tag{B-4.33}$$

where $G(x)$, or, more precisely, $G_t(x)$, is the kernel of the Weierstrass transform defined by (B-4.33).

We next expand the kernel $G_t(x-\xi)$ to express the Weierstrass transform (B-4.33) of $f(\xi)$ in terms of the two-sided Laplace transform (B-4.28) so that (B-4.33) reduces to

$$\begin{aligned} F(x) &= \mathcal{W}\left[f(\xi)\right](x) \\ &= \frac{\exp(-\frac{x^2}{4\kappa t})}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty}\exp\left[-\left(-\frac{x}{2\kappa t}\right)\xi\right]\exp\left(-\frac{\xi^2}{4\kappa t}\right)f(\xi)\,d\xi \\ &= \frac{\exp(-\frac{x^2}{4\kappa t})}{\sqrt{4\pi\kappa t}}\mathcal{L}\left\{\exp\left(-\frac{\xi^2}{4\kappa t}\right)f(\xi)\right\}\left(-\frac{x}{2\kappa t}\right), \end{aligned} \tag{B-4.34}$$

where $\mathcal{L}$ is the two-sided Laplace transform (B-4.28).

We use (B-4.32) with $\kappa t = a$ and (B-4.30) to derive the *inverse Weierstrass transform* so that

$$e^{as^2} = \frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{\infty} e^{-s\xi}\exp\left(-\frac{\xi^2}{4a}\right)d\xi. \tag{B-4.35}$$

Replacing s by the operator $D = \frac{d}{dx}$ and using the fact that

$$e^{-\xi D}f(x) = \sum_{k=0}^{\infty}\frac{(-\xi)^k D^k f(x)}{k!} = f(x-\xi), \tag{B-4.36}$$

equation (B-4.35) becomes

$$\begin{aligned} e^{aD^2} f(x) &= \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{\infty} \left[e^{-\xi D} f(x)\right] \exp\left(-\frac{\xi^2}{4a}\right) d\xi \\ &= \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{\infty} f(x-\xi) \exp\left(-\frac{\xi^2}{4a}\right) d\xi \\ &= \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4a}\right] f(\xi)\, d\xi \\ &= \int_{-\infty}^{\infty} G(x-\xi) f(\xi)\, d\xi = F(x). \end{aligned} \tag{B-4.37}$$

This gives the formula for the inverse Weierstrass transform

$$f(x) = e^{-aD^2} F(x), \quad x \in \mathbb{R}. \tag{B-4.38}$$

The celebrated Riemann–Liouville fractional integral of order α of a function $f(t)$ is usually defined by

$$D^{-\alpha} f(t) = {}_0 D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)\, dx, \quad \operatorname{Re}\alpha > 0. \tag{B-4.39}$$

Clearly, $D^{-\alpha}$ is a linear integral operator, and (B-4.39) can be expressed in terms of the convolution product

$$D^{-\alpha} f(t) = f(t) * g(t), \tag{B-4.40}$$

where $g(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $\overline{g}(s) = \mathcal{L}\{g(t)\} = s^{-\alpha}$. Using the Laplace convolution theorem to (B-4.40) gives

$$\begin{aligned} \mathcal{L}\{D^{-\alpha} f(t)\} &= \mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} \\ &= s^{-\alpha}\overline{f}(s), \end{aligned} \tag{B-4.41}$$

or equivalently,

$$D^{-\alpha} f(t) = \mathcal{L}^{-1}\{s^{-\alpha}\overline{f}(s)\}. \tag{B-4.42}$$

This can be used to evaluate the fractional integral of a given function $f(t)$. For example, if $f(t) = t^\beta$, then

$$D^{-\alpha} t^\beta = \mathcal{L}^{-1}\left\{\frac{\Gamma(\beta+1)}{s^{\alpha+\beta+1}}\right\} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \tag{B-4.43}$$

Another consequence of (B-4.39) for the *fractional derivative*, $D^{-\alpha} f(t)$, is that it can be defined as the solution $\phi(t)$ of the integral equation

$$D^{-\alpha}\phi(t) = f(t) \tag{B-4.44}$$

so that its Laplace transform gives

$$\overline{\phi}(s) = s^{\alpha}\overline{f}(s). \tag{B-4.45}$$

Consequently, the inverse Laplace transform yields

$$\phi(t) = D^{-\alpha}f(t) = \mathcal{L}^{-1}\{s^{\alpha}\overline{f}(s)\} \tag{B-4.46}$$

$$= \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} * f(t) = \frac{1}{\Gamma(-\alpha)}\int_0^t (t-x)^{-\alpha-1}f(x)\,dx. \tag{B-4.47}$$

This is known as the *Cauchy integral formula* for the fractional derivative of $f(t)$. In fact, (B-4.46) can often be used to obtain the fractional derivative of $f(t)$. For example, if $f(t) = t^{\beta}$ then

$$D^{\alpha}t^{\beta} = \mathcal{L}^{-1}\left\{\frac{\Gamma(\beta+1)}{s^{\beta-\alpha+1}}\right\} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1}t^{(\beta-\alpha)}. \tag{B-4.48}$$

In general, the Laplace convolution integral equation is of the form

$$f(t) = h(t) + \lambda\int_0^t g(t-\tau)f(\tau)\,d\tau, \tag{B-4.49}$$

where λ is a given constant parameter, $f(t)$ is the unknown function, $h(t)$ and $g(t)$ are given functions.

Applications of the Laplace transform to (B-4.49) combined with the convolution theorem yields

$$\overline{f}(s) = \frac{\overline{h}(s)}{1-\lambda\overline{g}(s)}. \tag{B-4.50}$$

The inverse Laplace transform gives the formal solution of $f(t)$ in the form

$$f(t) = \mathcal{L}^{-1}\left\{\frac{\overline{h}(s)}{1-\lambda\overline{g}(s)}\right\}. \tag{B-4.51}$$

For example, the *Abel integral equation of the first kind* in the form

$$\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,d\tau = g(t), \tag{B-4.52}$$

or equivalently,

$$t^{\alpha-1} * f(t) = g(t), \tag{B-4.53}$$

can be solved by application of the Laplace transform so that

$$\overline{f}(s) = \frac{1}{\Gamma(\alpha)}s^{\alpha}\overline{g}(s) = \frac{1}{\Gamma(\alpha)}s\left[s^{\alpha-1}\overline{g}(s)\right]. \tag{B-4.54}$$

This leads to the solution

$$f(t) = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} g(\tau)\, d\tau. \tag{B-4.55}$$

The *Abel integral equation of the second kind* is given by

$$f(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau = g(t), \quad \alpha > 0, \tag{B-4.56}$$

where λ is a real or complex parameter and $g(t)$ is a given function.

Application of the Laplace transform to (B-4.56) leads to the solution

$$\overline{f}(s) = \left(\frac{s^\alpha}{s^\alpha + \lambda}\right)\overline{g}(s) = \left[s \cdot \frac{s^{\alpha-1}}{s^\alpha + \lambda} \cdot \overline{g}(s)\right], \tag{B-4.57}$$

so that the inverse Laplace transform gives the solution of (B-4.56) as

$$f(t) = \frac{d}{dt} \int_0^t E_{\alpha,1}\left(-\lambda \tau^\alpha\right) g(t-\tau)\, d\tau, \tag{B-4.58}$$

where the Mittag-Leffler function, $E_{\alpha,\beta}(z)$, is given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad \alpha, \beta > 0. \tag{B-4.59}$$

C

Answers and Hints to Selected Exercises

1.15 Exercises

1. (a) $A = 4$, $B = 5$, $C = 1$, $B^2 - 4AC = 9 > 0$. Hyperbolic, $\frac{dy}{dx} = \frac{B \pm \sqrt{B^2-4AC}}{2A} = 1, \frac{1}{4}$. Integrating gives $y - x = c_1$ and $y = \frac{1}{4}x + c_2$, $u_{\xi\eta} = \frac{1}{3}(u_\eta - \frac{8}{3})$; $\alpha = \xi + \eta$, $\beta = \xi - \eta$, $u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{3}(u_\alpha - u_\beta - \frac{8}{3})$.

   (b) $A = 2$, $B = -3$, $C = 1$, $B^2 - 4AC = 1 > 0$. Hyperbolic, $\frac{dy}{dx} = -\frac{1}{2}, -1$.

   Integrating gives $x + 2y = c_1$, $x + y = c_2$; $\quad u_{\xi\eta} = \eta - \xi$.

   (c) Hyperbolic, $\xi = y^2 - x^2 = c_1$, $\eta = y - x = c_2$; $\quad u_{\xi\eta} + \frac{1}{\eta}u_\xi = 0$.

   (d) Hyperbolic for $y < 0$ and elliptic for $y > 0$. Parabolic for $y = 0$.

   For the hyperbolic case, $\xi = x + 2\sqrt{-y}$, $\eta = x - 2\sqrt{-y}$, $y = -\frac{1}{4}(x-c)^2$, and $u_{\xi\eta} = \frac{1}{2(\xi-\eta)}(u_\eta - u_\xi)$.

   For the elliptic case, $\alpha = x$, $\beta = 2\sqrt{y}$; $u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{\beta}u_\beta$.

   (e) Elliptic for $y > e^{-x}$, parabolic for $y = e^{-x}$, and hyperbolic for $y < e^{-x}$.

   (f) Hyperbolic for $x < 0$ and elliptic for $x > 0$.

   For $x < 0$; $\xi, \eta = \frac{3}{2}y \pm (-x)^{3/2}$, $(y - c) = \pm\frac{2}{3}(-x)^{3/2}$ (cubic parabolas). $u_{\xi\eta} = \frac{1}{6(\xi-\eta)}(u_\xi - u_\eta)$.

   For $x > 0$, $\alpha = \frac{3}{2}y$, $\beta = -x^{3/2}$; $u_{\alpha\alpha} + u_{\beta\beta} + (\frac{1}{3\beta})u_\beta = 0$, where α and β satisfy the Beltrami equations $\beta_x = -\sqrt{x}\,\alpha_y$, $\beta_y = \frac{1}{\sqrt{x}}\alpha_x$.

   (g) Hyperbolic for $|x| < 2|y|$, elliptic for $|x| > 2|y|$, and parabolic for $|x| = 2|y|$.

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(h) Hyperbolic in the first and third quadrants, elliptic in the second and fourth quadrants.

(i) Elliptic for $|x| < a$, parabolic for $|x| = a$, and hyperbolic for $|x| > a$.

(j) $\xi = y^2$, $\eta = x^2$; $2\xi\eta(u_{\xi\xi} + u_{\eta\eta}) + \eta u_\xi + \xi u_\eta = 0$.

2. (d) Elliptic for all x and y. In this case, the characteristics equations are $\frac{dy}{dx} = \pm i \operatorname{sech}^2 x$ so that the characteristics are $y \mp i \tanh x = \text{constant}$ so that $\alpha = y$, $\beta = \tanh x$. Thus, the canonical form of the given equation is $u_{\alpha\alpha} + u_{\beta\beta} = 2\beta(1-\beta^2)^{-1}u_\beta$.

(f) Hyperbolic for all x and y. The characteristic equations are $\frac{dy}{dx} = \pm\operatorname{sech}^2 x$ so that the characteristics are $y \mp \tanh x = \text{const.}$, $\xi = y + \tanh x$, and $\eta = y - \tanh x$. Thus, the canonical form of the given equation is $u_{\xi\eta} = (\eta - \xi) \times (u_\xi - u_\eta)\{4 - (\xi - \eta)^2\}^{-1}$.

(g) $\frac{dy}{dx} = \operatorname{cosec} y$; $\xi = x + \cos y$, $\eta = y$; $u_{\eta\eta} = \sin^2 \eta \cos \eta u_\xi$.

4. (a) $A = u^2$, $B = 2u_x u_y$, $C = -u^2$. Hence, $B^2 - 4AC = 4(u_x^2 u_y^2 + u^4) > 0$ for all $u(x,y)$. Hyperbolic for all $u(x,y)$.

(b) $|\nabla u| = \sqrt{u_x^2 + u_y^2}$. $A = 1 - u_x^2$, $B = -2u_x u_y$, $C = 1 - u_y^2$. Hence, $B^2 - 4AC = -4 + 4(u_x^2 + u_y^2) = -4 + 4(\Delta u)^2 > 0, = 0$, or < 0 for $|\Delta u| > 1$, $= 1$, or, < 1.

5. $\alpha_n = (\frac{n\pi}{\ell})$, $n = 0, 1, 2, 3, \ldots$; $u_n(x,t) = a_n \exp(-\alpha_n^2 kt)\cos(\frac{n\pi x}{\ell})$, and $u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{\ell})$ is the cosine Fourier series, where $a_0 = \frac{1}{\ell}\int_0^\ell f(x)\,dx$ and $a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos(\frac{n\pi x}{\ell})\,dx$.

6. For $\lambda = -\alpha^2$, $u(x,y) = (A\cos\alpha x + B\sin\alpha x)(C\cosh\alpha y + D\sinh\alpha y)$, $A = 0$, and $\alpha = \frac{n\pi}{a}$, $n = 1, 2, 3, \ldots$; $\frac{C}{D} = -\tanh\alpha b$, and $u(x,y) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{a})\sinh\frac{n\pi(b-y)}{a}$, where $a_n = (\sinh\frac{n\pi b}{a})^{-1}(\frac{2}{a})\int_0^a f(x) \times \sin(\frac{n\pi x}{a})\,dx$.

7. Hint: We assume $u(x,t) = X(x)T(t) \neq 0$ and substitute in the given equation to obtain $XT''(t) + c^2 TX''''(x) = 0$, or equivalently, $-\frac{1}{c^2}\frac{T''}{T} = \frac{X''''}{X} = \lambda^4$.

We have $X'''' = \lambda^4 X$, $T'' = -\lambda^4 c^2 T$, where λ^4 is a separation constant. Thus,

$$X(x) = A\cosh\lambda x + B\cos\lambda x + C\sinh\lambda x + D\sin\lambda x,$$
$$T(t) = E\cos(\lambda^2 ct) + F\sin(\lambda^2 ct), \qquad B = -A, \qquad D = -C,$$
$$X(x) = A(\cosh\lambda x - \cos\lambda x) + C(\sinh\lambda x - \sin\lambda x).$$

From conditions $X(\ell) = 0 = X'(\ell)$, we obtain

$$\begin{vmatrix} \cos\lambda\ell - \cos\lambda\ell & \sinh\alpha\ell - \sin\alpha\ell \\ \sinh\lambda\ell + \sin\lambda\ell & \cosh\alpha\ell - \cos\alpha\ell \end{vmatrix} = 0 \quad \Longrightarrow \quad \cos\lambda\ell\cosh\lambda\ell = 1.$$

8. We seek a nontrivial separable solution $u(x,t) = X(x)T(t)$ so that $\sin\alpha\ell = 0$, $\alpha = (\frac{n\pi}{\ell})$, $n = 1, 2, 3, 4, \ldots$; $\sinh\alpha\ell \neq 0$. $u(x,t) = \sum_{n=1}^{\infty}[a_n\cos\{(\frac{n\pi}{\ell})^2 ct\} + b_n\sin\{(\frac{n\pi}{\ell})^2 ct\}]\sin(\frac{n\pi x}{\ell})$.

9. $a_n = 0$ for all even n, $a_n = (\frac{4\ell}{n^2\pi^2})$, $n = 1, 5, 9, \ldots$, and $a_n = -(\frac{4\ell}{n^2\pi^2})$, $n = 3, 7, 11, \ldots$.

   Thus, $u(x,t) = \sum_{n=1}^{\infty} a_n \exp(-\frac{n^2\pi^2\kappa t}{\ell^2})\sin(\frac{n\pi x}{\ell})$.

11. Application of the Fourier transform, $\mathcal{F}\{u(x,t)\} = U(k,t)$, to the problem gives $\frac{d^2U}{dt^2} + k^4 U = 0$, $U(k,0) = F(k)$, and $U_t(k,0) = 0$. Thus, the solution of the transformed system is $U(k,t) = F(k)\cos(k^2 t)$. Consequently, the inverse Fourier transform together with the Fourier convolution yields the solution $u(x,t) = \mathcal{F}^{-1}\{F(k)\cos(k^2 t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)h(\xi,t)\,d\xi$, where $\mathcal{F}^{-1}\{\cos(k^2 t)\} = \frac{1}{\sqrt{2t}}\cos(\frac{x^2}{4t} - \frac{\pi}{4}) = h(x,t)$.

12. Apply the joint Fourier and Laplace transforms (1.9.12) to obtain $U(k,s) = \frac{(s+\alpha)F(k)+G(k)}{(s^2+s\alpha+c^2k^2)}$. Application of the joint inverse transforms combined with the convolution of the Fourier transform gives the solution for $u(x,t)$.

13. $u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{A(k)\exp[i(kx+\omega t)] + B(k)\exp[i(kx-\omega t)]\}\,dk$, where $\omega = \sqrt{c^2k^2 + a^2}$, and $A(k) = \frac{1}{2}[F(k) + \frac{1}{i\omega}G(k)]$, and $B(k) = \frac{1}{2}[F(k) - \frac{1}{i\omega}G(k)]$.

15. Hint: $\mathcal{F}^{-1}\left\{\begin{matrix}\cos\\ \sin\end{matrix}(atk^2)\right\} = \frac{1}{2\sqrt{at}}[\cos(\frac{x^2}{4at}) \pm \sin(\frac{x^2}{4at})]$.

17.

$$\phi(x,z,t) = -\frac{P}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\omega t}{\omega}\exp\bigl(ikx + |k|z\bigr)\,dk, \quad \omega^2 = g|k|,$$
$$\eta(x,t) = \frac{P}{2\pi}\int_{-\infty}^{\infty}\cos\omega t\exp(ikx)\,dk \approx \frac{Pt}{2\sqrt{2\pi}}\cdot\frac{\sqrt{g}}{x^{3/2}}\cos\left(\frac{gt^2}{4x}\right)$$

for $gt^2 \gg 4x$.

19.

$$\phi(x,z,t) = \frac{iP\exp(\varepsilon t)}{2\pi\rho}\int_{-\infty}^{\infty}\frac{(Uk - i\varepsilon)\exp(|k|z + ikx)}{(Uk - i\varepsilon)^2 - g|k|}\,dk,$$
$$\eta(x,t) = \frac{P\exp(\varepsilon t)}{2\pi\rho}\int_{-\infty}^{\infty}\frac{|k|\exp(ikx)\,dk}{(Uk - i\varepsilon)^2 - g|k|}.$$

23. Using the Fourier transform, $U(k,y) = \mathcal{F}\{u(x,y)\}$, gives the solution of the transformed system in the form $U(k,y) = F(k)\cos(k^2y)$. The inverse Fourier transform together with the Fourier convolution gives the solution $u(x,y) = \frac{1}{\sqrt{4\pi y}}\int_{-\infty}^{\infty} f(x-\xi)\cos(\frac{\xi^2}{4y} - \frac{\pi}{4})\,d\xi$.

24.

$$u(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k,\ell)\cos\bigl\{c\bigl(k^2+\ell^2\bigr)^{\frac{1}{2}}t\bigr\} \times \exp\bigl\{i(kx + \ell y)\bigr\}\,dk\,d\ell.$$

25. Application of the joint Laplace and Fourier transform gives $\bar{U}(k,s) = (s + \kappa k^2)^{-1}\bar{Q}(k,s)$. The use of the inverse Laplace transform combined with the convolution theorem yields

$$U(k,t) = \exp\bigl(-\kappa k^2 t\bigr) * Q(k,t) = \int_0^t \exp\bigl[-\kappa k^2(t-\tau)\bigr]Q(k,\tau)\,d\tau.$$

Application of the inverse Fourier transform and its convolution theorem gives

$$\begin{aligned} u(x,t) &= \int_0^t \mathcal{F}^{-1}\{P(k,t-\tau)Q(k,\tau)\}\, d\tau \\ &= \int_0^t p(x,t-\tau) * q(x,\tau)\, d\tau \\ &= \frac{1}{\sqrt{2\pi}} \int_0^t \left[\int_{-\infty}^{\infty} p(x-\xi,t-\tau) q(\xi,\tau)\, d\xi\right] d\tau \\ &= \frac{1}{\sqrt{4\pi\kappa}} \int_0^t (t-\tau)^{-\frac{1}{2}}\, d\tau \int_{-\infty}^{\infty} q(\xi,\tau) \exp\left[-\frac{(x-\xi)^2}{4\kappa(t-\tau)}\right] d\xi, \end{aligned}$$

where $p(x,\tau) = \mathcal{F}^{-1}\{\exp(-\kappa k^2 \tau)\} = \frac{1}{\sqrt{2\kappa\tau}} \exp(-\frac{x^2}{4\kappa\tau})$.

27. $u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{F(k)\cos(x\alpha) + \frac{G(k)}{\alpha}\sin(x\alpha)\}e^{ikt}\, dk$, where $\alpha^2 = \alpha^2(k) = \frac{1}{c^2}(k^2 - iak - b)$.

28. Hint: Seek a solution of the form $\psi(x,y,t) = \phi_n(x,t)\sin(n\pi y)$ with $\psi_0(x,y) = \psi_{0n}(x)\sin(n\pi y)$, so that $\phi_n(x,t)$ satisfies the equation $\frac{\partial}{\partial t}[\frac{\partial^2\phi_n}{\partial x^2} - \alpha^2\phi_n] + \beta\frac{\partial\phi_n}{\partial x} = 0$, where $\alpha^2 = (n\pi)^2 + \kappa^2$. Apply the Fourier transform of $\phi_n(x,t)$ with respect to x, and use $\Psi_n(k,0) = \mathcal{F}\{\psi_{0n}(x)\}$. $\phi_n(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi_n(k,0) \times \exp[i\{kx - \omega(k)t\}]\, dk$, where $\omega(k) = -\beta k(k^2 + \alpha^2)^{-1}$.

29. Divide the first equation by L and the second equation by C to obtain the third and the fourth equations $I_t + \frac{R}{L}I = -\frac{1}{L}V_x$, and $V_t + \frac{G}{C}V = -\frac{1}{C}I_x$. Differentiate the third equation with respect to time t and replace V_t and V_x on the right-hand side to derive a second order equation telegraph equation for I in the form $I_{tt} - c^2 I_{xx} + (p+q)I_t + pqI = 0$, where $c^2 = (LC)^{-1}$, $p = \frac{R}{L}$, and $q = \frac{G}{C}$. Similar calculation can be used to obtain the same equation for V. Thus, $u = I$ or V satisfies the above telegraph equation with coefficients c^2, p, and q, or with coefficients c^2, a, and b.

30. (a) $V(x,t) = V_0\,\mathrm{erfc}(\frac{x}{2\sqrt{\kappa t}})$, $\kappa = a^{-1} = (RC)^{-1}$. $I(x,t) = \frac{V_o}{R}\frac{1}{\sqrt{\pi\kappa t}} \times \exp(-\frac{x^2}{4\kappa t})$.

(b) $V(x,t) = \frac{1}{2}e^{-x\sqrt{b}}\,\mathrm{erfc}\{\frac{x}{2}\sqrt{\frac{a}{t}} + \sqrt{\frac{bt}{a}}\} + \frac{1}{2}e^{-x\sqrt{b}}\,\mathrm{erfc}\{\frac{x}{2}\sqrt{\frac{a}{t}} - \sqrt{\frac{bt}{a}}\}$.

31. $V(x,t) = V_0 \exp(-\frac{kx}{c}) f(t - \frac{x}{c}) H(t - \frac{x}{c})$.

32. $u(x,t) = x \cos \omega t$.

33. $u(x,t) = \frac{k}{(\pi c)^2}[1 - \cos(\frac{\pi ct}{a})] \sin(\frac{\pi x}{a})$.

34. $u(z,t) U \exp[i\omega t - (\frac{\omega}{2\nu})^{\frac{1}{2}}(1+i)z]$, $u(z,t) = U \text{erfc}(\frac{z}{2\sqrt{\nu t}})$.

35. $u(z,t) = Ut[(1+2\zeta^2)\text{erfc}(\zeta) - \frac{2\zeta}{\sqrt{\pi}} \exp(-\zeta^2)]$, $\zeta = \frac{z}{2\sqrt{\nu t}}$.

36.

$$\begin{aligned} q(z,t) = \frac{a}{2} e^{i\omega t} \big[& e^{-\lambda_1 z} \text{erfc}\{\zeta - [it(2\Omega + \omega)]^{\frac{1}{2}}\} \\ & + e^{\lambda_1 z} \text{erfc}\{\zeta + [it(2\Omega + \omega)]^{\frac{1}{2}}\}\big] \\ & + \frac{b}{2} e^{-i\omega t} \big[e^{-\lambda_2 z} \text{erfc}\{\zeta - [it(2\Omega - \omega)]^{\frac{1}{2}}\} \\ & + e^{\lambda_2 z} \text{erfc}\{\zeta + [it(2\Omega + \omega)]^{\frac{1}{2}}\}\big], \end{aligned}$$

where $\lambda_{1,2} = \{\frac{i(2\Omega \pm \omega)}{\nu}\}^{\frac{1}{2}}$. Thus,

$$q(z,t) \sim a \exp(i\omega - \lambda_1 z) + b \exp(-i\omega t - \lambda_2 z), \quad \delta_{1,2} = \left\{ \frac{\nu}{|2\Omega \pm \omega|} \right\}^{\frac{1}{2}}.$$

43. (b) Hint: $\tilde{f}(k) = (\frac{Q}{\pi ak}) J_1(ak)$.

45. $u(r,t) = \int_0^\infty k \, \tilde{f}(k) \cos(btk^2) J_0(kr) \, dk$.

If $f(r) = \exp(-\frac{r^2}{a^2})$, then $\tilde{f}(k) = \frac{a^2}{2} \exp(-\frac{a^2k^2}{4})$.

Using the self-reciprocity of the Hankel transform gives the solution

$$\begin{aligned} u(r,t) &= \frac{1}{2} a^2 \mathcal{H}_0 \left[\exp\left(-\frac{a^2k^2}{4}\right) \cos\left(btk^2\right) \right](r) \\ &= a^2 \Omega(t) \exp\left[-a^2 \Omega(t) r^2\right] \left[a^2 \cos\left(4bt\Omega(t) r^2\right) + 4bt \sin\left(4bt\Omega(t) r^2\right)\right], \end{aligned}$$

where $\Omega(t) = (a^4 + 16b^2t^2)^{-1}$.

48. Hint: The solution of the dual integral equations

$$\int_0^\infty k J_0(kr) A(k) \, dk = u_0, \quad 0 \le r \le a,$$
$$\int_0^\infty k^2 J_0(kr) A(k) \, dk = 0, \quad a < r < \infty,$$

is given by $A(k) = (\frac{2u_0}{\pi}) \frac{\sin(ak)}{k^2}$.

49. Hint: See Debnath (1994, pp. 103–105).

50. $u(r,z) = \left(\frac{1}{\pi a}\right) \int_0^\infty k^{-1} J_1(kr) J_0(kr) \exp(-kz)\, dk$.

52. Hint: $\mathcal{L}^{-1}[(s^2+a^2)^{-\frac{1}{2}} \exp\{-k(s^2+a^2)^{\frac{1}{2}}\}] = H(t-k) J_0(a\sqrt{t^2-k^2})$.

Application of the joint Laplace and Hankel transforms

$$\tilde{\bar{u}}(k,z,s) = \int_0^\infty r J_0(kr)\, dr \int_0^\infty e^{-st} u(r,z,t)\, dt$$

to the given equation and the general boundary condition $u_z(r,0,t) = f(r,t) = H(a-r)g(t)$ gives

$$\frac{d^2\tilde{\bar{u}}}{dz^2} - \frac{1}{c^2}\left(s^2 + k^2c^2\right)\tilde{\bar{u}} = 0, \qquad \tilde{\bar{u}}_z(k,0,s) = \frac{a}{k} J_1(ak)\bar{g}(s).$$

The bounded solution of the equation is

$$\tilde{\bar{u}}(k,z,s) = \tilde{\bar{A}}(k,s) \exp\left[-\frac{z}{c}\sqrt{s^2+c^2k^2}\right]$$
$$\text{with } \tilde{\bar{A}}(k,s) = -\left(\frac{ac}{k}\right)\frac{\bar{g}(s)J_1(ak)}{\sqrt{s^2+c^2k^2}}.$$

Thus, $\tilde{\bar{u}}(k,z,s) = -\left(\frac{ac}{k}\right)\frac{J_1(ak)}{\sqrt{s^2+c^2k^2}}\bar{g}(s)\exp[-\frac{z}{c}\sqrt{s^2+c^2k^2}]$.

The inverse Laplace transform gives

$$\begin{aligned}\tilde{u}(k,z,t) &= -\left(\frac{ac}{k}\right) J_1(ak)\mathcal{L}^{-1}\left[\frac{\bar{g}(s)}{\sqrt{s^2+c^2k^2}}\exp\left(-\frac{z}{c}\sqrt{s^2+c^2k^2}\right)\right] \\ &= -\frac{ac}{k} J_1(ak)\int_0^t g(t-\tau)H\left(\tau - \frac{z}{c}\right) J_0\left(ck\sqrt{\tau^2 - \frac{z^2}{c^2}}\right) d\tau.\end{aligned}$$

The inverse Hankel transform leads to the final solution

$$\begin{aligned}u(r,z,t) = (-ac)\int_0^\infty J_0(kr)J_1(ak)\, dk \int_0^t g(t-\tau)H\left(\tau - \frac{z}{c}\right) \\ \times J_0\left[k\sqrt{(c^2\tau^2 - z^2)}\right] d\tau.\end{aligned}$$

This, for $g(t)=\delta(t)$, becomes

$$u(r,z,t)=(-ac)H\left(t-\frac{z}{c}\right)\int_0^\infty J_0(kr)J_1(ak)J_0\left[k\sqrt{c^2t^2-z^2}\right]dk.$$

53. $u(r,z)=b\int_0^\infty k^{-1}\left(\frac{\sinh kz}{\cosh ka}\right)J_1(bk)J_0(kr)\,dk.$

54. Hint: $\mathcal{H}_0[(a^2-r^2)^{-\frac{1}{2}}H(a-r)]=\frac{\sin(ak)}{k}$, and

$$\begin{aligned}&\mathcal{L}^{-1}\left[\left\{\sqrt{s}(\sqrt{s}-a)\right\}^{-1}\exp\left(-k\sqrt{s}\right)\right]\\&\quad=\exp\left(-ak-a^2t\right)\operatorname{erfc}\left(\frac{k}{2\sqrt{t}}-a\sqrt{t}\right).\end{aligned}$$

55. Hint: Use the joint Hankel and Laplace transform method.

57. Use the Hankel transform and derive

$$\begin{aligned}u(r,z,t)=\frac{1}{\rho}\int_0^\infty k\exp(kz)J_0(kz)\left[\int_0^\infty\left\{\int_0^{r_0(t)}\alpha p(\alpha,\tau)\right.\right.\\\left.\left.\times J_0(k\alpha)\,d\alpha\right\}\cos\left\{\omega(t-\tau)\right\}d\tau\right],\end{aligned}$$

where $\omega^2=gk$.

59. Hint: Use the joint Laplace and Fourier transform.

$$G(x,t)=\frac{W}{2\pi m}\int_{-\infty}^\infty\frac{\sin\alpha t}{\alpha}\exp(ikx)\,dk,\quad\alpha=\left(a^2k^4+\omega^2\right)^{\frac{1}{2}},$$

where $a^2=\frac{EI}{m}$ and $\omega^2=\frac{\kappa}{m}$.

61. Hint: Set

$$G(x,y;\xi,\eta)=\sum_{m=1}^\infty\sum_{n=1}^\infty a_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

in the original equation to find a_{mn} from

$$\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\left(\frac{ab}{4}\right)a_{mn}=\sin\left(\frac{m\pi\xi}{a}\right)\sin\left(\frac{n\pi\eta}{b}\right).$$

62. Apply the joint Laplace and double Fourier transform.

63. Hint: $G_{tt} - c^2 G_{xx} + d^2 G = \delta(x)\delta(t)$.

The joint Laplace and Fourier transform gives

$$\tilde{\bar{G}}(k,s) = \frac{1}{\sqrt{2\pi}} \frac{1}{(s^2+\alpha^2)}, \quad \alpha = \left(c^2k^2 + d^2\right)^{\frac{1}{2}}.$$

The inverse transforms give

$$\begin{aligned} G(x,t) &= \frac{1}{2\pi c} \int_{-\infty}^{\infty} \left(k^2 + \frac{d^2}{c^2}\right)^{-\frac{1}{2}} \sin\left\{ct\sqrt{k^2 + \frac{d^2}{c^2}}\right\} \exp(ikx)\, dk \\ &= \left(\frac{1}{2c}\right) J_0\left[\frac{d}{c}\sqrt{c^2t^2 - x^2}\right] H\left(ct - |x|\right). \end{aligned}$$

64. Hint: Eigenvalues are $\lambda_n = (\frac{n\pi}{\ell})^2$, $n = 0, 1, 2, 3, \ldots$. Eigenfunctions are $X_n(x) = A_n \cos(\frac{n\pi x}{\ell}) + B_n \sin(\frac{n\pi x}{\ell})$.

Thus,

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} X_n(x) T_n(t) = \sum_{n=0}^{\infty} u_n(x,t) \\ &= \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right] \exp\left(-\frac{n\pi k t}{\ell}\right), \\ f(x) = u(x,0) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right)\right], \end{aligned}$$

where

$$\begin{aligned} a_n &= \frac{1}{\ell} \int_{-\ell}^{\ell} f(\xi) \cos\left(\frac{n\pi\xi}{\ell}\right) d\xi, \quad n = 0, 1, 2, \ldots, \\ b_n &= \frac{1}{\ell} \int_{-\ell}^{\ell} f(\xi) \sin\left(\frac{n\pi\xi}{\ell}\right) d\xi, \quad n = 1, 2, 3, \ldots. \end{aligned}$$

65. (b) Initial data tend to zero as $n \to \infty$, but the solution

$$u_n(x,y) \to \infty \quad \text{as } n \to \infty.$$

66. (a) Hint: $\lambda = 0$, $u(x) = A + Bx$ with the boundary conditions leads to the trivial solution.

If $\lambda < 0$, $u(x) = A \cosh kx + B \sinh kx (\lambda = -k^2)$. The first boundary condition gives $A = 0$, and the second boundary condition yields $k = -\tanh k$. The graphical representation of $y = -k$ and $y = \tanh k$ shows that there is no intersection of these curves for $k > 0$. Hence, there are no negative eigenvalues. For $\lambda = k^2 > 0$, $u(x) = A \cos kx + B \sin kx$. The boundary conditions give $A = 0$, and $B \neq 0$, $k = -\tan k$. The graphical representation of $y = -k$ and $y = \tanh k$ shows that there is an infinite number of values $k = k_n$ $(n = 1, 2, 3, \ldots)$ so that there is an infinite number of eigenvalues $\lambda = k_n^2 \approx \frac{1}{4}(2n+1)^2\pi^2$ for large n. The corresponding eigenfunctions are $u_n(x) = \sin k_n x$, where $n = 1, 2, 3, \ldots$.

(b) $\lambda_n = k_n^2 = (n\pi)^2$, $u_n(x) = A_n \cos n\pi x$, $n = 0, 1, 2, \ldots$.

(c) $\lambda_n = k_n^2 = (2n\pi)^2$, $u_n(x) = \{\sin 2\pi nx, \cos 2\pi nx\}$, $(n = 0, 1, 2, \ldots)$ are eigenfunctions. To each eigenvalue, there correspond two eigenfunctions. Eigenvalues are degenerate.

(d) For $1 - 4\lambda = 0$, or > 0, only trivial solutions.

For $1 - 4\lambda = -k^2 < 0$, we obtain eigenvalues $\lambda_n = \frac{1}{4}(1 + \frac{n^2\pi^2}{a^2})$ and eigenfunctions $u_n(x) = B_n e^{-x} \sin k_n x$, $(k_n = \frac{n\pi}{a}$, $n = 1, 2, 3, \ldots)$.

67. Hint: Multiply the equation by $\frac{p(x)}{a_2(x)}$ so that

$$p(x)u'' + p'(x)u' + \big[q(x) + \lambda\rho(x)\big]u = 0,$$

where $p'(x) = \frac{p(x)a_1(x)}{a_2(x)}$, $q(x) = \frac{p(x)a_0(x)}{a_2(x)}$, and $\rho(x) = \frac{p(x)}{a_2(x)}$.

69. Hint: $\lambda = k^2 > 0$. The general solution is

$$y(x) = A + Bx + C \cos kx + D \sin kx.$$

The first boundary conditions at $x = 0$ give $A = C = 0$.

The remaining boundary conditions at $x = a$ give

$$Ba + D \sin ka = 0, \qquad k^2 D \sin ak = 0.$$

For nontrivial solutions, $B = C$, $D \neq 0$, $k = \frac{n\pi}{a}$.

Thus, $\lambda_n = \frac{n^2\pi^2}{a^2}$, and $y_n(x) = D_n \sin(\frac{n\pi x}{a})$.

The critical buckling loads are $P_n = EI(\frac{n^2\pi^2}{a^2})$. The Euler load is the largest load which the beam can withstand before possible buckling: $P_1 = \frac{\pi^2}{a^2}(EI)$. The corresponding fundamental bucking mode is $y_1(x) = D_1 \sin(\frac{\pi x}{a})$.

71. $X'' + 2bX' + k^2 X = 0$, $\dot{T} + k^2 T = 0$, where $-k^2$ is a separation constant. $m = -b \pm i\alpha$, $\alpha^2 = k^2 - b^2 > 0$.

$X(x) = \exp(-bx)(A \cos \alpha x + B \sin \alpha x)$. Using boundary conditions, we obtain $A = 0$, $\sin \alpha a = 0$, $B \neq 0$, $\alpha = (\frac{n\pi}{a})$, $n = 1, 2, 3, \ldots$.

Hence, $T(t) = C \exp(-k^2 t)$. Thus, the final solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\bigl(-k_n^2 t - bx\bigr) \sin \alpha_n x, \quad k_n^2 = b^2 + \alpha_n^2.$$

72. (a) Eigenvalues are $k^2 = (\frac{\lambda}{a})^2$ and eigenfunctions are

$$R(r) = A J_n\left(\frac{\lambda r}{a}\right), \quad J_n(ka) = 0,$$

$$T(t) = B \cos\left(\frac{\lambda ct}{a}\right) + C \sin\left(\frac{\lambda ct}{a}\right), \qquad \Theta(\theta) = D \cos n\theta + E \sin n\theta,$$

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (a_{mn} \cos n\theta + b_{nm} \sin n\theta) J_n\left(\frac{\lambda_{mn} r}{a}\right) \cos\left(\frac{\lambda_{mn} ct}{a}\right),$$

where λ_{mn} are positive roots of $J_n(x) = 0$.

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (a_{nm} \cos n\theta + b_{nm} \sin n\theta) J_n\left(\frac{\lambda_{mn} r}{a}\right), \quad \text{where}$$

$$a_{0n} = \frac{1}{\pi a^2 J_1^2(\lambda_{n0})} \int_0^a \int_{-\pi}^{\pi} f(r,\theta) J_0\left(\frac{\lambda_{m0} r}{a}\right) r\, dr\, d\theta \quad \text{and } n \geq 1,$$

$$(a_{nm}, b_{nm}) = \frac{2}{\pi a^2 J_{n+1}^2(\lambda_{mn})} \int_{-\pi}^{\pi} \int_0^a f(r,\theta) J_n\left(\frac{\lambda_{mn} r}{a}\right) \times (\cos n\theta, \sin n\theta) r\, dr\, d\theta.$$

The set of frequencies is $\{\frac{\pi c}{a}\lambda_{mn} : n \geq 0, m \geq 1\}$.

(b) Since f is independent of θ, so is u. The solution is

$$u(r,t) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\lambda_m r}{a}\right) \cos\left(\frac{\lambda_m ct}{a}\right), \quad \lambda_m = \lambda_{m0},$$

$$f(r) = \left(a^2 - r^2\right) = u(r,0) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\lambda_m r}{a}\right)$$

so that $a_m = \frac{8a^2}{\lambda_m^3 J_1(\lambda_m)}$.

73. (a) Hint: Use $u(r,z) = R(r)Z(z)$ so that

$$r^2 R'' + rR' + k^2 r^2 R = 0, \quad R(a) = 0; \qquad Z'' = k^2 Z, \quad Z(0) = 0.$$

The eigenvalues of this Sturm–Liouville problem are $k_n^2 = (\lambda_n/a)^2$ with the corresponding eigenfunctions $R_n(r) = J_0(\frac{\lambda_n r}{a})$, where λ_n are the positive roots of $J_0(x) = 0$.

Thus, $Z_n(z) = \sinh(\frac{z\lambda_n}{a})$. $u(r,z) = \sum_{n=1}^{\infty} a_n J_0(\frac{r\lambda_n}{a}) \sinh(\frac{z\lambda_n}{a})$, and $g(r) = u(r,h) = \sum_{n=1}^{\infty} a_n \sinh(\frac{h\lambda_n}{a}) J_0(\frac{r\lambda_n}{a})$, where $b_n = a_n \sinh(\frac{h\lambda_n}{a}) = \int_0^a r g(r) J_0(\frac{r\lambda_n}{a})\, dr$.

(b) $a_n = \operatorname{cosech}(\frac{r\lambda_n}{a}) \int_0^a r J_0(\frac{r\lambda_n}{a})\, dr = \frac{2\operatorname{cosech}(\frac{h\lambda_n}{a})}{\lambda_n J_1(\lambda_n)}$.

Hence, the solution $u(r,z)$ follows.

74. (a) Hint: Multiply the left-hand side of the wave equation by $2u_t$, and the resulting expression follows from performing the indicated differentiations on the right-hand side of the differential equality.

(b)

$$2u_t\left(c^2 \nabla_n^2 u - u_{tt}\right) = 2c^2\left[(u_t u_{x_1})_{x_1} + (u_t u_{x_2})_{x_2} + \cdots + (u_t u_{x_n})_{x_n}\right] - c^2\left[\left(u_{x_1}^2 + u_{x_2}^2 + \cdots + u_{x_n}^2\right) + u_t^2\right]_t.$$

(c) Perform the indicated differentiation on the right-hand side of (b) so that (c) can be obtained.

75. Hint: (a) In the context of a stretched string, $-d^2u$ can be interpreted as an additional spring force normal to the string.

Multiply the Klein–Gordon equation by u_t to obtain

$$u_t u_{tt} - c^2 u_t u_{xx} + d^2 u_t u = 0,$$
$$\frac{1}{2}\frac{\partial}{\partial t}\left(u_t^2\right) + \frac{c^2}{2}\frac{\partial}{\partial x}\left(u_x^2\right) - c^2\frac{\partial}{\partial x}(u_t u_x) + \frac{1}{2}d^2\left(u^2\right) = 0.$$

(c) Assume $w = u - v$ and u, v or (u_x, v_x) are given at $x = a, b$.

$$w_{tt} = c^2 w_{xx}, a \leq x \leq b,$$

$w(x,0) = 0 = w_t(x,0)$; $w(\text{or}w_x) = 0$ at $x = a, b$.

Using (b), $E(t) = E_0 = 0$, and noting $w(x,0) = 0$ implies $w_x(x,0) = 0$.

Therefore, $E(t) = \frac{1}{2}\int_a^b (w_t^2 + c^2 w_x^2 + d^2 w^2)\,dx = 0$.

Since the integrand is positive, $w \equiv 0$.

76. (a) Hint: Seek a separable solution $u(r,t) = R(r)T(t)$ so that

$$r^2 R'' + 2rR' + k^2 r^2 R = 0, \qquad T'' + c^2 k^2 T = 0,$$
$$R(r) = \frac{1}{\sqrt{r}}\left[A_1 J_{\frac{1}{2}}(kr) + B_1 Y_{\frac{1}{2}}(kr)\right],$$
$$T(t) = C\cos(ckt) + D\sin(ckt) = \frac{1}{r}(A\sin kr + B\cos kr).$$

We assume that $R(r)$ is finite at $r = 0$, and hence, $B = 0$.

The eigenvalues are $k = k_n = \frac{n\pi}{a}$, $n = 1, 2, \ldots$.

$u(r,t) = \sum_{n=1}^{\infty}(a_n \cos ck_n t + b_n \sin ck_n t)\frac{1}{r}\sin(\frac{n\pi r}{a})$.

The initial conditions $u(r,0) = f(r)$ and $u_t(r,0) = g(r)$ can be satisfied by the Fourier series expansion of $rf(r)$ and $rg(r)$ on $(0,a)$.

(b) $a_n = 0$, $b_{2n} = 0$, $b_{2n-1} = \frac{8a^3}{c\pi^4(2n-1)^4}$.

77. We seek a separable solution $u(r,z) = R(r)Z(z)$ so that $r^2R''+rR'-k^2r^2R = 0$, and $Z'' + k^2Z = 0$ with $Z(0) = Z(h) = 0$. $k^2 = \frac{n^2\pi^2}{h^2}$ and $Z(z) = a_n \sin(\frac{n\pi z}{h})$. We assume $R(r)$ is finite at $r = 0$, and hence $R(r) = a_n I_0(\frac{n\pi r}{h})$, and thus, the solution is $u(r,z) = \sum_{n=1}^{\infty} a_n I_0(\frac{n\pi r}{h}) \sin(\frac{n\pi z}{h})$. We have $f(z) = \sum_{n=1}^{\infty} a_n I_0(\frac{n\pi a}{h}) \sin(\frac{n\pi z}{a})$ so that $a_n = [I_0(\frac{n\pi a}{h})]^{-1} \frac{2}{h} \int_0^h f(z) \sin(\frac{n\pi z}{h})\, dz$.

78. (a) Hint: $u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) = 1 + \cos 2\theta$.

$a_0 = 2$, $a_2 = 1$, $a_n = 0$, $n \neq 0, 2$.

$b_n = 0$ for all n, $u(r,\theta) = 1 + r^2 \cos 2\theta$.

(b) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |2\theta|\, d\theta = \frac{2}{\pi}[-\int_{-\pi}^{0} \theta d\theta + \int_0^{\pi} \theta d\theta] = 2\pi$.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |2\theta| \cos\theta\, d\theta = -\frac{8}{\pi n^2}$ for odd n, and 0 for even n.

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |2\theta| \sin n\theta\, d\theta = 0$.

Thus, $u(r,\theta) = \pi - \frac{8}{\pi}\sum_{n=0}^{\infty}(2n+1)^{-2} r^{2n+1} \cos(2n+1)\theta$.

(c) Hint: $\sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) = 2\cos 2\theta$.

$a_2 = 1$, $a_n = 0$, $n \neq 2$; $b_n = 0$ for all n.

Therefore, $u(r,\theta) = \frac{a_0}{2} + r^2 \cos 2\theta$.

(d) $u(r,\theta) = \frac{a_0}{2} + r(\cos\theta + \sin\theta)$.

79. (a) Hint: Seek a separable solution $u(r,\theta) = R(r)\Theta(\theta) \neq 0$ so that

$r^2R''(r) + rR'(r) - \lambda^2 R = 0$, $\Theta''(\theta) + \lambda^2\Theta(\theta) = 0$.

Since $u(r,\theta)$ is periodic with period 2π, so is $\Theta(\theta)$.

Eigenvalues: $\lambda_n = \lambda = n$, $n = 1, 2, \ldots$.

The solution of the Cauchy–Euler equation for $R(r)$ is

$$n = 0: \quad R_0(r) = \frac{1}{2}(A_0 + B_0 \ln r),$$

$$n \geq 1: \quad R_n(r) = A_n r^n + B_n r^{-n}, \qquad \Theta_n(\theta) = C_n \cos n\theta + D_n \sin n\theta,$$

$$u(r,\theta) = \frac{1}{2}(a_0 + b_0 \ln r) + \sum_{n=1}^{\infty}\left(a_n r^n + b_n r^{-n}\right)\cos n\theta + \left(c_n r^n + d_n r^{-n}\right)\sin n\theta.$$

The Fourier coefficients for $r = a$ are

$$(a_0 + b_0 \ln a) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta,$$
$$\left(a_n a^n + b_n a^{-n}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta,$$
$$\left(c_n a^n + d_n a^{-n}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta.$$

The process is similar, when $r = b$ gives the Fourier coefficients.

(b) Use the same method described in Exercise 76(a) to find

$$r = 1: \quad a_0 = 1, \qquad a_n + b_n = 0, \quad n \ge 1; \qquad c_1 + d_1 = 1,$$
$$c_n + b_n = 0, \quad n > 1.$$
$$r = 2: \quad b_0 = -1, \qquad 2a_1 + \frac{b_1}{2} = 1, \qquad 2^n a_n + 2^{-n} b_n = 0 \quad \text{for } n > 1,$$
$$2^n c_n + 2^{-n} d_n = 0, \quad n \ge 1; \qquad a_1 = \frac{2}{3}, \qquad b_1 = -\frac{2}{3},$$
$$c_1 = -\frac{1}{3}, \qquad d_1 = \frac{4}{3},$$
$$u(r,\theta) = \frac{1}{2}(1 - \ln r) + \frac{2}{3}\left(r - r^{-1}\right)\cos\theta - \frac{1}{3}\left(r - 4r^{-1}\right)\sin\theta.$$

(c) Use the solution obtained in Exercise 76(a) with the given boundary conditions.

$$r = 1: \quad b_0 = 0, \qquad a_n - b_n = 0, \quad n \ge 1; \qquad c_n - d_n = 0, \quad n \ge 1,$$
$$r = 2: \quad b_0 = 0, \qquad a_1 - \frac{1}{4}b_1 = \frac{3}{4}, \qquad c_1 - \frac{1}{4}d_1 = \frac{3}{4}.$$

Thus, $a_1 = b_1 = 1$ and $c_1 = d_1 = -1$.

Hence, the solution is $u(r,\theta) = \frac{a_0}{2} + \left(r + r^{-1}\right)\cos\theta - \left(r + r^{-1}\right)\sin\theta$.

81. Define $v(x,y) = \frac{1}{4}(x^2 + y^2)$ so that $\nabla^2\phi = 1$ and assume $w = u + v$ so that $\nabla^2 w = 0$, $w = v$ on ∂D, and $w(0,0) = u(0,0)$. Use the result $\min_{\partial D} v \le u(0,0) \le \max_{\partial D} v$.

Max $v = \frac{a^2}{4}$ and the minimum of $v = \frac{r^2}{4}$, where r is determined from the circle $x^2 + y^2 = r^2$ which touches ∂D: $\frac{x}{a} + \frac{y}{b} = 1$, $(x > 0,\ y > 0)$.

Since $(x, y) = (\frac{r^2}{4},\ \frac{r^2}{4})$, one has $r^2 = a^2b^2(a^2 + b^2)^{-1}$.

82. Hint: We take $v = u_1 - u_2$ where u_1 and u_2 are solutions of $\nabla^2 u = 0$ on D.

    $\nabla^2 v = 0$ for $\mathbf{x} \in D$ and $v(\mathbf{x}) = 0$ on ∂D.

    By the min–max principle, v attains its maximum and minimum on ∂D so that $0 \le v \le 0$ for $\mathbf{x} \in D$, and hence, $v = 0$ on D.

91. Hint: At the leading order $\nabla^2 u_0 = 0, u_0(1, \theta) = \sin\theta$.

    Seek a separable solution $u_0 = f(r)\sin\theta$ so that $f'' + \frac{1}{r}f' - \frac{1}{r^2}f = 0, f(1) = 1$. The solution is $f(r) = Ar + Br^{-1}$. For the bounded solution, $B \equiv 0$. Hence $f(r) = r$, and $u_0 = r\sin\theta$. At $O(\varepsilon^2)$, u_2 satisfies the equation $\nabla^2 u_2 = -r\sin\theta, \quad u_2(1, \theta) = 0$. For the separable solution $u_2 = g(r)\sin\theta$, $g(r)$ satisfies $g'' + \frac{1}{r}g' - \frac{1}{r^2}g = -\frac{r}{A}, \quad g(1) = 0$. Using the variation of parameters, the bounded solution is $g(r) = \frac{1}{8}(r - r^3)$.

92. Hint: $u = 0$ satisfies the given equation and the far field boundary condition, but not the boundary condition at $r = 1$. We need a boundary layer near $r = 1$ so that we can define $r = 1 + \varepsilon r'$ with $r' = O(1)$ in the boundary layer for $\varepsilon \ll 1$. At the leading order $u_{r'r'} - u = 0$, which gives solution $u = A(\theta)e^{r'} + B(\theta)e^{-r'}$. This matches with the far field if $A \equiv 0$ and satisfies the condition at $r' = 0$ when $B = 1$. Thus the inner solution and a composite solution are valid at the leading order for $r \ge a$. Hence the solution follows.

93. Hint: Seek a separable solution $u(x, t) = X(x)T(t) \ne 0$, where $X(x)$ and $T(t)$ satisfy $X'' + \lambda X = 0$, $0 < x < \ell$, $X'(0) = 0 = X'(\ell)$, and $T'(t) + \lambda\kappa T = 0, \quad t > 0$. The eigenvalues are $\lambda_n = (\frac{n\pi}{\ell})^2$ and the eigenfunctions are $X_n(x) = A_n \cos(\frac{n\pi x}{\ell})$, $n = 0, 1, 2, \ldots$. Thus, the solution is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} \exp\left(-\frac{\kappa n^2\pi^2}{\ell^2}t\right),$$

$$f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}, \quad \text{where } a_0 = \frac{1}{\ell}\int_0^{\ell} f(x)\,dx,$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx, \quad n = 1,2,3,\ldots.$$

(b) $a_0 = \frac{1}{2}$, $a_n = \frac{2}{n^2\pi^2}[(-1)^n - 1]$.

96. The solution follows from (1.9.15), when $q(\xi,\tau) = \sin(k\xi - \omega\tau)$.

In case (a), when $c \neq \frac{\omega}{k}$,

$$\begin{aligned} u(x,t) &= \left(k^2c^2 - \omega^2\right)^{-1}\sin(kx - \omega t) \\ &\quad + (kc - \omega)\left[2kc\left(\omega^2 - k^2c^2\right)\right]^{-1}\sin(kt + kct) \\ &\quad + (kc + \omega)\left[2kc\left(\omega^2 - k^2c^2\right)\right]^{-1}\sin(kt - kct). \end{aligned}$$

Thus, the solution consists of three sinusoidal waves that propagate with different amplitudes and with velocities $\pm c$ and the phase velocity (ω/k).

In case (b), when $c = \frac{\omega}{k}$, the solution is given by

$$u(x,t) = \frac{1}{4}\sin(x - t) - \frac{1}{4}\sin(x + t) + \frac{t}{2}\cos(x - t).$$

Thus, the solution consists of two harmonic waves that propagate with velocities ± 1 and another harmonic wave whose amplitude grows linearly with time t.

98. Use the joint Laplace and finite Hankel transform of order one defined by

$$\bar{V}(k_m, s) = \int_0^a rJ_1(rk_m)\,dr\int_0^{\infty} e^{-st}v(r,t)\,dt,$$

where $\bar{V}(k_m, s)$ is the Laplace transform of $V(k_m, t)$, and k_m are the roots of the equation $J_1(ak_m) = 0$.

The solution of the transformed system is

$$\bar{V}(k_m, s) = \frac{a^2 \nu k_m \Omega \bar{f}(s) J_1'(ak_m)}{(s + \nu k_m^2)}.$$

(See Myint-U and Debnath 2007, Chapter 12, p. 507.)

The solution is

$$v(r,t) = -2\nu\Omega \sum_{m=1}^{\infty} \frac{k_m J_1(rk_m)}{J_1'(ak_m)} \int_0^t f(t-\tau) \exp\left(-\nu k_m^2 \tau\right) d\tau.$$

When $f(t) = \cos \omega t$, the solution is given by

$$v(r,t) = -2\nu\Omega \sum_{m=1}^{\infty} \frac{k_m J_1(rk_m)}{J_1'(ak_m)} \int_0^t \cos\omega(t-\tau) \exp\left(-\nu k_m^2 \tau\right) d\tau.$$

When $\omega = 0$, the solution becomes

$$v(r,t) = r\Omega - 2\Omega \sum_{m=1}^{\infty} \frac{J_1(rk_m) \exp(-\nu t k_m^2)}{k_m J_2(ak_m)}.$$

In the limit as $t \to \infty$, the transients die out and the ultimate steady-state is attained as the rigid body rotation about the axis of the cylinder.

99. (b) The Cauchy data tend to zero as $n \to \infty$. But, for $t > 0$, the solution $u_n(x,t) \to \infty$ as $n \to \infty$ for certain values of x and t. So, the problem is ill-posed.

100. (b) The initial data tend to zero as $n \to \infty$. However, for $y \neq 0$, the solution $u_n(x,y) \to \infty$ as $n \to \infty$. Thus, the problem is not well-posed.

101. (b) For $y \neq 0$, the amplitude of the solution tends to infinity as $n \to \infty$ due to the factor $\sinh(ny)$, even though the initial data $u_y(x,0) = \exp(-\sqrt{n}) \times \sin nx \to 0$ as $n \to \infty$. However, the solution $u_n(x,y)$ and all of its derivatives tend to zero as $n \to \infty$ uniformly throughout the half-strip in the (x,y) plane.

102. Without the last term in the first equation, the equation is known as the Euler equation. The second equation is the continuity equation for a certain vir-

tual fluid. The square of the amplitude a^2 plays the role of density ρ and $aV(x)$ plays the role of pressure. The second term on the right-hand side of the first equation plays the role of dispersion. Thus, the transformation of the Schrödinger equation into two equations in fluid dynamics is usually referred to as the *hydrodynamic analogy of quantum mechanics*.

103. (b) $u_{xx} = -n \cos nx \cosh ny$, $u_{yy} = n \cos nx \cosh ny$, and hence, $u_{xx} + u_{yy} = 0$. The boundary data is changed by only a small amount as $u(0, y) \to 0$ as $n \to \infty$. Yet the solution $u(x, y)$ is changed from zero by a large amount. Along the line $x = 0$, the solution is $u(0, y) = \frac{1}{n} \cosh ny \to \infty$ as $y \to \infty$. Thus, a small change of the boundary data produces a large change in the solution. So, the problem is ill-posed.

104. (a) We have $u_t = \kappa n^2 \exp(\kappa n^2 t) \sin nx$ and $u_{xx} = -n^2 \exp(\kappa n^2 t) \sin nx$, and hence, $u_t + \kappa u_{xx} = 0$.

     (b) Thus, $u(x, 0) = \frac{1}{n} \sin nx \to 0$ as $n \to \infty$, yet $u(x, t) \to \infty$ as $n \to \infty$ for any positive t. Hence, the problem is ill-posed.

105. (b) It is easy to verify that $u(x, t) = \frac{1}{n} \sin nx \exp(\kappa n^2 - \delta n^4)t$ is the solution of the equation. It also satisfies the initial condition. The solution is well behaved for large n, and also it is bounded for all n for any finite t. However, the solution is unstable as $n \to 0$. For small δ, the negative diffusion equation is obviously ill-posed.

2.9 Exercises

1. (i) Hint: $0 = \delta \int_{t_1}^{t_2} (T - V)\, dt = \int_{t_1}^{t_2} (\delta T - \delta V)\, dt = \int_{t_1}^{t_2} (m\dot{\mathbf{r}} \cdot \delta\dot{\mathbf{r}} + \mathbf{F} \cdot \delta\mathbf{r})\, dt$.

   (ii) $0 = \delta \int_{t_1}^{t_2} (\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2)\, dt$, or $\ddot{x} + \omega^2 x = 0$.

2. Hint: Apply the variational principle

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0, \quad \text{where } T = \frac{1}{2}\rho \int_0^{\ell} \dot{y}^2\, dx \text{ and}$$
$$V = \frac{1}{2} \int_0^{\ell} EI y''^2\, dx.$$

Thus, $\rho\ddot{y} + EIy^{(iv)} = 0$.

3. $\eta_t + 6\eta\eta_x + \eta_{xxx} = 0$.

10. (a) We have $\nabla^4 u = 0$ (biharmonic equation).

(b) $u_{tt} - \alpha^2\nabla^2 u + \beta^2 u = 0$ (Klein–Gordon equation).

(c) $\phi_t + \alpha\phi_x + \beta\phi_{xxx} = 0$ $(\phi = u_x$, KdV equation).

(d) $u_{tt} + \alpha^2 u_{xxxx} = 0$ (elastic beam equation).

(e) $\frac{d}{dx}(pu') + (r + \lambda s)u = 0$ (Sturm–Liouville equation).

11. $(\frac{\hbar^2}{2m})\nabla^2\psi + (E - V)\psi = 0$.

14. $u_{tt} - c^2 u_{xx} = F(x,t)$, $c^2 = \frac{T^*}{\rho}$.

18. Hint: (a) Use the vector identity

$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\nabla\times\mathbf{u})$.

(b) Use the fact that $\mathbf{u}\cdot(\mathbf{u}\times\boldsymbol{\omega}) = 0$ to get the desired result.

(c) Use $\nabla\cdot\mathbf{u} = 0$ and add the following result:

$(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{p}{\rho} + \Omega)(\nabla\cdot\mathbf{u}) = 0$ to the equation in (b) to find $\frac{\partial}{\partial t}(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}) + \nabla\cdot[\mathbf{u}(\frac{1}{2}\mathbf{u}\cdot\mathbf{u} + \frac{p}{\rho} + \Omega)] = 0$, where $(\mathbf{u}\cdot\nabla)\phi + (\mathbf{u}\cdot\nabla)\phi = \nabla\cdot(\phi\mathbf{u})\cdot$

Adding a zero contribution $\frac{\partial\Omega}{\partial t}$ gives the energy equation.

The first term represents the rate of change of the total energy (kinetic and potential), and the second term describes the energy flow carried by the velocity combined with the contribution from the rate of working of the pressure forces.

19. Hint: Follow Example 2.4.2 to obtain

$$\begin{aligned}(1+q^2+r^2)p_x + (1+p^2+r^2)q_y + (1+p^2+q^2)r_z \\ - 2(pqp_y + qrq_z + rpr_x) = 0.\end{aligned}$$

22. (a) $u_y - u_x y' - \frac{uy''}{1+y'^2} = 0$, (b) $y'^2 = \frac{1-A^2(y_1-y)}{A^2(y_1-y)}$.

24. (a) Hint: For a conservative system, $T + V = C$.

Putting $V = C - T$ in (2.4.25) gives the principle of least action.

(b) The principle of least action asserts that time action is stationary for any conservative system.

27. (a) In one dimension, the Lagrangian is $L = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$ so that $\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}$ and $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$. The Euler–Lagrange equation is $\frac{\partial L}{\partial x} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = 0$, or equivalently, $m\ddot{x} = -\frac{\partial V}{\partial x} = F$. This is Newton's law of motion.

(b) The Lagrangian $L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2)$. The Euler–Lagrange equations are $\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = \frac{\partial L}{\partial x}$ and $\frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) = \frac{\partial L}{\partial y}$. In this case, these equations give $m\ddot{x} = -\frac{\partial V}{\partial x}$ and $m\ddot{y} = -\frac{\partial V}{\partial y}$. In this formulation, conserved quantities are not obvious.

(c) In polar coordinates (r, θ), $x = r\cos\theta$, $y = r\sin\theta$, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$, $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\sin\theta$. Thus, $x^2 + y^2 = r^2$ and $\dot{x}^2 + \dot{y}^2 = \dot{x}^2 + r^2\dot{\theta}^2$. Thus, the Lagrangian becomes $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r^2)$. The Euler–Lagrange equations are $\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) = \frac{\partial L}{\partial r}$, or, $\frac{dp_r}{dt} = m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$, $\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = \frac{\partial L}{\partial r}$, or, $\frac{dp_\theta}{dt} = \frac{d}{dt}(mr^2\dot{\theta}) = 0$ because $\frac{\partial L}{\partial r} = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2$, $\frac{\partial L}{\partial \dot{\theta}} = 0$, and $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$. Since $\frac{dp_\theta}{dt} = 0$, one gets $mr^2\dot{\theta} = \text{constant}$ which shows that the angular momentum is conserved and the radial equation of motion is $m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$. The first term in the radial equation on the right-hand side represents the centrifugal force, while the second term gives the dynamical radial force.

28. In the linearized limit, $f(r) = ae^{-br_n} - a \approx -abr_n$ so that the nonlinear Toda lattice equation $m\ddot{r} = 2f(r_n) - f(r_{n-1}) - f(r_{n+1})$ becomes $m\ddot{r}_n = ab(r_{n+1} + r_{n-1} - 2r_n)$. Substituting the traveling wave solution into the linearized Toda lattice equation gives $m\omega^2\cos\theta = 2ab\cos\theta(1 - \cos k) = 4ab\sin^2\frac{k}{2}$. Thus, the dispersion relation is $\omega^2 = \frac{4ab}{m}\sin^2\frac{k}{2}$, where k corresponds to the discrete wavenumber that is similar to the continuous wavenumber.

3.6 Exercises

2. (a) $xp - yq = x - y$, (d) $yp - xq = y^2 - x^2$.

3. (a) $u = f(y)$, (b) $u = f(bx - ay)$, (c) $u = f(ye^{-x})$,

   (d) $u = f(y - \tan^{-1} x)$, (e) $u = f(\frac{x^2-y^2}{x})$,

   (f) $\frac{x+u}{y} = C_1$, $(u+y)^2 - x^2 = C_2$, $f(\frac{x+u}{y}, (u+y)^2 - x^2) = 0$,

   $x^2 + y^2 = C_1, y(u - y) = C_2$,

   (g) We have $\frac{dx}{y^2} = -\frac{dy}{xy} = \frac{du}{xu-2xy} = \frac{d(u-y)}{x(u-y)}$, $u = y + y^{-1} f(x^2 + y^2)$,

   (h) $u + \log x = f(xy)$, (i) $f(x^2 + u^2,\ y^3 + u^3) = 0$.

5. (a) $u = \sin(x - \frac{3}{2}y)$, (b) $u = \exp(x^2 - y^2)$,

   (c) $u = xy + f(\frac{y}{x})$, $u = xy + 2 - (\frac{y}{x})^3$, (d) $u = \sin(y - \frac{1}{2}x^2)$.

   (e) $u = \begin{cases} \frac{1}{2}y^2 + \exp[-(x^2 - y^2)] & \text{for } x > y, \\ \frac{1}{2}x^2 + \exp[-(y^2 - x^2)] & \text{for } x < y. \end{cases}$

   (f) Hint: $y = \frac{1}{2}x^2 + C_1$, $u = C_1^2 x + C_2$,

   $u = x(y - \frac{1}{2}x^2)^2 + f(y - \frac{1}{2}x^2)$, $u = x(y - \frac{1}{2}x^2)^2 + \exp(y - \frac{1}{2}x^2)$.

   (g) $\frac{y}{x} = C_1$ and $\frac{u+1}{y} = C_2$, $C_2 = 1 + \frac{1}{C_1^2}$, $u = y + \frac{x^2}{y} - 1$, $y \neq 0$.

   (h) Hint: $x + y = C_1$, $\frac{dy}{-u} = \frac{du}{u^2+C_1^2}$, $u^2 + C_1^2 = C_2 \exp(-2y)$.

   From the Cauchy data, it follows that $1 + C_1^2 = C_2$, and hence,

$$u = \left[\left\{1 + (x+y)^2\right\}e^{-2y} - (x+y)^2\right]^{\frac{1}{2}}.$$

   (i) $\frac{dy}{dx} - \frac{y}{x} = 1$, $\frac{d}{dx}(\frac{y}{x}) = \frac{1}{x}$ which implies that $x = C_1 \exp(\frac{y}{x})$.

   $\frac{u+1}{x} = C_2$, $f(\frac{u+1}{x}, x\exp(-\frac{y}{x})) = 0$.

   Initial data imply $x = C_1$ and $\frac{x^2+1}{x} = C_2$. Hence $C_2 = C_1 + \frac{1}{C_1}$.

   $\frac{u+1}{x} = x\exp(-\frac{y}{x}) + \frac{1}{x}\exp(\frac{y}{x})$. Thus, $u = x^2 \exp(-\frac{y}{x}) + \exp(\frac{y}{x}) - 1$.

6. We find $u^2 - 2ut + 2x = 0$, and hence, $u = t \pm \sqrt{t^2 - 2x}$.

7. $u(x, y) = \exp(\frac{x}{x^2-y^2})$.

8. (a) $u = f(\frac{y}{x}, \frac{z}{x})$, (b) Hint: $u_1 = \frac{x-y}{xy} = C_1$, $\frac{d(x-y)}{x^2-y^2} = \frac{dz}{z(x+y)}$ gives $u = \frac{x-z}{z} = C_2$, $u = f(\frac{x-y}{xy}, \frac{x-y}{z})$. (c) $\phi = (x+y+z) = C_1$.
Hint: $\frac{(\frac{dx}{z})}{y-z} = \frac{(\frac{dy}{y})}{z-x} = \frac{(\frac{dz}{z})}{x-y} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0} = \frac{d\log(xyz)}{0}$, $\psi = xyz = C_2$, $u = f(x+y+z, xyz)$ is the general solution.
(d) Hint: $x\,dx + y\,dy = 0$, $x^2+y^2 = C_1$, $z\,dz = -(x^2+y^2)y\,dy = -C_1 y\,dy$, $z^2 + (x^2+y^2)y^2 = C_2$, $u = f(x^2+y^2, z^2+(x^2+y^2)y^2)$.
(e) $\frac{x^{-1}dx}{y^2-z^2} = \frac{y^{-1}dy}{z^2-x^2} = \frac{z^{-1}dz}{y^2-x^2} = \frac{d(\log xyz)}{0}$. Thus, $u = f(x^2+y^2+z^2, xyz)$ is a general solution.

9. (a) Hint: $y - \frac{x^2}{2} = C_1$, $u = xy - \frac{x^3}{3} + C_2$, $\phi(u - xy + \frac{x^3}{3}, y - \frac{x^2}{2}) = 0$.
$u = xy - \frac{x^3}{3} + f(y - \frac{x^2}{2})$, $u = xy - \frac{x^3}{3} + (y - \frac{x^2}{2})^2$.
(b) $u = xy - \frac{1}{3}x^3 + y - \frac{x^2}{2} + \frac{5}{6}$.

11. $\frac{x+u}{y} = C_1$, $u^2 - (x-y)^2 = C_2$, $u^2 - \frac{2u}{y} - (x-y)^2 - \frac{2}{y}(x-y) = 0$.
Thus, $u = \frac{2}{y} + (x-y)$, $y > 0$.

12. (a) $x = \frac{\tau^2}{2} + \tau s + s$, $y = \tau + 2s$, $u = \tau + s = \frac{(2x-2y+y^2)}{2(y-1)}$.
(b) $x = \frac{\tau^2}{2} + \tau s + s^2$, $y = \tau + 2s$, $u = \tau + s$, $(y-s)^2 = 2x - s^2$, a set of parabolas.
(c) $x = \frac{1}{2}(\tau+s)^2$, $y = u = \tau + s$.

13. Hint: The initial curve is a characteristic, and hence, no solution exists.

14. (a) $u = \exp(\frac{xy}{x+y})$, (b) $u = \sin[(\frac{x^2-y^2+1}{2})^{\frac{1}{2}}]$,
(c) $\frac{dx}{x} = \frac{dy}{-y} = \frac{du}{-1}$ gives $xy = c_1$ and $u - \ln y = c_2 = f(xy)$. Hence, $2x - \ln(3x) = f(3x^2)$. Or, $f(t) = 2\sqrt{\frac{t}{3}} - \ln(\sqrt{3t})$. Thus, $u = 2(\frac{xy}{3})^{\frac{1}{2}} + \frac{1}{2}\log(\frac{y}{3x})$,
(e) $\frac{dx}{d} = \frac{dy}{y} = \frac{du}{x^2+y^2}$ gives $\frac{x}{y} = c_1$, and $\frac{xdx+ydy}{x^2+y^2} = \frac{du}{x^2+y^2}$ yields $2u = x^2+y^2+c_2$. Or, $2u = x^2+y^2+f(\frac{x}{y})$. Hence, $f(x) = x^2 - 1$. Thus, $2u = x^2 + y^2 + \frac{x^2}{y^2} - 1$.
(f) $\frac{dx}{y^2} = \frac{dy}{xy} = \frac{du}{x}$ gives $x^2 - y^2 = c_1$ and $u - \ln y = c_2 = f(x^2-y^2)$. Hence, $f(t) = t + 1$. Thus, $u = \ln y + (x^2 - y^2) + 1$.

(g) $\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xy}$ gives $\frac{dy}{x} = c_1$ and $du = ydx = (c_1 x)dx$, $u = \frac{c_1 x^2}{2} + c_2 = \frac{xy}{2} + c_2$. $u = \frac{xy}{2} + f(\frac{y}{x})$. Hence, $f(1) = 0$. Thus, $u = \frac{xy}{2} + f(\frac{y}{x})$, where f is an arbitrary function such that $f(1) = 0$.

15. $\frac{xdx}{x(u^2-y^2)} = \frac{ydy}{xy^2} = \frac{udu}{-xu^2} = \frac{xdx+ydy+udu}{0}$, and hence, $x^2 + y^2 + u^2 = c_1$. And $\frac{dy}{y} = -\frac{du}{u}$ gives $uy = c_2$. Thus, $x^2 + y^2 + u^2 = f(uy)$, and hence, $3u^2 = f(u^2)$. Thus, $3uy = u^2 + x^2 + y^2$.

16. (a) $x(s,\tau) = \tau$, $y(s,\tau) = \frac{\tau^2}{2} + a\tau s + s$, $u(s,\tau) = \tau + as$. $\tau = x$, $s = (1 + ax)^{-1}(y - \frac{1}{2}x^2)a$, and hence, $u(x,y) = x + as = (1+ax)^{-1}\{x + a(y + \frac{1}{2}x^2)\}$, singular at $x = -\frac{1}{a}$.

(b) $y = \frac{u^2}{2} + f(u - x)$, $2y = u^2 + (u - x)^2$, $u(0, y) = \sqrt{y}$.

17. (a) Hint: $\frac{d(x+y+u)}{2(x+y+u)} = \frac{d(y-u)}{-(y-u)} = \frac{d(u-x)}{-(u-x)}$, $(x + y + u)(y - u)^2 = c_1$ and $(x + y + u)(u - x)^2 = c_2$.

(b) Hint: $\frac{dx}{x} = \frac{dy}{-y}$. Hence, $xy = a$. $\frac{dx}{xu(u^2+a)} = \frac{du}{x^4}$. So, $\frac{dx}{du} = \frac{u(u^2+a)}{x^3}$ giving $x^4 = u^4 + 2au^2 + b$ and, thus, $x^4 - u^4 - 2u^2xy = b$.

(c) $\frac{dx}{x+y} = \frac{dy}{x-y} = \frac{dy}{0}$ (exact equation). $u = f(x^2 - 2xy - y^2)$.

(d) $f(x^2 - y^2,\ u - \frac{1}{2}y^2(x^2 - y^2)) = 0$.

(e) $f(x^2 + y^2 + z^2,\ ax + by + cz) = 0$.

18. Hint: $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$, and hence, $\frac{x}{z} = c$, $\frac{y}{z} = d$. $x^2+y^2 = a^2$ and $z = \tan^{-1}(\frac{y}{x})$ give $(c^2 + d^2)z^2 = a^2$ and $z = b\tan^{-1}(\frac{d}{c})$. $c = (\frac{a}{z})\cos\theta$, $d = (\frac{a}{z})\sin\theta$, and $z = b\tan^{-1}(\tan\theta) = b\theta$. Thus, the curves are $x\ b\ \theta = az\cos\theta$ and $y\ b\ \theta = az\sin\theta$.

19. Hint: $\frac{(dx-2dy)}{9u} = \frac{du}{-3(x-2y)}$. $F\{x + y + u, (x - 2y)^2 + 3u^2\} = 0$. Thus, $(x - 2y)^2 + 3u^2 = (x + y + u)^2$.

20. $F(x^2 + y,\ yu) = 0$, $(x^2 + y)^4 = yu$.

21. Hint: $x - y + z = c_1$, $\frac{dz}{-(x+y+z)} = \frac{(dx+dy+dz)}{8z}$, and hence, $8z^2 + (x + y + z)^2 = c_2$. $F\{(x - y + z),\ 8z^2 + (x + y + z)^2\} = 0$.

$c_1^2 + c_2 = 2a^2$, or $(x - y + z)^2 + (x + y + z)^2 + 8z^2 = 2a^2$.

22. $F(x^2+y^2+z^2,\ y^2-2yz-z^2)=0$.

(a) $y^2-2yz-z^2=0$, two planes are $y=(1\pm\sqrt{2})z$.

(b) $x^2+2yz+2z^2=0$, a quadric cone with vertex at the origin.

(c) $x^2-2yz+2y^2=0$, a quadric cone with vertex at the origin.

23. Use the Hint of 17(c).

$\frac{dx}{dt}=x+y$, $\frac{dy}{dt}=x-y$, $\frac{d^2x}{dt^2}=2x$.

$(\frac{dx}{dt})^2=2x^2+c$. When $x=0=y$, $\frac{dx}{dt}=\sqrt{2}x$.

$\sqrt{2}\,u=\ln x+x^2-2xy+2y$.

24. (a) $a=f(x+\frac{3}{2}y)$.

(b) $x=at+c_1$, $y=bt$, $u=c_2e^{ct}$, $c_2=f(c_1)$, $u(x,y)=f(x-\frac{a}{b}y)\exp(\frac{cy}{b})$.

(c) $u=f(\frac{x}{1-y})(1-y)^c$.

(d) $x=\frac{1}{2}t^2+\alpha st+s$, $y=t$, and $u=y+\frac{1}{2}\alpha(\alpha y+1)^{-1}(2x-y^2)$.

26. (a) Hint: $(f')^2=1-(g')^2=\lambda^2$; $f'(x)=\lambda$ and $g'(y)=\sqrt{1-\lambda^2}$.

$f(x)=\lambda x+c_1$ and $g(y)=y\sqrt{1-\lambda^2}+c_2$. $u(x,y)=\lambda x+y\sqrt{1-\lambda^2}+c$.

(b) Hint: $(f')^2+(g')^2=f(x)+g(y)$ or $(f')^2-f(x)=g(y)-(g')^2=\lambda$.

Hence, $(f')^2=f(x)+\lambda$ and $(g')=\sqrt{g(y)-\lambda}$.

Or, $\frac{df}{\sqrt{f+\lambda}}=dx$ and $\frac{dg}{\sqrt{g-\lambda}}=dy$.

$f(x)+\lambda=(\frac{x+c_1}{2})^2$ and $g(y)-\lambda=(\frac{y+c_2}{2})^2$.

$u(x,y)=(\frac{x+c_1}{2})^2+(\frac{y+c_2}{2})^2$.

(c) Hint: $(f')^2+x^2=-g'(y)=\lambda^2$.

Or, $f'(x)=\sqrt{\lambda^2-x^2}$, and $g(y)=-\lambda^2y+c_2$.

Putting $x=\lambda\sin\theta$, we obtain

$f(x)=\frac{1}{2}\lambda^2\sin^{-1}(\frac{x}{\lambda})+\frac{x}{2}\sqrt{\lambda^2-x^2}+c_1$,

$u(x,y)=\frac{1}{2}\lambda^2\sin^{-1}(\frac{x}{\lambda})+\frac{x}{2}\sqrt{\lambda^2-x^2}-\lambda^2y+(c_1+c_2)$.

(d) Hint: $x^2(f')^2=\lambda^2$ and $1-y^2(g')^2=\lambda^2$.

Or, $f(x)=\lambda\ln x+c_1$ and $g(y)=\sqrt{1-\lambda^2}\ln y+c_2$.

27. (a) Hint: $v=\ln u$ gives $v_x=\frac{1}{u}\cdot u_x$, $v_y=\frac{1}{u}\cdot u_y$.

$x^2(\frac{u_x}{u})^2 + y^2(\frac{u_y}{u})^2 = 1$.

Or, $x^2v_x^2 + y^2v_y^2 = 1$ gives $x^2(f')^2 + y^2(g')^2 = 1$.

$x^2\{f'(x)\}^2 = 1 - y^2(g')^2 = \lambda^2$.

Or, $f(x) = \lambda \ln x + c_1$ and $g(y) = \sqrt{1-\lambda^2}(\ln y) + c_2$.

Thus $v(x,y) = \lambda \ln x + \sqrt{1-\lambda^2}(\ln y) + \ln c$, $(c_1 + c_2 = \ln c)$.

$u(x,y) = cx^\lambda y^{\sqrt{1-\lambda^2}}$.

(b) Hint: $v = u^2$ and $v(x,y) = f(x) + g(y)$ may not work.

Try $u = u(s)$, $s = \lambda xy$, so that $u_x = u'(y)\cdot(\lambda y)$ and $u_y = u'(s)\cdot(\lambda x)$.

Consequently, $2\lambda^2(\frac{1}{u}\frac{du}{ds})^2 = 1$.

Or, $\frac{1}{u}\frac{du}{ds} = \frac{1}{\sqrt{2}}\frac{1}{\lambda}$. Hence, $u(s) = c_1 \exp(\frac{s}{\lambda\sqrt{2}})$.

Thus, $u(x,y) = c_1 \exp(\frac{xy}{\sqrt{2}})$.

28. Hint: $v_x = \frac{1}{2}\frac{u_x}{\sqrt{u}}$, $v_y = \frac{1}{2}\frac{u_y}{\sqrt{u}}$. This gives $x^4(f')^2 + y^2(g')^2 = 1$.

Or, $x^4(f')^2 = 1 - y^2(g')^2 = \lambda^2$.

Or, $x^4(f')^2 = \lambda^2$ and $y^2(g')^2 = 1 - \lambda^2$.

Hence, $f(x) = -\frac{\lambda}{x} + c_1$ and $g(y) = \sqrt{1-\lambda^2}\ln y + c_2$.

Thus, $u(x,y) = (-\frac{\lambda}{x} + \sqrt{1-\lambda^2}\ln y + c)^2$.

29. Hint: $v_x = \frac{u_x}{u}$, $v_y = \frac{u_y}{u}$. $\frac{v_x^2}{x^2} + \frac{v_y^2}{y^2} = 1$, and $v = f(x) + g(y)$.

Or, $\frac{(f')^2}{x^2} = 1 - \frac{1}{y^2}(g')^2 = \lambda^2$.

$f'(x) = \lambda x$, and $g'(y) = \sqrt{1-\lambda^2}y$.

Or, $f(x) = \frac{\lambda}{2}x^2 + c_1$, and $g(y) = \frac{1}{2}y^2\sqrt{1-\lambda^2} + c_2$.

$v(x,y) = \frac{\lambda}{2}x^2 + \frac{y^2}{2}\sqrt{1-\lambda^2} + c = \ln u$.

$u(x,y) = c\exp[\frac{\lambda}{2}x^2 + \frac{y^2}{2}\sqrt{1-\lambda^2}]$, $c_1 + c_2 = \ln c$.

$e^{x^2} = u(x,0) = ce^{\frac{\lambda}{2}x^2}$, which gives $c = 1$ and $\lambda = 2$.

31. (b) $v(x,t) = x + at$, $u(x,t) = \frac{6x+3at^2+5at^3}{6(1+2t)}$.

32. (a) We have $v_t - av_x = 0$, $v(x,0) = e^x$. So, $\frac{dt}{1} = \frac{dx}{-a} = \frac{dv}{0}$ gives $dv = 0$, or, $v = \text{const.} = c_1$. Also, $dx + a\,dt = 0$ yields $x + at = \text{const.} = c_2$. Thus,

$v = f(x+at)$, and $e^x = v(x,0) = f(x)$. Or, $v = e^{x+at}$. The first equation becomes $u_t + uu_x = e^{-x}v = e^{at}$. This gives $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{e^{at}}$.

Or, $u - \frac{e^{at}}{a} = c_1$. Thus, $dx = udt = (\frac{e^{at}}{a} + c_1)dt$.

Or, $x = \frac{e^{at}}{a^2} + c_1 t + c_2 = \frac{e^{at}}{a^2} + (u - \frac{e^{at}}{a})t + c_2 = \frac{e^{at}}{a^2} + ut - \frac{te^{at}}{a} + c_2$.

Or, $x - ut + \frac{t}{a}e^{at} - \frac{e^{at}}{a^2} = c_2$. Thus, $u(x,t) - \frac{e^{at}}{a} = f(x - ut + \frac{t}{a}e^{at} - \frac{e^{at}}{a^2})$.

Hence, $u - \frac{1}{a}e^{at} = c_1$ and $u(x,t) - \frac{1}{a}e^{at} = f(x - ut + \frac{t}{a}e^{at} - \frac{1}{a^2}e^{at})$.

Using the initial condition gives $x - \frac{1}{a} = f(x - \frac{1}{a^2})$, or $f(x) = (x + \frac{1}{a^2} - \frac{1}{a})$.

Hence, $u(x,t) = \frac{1}{a}e^{at} + (x - ut + \frac{t}{a}e^{at} - \frac{1}{a^2}e^{at} + \frac{1}{a^2} - \frac{1}{a})$.

Thus, $u(x,t) = (1+t)^{-1}[x + (\frac{1}{a} + \frac{t}{a} - \frac{1}{a^2})e^{at} + (\frac{1}{a^2} - \frac{1}{a})]$.

33. (a) We have $\frac{dx}{\sqrt{x}} = \frac{dy}{u} = \frac{du}{-u^2}$ leads to $2\sqrt{x} - \frac{1}{u} = C_1$ and $y = -\ln(C_2 u)$.

At $(x_0, 0)$, $u = 1$ so that $C_1 = 2\sqrt{x_0} - 1$ and $C_2 = 1$. Thus, $u^{-1} = 2(\sqrt{x} - \sqrt{x_0}) + 1$ and $u = e^{-y}$, $y = \ln u^{-1} = \ln[2(\sqrt{x} - \sqrt{x_0}) + 1]$.

(b) We have $\frac{dx}{ux^2} = \frac{dy}{e^{-y}} = \frac{du}{-u^2}$ so that $\frac{1}{x} = \ln u + C_1$ and $e^y = \frac{1}{u} + C_2$.

At $(x_0, 0)$, $u = 1$, that is, $C_2 = 0$ and $u = e^{-y}$.

Hence, $C_1 = \frac{1}{x_0}$. Thus, $\ln u = (\frac{1}{x} - \frac{1}{x_0})$ and $y = \frac{1}{x_0} - \frac{1}{x}$.

34. For any function $u(\xi, \tau)$, we have $\frac{\partial^n u}{\partial x^n} = \frac{\partial^n u}{\partial \xi^n}$, $\frac{\partial u}{\partial t} = \mu\frac{\partial u}{\partial \tau} - c\frac{\partial u}{\partial \xi}$, $u_{tt} = \mu^2 u_{\tau\tau} - 2\mu c u_{\xi,\tau} + c^2 u_{\xi\xi}$, etc.

Thus, the equation assumes the form $u_\tau + (\alpha u^n)u_\xi = 0$. We have $\frac{d\tau}{1} = \frac{d\xi}{\alpha u^n} = \frac{du}{0}$ or $\frac{du}{d\tau} = 0$ and $\frac{d\xi}{d\tau} = \alpha u^n$. Thus, $u = \text{const.} = C_1$ and $\xi = \alpha u^n \tau + C_2$. Then the general solution is $f(u, \xi - \alpha u^n \tau) = 0$, or $u = \phi(\xi - \alpha u^n \tau)$ which is a Riemann simple wave. Hence, $u(\xi, 0) = \phi(\xi) = u_0 \sin k\xi$. Thus, $u(\xi, \tau) = u_0 \sin[k\xi - (\frac{u}{u_0})^n(\alpha k u_0^n \tau)]$.

35. (a) We have $\frac{dx}{x+y} = \frac{dy}{x-y} = \frac{du}{0}$. So, $u = \text{const.} = c_1$. Hence, $\frac{dy}{dx} = \frac{x-y}{x+y}$. This is an exact equation and hence, $f(x,y) = c_2$, or, $x^2 - 2xy - y^2 = c_2$.

Thus, the general solution is $u = f(x^2 - 2xy - y^2)$, f is an arbitrary function.

(b) We have $\frac{dx}{1} = \frac{dy}{-(ax+by)} = \frac{du}{0}$. Thus, $u = \text{const.} = c_1$. Hence, $\frac{dy}{dx} + by = -ax$, giving $y(b^2y - abx + a) = c_2$. Hence, $u = f(y(b^2y - abx + a))$ is the general solution.

(c) We have $\frac{dx}{x^{-1}} = \frac{dy}{y^{-1}} = \frac{du}{x^2-y^2}$. Hence, $x^2 - y^2 = c_1$. $\frac{du}{dx} = x(x^2 - y^2) = c_1 x$ giving $u = \frac{1}{2}c_1x^2 + c_2$. Thus, $u - \frac{1}{2}x^2(x^2 - y^2) = c_2$ and $u - \frac{1}{2}x^2(x^2 - y^2) = f(x^2 - y^2)$.

(e) We find that $\frac{dx}{y} = \frac{dy}{-x} = \frac{du}{0}$. This gives $x^2 - y^2 = C_1$ and $u = C_2$. So, $u = f(x^2 + y^2)$. Hence, $y = f(a^2 + y^2) = f(t)$, $a^2 + y^2 = t$. Therefore, $f(t) = \sqrt{t - a^2}$. Thus, $u = f(x^2 + y^2) = \sqrt{x^2 + y^2 - a^2}$.

(f) We find that $\frac{dx}{x} = \frac{dy}{x+y} = udu$, or, $\frac{du}{dx} = \frac{1}{xu}$, or $\frac{u^2}{2} - \log x = c_1$. Thus, $\frac{dy}{dx} = \frac{x+y}{x}$, and hence, $\frac{y}{x} = \log x + c_2$. So, $\frac{y}{x} - \log x = f(\frac{u^2}{2} - \log x)$. Thus, $x - \log x = f(-\log x)$. Putting $t = -\log x$, $f(t) = t - e^{-t}$. Thus, $\frac{y}{x} = \frac{u^2}{2} - x\exp(-\frac{u^2}{2})$.

36. (a) We have, $\frac{dt}{1} = \frac{dx}{-x} = \frac{du}{u}$ gives $t + \ln x = C_1$ and $xu = C_2$. Or, $g(xu, t + \ln x) = 0$, or, $u = \frac{1}{x}h(t + \ln x)$ is the general solution, where h is an arbitrary function. Hence, $u(x, 0) = f(x) = \frac{1}{x}h(\ln x)$, or, $h(\ln x) = xf(x)$, that is, $h(x) = e^x f(e^x)$. Thus, $u(x, t) = e^t f(xe^t)$.

4.6 Exercises

1. (a) $16u = (x + 4y)^2$, (b) $u = \alpha(x + y)$.
2. (a) $\sqrt{a}\,\log u = x + ay + b$, (b) $2\sqrt{u(1 + a^2)} = x + ay + b$,
   (d) $u = ay + \frac{1}{4}(x + b)^2$, (g) $u = a\log x + (4 - a^2)^{\frac{1}{2}}\log y + b$,
   (h) $u = a(1 + x^2)^{\frac{1}{2}} + \frac{1}{2}(ay)^2 + b$.
3. (a) $\log u = x + y - 1$, (c) $u = \frac{xy}{2(y-2)}$ which is singular at $y = 2$,
   (e) $u(x, y) = x + y$, (f) $4u = (x + y + 2)^2$.
   (i) $u = \exp(-act)f(x - ct)$. Use equation (4.2.19)
   $x(\tau, 0) = x_0(\tau) = \tau$, $t(\tau, 0) = 0$, $u(\tau, 0) = f(\tau)$.

4. Hint: At $s = 0$, $p(t, x) = q(t, s) = 1$, then, $p(t, s) = q(t, s) = 1$.

   $x(t, s) = (t - 2)e^s + 2$, $y(t, s) = 1 + e^s$,

   $u(t, s) = (t - 1)e^s + 2$, $u(x, y) = x + y - 1$.

8. Hint: $x(0, s) = 0$, $y(0, s) = s$, $u(0, s) = -s$.

   $x(t, s) = -\sinh t$, $y(t, s) = s \cosh t$,

   $u(t, s) = -\frac{s}{2}(1 + \cosh t),\ u(x, y) = -y(x^2 + 1)^{\frac{1}{2}}$.

10. Hint: Characteristics are $x - 2ct = s$ where s is a parameter. The points on the characteristic lines travel with speed $2c$, whereas points on the wave profile $u = c(x - ct)$ move with speed c. The wave profile u is not constant on the characteristic lines.

11. (a) $u = \frac{1}{2}xy + f(\frac{y}{x})$ where f is an arbitrary function such that $f(1) = 0$.

    (b) Characteristics are $xy = \text{const.}$ (a family of hyperbolas).

    $u = a \log y + f(xy)$, $u = \frac{a}{2} \log(\frac{y}{3x}) + 2(\frac{xy}{3})^{\frac{1}{2}}$.

    (c) At $t = 0$, $p = 0$, $q = \frac{1}{s}$; $sp = \tanh(\frac{t}{s})$, $sq = \text{sech}(\frac{t}{s})$.

    $x(s, t) = -2s^2 \tan(\frac{t}{s}) \,\text{sech}(\frac{t}{s}), \quad y(s, t) = 2s^2 \,\text{sech}^2(\frac{t}{s}) - s^2$.

    $u(s, t) = 2s \,\text{sech}(\frac{t}{s})$, $(u^2 - 2y)^2 = 4(x^2 + y^2)$.

    (d) $u(x, y) = x - y$, (e) $u(x, t) = x^2 \tan t$.

12. Hint: Write the characteristic strip equations $\frac{dx}{dt} = F_p, \ldots$. Since $\frac{1}{p}\frac{dp}{dt} = \frac{1}{q}\frac{dq}{dt}$, p and q are proportional to each other. The initial data $p_0(s)$ and $q_0(s)$ are the solutions of $p_0(s) = 0$ and $(p_0^2 + q_0^2)^{\frac{1}{2}} = \tan\theta$ so that $p_0(s) = 0$ and $q_0(s) = \tan\theta$. Since $p_0(s) = 0$, the constant of proportionality must be zero, and so $p(t, s) = 0$. We then find $x(t, s) = s$ and $z(t, s) = \cos(\theta - t)$. Then $\frac{dq}{dt} = -[q^2 + q\cos(\theta - t)]$ gives $q(t, s) = \tan(\theta - t)$ $y(t, s) = -\sin(\theta - t) + 2\sin\theta$, or equivalently, $(y - 2\sin\theta)^2 + z^2 = 1$, which represents a cylinder.

14. Hint: $(\frac{dz}{d\xi}) = \pm\frac{1}{z}(\frac{1-z^2}{1+a^2})^{\frac{1}{2}}$,

    $d(1 - z^2)^{\frac{1}{2}} = \pm d(\frac{x+ay}{\sqrt{a^2+1}}) = \pm d(x\cos\theta + y\sin\theta)$.

15. (a) Hint: $\frac{dp}{dq} = \frac{p}{q}$ gives $p = c_1 q$.

$\frac{dx}{dt} + (u-a)\frac{dp}{dt} + p\frac{du}{dt} = 0$, or $dx + (u-a)\,dp + p\,du = 0$,

or $d(x + (u-a)p) = 0$ gives $x + p(u-a) = \text{const}$.

Similarly, $y + q(u-a) = \text{const}$.

These equations and $F = 0$ give the integral surfaces

$x^2 + y^2 + (u-a)^2 - 1 = 0$, $u(x,t) = \frac{(x-ct)}{1+t(x-ct)}$.

(b) Hint: $\frac{dx}{dt} = 2pu^2$, $\frac{dy}{dt} = 2qu^2$, $\frac{du}{dt} = 2(1-u^2)$,

$\frac{dp}{dt} = -\frac{2p}{u}$, and $\frac{dq}{dt} = -\frac{2q}{u}$.

$(\dot{p}u + \dot{u}p) = \frac{d}{dt}(up) = -2pu^2$, $\frac{d}{dt}(uq) = -2qu^2$.

Consequently, $\frac{dx}{dt} = 2pu^2 = -\frac{d}{dt}(up)$ and $\frac{dy}{dt} = -\frac{d}{dt}(uq)$, and hence,

$(x-a)^2 + (y-b)^2 = u^2(p^2+q^2) = 1 - u^2$, where a and b are constants.

(c) $(x-x_0)^2 + (y-y_0)^2 = (u-u_0)^2$, where x_0, y_0, and u_0 are constants. This is a three-parameter family of solutions. Use equation (4.3.1).

16. (a) $u(x,y) = \begin{cases} x^2 & \text{if } y = 0, \\ (2y^2)^{-1}(1 + 2xy - \sqrt{1+4xy}) & \text{if } y \neq 0. \end{cases}$

(b) $u(x,y) = \begin{cases} x & \text{if } y = 0, \\ (2y)^{-1}(\sqrt{1+4xy} - 1) & \text{if } y \neq 0. \end{cases}$

5.5 Exercises

1. (a) $u(x,t) = \sin\xi$, $\xi = x - t\sin\xi$.

   (b) $u(x,t) = \frac{x}{1-t}$, $u \to \infty$ as $t \to 1$.

2. Hint: $(u)_t + (\frac{1}{2}u^2)_x = 0$. Multiplying it by nu^{n-1} gives the first result.

4. $\xi = x - t\exp(-\frac{1}{\xi})$ for $x > 0$; $u(x,t) = \exp(-\frac{1}{\xi})$.

6. The characteristic diagram in the (x,t)-plane shows that the initial signal $f(x)$ is focused along the characteristics to a region near $x = 0$ as t increases.

7. $u = t + A$, $x = ut - \frac{t^2}{2} + B$, where A and B are constants.

8. Hint: Shock $t = 2(x-1)$ from $(1,0)$ to $(2,2)$ and shock $x = \sqrt{2}t$ from $(2,2)$. $u = 0$ for $x < 0$, and, when $x \geq 0$, $u = \frac{x}{t}$ to the left of $t = x$ and $x = \sqrt{2}t$, $u = 0$ to the right of shocks, and $u = 1$ in a triangular region bounded by $t = 0$, $t = x$, and $t = 2(x-1)$.

9. $u = -1$ to the left of the line $t + x = 0$, and $u = 1$ to the right of the line $x - t = 1$. Between these lines, $u = (2x-1)/(1+2t)$.

10. Characteristics from $0 \leq x \leq t$, $t = 0$ intersect at $(\frac{1}{2}, \frac{1}{2})$. Characteristics from $x \leq 0$, $x \geq 1$ also intersect in the region $t \geq \frac{1}{2}$, and hence, no single-valued solution. Shock parallel to the t-axis from $(\frac{1}{2}, \frac{1}{2})$.

    We find $u(x,t) = (2x-1)/(1-2t)$ in the triangular region bounded by $x - t = 0$, $x + t = 1$, $t = 0$.

$$u(x,t) = \begin{cases} +1 & \text{to the left of shock,} \\ -1 & \text{to the right of shock.} \end{cases}$$

11. When $a < 0$, shock along the line $at = 2x$, $-a$ to the left of shock, $2a$ to the right. When $a > 0$, $-a$ to the left of the line $x = -at$ and $2a$ to the right of the line $x = 2at$. Between these lines, $u = (\frac{x}{t})$.

15. Hint: Here $c(u) = u^2$, $f(x) = \sqrt{x}$, $\frac{dt}{1} = \frac{dx}{c(u)} = \frac{du}{0}$.

    Thus, $du = 0$ gives $u(x,t) = \text{const.}$ on the characteristics.

    $u(x,t) = f(\xi) = \sqrt{\xi}$.

    $\xi = x - tF(\xi) = x - tc(f(\xi)) = x - tc(\sqrt{\xi}) = x - t\xi$.

    Thus, $\xi = \frac{x}{1+t}$. Hence, $u(x,t) = (\frac{x}{1+t})^{\frac{1}{2}}$.

16. $c(u) = u^3$, $f(x) = x^{\frac{1}{3}}$, we have $\frac{dt}{1} = \frac{dx}{u^3} = \frac{du}{0}$.

    $u(x,t) = \text{const.}$ on the characteristics.

    Thus, $u(x,t) = f(\xi) = \xi^{\frac{1}{3}}$.

    $\xi = x - tF(\xi) = x - tc(f(\xi)) = x - tc(\xi^{\frac{1}{3}}) = x - t\xi$.

    Hence, $\xi = \frac{x}{1+t}$ and $u(x,t) = (\frac{x}{1+t})^{\frac{1}{3}}$.

17. $c(u) = u$, $f(x) = x$; $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{2t}$ gives $\frac{du}{dt} = u$ and $\frac{du}{dt} = 2t$. Hence, $u = t^2 + A$ on the characteristic curve $x = x(t)$ joining (x, t) and $(\xi, 0)$.

    Also, $\frac{dx}{dt} = [u(x,t) - t^2] + t^2 = [u(\xi, 0) - 0] + t^2 = \xi + t^2$ with $\xi = x(0)$.

    The equation of the characteristic curve is $x(t) = \xi t + \frac{t^3}{3} + B$ with $x(0) = \xi$.

    $x(t) = \xi(1 + t) + \frac{t^3}{3}$. $u(x(0), 0) = u(\xi, 0) = f(\xi) = \xi$.

    Thus, $A = \xi$, $u(x,t) = t^2 + \xi = t^2 + \frac{3x - t^3}{3(1+t)}$.

18. Hint: $\frac{dt}{1} = \frac{dx}{3t} = \frac{du}{u}$, $u(x, 0) = f(x) = \cos x$.

    Or, $\frac{dx}{dt} = 3t$ and $\frac{du}{u} = dt$. Or, $x = \frac{3}{2}t^2 + A$ and $u(x,t) = Be^t$.

    $B = u(\xi, 0) = \cos\xi$, with $x(0) = \xi$. Then $A = \xi$ and $x - \frac{3}{2}t^2 = \xi$.

    Consequently, $u(x,t) = e^t \cos\xi = e^t \cos(x - \frac{3}{2}t^2)$.

19. Hint: $\frac{dt}{2} = \frac{dx}{1} = \frac{du}{0}$, and hence $u(x,t) = \text{const.}$ on the characteristic curve $\frac{dx}{dt} = \frac{1}{2}$, or equivalently, $x = \frac{t}{2} + A$.

    If $x(0) = \xi$, then $x = \xi + \frac{t}{2}$.

    $$u = (x,t) = \text{const.} = u(\xi, 0) = \begin{cases} \sin\xi & \text{if } 0 \le \xi \le \pi, \\ 0 & \text{otherwise.} \end{cases}$$

    Or equivalently, $$u(x,t) = \begin{cases} \sin(x - \frac{t}{2}) & \text{if } 0 \le x - \frac{1}{2}t \le \pi, \\ 0 & \text{otherwise.} \end{cases}$$

20. Hint: $\frac{dt}{1} = \frac{dx}{1} = \frac{du}{x}$ gives $x = t + A$, and hence, $x = t + \xi$, if $x(0) = \xi$.

    $\frac{du}{dt} = x(t) = t + \xi$, and hence, $u(x,t) = \frac{1}{2}t^2 + \xi t + B$.

    Or, $B = u(x,t) - \frac{1}{2}t^2 - \xi t = u(\xi, 0) = \sin\xi$.

    If $x > t$, $u(x,t) = \frac{1}{2}t^2 + t(x - t) + \sin(x - t)$.

    If (x,t) lies above the line $x = t$, characteristic passing through this point never reaches the x-axis, but it reaches the t-axis so that the boundary condition $u(0,t) = t$ can be used (the initial condition cannot be used in this case, $x < t$). For $x < t$, $\frac{du}{dt} = x = t - \eta$ or $\eta = t - x$.

    Integrating gives $u(x,t) = \frac{1}{2}t^2 - \eta t + B$. Using the boundary condition, we find $B = u(x,t) - \frac{1}{2}t^2 + \eta t = u(0, \eta) - \frac{1}{2}\eta^2 + \eta^2 = \eta + \frac{1}{2}\eta^2$.

With $\eta = t - x$, the solution is given by

$u(x,t) = \frac{1}{2}t^2 - (t-x)t + (t-x) + \frac{1}{2}(t-x)^2 = \frac{1}{2}x^2 + t - x.$

On $x = t$, $u(x,t) = \frac{t^2}{2}$.

21. We have $\frac{dx}{x^2} = \frac{dt}{u} = \frac{du}{1} = dz$ gives $\frac{dx}{dz} = x^2$, $\frac{du}{dz} = 1$, $\frac{dt}{dz} = u$, whence $x = (a-z)^{-1}$, $u = z + b$, $t = \frac{1}{2}z^2 + bz + c$, when a, b, and c are constants. The characteristic equation originating from the initial curve at $x = s$, $t = 1 - s (z = 0)$ is $x = (\frac{s}{1-sz})$, $u = z (b = 0)$, $t = \frac{1}{2}z^2 + 1 - s$.

For each x, t, the solution u depends on the parameters s and z. Eliminating s and z gives the answer as follows: $s = \frac{1}{2}z^2 + 1 - t = (\frac{1}{2}u^2 + 1 - t)$ and $s = \frac{x}{1+zx} = (\frac{x}{1+ux})$. Thus, $(\frac{1}{2}u^2 + 1 - t)(1 + ux) = x$.

22. The Darcy law for water is $V_d = -KH_x$, K is the hydraulic conductivity and H is the hydraulic head, $H = \frac{p}{g\rho} + z + \frac{v^2}{2g} \approx \frac{p}{g\rho} + z$ for small $(v^2/2g)$, p is the pressure, g is the acceleration to gravity, and ρ is the density. The Darcy law describes the assumption that water flows slowly in the soil pores for the head loss to be proportional to the velocity.

The characteristic equations associated with Buckley–Leverett equation are $\frac{dt}{1} = \frac{dx}{c} = \frac{ds}{0}$. Or equivalently, $\frac{ds}{dt} = 0$, and hence, $s = \text{const.}$ and it is a conserved quantity. Hence, $\frac{dx}{dt} = c$ is the speed at which the interface between water and hydrocarbon moves. Thus, $c = \frac{\partial F}{\partial s} = 0$ when $s = 0$ and $s = 1$, and c attains its maximum when $0 < s < 1$. The case $c = 0$ for $s = 0$ reflects a well known property of hydrocarbons as non-wetting liquids, the mobility of which is very low for small water saturation.

6.12 Exercises

1. Hint: Use $h_t + (uh)_x = 0$.

2. $$\rho(x,t) = f(\xi) = \begin{cases} 200 & \text{if } \xi < -\varepsilon, \\ 100(1 - \frac{\xi}{\varepsilon}) & \text{if } -\varepsilon < \xi < \varepsilon, \\ 0 & \text{if } \xi > \varepsilon, \end{cases}$$

$$x = \xi + [60 - \tfrac{3}{5} f(\xi)]t = \begin{cases} \xi - 60t & \text{if } \xi < -\varepsilon, \\ (60t - \varepsilon)(\frac{\xi}{\varepsilon}) & \text{if } -\varepsilon < \xi < \varepsilon, \\ \xi + 60t & \text{if } \xi > \varepsilon, \end{cases}$$

$$\rho(x,t) = \begin{cases} 200 & \text{if } x < -(60t + \varepsilon), \\ 100(1 - \frac{x}{60t+\varepsilon}) & \text{if } -(60t+\varepsilon) < x < 60t + \varepsilon. \end{cases}$$

The graph of $\rho(x,0)$ is linearly increasing in x from $-\varepsilon$ to $+\varepsilon$.

3. (a) $$\rho = f(\xi) = \begin{cases} 25(3 - |\xi|) & \text{if } |\xi| \le 1, \\ 50 & \text{if } |\xi| \ge 1, \end{cases} \quad \text{and}$$

$$x = \begin{cases} \xi = 15(1 + |\xi|)t & \text{if } |\xi| \le 1, \\ \xi + 30t & \text{if } |\xi| \ge 1. \end{cases}$$

(b) For $t < \frac{1}{15}$, we eliminate ξ from the above solution to obtain

$$\rho(x,t) = \begin{cases} 25(3 + \frac{x-15t}{1-15t}) & \text{if } -1 + 30t \le x \le 15t, \\ 25(3 - \frac{x-15t}{x+15t}) & \text{if } 15t \le x \le 1 + 30t, \\ 50 & \text{if } |x - 30t| \ge 1. \end{cases}$$

The solution $\rho(x,t)$ for $t = \frac{1}{15}(1-\varepsilon)$ can be found from the result given in Exercise 3(a). Use results in 3(a) to find ρ and x for $t = \frac{1}{15}$.

$$\rho = \begin{cases} 25(3 - |\xi|) & \text{if } |\xi| \le 1, \\ 50 & \text{if } |\xi > 1|, \end{cases} \quad \text{and}$$

$$x = \begin{cases} 2+\xi & \text{if } \xi > 1 \text{or} \xi < -1, \\ 1+2\xi & \text{if } 0 \le \xi \le 1, \\ 1 & \text{if } -1 \le \xi < 0. \end{cases}$$

Use these results to draw a graph of $\rho(x, \frac{1}{15})$.

4. Hint: Use $u = x - c_0 t$ and $v = t$ as independent variables.

12. Hint: The equation in matrix form is

$$\begin{pmatrix} u & c^2\rho & 0 \\ \frac{1}{\rho} & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} p_x \\ u_x \\ S_x \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_t \\ u_t \\ S_t \end{pmatrix} = 0.$$

The eigenvalues are the roots of the equation

$$\begin{vmatrix} u-\lambda & c^2\rho & 0 \\ \frac{1}{\rho} & u-\lambda & 0 \\ 0 & 0 & u-\lambda \end{vmatrix} = 0 \quad \text{which gives } \lambda = \frac{dx}{dt} = u,\ u \pm c.$$

13. Hint: $du + \lambda_{2,1}\, dv = 0$ along the characteristics C_1 and C_2, respectively.

$a(du)^2 + 2b\, du\, dv + c(dv)^2 = 0$, which corresponds to the characteristics

$dx = \lambda\, dt$, $a(dx)^2 - 2b\, dx\, dt + c(dt)^2 = 0$.

14. Hint: Along the characteristics C_+: $\eta = x - ct = \text{const.}$ and $\frac{du_+}{dt} + \frac{1}{2} a u_+ = 0$ with the solution $u_+(t) = u_+(0)\exp(-\frac{1}{2}at)$ with $\eta = x - ct$ and $u_-(t) = u_-(0)\exp(-\frac{1}{2}at)$ with $\xi = x + ct$.

15. Hint: $\rho = \rho(x(t), t)$, $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt} \cdot \frac{\partial \rho}{\partial x}$, and then use equations (6.3.1) and (6.3.2).

16. Hint: $\lambda_{1,2} = u \pm c$, $L_{1,2} = (1, \pm 2)$.

$(\frac{\partial}{\partial t} + \lambda_1 \frac{\partial}{\partial x})(u + 2c) = H_x$ along C_+: $\frac{dx}{dt} = u + c$,

$(\frac{\partial}{\partial t} + \lambda_2 \frac{\partial}{\partial x})(u - 2c) = H_x$ along C_-: $\frac{dx}{dt} = u - c$.

18. Hint: See Jeffrey and Taniuti (1964, pp. 72, 73).
19. Hint: Equate x_{sr} and x_{rs} to derive the Euler–Poisson–Darboux equation.
22. Hint: The first equation can be written as $\frac{\partial\rho}{\partial t} + \text{div}(\rho u, \rho v, \rho w) = 0$.

    The first and second equations can be combined to find the result

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho u \\ \rho v \\ \rho w \end{pmatrix} + \text{div}\begin{pmatrix} p+\rho u^2 & \rho uv & \rho uw \\ \rho uv & p+\rho v^2 & \rho vw \\ \rho uw & \rho vw & p+\rho w^2 \end{pmatrix} = 0.$$

    The above two results lead to the desired conservation form.

25. For (i), $A = \begin{pmatrix} 0 & \frac{c}{a} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{u}{a} \\ 0 \\ -v-w \end{pmatrix}$.

    For (ii), $A = \begin{pmatrix} & -\frac{c}{a} & \frac{c}{a} & 0 \\ -1 & -\frac{c}{a} & \frac{c}{a} & 0 \\ & 0 & 0 & \frac{1}{2} \end{pmatrix}$ and $B = \begin{pmatrix} \frac{u}{a} \\ \frac{u}{a} \\ -\frac{v}{2}-\frac{w}{2} \end{pmatrix}$,

    and thus, for both cases (i) and (ii),

$$U = \begin{pmatrix} v \\ w \\ u \end{pmatrix}.$$

26. Characteristics drawn from the negative x-axis are vertical lines $x = \text{const.}$, and along these characteristics, $\rho = \rho_0$ and $\rho(x,t) = \rho_0$ for $x < 0$, $t > 0$. Characteristics drawn from points $x > a$ satisfy $\dot{x}(t) = c(\rho(x,0),0) = c(0) = \rho_m$. If $x > a + \rho_m t$, $t > 0$, $\rho(x,t) = 0$.

    If $0 < x < a$, $\rho(x,t) = f(x - tc(\rho)) = \rho_0[a - \{x - t\rho_m(1 - \frac{\rho}{\rho_0})\exp(-\frac{\rho}{\rho_0})\}]$.

    Hence, $x = (a - \frac{\rho}{\rho_0}) + \rho_m(a - \frac{\rho}{\rho_0})t\exp(-\frac{\rho}{\rho_0})$.

27. (a) This system of two equations is known to be hyperbolic. The first equation is already in the one-dimensional divergence form because it becomes $\rho_t + (\rho u)_x = 0$. The one-dimensional form of the second equation is not in divergence form, but it is multiplied by ρ and added to u times the first equation to obtain the conservation form $(\rho u)_t + (\rho u^2 + p)_x = 0$.

(c) We have $\rho_t + (\rho u)_x = 0$, or equivalently, $\frac{\partial \rho}{\partial u}\frac{\partial u}{\partial t} + (\rho_x u + \rho u_x) = 0$.
Thus, $\frac{d\rho}{du} \cdot \frac{\partial u}{\partial t} + (\rho + u\frac{d\rho}{du})(\frac{\partial u}{\partial x}) = 0$. It follows that $\frac{\partial u}{\partial t} + (uu_x) + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$, or equivalently, $\frac{\partial u}{\partial t} + (uu_x + \frac{1}{\rho}\frac{\partial p}{\partial x}) = 0$.
Thus, $\frac{\partial u}{\partial t} + (u + \frac{c^2(\rho)}{\rho}\frac{d\rho}{du})\frac{\partial u}{\partial x} = 0$.

28. We have $xv_t + w_x = 0$ and close the system by using $v_x = w_t$. Thus, the Tricomi equation becomes $U_t + AU_x = 0$, where $U = \begin{bmatrix} v \\ w \end{bmatrix}$ and $A = \begin{bmatrix} 0 & \frac{1}{x} \\ -1 & 0 \end{bmatrix}$. This has eigenvalues $\lambda^2 = -\frac{1}{x}$ for $x \neq 0$.
When $x < 0$, the eigenvalue are real and distinct, and the system is hyperbolic.
When $x > 0$, the system is elliptic, and the system is parabolic when $x = 0$.

29. The first equation is in conservation form: $h_t + (uh)_x = 0$. Multiply the second equation by h and add the result to u times the first equation to obtain the conservation form.

7.9 Exercises

5. Hint:

$$\begin{aligned}
\dot{I}_4 &= \int_{-\infty}^{\infty} xu_t\,dx = -\int_{-\infty}^{\infty} x\,\partial_x\left(\frac{u^2}{2} + \tilde{K}u\right)dx \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{2}u^2 + \tilde{K}u\right)dx = \frac{1}{2}I_2(u) + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} K(x-s)u(s,t)\,ds \\
&= \frac{1}{2}I_2(u) + c(0)I_1(u).
\end{aligned}$$

Integrating with respect to t gives $I_4(u) - \frac{1}{2}t\,I_2(u) - tc(0)I_1(u) = \text{const}.$

6. Hint: (a) Multiply the given equation by $\sqrt{t}$ to get

$$\left(\sqrt{t}u\right)_t + \left[\sqrt{t}\left(u_{xx} - 3u^2\right)\right]_x = 0.$$

(b) Multiply the given equation by $2tu$ to obtain

$$\left(tu^2\right)_t + \left[t\left(2uu_{xx} - u_x^2 - 4u^3\right)\right]_x = 0.$$

7. Hint: Multiply equation (2.7.67) by u and follow the procedure described by Benney (1974) and Miura (1974) to obtain

$$\left(\frac{1}{3}u^3\right)_t + \left(\frac{1}{4}u^4\right)_x + wu^2 u_z + u^2 h_x = 0,$$

or equivalently, $(\frac{1}{3}u^3)_t + (\frac{1}{4}u^4 + u^2 h)_x + (\frac{1}{3}u^3 w)_z - \frac{1}{3}u^3 w_z - 2huu_x = 0$. Multiply equation (2.7.67) by h to obtain

$$(hu)_t - uh_t + huu_{xx} + hwu_z + \frac{1}{2}\left(h^2\right)_x = 0.$$

This equation is added to the previous equation to generate

$$\left(hu + \frac{1}{3}u^3\right)_t + \left(u^2 h + \frac{1}{4}h^4 + \frac{1}{2}h^2\right)_x + \frac{1}{3}\left(u^3 w\right)_z - huu_x$$
$$+ hwu_z - huu_x - uh_t = 0.$$

Write $m = \int_0^h u\,dz$ so that $h_t + m_x = 0$, and then

$$\left(hu + \frac{1}{3}u^3\right)_x + \left(hu^2 + um + \frac{1}{3}u^4 + \frac{1}{2}h^2\right)_x$$
$$+ \left[w\left(hu + m + \frac{1}{3}u^3\right)\right]_z = 0.$$

This gives the answer.

8. Hint: (a) $(\int_{-\infty}^{\infty} u\,dx)_y = 0$, and hence, $\int_{-\infty}^{\infty} u\,dx = c(t)$.

   Under suitable conditions, $c(t)$ is a constant, and hence, it can be set equal to zero. To prove (c), we find

$$\left(\int_{-\infty}^{\infty} u\,dx\right)_t + \left[u_{xx} - 3u^2\right]_{-\infty}^{\infty} + 3\left[\int_{-\infty}^{\infty} v\,dx\right]_y = 0.$$

   Both (b) and (c) represent the conservation of momentum in both x- and y-directions.

10. It follows from the definition of $\mathcal{H}$ that $\frac{\partial \mathcal{H}}{\partial J} = \omega$, which means that the frequency of the modulated wave is determined by the partial derivative of $\mathcal{H}$ with respect to the wave action J. Thus, $\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0$ reduces to the Hamiltonian form $\frac{\partial k}{\partial t} + \frac{\partial}{\partial x}(\frac{\partial \mathcal{H}}{\partial J}) = 0$. We next replace $\mathcal{L}_\omega$ by J in the Whitham equation to obtain $\frac{\partial J}{\partial t} + \frac{\partial}{\partial x}(\frac{\partial \mathcal{H}}{\partial k}) = 0$ because $\frac{\partial \mathcal{H}}{\partial k} = -\frac{\partial \mathcal{L}}{\partial k}$. Thus, the above equation describes conservation of the wave action J, and $\frac{\partial \mathcal{H}}{\partial k}$ represents the density of its flux. The quantity $I = \int_{-\infty}^{\infty} J\,dx$ is conserved in a localized wave packet. It is noted that variables J and k are introduced to describe the modulated wave, and they play the role similar to the canonical variables of the action-angle in classical mechanics.

8.14 Exercises

1. (a) $\alpha = \beta = 1$ and $\gamma = 2$, (b) $\beta = 2\alpha$,

   (c) $\gamma = \alpha - \beta$, (d) $\beta = 2\alpha$.

2. In D_1 and for $c \geq 2$, $u(\xi) = A\exp(m_1\xi) + B\exp(m_2\xi)$, where $m_1, m_2 = \frac{1}{2}[-c_1 \pm \sqrt{c^2 - 4}]$. In D_2, $u(\xi) = 1 + A_1\exp(n_1\xi) + B_1\exp(n_2\xi)$, where $n_1, n_2 = \frac{1}{2}[-c \pm \sqrt{c^2 + 4}]$.

   To determine A, B, A_1, and B_1, we use the facts that (i) $u(0) = \frac{1}{2}$, (ii) the values of $u'(0)$ in D_1 and D_2 are equal, and (iii) $u(-\infty) = 1$ and $u(\infty) = 0$. In D_1, for $\xi > 0$, we have

$$u(\xi) = \frac{1}{4\sqrt{c^2-4}}\left[\left\{\sqrt{c^2-4}+2c-\sqrt{c^2+4}\right\}\exp(m_1\xi) + \left\{\sqrt{c^2-4}-2c+\sqrt{c^2+4}\right\}\exp(m_2\xi)\right].$$

In D_2, for $\xi < 0$, we find $u(\xi) = 1 - \frac{1}{2}\exp(n_1\xi)$.

When $c = 2$, we find solutions in D_1 and in D_2.

In D_1, for $\xi > 0$, $u(\xi) = \frac{1}{2}\exp(-\xi) + (1 - \frac{1}{\sqrt{2}})\xi\exp(-\xi)$, and in D_2, for $\xi < 0$, $u(\xi) = 1 - \frac{1}{2}\exp[(\sqrt{2}-1)\xi]$.

8. Hint: $\tilde{\psi} = a^{\beta+1}\psi$, and set $\beta = 0$ to find the similarity solutions.
9. Hint: The boundary conditions are $u(\eta = 0) = 0$ and $u(\eta \to \infty) = 1$, $v(\eta = 0) = v_\omega$.
11. $v'''' + \frac{c^2}{4}(\eta^2\, v'' + 5\,\eta\, v' + 3v) = 0$.

Hint: $\eta_t = -(\frac{\alpha}{\beta})\frac{\eta}{t}$, $u_t = \frac{1}{\beta}\, t^{\frac{\gamma}{\beta}-1}(\gamma v - \alpha\,\eta\, v')$,
$u_{tt} = t^{\frac{\gamma}{\beta}-2}\{(\frac{\alpha}{\beta})^2\eta^2 v'' + \frac{\alpha}{\beta}(-\frac{2\gamma}{\beta} + \frac{\alpha}{\beta} + 1)\eta\, v' + (\frac{\gamma}{\beta} - 1)(\frac{\gamma}{\beta})v\}$,
$u_x = t^{\frac{\gamma}{\beta}-\frac{1}{2}}v'$, $u_{xxxx} = t^{\frac{\gamma}{\beta}-2}v''''$.

13. Hint:

$$\begin{aligned}\dot{Q}(t) &= 2\int_0^a u_x u_{xt}\, dx = 2[u_x u_t]_0^a - 2\int_0^a u_{xx} u_t\, dx \\ &= -2\int_0^a u_{xx}\big(u_{xx} + F(u)\big)\, dx \\ &= -2\int_0^a u_{xx}^2\, dx + 2\int_0^a u_x^2 F'(u)\, dx.\end{aligned}$$

Using the boundary condition gives the Poincaré inequality

$$\int_0^a u_{xx}^2\, dx \ge \pi^2 \int_0^a u_x^2\, dx$$

and

$$\int_0^a u_x^2 F'(u)\,dx \le \int_0^a u_x^2 \big|F'(u)\big|\,dx \le A\int_0^a u_x^2\,dx,$$
$$\dot{Q}(t) \le 2\big(A-\pi^2\big)Q(t) = \mu Q(t).$$

Integrating from 0 to a gives the inequality.

14. See Murray (1997), p. 357.

20. Combining $u_t + (\rho u)_x = 0$ with $(\rho u)_t + (\rho u^2)_x = 0$ gives $u_t + uu_x = 0$. Or in other words, $u_t + (\frac{1}{2}u^2)_x = 0$. The characteristic equations are $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$, which leads to $u =$ const. for $\frac{dx}{dt} = u$. Thus, u is invariant, and ρ and ρu are conserved quantities. Writing $U = u$ and $F = \frac{1}{2}u^2 = \frac{1}{2}U^2$ gives $U_t + F_x = 0$. Or equivalently, $U_t + F_U U_x = 0$, that is, $U_t + cU_x = 0$, where $c = \frac{\partial F}{\partial U} = U = u$ is the propagation velocity. Differentiating the inviscid Burgers equation $u_t + uu_x = 0$ with respect to x gives $(u_x)_t + u(u_x)_x = -u_x^2$. Thus, the characteristic equations for u_x are $\frac{dt}{1} = \frac{dx}{u} = \frac{du_x}{-u_x^2}$, that is, $\frac{d}{dt}u_x = -u_x^2$ for $\frac{dx}{dt} = u$. Integrating gives $-\frac{1}{u_x} = -t + A$, where A is a constant. When $t = t_0$, $(u_x) = u_0'$, and $A = t_0 - \frac{1}{u_0'}$. Thus, the solution becomes $(u_x)(t) = u_0'[1 + (t - t_0)u_0']^{-1}$.

9.14 Exercises

20. Hint: Add two equations in 20(d) and integrate the result to show $\frac{dE}{dt} = 0$.

21. Hint: If $u_0(x) = (c_1 - \gamma)\exp(-|x - x_1|) + (c_2 - \gamma)\exp(-|x - x_2|)$, $x \in \mathbb{R}$ with $c_1 < \gamma$, $c_2 > \gamma$ and $x_1 < x_2$, verify that

$$m_0 = u_0 - u_{0xx} = 2c_1\delta(x - x_1) + 2c_2\delta(x - x_2),$$

where $\delta(x)$ is the Dirac delta function.

22. Hint: $u(x,t)$ is a solution of

$$\int_0^T \int_{\mathbb{R}} (u\phi_t + F(u)\phi_x)\,dx\,dt + \int_{\mathbb{R}} u(x,0)\phi(x,0)\,dx = 0,$$

for any $T > 0$ and $\phi \in C_0^\infty([0,T] \times \mathbb{R})$ with $F(u) = \frac{1}{2}u^2 + p * (\frac{3}{2}u^2) + \gamma u$, where $p(x)$ is defined in 21(a). (See Guo 2009.)

10.9 Exercises

10. We seek the solution $\psi(x,t) = \Psi(x)e^{-i\omega t}$ where $\omega = (E/\hbar)$ is the frequency of the De Broglie wave and the function $\Psi(x)$ satisfies the equation

$$\Psi''(x) + \left(\frac{2mE}{\hbar^2}\right)(1-x)\Psi = 0.$$

Making the change of variable $z = -(1-x)(2mE/\hbar^2)^{\frac{1}{3}}$ reduces the equation into the form $\Psi'' - z\Psi = 0$. The bounded solution at $|z| \to \infty$ can be expressed in terms of the Airy function

$$\Psi(z) = C\,Ai(z) = C\int_0^\infty \cos\left(sz + \frac{s^3}{3}\right) ds,$$

where C is a constant. Thus, the solution describing the behavior of the electron is $\psi(z,t) = C\,Ai(z)\exp(-i\omega t)$. The function $|\psi|^2$ describes the probability of the electron at a point z. For $z > 0$, the probability drops to zero suddenly as if there were an obstacle reflecting the electron at $z = 0$ which is referred to as the turning point. For $z < 0$, the solution $\psi(z,t)$ represents a nonuniform standing wave. Using the asymptotic expansion of $Ai(z)$ for $|z| \to \infty$, in the form

$$Ai(z) \sim \begin{cases} \frac{1}{\sqrt{\pi}z^{\frac{1}{4}}}\exp(-\frac{2}{3}z^{3/2}), & \text{when } z \to \infty, \\ \frac{1}{\sqrt{\pi}z^{\frac{1}{4}}}\cos(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}), & \text{when } z \to -\infty. \end{cases}$$

The solution for $\psi(z,t)$ can be written as the superposition of two progressive waves in the form

$$\psi(z,t) \sim \frac{C}{\sqrt[4]{z}}\left\{\exp\left[-i\left(\omega t + \frac{2}{3}|z|^{3/2} - \frac{\pi}{4}\right)\right]\right.$$
$$\left. + \exp\left[-i\left(\omega t - \frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right)\right]\right\}.$$

This represents the De Broglie wave for the particle in free space.

11.14 Exercises

6. Hint: $\phi(x,t) = \tan(\frac{u}{4})$,

$$\phi(x,t) \sim \pm U \exp\left\{\pm\frac{x+Ut}{\sqrt{1-U^2}}\right\} \mp U \exp\left\{\mp\frac{x-Ut}{\sqrt{1-U^2}}\right\} \quad \text{as } t \to \mp\infty.$$

Note that $\mp U \exp(\theta) = \pm \exp\{\theta + \frac{1}{2}\frac{\gamma}{\sqrt{1-U^2}}\}$, where $\frac{\gamma}{2} = 1(1-U)^{\frac{1}{2}} \log U$.

14. (b) $u_4 = 4\tan^{-1}[(\frac{a_1+a_2}{a_1-a_2})\tan\{\frac{1}{4}(u_2-u_3)\}]$ for any solutions $u_2,\ u_3$.

$$\tan\left(\frac{u_r}{4}\right) = \exp(\theta_r), \quad \theta_r = (a_r x + a_r^{-1} t + \varepsilon_r),\ r = 2, 3.$$

15. Hint: Use $u_x(x,t)$ instead of $u(x,t)$.

22. $v_{tt} - c^2 v_{xx} - a^2 v = 0$.

24. $u(x,t) = \frac{1}{2c} J_0[d\sqrt{t^2 - \frac{x^2}{c^2}}]H(ct - |x|)$.

12.10 Exercises

1. (a) The continuity and the momentum equations for the inviscid Burgers equation can be written in the matrix form $U_t + AF_x = 0$ by defining $U = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} =$

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 \end{bmatrix} = \begin{bmatrix} u_2 \\ u_2^2/u_1 \end{bmatrix}$.

Thus, $A = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(\frac{u_2}{u_1})^2 & \frac{2u_2}{u_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -u^2 & 2u \end{bmatrix}$.

The eigenvalues of A are the roots of $0 = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -u^2 & 2u-\lambda \end{vmatrix} = \lambda^2 - 2u\lambda + u^2$. Thus, A has two equal eigenvalues $(\lambda = u, u)$ which are not distinct. Hence, the system is *not* hyperbolic.

(b) $A = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{\partial P}{\partial u_1} - (\frac{u_2}{u_1})^2 & \frac{2u_2}{u_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix}$, where $c = \sqrt{\frac{\partial P}{\partial u_1}}$. The eigenvalues are real and distinct roots of $|A - \lambda I| = 0$.

Or, $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ c^2 - u^2 & 2u - \lambda \end{vmatrix} = 0$, that is, $\lambda = \lambda_1, \lambda_2 = u \mp c$.

The corresponding eigenvectors are $\begin{bmatrix} 1 \\ u-c \end{bmatrix}$ and $\begin{bmatrix} 1 \\ u+c \end{bmatrix}$, and a new matrix B can be defined by $B = \begin{bmatrix} 1 & 1 \\ u-c & u+c \end{bmatrix}$, and hence, $B^{-1} = \frac{1}{2c}\begin{bmatrix} c+u & -1 \\ c-u & 1 \end{bmatrix}$ with $B^{-1}AB =$ diagonal matrix, $D = \begin{bmatrix} u-c & 0 \\ 0 & u+c \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

The Riemann invariants are $dR = B^{-1}dU$, and hence, $dR_1 = (\frac{1}{2} + \frac{u}{2c})dU_1 - \frac{1}{2}dU_2$ and $dR_2 = (\frac{1}{2} - \frac{u}{2c})dU_1 + \frac{1}{2}dU_2$, where the matrix $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$.

(c) The 2×2 matrix system can be generalized for an $n \times n$ system as $U_t + F_x = S$. Or equivalently, $U_t + AU_x = S$, where U, F, and S are $n \times 1$ matrices given by

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \qquad F = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}, \qquad S = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{bmatrix},$$

A is the $n \times n$ *Jacobian matrix* of F:

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \cdots & \frac{\partial F_1}{\partial u_n} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \cdots & \frac{\partial F_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial u_1} & \frac{\partial F_n}{\partial u_2} & \cdots & \frac{\partial F_n}{\partial u_n} \end{bmatrix}.$$

A hyperbolic system of conservation laws is a system of conservation laws with the following properties: (i) The components of F and S are functions of components of U and possibly of x and t, but F does not contain any derivative of U with respect to x and t, and (ii) the matrix A has n real and distinct eigenvalues which are roots of $|A - \lambda I| = 0$. The theory of hyperbolic systems of conservation laws with many examples of applications is available in many books including Defermos (2000), Holden and Risebro (2002), Jeffrey (1976), Lax (1973, 2006), Serre (2000), Sharma (2010), and Zheng (2001).

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