VLADIMIR MAZ'YA

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SOBOLEV SPACES

WITH APPLICATIONS TO ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Sobolev Spaces

with Applications to Elliptic Partial Differential Equations

2nd, revised and augmented Edition

Springer

Professor Vladimir Maz'ya
Department of Mathematics Sciences
University of Liverpool
Liverpool L69 7ZL,
UK
and
Department of Mathematics
Linköping University
Linköping 581 83,
Sweden
vlmaz@mai.liu.se

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To Tatyana

Preface

Sobolev spaces, i.e., the classes of functions with derivatives in L_p, occupy an outstanding place in analysis. During the last half-century a substantial contribution to the study of these spaces has been made; so now solutions to many important problems connected with them are known.

In the present monograph we consider various aspects of theory of Sobolev spaces in particular, the so-called embedding theorems. Such theorems, originally established by S.L. Sobolev in the 1930s, proved to be a useful tool in functional analysis and in the theory of linear and nonlinear partial differential equations.

A part of this book first appeared in German as three booklets of Teubner-Texte für Mathematik [552, 555]. In the Springer volume of "Sobolev Spaces" [556] published in 1985, the material was expanded and revised.

As the years passed the area became immensely vast and underwent important changes, so the main contents of the 1985 volume had the potential for further development, as shown by numerous references. Therefore, and since the volume became a bibliographical rarity, Springer-Verlag offered me the opportunity to prepare the second, updated edition of [556].

As in [556], the selection of topics was mainly influenced by my involvement in their study, so a considerable part of the text is a report on my work in the field. In comparison with [556], the present text is enhanced by more recent results. New comments and the significantly augmented list of references are intended to create a broader and modern view of the area. The book differs considerably from the monographs of other authors dealing with spaces of differentiable functions that were published in the last 50 years.

Each of the 18 chapters of the book is divided into sections and most of the sections consist of subsections. The sections and subsections are numbered by two and three numbers, respectively (3.1 is Sect. 1 in Chap. 3, 1.4.3 is Subsect. 3 in Sect. 4 in Chap. 1). Inside subsections we use an independent numbering of theorems, lemmas, propositions, corollaries, remarks, and so on. If a subsection contains only one theorem or lemma then this theorem or lemma has no number. In references to the material from another section or subsection we first indicate the number of this section or subsection. For example, Theorem 1.2.1/1 means Theorem 1 in Subsect. 1.2.1, (2.6.6) denotes formula (6) in Sect. 2.6.

The reader can obtain a general idea of the contents of the book from the Introduction. Most of the references to the literature are collected in the Comments. The list of notation is given at the end of the book.

The volume is addressed to students and researchers working in functional analysis and in the theory of partial differential operators. Prerequisites for reading this book are undergraduate courses in these subjects.

Acknowledgments

My cordial thanks are to S. Bobkov, Yu.D. Burago, A. Cianchi, E. Milman, S. Poborchi, P. Shvartsman, and I. Verbitsky who read parts of the book and supplied me with their comments.

I wish to express my deep gratitude to M. Nieves for great help in the technical preparation of the text.

The dedication of this book to its translator and my wife T.O. Shaposhnikova is a weak expression of my gratitude for her infinite patience, useful advice, and constant assistance.

Liverpool–Linköping　　　　Vladimir Maz'ya
January 2010

Contents

Introduction

In [711–713] Sobolev proved general integral inequalities for differentiable functions of several variables and applied them to a number of problems of mathematical physics. Sobolev considered the Banach space $W_p^l(\Omega)$ of functions in $L_p(\Omega)$, $p \geq 1$, with generalized derivatives of order l integrable with power p. In particular, using these theorems on the potential-type integrals as well as an integral representation of functions, Sobolev established the embedding of $W_p^l(\Omega)$ into $L_q(\Omega)$ or $C(\Omega)$ under certain conditions on the exponents p, l, and q.[1]

Later the Sobolev theorems were generalized and refined in various ways (Kondrashov, Il'in, Gagliardo, Nirenberg, et al.). In these studies the domains of functions possess the so-called cone property (each point of a domain is the vertex of a spherical cone with fixed height and angle which is situated inside the domain). Simple examples show that this condition is precise, e.g., if the boundary contains an outward "cusp" then a function in $W_p^1(\Omega)$ is not, in general, summable with power $pn/(n-p)$, $n > p$, contrary to the Sobolev inequality. On the other hand, looking at Fig. 1, the reader can easily see that the cone property is unnecessary for the embedding $W_p^1(\Omega) \subset L_{2p/(2-p)}(\Omega)$, $2 > p$. Indeed, by unifying Ω with its mirror image, we obtain a new domain with the cone property for which the above embedding holds by the Sobolev theorem. Consequently, the same is valid for the initial domain although it does not possess the cone property.

Now we note that, even before the Sobolev results, it was known that certain integral inequalities hold under fairly weak requirements on the domain. For instance, the Friedrichs inequality ([292], 1927)

$$\int_\Omega u^2 \, \mathrm{d}x \leq K \left(\int_\Omega (\operatorname{grad} u)^2 \, \mathrm{d}x + \int_{\partial\Omega} u^2 \, \mathrm{d}s \right)$$

[1] A sketch of a fairly rich prehistory of Sobolev spaces can be found in Naumann [624].

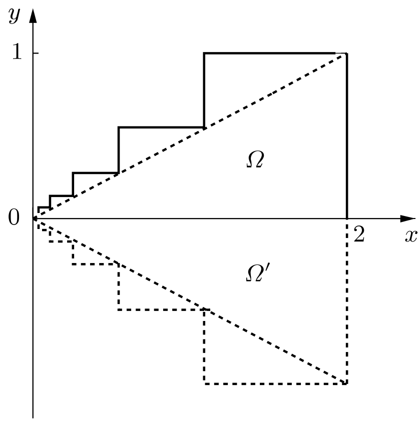

Fig. 1.

was established under the sole assumption that Ω is a bounded domain for which the Gauss–Green formula holds. In 1933, Nikodým [637] gave an example of a domain Ω such that the square integrability of the gradient does not imply the square integrability of the function defined in Ω. The monograph of Courant and Hilbert [216], Chap. 7, contains sufficient conditions for the validity of the Poincaré inequality

$$\int_\Omega u^2\,\mathrm{d}x \le K \int_\Omega (\operatorname{grad} u)^2\,\mathrm{d}x + \frac{1}{m_n \Omega}\left(\int_\Omega u\,\mathrm{d}x\right)^2$$

(see [663, p. 76] and [664, pp. 98–104]) and of the Rellich lemma [672] on the compactness in $L_2(\Omega)$ of the set bounded in the metric

$$\int_\Omega \left[(\operatorname{grad} u)^2 + u^2\right]\mathrm{d}x.$$

The previous historical remarks naturally suggest the problem of describing the properties of domains that are equivalent to various properties of embedding operators.

Starting to work on this problem in 1959 as a fourth-year undergraduate student, I discovered that Sobolev-type theorems for functions with gradients in $L_p(\Omega)$ are valid if and only if some isoperimetric and isocapacitary inequalities hold. Such necessary and sufficient conditions appeared in the early 1960s in my works [527–529, 531, 533, 534]. For $p = 1$ these conditions coincide with isoperimetric inequalities between the volume and the area of a part of the boundary of an arbitrary subset of the domain.

For $p > 1$, geometric functionals such as volume and area prove to be insufficient for an adequate description of the properties of domains. Here inequalities between the volume and the p-capacity or the p-conductivity arise.

Similar ideas were applied to complete characterizations of weight functions and measures in the norms involved in embedding theorems. Moreover, the method of proof of the criteria does not use specific properties of the Euclidean space. The arguments can be carried over to the case of Riemannian manifolds and even abstract metric spaces. A considerable part of the present book (Chaps. 2–9 and 11) is devoted to the development of this isoperimetric and isocapacitary ideology.

However, this theory does not exhaust the material of the book even conceptually. Without aiming at completeness, I mention that other areas of the study in the book are related to the following questions. How massive must a subset e of a domain Ω be in order that the inequality

$$\|u\|_{L_q(\Omega)} \leq C\|\nabla_l u\|_{L_p(\Omega)}$$

holds for all smooth functions vanishing on e? How does the class of domains admissible for integral inequalities depend upon additional requirements imposed upon the behavior of functions at the boundary? What are the conditions on domains and measures involved in the norms ensuring density of a space of differentiable functions in another one? We shall study the criteria of compactness of Sobolev-type embedding operators. Sometimes the best constants in functional inequalities will be discussed. The embedding and extension operators involving Birbaum–Orlicz spaces, the space BV of functions whose gradients are measures, and Besov and Bessel potential spaces of functions with fractional smoothness will also be dealt with.

The investigation of the above-mentioned and similar problems is not only of interest in its own right. By virtue of well-known general considerations it leads to conditions for the solvability of boundary value problems for elliptic equations and to theorems on the structure of the spectrum of the corresponding operators. Such applications are also included.

I describe briefly the contents of the book. More details can be found in the Introductions to the chapters.

Chapter 1 gives prerequisites to the theory. Along with classical facts this chapter contains certain new results. It addresses miscellaneous topics related to the theory of Sobolev spaces. Some of this material is of independent interest and some (Sects. 1.1–1.3) will be used in the sequel. The core of the chapter is a generalized version of Sobolev embedding theorems (Sect. 1.4). We also deal with various extension and approximation theorems (Sects. 1.5 and 1.7), and with maximal algebras in Sobolev spaces (Sect. 1.8). Section 1.6 is devoted to inequalities for functions vanishing on the boundary along with their derivatives up to some order.

The idea of the equivalence of isoperimetric and isocapacitary inequalities on the one hand and embedding theorems on the other hand is crucial for Chap. 2. Most of this chapter deals with the necessary and sufficient conditions for the validity of integral inequalities for gradients of functions that vanish at the boundary. Of special importance for applications are multidimensional inequalities of the Hardy–Sobolev type proved in Sect. 2.1. The basic results of Chap. 2 are applied to the spectral theory of the Schrödinger operator in Sect. 2.5.

Chapters 3 and 4 briefly address the so-called conductor and capacitary inequalities, which are stronger than inequalities of the Sobolev type and are valid for functions defined on quite general topological spaces.

The space $L^1_p(\Omega)$ of functions with gradients in $L_p(\Omega)$ is studied in Chaps. 5–8. Chapter 5 deals with the case $p = 1$. Here, the necessary and sufficient conditions for the validity of embedding theorems stated in terms of the classes $\mathscr{J}_\alpha$ characterized by isoperimetric inequalities are found. We also check whether some concrete domains belong to these classes. In Chaps. 6 and 7 we extend the presentation to the case $p > 1$. Here the criteria are formulated in terms of the p-conductivity. In Chap. 6 we discuss theorems on embeddings into $L_q(\Omega)$ and $L_q(\partial\Omega)$. Chapter 7 concerns embeddings into $L_\infty(\Omega) \cap C(\Omega)$. In particular, we present the necessary and sufficient conditions for the validity of the previously mentioned Friedrichs and Poincaré inequalities and of the Rellich compactness lemma. In Chap. 9 we study the essential norm and other noncompactness characteristics of the embedding operator $L^1_p(\Omega) \to L_q(\Omega)$.

Throughout the book and especially in Chaps. 5–8 we include numerous examples of domains that illustrate possible pathologies of embedding operators. For instance, in Sect. 1.1 we show that the square integrability of second derivatives and of the function do not imply the square integrability of the first derivatives. In Sect. 7.5 we consider the domain for which the embedding operator of $W^1_p(\Omega)$ into $L_\infty(\Omega) \cap C(\Omega)$ is continuous without being compact. This is impossible for domains with "good" boundaries. The results of Chaps. 5–7 show that not only the classes of domains determine the parameters p, q, and so on in embedding theorems, but that a feedback takes place. The criteria for the validity of integral inequalities are applied in Chap. 6 to the theory of elliptic boundary value problems. The exhaustive results on embedding operators can be restated as necessary and sufficient conditions for the unique solvability and for the discreteness of the spectrum of boundary value problems, in particular, of the Neumann problem.

Chapter 9, written together with Yu.D. Burago, is devoted to the study of the space $BV(\Omega)$ consisting of the functions whose gradients are vector charges. Here we present a necessary and sufficient condition for the existence of a bounded nonlinear extension operator $BV(\Omega) \to BV(\mathbb{R}^n)$. We find necessary and sufficient conditions for the validity of embedding theorems for the space $BV(\Omega)$, which are similar to those obtained for $L^1_1(\Omega)$ in Chap. 5. In some integral inequalities we obtain the best constants. The results of Sects. 9.5 and 9.6 on traces of functions in $BV(\Omega)$ make it possible to discuss boundary values of "bad" functions defined on "bad" domains. Along with the results due to Burago and the author in Chap. 9 we present the De Giorgi–Federer theorem on conditions for the validity of the Gauss–Green formula.

Chapters 2–9 mainly concern functions with *first* derivatives in L_p or in C^*. This restriction is essential since the proofs use the truncation of functions along their level surfaces. The next six chapters deal with functions that have derivatives of any integer, and sometimes of fractional, order.

In Chap. 10 we collect (sometimes without proofs) various properties of Bessel and Riesz potential spaces and of Besov spaces in $\mathbb{R}^n$. In Chap. 10 we also present a review of the results of the theory of (p,l)-capacities and of nonlinear potentials.

In Chap. 11 we investigate necessary and sufficient conditions for the validity of the trace inequality

$$\|u\|_{L_q(\mu)} \le C\|u\|_{S^l_p}, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{0.0.1}$$

where $L_q(\mu)$ is the space with the norm $(\int |u|^q \,\mathrm{d}\mu)^{1/q}$, μ is a measure, and S^l_p is one of the spaces just mentioned. For $q \ge p$, (0.0.1) is equivalent to the isoperimetric inequality connecting the measure μ and the capacity generated by the space S^l_p. This result is of the same type as the theorems in Chaps. 2–9. It immediately follows from the capacitary inequality

$$\int_0^\infty \operatorname{cap}(\mathscr{N}_t; S^l_p)\, t^{p-1}\,\mathrm{d}t \le C\|u\|^p_{S^l_p},$$

where $\mathscr{N}_t = \{x : |u(x)| \ge t\}$. Inequalities of this type, initially found by the author for the spaces $L^1_p(\Omega)$ and $\mathring{L}^2_p(\mathbb{R}^n)$ [543], have proven to be useful in a number of problems of function theory and were intensively studied.

For $q > p \ge 1$ the criteria for the validity of (0.0.1), presented in Chap. 11 do not contain a capacity. In this case the measure of any ball is estimated by a certain function of the radius.

Chapter 12 is devoted to pointwise interpolation inequalities for derivatives of integer and fractional order.

Further, in Chap. 13 we introduce and study a certain kind of capacity. In comparison with the capacities defined in Chap. 10, here the class of admissible functions is restricted, they equal the unity in a neighborhood of a compactum. (In the case of the capacities in Chap. 10, the admissible functions majorize the unity on a compactum.) If the order l of the derivatives in the norm of the space equals 1, then the two capacities coincide. For $l \ne 1$ they are equivalent, which is proved in Sect. 13.3.

The capacity introduced in Chap. 13 is applied in subsequent chapters to prove various embedding theorems. An auxiliary inequality between the L_q-norm of a function on a cube and a certain Sobolev seminorm is studied in detail in Chap. 14. This inequality is used to justify criteria for the embedding of $\mathring{L}^l_p(\Omega)$ into different function spaces in Chap. 15. By $\mathring{L}^l_p(\Omega)$ we mean the completion of the space $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla_l u\|_{L_p(\Omega)}$. It is known that this completion is not embedded, in general, into the distribution space $\mathscr{D}'$. In Chap. 15 we present the necessary and sufficient conditions for the embeddings of $\mathring{L}^l_p(\Omega)$ into $\mathscr{D}'$, $L_q(\Omega, \mathrm{loc})$, and $L_p(\Omega)$. For $p = 2$, these results can be interpreted as necessary and sufficient conditions for the solvability of the Dirichlet problem for the polyharmonic equation in unbounded domains provided the right-hand side is contained in $\mathscr{D}'$ or in $L_q(\Omega)$. In Chap. 16 we find criteria for the boundedness and the compactness of the embedding operator of the space $\mathring{L}^l_p(\Omega, \nu)$ into $W^r_q(\Omega)$, where ν is a measure and $\mathring{L}^l_p(\Omega, \nu)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\left(\int_\Omega |\nabla_l u|^p \, \mathrm{d}x + \int_\Omega |u|^p \, \mathrm{d}\nu \right)^{1/p}.$$

The topic of Chap. 17 is a necessary and sufficient condition for density of $C_0^\infty(\Omega)$ in a certain weighted Sobolev space which appears in applications. Finally, Chap. 18 contains variations on the theme of Molchanov's discreteness criterion for the spectrum of the Schrödinger operator as well as two-sided estimates for the first Dirichlet–Laplace eigenvalue.

Obviously, it is impossible to describe such a vast area as Sobolev spaces in one book. The treatment of various aspects of this theory can be found in the books by Sobolev [713]; R.A. Adams [23]; Nikolsky [639]; Besov, Il'in, and Nikolsky [94]; Gel'man and Maz'ya [305]; Gol'dshtein and Reshetnyak [316]; Jonsson and Wallin [408]; Ziemer [813]; Triebel [758–760]; D.R. Adams and Hedberg [15]; Maz'ya and Poborchi [576]; Burenkov [155]; Hebey [361]; Haroske, Runst, and Schmeisser [354]; Hajłasz [342]; Saloff-Coste [687]; Attouch, Buttazzo, and Michaille [54]; Tartar [744]; Haroske and Triebel [355]; Leoni [486]; Maz'ya and Shaposhnikova [588]; Maz'ya [565]; and A. Laptev (Ed.) [479].

1

Basic Properties of Sobolev Spaces

The plan of this chapter is as follows. Sections 1.1 and 1.2 contain the prerequisites on Sobolev spaces and other function analytic facts to be used in the book. In Sect. 1.3 a complete study of the one-dimensional Hardy inequality with two weights is presented. The case of a weight of unrestricted sign on the left-hand side is also included here, following Maz'ya and Verbitsky [593]. Section 1.4 contains theorems on necessary and sufficient conditions for the L_q integrability with respect to an arbitrary measure of functions in $W_p^l(\Omega)$. These results are due to D.R. Adams, $p > 1$, [2, 3] and the author, $p = 1$, [551]. Here, as in Sobolev's papers, it is assumed that the domain is "good," for instance, it possesses the cone property. In general, in requirements on a domain in Chap. 1 we follow the "all or nothing" principle. However, this rule is violated in Sect. 1.5 which concerns the class preserving extension of functions in Sobolev spaces. In particular, we consider an example of a domain for which the extension operator exists and which is not a quasicircle.

In Sect. 1.6 an integral representation of functions in $W_p^l(\Omega)$ that vanish on $\partial\Omega$ along with all their derivatives up to order $k-1$, $2k \geq l$, is obtained. This representation entails the embedding theorems of the Sobolev type for any bounded domain Ω. In the case $2k < l$ it is shown by example that some requirements on $\partial\Omega$ are necessary. Section 1.7 is devoted to an approximation of Sobolev functions by bounded ones. Here we reveal a difference between the cases $l = 1$ and $l > 1$. The chapter finishes with a discussion in Sect. 1.8 of the maximal subalgebra of $W_p^l(\Omega)$ with respect to multiplication.

1.1 The Spaces $L_p^l(\Omega)$, $V_p^l(\Omega)$ and $W_p^l(\Omega)$

1.1.1 Notation

Let Ω be an open subset of n-dimensional Euclidean space $\mathbb{R}^n = \{x\}$. Connected open sets Ω will be called domains. The notations $\partial\Omega$ and $\bar{\Omega}$ stand for the boundary and the closure of Ω, respectively. Let $C^\infty(\Omega)$ denote the space

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of infinitely differentiable functions on Ω; by $C^\infty(\bar{\Omega})$ we mean the space of restrictions to Ω of functions in $C^\infty(\mathbb{R}^n)$.

In what follows $\mathscr{D}(\Omega)$ or $C_0^\infty(\Omega)$ is the space of functions in $C^\infty(\mathbb{R}^n)$ with compact supports in Ω. The classes $C^k(\Omega)$, $C^k(\bar{\Omega})$, and $C_0^k(\Omega)$ of functions with continuous derivatives of order k and the classes $C^{k,\alpha}(\Omega)$, $C^{k,\alpha}(\bar{\Omega})$, and $C_0^{k,\alpha}(\Omega)$ of functions for which the derivatives of order k satisfy a Hölder condition with exponent $\alpha \in (0,1]$ are defined in an analogous way.

Let $\mathscr{D}'(\Omega)$ be the space of distributions dual to $\mathscr{D}(\Omega)$ (cf. L. Schwartz [695], Gel'fand and Shilov [304]). Let $L_p(\Omega)$, $1 \le p < \infty$, denote the space of Lebesgue measurable functions, defined on Ω, for which

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p \, \mathrm{d}x \right)^{1/p} < \infty.$$

We use the notation $L_\infty(\Omega)$ for the space of essentially bounded Lebesgue measurable functions, i.e., uniformly bounded up to a set of measure zero. As a norm of f in $L_\infty(\Omega)$ one can take its essential supremum, i.e.,

$$\|f\|_{L_\infty(\Omega)} = \inf\bigl\{c > 0 : |f(x)| \le c \text{ for almost all } x \in \Omega\bigr\}.$$

By $L_p(\Omega, \mathrm{loc})$ we mean the space of functions locally integrable with power p in Ω. The space $L_p(\Omega, \mathrm{loc})$ can be naturally equipped with a countable system of seminorms $\|u\|_{L_p(\omega_k)}$, where $\{\omega_k\}_{k\ge 1}$ is a sequence of domains with compact closures $\bar{\omega}_k$, $\bar{\omega}_k \subset \omega_{k+1} \subset \Omega$, and $\bigcup_k \omega_k = \Omega$. Then $L_p(\Omega, \mathrm{loc})$ becomes a complete metrizable space.

If $\Omega = \mathbb{R}^n$ we shall often omit Ω in notations of spaces and norms. Integration without indication of limits extends over $\mathbb{R}^n$. Further, let $\operatorname{supp} f$ be the support of a function f and let $\operatorname{dist}(F, E)$ denote the distance between the sets F and E. Let $B(x, \varrho)$ or $B_\varrho(x)$ denote the open ball with center x and radius ϱ, $B_\varrho = B_\varrho(0)$. We shall use the notation m_n for n-dimensional Lebesgue measure in $\mathbb{R}^n$ and v_n for $m_n(B_1)$.

Let c, c_1, $c_2, \dots,$ denote positive constants that depend only on "dimensionless" parameters n, p, l, and the like. We call the quantities a and b equivalent and write $a \sim b$ if $c_1 a \le b \le c_2 a$. If α is a multi-index $(\alpha_1, \dots, \alpha_n)$, then, as usual, $|\alpha| = \sum_j \alpha_j$, $\alpha! = \alpha_1!, \dots, \alpha_n!$, $D^\alpha = D^{\alpha_1}_{x_1}, \dots, D^{\alpha_n}_{x_n}$, where $D_{x_i} = \partial/\partial x_i$, $x^\alpha = x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$. The inequality $\beta \ge \alpha$ means that $\beta_i \ge \alpha_i$ for $i = 1, \dots, n$. Finally, $\nabla_l = \{D^\alpha\}$, where $|\alpha| = l$ and $\nabla = \nabla_1$.

1.1.2 Local Properties of Elements in the Space $L_p^l(\Omega)$

Let $L_p^l(\Omega)$ denote the space of distributions on Ω with derivatives of order l in the space $L_p(\Omega)$. We equip $L_p^l(\Omega)$ with the seminorm

$$\|\nabla_l u\|_{L_p(\Omega)} = \left(\int_\Omega \Bigl(\sum_{|\alpha|=l} \bigl|D^\alpha u(x)\bigr|^2 \Bigr)^{p/2} \right)^{1/p}.$$

Theorem. *Any element of $L_p^l(\Omega)$ is in $L_p(\Omega, \mathrm{loc})$.*

Proof. Let ω and g be bounded open subsets of $\mathbb{R}^n$ such that $\omega \subset g \subset \Omega$. Moreover, we assume that the sets ω and g are contained in g and Ω along with their ε neighborhoods. We introduce $\varphi \in \mathscr{D}(\Omega)$ with $\varphi = 1$ on g, take an arbitrary $u \in L_p^l(\Omega)$, and set $T = \varphi u$. Further, let $\eta \in \mathscr{D}$ be such that $\eta = 1$ in a neighborhood of the origin and $\operatorname{supp} \eta \subset B_\varepsilon$.

It is well known that the fundamental solution of the polyharmonic operator Δ^l is

$$\Gamma(x) = \begin{cases} c_{n,l}(-1)^l |x|^{2l-n}, & \text{for } 2l < n \text{ or for odd } n \le 2l, \\ c_{n,l}(-1)^{l-1}|x|^{2l-n} \log|x|, & \text{for even } n \le 2l. \end{cases}$$

Here the constant $c_{n,l}$ is chosen so that $\Delta^l \Gamma = \delta(x)$ holds.

It is easy to see that $\Delta^l(\eta\Gamma) = \zeta + \delta$ with $\zeta \in \mathscr{D}(\mathbb{R}^n)$ and δ denoting Dirac's function. Therefore,

$$T + \zeta * T = \sum_{|\alpha|=l} \frac{l!}{\alpha!} D^\alpha(\eta\Gamma) * D^\alpha T,$$

where the star denotes convolution. We note that $\zeta * T \in C^\infty(\mathbb{R}^n)$. So, we have to examine the expression $D^\alpha(\eta\Gamma) * D^\alpha T$. Using the formula

$$D^\alpha(\varphi u) = \sum_{\alpha \ge \beta} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\alpha \varphi D^{\alpha-\beta} u,$$

we obtain

$$D^\alpha T = D^\alpha(\varphi u) = \varphi D^\alpha u,$$

in g. Hence,

$$D^\alpha(\eta\Gamma) * D^\alpha T = D^\alpha(\zeta\Gamma) * \varphi D^\alpha u,$$

in ω. To conclude the proof, we observe that the integral operator with a weak singularity, applied to $\varphi D^\alpha u$, is continuous in $L_p(\omega)$. □

Corollary. *Let $u \in L_p^l(\Omega)$. Then all distributional derivatives $D^\alpha u$ with $|\alpha| = 0, 1, \ldots, l-1$ belong to the space $L_p(\Omega, \mathrm{loc})$.*

The proof follows immediately from the inclusion $D^\alpha u \in L_p^{l-|\alpha|}(\Omega)$ and the above theorem.

Remark. By making use of the results in Sect. 1.4.5 we can refine the theorem to obtain more information on local properties of elements in $L_p^l(\Omega)$.

By the above theorem, if Ω is connected, we can supply $L_p^l(\Omega)$ with the norm

$$\|u\|_{L_p^l(\Omega)} = \|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\omega)}, \tag{1.1.1}$$

where ω is an arbitrary bounded open nonempty set with $\bar{\omega} \subset \Omega$.

1.1.3 Absolute Continuity of Functions in $L_p^1(\Omega)$

First we mention some simple facts concerning the approximation of functions in $L_p(\Omega)$ by smooth functions.

Let $\varphi \in \mathscr{D}, \varphi \geq 0$, supp $\varphi \subset B_1$, and

$$\int \varphi(x)\, \mathrm{d}x = 1.$$

With any $u \in L_p(\Omega)$ that vanishes on $\mathbb{R}^n \backslash \Omega$, we associate the family of its *mollifications*

$$(\mathscr{M}_\varepsilon u) = \varepsilon^{-n} \int \varphi\Big(\frac{x-y}{\varepsilon}\Big) u(y)\, \mathrm{d}y.$$

The function φ is called a *mollifier* and ε is called *a radius of mollification.*

We formulate some almost obvious properties of a mollification:

1. $\mathscr{M}_\varepsilon u \in C^\infty(\mathbb{R}^n)$;
2. If $u \in L_p(\Omega)$, then $\mathscr{M}_\varepsilon u \to u$ in $L_p(\Omega)$ and $\|\mathscr{M}_\varepsilon u\|_{L_p(\mathbb{R}^n)} \leq \|u\|_{L_p(\Omega)}$;
3. If ω is a bounded domain $\bar{\omega} \subset \Omega$, then for sufficiently small ε

$$D^\alpha \mathscr{M}_\varepsilon u = \mathscr{M}_\varepsilon D^\alpha u,$$

in ω. Hence, for $u \in L_p^l(\Omega)$,

$$D^\alpha \mathscr{M}_\varepsilon u \to D^\alpha u \quad \text{in } L_p(\omega),\ |\alpha| \leq l.$$

The properties of a mollification enable us to prove easily that $\|\nabla_l u\|_{L_p(\Omega)} = 0$ is equivalent to asserting that u is a polynomial of a degree not higher than $l-1$.

We now discuss a well-known property of $L_p^1(\Omega)$, $p \geq 1$. A function defined on Ω is said to be absolutely continuous on the straight line l if this function is absolutely continuous on any segment of l, contained in Ω.

Theorem 1. *Any function in $L_p^1(\Omega)$ (possibly modified on a set of zero measure m_n) is absolutely continuous on almost all straight lines that are parallel to the coordinate axes. The distributional gradient of a function in $L_p^1(\Omega)$ coincides with the usual gradient almost everywhere.*

In the proof of this assertion we use the following lemma.

Lemma. *There is a sequence $\{\eta_k\}$ of functions in $\mathscr{D}(0,1)$ such that inclusion $g \in L_1(0,1)$ and equations*

$$\int_0^1 g(t)\eta_k'(t)\, \mathrm{d}t = 0$$

for all $k = 1, 2, \ldots,$ imply that $g(t) = \mathrm{const}$ a.e. on $(0,1)$.

Proof. Let $\varDelta$ be any interval with rational endpoints such that $\bar{\varDelta} \subset (0,1)$. Let $\varPhi(\varDelta)$ denote the collection of mollifications of the characteristic function $\chi_\varDelta$ with the radii $\operatorname{dist}(\varDelta, \mathbb{R}^1 \backslash (0,1))/2^i$, $i = 1, 2, \dots$. Clearly the union $\varPhi = \bigcup_\varDelta \varPhi(\varDelta)$ is a countable subset of $\mathscr{D}(0,1)$, hence $\varPhi$ is a sequence $\varPhi = \{\varphi_k\}_{k=1}^\infty$. We observe that if $f \in L_1(0,1)$ and

$$\int_0^1 f(t)\varphi_k(t)\,\mathrm{d}t = 0$$

for all $k = 1, 2, \dots$, then

$$\int_e f(t)\,\mathrm{d}t = 0$$

for any interval $e \subset (0,1)$ and hence for any measurable subset e of $(0,1)$. Thus $f = 0$ a.e. on $(0,1)$.

Now η_k can be defined by

$$\eta_k(t) = \int_0^t \left(\varphi(s) - \alpha(s) \int_0^1 \varphi_k(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s,$$

where $\alpha \in \mathscr{D}(0,1)$ and

$$\int_0^1 \alpha(t)\,\mathrm{d}t = 1.$$

Indeed, if $g \in L_1(0,1)$ and

$$\int_0^1 g(t)\eta_k'(t)\,\mathrm{d}t = 0 \quad \text{for } k = 1, 2, \dots,$$

we have

$$\begin{aligned} 0 &= \int_0^1 g(t) \left(\varphi_k(t) - \alpha(t) \int_0^1 \varphi_k(s)\,\mathrm{d}s \right) \mathrm{d}t \\ &= \int_0^1 \left(g(t) - \int_0^1 g(s)\alpha(s)\,\mathrm{d}s \right) \varphi_k(t)\,\mathrm{d}t. \end{aligned}$$

Therefore

$$g(t) = \int_0^1 g(s)\alpha(s)\,\mathrm{d}s \quad \text{a.e. on } (0,1).$$

□

For *the proof of Theorem* 1.1.3/1 it suffices to assume that $\Omega = \{x : 0 < x_i < 1, 1 \le i \le n\}$. Let $x' = (x_1, \dots, x_{n-1})$. By Fubini's theorem

$$\int_0^1 \left| \frac{\partial u}{\partial t}(x', t) \right| \mathrm{d}t < \infty \quad \text{for almost all } x' \in \omega,$$

where $\partial u / \partial t$ is the distributional derivative. Therefore, the function

$$x \mapsto v(x) = \int_0^{x_n} \frac{\partial u}{\partial t}(x', t)\,\mathrm{d}t$$

is absolutely continuous on the segment $[0, 1]$ for almost all $x' \in \omega$ and its classical derivative coincides with $\partial u/\partial x_n$ for almost all $x_n \in (0, 1)$.

Let ζ be an arbitrary function in $\mathscr{D}(\omega)$ and let $\{\eta_k\}$ be the sequence from the above lemma. After integration by parts we obtain

$$\int_0^1 v(x', t)\eta_k'(t)\,\mathrm{d}t = -\int_0^1 \eta_k(t)\frac{\partial v}{\partial t}(x', t)\,\mathrm{d}t, \quad k = 1, 2, \ldots .$$

Multiplying both sides of the last equality by $\zeta(x')$ and integrating over ω, we obtain

$$\int_\Omega v(x)\eta_k'(x_n)\zeta(x')\,\mathrm{d}x = -\int_\Omega \eta_k(x_n)\zeta(x')\frac{\partial v}{\partial x_n}\,\mathrm{d}x.$$

By the definition of distributional derivative,

$$\int_\Omega u(x)\eta_k'(x_n)\zeta(x')\,\mathrm{d}x = -\int_\Omega \eta_k(x_n)\zeta(x')\frac{\partial v}{\partial x_n}\,\mathrm{d}x.$$

Hence the left-hand sides of the two last identities are equal. Since $\zeta \in \mathscr{D}(\omega)$ is arbitrary, we have for almost all $x' \in \omega$

$$\int_0^1 \big[u(x', x_n) - v(x', x_n)\big]\eta_k'(x_n)\,\mathrm{d}x_n = 0, \quad k = 1, 2, \ldots .$$

By the Lemma, for the same $x' \in \omega$ the difference $u(x', x_n) - v(x', x_n)$ does not depend on x_n. In other words, for almost any fixed $x' \in \omega$

$$u(x) = \int_0^{x_n} \frac{\partial u}{\partial t}(x', t)\,\mathrm{d}t + \mathrm{const},$$

which completes the proof. □

The converse assertion is contained in the following theorem.

Theorem 2. *If a function u defined on Ω is absolutely continuous on almost all straight lines that are parallel to coordinate axes and the first classical derivatives of u belong to $L_p(\Omega)$. Then these derivatives coincide with the corresponding distributional derivatives, and hence $u \in L_p^1(\Omega)$.*

Proof. Let v_j be the classical derivative of u with respect to x_j and let $\eta \in \mathscr{D}(\Omega)$. After integration by parts we obtain

$$\int_\Omega \eta v_j\,\mathrm{d}x = -\int_\Omega \frac{\partial \eta}{\partial x_j} u\,\mathrm{d}x,$$

which shows that v_j is the distributional derivative of u with respect to x_j. □

1.1.4 Spaces $W_p^l(\Omega)$ and $V_p^l(\Omega)$

We introduce the spaces

$$W_p^l(\Omega) = L_p^l(\Omega) \cap L_p(\Omega) \quad \text{and} \quad V_p^l(\Omega) = \bigcap_{k=0}^{l} L_p^k(\Omega),$$

equipped with the norms

$$\|u\|_{W_p^l(\Omega)} = \|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)},$$
$$\|u\|_{V_p^l(\Omega)} = \sum_{k=0}^{l} \|\nabla_k u\|_{L_p(\Omega)}.$$

We present here two examples of domains which show that, in general, the spaces $L_p^l(\Omega)$, $W_p^l(\Omega)$, and $V_p^l(\Omega)$ may be nonisomorphic if $\partial\Omega$ is not sufficiently regular.

In his paper of 1933 Nikodým [637] studied functions with a finite Dirichlet integral. There he gave an example of a domain for which $W_2^1(\Omega) \neq L_2^1(\Omega)$.

Example 1. The domain Ω considered by Nikodým is the union of the rectangles (cf. Fig. 2)

$$A_m = \{(x,y) : 2^{1-m} - 2^{-1-m} < x < 2^{1-m},\ 2/3 < y < 1\},$$
$$B_m = \{(x,y) : 2^{1-m} - \varepsilon_m < x < 2^{1-m},\ 1/3 \le y \le 2/3\},$$
$$C = \{(x,y) : 0 < x < 1,\ 0 < y < 1/3\},$$

where $\varepsilon_m \in (0, 2^{-m-1})$ and $m = 1, 2, \ldots$.

Positive numbers α_m are chosen so that the series

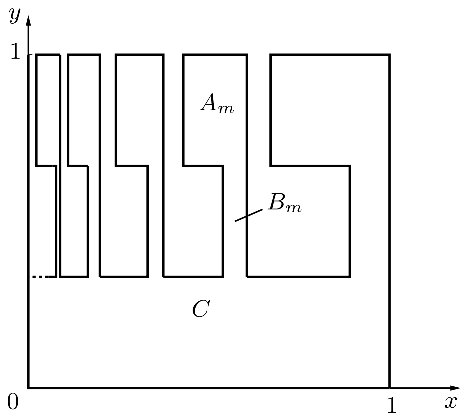

Fig. 2.

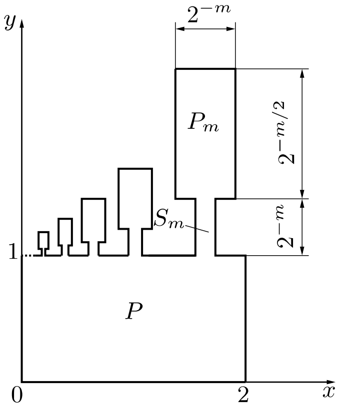

Fig. 3.

$$\sum_{m=1}^{\infty} \alpha_m^2 m_2(A_m), \tag{1.1.2}$$

diverges. Let u be a continuous function on Ω that is equal to α_m on A_m, zero on C, and linear on B_m. Since the series (1.1.2) diverges, u does not belong to $L_2(\Omega)$. On the other hand, the numbers ε_m can be chosen to be so small that the Dirichlet integral

$$\sum_{m=1}^{\infty} \iint_{B_m} \left(\frac{\partial u}{\partial y}\right)^2 \mathrm{d}x\,\mathrm{d}y,$$

converges.

Example 2. The spaces $W_2^2(\Omega)$ and $V_2^2(\Omega)$ do not coincide for the domain shown in Fig. 3. Let

$$u(x,y) = \begin{cases} 0 & \text{on } P, \\ 4^m (y-1)^2 & \text{on } S_m \ (m = 1,2,\ldots), \\ 2^{m+1}(y-1) - 1 & \text{on } P_m \ (m = 1,2,\ldots). \end{cases}$$

We can easily check that

$$\iint_{S_m} (\nabla_2 u)^2 \,\mathrm{d}x\,\mathrm{d}y = 2^{2-m},$$

$$\iint_{S_m} u^2 \,\mathrm{d}x\,\mathrm{d}y = 2^{-5m},$$

$$\iint_{P_m} u^2 \,\mathrm{d}x\,\mathrm{d}y \sim 2^{-m/2},$$

$$\iint_{S_m} (\nabla u)^2 \,\mathrm{d}x\,\mathrm{d}y \sim 2^{-3m},$$

$$\iint_{P_m} (\nabla u)^2 \,\mathrm{d}x\,\mathrm{d}y \sim 2^{m/2}.$$

Therefore, $\|\nabla u\|_{L_2(\Omega)} = \infty$ whereas $\|u\|_{W_2^2(\Omega)} < \infty$.

1.1.5 Approximation of Functions in Sobolev Spaces by Smooth Functions in Ω

Let $1 \le p < \infty$. The following two theorems show the possibility of approximating any function in $L_p^l(\Omega)$ and $W_p^l(\Omega)$ by smooth functions on Ω.

Theorem 1. *The space $L_p^l(\Omega) \cap C^\infty(\Omega)$ is dense in $L_p^l(\Omega)$.*

Proof. Let $\{\mathscr{B}_k\}_{k\ge1}$ be a locally finite covering of Ω by open balls $\mathscr{B}_k$ with radii r_k, $\bar{\mathscr{B}}_k \subset \Omega$, and let $\{\varphi_k\}_{k\ge1}$ be a partition of unity subordinate to this covering. Let $u \in L_p^l(\Omega)$ and let $\{\varrho_k\}$ be a sequence of positive numbers which monotonically tends to zero so that the sequence of balls $\{(1+\varrho_k)\mathscr{B}_k\}$ has the same properties as $\{\mathscr{B}_k\}$. If $\mathscr{B}_k = B_\varrho(x)$, then by definition we put $c\mathscr{B}_k = B_{c\varrho}(x)$. Let w_k denote the mollification of $u_k = \varphi_k u$ with radius $\varrho_k r_k$. Clearly, $w = \sum w_k$ belongs to $C^\infty(\Omega)$. We take $\varepsilon \in (0, 1/2)$ and choose ϱ_k to satisfy

$$\|u_k - w_k\|_{L_p^l(\Omega)} \le \varepsilon^k.$$

On any bounded open set ω, $\bar\omega \subset \Omega$, we have

$$u = \sum u_k,$$

where the sum contains a finite number of terms. Hence,

$$\|u - w\|_{L_p^l(\Omega)} \le \sum \|u_k - w_k\|_{L_p^l(\Omega)} \le \varepsilon(1-\varepsilon)^{-1}.$$

Therefore, $w \in L_p^l(\Omega) \cap C^\infty(\Omega)$ and

$$\|u - w\|_{L_p^l(\Omega)} \le 2\varepsilon.$$

The theorem is proved. □

The next theorem is proved similarly.

Theorem 2. *The space $W_p^l(\Omega) \cap C^\infty(\Omega)$ is dense in $W_p^l(\Omega)$ and the space $V_p^l(\Omega) \cap C^\infty(\Omega)$ is dense in $V_p^l(\Omega)$.*

Remark. It follows from the proof of Theorem 1 that the space $L_p^l(\Omega) \cap C^\infty(\Omega) \cap C(\bar\Omega)$ is dense in $L_p^l(\Omega) \cap C(\bar\Omega)$ if Ω has a compact closure. The same is true if L_p^l is replaced by W_p^l or by V_p^l.

In fact, let ϱ_k be such that

$$\|u_k - w_k\|_{C(\bar\Omega)} \le \varepsilon^k.$$

We put

$$\mathscr{V}_N = \sum_{k=1}^{N} w_k + \sum_{k=N+1}^{\infty} u_k.$$

Then

$$\sup_{x\in\Omega} \big|w(x) - \mathscr{V}_N(x)\big| \le \sum_{k=N+1}^{\infty} \|u_k - w_k\|_{C(\bar\Omega)} \le 2\,\varepsilon^{N+1},$$

and hence $w \in C(\bar\Omega)$ since w is the limit of a sequence in $C(\bar\Omega)$. On the other hand,

$$\|u - w\|_{C(\bar\Omega)} \le \sum_{k=1}^{\infty} \|u_k - w_k\|_{C(\bar\Omega)} \le 2\,\varepsilon,$$

which completes the proof. □

1.1.6 Approximation of Functions in Sobolev Spaces by Functions in $C^\infty(\bar\Omega)$

We consider a domain $\Omega \subset \mathbb{R}^2$ for which $C^\infty(\Omega)$ cannot be replaced by $C^\infty(\bar\Omega)$ in Theorems 1.1.5/1 and 1.1.5/2. We introduce polar coordinates (ϱ, θ) with $0 \le \theta < 2\pi$. The boundary of the domain $\Omega = \{(\varrho, \theta) : 1 < \varrho < 2, 0 < \theta < 2\pi\}$ consists of the circles $\varrho = 1$, $\varrho = 2$, and the interval $\{(\varrho, \theta) : 1 < \varrho < 2,\ \theta = 0\}$. The function $u = \theta$ is integrable on Ω along with all its derivatives, but it is not absolutely continuous on segments of straight lines $x = \text{const} > 0$, which intersect Ω. According to Theorem 1.1.3/1 the function u does not belong to $L^l_p(\Omega_1)$, where Ω_1 is the annulus $\Omega = \{(\varrho, \theta) : 1 < \varrho < 2, 0 \le \theta < 2\pi\}$. Hence, the derivatives of this function cannot be approximated in the mean by functions in $C^\infty(\bar\Omega)$.

A necessary and sufficient condition for the density of $C^\infty(\bar\Omega)$ in Sobolev spaces is unknown. The following two theorems contain simple sufficient conditions.

Definition. A domain $\Omega \subset \mathbb{R}^n$ is called *starshaped with respect to a point* O if any ray with origin O has a unique common point with $\partial\Omega$.

Theorem 1. *Let $1 \le p < \infty$. If Ω is a bounded domain, starshaped with respect to a point, then $C^\infty(\bar\Omega)$ is dense in $W^l_p(\Omega)$ and $V^l_p(\Omega)$, $p \in [1, \infty)$. The same is true for the space $L^l_p(\Omega)$, i.e., for any $u \in L^l_p(\Omega)$ there is a sequence $\{u_i\}_{i\ge 1}$ of functions in $C^\infty(\bar\Omega)$ such that*

$$u_i \to u \quad \textit{in } L_p(\Omega, \text{loc}) \qquad \textit{and} \qquad \big\|\nabla_l(u_i \to u)\big\|_{L_p(\Omega)} \to 0.$$

Proof. Let $u \in W^l_p(\Omega)$. We may assume that Ω is starshaped with respect to the origin. We introduce the notation $u_\tau(x) = u(\tau x)$ for $\tau \in (0, 1)$. We can easily see that $\|u - u_\tau\|_{L_p(\Omega)} \to 0$ as $\tau \to 1$.

From the definition of the distributional derivative it follows that $D^\alpha(u_\tau) = \tau^l(D^\alpha u)_\tau$, $|\alpha| = l$. Hence $u_\tau \in W_p^l(\tau^{-1}\Omega)$ and

$$\begin{aligned}\|D^\alpha(u-u_\tau)\|_{L_p(\Omega)} &\le \big\|\big(D^\alpha u\big)_\tau - D^\alpha(u_\tau)\big\|_{L_p(\Omega)} + \big\|D^\alpha u - \big(D^\alpha u\big)_\tau\big\|_{L_p(\Omega)} \\ &\le \big(1-\tau^l\big)\big\|\big(D^\alpha u\big)_\tau\big\|_{L_p(\Omega)} + \big\|D^\alpha u - \big(D^\alpha u\big)_\tau\big\|_{L_p(\Omega)}.\end{aligned}$$

The right-hand side tends to zero as $\tau \to 1$. Therefore, $u_\tau \to u$ in $W_p^l(\Omega)$.

Since $\bar\Omega \subset \tau^{-1}\Omega$, the sequence of mollifications of u_τ converges to u_τ in $W_p^l(\Omega)$. Now, using the diagonalization process, we can construct a sequence of functions in $C^\infty(\bar\Omega)$ that approximates u in $W_p^l(\Omega)$. Thus we proved the density of $C^\infty(\bar\Omega)$ in $W_p^l(\Omega)$. The spaces $L_p^l(\Omega)$ and $V_p^l(\Omega)$ can be considered in an analogous manner.

Theorem 2. *Let $1 \le p < \infty$. Let Ω be a domain with compact closure of the class C. This means that every $x \in \partial\Omega$ has a neighborhood $\mathscr{U}$ such that $\Omega \cap \mathscr{U}$ has the representation $x_n < f(x_1, \dots, x_{n-1})$ in some system of Cartesian coordinates with a continuous function f. Then $C^\infty(\bar\Omega)$ is dense in $W_p^l(\Omega)$, $V_p^l(\Omega)$, and $L_p^l(\Omega)$.*

Proof. We limit consideration to the space $V_p^l(\Omega)$. By Theorem 1.1.5/2 we may assume that $u \in C^\infty(\Omega) \cap V_p^l(\Omega)$.

Let $\{\mathscr{U}\}$ be a small covering of $\partial\Omega$ such that $\mathscr{U} \cap \partial\Omega$ has an explicit representation in Cartesian coordinates and let $\{\eta\}$ be a smooth partition of unity subordinate to this covering. It is sufficient to construct the required approximation for $u\eta$.

We may specify Ω by

$$\Omega = \big\{x = (x', x_n) : x' \in G,\ 0 < x_n < f(x')\big\},$$

where $G \subset \mathbb{R}^{n-1}$ and $f \in C(\bar G)$, $f > 0$ on G. Also we may assume that u has a compact support in $\Omega \cup \{x : x' \in G, x_n = f(x')\}$.

Let ε denote any sufficiently small positive number. Obviously, $u_\varepsilon(x) = u(x', x_n - \varepsilon)$ is smooth on $\bar\Omega$. It is also clear that for any multi-index α, $0 \le |\alpha| \le l$, we have

$$\|D^\alpha(u_\epsilon - u)\|_{L_p(\Omega)} = \big\|\big(D^\alpha u\big)_\varepsilon - D^\alpha u\big\|_{L_p(\Omega)} \to 0$$

as $\varepsilon \to +0$. The result follows. □

Remark. The domain Ω, considered at the beginning of this section, for which $C^\infty(\bar\Omega)$ is not dense in Sobolev spaces, has the property $\partial\Omega \ne \partial\bar\Omega$. We might be tempted to suppose that the equality $\partial\Omega = \partial\bar\Omega$ provides the density of $C^\infty(\bar\Omega)$ in $L_p^l(\Omega)$. The following example shows that this conjecture is not true.

Example. We shall prove the existence of a bounded domain $\Omega \subset \mathbb{R}^n$ such that $\partial\Omega = \partial\bar\Omega$ and $L_p^1(\Omega) \cap C(\bar\Omega)$ is not dense in $L_p^1(\Omega)$.

We start with the case $n = 2$. Let K be a closed nowhere dense subset of the segment $[-1, 1]$ and let $\{B_i\}$ be a sequence of open disks constructed on adjacent intervals of K taken as their diameters. Let B be the disk $x^2+y^2 < 4$ and let $\Omega = B\backslash\overline{\cup B_i}$. We can choose K so that the linear measure of $\Gamma = \{x \in K : |x| < 1/2\}$ is positive. Consider the characteristic function θ of the upper halfplane $y > 0$ and a function $\eta \in C_0^\infty(-1, 1)$ which is equal to unity on $(-1/2, 1/2)$.

The function U, defined by

$$U(x, y) = \eta(x)\theta(x, y),$$

belongs to the space $L_p^1(B)$ for all $p \geq 1$. Suppose that $u_j \to U$ in $L_p^1(\Omega)$, where $\{u_j\}_{j\geq 1}$ is a sequence of functions in $C(\bar{\Omega}) \cap L_p^1(\Omega)$. According to our assumption, for almost all $x \in \Gamma$ and for all $\delta \in (0, 1/2)$,

$$u_j(x, \delta) - u_j(x, -\delta) = \int_{-\delta}^{\delta} \frac{\partial u_j(x, y)}{\partial y}\, \mathrm{d}y.$$

Hence

$$\int_\Gamma \big|u_j(x, \delta) - u_j(x, -\delta)\big|\, \mathrm{d}x \leq \iint_{\Gamma(\delta)} \big|\operatorname{grad} u_j(x, y)\big|\, \mathrm{d}x\, \mathrm{d}y,$$

where $\Gamma(\delta) = \Gamma \times (-\delta, \delta)$.

Since $u_j \to U$ in $L_1^1(\Omega)$, the integrals

$$\iint_{\Gamma(\delta)} \big|\operatorname{grad} u_j(x, y)\big|\, \mathrm{d}x\, \mathrm{d}y, \quad j \geq 1,$$

are uniformly small. Therefore, for each $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$

$$\int_\Gamma \big|u_j(x, \delta) - u_j(x, -\delta)\big|\, \mathrm{d}x < \varepsilon.$$

Applying Fubini's theorem, we obtain that the sequence in the left-hand side converges to

$$\int_\Gamma \big|U(x, \delta) - U(x, -\delta)\big|\, \mathrm{d}x = m_1(\Gamma)$$

as $j \to \infty$ for almost all small δ. Hence $m_1(\Gamma) \leq \varepsilon$ which contradicts the positiveness of $m_1(\Gamma)$. Since $\partial\Omega = \partial\bar{\Omega}$, the required counterexample has been constructed for $n = 2$.

In the case $n > 2$, let Ω_2 denote the plane domain considered previously, put $\Omega = \Omega_2 \times (0, 1)^{n-2}$, and duplicate the above argument.

1.1.7 Transformation of Coordinates in Norms of Sobolev Spaces

Let H and G be domains in $\mathbb{R}^n$ and let

$$T : y \to x(y) = \big(x_1(y), \dots, x_n(y)\big),$$

be a homeomorphic map of H onto G.

We say that T is a *quasi-isometric map*, if for any $y_0 \in H, x_0 \in G$,

$$\limsup_{y \to y_0} \frac{|x(y) - x(y_0)|}{|y - y_0|} \le L, \qquad \limsup_{x \to x_0} \frac{|y(x) - y(x_0)|}{|x - x_0|} \le L, \tag{1.1.3}$$

and the Jacobian $\det x'(y)$ preserves its sign in H.

We can check that the estimates (1.1.3) are equivalent to

$$\|x'(y)\| \le L \quad \text{a.e. on } H, \qquad \|y'(x)\| \le L \quad \text{a.e. on } G,$$

where x', y' are the Jacobi matrices of the mappings $y \to x(y)$, $x \to y(x)$ and $\|\cdot\|$ is the norm of the matrix. This immediately implies that the quasi-isometric map satisfies the inequalities

$$L^{-n} \le \big|\det x'(y)\big| \le L^n. \tag{1.1.4}$$

By definition, the map T belongs to the class $C^{l-1,1}(\bar{H})$, $l \ge 1$, if the functions $y \to x_i(y)$ belong to the class $C^{l-1,1}(\bar{H})$. It is easy to show that if T is a quasi-isometric map of the class $C^{l-1,1}(\bar{H})$, then T^{-1} is of the class $C^{l-1,1}(\bar{G})$.

Theorem. *Let T be a quasi-isometric map of the class $C^{l-1,1}(\bar{H})$, $l \ge 1$, which maps H onto G. Let $u \in V_p^l(G)$ and $v(y) = u(x(y))$. Then $v \in V_p^l(H)$ and for almost all $y \in H$ the derivatives $D^\alpha v(y), |\alpha| \le l$ exist and are expressed by the classical formula*

$$D^\alpha v(y) = \sum_{1 \le |\beta| \le |\alpha|} \varphi_\beta^\alpha(y) \big(D^\beta u\big)\big(x(y)\big). \tag{1.1.5}$$

Here

$$\varphi_\beta^\alpha(y) = \sum_s c_s \prod_{i=1}^n \prod_j \big(D^{s_{ij}} x_i\big)(y),$$

and the summation is taken over all multi-indices $s = (s_{ij})$ satisfying the conditions

$$\sum_{i,j} s_{i,j} = \alpha, \quad |s_{ij}| \ge 1, \qquad \sum_{i,j} \big(|s_{ij}| - 1\big) = |\alpha| - |\beta|.$$

Moreover, the norms $\|v\|_{V_p^l(H)}$ and $\|u\|_{V_p^l(G)}$ are equivalent.

Proof. Let $u \in C^\infty(G) \cap V_p^l(G)$. Then v is absolutely continuous on almost all straight lines that are parallel to coordinate axes. The first partial derivatives of v are expressed by the formula

$$\frac{\partial v(y)}{\partial y_m} = \sum_{i=1}^{n} \frac{\partial x_i(y)}{\partial y_m} \left(\frac{\partial u}{\partial x_i} \right) (x(y)), \tag{1.1.6}$$

for almost all y. Since

$$\|\nabla v\|_{L_p(H)} \leq c \|\nabla u\|_{L_p(G)},$$

it follows by Theorem 1.1.3/2 that $v \in V_p^1(H)$. After the approximation of an arbitrary $u \in V_p^l(G)$ by functions in $C^\infty(G) \cap V_p^1(G)$ (cf. Theorem 1.1.5/2) the result follows in the case $l = 1$.

For $l > 1$ we use induction. Let (1.1.5) hold for $|\alpha| = l - 1$. Since $D^\beta u \in V_p^1(G)$, the functions $y \to (D^\beta u)(x(y))$ belong to the space $V_p^l(H)$. This and the inclusion $\varphi_\beta^\alpha \in C^{0,1}(\bar{H})$ imply that each term on the right-hand side of (1.1.4) with $|\alpha| = l - 1$ belongs to $V_p^1(H)$. Applying (1.1.6) to (1.1.5) with $|\alpha| = l$, we obtain

$$\|\nabla_l v\|_{L_p(H)} \leq c \|u\|_{V_p^l(G)}.$$

The result follows. □

1.1.8 Domains Starshaped with Respect to a Ball

Definition. Ω is *starshaped with respect to a ball* contained in Ω if Ω is starshaped with respect to each point of this ball.

Lemma. *Let Ω be a bounded domain starshaped with respect to a ball B_ϱ with radius ϱ and with center at the origin of spherical coordinates (r, ω). If $\partial\Omega$ has a representation $r = r(\omega)$, then $r(\omega)$ satisfies a Lipschitz condition.*

Proof. We show that for all $x, y \in \partial\Omega$ with

$$|\omega_x - \omega_y| < 1, \tag{1.1.7}$$

the inequality

$$|x - y| \leq 2\mathbf{D}^2 \varrho^{-1} |\omega_x - \omega_y|$$

holds where $\mathbf{D}$ is the diameter of Ω.

The inequality (1.1.7) means that the angle φ between the vectors x and y is less than $\pi/3$. We shall show that the straight line l, passing through the points x, y, cannot intersect the ball $B_{\varrho/2}$.

In fact, if there exists a point $z \in l \cap B_{\varrho/2}$, then z belongs to the segment xy since Ω is starshaped with respect to z. Consider the triangles Oxz, Oyz. The inequalities $|x| \geq \varrho$, $|y| \geq \varrho$, $|z| \leq \varrho/2$ imply $|z| \leq |y - z|, |z| \leq |x - z|$. Hence $\sphericalangle Oxz \leq \pi/3$, $\sphericalangle Oyz \leq \pi/3$, and $\varphi = \pi - \sphericalangle Oxz - \sphericalangle Oyz \geq \pi/3$, which contradicts (1.1.7).

The distance from the origin O to the line l is $|x||y||x - y|^{-1} \sin\varphi$ which is less than $\varrho/2$ since $l \cap B_{\varrho/2} \neq \varnothing$. Therefore

$$|x - y| \leq 2\varrho^{-1} |x||y| \sin\varphi \leq 4\varrho^{-1} \mathbf{D}^2 \sin(\varphi/2) = 2\varrho^{-1} \mathbf{D}^2 |\, \omega_x - \omega_y|,$$

and the result follows. □

Remark. It is easy to see that the converse assertion also holds. Namely, if Ω is a bounded domain and $\partial\Omega$ has the representation $r = r(\omega)$ in spherical coordinates with $r(\omega)$ satisfying a Lipschitz condition, then Ω is starshaped with respect to a ball with its center at the origin.

1.1.9 Domains of the Class $C^{0,1}$ and Domains Having the Cone Property

Definition 1. We say that a bounded domain Ω belongs to *the class* $C^{0,1}$ if each point $x \in \partial\Omega$ has a neighborhood $\mathscr{U}$ such that the set $\mathscr{U} \cap \Omega$ is represented by the inequality $x_n < f(x_1, \dots, x_{n-1})$ in some Cartesian coordinate system with function f satisfying a Lipschitz condition.

By Lemma 1.1.8 any bounded domain starshaped with respect to a ball belongs to the class $C^{0,1}$.

Definition 2. A domain Ω possesses *the cone property* if each point of Ω is the vertex of a cone contained in Ω along with its closure, the cone being represented by the inequalities $x_1^2 + \dots + x_{n-1}^2 < b\,x_n^2$, $0 < x_n < a$ in some Cartesian coordinate system, $a, b = \text{const}$.

Remark 1. It is easy to show that bounded domains of the class $C^{0,1}$ have the cone property. The example of a ball with deleted center shows that the converse assertion is not true.

Lemma 1. *Let Ω be a bounded domain having the cone property. Then Ω is the union of a finite number of domains starshaped with respect to a ball.*

Since a domain having the cone property is a union of congruent cones and hence it is a union of domains starshaped with respect to balls of a fixed radius, then Lemma 1 follows immediately from the next lemma.

Lemma 2. *If a bounded domain Ω is a union of an infinite number of domains G_α starshaped with respect to balls $\mathscr{B}_\alpha \subset G_\alpha$ of a fixed radius $R > 0$, then for each $r < R$ there exists a finite number of domains Ω_k $(1 \le k \le N)$ starshaped with respect to balls of radius r, contained in Ω_k, and such that $\bigcup_k \Omega_k = \Omega$.*

Proof. Let G_1 be a domain in the collection $\{G_\alpha\}$. Consider the domain $\Omega_1 = \bigcup_\beta G_\beta$, where the union is taken over all domains G_β for which the distance between the centers of the balls $\mathscr{B}_\beta$ and $\mathscr{B}_1$ is $\varrho \le R - r$. Obviously, any of the balls $\mathscr{B}_\beta$ contain the ball C_1 of radius r concentric with $\mathscr{B}_1$. Since any G_β is starshaped with respect to C_1, then Ω_1 is starshaped with respect to C_1.

We define G_2 to be any of the domains G_α such that $G_\alpha \cap \Omega_1 = \varnothing$. Repeating the preceding construction, we define a domain Ω_2 starshaped with respect to the ball C_2 of radius r with the center situated at a distance $d > R - r$ from the center of the ball C_1. Analogously, we construct a domain Ω_3 starshaped with respect to a ball C_3 of radius r with the center situated at a distance $d > R - r$ from the centers of the balls C_1 and C_2, and so on.

Clearly this process will stop after a finite number of steps since the centers of the balls $C_1, C_2, \dots,$ are contained in a bounded domain and the distance between centers is more than $R - r > 0$. The result follows.

Remark 2. Domains of the class $C^{0,1}$ are sometimes called domains having *the strong Lipschitz property*, whereas *Lipschitz domains* are defined as follows.

Definition 3. A bounded domain Ω is called a *Lipschitz domain* if each point of its boundary has a neighborhood $\mathscr{U} \subset \mathbb{R}^n$ such that a quasi-isometric transformation maps $\mathscr{U} \cap \Omega$ onto a cube.

Clearly, domains of the class $C^{0,1}$ are Lipschitz domains. The following example shows that the converse is not true, i.e., a Lipschitz domain may not have a strong Lipschitz property. Moreover, the Lipschitz domain considered in the next example fails to have the cone property (cf. Remark 1).

Example. Let $\Omega \subset \mathbb{R}^2$ be the union of the rectangles $P_k = \{x : |x_1 - 2^{-k}| < 2^{-k-2},\ 0 \le x_2 < 2^{-k-2}\}$, $k = 1, 2, \dots,$ and the square $Q = \{x : 0 < x_1 < 1,\ -1 < x_2 < 0\}$. Obviously, Ω does not have the cone property. We shall show that Ω can be mapped onto the square Q by a quasi-isometric map.

We can easily check that the mapping $T_0 : x \to y = (y_1, y_2)$, being the identity on Q and defined on P_k by

$$y_1 = \big(x_1 - 2^{-k}\big)\big(1 - 2^k x_2\big) + 2^{-k}, \qquad y_2 = x_2,$$

is quasi-isometric. The image $T_0\Omega$ is the union of the square $T_0 Q$ and the set $\{y : 0 < y_1 < 1, 0 \le y_2 < f(y_1)\}$ with f satisfying the Lipschitz condition with the constant 4 and $0 \le f(y_1) \le 1/8$ (cf. Fig. 4).

Let η be a piecewise linear function, $\eta = 1$ for $y > 0$ and $\eta = 0$ for $y < -1$. The Lipschitz transformation $T_1 : y \to z$, defined by

$$z_1 = y_1, \qquad z_2 = y_2 - f(y_1)\eta(y_2),$$

maps $T_0\Omega$ onto the square $\{z : 0 < z_1 < 1, -1 < z_2 < 0\}$. The Jacobian of T_1 is greater than $1/2$; therefore T_1 is a quasi-isometric mapping.

Thus, Ω is mapped onto Q by the quasi-isometric mapping $T_0 T_1$.

1.1.10 Sobolev Integral Representation

Theorem 1. *Let Ω be a bounded domain starshaped with respect to a ball B_δ, $B_\delta \subset \Omega$, and let $u \in L^l_p(\Omega)$. Then for almost all $x \in \Omega$*

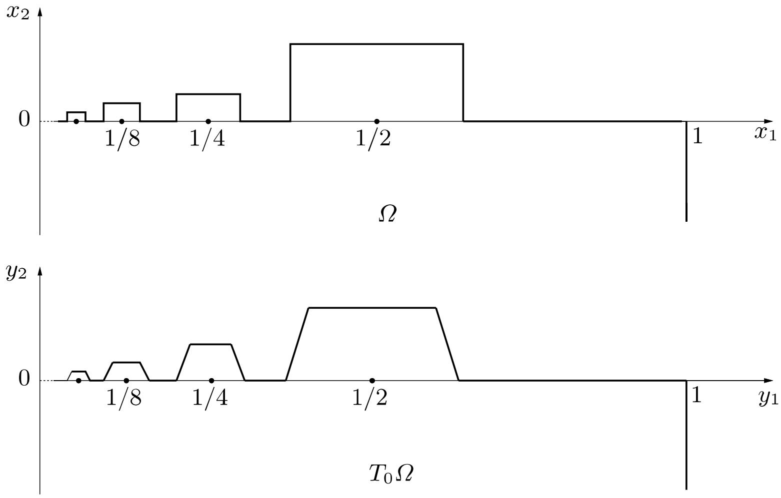

Fig. 4.

$$u(x) = \delta^{-n} \sum_{|\beta|<l} \left(\frac{x}{\delta}\right)^{\beta} \int_{B_\delta} \varphi_\beta\left(\frac{y}{\delta}\right) u(y)\, \mathrm{d}y + \sum_{|\alpha|=l} \int_{\Omega} \frac{f_\alpha(x; r, \theta)}{r^{n-1}} D^\alpha u(y)\, \mathrm{d}y, \tag{1.1.8}$$

where $r = |y-x|$, $\theta = (y-x)r^{-1}$, $\varphi_\beta \in \mathscr{D}(B_1)$, f_α *are infinitely differentiable functions in* x, r, θ *such that*

$$|f_\alpha| \le c\,(\mathbf{D}/\delta)^{n-1},$$

where c *is a constant that is independent of* Ω *and* $\mathbf{D}$ *is the diameter of* Ω.

Proof. It suffices to put $\delta = 1$. Let $\varphi \in \mathscr{D}(B_1)$ and

$$\int_{B_1} \varphi(z)\, \mathrm{d}z = 1.$$

First we assume that $u \in C^\infty(\bar{\Omega})$. If $x \in \Omega, z \in B_1$, the line segment $[z, x]$ is contained in Ω and so the following Taylor's formula applies:

$$u(x) = \sum_{|\beta|<l} \frac{D^\beta u(z)}{\beta!}(x-z)^\beta + l \int_0^1 (1-t)^{l-1} \sum_{|\alpha|=l} \frac{1}{\alpha!} D^\alpha u\big(z+t(x-z)\big)(x-z)^\alpha\, \mathrm{d}t.$$

Multiplying this equality by $\varphi(z)$ and integrating over $z \in B_1$, we obtain

$$\begin{aligned} u(x) = &\sum_{|\beta|<l} \int_{B_1} \frac{D^\beta u(z)}{\beta!}(x-z)^\beta \varphi(z)\,\mathrm{d}z \\ &+ l \sum_{|\alpha|=l} \frac{1}{\alpha!} \int_0^1 \mathrm{d}t \\ &\times \int_{B_1} (1-t)^{l-1} D^\alpha u\big(z+t(x-z)\big)(x-z)^\alpha \varphi(z)\,\mathrm{d}z. \end{aligned} \tag{1.1.9}$$

Since

$$\begin{aligned} &\int_{B_1} D^\beta u(z)(x-z)^\beta \varphi(z)\,\mathrm{d}z \\ &\quad = (-1)^{|\beta|} \int_{B_1} u(z) D_z^\beta\big((x-z)^\beta \varphi(z)\big)\,\mathrm{d}z, \quad |\beta|<l, \end{aligned}$$

the former of the last two sums in (1.1.9) is a polynomial of degree $l-1$ in $\mathbb{R}^n$, which can be written in the form of the first term on the right-hand side of (1.1.8). We extend $D^\alpha u$ to be zero outside Ω for $|\alpha| = l$ and perform the change of variable $y = z + t(x-z)$ in the last integral over B_1 in (1.1.9). Then

$$x - z = (1-t)^{-1}(x-y), \qquad \mathrm{d}z = (1-t)^{-n}\,\mathrm{d}y,$$

and the general term of the second sum in (1.1.9) is

$$\frac{1}{\alpha!} \int_{\mathbb{R}^n} \mathrm{d}y \int_0^1 D^\alpha u(y)(x-y)^\alpha \varphi\Big(\frac{y-tx}{1-t}\Big) \frac{\mathrm{d}t}{(1-t)^{n+1}}.$$

The last expression can be written as

$$\frac{1}{\alpha!} \int_{\mathbb{R}^n} D^\alpha u(y)(x-y)^\alpha K(x,y)\,\mathrm{d}y,$$

with

$$K(x,y) = \int_0^1 \varphi\Big(\frac{y-tx}{1-t}\Big) \frac{\mathrm{d}t}{(1-t)^{n+1}} = r^{-n} \int_r^\infty \varphi(x+\varrho\theta)\varrho^{n-1}\,\mathrm{d}\varrho,$$

where $r = |x-y|$ and $\theta = (y-x)/|x-y|$. Here we have made the change of variables

$$\frac{y-tx}{1-t} = x + \varrho\theta,$$

that is, $y - x = (1-t)\varrho\theta$ and $r = (1-t)\varrho$.

Note that the function $\mathbb{R}^n \setminus \{y\} \ni x \mapsto K(x,y)$ is in $C^\infty(\mathbb{R}^n \setminus \{y\})$ for any fixed $y \in \mathbb{R}^n$. Moreover, $K(x,y) = 0$ if the line segment $[x,y]$ is not contained in the cone $V_x = \bigcup_{z\in B_1}[x,z]$. Thus, formula (1.1.8) holds where

$$f_\alpha(x,r,\theta) = \frac{(-1)^l l}{\alpha!} \theta^\alpha \int_r^\infty \varphi(x+\varrho\theta)\varrho^{n-1}\,\mathrm{d}\varrho.$$

To check the estimate $|f_\alpha| \le c\,\mathbf{D}^{n-1}$, we observe that

$$\begin{aligned}\big|f_\alpha(x,r,\theta)\big| &\le \frac{l}{\alpha!}\int_r^{\mathbf{D}} \big|\varphi(x+\varrho\theta)\big|\varrho^{n-1}\,\mathrm{d}\varrho\\ &\le \frac{l}{\alpha!}\mathbf{D}^{n-1}\int_{x+\varrho\theta\in B_1}\big|\varphi(x+\varrho\theta)\big|\,\mathrm{d}\varrho.\end{aligned}$$

The last integral is dominated by $2\sup\{|\varphi(z)| : z\in B_1\}$, which gives the required estimate.

The representation (1.1.8) has been proved for $u\in C^\infty(\bar\Omega)$. Suppose $u\in L_p^l(\Omega)$ and $p\in[1,\infty)$. By Theorem 1.1.6/1, there is a sequence $\{u_i\}_{i\ge1}$ such that

$$u_i\in C^\infty(\bar\Omega), u_i\to u \quad \text{in } L_p(\Omega,\mathrm{loc}), \quad \|\nabla_l(u_i-u)\|_{L_p(\Omega)}\to 0.$$

Passing to the limit in (1.1.8) for u_i and using the continuity of the integral operator with a weak singularity in $L_p(\Omega)$, we arrive at (1.1.8) in the general case. This completes the proof of the theorem. □

For $\Omega=\mathbb{R}^n$ we obtain a simpler integral representation of $u\in\mathscr{D}$.

Theorem 2. *If $u\in\mathscr{D}$, then*

$$u(x)=\frac{(-1)^l l}{\omega_n}\sum_{|\alpha|=l}\int_{\mathbb{R}^n}\frac{\theta^\alpha}{\alpha!}D^\alpha u(y)\frac{\mathrm{d}y}{r^{n-l}}, \tag{1.1.10}$$

where ω_n is the $(n-1)$-dimensional measure of S^{n-1}, i.e.,

$$\omega_n=\frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{1.1.11}$$

and as in Theorem 1, $r=|y-x|$, $\theta=(y-x)\,r^{-1}$.

Proof. For fixed $x\in\mathbb{R}^n$ and $\theta\in S^{n-1}$, we have

$$u(x)=\frac{(-1)^l}{(l-1)!}\int_0^\infty r^{l-1}\frac{\partial^l}{\partial r^l}u(x+r\theta)\,\mathrm{d}r.$$

Since

$$\frac{\partial^l}{\partial r^l}u(x+r\theta)=\sum_{|\alpha|=l}\frac{l!}{\alpha!}\theta^\alpha(D^\alpha u)(x+r\theta),$$

it follows that

$$u(x)=(-1)^l l\int_0^\infty r^{l-1}\sum_{|\alpha|=l}\frac{\theta^\alpha}{\alpha!}(D^\alpha u)(x+r\theta)\,\mathrm{d}r.$$

Integration with respect to θ implies (1.1.10). □

1.1.11 Generalized Poincaré Inequality

The following assertion, based on Lemma 1.1.9/1 and Theorem 1.1.10/1, will be used in Sect. 1.1.13.

Lemma. *Let Ω be a bounded domain having the cone property and let ω be an arbitrary open set, $\bar{\omega} \subset \Omega$. Then for any $u \in L_p^l(\Omega)$, $p \geq 1$, there exists a polynomial*

$$\Pi(x) = \sum_{|\alpha| \leq l-1} (u, \varphi_\alpha) x^\alpha, \tag{1.1.12}$$

such that

$$\sum_{k=0}^{l-1} \big\| \nabla_k (u - \Pi) \big\|_{L_p(\Omega)} \leq C \|\nabla_l u\|_{L_p(\Omega)}. \tag{1.1.13}$$

Here $\varphi_\alpha \in \mathscr{D}(\omega)$ and C is a constant independent of u.

Proof. Clearly, we may assume that ω is a ball.

Let G be any subdomain of Ω starshaped with respect to a ball referred to in Lemma 1.1.9/1 and let $\mathscr{B}$ be the corresponding ball. We construct a finite family of open balls $\{\mathscr{B}_i\}_{i=0}^M$ such that $\mathscr{B}_0 = \mathscr{B}$, $\mathscr{B}_i \cap \mathscr{B}_{i+1} \neq \varnothing$, $\mathscr{B}_M = \omega$. Since G is starshaped with respect to any ball contained in $\mathscr{B}_0 \cap \mathscr{B}_1$, then by the integral representation (1.1.8) and by continuity of the integral operator with the kernel $|x-y|^{l-k-n}$ in $L_p(G)$ we obtain

$$\|\nabla_k u\|_{L_p(G)} \leq C \big(\|\nabla_l u\|_{L_p(G)} + \|u\|_{L_p(\mathscr{B}_0 \cap \mathscr{B}_1)} \big), \quad 0 \leq k < l. \tag{1.1.14}$$

Also, for $i = 1, \dots, M-1$,

$$\|u\|_{L_p(\mathscr{B}_i)} \leq C \big(\|\nabla_l u\|_{L_p(\mathscr{B}_i)} + \|u\|_{L_p(\mathscr{B}_i \cap \mathscr{B}_{i+1})} \big).$$

Therefore,

$$\|\nabla_k u\|_{L_p(G)} \leq C \big(\|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\omega)} \big).$$

Summing over all G we obtain

$$\|\nabla_k u\|_{L_p(\Omega)} \leq C \big(\|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\omega)} \big). \tag{1.1.15}$$

From the integral representation (1.1.8) for a function on ω it follows that

$$\|u - \Pi\|_{L_p(\omega)} \leq C \|\nabla_l u\|_{L_p(\omega)},$$

where

$$\Pi(x) = \sum_{|\beta| < l} x^\beta \int_\omega \varphi_\beta(y) u(y) \, \mathrm{d}y, \quad \varphi_\beta \in \mathscr{D}(\omega).$$

It remains to replace u by $u - \Pi$ in (1.1.15). The Lemma is proved. □

The Lemma implies the following obvious corollary.

Corollary. *The spaces $V_p^l(\Omega)$, $W_p^l(\Omega)$, and $L_p^l(\Omega)$ coincide for Ω having the cone property.*

Remark. In this section we deliberately do not give a general formulation of the lemma for functions on domains that have the cone property. Such a formulation follows immediately from the Lemma combined with Theorem 1.4.5, which will appear later.

On the other hand, the class of domains, considered in the Lemma, is also not maximal. The statements of the Lemma and of the Corollary are true for any bounded domain which is a union of domains of the class C, defined in Theorem 1.1.6/2. The proof is essentially the same, but we must use the following simple property of domains of the class C instead of (1.1.14).

Let

$$\Omega = \big\{x : x_1^2 + \cdots + x_{n-1}^2 < \varrho^2,\ 0 < x_n < f(x_1, \dots, x_{n-1})\big\},$$

with a continuous function f defined on the ball $x_1^2 + \cdots + x_{n-1}^2 \le \varrho^2$. Let G denote the "base" of Ω, i.e., the cylinder

$$\big\{x : x_1^2 + \cdots + x_{n-1}^2 < \varrho^2,\ 0 < x_n < \min f(x_1, \dots, x_{n-1})\big\}.$$

Then, for all $u \in C^1(\bar{\Omega})$,

$$\|u\|_{L_p(\Omega)} \le C\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_p(G)}\big).$$

The last inequality follows from the elementary inequality

$$\int_0^a \big|f(t)\big|^p \,\mathrm{d}t \le c\bigg(a^p \int_0^a \big|f'(t)\big|^p \,\mathrm{d}t + \frac{a}{b}\int_0^b \big|f(t)\big|^p \,\mathrm{d}t\bigg), \tag{1.1.16}$$

where $f \in C^1[0, a]$, $0 < b < a$,k and c depends only on p.

The proof of (1.1.16) runs as follows. Let

$$\Phi_a = \bigg(a^{-1}\int_0^a \big|f(t)\big|^p \,\mathrm{d}t\bigg)^{1/p},$$

be the value of $|f|$ at some point of $(0, a)$. We have

$$|\Phi_a - \Phi_b| \le \int_0^a |f'(t)|\,\mathrm{d}t \le a^{(p-1)/p}\|f'\|_{L_p(0,a)},$$

and (1.1.16) follows.

Example. Considering the domain

$$\Omega = \big\{(x, y) \in \mathbb{R}^2 : |y| < \exp(-1/x),\ 0 < x < 1\big\},$$

and the function $u(x, y) = x^{2l}\exp(1/(p\,x))$, we may easily check that, in general, the space $L_p(\Omega)$ on the left-hand side of (1.1.13) cannot be replaced by any of the spaces $L_q(\Omega)$, $q > p$, for domains of the class C.

1.1.12 Completeness of $W_p^l(\Omega)$ and $V_p^l(\Omega)$

In the following theorem Ω is an arbitrary open subset of $\mathbb{R}^n$.

Theorem. *The spaces $W_p^l(\Omega)$ and $V_p^l(\Omega)$ are complete.*

Proof. Let $\{u_k\}_{k\geq 1}$ be a Cauchy sequence in $W_p^l(\Omega)$. Let $u_k \to u$ in $L_p(\Omega)$ and let $D^\alpha u_k \to v_\alpha$, $|\alpha| = l$ in $L_p(\Omega)$. For any $\varphi \in \mathscr{D}(\Omega)$ we have

$$\int_\Omega u D^\alpha \varphi \,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega u_k D^\alpha \varphi \,\mathrm{d}x$$
$$= (-1)^l \lim_{k\to\infty} \int_\Omega \varphi D^\alpha u_k \,\mathrm{d}x = (-1)^l \int_\Omega v_\alpha \varphi \,\mathrm{d}x.$$

Thus, $v_\alpha = D^\alpha u$ and the sequence $\{u_k\}$ converges to $u \in W_p^l(\Omega)$. The result follows for the space $W_p^l(\Omega)$. The case of $V_p^l(\Omega)$ can be considered in the same way. □

1.1.13 The Space $\dot{L}_p^l(\Omega)$ and Its Completeness

Let Ω be a domain.

Definition. $\dot{L}_p^l(\Omega)$ is the factor space $L_p^l(\Omega)/\mathscr{P}_{l-1}$, where $\mathscr{P}_k$ is the subspace of polynomials of degree not higher than k.

We equip $\dot{L}_p^l(\Omega)$ with the norm $\|\nabla_l u\|_{L_p(\Omega)}$. The elements of $\dot{L}_p^l(\Omega)$ are classes $\dot{u} = \{u + \Pi\}$ where $\Pi \in \mathscr{P}_{l-1}$, $u \in L_p^l(\Omega)$.

Theorem 1. *The space $\dot{L}_p^l(\Omega)$ is complete.*

Proof. Let $\{\dot{u}_k\}_{k\geq 1}$ be a Cauchy sequence in $\dot{L}_p^l(\Omega)$. This means that for any u_k in the class $\dot{u}_k$ and any multi-index α, $|\alpha| = l$, $D^\alpha u_k \to T_\alpha$ in $L_p(\Omega)$. We shall show that there exists a $u \in L_p^l(\Omega)$ such that $D^\alpha u = T_\alpha$.

Let B be an open ball $\bar{B} \subset \Omega$, and let $\{\omega_j\}_{j\geq 0}$ be a sequence of domains with compact closures and smooth boundaries such that

$$\bar{B} \subset \omega_0, \qquad \bar{\omega}_j \subset \omega_{j+1}, \qquad \bigcup_j \omega_j = \Omega.$$

Let Π_k be a polynomial specified by the set $\omega = B$ and by the function u_k in Lemma 1.1.11. Since u_k is in the class $\dot{u}_k$, then

$$v_k = u_k - \Pi_k$$

is in the same class. By Lemma 1.1.11, $\{v_k\}$ is a Cauchy sequence in $L_p(\omega_j)$ for any j. We denote the limit function by u. Clearly, for any $\varphi \in \mathscr{D}(\Omega)$ and any multi-index α with $|\alpha| = l$,

$$\big(u, D^\alpha\varphi\big) = \lim_{k\to\infty} \big(v_k, D^\alpha\varphi\big) = \lim_{k\to\infty} (-1)^l \big(D^\alpha u_k, \varphi\big) = (-1)^l (T_\alpha, \varphi).$$

The result follows. □

The proof of Theorem 1 also contains the following assertion.

Theorem 2. *Let $\{u_k\}$ be a sequence of functions in $L_p^l(\Omega)$ such that*

$$\left\|\nabla_l(u_k - u)\right\|_{L_p(\Omega)} \xrightarrow{k\to\infty} 0$$

for some $u \in L_p^l(\Omega)$. Then there exists a sequence of polynomials $\Pi_k \in \mathscr{P}_{l-1}$ with $u_k - \Pi_k \to u$ in $L_p^l(\Omega, \mathrm{loc})$.

1.1.14 Estimate of Intermediate Derivative and Spaces $\mathring{W}_p^l(\Omega)$ and $\mathring{L}_p^l(\Omega)$

We start with an optimal estimate of the first derivative by the L_1 norms of the function and its second derivative.

Lemma. *Let $-\infty < a < b < \infty$. For all $f \in W_1^2(a,b)$ and for every $x \in [a,b]$*

$$\left|f'(x)\right| \le \int_a^b \left|f''(t)\right| \mathrm{d}t + \frac{4}{(b-a)^2}\int_a^b \left|f(t)\right| \mathrm{d}t. \tag{1.1.17}$$

Both constants $4(a-b)^{-2}$ and 1 in front of the integrals on the right-hand side are sharp.

Proof. By dilation, (1.1.17) is equivalent to the inequality

$$\left|f'(x)\right| \le \int_{-1}^1 \left|f''(t)\right| \mathrm{d}t + \int_{-1}^1 \left|f(t)\right| \mathrm{d}t. \tag{1.1.18}$$

Integrating by parts, one checks the identity

$$f'(x) = \int_{-1}^1 \mathscr{K}(x,t) f''(t)\,\mathrm{d}t + \int_{-1}^1 (\mathrm{sign}\ t) f(t)\,\mathrm{d}t,$$

where

$$\mathscr{K}(x,t) = \begin{cases} \int_{-1}^t (1-|y|)\,\mathrm{d}y & \text{for } t < x,\\ \int_t^1 (|y|-1)\,\mathrm{d}y & \text{for } t > x.\end{cases}$$

Inequality (1.1.18) follows from

$$\int_{-1}^1 \bigl(1-|y|\bigr)\,\mathrm{d}y = 1.$$

Putting $f(t) = t$ in (1.1.18), we see that $4(b-a)^{-2}$ is the best constant in (1.1.17). To show the sharpness of the constant 1, we choose f as a smooth approximation of the function $\varepsilon^{-1}(t-1+\varepsilon)_+$, where ε is an arbitrarily small positive number. □

Remark. The just proved lemma leads directly to the inequality

$$\|f'\|^p_{L_p(a,b)} \le c(p)\big((b-a)\|f''\|^p_{L_p(a,b)} + (b-a)^{-1}\|f\|^p_{L_p(a,b)}\big), \tag{1.1.19}$$

where $p \in [1,\infty]$. In its turn, (1.1.19) implies

$$\|f'\|^p_{L_p(\mathbb{R})} \le c(p)\big(\varepsilon\|f''\|^p_{L_p(\mathbb{R})} + \varepsilon^{-1}\|f\|^p_{L_p(\mathbb{R})}\big), \tag{1.1.20}$$

with an arbitrary $\varepsilon > 0$. Clearly, (1.1.20) is equivalent to the inequality

$$\|f'\|_{L_p(\mathbb{R})} \le \big(2c(p)\big)^{1/p}\|f''\|^{1/2}_{L_p(\mathbb{R})}\|f\|^{1/2}_{L_p(\mathbb{R})}, \tag{1.1.21}$$

which can be found in Hardy, Littlewood, and Pólya [351], Sect. 7.8 (see also Lemma 1.8.1/2 in the present book).

Let, as before, Ω denote an open subset of $\mathbb{R}^n$. We conclude this section by introducing the spaces $\mathring{V}^l_p(\Omega)$ and $\mathring{W}^l_p(\Omega)$, as completions of $C_0^\infty(\Omega)$ in the norms of $V^l_p(\Omega)$ and $W^l_p(\Omega)$, $1 \le p < \infty$. In fact, these new spaces coincide, which results from the one-dimensional inequality (1.1.21).

Another space $\mathring{L}^l_p(\Omega)$, to be used frequently in what follows, is defined as the completion of $C_0^\infty(\Omega)$ in the norm $\|\nabla_l u\|_{L_p(\Omega)}$. Important properties of $\mathring{L}^l_p(\Omega)$ will be studied in Chap. 15.

1.1.15 Duals of Sobolev Spaces

Theorem 1. *Let $1 \le p < \infty$. Any linear functional on $L^l_p(\Omega)$ can be expressed as*

$$f(u) = \int_\Omega \sum_{|\alpha|=l} g_\alpha(x) D^\alpha u(x)\,\mathrm{d}x, \tag{1.1.22}$$

where $g_\alpha \in L_{p'}(\Omega)$, $pp' = p + p'$, and

$$\|f\| = \inf\bigg\|\bigg(\sum_{|\alpha|=l} g^2_\alpha\bigg)^{1/2}\bigg\|_{L_{p'}(\Omega)}. \tag{1.1.23}$$

Here the infimum is taken over all collections $\{g_\alpha\}_{|\alpha|=l}$, for which (1.1.22) holds with any $u \in L^l_p(\Omega)$.

Proof. Obviously, the right-hand side of (1.1.22) is a linear functional on $L^l_p(\Omega)$ and

$$\|f\| \le \bigg\|\bigg(\sum_{|\alpha|=l} g^2_\alpha\bigg)^{1/2}\bigg\|_{L_{p'}(\Omega)}.$$

To express $f(u)$ as (1.1.22), consider the space $L_p(\Omega)$ of vectors $v = \{v_\alpha\}_{|\alpha|=l}$ with components in $L_p(\Omega)$, equipped with the norm

$$\Big\| \Big(\sum_{|\alpha|=l} v_\alpha^2 \Big)^{1/2} \Big\|_{L_p(\Omega)}.$$

Since the space $L_p^l(\Omega)$ is complete, the range of the operator $\nabla_l : L_p^l(\Omega) \to L_p(\Omega)$ is a closed subspace of $L_p(\Omega)$. For any vector $v = \nabla_l u$ we define $\Phi(v) = f(u)$. Then

$$\|\Phi\| = \|f\|,$$

and by the Hahn–Banach theorem Φ has a norm-preserving extension to $L_p(\Omega)$. The proof is complete. □

As before, let $W_p^l(\Omega) = L_p^l(\Omega) \cap L_p(\Omega)$ and let $\mathring{W}_p^l(\Omega)$ be the completion of $\mathscr{D}(\Omega)$ with respect to the norm in $W_p^l(\Omega)$.

The following assertion is proved in the same manner as Theorem 1.

Theorem 2. *Any linear functional on $W_p^l(\Omega)$ (or on $\mathring{W}_p^l(\Omega)$) has the form*

$$f(u) = \int_\Omega \Big(\sum_{|\alpha|=l} g_\alpha(x) D^\alpha u(x) + g(x) u(x) \Big) \, dx, \tag{1.1.24}$$

where $g_\alpha \in L_{p'}(\Omega)$, $g \in L_{p'}(\Omega)$, and

$$\|f\| = \inf \Big\| \Big(\sum_{|\alpha|=l} g_\alpha^2 + g^2 \Big)^{1/2} \Big\|_{L_{p'}(\Omega)}. \tag{1.1.25}$$

Here the infimum is taken over all collections of functions g_α, $g \in L_{p'}(\Omega)$ for which (1.1.24) holds with any $u \in W_p^l(\Omega)$ (or $u \in \mathring{W}_p^l(\Omega)$).

The following simpler characterization of the space of linear functionals on $\mathring{W}_p^l(\Omega)$ is a corollary of Theorem 2.

Corollary. *Any linear functional on $\mathring{W}_p^l(\Omega)$ can be identified with a generalized function $f \in \mathscr{D}'(\Omega)$ given by*

$$f = \sum_{|\alpha|=l} (-1)^l D^\alpha g_\alpha + g, \tag{1.1.26}$$

where g_α and g belong to $L_{p'}(\Omega)$. The norm of this functional is equal to the right-hand side of (1.1.25), where the infimum is taken over all collections of functions g_α, g, entering the expression (1.1.26).

1.1.16 Equivalent Norms in $W_p^l(\Omega)$

The following theorem describes a wide class of equivalent norms in $W_p^l(\Omega)$.

Theorem. *Let Ω be a bounded domain such that $L_p^l(\Omega) \subset L_p(\Omega)$ (for example, Ω has the cone property). Let $\mathscr{F}(u)$ be a continuous seminorm in $W_p^l(\Omega)$ such that $\mathscr{F}(\Pi_{l-1}) \neq 0$ for any nonzero polynomial Π_{l-1} of degree not higher than $l-1$. Then the norm*

$$\|\nabla_l u\|_{L_p(\Omega)} + \mathscr{F}(u), \tag{1.1.27}$$

is equivalent to the norm in $W_p^l(\Omega)$.

Proof. Let $\mathscr{I}$ be the identity mapping of $W_p^l(\Omega)$ into the space $B(\Omega)$ obtained by the completion of $W_p^l(\Omega)$ with respect to the norm (1.1.27). This mapping is one-to-one, linear, and continuous. By Theorem 1.1.13/1 on the completeness of $\mathring{L}_p^l(\Omega)$ it follows that $B(\Omega) \subset L_p^l(\Omega)$. Since $L_p^l(\Omega) = W_p^l(\Omega)$, then $\mathscr{I}$ maps $W_p^l(\Omega)$ onto $B(\Omega)$. By the Banach theorem (cf. Bourbaki [128], Chap. 1, §3, Sect. 3), $\mathscr{I}$ is an isomorphism. The result follows. □

Remark. The functional

$$\mathscr{F}(u) = \sum_{0 \le |\alpha| < l} |f_\alpha(u)|,$$

where f_α are linear functionals in $W_p^l(\Omega)$ such that

$$\det\bigl(f_\alpha(x^\beta)\bigr) \neq 0, \quad |\alpha|, |\beta| \le l-1,$$

satisfies the conditions of the Theorem. For example, we can put

$$f_\alpha(u) = \int_\Omega u D^\alpha \varphi \, dx,$$

where $\varphi \in \mathscr{D}(\Omega)$ and

$$\int_\Omega \varphi(x) \, dx \neq 0.$$

Let $\mathfrak{P}$ be a projector of $W_p^l(\Omega)$ onto the polynomial subspace $\mathscr{P}_{l-1}$, i.e., a linear continuous mapping of $W_p^l(\Omega)$ onto $\mathscr{P}_{l-1}$ such that $\mathfrak{P}^2 = \mathfrak{P}$. Then we may take $|\mathfrak{P}(u)|$ as $\mathscr{F}(u)$.

Since $\mathfrak{P}(u - \mathfrak{P}(u)) = 0$, by the Theorem we have the equivalence of the seminorms $\|\nabla_l u\|_{L_p(\Omega)}$ and $\|u - \mathfrak{P}(u)\|_{W_p^l(\Omega)}$ (cf. Lemma 1.1.11).

1.1.17 Extension of Functions in $V_p^l(\Omega)$ onto $\mathbb{R}^n$

In this section we discuss space-preserving extensions of functions in $V_p^l(\Omega)$ onto the exterior of Ω. We begin with the well-known procedure of "finite-order reflection."

We introduce the notation. Let $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1})$, let π be an n-dimensional parallelepiped, $P = \pi \times (-a/(l+1), a)$, $P_+ = \pi \times (0, a)$, $P_- = P \backslash P_+$.

Theorem. *For any integer $l \geq 0$ there exists a linear mapping*

$$C^\infty(\bar{P}_+) \ni u^* \to u \in C^l(\bar{P}),$$

such that $u^* = u$ on P_+. It can be uniquely extended to a continuous mapping $L_p^k(P_+) \to L_p^k(P)$, $p \geq 1$, for $k = 0, 1, \dots, l$.

The extension $u \to u^*$ can be defined on the space $C^{k,1}(\bar{P}_+)$, $k = 0, 1, \dots, l-1$. It is a continuous mapping into $C^{k,1}(\bar{P})$.

This mapping has the following property: if $\operatorname{dist}(\operatorname{supp} u, F) > 0$ where F is a compactum in $\bar{P}_+$, then $u^* = 0$ in a neighborhood of F.

Proof. Let $u \in C^\infty(\bar{P}_+)$. We set $u^* = u$ in P_+ and

$$u^*(x) = \sum_{j=1}^{l+1} c_j u(x', -jx_n) \quad \text{in } P_-,$$

where the coefficients c_j satisfy the system

$$\sum_{j=1}^{l+1} (-j)^k c_j = 1, \quad k = 0, \dots, l.$$

The determinant of this system (the Vandermonde determinant) does not vanish.

Obviously, $u^* \in C^l(\bar{P})$. It is also clear that

$$\|\nabla_k u\|_{L_p(P)} \leq c \|\nabla_k u\|_{L_p(P_+)}.$$

Since $C^\infty(\bar{P}_+)$ is dense in $L_p^k(P_+)$ by Theorem 1.1.6/1, the mapping $u^* \to u$ admits a unique extension to a continuous mapping $L_p^k(P_+) \to L_p^k(P)$. The continuity of the mapping $C^{k,l}(\bar{P}_+) \ni u \to u^* \in C^{k,l}(\bar{P})$ can be checked directly.

Let $\operatorname{dist}(\operatorname{supp} u, F) > 0$. We denote an arbitrary point of $F \cap \pi$ by x and introduce a small positive number δ such that $(l+1)\delta < \operatorname{dist}(\operatorname{supp} u, F)$. Since $u = 0$ on $\{x \in \bar{P}_+ : |x'| < (l+1)x_n\}$, we have $u^* = 0$ on $\{x \in P_- : |x'|^2 + (l+1)x_n^2 < (l+1)^2\delta^2\}$. Thus, $u^* = 0$ if $|x| < \delta$. This completes the proof. □

Let Ω be a domain in $\mathbb{R}^n$ with compact closure $\bar{\Omega}$ and with sufficiently smooth boundary $\partial\Omega$. Using the described extension procedure along with a partition of unity and a local mapping of Ω onto the halfspace, it is possible to construct a linear continuous operator $\mathscr{E} : V_p^l(\Omega) \to V_p^l(\mathbb{R}^n)$ such that $\mathscr{E}u|_\Omega = u$ for all $u \in V_p^l(\Omega)$.

If such an operator exists for a domain then, by definition, this domain belongs to *the class* EV_p^l.

Thus, domains with smooth boundaries are contained in EV_p^l. The bounded domains of the class $C^{0,1}$ turn out to have the same property. The last assertion is proved in Stein [724], §3, Ch.VI (cf. also Comments to the present section).

In Sect. 1.5 we shall return to the problem of description of domains in EV_p^l.

1.1.18 Removable Sets for Sobolev Functions

Here we describe a class of sets of removable singularities for elements in $V_p^l(\Omega)$, $\Omega \subset \mathbb{R}^n$. The description will be given in terms of the $(n-1)$-dimensional Hausdorff measure. We begin with the definition of the s-dimensional Hausdorff measure.

Let E be a set in $\mathbb{R}^n$. Consider various coverings of E by countable collections of balls of radii $\le \varepsilon$. For each $s \ge 0$, let

$$\sigma_s(\varepsilon) = v_s \inf \sum_i r_i^s,$$

where r_i is the radius of the ith ball, $v_s > 0$, and the infimum is taken over all such coverings. By the monotonicity of σ_s, there exists the limit (finite or infinite)

$$H_s(E) = \lim_{\varepsilon \to +0} \sigma_s(\varepsilon).$$

This limit is called the *s-dimensional Hausdorff measure* of E. If s is a positive integer, then $v_s = m_s(B_1^{(s)})$, otherwise v_s is any positive constant. For example, one may put $v_s = \pi^{s/2}/\Gamma(1+s/2)$ for any $s \ge 0$. In the case where s is a positive integer, $s \le n$, the Hausdorff measure H_s agrees with the s-dimensional area of an s-dimensional smooth manifold in $\mathbb{R}^n$. In particular, $H_n(E) = m_n(E)$ for Lebesgue measurable sets $E \subset \mathbb{R}^n$ (see, e.g., Federer [271, 3.2], Ziemer [812, 1.4.2]).

Theorem. *Let $u \in V_p^l(\Omega \setminus F)$, where $\Omega \subset \mathbb{R}^n$ is an open set and $F \subset \Omega$ is a closed set in Ω satisfying $H_{n-1}(F) = 0$. Then $u \in V_p^l(\Omega)$.*

Proof. If $u \in V_p^l(\Omega \setminus F)$, then u and $D^\alpha u, |\alpha| \le l$ are defined almost everywhere on Ω and hence $D^\alpha u \in L_p(\Omega)$. It suffices to verify that $D^\alpha u$ is the generalized derivative of u on Ω, i.e., for all $\eta \in \mathscr{D}(\Omega)$,

$$\int_\Omega u D^\alpha \eta \, dx = (-1)^{|\alpha|} \int_\Omega \eta D^\alpha u \, dx. \tag{1.1.28}$$

Let

$$p_i(x) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \quad 1 \le i \le n,$$

denote the projection of a point $x \in \mathbb{R}^n$ on the coordinate hyperplane orthogonal to the x_i axis. By assumptions, each set $p_i(F)$ has $(n-1)$-dimensional Lebesgue measure zero. Thus, almost every straight line that is parallel to the x_i axis is disjointed from F. According to Fubini's theorem, we have

$$\int_\Omega \eta \frac{\partial u}{\partial x_i}\, \mathrm{d}x = \int_{\Omega'} \mathrm{d}x' \int_{\ell(x')} \eta \frac{\partial u}{\partial x_i}\, \mathrm{d}x_i, \quad 1 \le i \le n,$$

where $\Omega' = p_i(\Omega), x' = p_i(x)$ and $\ell(x')$ is the intersection of Ω with the line $x' = \text{const}$. Note that $\ell(x')$ is in $\Omega \setminus F$ for almost all $x' \in \Omega'$. An application of Theorem 1.1.3/2 leads to

$$\int_{\Omega'} \mathrm{d}x' \int_{\ell(x')} \eta \frac{\partial u}{\partial x_i}\, \mathrm{d}x_i = -\int_{\Omega'} \mathrm{d}x' \int_{\ell(x')} u \frac{\partial \eta}{\partial x_i}\, \mathrm{d}x_i - \int_\Omega u \frac{\partial \eta}{\partial x_i}\, \mathrm{d}x.$$

Hence (1.1.28) follows for $|\alpha| = l = 1$. The general case can be concluded by induction on l. Indeed, let $l \ge 2$ and let (1.1.28) hold for $u \in V_p^{l-1}(\Omega \setminus F)$ with $|\alpha| \le l-1$. If $u \in V_p^l(\Omega \setminus F)$, $|\alpha| = l$, and $D^\alpha = D_i D^\beta$ for some $1 \le i \le n$, the left-hand side of (1.1.28) equals

$$-\int_\Omega \frac{\partial u}{\partial x_i} D^\beta \eta\, \mathrm{d}x, \tag{1.1.29}$$

since $u \in V_p^1(\Omega \setminus F)$. By the induction hypothesis, expression (1.1.29) coincides with the right-hand side of (1.1.28) (because $\partial u/\partial x_i \in V_p^{l-1}(\Omega \setminus F)$). This completes the proof. □

Let us show that the condition $H_{n-1}(F) = 0$ in the above theorem cannot be replaced by the finiteness of $H_{n-1}(F)$.

Example. Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 2\}$ and let F be the segment $\{x \in \Omega : x_2 = 0, |x_1| \le 1\}$. We introduce the set $S = \{x \in \Omega : |x_1| \le 1, x_2 > 0\}$ and define the function u on Ω by

$$u(x) = \begin{cases} 0 & \text{on } \Omega \setminus S, \\ \exp(-(1-x_1^2)^{-1}) & \text{on } S. \end{cases}$$

We have $H_1(F) = 2$, $u \in V_p^l(\Omega \setminus S)$ for any l, but $u \notin V_p^l(\Omega)$.

1.1.19 Comments to Sect. 1.1

The space $W_p^l(\Omega)$ was introduced and studied in detail by Sobolev [711–713]. (Note that as early as in 1935 he also developed a theory of distributions in $(C_0^l)^*$ [710].) The definitions of the spaces $L_p^l(\Omega)$ and $\dot{L}_p^l(\Omega)$ are borrowed from the paper by Deny and Lions [234]. The proofs of Theorems 1.1.2, 1.1.5/1, 1.1.12, 1.1.13/1, and 1.1.13/2 follow the arguments of this paper where similar results are obtained for $l = 1$. Theorem 1.1.5/1 was also proved by Meyers and Serrin [599]. Concerning the contents of Sect. 1.1.3 we note that the mollification operator $\mathscr{M}_\varepsilon$ was used by Leray in 1934 [487] and independently by Sobolev in 1935 [709]. A detailed exposition of properties of mollifiers including those with a variable radius can be found in Chap. 2 of Burenkov's book [155]. For a history of the mollifiers see Naumann [624].

The property of absolute continuity on almost all straight lines parallel to coordinate axes served as a foundation for the definition of spaces similar to L_p^1 in papers by Levi [489], Nikodým [637], Morrey [612], and others. The example of a domain for which $W_2^2(\Omega) \neq V_2^2(\Omega)$ (see Sect. 1.1.4) is due to the author. The example considered at the beginning of Sect. 1.1.6 is borrowed from the paper by Gagliardo [299]. Remark 1.1.6 and the subsequent example are taken from the paper by Kolsrud [440]. Theorem 1.1.6/1 is given in the textbook by Smirnov [705] and Theorem 1.1.6/2 was proved in the above-mentioned paper by Gagliardo. The topic of Sect. 1.1.6 was also discussed by Fraenkel [285] and Amick [46]. The following deep approximation result was obtained by Hedberg (see [370]).

Theorem. *Let Ω be an arbitrary open set in $\mathbb{R}^n$ and let $f \in \mathring{W}_p^l(\Omega)$ for some positive integer l and some p, $1 < p < \infty$. Then there exists a sequence of functions $\{\omega_\nu\}_{\nu \geq 1}$, $0 \leq \omega_\nu \leq 1$, such that $\operatorname{supp} \omega_\nu$ is a compact subset of Ω, $\omega_\nu f \in (L_\infty \cap \mathring{W}_p^l)(\Omega)$, and*

$$\lim_{\nu \to \infty} \|f - \omega_\nu f\|_{W_p^l(\Omega)} = 0.$$

An ingenious approximation construction for functions in a two-dimensional bounded Jordan domain was proposed by Lewis in 1987 [492] (see also [491]). He proved that $C^\infty(\bar{\Omega})$ is dense in $W_p^1(\Omega)$ for $1 < p < \infty$. Unfortunately, this construction does not work for higher dimensions, for $p = 1$, and for the space $W_p^l(\Omega)$ with $l > 1$. If a planar domain is not Jordan, it may happen that for $l > 1$ bounded functions in the space $L_p^l(\Omega)$ are not dense in $L_p^l(\Omega)$ (Maz'ya and Netrusov [572], see Sect. 1.7 in the sequel).

The problem of the approximation of Sobolev functions on planar domains by C^∞ functions, which together with all derivatives are bounded on Ω, was treated by Smith, Stanoyevitch, and Stegenga in [707], where some interesting counterexamples are given as well.

In [346] Hajłasz and Malý studied the approximation of mappings $u : \Omega \to \mathbb{R}^m$ from the Sobolev space $[W_p^1(\Omega)]^m$ by a sequence $\{u_\nu\}_{\nu \geq 1}$ of more regular mappings in the sense of convergence

$$\int_\Omega f(x, u_\nu, \nabla u_\nu)\, \mathrm{d}x \to \int_\Omega f(x, u, \nabla u)\, \mathrm{d}x$$

for a large class of nonlinear integrands.

During the last two decades a theory of variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ was developed, where the Lebesgue L^p-space is replaced by the

Lebesgue space with variable exponent $p(x)$. This topic in not touched in the present book, but we refer to Kovácǐk and Rákošnik [461] where such spaces first appeared and to the surveying papers by Diening, Hästö and Nekvinda [236], Kokilashvili and Samko [439] and S. Samko [690]. In particular, the density of $C_0^\infty(\mathbb{R}^n)$ in $W^{1,p(\cdot)}(\mathbb{R}^n)$ was proved by S. Samko [688] under the so-called log-condition on $p(x)$, standard for the variable exponent analysis:

$$\big|p(x) - p(y)\big| \leq \frac{\text{const}}{|\log|x-y||}$$

for small $|x-y|$. This question is nontrivial because of impossibility to use mollifiers directly. The Hardy type inequality in variable Lebesgue spaces are studied by Rafeiro and Samko [669].

In connection with Sect. 1.1.7 see the books by Morrey [613], Reshetnyak [677], and the article by the author and Shaposhnikova [578], see also Sect. 9.4 in the book [588].

The condition of being starshaped with respect to a ball and having the cone property were introduced into the theory of W_p^l spaces by Sobolev [711–713]. Lemma 1.1.9/2 was proved by Glushko [312]. The example given in Sect. 1.1.9 is due to the author; another example of a Lipschitz domain that does not belong to $C^{0,1}$ can be found in the book by Morrey [613]. Properties of various classes of domains appearing in the theory of Sobolev spaces were investigated by Fraenkel in [285]. Fraenkel's paper [286] contains a thorough study of the conditions on domains Ω guaranteeing the embedding of the space $C^1(\bar{\Omega})$ in $C^{(0,\alpha)}(\bar{\Omega})$ when $\alpha > 0$.

Integral representations (1.1.8) and (1.1.10) were obtained by Sobolev [712, 713] and used in his proof of embedding theorems. Various generalizations of such representations are due to Il'in [396, 397], Smith [706] (see also the book by Besov, Il'in, and Nikolsky [94]), and to Reshetnyak [675]. We follow Burenkov [154] in the proof of Theorem 1.1.10/1.

The Poincaré inequality for bounded domains that are the unions of domains of the class C was proved by Courant and Hilbert [216]. Properties of functions in $L_p^l(\Omega)$ for a wider class of domains were studied by J.L. Lions [499].

Stanoyevitch showed [720] (see [721] for the proof) that the best constant in the one-dimensional Poincaré inequality

$$\|u - \bar{u}\|_{L_p(-1,1)} \leq C\|\nabla u\|_{L_p(-1,1)}$$

is equal to 1 if $p = 1$ and

$$\frac{(p')^{1/p}p^{1/p'}}{\Gamma(1/p)\Gamma(1/p')} = \frac{p\sin(\pi/p)}{\pi(p-1)^{1/p}},$$

if $1 < p < \infty$. Moreover, a unique extremal function exists if and only if $1 < p \leq \infty$.

Theorem 1.1.16 on equivalent norms in $W_p^l(\Omega)$ is due to Sobolev [713, 714].

The extension procedure described at the beginning of Sect. 1.1.17 was proposed by Hestenes [378] for the space $C^k(\bar{\Omega})$ (see also Lichtenstein [494]). The same method was used by Nikolsky [638] and Babich [59] for $W_p^l(\Omega)$. The fact that a space-preserving extension for functions in $W_p^l(\Omega)$ $(1 < p < \infty)$ onto $\mathbb{R}^n$ is possible for domains of the class $C^{0,1}$ was discovered by Calderón [163]. His proof is based on the integral representation (1.1.8) along with the theorem on the continuity of the singular integral operator in L_p. A method that is appropriate for $p = 1$ and $p = \infty$ was given by Stein [724]. The main part of his proof was based on the extension of functions defined in a neighborhood of a boundary point. Then, using a partition of unity, he constructed a global extension. For the simple domain

$$\Omega = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n > f(x')\},$$

where f is a function on $\mathbb{R}^{n-1}$ satisfying a Lipschitz condition, the extension of u is defined by

$$u^*(x', x_n) = \int_1^\infty u\big(x', x_n + \lambda\delta(x', x_n)\big)\psi(\lambda)\,\mathrm{d}\lambda, \quad x_n < f(x').$$

Here δ is an infinitely differentiable function, equivalent to the distance to $\partial\Omega$. The function ψ is defined and continuous on $[1, \infty)$, decreases as $\lambda \to \infty$ more rapidly than any power of λ^{-1}, and satisfies the conditions

$$\int_1^\infty \psi(\lambda)\,\mathrm{d}\lambda = 1, \qquad \int_1^\infty \lambda^k\psi(\lambda)\,\mathrm{d}\lambda = 0, \quad k = 1, 2, \dots .$$

More information on extension operators acting on Sobolev spaces can be found in Sect. 1.5.

Theorem 1.1.18 on removable sets for Sobolev functions is due to Väisälä [771] (see also Reshetnyak [677, Chap. 1, 1.3]). Koskela showed that W_p^1 removability of sets lying in a hyperplane depends on their thickness measured in terms of a so-called p porosity [455].

1.2 Facts from Set Theory and Function Theory

In this section we collect some known facts from set theory and function theory that will be used later.

1.2.1 Two Theorems on Coverings

The following assertion is a generalization of the classical Besicovitch covering theorem, see Guzman [332] or DiBenedetto [235] for a proof.

Theorem 1. *Let $\mathscr{S}$ be a set in $\mathbb{R}^n$. With each point $x \in \mathscr{S}$ we associate a ball $B_{r(x)}(x)$, $r(x) > 0$, and denote the collection of these balls by $\mathfrak{B}$. We assume that*

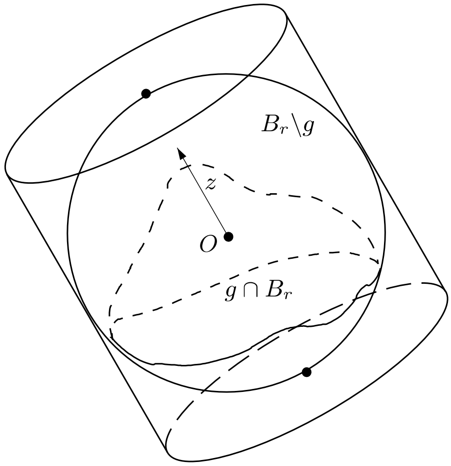

Fig. 5.

(α) *the radii of balls in* $\mathfrak{B}$ *are totally bounded,*

(β) *the sequence of radii of any disjoint sequence of balls in* $\mathfrak{B}$ *tends to zero.*

Then we can choose a sequence of balls $\{\mathscr{B}_m\}$ *in* $\mathfrak{B}$ *such that*:

1. $\mathscr{S} \subset \bigcup_m \mathscr{B}_m$;
2. *There exists a number* M*, depending on the dimension of the space only, such that every point of* $\mathbb{R}^n$ *belongs to at most* M *balls in* $\{\mathscr{B}_m\}$;
3. *The balls* $(1/3)\mathscr{B}_m$ *are disjoint*;
4. $\bigcup_{B\in\mathfrak{B}} B \subset \bigcup_m 4\mathscr{B}_m$.

Remark 1. Theorem 1 remains valid if balls are replaced by cubes with edges parallel to coordinate planes. This result is contained in the paper by Morse [615]. It also follows from [615] that balls and cubes can be replaced by other bodies. The best value of M was studied by Sullivan [733] as well as Füredi and Loeb [297].

We anticipate another covering theorem by the next auxiliary assertion.

Lemma. *Let* g *be an open subset of* $\mathbb{R}^n$ *with a smooth boundary and let* $2m_n(B_r \cap g) = m_n(B_r)$*. Then*

$$s(B_r \cap \partial g) \geq c_n r^{n-1},$$

where c_n *is a positive constant which depends only on* n*, and* s *is the* $(n-1)$*-dimensional area.*

Proof. Let χ and ψ be the characteristic functions of the sets $g \cap B_r$ and $B_r\backslash g$. For any vector $z \neq 0$ we introduce a projection mapping P_z onto the $(n-1)$-dimensional subspace orthogonal to z (see Fig. 5). By Fubini's theorem,

$$
\begin{aligned}
(1/4)v_n^2 r^{2n} &= m_n(g \cap B_r)\, m_n(B_r \backslash g) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi(x)\psi(y)\, \mathrm{d}x\, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi(x)\psi(x+z)\, \mathrm{d}z\, \mathrm{d}x \\
&= \int_{|z| \le 2r} m_n\big(\{x : x \in B_r \cap g, x+z \in B_r \backslash g\}\big)\, \mathrm{d}z.
\end{aligned}
$$

Since every interval connecting $x \in g \cap B_r$ with $(x+z) \in B_r \backslash g$ intersects $B_r \cap \partial g$, the last integral does not exceed

$$
2r \int_{|z| \le 2r} m_{n-1}\big[P_z(B_r \cap \partial g)\big]\, \mathrm{d}z \le (2r)^{n+1} v_n s(B_r \cap \partial g).
$$

The lemma is proved. □

Remark 2. The best value of c_n equals the volume of the $(n-1)$-dimensional unit ball (cf. Lemma 5.2.1/1 below).

Theorem 2. *Let g be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. There exists a covering of g by a sequence of balls with radii ϱ_i, $i = 1, 2, \dots$, such that*

$$
\sum_j \varrho_j^{n-1} \le c\, s(\partial g), \tag{1.2.1}
$$

where c is a constant which depends only on n.

Proof. Each point $x \in g$ is the center of a ball $B_r(x)$ for which

$$
\frac{m_n(B_r(x) \cap g)}{m_n(B_r(x))} = \frac{1}{2}. \tag{1.2.2}
$$

(This ratio is a continuous function of r, which is equal to 1 for small values of r and converges to zero as $r \to \infty$.) Let us put $\mathscr{B} = \{B_{3r(x)}(x)\}$, where $B_{r(x)}(x)$ satisfies (1.2.2). By Theorem 1 there exists a sequence of disjoint balls $B_{r_j}(x_j)$ such that

$$
g \subset \bigcup_{j=1}^{\infty} B_{3r_j}(x_j).
$$

(Here we actually use a weaker variant of Theorem 1 (cf. Dunford and J. Schwartz [244], III.12.1).)

The Lemma together with (1.2.2) implies

$$
s(B_{r_j}(x_j) \cap \partial g) \ge c_n r_j^{n-1}.
$$

Therefore,

$$
s(\partial g) \ge \sum_j s(B_{r_j}(x_j) \cap \partial g) \ge 3^{1-n} c_n \sum_j (3r_j)^{n-1}.
$$

Thus, $\{B_{3r_j}(x_j)\}$ is the required covering. □

1.2.2 Theorem on Level Sets of a Smooth Function

We recall the Vitali covering theorem (see Dunford and J. Schwartz [244], III.12.2).

Let $E \subset \mathbb{R}^1$ and let $\mathscr{M}$ be a collection of intervals. We say that $\mathscr{M}$ forms a covering of E in the sense of Vitali if for each $t \in E$ and any $\varepsilon > 0$ there exists an interval $i \in \mathscr{M}$ such that $t \in i$, $m_1(i) < \varepsilon$.

Theorem 1. *If E is covered by a collection $\mathscr{M}$ of intervals in the sense of Vitali, then we can select a countable or finite set of intervals $\{i_k\}$ such that $i_k \cap i_l = \varnothing$ for $k \neq l$, $m_1(E \setminus \bigcup_k i_k) = 0$.*

Consider a function f

$$\Omega \ni x \to f(x) = t \in \mathbb{R}^1.$$

The set

$$K_1 = \{x : \nabla f(x) = 0\}$$

is called critical.

If $E \subset \Omega$ then $f(E)$ is the image of E under the mapping f. If $A \subset \mathbb{R}^1$ then $f^{-1}(A)$ is the pre-image of A in Ω. We shall briefly denote $f^{-1}(t)$ by $\mathscr{E}_t$.

Theorem 2. *Let Ω be an open set in $\mathbb{R}^n$ and $f \in C^\infty(\Omega)$. Then*

$$m_1\big[f(K_1)\big] = 0.$$

Proof. It is sufficient to assume that Ω is a bounded set.

1. We introduce the notation

$$K_n = \big\{x : (\nabla f)(x) = 0, \dots, (\nabla_n f)(x) = 0\big\}.$$

First we show that

$$m_1\big[f(K_n)\big] = 0.$$

For any $\varepsilon > 0$ and each $x \in K_n$ we choose a number $r_x > 0$ such that $B(x, r_x) \subset \Omega$ and

$$\underset{B(x,r_x)}{\operatorname{osc}} f < \varepsilon r_x^n.$$

We fix a point $t \in f(K_n)$ and consider any point

$$x(t) \in \mathscr{E}_t \cap K_n.$$

Then we cover t by intervals $(t - \delta, t + \delta)$ with

$$\delta < \varepsilon r_{x(t)}^n. \tag{1.2.3}$$

The collection of all these intervals forms a covering of $f(K_n)$ in the sense of Vitali. We choose a countable system of disjoint intervals $i_1, i_2, \dots$, which covers $f(K_n)$ up to a set of linear measure zero.

Let

$$i_m = (t_m - \delta_m, t_m + \delta_m),$$

and $x_m \in \mathscr{E}_{t_m} \cap K_n$. By (1.2.3), $\delta < \varepsilon\, r^n_{x_m}$. Hence

$$f^{-1}(i_m) \supset B\big(x_m, (\delta_m/\varepsilon)^{1/n}\big),$$

and therefore,

$$m_n\big[f^{-1}(i_m)\big] \geq v_n \frac{\delta_m}{\varepsilon}.$$

Since the intervals i_m are mutually disjointed, their pre-images have the same property. Thus,

$$\sum \delta_m \leq \frac{\varepsilon}{v_n} \sum_{m=1}^{\infty} m_n\big[f^{-1}(i_m)\big] \leq m_n(\Omega),$$

i.e., $m_1[f(K_n)] \leq c\varepsilon m_n(\Omega)$, hence $m_1[f(K_n)] = 0$.

2. Now we use an induction on n. The theorem holds for $n = 1$. Assume it holds for $n - 1$.

Consider the set $K_1 \backslash K_n$. For any $x \in K_1 \backslash K_n$ there exists a multi-index α, $|\alpha| < n$, and an integer $i \leq n$ such that

$$\big(D^\alpha f\big)(x) = 0, \qquad \left(\frac{\partial}{\partial x_i} D^\alpha f\right)(x) \neq 0. \tag{1.2.4}$$

Let H be a set of points for which (1.2.4) holds. This set is obviously defined by the pairs (α, i). We show that

$$m_1\big[f(H)\big] = 0.$$

Without loss of generality we may assume that $i = n$. With the notation $g = D^\alpha f$, we have

$$g(x) = 0, \qquad \frac{\partial g}{\partial x_n} \neq 0 \quad \text{for } x \in H.$$

By the implicit function theorem, for any $x_0 \in H$ there exists a neighborhood $\mathscr{U}$ such that

$$\mathscr{U} \cap \big\{x : g(x) = 0\big\} \subset \big\{x : x_n = \varphi(X)\big\},$$

where $X = (x_1, \dots, x_{n-1})$ and φ is an infinitely differentiable function in some domain $G \subset \mathbb{R}^{n-1}$. Since we can select a countable covering from any covering of H, it is sufficient to prove that

$$m_1\big[f(H \cap \mathscr{U})\big] = 0.$$

If $x \in H \cap \mathscr{U}$, then

$$f(x) = f\bigl(X, \varphi(X)\bigr) \stackrel{\text{def}}{=} h(X),$$

where $X \in G$. Let P denote the projection of $H \cap \mathscr{U}$ onto the plane $x_n = 0$. Since $\nabla h = 0$ for $X \in P$, by the induction hypothesis we have

$$m_1\bigl[h(P)\bigr] = 0.$$

Taking into account that $h(P) = f(H \cap \mathscr{U})$, we complete the proof. □

From Theorem 2 and the implicit function theorem we immediately obtain the following corollary.

Corollary. *Let $f \in C^\infty(\Omega)$ ($f \in \mathscr{D}(\Omega)$), then for almost all t the sets $\mathscr{E}_t = \{x : f(x) = t\}$ are C^∞-manifolds (C^∞-compact manifolds).*

1.2.3 Representation of the Lebesgue Integral as a Riemann Integral along a Halfaxis

Lemma. *Let $(X, \mathfrak{B}, \mu)$ be a space with a (nonnegative) σ-finite measure and let $u : X \to \mathbb{R}^1$ be a μ-measurable nonnegative function. Then*

$$\int_X u(x)\mu(\mathrm{d}x) = \int_0^\infty \mu(\mathscr{L}_t)\,\mathrm{d}t = \int_0^\infty \mu(\mathscr{M}_t)\,\mathrm{d}t, \tag{1.2.5}$$

where

$$\mathscr{L}_t = \bigl\{x \in X : u(x) > t\bigr\}, \qquad \mathscr{M}_t = \bigl\{x \in X : u(x) \geq t\bigr\}.$$

Proof. Let $\chi_{(a,b)}$ denote the characteristic function of the interval (a, b) of real axis. Writing

$$u(x) = \int_0^\infty \chi_{(0,u(x))}(t)\,\mathrm{d}t,$$

and using Fubini's theorem on the product space $X \times (0, \infty)$, we obtain

$$\int_X u(x)\mu(\mathrm{d}x) = \int_0^\infty \mathrm{d}t \int_X \chi_{(0,u(x))}(t)\mu(\mathrm{d}x) = \int_0^\infty \mu(\mathscr{L}_t)\,\mathrm{d}t.$$

Thus the first equality (1.2.5) is established. The second equality can be obtained in the same way. The lemma is proved. □

Remark. We can easily derive a generalization of (1.2.5) for the integral

$$\int_X u(x)\mu(\mathrm{d}x),$$

where μ is a charge, u is not necessarily of a definite sign, and

$$\int_X \bigl|u(x)\bigr||\mu|(\mathrm{d}x) < \infty.$$

1.2.4 Formula for the Integral of Modulus of the Gradient

Here we establish the following assertion (we refer the reader to the beginning of Sect. 1.1.18 for the definition of Hausdorff measure).

Theorem. *Let Φ be a Borel measurable nonnegative function on Ω and let $u \in C^{0,1}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. Then*

$$\int_\Omega \Phi(x)\big|\nabla u(x)\big|\,\mathrm{d}x = \int_0^\infty \mathrm{d}t \int_{\mathscr{E}_t} \Phi(x)\,\mathrm{d}s(x), \tag{1.2.6}$$

where s is the $(n-1)$-dimensional Hausdorff measure, $\mathscr{E}_t = \{x \in \Omega : |u(x)| = t\}$.

We shall derive (1.2.6) in the following weaker formulation, which will be used in this chapter.

If $\Phi \in C(\Omega)$, $\Phi \geq 0$, and $u \in C^\infty(\Omega)$, then (1.2.6) holds.

(*Here we may assume s to be the $(n-1)$-dimensional Lebesgue measure, since by Corollary 1.2.2 the sets $\mathscr{E}_t$ are smooth manifolds.*)

Proof. Let $\mathbf{w}$ be an n-tuple vector function in $\mathscr{D}(\Omega)$. Using integration by parts and applying Lemma 1.2.3, we obtain

$$\begin{aligned}\int_\Omega \mathbf{w}\nabla u\,\mathrm{d}x &= -\int_\Omega u \operatorname{div}\mathbf{w}\,\mathrm{d}x\\ &= -\int_0^\infty \mathrm{d}t \int_{u\geq t} \operatorname{div}\mathbf{w}\,\mathrm{d}x + \int_{-\infty}^0 \mathrm{d}t \int_{u\leq t} \operatorname{div}\mathbf{w}\,\mathrm{d}x.\end{aligned}$$

Since $u \in C^\infty(\Omega)$, the sets $\{x : u(x) = t\}$ are infinitely differentiable manifolds for almost all t. Therefore for almost all $t > 0$,

$$\int_{u>t} \operatorname{div}\mathbf{w}\,\mathrm{d}x = -\int_{u=t} \mathbf{w}\nu\,\mathrm{d}s = -\int_{u=t} \frac{\mathbf{w}\nabla u}{|\nabla u|}\,\mathrm{d}s,$$

where $\nu(x)$ is the normal to $\{x : u(x) = t\}$ directed into the set $\{x : u(x) \geq t\}$. The integral

$$\int_{u\leq t} \operatorname{div}\mathbf{w}\,\mathrm{d}x,$$

must be treated analogously. Consequently,

$$\int_\Omega \mathbf{w}\nabla u\,\mathrm{d}x = \int_0^\infty \mathrm{d}t \int_{\mathscr{E}_t} \frac{\mathbf{w}\,\nabla u}{|\nabla u|}\,\mathrm{d}x.$$

Setting

$$\mathbf{w} = \Phi \frac{\nabla u}{(|\nabla u|^2 + \varepsilon)^{1/2}},$$

where $\Phi \in \mathscr{D}(\Omega)$ and ε is a positive number, we obtain

$$\int_\Omega \Phi \frac{(\nabla u)^2}{((\nabla u)^2+\varepsilon)^{1/2}}\,\mathrm{d}x = \int_0^\infty \mathrm{d}t \int_{\mathscr{E}_t} \frac{\mathbf{w}\nabla u}{|\nabla u|}\,\mathrm{d}s.$$

Passing to the limit as $\varepsilon \downarrow 0$ and making use of Beppo Levi's monotone convergence theorem, we obtain (1.2.6) for all $\Phi \in \mathscr{D}(\Omega)$.

Let $\Phi \in C(\Omega)$, $\operatorname{supp}\Phi \subset \Omega$ and let $\mathscr{M}_h\Phi$ be a mollification of Φ with radius h. Since $\operatorname{supp} \mathscr{M}_h\Phi \subset \Omega$ for small values of h, we have

$$\int_\Omega (\mathscr{M}_h\Phi)(\nabla u)\,\mathrm{d}x = \int_0^\infty \mathrm{d}t \int_{\mathscr{E}_t} \mathscr{M}_h\Phi\,\mathrm{d}s. \tag{1.2.7}$$

Obviously, there exists a constant $C = C(\Phi)$ such that

$$\int_{\mathscr{E}_t} \mathscr{M}_h\Phi\,\mathrm{d}s \le C \int_{\mathscr{E}_t} \alpha\,\mathrm{d}s, \tag{1.2.8}$$

where $\alpha \in \mathscr{D}(\Omega)$, $\alpha = 1$ on $\bigcup_h \operatorname{supp}\mathscr{M}_h\Phi$, $\alpha \ge 0$. By (1.2.6), applied to $\Phi = \alpha$, the integral on the right-hand side of (1.2.8) is an integrable function on $(0, +\infty)$. Since $\mathscr{M}_h\Phi \to \Phi$ uniformly and since $s(\mathscr{E}_t \cap \operatorname{supp}\alpha) < \infty$ for almost all t, then also

$$\int_{\mathscr{E}_t} \mathscr{M}_h\Phi\,\mathrm{d}s \xrightarrow{h\to 0} \int_{\mathscr{E}_t} \Phi\,\mathrm{d}s$$

for almost all t. Now, Lebesgue's theorem ensures the possibility of passing to the limit as $h \to 0$ in (1.2.7).

Further, we remove the restriction $\operatorname{supp}\Phi \subset \Omega$. Let $\Phi \in C(\Omega)$, $\Phi \ge 0$ and let α_m be a sequence of nonnegative functions in $\mathscr{D}(\Omega)$ such that $\bigcup_m \operatorname{supp}\alpha_m = \Omega$, $0 \le \alpha_m \le 1$ and $\alpha_m(x) = 1$ for $x \in \operatorname{supp}\alpha_{m-1}$. Then $\operatorname{supp}(\alpha_m\Phi) \subset \Omega$ and

$$\int_\Omega \alpha_m\Phi|\nabla u|\,\mathrm{d}x = \int_0^\infty \mathrm{d}t \int_{\mathscr{E}_t} \alpha_m\Phi\,\mathrm{d}s.$$

Since the sequence $\alpha_m\Phi$ does not decrease, we may pass to the limit as $m \to \infty$ by Beppo Levi's theorem (see Natanson [627]). This completes the proof. □

1.2.5 Comments to Sect. 1.2

In Sect. 1.2 we collected auxiliary material most of which will be used in this chapter.

Theorem 1.2.1/1 is due to Besicovitch in the two-dimensional case for disks [86]. Morse generalized this result to more general spaces and shapes [616]. Theorem 1.2.1/2 is due to Gustin [331]. Here we presented a simple proof of Theorem 1.2.1/2 given by Federer [270]. Theorem 1.2.2/2 was proved by Morse [614] for functions in C^n. Here we followed the proof presented in the book by Landis [475] Chap. II, §2. Whitney showed in [795] that there exist functions $f \in C^{n-1}$ for which Theorem 1.2.2/2 fails.

Lemma 1.2.3 is contained in the paper by Faddeev [266]. The equality (1.2.6) was established by Kronrod [466] in the two-dimensional case for asymptotically differentiable functions. Federer proved a generalization of Theorem 1.2.4 for Lipschitz mappings $\mathbb{R}^n \to \mathbb{R}^m$, in [269]. This result, frequently called *the coarea formula*, is the identity

$$\int_{\Omega} \Phi(x)\big|J_m f(x)\big|\,\mathrm{d}x = \int_{\mathbb{R}^m} \int_{f^{-1}(y)} \Phi(x)\,\mathrm{d}H_{n-m}(x)\,\mathrm{d}y,$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $f : \Omega \to \mathbb{R}^m$ is Lipschitz, Φ is integrable $: \Omega \to \mathbb{R}$, $J_m f$ is its m-dimensional Jacobian, so that $|J_m f|$ is the square root of the sum of the squares of the determinants of the $m \times m$ minors of the differential of f, and $1 \le m < n$. Malý, Swanson, and Ziemer [514] generalized the co-area formula to the case $f \in [W_p^1(\Omega, \mathrm{loc})]^m$ with $p > m > 1$ or $p \ge m = 1$. Refinements and consequences of this result concerning the space W_1^1 can be found in the paper by Swanson [735].

In the form of an inequality with an appropriate definition of the modulus of the gradient of a function, Theorem 1.2.4 may be extended to abstract metric spaces, cf. Bobkov and Houdré [116, 117].

1.3 Some Inequalities for Functions of One Variable

1.3.1 Basic Facts on Hardy-type Inequalities

Most of this section is concerned with variants and extensions of the following Hardy inequality (cf. Hardy, Littlewood, and Pólya [351], Sect. 9.9).

If $f(x) \ge 0$, then

$$\int_0^\infty x^{-r} F(x)^p\,\mathrm{d}x \le \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}\big(xf(x)\big)^p\,\mathrm{d}x, \tag{1.3.1}$$

where $p > 1$, $r \ne 1$ and

$$F(x) = \int_0^x f(t)\,\mathrm{d}t \quad \text{for } r > 1,$$
$$F(x) = \int_x^\infty f(t)\,\mathrm{d}t \quad \text{for } r < 1.$$

In this section we make some remarks concerning (1.3.1) and related inequalities.

(i) First of all, (1.3.1) fails if $r = 1$ for each of the above definitions of F. One can see it by choosing either f or $1 - f$ as the characteristic function of the interval $[0, 1]$.

(ii) The constant factor in front of the integral on the right-hand side of (1.3.1) is sharp. Let, for example, $r > 1$, and let

$$\int_0^\infty x^{-r}|F|^p\,\mathrm{d}x \le C\int_0^\infty x^{-r+p}|f|^p\,\mathrm{d}x \tag{1.3.2}$$

hold. We take

$$f_\varepsilon(x) = \begin{cases} 0 & \text{for } x > 1,\\ x^{\frac{r-p-1}{p}-\varepsilon} & \text{for } x > 1,\end{cases}$$

where ε is a positive sufficiently small number. Then

$$F_\varepsilon(x) = \int_0^x f_\varepsilon(t)\,\mathrm{d}t = \begin{cases} 0 & \text{for } x < 1,\\ \frac{1}{\frac{r-1}{p}-\varepsilon}(x^{\frac{r-1}{p}-\varepsilon}-1) & \text{for } x > 1.\end{cases}$$

We have

$$\int_0^\infty x^{-r}|F_\varepsilon|^p\,\mathrm{d}x = \left(\frac{p}{r-1-p\varepsilon}\right)^p\int_1^\infty \left(1-x^{\frac{1-r}{p}+\varepsilon}\right)^p\frac{\mathrm{d}x}{x^{1+p\varepsilon}}.$$

Since $(1-x^{\frac{1-r}{p}+\varepsilon})^p = 1+O(x^{\frac{1-r}{p}+\varepsilon})$ for $x > 1$, the last integral is equal to

$$\int_1^\infty \frac{\mathrm{d}x}{x^{1+p\varepsilon}} + O(1) = (p\varepsilon)^{-1} + O(1).$$

The right-hand side of (1.3.2) is

$$\int_0^\infty x^{r+p}|f_\varepsilon|^p\,\mathrm{d}x = \int_1^\infty x^{-1-p\varepsilon}\,\mathrm{d}x = (p\varepsilon)^{-1}.$$

Hence (1.3.2) becomes

$$\frac{p^p}{(r-1-p\varepsilon)^p}\left(\frac{1}{p\varepsilon}+O(1)\right) \le C(p\varepsilon)^{-1},$$

which implies

$$C \ge \frac{p^p}{(r-1)^p}.$$

The case $r < 1$ is quite similar.

(iii) A multidimensional variant of Hardy's inequality is

$$\int_{\mathbb{R}^m}\big|u(y)\big|^p|y|^{-s}\,\mathrm{d}y \le \left(\frac{p}{|s-m|}\right)^p\int_{\mathbb{R}^m}\big|\nabla u(y)\big|^p|y|^{p-s}\,\mathrm{d}y, \tag{1.3.3}$$

where $u \in C_0^\infty(\mathbb{R}^m)$ and $s \ne m$. In the case $s > m$ we require $u(0) = 0$. This inequality follows from the one-dimensional inequality (1.3.3) written as

$$\int_0^\infty \frac{|v(r)|^p}{r^{s-m+1}}\,\mathrm{d}r \le \left(\frac{p}{|s-m|}\right)^p\int_0^\infty \big|v'(r)\big|^p r^{m+p-s-1}\,\mathrm{d}r,$$

if one passes to spherical coordinates $r = |x|$, $\omega = x/r$ and remarks that $|\frac{\partial u}{\partial r}| \le |\nabla u|$. One should use, of course, that

$$\int_{\mathbb{R}^m} \cdots \, \mathrm{d}x = \int_{S^{m-1}} \mathrm{d}s_\omega \int_0^\infty \cdots r^{m-1} \, \mathrm{d}r.$$

The constant in (1.3.3) is sharp, which can be shown by radial functions.

(iv) A more general Hardy's inequality

$$\int_{\mathbb{R}^{m+n}} \big|u(z)\big|^p |y|^{-s} \, \mathrm{d}z \le \left(\frac{p}{|s-m|}\right)^p \int_{\mathbb{R}^{m+n}} \big|\nabla u(z)\big|^p |y|^{p-s} \, \mathrm{d}z,$$

where $u \in C_0^\infty(\mathbb{R}^{m+n})$, $z = (x, y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $s \ne m$, and additionally $u(x, 0) = 0$ for all $x \in \mathbb{R}^n$ if $s > m$, is obtained by integration of (1.3.3), with $u(z)$ instead of $u(y)$, over $\mathbb{R}^n$.

(v) Finally, a few words about the critical value $s = m$ excluded in (1.3.3). The one-dimensional inequality holds

$$\int_1^\infty \frac{|u(t)|^p}{t(\log t)^p} \, \mathrm{d}t \le \left(\frac{p}{p-1}\right)^p \int_1^\infty |u'(t)|^p t^{p-1} \, \mathrm{d}t, \tag{1.3.4}$$

where $u(1) = 0$ and $p > 1$. In fact, by introducing the new variable $x = \log t$, we rewrite (1.3.4) as the Hardy inequality

$$\int_0^\infty \frac{|\tilde{u}(x)|^p}{x^p} \, \mathrm{d}x \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |\tilde{u}'(x)|^p \, \mathrm{d}x$$

with $\tilde{u}(x) = u(e^x)$.

As a consequence of (1.3.4) we obtain

$$\int_{\mathbb{R}^{n+m}} \frac{|u(z)|^p \, \mathrm{d}z}{|y|^m (\log |y|)^p} \le \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}^{n+m}} \big|\nabla u(z)\big|^p |y|^{p-m} \, \mathrm{d}z$$

for all $u \in C_0^\infty(\mathbb{R}^{n+m})$ such that $u(z) = 0$ for $|y| \le 1$.

1.3.2 Two-weight Extensions of Hardy's Type Inequality in the Case $p \le q$

Theorem 1. *Let μ and ν be nonnegative Borel measures on $(0, \infty)$ and let ν^* be the absolutely continuous part of ν. The inequality*

$$\left[\int_0^\infty \left|\int_0^x f(t) \, \mathrm{d}t\right|^q \mathrm{d}\mu(x)\right]^{1/q} \le C \left[\int_0^\infty \big|f(x)\big|^p \, \mathrm{d}\nu(x)\right]^{1/p}, \tag{1.3.5}$$

holds for all Borel functions f and $1 \le p \le q \le \infty$ if and only if

$$B := \sup_{r>0} \big[\mu\big([r, \infty)\big)\big]^{1/q} \left[\int_0^r \left(\frac{d\nu^*}{\mathrm{d}x}\right)^{-1/(p-1)} \mathrm{d}x\right]^{(p-1)/p} < \infty. \tag{1.3.6}$$

Moreover, if C is the best constant in (1.3.5), then

$$B \le C \le B\left(\frac{q}{q-1}\right)^{(p-1)/p} q^{1/q}. \tag{1.3.7}$$

If $p = 1$ or $q = \infty$, then $B = C$.

In the case $q = \infty$ the condition (1.3.6) means that

$$B = \sup\bigl\{r > 0 : \mu([r, \infty)) > 0\bigr\} < \infty,$$

and $\frac{d\nu^*}{dx} > 0$ for almost all $x \in [0, B]$.

We begin with the proof of the following less general theorem on absolutely continuous measures μ and ν.

Theorem 2. *Let $1 \le p \le q \le \infty$. In order that there exists a constant C, independent of f, such that*

$$\left[\int_0^\infty \left|w(x)\int_0^x f(t)\,dt\right|^q dx\right]^{1/q} \le C\left[\int_0^\infty \bigl|v(x)f(x)\bigr|^p dx\right]^{1/p}, \tag{1.3.8}$$

it is necessary and sufficient that

$$B := \sup_{r>0}\left(\int_r^\infty \bigl|w(x)\bigr|^q dx\right)^{1/q}\left(\int_0^r \bigl|v(x)\bigr|^{-p'} dx\right)^{1/p'} < \infty, \tag{1.3.9}$$

where $p' = p/(p-1)$. Moreover, if C is the best constant in (1.3.8) and B is defined by (1.3.9), then (1.3.7) holds. If $p = 1$ or $p = \infty$, then $B = C$.

Proof. The case $1 < p \le q < \infty$. *Necessity.* If $f \ge 0$ and $\operatorname{supp} f \subset [0, r]$, then from (1.3.8) it follows that

$$\left(\int_r^\infty \bigl|w(x)\bigr|^q dx\right)^{1/q}\int_0^r f(t)\,dt \le C\left(\int_0^r \bigl|v(x)f(x)\bigr|^p dx\right)^{1/p}.$$

Let

$$\int_0^r \bigl|v(x)\bigr|^{-p/(p-1)} dx < \infty.$$

We set $f(x) = |v(x)|^{-p'}$ for $x < r$ and $f(x) = 0$ for $x > r$. Then

$$\left[\int_r^\infty \bigl|w(x)\bigr|^q dx\right]^{1/q}\left[\int_0^r \bigl|v(x)\bigr|^{-p'} dx\right]^{1/p'} \le C. \tag{1.3.10}$$

If

$$\int_0^r \bigl|v(x)\bigr|^{-p'} dx = \infty,$$

then we arrive at the same result, replacing $v(x)$ by $v(x) + \varepsilon \operatorname{sgn} v(x)$ with $\varepsilon > 0$ in (1.3.8) and passing to the limit as $\varepsilon \to 0$.

Sufficiency. We put

$$h(x) = \Big(\int_0^x \big| v(t) \big|^{-p'} \, dt \Big)^{1/qp'}.$$

By Hölder's inequality,

$$\begin{aligned} &\Big(\int_0^\infty \Big| w(x) \int_0^x f(t) \, dt \Big|^q \, dx \Big)^{p/q} \\ &\le \Big\{ \int_0^\infty \big| w(x) \big|^q \Big(\int_0^x \big| f(t) h(t) v(t) \big|^p \, dt \Big)^{q/p} \\ &\quad \times \Big(\int_0^x \big| h(t) v(t) \big|^{-p'} \, dt \Big)^{q/p'} \, dx \Big\}^{p/q}. \end{aligned} \tag{1.3.11}$$

Now we prove that

$$\Big(\int_0^\infty \varphi(x) \Big(\int_0^x f(y) \, dy \Big)^r \, dx \Big)^{1/r} \le \int_0^\infty f(y) \Big(\int_y^\infty \varphi(x) \, dx \Big)^{1/r} \, dy, \tag{1.3.12}$$

provided that $\varphi \ge 0$, $f \ge 0$ and $r \ge 1$. In fact, the left-hand side in (1.3.12) is equal to

$$\Big(\int_0^\infty \Big(\int_0^\infty \varphi(x)^{1/r} f(y) \chi_{[y,\infty)}(x) \, dy \Big)^r \, dx \Big)^{1/r},$$

where $\chi_{[y,\infty)}$ is the characteristic function of the halfaxis $[y, \infty)$. By Minkowski's inequality the last expression does not exceed

$$\begin{aligned} &\int_0^\infty \Big(\int_0^\infty \big[\varphi(x)^{1/r} f(y) \chi_{[y,\infty)}(x) \big]^r \, dx \Big)^{1/r} \, dy \\ &= \int_0^\infty f(y) \Big(\int_y^\infty \varphi(x) \, dx \Big)^{1/r} \, dy. \end{aligned}$$

By (1.3.12), the right-hand side in (1.3.11) is majorized by

$$\int_0^\infty \big| f(t) h(t) v(t) \big|^p \Big(\int_t^\infty \big| w(x) \big|^q \Big(\int_0^x \big| h(y) v(y) \big|^{-p'} \, dy \Big)^{q/p'} \, dx \Big)^{p/q} \, dt. \tag{1.3.13}$$

Using here the expression for h, we rewrite the integral in x as

$$\int_t^\infty \big| w(x) \big|^q \Big(\int_0^x \big| v(y) \big|^{-p'} \Big(\int_0^y \big| v(z) \big|^{-p'} \, dz \Big)^{-1/q} \, dy \Big)^{q/p'} \, dx. \tag{1.3.14}$$

Since

$$\int_0^x \big| v(y) \big|^{-p'} \Big(\int_0^y \big| v(z) \big|^{-p'} \, dz \Big)^{-1/q} \, dy = q' \Big(\int_0^x \big| v(x) \big|^{-p'} \, dy \Big)^{-1/q'},$$

(1.3.14) is equal to

$$(q')^{q/p'} \int_t^\infty \big|w(x)\big|^q \left(\int_0^x \big|v(y)\big|^{-p'} \,\mathrm{d}y \right)^{q/(p'q')} \mathrm{d}x.$$

By the definition of B this expression is majorized by

$$\begin{aligned} &B^{q/q'}(q')^{q/p'} \int_t^\infty \big|w(x)\big|^q \left(\int_x^\infty \big|w(y)\big|^q \,\mathrm{d}y \right)^{-1/q'} \mathrm{d}x \\ &\quad = B^{q-1}(q')^{q/p'} q \left(\int_t^\infty \big|w(x)\big|^q \,\mathrm{d}x \right)^{1/q'} \\ &\quad \le B^q (q')^{q/p'} q \left(\int_0^t \big|v(x)\big|^{-p'} \,\mathrm{d}x \right)^{-1/p'} \\ &\quad = B^q (q')^{q/p'} q h(t)^{-q}. \end{aligned} \tag{1.3.15}$$

Therefore, (1.3.11) has the following majorant

$$\begin{aligned} &\int_0^\infty \big|f(t)v(t)h(t)\big|^p \big(B^q (q')^{q/p'} q h(t)^{-q}\big)^{p/q} \,\mathrm{d}t \\ &\quad = B^q (q')^{p/p'} q^{p/q} \int_0^\infty \big|v(t)f(t)\big|^p \,\mathrm{d}t. \end{aligned}$$

Hence, (1.3.8) holds with the constant $B(q')^{(p-1)/p} q^{1/q}$.

Now we consider the limit cases.

If $p = \infty$ then $q = \infty$ and (1.3.8) follows from the obvious estimate

$$\begin{aligned} &\operatorname*{ess\,sup}_{0<x<\infty} \left| w(x) \int_0^x f(t)\,\mathrm{d}t \right| \\ &\quad \le \operatorname*{ess\,sup}_{0<x<\infty} \big|w(x)\big| \int_0^x \frac{\mathrm{d}t}{|v(t)|} \operatorname*{ess\,sup}_{0<t<x} \big|v(t)f(t)\big|. \end{aligned}$$

If $p = 1$, $q < \infty$, then from (1.3.12) it follows that

$$\begin{aligned} &\left(\int_0^\infty \left| w(x) \int_0^x f(t)\,\mathrm{d}t \right|^q \mathrm{d}x \right)^{1/q} \\ &\quad \le \int_0^\infty \big|f(t)\big| \left(\int_t^\infty \big|w(x)\big|^q \,\mathrm{d}x \right)^{1/q} \frac{1}{|v(t)|} \big|v(t)\big| \,\mathrm{d}t \\ &\quad \le B \int_0^\infty \big|v(t)f(t)\big| \,\mathrm{d}t. \end{aligned}$$

Let $q = \infty$, $p = 1$. Then

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0<x<\infty}\Big|w(x)\int_0^x f(t)\,\mathrm{d}t\Big| \\
&\quad\le \operatorname*{ess\,sup}_{0<x<\infty}\bigg(\big|w(x)\big| \operatorname*{ess\,sup}_{0<t<x}\frac{1}{|w(t)|}\int_0^x \big|v(t)f(t)\big|\,\mathrm{d}t\bigg) \\
&\quad\le B\int_0^\infty \big|v(t)f(t)\big|\,\mathrm{d}t.
\end{aligned}$$

If $p>1$, then

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0<x<\infty}\Big|w(x)\int_0^x f(t)\,\mathrm{d}t\Big| \\
&\quad\le \operatorname*{ess\,sup}_{0<x<\infty}\bigg[\big|w(x)\big|\Big(\int_0^x \big|v(t)\big|^{-p'}\,\mathrm{d}t\Big)^{1/p'}\Big(\int_0^x \big|v(t)f(t)\big|^p\,\mathrm{d}t\Big)^{1/p}\bigg] \\
&\quad\le B\Big(\int_0^x \big|v(t)f(t)\big|^p\,\mathrm{d}t\Big)^{1/p}.
\end{aligned}$$

This completes the proof of Theorem 2. □

Proof of Theorem 1. Setting $f=0$ on the support of the singular part of the measure ν, we obtain that (1.3.5) is the equivalent to

$$\bigg[\int_0^\infty\Big|\int_0^x f(t)\,\mathrm{d}t\Big|^q\,\mathrm{d}\mu(x)\bigg]^{1/q} \le C\bigg[\int_0^x \big|f(x)\big|^p\frac{d\nu^*}{\mathrm{d}x}\,\mathrm{d}x\bigg]^{1/p}.$$

The estimate $B\le C$ can be derived in the same way as in the proof of Theorem 2, if $|v(x)|^p$ is replaced by $\mathrm{d}\nu^*/\mathrm{d}x$ and

$$\int_r^\infty \big|w(x)\big|^q\,\mathrm{d}x$$

by $\mu([r,\infty))$.

Now we establish the upper bound for C. We may assume $f\ge 0$. Let $\{g_n\}$ be a sequence of decreasing absolutely continuous functions on $[0,\infty)$ satisfying

$$\begin{aligned}
&0\le g_n(x)\le g_{n+1}(x)\le \mu\big([x,\infty)\big), \\
&\lim_{n\to\infty} g_n(x)=\mu\big([x,\infty)\big),
\end{aligned}$$

for almost all x. We have

$$\bigg[\int_0^\infty\Big(\int_0^x f(t)\,\mathrm{d}t\Big)^q\,\mathrm{d}\mu(x)\bigg]^{1/q} = \bigg[\int_0^\infty \mu\big([x,\infty)\big)\,\mathrm{d}\Big(\int_0^x f(t)\,\mathrm{d}t\Big)^q\bigg]^{1/q}.$$

By the monotone convergence theorem the right-hand side is equal to

$$\sup_n \Big[\int_0^\infty g_n(x)\,\mathrm{d}\Big(\int_0^x f(t)\,\mathrm{d}t\Big)^q\Big]^{1/q}$$
$$= \sup_n \Big[\int_0^\infty \Big(\int_0^x f(t)\,\mathrm{d}t\Big)^q \big[-g_n'(x)\big]\,\mathrm{d}x\Big]^{1/q}. \tag{1.3.16}$$

The definition of the constant B and the sequence $\{g_n\}$ imply

$$\Big[\int_r^\infty \big[-g_n'(x)\big]\,\mathrm{d}x\Big]^{1/q}\Big[\int_0^r \Big(\frac{\mathrm{d}\nu^*}{\mathrm{d}x}\Big)^{-p'/p}\mathrm{d}x\Big]^{1/p'} \le B.$$

From this and Theorem 2 we conclude that the right-hand side in (1.3.16) is not greater than

$$B(q')^{(p-1)/p}q^{1/q}\Big(\int_0^\infty (f(x))^p\frac{\mathrm{d}\nu^*}{\mathrm{d}x}\,\mathrm{d}x\Big)^{1/p},$$

which completes the proof. □

Replacing x by x^{-1} we derive the following assertion from Theorem 1.

Theorem 3. *Let $1 \le p \le q \le \infty$. In order that there exist a constant C, independent of f and such that*

$$\Big[\int_0^\infty \Big|\int_x^\infty f(t)\,\mathrm{d}t\Big|^q \mathrm{d}\mu(x)\Big]^{1/q} \le C\Big[\int_0^\infty |f(x)|^p\,\mathrm{d}\nu(x)\Big]^{1/p}, \tag{1.3.17}$$

it is necessary and sufficient that the value

$$B := \sup_{r>0}\big[\mu\big((0,r)\big)\big]^{1/q}\Big[\int_r^\infty \Big(\frac{\mathrm{d}\nu^*}{\mathrm{d}x}\Big)^{-1/(p-1)}\mathrm{d}x\Big]^{(p-1)/p},$$

be finite. The best constant in (1.3.17) *satisfies the same inequalities as in Theorem* 1.

Analogously, by the change of variable

$$(0,\infty) \ni x \to y = x - x^{-1} \in (-\infty,+\infty),$$

from Theorem 1 we obtain the next assertion.

Theorem 4. *Let $1 \le p \le q \le \infty$. In order that there exist a constant C, independent of f and such that*

$$\Big[\int_{-\infty}^{+\infty} \Big|\int_x^\infty f(t)\,\mathrm{d}t\Big|^q \mathrm{d}\mu(x)\Big]^{1/q} \le C\Big[\int_{-\infty}^{+\infty} |f(x)|^p\,\mathrm{d}\nu(x)\Big]^{1/p}, \tag{1.3.18}$$

it is necessary and sufficient that

$$B := \sup_{r\in(-\infty,+\infty)}\big[\mu\big((-\infty,r)\big)\big]^{1/q}\Big[\int_r^\infty \Big(\frac{\mathrm{d}\nu^*}{\mathrm{d}x}\Big)^{-1/(p-1)}\mathrm{d}x\Big]^{(p-1)/p} < \infty.$$

Constants B and C are related in the same way as in Theorem 1.

1.3.3 Two-Weight Extensions of Hardy's Inequality in the Case $p > q$

Lemma. *Let $1 \le q < p \le \infty$ and let ω be a nonnegative Borel function on $(0,b)$, where $b \in (0,\infty]$. In order that there exist a constant C, independent of ψ and such that*

$$\Big(\int_0^b \omega(t)\Big|\int_0^t \psi(\tau)\,\mathrm{d}\tau\Big|^q \mathrm{d}t\Big)^{1/q} \le C\Big(\int_0^b \big|\psi(t)\big|^p \,\mathrm{d}t\Big)^{1/p}, \tag{1.3.19}$$

it is necessary and sufficient that

$$B := \Big(\int_0^b \Big(\int_t^b \omega(\tau)\,\mathrm{d}\tau\Big)^{p/(p-q)} t^{(q-1)p/(p-q)}\,\mathrm{d}t\Big)^{(p-q)/pq} < \infty. \tag{1.3.20}$$

If C is the best constant in (1.3.19), then

$$\Big(\frac{p-q}{p-1}\Big)^{(q-1)/q} q^{1/q} B \le C \le \Big(\frac{p}{p-1}\Big)^{(q-1)/q} q^{1/q} B$$

for $q > 1$ and $B = C$ for $q = 1$.

Proof. Sufficiency. First consider the case $q > 1$. We may assume $\psi(t) \ge 0$. Integrating by parts on the left-hand side of (1.3.19) and using Hölder's inequality with exponents $p/(p-q)$, $p/(q-1)$, and p, we obtain

$$\begin{aligned}
&\Big(\int_0^b \omega(t)\Big(\int_0^t \psi(\tau)\,\mathrm{d}\tau\Big)^q \mathrm{d}t\Big)^{1/q} \\
&= q^{1/q}\Big(\int_0^b \int_t^b \omega(\tau)\,\mathrm{d}\tau\, \psi(t)\Big(\int_0^t \psi(\tau)\,\mathrm{d}\tau\Big)^{q-1} \mathrm{d}t\Big)^{1/q} \\
&\le q^{1/q}\Big[\Big(\int_0^b \psi(t)^p\,\mathrm{d}t\Big)^{1/p}\Big(\int_0^b \Big(\int_0^t \psi(\tau)\,\mathrm{d}\tau\Big)^p t^{-p}\,\mathrm{d}t\Big)^{(q-1)/p} \\
&\quad \times \Big(\int_0^b t^{(q-1)p/(p-q)}\Big(\int_t^b \omega(\tau)\,\mathrm{d}\tau\Big)^{p/(p-q)} \mathrm{d}t\Big)^{(p-q)/p}\Big]^{1/q}.
\end{aligned} \tag{1.3.21}$$

From (1.3.20) and Hardy's inequality (1.3.2), it follows that (1.3.21) is majorized by

$$B\Big(\frac{p}{p-1}\Big)^{(q-1)/q} q^{1/q}\Big(\int_0^b \psi(t)^p\,\mathrm{d}t\Big)^{1/p}.$$

Necessity. Consider, for example, the case $b = \infty$. The proof is similar for $b < \infty$. If (1.3.19) holds for the weight ω with the constant C, then it holds for the weight $\omega_n = \omega\chi_{[0,N]}$, where $\chi_{[0,N]}$ is the characteristic function of the segment $[0, N]$, with the same constant. We put

$$f_N(x) = \left(\int_x^\infty \omega_N(t)\,\mathrm{d}t\right)^{1/(p-q)} x^{(q-1)/(p-q)},$$

$$B_N = \left(\int_0^\infty \left(\int_t^\infty \omega_N(\tau)\,\mathrm{d}\tau\right)^{p/(p-q)} t^{(q-1)p/(p-q)}\,\mathrm{d}t\right)^{(p-q)/p}.$$

From (1.3.19) we have

$$\begin{aligned} CB_N^{q/(p-q)} &= C\left(\int_0^\infty f_N(x)^p\,\mathrm{d}x\right)^{1/p} \\ &\geq \left(\int_0^\infty \omega_N(t)\left(\int_0^t f_N(\tau)\,\mathrm{d}\tau\right)^q \mathrm{d}t\right)^{1/q}. \end{aligned} \tag{1.3.22}$$

Integrating by parts, we find that the right-hand side in (1.3.22) is equal to

$$\left(q\int_0^\infty f_N(t)\int_t^\infty \omega_N(\tau)\,\mathrm{d}\tau\left(\int_0^t f_N(\tau)\,\mathrm{d}\tau\right)^{q-1}\mathrm{d}t\right)^{1/q}. \tag{1.3.23}$$

Since

$$\begin{aligned} &\left(\int_0^t f_N(\tau)\,\mathrm{d}\tau\right)^{q-1} \\ &\quad = \left(\int_0^t x^{(q-1)/(p-q)}\left(\int_x^\infty \omega_N(\tau)\,\mathrm{d}\tau\right)^{1/(p-q)}\mathrm{d}x\right)^{q-1} \\ &\quad \geq \left(\int_t^\infty \omega_N(\tau)\,\mathrm{d}\tau\right)^{(q-1)/(p-q)} t^{(p-1)(q-1)/(p-q)}\left(\frac{p-1}{p-q}\right)^{1-q}, \end{aligned}$$

we see that (1.3.23) is not less than

$$\begin{aligned} &q^{1/q}\left(\frac{p-1}{p-q}\right)^{(1-q)/q}\left(\int_0^\infty\left(\int_t^\infty \omega_N(\tau)\,\mathrm{d}\tau\right)^{p/(p-q)} t^{(q-1)p/(p-q)}\,\mathrm{d}t\right)^{1/q} \\ &= q^{1/q}\left(\frac{p-1}{p-q}\right)^{(1-q)/q} B_N^{p/(p-q)}. \end{aligned}$$

Therefore,

$$B_N \leq q^{-1/q}\left(\frac{p-q}{p-1}\right)^{(1-q)/q} C,$$

and the same estimate is valid for B.

In the case $q = 1$ the condition (1.3.6) becomes especially simple:

$$B = \left(\int_0^b \left(\int_t^b \omega(\tau)\,\mathrm{d}\tau\right)^{p'} \mathrm{d}t\right)^{1/p'} < \infty.$$

To prove that in this case $C \le B$ we integrate by parts on the left-hand side of (1.3.19) and apply Hölder's inequality with exponents p and p' (cf. (1.3.21)). Then the right-hand side of (1.3.19) has the upper bound

$$\left(\int_0^b \left(\int_t^\infty \omega(\tau)\,\mathrm{d}\tau\right)^{p'} \mathrm{d}t\right)^{1/p'} \left(\int_0^b \left|\psi(t)\right|^p \mathrm{d}t\right)^{1/p}.$$

Thus, we proved that $C \le B$.

To derive the inequality $B \le C$ we substitute

$$f_N(x) = \left(\int_x^\infty \omega_N(t)\,\mathrm{d}t\right)^{1/(p-1)},$$

into (1.3.23). This yields $B_N \le C$ and hence $B \le C$. The lemma is proved.□

Theorem 1. *Let $1 \le q < p \le \infty$. Then (1.3.8) holds if and only if*

$$\left(\int_0^\infty \left[\left(\int_0^x \frac{\mathrm{d}y}{|v(y)|^{p'}}\right)^{q-1} \int_x^\infty \left|w(y)\right|^q \mathrm{d}y\right]^{p/(p-q)} \frac{\mathrm{d}x}{|v(x)|^{p'}}\right)^{(p-q)/pq} < \infty. \tag{1.3.24}$$

If C is the best constant in (1.3.8) and B stands for the left-hand side of (1.3.24), then

$$\left(\frac{p-q}{p-1}\right)^{(q-1)/q} q^{1/q} B \le C \le \left(\frac{p}{p-1}\right)^{(q-1)/q} q^{1/q} B$$

for $1 < q < p \le \infty$ and $B = C$ for $q = 1$, $1 < p \le \infty$.

Proof. We may assume that $f \ge 0$, since the right-hand side in (1.3.8) does not change and the left-hand side increases if f is replaced by $|f|$. We may as well assume $f(x) = 0$ for sufficiently large values of x. Let us put

$$t(x) = \int_0^x \left|v(y)\right|^{-p'} \mathrm{d}y.$$

Then (1.3.8) becomes

$$\left(\int_0^b \left|\tilde{w}(t)\right|^q \left|\tilde{v}(t)\right|^{p'} \left|\varphi(t)\right|^q \mathrm{d}t\right)^{1/q} \le C\left(\int_0^b \left|\varphi'(t)\right|^p \mathrm{d}t\right)^{1/p},$$

where $\tilde{w}(t(x)) = w(x)$, $\tilde{v}(t(x)) = v(x)$,

$$\varphi\big(t(x)\big) = \int_0^x f(y)\,\mathrm{d}y, \qquad b = \int_0^\infty \left|v(y)\right|^{-p'} \mathrm{d}y.$$

Now, in the case $1 \le q < p < \infty$ the result follows from the Lemma.

Let $p = \infty$. Then

$$B = \left(\int_0^\infty \left(\int_0^x \frac{\mathrm{d}y}{|v(y)|} \right)^{q-1} \int_x^\infty |w(y)|^q \,\mathrm{d}y \frac{\mathrm{d}x}{|v(x)|} \right)^{1/q}$$
$$= q^{-1/q} \left(\int_0^\infty |w(x)|^q \left(\int_0^x \frac{\mathrm{d}y}{|v(y)|} \right)^q \mathrm{d}x \right)^{1/q}.$$

Hence

$$\left(\int_0^\infty \left| w(x) \int_0^x f(t)\,\mathrm{d}t \right|^q \mathrm{d}x \right)^{1/q} \le B q^{1/q} \operatorname*{ess\,sup}_{0<x<\infty} |vf|.$$

To prove the necessity we note that v does not vanish on the set of positive measure and put $f = 1/v$. The theorem is proved. □

The following more general assertion can be derived from Theorem 1 in the same way as Theorem 1.3.2/1 was derived from Theorem 1.3.2/2.

Theorem 2. *Let μ and ν be nonnegative Borel measures on $(0, \infty)$ and let ν^* be the absolutely continuous part of ν. Inequality (1.3.8) with $1 \le q < p \le \infty$ holds for all Borel functions f if and only if*

$$B = \left(\int_0^\infty \left[\mu\big([x,\infty)\big) \left(\int_0^x \left(\frac{\mathrm{d}\nu^*}{\mathrm{d}y} \right)^{-p'} \mathrm{d}y \right)^{q-1} \right]^{p/(p-q)} \left(\frac{\mathrm{d}\nu^*}{\mathrm{d}x} \right)^{-p'} \mathrm{d}x \right)^{(p-q)/pq}$$
$$< \infty.$$

The best constant C in (1.3.17) is related with B in the same manner as in Theorem 1.

The change of variable $(0, \infty) \ni x \to y = x - x^{-1} \in (-\infty, +\infty)$ leads to the following necessary and sufficient condition for the validity of (1.3.18):

$$\int_{-\infty}^{+\infty} \left[\mu\big((-\infty, x]\big) \left(\int_x^\infty \left(\frac{\mathrm{d}\nu^*}{\mathrm{d}y} \right)^{-p'} \mathrm{d}y \right)^{q-1} \right]^{p/(p-q)} \left(\frac{\mathrm{d}\nu^*}{\mathrm{d}x} \right)^{-p'} \mathrm{d}x < \infty,$$

where $1 \le q < p \le \infty$.

1.3.4 Hardy-Type Inequalities with Indefinite Weights

Here we are concerned with inequalities similar to those in Sect. 1.3 with weights of unrestricted sign. We start with the estimate

$$\left| \int_{\mathbb{R}_+} u(x) \overline{v(x)} Q(x)\,\mathrm{d}x \right| \le \mathrm{const} \|u'\|_{L_p(\mathbb{R}_+)} \|v'\|_{L_{p'}(\mathbb{R}_+)} \tag{1.3.25}$$

for all $u, v \in C_0^\infty(\mathbb{R}_+)$.

Let us assume that Q is a locally integrable real- or complex-valued function such that

$$\lim_{b\to+\infty}\int_a^b Q(x)\,\mathrm{d}x = \int_a^\infty Q(x)\,\mathrm{d}x \tag{1.3.26}$$

exists for every $a > 0$.

Theorem 1. *Under the above assumptions on Q, let*

$$\Gamma(x) = \int_x^\infty Q(t)\,\mathrm{d}t, \quad x > 0.$$

Let $1 < p < \infty$, and $p^* = \max(p, p')$. Then (1.3.25) is valid if and only if

$$\sup_{a>0} a^{p^*-1}\int_a^\infty \big|\Gamma(x)\big|^{p^*}\,\mathrm{d}x < \infty. \tag{1.3.27}$$

It is not difficult to see that (1.3.27) is equivalent to the pair of conditions

$$\sup_{a>0} a^{p-1}\int_a^\infty \big|\Gamma(x)\big|^{p}\,\mathrm{d}x < \infty, \qquad \sup_{a>0} a^{p'-1}\int_a^\infty \big|\Gamma(x)\big|^{p'}\,\mathrm{d}x < \infty. \tag{1.3.28}$$

Proof. For $u, v \in C_0^\infty(\mathbb{R}_+)$, let

$$\langle Qu, v\rangle = \int_0^\infty Q(x)u(x)\overline{v(x)}\,\mathrm{d}x.$$

We can extend $\langle Qu, v\rangle$ by continuity to the case where

$$u(x) = \int_0^x f(t)\,\mathrm{d}t, \qquad v(x) = \int_0^x g(\tau)\,\mathrm{d}\tau,$$

for $f, g \in C_0^\infty(\mathbb{R}_+)$, by setting

$$\langle Qu, v\rangle = \lim_{a\to+\infty}\int_0^a Q(x)u(x)\overline{v(x)}\,\mathrm{d}x.$$

To show that the limit on the right-hand side exists, assume that both f and g are supported in $(\delta, b) \subset \mathbb{R}_+$. Then clearly,

$$\lim_{a\to+\infty}\int_0^a Q(x)u(x)\overline{v(x)}\,\mathrm{d}x = \int_\delta^b Q(x)\Big(\int_\delta^x f(t)\,\mathrm{d}t\int_\delta^x \overline{g(\tau)}\,\mathrm{d}\tau\Big)\,\mathrm{d}x + \int_b^\infty Q(x)\,\mathrm{d}x\int_\delta^b f(t)\,\mathrm{d}t\int_\delta^b \overline{g(\tau)}\,\mathrm{d}\tau.$$

Observe that we have to be careful here: In what follows one cannot estimate the two terms on the right-hand side of the preceding equation separately because this would lead to the restriction

$$\sup_{b>0} b\left|\int_b^\infty Q(x)\,\mathrm{d}x\right| < \infty,$$

which is not necessary for the boundedness of the bilinear form.

Using Fubini's theorem, we obtain

$$\begin{aligned}\langle Qu, v\rangle &= \int_\delta^b \int_\delta^b f(t)\overline{g(\tau)} \int_{\max(t,\tau)}^b Q(x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau \\ &\quad + \int_b^\infty Q(x)\,\mathrm{d}x \int_\delta^b f(t)\,\mathrm{d}t \int_\delta^b \overline{g(\tau)}\,\mathrm{d}\tau \\ &= \int_\delta^b \int_\delta^b f(t)\overline{g(\tau)} \int_{\max(t,\tau)}^\infty Q(x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau.\end{aligned}$$

By definition

$$\Gamma\bigl(\max\{t,\tau\}\bigr) = \int_{\max(t,\tau)}^\infty Q(x)\,\mathrm{d}x.$$

Thus, (1.3.25) is equivalent to the inequality

$$\begin{aligned}&\left|\int_0^\infty \int_0^\infty f(t)\overline{g(\tau)}\Gamma\bigl(\max\{t,\tau\}\bigr)\,\mathrm{d}t\,\mathrm{d}\tau\right| \\ &\quad \le \mathrm{const}\|f\|_{L_p(\mathbb{R}_+)}\|g\|_{L_{p'}(\mathbb{R}_+)}\end{aligned} \tag{1.3.29}$$

for compactly supported f, g.

Using the reverse Hölder inequality, the preceding estimate can be rewritten in the equivalent form

$$\int_0^\infty \left|\int_0^\infty \Gamma\bigl(\max\{t,\tau\}\bigr) f(t)\,\mathrm{d}t\right|^p \mathrm{d}\tau \le c\|f\|^p_{L_p(\mathbb{R}_+)}. \tag{1.3.30}$$

Clearly,

$$\int_0^\infty \Gamma\bigl(\max\{t,\tau\}\bigr) f(t)\,\mathrm{d}t = \Gamma(\tau)\int_0^\tau f(t)\,\mathrm{d}t + \int_\tau^\infty f(t)\Gamma(t)\,\mathrm{d}t. \tag{1.3.31}$$

Suppose now that (1.3.27), or equivalently, both inequalities in (1.3.28) hold. Then the estimate involving the first term in (1.3.31) is established by means of the weighted Hardy inequality

$$\int_0^\infty \left|\int_0^\tau f(t)\,\mathrm{d}t\right|^p \bigl|\Gamma(\tau)\bigr|^p\,\mathrm{d}\tau \le C\|f\|^p_{L_p(\mathbb{R}_+)}, \tag{1.3.32}$$

which holds if and only if the first part of condition (1.3.28) is valid (see Theorem 1.3.2/1).

The second term in (1.3.31) is estimated by using a similar weighted Hardy inequality

$$\int_0^\infty \left| \int_\tau^\infty f(t) \Gamma(t) \, \mathrm{d}t \right|^p \mathrm{d}\tau \le C \|f\|^p_{L_p(\mathbb{R}_+)},$$

which, by Theorem 1.3.2/3, is equivalent to the second part of condition (1.3.28). This proves the "if" part of the theorem.

To prove the "only if" part, it suffices to assume that $f(x)$ in (1.3.30) is supported on an interval $[\delta, a]$, $a > 0$, and restrict the domain of integration in τ on the left-hand side of (1.3.30) to $\tau \in (a, +\infty)$. Taking into account that the second term in (1.3.31) vanishes, we get

$$\begin{aligned} &\int_a^\infty \left| \int_\delta^a \Gamma\big(\max\{t, \tau\}\big) f(t) \, \mathrm{d}t \right|^p \mathrm{d}\tau \\ &\quad = \left| \int_\delta^a f(t) \, \mathrm{d}t \right|^p \int_a^\infty \big|\Gamma(\tau)\big|^p \, \mathrm{d}\tau \le C \|f\|^p_{L_p(\mathbb{R}_+)}. \end{aligned}$$

Applying the reverse Hölder inequality again, we obtain the first part of (1.3.28)

$$a^{p-1} \int_a^\infty \big|\Gamma(\tau)\big|^p \, \mathrm{d}\tau \le C.$$

Since (1.3.30) is symmetric, a dual estimate in the $L_{p'}$ norm yields the second part of (1.3.28)

$$a^{p'-1} \int_a^\infty \big|\Gamma(\tau)\big|^{p'} \, \mathrm{d}\tau \le C.$$

Hence (1.3.27) holds. □

Remark 1. Notice that a similar argument works with minor changes if the integration (1.3.25) is performed against real- or complex-valued measure $\mathrm{d}Q$ in the place of $Q(x)\,\mathrm{d}x$. However, the general case where Q is a distribution requires taking care of some technical problems which are considered in detail by the author and Verbitsky in [592], Sect. 2.

Remark 2. For any $p \in (1, \infty)$, a simple condition

$$\sup_{a>0} a\big|\Gamma(a)\big| < \infty, \tag{1.3.33}$$

is sufficient, but generally not necessary for (1.3.25) to hold. However, for *nonnegative* Q, condition (1.3.33) is equivalent to (1.3.27).

Theorem 1 is easily carried over to the two-weight setting.

Theorem 2. *Let $W_1, W_2 \ge 0$ be locally integrable weight functions on $\mathbb{R}_+$ such that, respectively,*

$$\int_0^a W_1(x)^{1-p'} \, \mathrm{d}x < +\infty \quad \text{and} \quad \int_0^a W_2(x)^{1-p} \, \mathrm{d}x < +\infty$$

for every $a > 0$. Then the two-weight bilinear inequality

$$\left|\int_0^\infty u(x)v(x)Q(x)\,\mathrm{d}x\right| \le \mathrm{const}\left(\int_0^\infty |u'(x)|^p W_1(x)\,\mathrm{d}x\right)^{1/p} \times\left(\int_0^\infty \left|v'(x)\right|^{p'} W_2(x)\,\mathrm{d}x\right)^{1/p'} \tag{1.3.34}$$

holds for all $u, v \in C_0^\infty(\mathbb{R}_+)$ if and only if the following pair of conditions hold:

$$\sup_{a>0}\left(\int_0^a W_1(x)^{1-p'}\,\mathrm{d}x\right)^{p-1}\int_a^\infty \left|\Gamma(x)\right|^p W_2(x)^{1-p}\,\mathrm{d}x < \infty \tag{1.3.35}$$

and

$$\sup_{a>0}\left(\int_0^a W_2(x)^{1-p}\,\mathrm{d}x\right)^{p'-1}\int_a^\infty \left|\Gamma(x)\right|^{p'} W_1(x)^{1-p}\,\mathrm{d}x < \infty, \tag{1.3.36}$$

where $\Gamma(x) = \int_x^\infty Q(t)\,\mathrm{d}t$.

For functions defined on the interval $(0, 1)$, Theorem 2 can be recast in a similar way.

Theorem 3. *The inequality*

$$\left|\int_0^1 u(x)v(x)Q(x)\,\mathrm{d}x\right| \le \mathrm{const}\left\|u'(x)\right\|_{L_p(0,1)}\left\|v'(x)\right\|_{L_{p'}(0,1)} \tag{1.3.37}$$

holds for all $u, v \in C^\infty(0, 1)$ such that $u(0) = 0, v(0) = 0$ if and only if Q can be represented in the form $Q = \Gamma'$, where

$$\sup_{a>0} a^{p-1}\int_0^1 \left|\Gamma(x)\right|^{p^*}\,\mathrm{d}x < \infty \tag{1.3.38}$$

as $a \to 0^+$. The corresponding compactness criterion holds with the preceding condition replaced by

$$\limsup_{a\to 0^+} a^{p^*-1}\int_0^1 \left|\Gamma(x)\right|^{p^*}\,\mathrm{d}x = 0.$$

For functions with zero boundary values at both endpoints, one only has to add similar conditions at $a = 1$.

We now state the analog of Theorem 1 on the whole line $\mathbb{R}$ for the Sobolev space $W_p^1(\mathbb{R})$ which consists of absolutely continuous functions $u : \mathbb{R} \to \mathbb{C}$ such that

$$\|u\|_{W^1_p(\mathbb{R})} = \left[\int_{\mathbb{R}} \left(\left|u(x)\right|^p + \left|u'(x)\right|^p\right) \mathrm{d}x\right]^{1/p} < \infty.$$

Theorem 4. *Let $1 < p < \infty$, and $p^* = \max(p, p')$. The inequality*

$$\left|\int_{\mathbb{R}} u(x)v(x)Q(x)\,\mathrm{d}x\right| \le \mathrm{const}\, \|u\|_{W^1_p(\mathbb{R})} \|v\|_{W^1_{p'}(\mathbb{R})} \tag{1.3.39}$$

holds for all $u, v \in C_0^\infty(\mathbb{R})$, if and only if Q can be represented in the form $Q = \Gamma' + \Gamma_0$, where Γ and Γ_0 satisfy the following conditions:

$$\sup_{a>0} \int_0^{a+1} \left|\Gamma(x)\right|^{p^*} \mathrm{d}x < \infty, \qquad \sup_{a>0} \int_0^{a+1} \left|\Gamma_0(x)\right| \mathrm{d}x < \infty. \tag{1.3.40}$$

The proofs of Theorems 3 and 4 are similar to the proof of Theorem 1.

The usual approach to inequality (1.3.25), in the case where Q is real valued, is to represent it in the form $Q = Q_+ - Q_-$, where Q_+ and Q_- are, respectively, the positive and negative parts of Q, and then treat them separately. However, this procedure ignores a possible cancellation between Q_+ and Q_- and diminishes the class of admissible potentials Q.

The following examples demonstrate the difference between sharp results which follow from Theorem 1, and the usual approach where Q_+ and Q_- are treated separately.

Example 1. Let

$$Q(x) = \frac{\sin x}{x^{1+\epsilon}}, \quad \epsilon > 0.$$

Then

$$\Gamma(x) = \int_x^{+\infty} \frac{\sin t}{t^{1+\epsilon}}\,\mathrm{d}t = \frac{\cos x}{x^{1+\epsilon}} + O\left(\frac{1}{x^{2+\epsilon}}\right) \quad \text{as } x \to +\infty.$$

As $x \to 0+$, clearly, $\Gamma(x) = O(1)$ for $\epsilon < 1$, $\Gamma(x) = O(\log x)$ for $\epsilon = 1$, and $\Gamma(x) = O(x^{1-\epsilon})$ for $\epsilon > 1$. From this it is easy to see that (1.3.27) is valid if and only if $0 \le \epsilon \le 2$, and hence by Theorem 1, $\mathcal{L} : \mathring{L}^1_p(\mathbb{R}_+) \to L^{-1}_p(\mathbb{R}_+)$ is bounded for $1 < p < \infty$. Moreover, the multiplication operator $Q : \mathring{L}^1_p(\mathbb{R}_+) \to L^{-1}_p(\mathbb{R}_+)$ is compact if and only if $0 < \epsilon < 2$.

Note that the same Theorem 1 applied separately to Q_+ and Q_- gives a satisfactory result only for $1 \le \epsilon \le 2$.

In the next example, Q is a charge on $\mathbb{R}_+$, and the condition imposed on Q depends explicitly on p.

Example 2. Let

$$Q = \sum_{j=1}^{\infty} c_j(\delta_j - \delta_{j+1}),$$

where δ_a is a unit point mass at $x = a$. Then clearly

$$\Gamma(x) = \sum_{j=1}^{\infty} c_j \chi_{(j,j+1)}(x).$$

It follows that (1.3.25) holds if and only if

$$\sup_{n \geq 1} n^{p^*-1} \sum_{j=n}^{\infty} |c_j|^{p^*} < \infty.$$

In particular, for $1 < r \leq 2$, let $c_j = j^{-1/r}$ if $j = 2^m$, and $c_j = 0$ otherwise. Then $\mathcal{L} : \mathring{L}_p^1(\mathbb{R}_+) \to L_p^{-1}(\mathbb{R}_+)$ if and only if $r \leq p \leq r/(r-1)$. Note that in this example condition (1.3.27) fails for all $r > 1$.

1.3.5 Three Inequalities for Functions on $(0, \infty)$

Lemma 1. *If f is a nonnegative nonincreasing function on $(0, \infty)$ and $p \geq 1$, then*

$$\int_0^{\infty} \big[f(x)\big]^p \,\mathrm{d}\big(x^p\big) \leq \left(\int_0^{\infty} f(x)\,\mathrm{d}x\right)^p. \tag{1.3.41}$$

Proof. Obviously,

$$\begin{aligned} p\int_0^{\infty} \big[xf(x)\big]^{p-1} f(x)\,\mathrm{d}x &\leq p\int_0^{\infty} \left[\int_0^x f(t)\,\mathrm{d}t\right]^{p-1} f(x)\,\mathrm{d}x \\ &= \left(\int_0^{\infty} f(x)\,\mathrm{d}x\right)^p. \end{aligned}$$

The result follows.

Lemma 2. *If $f(x) \geq 0$, then*

$$\begin{aligned} &\left[\int_0^{\infty} f(x)\,\mathrm{d}x\right]^{a\mu+b\lambda} \\ &\quad \leq c(a,b,\lambda,\mu)\left[\int_0^{\infty} x^{a-1-\lambda} f(x)^a\,\mathrm{d}x\right]^{\mu}\left[\int_0^{\infty} x^{b-1+\mu} f(x)^b\,\mathrm{d}x\right]^{\lambda}, \end{aligned} \tag{1.3.42}$$

where $a > 1$, $b > 1$, $0 < \lambda < a$, $0 < \mu < b$.

Proof. Obviously,

$$\begin{aligned} \int_0^{\infty} f(x)\,\mathrm{d}x = &\int_0^{\infty} x^{(a-1-\lambda)/a} f(x) \frac{\mathrm{d}x}{x^{(a-1-\lambda)/a}(1+x)} \\ &+ \int_0^{\infty} x^{(b-1+\mu)/b} f(x) \frac{\mathrm{d}x}{x^{(b-1+\mu)/b}(1+x^{-1})}. \end{aligned}$$

By Hölder's inequality

$$\int_0^\infty x^{(a-1-\lambda)/a} f(x) \frac{\mathrm{d}x}{x^{(a-1-\lambda)/a}(1+x)} \le L\Big(\int_0^\infty x^{a-1-\lambda} f(x)^a \,\mathrm{d}x\Big)^{1/a},$$

$$\int_0^\infty x^{(b-1+\mu)/b} f(x) \frac{\mathrm{d}x}{x^{(b-1+\mu)/b}(1+x^{-1})} \le M\Big(\int_0^\infty x^{b-1+\mu} f(x)^b \,\mathrm{d}x\Big)^{1/b},$$

where

$$L = \Big(\int_0^\infty \frac{\mathrm{d}x}{x^{(a-1-\lambda)/(a-1)}(1+x)^{a/(a-1)}}\Big)^{(a-1)/a},$$

$$M = \Big(\int_0^\infty \frac{\mathrm{d}x}{x^{(b-1+\mu)/(b-1)}(1+x^{-1})^{b/(b-1)}}\Big)^{(b-1)/b}.$$

Hence

$$\int_0^\infty f(x)\,\mathrm{d}x \le L\Big(\int_0^\infty x^{a-1-\lambda} f(x)^a \,\mathrm{d}x\Big)^{1/a} + M\Big(\int_0^\infty x^{b-1+\mu} f(x)^b \,\mathrm{d}x\Big)^{1/b}.$$

Replacing $f(x)$ by $f(z/\varrho)$, where $\varrho > 0$, and setting $z = \varrho x$, we obtain

$$\varrho^{-1}\int_0^\infty f(z/\varrho)\,\mathrm{d}z \le L\varrho^{(\lambda-a)/a}\Big(\int_0^\infty z^{a-1-\lambda} f(z/\varrho)^a \,\mathrm{d}z\Big)^{1/a} + M\varrho^{-(\mu+b)/b}\Big(\int_0^\infty z^{b-1+\mu} f(z/\varrho)^b \,\mathrm{d}z\Big)^{1/b}.$$

Thus for all measurable nonnegative functions on $(0,\infty)$ and for any $\varrho > 0$,

$$\int_0^\infty \varphi(z)\,\mathrm{d}z \le L\varrho^{\lambda/a}\Big(\int_0^\infty z^{a-1-\lambda} \varphi(z)^a \,\mathrm{d}z\Big)^{1/a} + M\varrho^{-\mu/b}\Big(\int_0^\infty z^{b+\mu-1} \varphi(z)^b \,\mathrm{d}z\Big)^{1/b}.$$

Taking the minimum of the right-hand side over ϱ, we obtain (1.3.42). □

Lemma 3. *If f is a nonnegative nonincreasing function on $(0,\infty)$ and $p \ge 1$, then*

$$\frac{(p-1)^{p-1}}{p^p} \sup_{x>0} x^p f(x) \le \sup_{x>0} x^{p-1} \int_x^\infty f(t)\,\mathrm{d}t. \tag{1.3.43}$$

The characteristic function of the interval $(0,1)$ turns (1.3.43) into an equality.

Proof. Let c be an arbitrary positive number. Since f does not increase, we have

$$\left(\frac{p}{p-1}\,c\right)^p f\left(\frac{p}{p-1}c\right) \le \frac{p^p}{(p-1)^{p-1}} c^{p-1} \int_c^{cp/(p-1)} f(t)\,\mathrm{d}t$$
$$\le \frac{p^p}{(p-1)^{p-1}} \sup_{x>0} x^{p-1} \int_x^\infty f(t)\,\mathrm{d}t.$$

Setting $x = \frac{p}{p-1}\,c$, we arrive at (1.3.43).

If f is equal to unity for $0 < x < 1$ and to zero for $x \ge 1$, then

$$\sup_x x^{p-1} \int_x^\infty f(t)\,\mathrm{d}t = \sup_{0\le x\le 1} x^{p-1}(1-x)$$
$$= \frac{(p-1)^{p-1}}{p^p} = \frac{(p-1)^{p-1}}{p^p} \sup_{x>0} x^p f(x).$$

The lemma is proved. □

Remark. If f is an arbitrary nonnegative measurable function on $(0,\infty)$, then the inequality, opposite to (1.3.43), holds:

$$\sup_x x^{p-1} \int_x^\infty f(t)\,\mathrm{d}t \le \frac{1}{p-1} \sup_x x^p f(x), \tag{1.3.44}$$

the equality being attained for $f(x) = x^{-p}$.

In fact,

$$x^{p-1} \int_x^\infty f(t)\,\mathrm{d}t \le x^{p-1} \int_x^\infty \frac{\mathrm{d}t}{t^p} \sup_x x^p f(x) = \frac{1}{p-1} \sup_x x^p f(x).$$

1.3.6 Estimates for Differentiable Nonnegative Functions of One Variable

Let ω be a strictly increasing continuous function on $[0,\infty)$ such that $\omega(0) = 0$ and $\omega(x) \to \infty$ as $x \to \infty$. By ω^{-1} we mean the inverse of ω. We introduce the functions

$$T_\omega(v;x) = \sup_{y\in\mathbb{R}} \frac{|v(x)-v(y)|}{\omega(|x-y|)} \tag{1.3.45}$$

and

$$T^{\pm}_\omega(v;x) = \sup_{\tau>0} \frac{|v(x\pm\tau)-v(x)|}{\omega(\tau)}.$$

Lemma. *Let f be an absolutely continuous nonnegative function on $\mathbb{R}$ and let*

$$\psi(t) = \int_0^t \omega^{-1}(y)\,\mathrm{d}y. \tag{1.3.46}$$

Then for almost all $x \in \operatorname{supp} f$

$$\left|f'(x)\right| \le T^{\pm}_{\omega}(f';x)\psi^{-1}\left(\frac{f(x)}{T^{\pm}_{\omega}(f';x)}\right), \tag{1.3.47}$$

where ψ^{-1} *is the inverse of* ψ *and the sign* $+$ *or* $-$ *is taken if* $f'(x) \le 0$ *or* $f'(x) \ge 0$*, respectively.*

Proof. It suffices to consider the case $f'(x) \le 0$. Let $x \in \mathbb{R}$ be fixed. For any $t \ge 0$ we have

$$\begin{aligned} f(x+t) &= f(x) + \int_0^t f'(x+\tau)\,\mathrm{d}\tau \\ &= f(x) + f'(x)t + \int_0^t \frac{f'(x+\tau) - f'(x)}{\omega(\tau)}\omega(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Since f is nonnegative it follows that

$$0 \le f(x) - \left|f'(x)\right|t + T^{+}_{\omega}(f';x)\int_0^t \omega(\tau)\,\mathrm{d}\tau. \tag{1.3.48}$$

The right-hand side attains its minimal value at

$$t_* = \omega^{-1}\left(\frac{|f'(x)|}{T^{+}_{\omega}(f';x)}\right).$$

Therefore by (1.3.48)

$$\begin{aligned} 0 &\le f(x) - \left|f'(x)\right|\omega^{-1}\left(\frac{|f'(x)|}{T^{+}_{\omega}(f';x)}\right) + T^{+}_{\omega}(f';x)\int_0^{\omega^{-1}}\left(\frac{|f'(x)|}{T^{+}_{\omega}(f';x)}\right)\omega(\tau)\,\mathrm{d}\tau \\ &= f(x) - T^{+}_{\omega}(f';x)\int_0^{\omega^{-1}\left(\frac{|f'(x)|}{T^{+}_{\omega}(f';x)}\right)} \tau\,\mathrm{d}\omega(\tau), \end{aligned}$$

which is equivalent to (1.3.47). □

Theorem. *Let* f *be an absolutely continuous nonnegative function on* $\mathbb{R}$*. Then for all* $x \in \operatorname{supp} f$

$$\left|f'(x)\right| \le T_{\omega}(f';x)\psi^{-1}\left(\frac{f(x)}{T_{\omega}(f';x)}\right), \tag{1.3.49}$$

where ψ^{-1} *is the inverse of* (1.3.46)*. This inequality with* $x = 0$ *becomes an equality for the function*

$$f(x) = \psi(1) - x + \int_0^x \omega(\tau)\,\mathrm{d}\tau. \tag{1.3.50}$$

Proof. The estimate (1.3.49) follows from (1.3.47) if we notice that $T^{\pm}_{\omega}(v;x) \le T_{\omega}(v;x)$ and that the function

$$t \to \frac{1}{t} \int_0^t \omega^{-1}(y)\, \mathrm{d}y, \quad t > 0, \tag{1.3.51}$$

is increasing.

To show the sharpness of inequality (1.3.49) we first notice that $T_\omega(f';0) = 1$. Further, the left-hand side of (1.3.49) is $|f'(0)| = 1$ and its right-hand side is

$$\psi^{-1}\big(f(0)\big) = \psi^{-1}\big(\psi(1)\big) = 1.$$

Thus, the equality sign is attained in (1.3.49) with $x = 0$ for f defined by (1.3.50). The proof is complete. □

For the particular case $\omega(t) = t^\alpha$ we immediately have:

Corollary. *Let f be a differentiable nonnegative function on $\mathbb{R}$. Then for all $x \in \mathbb{R}$ and $\alpha > 0$*

$$\big|f'(x)\big|^{\alpha+1} \le \left(\frac{\alpha+1}{\alpha}\right)^\alpha \big(f(x)\big)^\alpha \sup_{y\in\mathbb{R}} \frac{|f'(x) - f'(y)|}{|x-y|^\alpha}. \tag{1.3.52}$$

The inequality (1.3.52) with $x = 0$ becomes an equality for the function

$$f(x) = \frac{x^{\alpha+1} + \alpha}{\alpha + 1} - x. \tag{1.3.53}$$

Remark 1. We introduce the seminorm

$$\langle v \rangle_\omega = \sup_{x,y\in\mathbb{R}} \frac{|v(x) - v(y)|}{\omega(|x-y|)}.$$

Since $T_\omega(f';x) \le \langle f' \rangle_\omega$, it follows from the above Theorem that

$$\big|f'(x)\big| \le \langle f' \rangle_\omega \psi^{-1}\left(\frac{f(x)}{\langle f' \rangle_\omega}\right). \tag{1.3.54}$$

If ω is concave then this inequality with $x = 0$ becomes an equality for the function (1.3.50). In fact, it suffices to show that $\langle f' \rangle_\omega = 1$. In view of the concavity of ω

$$\langle f' \rangle_\omega = \sup_{x,y\in\mathbb{R}} \frac{|\omega(x) - \omega(y)|}{\omega(|x-y|)} = \sup_{x>y} \frac{\omega(x) - \omega(y)}{\omega(x-y)} \le 1.$$

On the other hand, the last ratio equals 1 for $y = 0$.

For the particular case $\omega(x) = x^\alpha$, $\alpha \in (0,1]$, this gives a rougher variant of (1.3.52)

$$\big|f'(x)\big|^{\alpha+1} \le \left(\frac{\alpha+1}{\alpha}\right)^\alpha \big(f(x)\big)^\alpha \sup_{x,y\in\mathbb{R}} \frac{|f'(x) - f'(y)|}{|x-y|^\alpha}, \tag{1.3.55}$$

where the constant factor is still sharp since (1.3.55) with $x = 0$ becomes an equality for the function (1.3.53).

Remark 2. Taking $\alpha = 1$ in (1.3.55) we immediately arrive at the classical inequality

$$\left|f'(x)\right|^2 \leq 2f(x) \sup |f''|. \tag{1.3.56}$$

We can easily improve (1.3.56) using (1.3.52) and the right and left maximal functions defined by

$$\mathcal{M}_+\varphi(x) = \sup_{\tau>0} \frac{1}{\tau} \int_x^{x+\tau} \left|\varphi(y)\right| \mathrm{d}y, \tag{1.3.57}$$

$$\mathcal{M}_-\varphi(x) = \sup_{\tau>0} \frac{1}{\tau} \int_{x-\tau}^{x} \left|\varphi(y)\right| \mathrm{d}y. \tag{1.3.58}$$

Clearly,

$$\tau^{-1}\left|f'(x \pm \tau) - f'(x)\right| \leq (\mathcal{M}_\pm f'')(x), \quad \tau > 0.$$

Hence by (1.3.52) with $\alpha = 1$ we have the estimate

$$\left|f'(x)\right|^2 \leq 2f(x)(\mathcal{M}_\pm f'')(x), \tag{1.3.59}$$

where the sign $+$ or $-$ is taken if $f'(x) \leq 0$ or $f'(x) \geq 0$, respectively. By Corollary the constant 2 in (1.3.59) is the best.

1.3.7 Comments to Sect. 1.3

Concerning Sect. 1.3.1 we mention the following partial generalization of (1.3.1) with a sharp constant where the role of F is played by the Riemann–Liouville integrals of any positive order l:

$$F_l(x) = \frac{1}{\Gamma(l)} \int_0^x (x-t)^{l-1} f(t)\,\mathrm{d}t \tag{1.3.60}$$

and

$$F_l(x) = \frac{1}{\Gamma(l)} \int_x^\infty (t-x)^{l-1} f(t)\,\mathrm{d}t, \tag{1.3.61}$$

where $f \in L_1(\mathbb{R}_+, \mathrm{loc})$ and $f \geq 0$.

Theorem. (Section 329 in Hardy, Littlewood, and Pólya [351]) *Let $p > 1$ and $l > 0$. If F_l is defined by (1.3.60), then*

$$\int_0^\infty F_l^p \frac{\mathrm{d}x}{x^{lp}} < \left(\frac{\Gamma(1-\frac{1}{p})}{\Gamma(l+1-\frac{1}{p})}\right)^p \int_0^\infty f^p\,\mathrm{d}x,$$

except for the case when f vanishes almost everywhere.

If F_l is defined by (1.3.61), then

$$\int_0^\infty F_l^p \, dx < \left(\frac{\Gamma(\frac{1}{p})}{\Gamma(l + \frac{1}{p})} \right)^p \int_0^\infty (x^l f)^p \, dx,$$

except for the case when f vanishes almost everywhere. In both inequalities the constants are sharp.

There are a number of papers where particular cases of the theorems in Sects. 1.3.2 and 1.3.3 are considered. The first criterion of such a type was obtained by Kac and Krein [409] who dealt with the inequality (1.3.8) for $q = p = 2$. For $p = q$, Theorems 1.3.2/1 and 1.3.2/2 are due to Muckenhoupt [620]. A different proof of Theorem 1.3.2/2 in the case $p = q = 2$ can be found in Bobkov and Götze [115]. The generalizations for $p \neq q$ presented in Sects. 1.3.2 and 1.3.3 were obtained by Rosin and the author (see Maz'ya [552]). The case $p < q$ was independently investigated by Kokilashvili [438]. A new proof for the case $0 < q < p$ was given in Sinnamon and Stepanov [702] where the case $p = 1$ is included. For the history of Hardy's inequality (1.3.1) and its extensions see the book by Kufner, Maligranda, and Persson [468].

Stepanov found necessary and sufficient conditions on the weight functions u and v subject to the inequality

$$\int_0^\infty \left| f(x) u(x) \right|^p dx \le C \int_0^\infty \left| f^{(k)}(x) v(x) \right|^p dx,$$

where $k \ge 1$ and f vanishes together with all its derivatives up to order $k - 1$ at $x = 0$ or at infinity, [725, 726].

The material of Sect. 1.3.4 is borrowed from the article [593] by Maz'ya and Verbitsky.

Inequality (1.3.41) is proved in the paper by Hardy, Littlewood, and Pólya [350] and (1.3.42) is presented in the book [351] by the same authors. Levin [490] found the best constant in (1.3.42):

$$c(a, b, \lambda, \mu) = \left(\frac{1}{(as)^s (bt)^t} \left[\frac{\Gamma(\frac{s}{1-s-t}) \Gamma(\frac{t}{1-s-t})}{(\lambda + \mu) \Gamma(\frac{s+t}{1-s-t})} \right]^\alpha \right)^{a\mu + b\lambda},$$

where $s = \mu/(a\mu + b\lambda)$ and $t = \lambda/(a\mu + b\lambda)$. Lemma 1.3.5/3 was published in the author's book [552].

The results in Sect. 1.3.6 were obtained by Maz'ya and Kufner [571]. Historical remarks on multiplicative inequalities for differentiable functions are made in Sect. 13.5 of the book by Maz'ya and Shaposhnikova [579].

1.4 Embedding Theorems of Sobolev Type

This section deals with a generalization of the Sobolev embedding theorem. The heart of this result will be obtained as a corollary of estimates, in which the norms in the space of functions, integrable with power p with respect to an arbitrary measure, are majorized by the norms in Sobolev spaces. First we shall consider functions defined on $\mathbb{R}^n$ and then we shall proceed to the case of a bounded domain.

1.4.1 D.R. Adams' Theorem on Riesz Potentials

Let μ be a measure in $\mathbb{R}^n$, i.e., a nonnegative countably additive set function, defined on a Borel σ algebra of $\mathbb{R}^n$. Let $L_q(\mathbb{R}^n, \mu) = L_q(\mu)$ denote the space of functions on $\mathbb{R}^n$, which are integrable with power q with respect to μ. We put

$$\|u\|_{L_q(\mu)} = \left(\int |u|^q \, \mathrm{d}\mu \right)^{1/q}.$$

The space $L_q(\Omega, \mu)$, where μ is a measure on an open set Ω, is defined in an analogous manner.

To prove the basic result of this section, we need the classical *Marcinkiewicz interpolation theorem*, which is presented here without proof (cf. for example, Stein's book [724]).

Suppose p_0, p_1, q_0, and q_1 are real numbers, $1 \le p_i \le q_i < \infty$, $p_0 < p_1$, and $q_0 \ne q_1$. Let μ be a measure in $\mathbb{R}^n$ and let T be an additive operator defined on $\mathscr{D}$, its values being μ-measurable functions.

The operator T is said to be of weak type (p_i, q_i) if there exists a constant $\mathscr{A}_i$ such that for any $f \in \mathscr{D}$ and $\alpha > 0$,

$$\mu\big(\{x : |(Tf)(x)| > \alpha\}\big) \le \big(\alpha^{-1} \mathscr{A}_i \|f\|_{L_{p_i}}\big)^{q_i}.$$

Theorem 1. *Let T be an operator of the weak types (p_0, q_0) and (p_1, q_1). If $0 < \theta < 1$ and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then, for all $f \in \mathscr{D}$,

$$\|Tf\|_{L_q(\mu)} \le c \mathscr{A}_0^{1-\theta} \mathscr{A}_1^{\theta} \|f\|_{L_p},$$

and hence, T can be extended onto $L_p(\mathbb{R}^n)$ as a continuous operator: $L_p \to L_q(\mu)$. Here $c = c(p_1, p_2, q_1, q_2, \theta)$ is a constant independent of μ, T, and f.

Now, we proceed to the statement and proof of the basic theorem of this section.

Theorem 2. *Let $l > 0$, $1 < p < q < \infty$, and $lp < n$. The Riesz potential*

$$(I_l f)(x) = \int_{\mathbb{R}^n} |x-y|^{l-n} f(y) \, \mathrm{d}y$$

maps L_p *continuously into* $L_q(\mu)$ *if and only if the function*

$$\mathscr{M}(x) = \sup_{\varrho>0} \varrho^{-s}\mu\big(B(x,\varrho)\big),$$

where $s = q(\frac{n}{p} - l)$, *is bounded, that is,* $\mu(B(x,\varrho)) \le \text{const}\ \varrho^{q(\frac{n}{p}-l)}$.

Proof. Sufficiency. We show that

$$t\mu(\mathscr{L}_t)^{1/q} \le v_n^{1/p'} \frac{pq}{(n-pl)(q-p)} \sup \mathscr{M}(x)^{1/q}\|f\|_{L_p}, \tag{1.4.1}$$

where

$$p' = \frac{p}{p-1}, \qquad \mathscr{L}_t = \big\{y : \big(I_l|f|\big)(y) > t\big\}, \quad t>0.$$

Let μ_t be the restriction of μ to $\mathscr{L}_t$ and let r be a positive number, which will be specified later. Clearly,

$$\begin{aligned} t\mu(\mathcal{L}_t) &= t\int_{\mathbb{R}^n} \mathrm{d}\mu_t(y) \le \int_{\mathbb{R}^n} \big(I_l|f|\big)(y)\,\mathrm{d}\mu_t(y) \\ &= \int_{\mathbb{R}^n} \big|f(x)\big| \int_{\mathbb{R}^n} |x-y|^{l-n}\,\mathrm{d}\mu_t(y)\,\mathrm{d}x. \end{aligned}$$

By Lemma 1.2.3, the interior integral with respect to y can be expressed in the form

$$\int_0^\infty \mu_t\big(\big\{y : |x-y|^{l-n} \ge \tau\big\}\big)\,\mathrm{d}\tau = -\int_0^\infty \mu_t\big(B(x,\rho)\big)\,\mathrm{d}\big(\rho^{l-n}\big).$$

(The minus sign appears since $l<n$ and therefore $\rho=\infty$ corresponds to $\tau=0$ and vice versa.) Thus,

$$\begin{aligned} t\mu(\mathcal{L}_t) &\le (n-l)\int_0^\infty \int_{\mathbb{R}^n} \big|f(x)\big|\mu_t\big(B(x,\rho)\big)\,\mathrm{d}x\rho^{l-n-1}\,\mathrm{d}\rho \\ &= (n-l)\int_0^r (\cdots)\rho^{l-n-1}\,\mathrm{d}\rho + (n-l)\int_r^\infty (\cdots)\rho^{l-n-1}\,\mathrm{d}\rho = A_1 + A_2, \end{aligned}$$

where r is an arbitrary positive number.

Using the obvious inequality

$$\mu_t\big(B(x,\varrho)\big) \le \big(\mu_t\big(B(x,\varrho)\big)\big)^{1/p'}\mathscr{M}(x)^{1/p}\varrho^{s/p},$$

we obtain

$$A_1 \le (n-1)\sup \mathscr{M}(x)^{1/p}\|f\|_{L_p} \int_0^r \bigg(\int_{\mathbb{R}^n} \mu_t\big(B(x,\varrho)\big)\,\mathrm{d}x\bigg)^{1/p'} \varrho^{l-n-1+s/p}\,\mathrm{d}\varrho.$$

Since

$$\int_{\mathbb{R}^n} \mu_t\big(B(x,\varrho)\big)\,\mathrm{d}x = v_n\varrho^n\mu(\mathscr{L}_t),$$

we have

$$A_1 \le \frac{p(n-l)}{pl-n+s} v_n^{1/p'} \sup \mathscr{M}(x)^{1/p} \|f\|_{L_p}\mu(\mathscr{L}_t)^{1/p'} r^{l-(n-s)/p}.$$

Similarly,

$$A_2 \le (n-1)\|f\|_{L_p}\mu(\mathscr{L}_t)^{1/p}\int_r^\infty\Big(\int_{\mathbb{R}^n}\mu_t\big(B(x,\varrho)\big)\,\mathrm{d}x\Big)^{1/p'}\varrho^{l-n-1}\,\mathrm{d}\varrho$$
$$= \frac{p(n-l)}{n-pl} v_n^{1/p'}\|f\|_{L_p}\mu(\mathscr{L}_t) r^{l-n/p}.$$

Hence,

$$t\mu(\mathscr{L}_t)^{1/p} \le \|f\|_{L_p} v_n^{1/p'}(n-l)p\bigg(\frac{\sup\mathscr{M}(x)^{1/p}}{p\,l-n-s}\,r^{l-(n-s)/p} + \frac{\mu(\mathscr{L}_t)^{1/p}}{n-p\,l}\,r^{l-n/p}\bigg).$$

The right-hand side attains a minimum value at

$$r^s = \mu(\mathscr{L}_t)/\sup\mathscr{M}(x),$$

and this value is equal to

$$\frac{p(n-l)s}{(n-pl)(p\,l-n+s)}\,v_n^{1/p'}\|f\|_{L_p}\sup\mathscr{M}(x)^{1/q}\mu(\mathscr{L}_t)^{1/p-1/q}.$$

Thus, (1.4.1) is proved.

Applying interpolation Theorem 1, we find that the operator $I_l : L_p \to L_q(\mu)$ is continuous and

$$\|I_l f\|_{L_p(\mu)} \le c\sup\mathscr{M}(x)^{1/q}\|f\|_{L_p}. \tag{1.4.2}$$

Necessity. Let

$$\|I_l f\|_{Lq(\mu)} \le C\|f\|_{L_p}. \tag{1.4.3}$$

Let f denote the characteristic function of the ball $B(x,\varrho)$. Then, for $z \in B(x,\varrho)$,

$$(I_l f)(z) \ge (2\varrho)^{l-n}\int_{B(x,\varrho)}\mathrm{d}y = v_n 2^{l-n}\varrho^l.$$

This and (1.4.3) imply

$$\big(\mu\big(B(x,\varrho)\big)\big)^{1/q} \le 2^{n-l}v_n^{-1/p'}C\varrho^{-l+n/p}.$$

The theorem is proved. □

From Theorem 2 and the integral representation (1.1.10) we obtain the following criterion.

Corollary. *Let* $1 < p < q < \infty$ *and* $n > p\,l$.
1. *For all* $u \in \mathscr{D}$,

$$\|u\|_{L_q(\mu)} \le C \|\nabla_l u\|_{L_p}, \tag{1.4.4}$$

where

$$C^q \le c_1 \sup_{x;\varrho} \varrho^{(l-n/p)q} \mu\big[B(x,\varrho)\big].$$

2. *If* (1.4.4) *holds for all* $u \in \mathscr{D}$, *then*

$$C^q \ge c_2 \sup_{x;\varrho} \varrho^{(l-n/p)q} \mu\big[B(x,\varrho)\big].$$

Proof. The first statement follows from Theorem 2 and the integral representation (1.1.10). The second assertion can be justified by setting

$$u(y) = \eta\big(\varrho^{-1}(y-x)\big), \tag{1.4.5}$$

where $\eta \in \mathscr{D}(B_2)$ and $\eta = 1$ on B_1, into (1.4.4). □

1.4.2 Estimate for the Norm in $L_q(\mathbb{R}^n, \mu)$ by the Integral of the Modulus of the Gradient

In the next theorem we meet for the first time the phenomenon of equivalence of integral and isoperimetric inequalities:

Theorem 1. 1. *Let*

$$\sup_{\{g\}} \frac{\mu(g)^{1/q}}{s(\partial g)} < \infty, \tag{1.4.6}$$

where $q \ge 1$ *and* $\{g\}$ *is a collection of subsets of an open set* Ω, $\bar{g} \subset \Omega$, *with compact closures and bounded by* C^∞ *manifolds. Then for all* $u \in \mathscr{D}(\Omega)$

$$\|u\|_{L_q(\Omega,\mu)} \le C \|\nabla u\|_{L_1(\Omega)}, \tag{1.4.7}$$

where

$$C \le \sup_{\{g\}} \frac{\mu(g)^{1/q}}{s(\partial g)}. \tag{1.4.8}$$

2. *Suppose that for all* $u \in \mathscr{D}(\Omega)$ *the inequality* (1.4.7) *holds. Then*

$$C \ge \sup_{\{g\}} \frac{\mu(g)^{1/q}}{s(\partial g)}. \tag{1.4.9}$$

Proof. 1. By Lemma 1.2.3

$$\|u\|_{L_q(\Omega,\mu)} = \left(\int_0^\infty \mu(\mathscr{L}_t)\, \mathrm{d}\big(t^q\big)\right)^{1/q}, \tag{1.4.10}$$

where $\mathscr{L}_t = \{x : |u(x)| > t\}$. Since $\mu(\mathscr{L}_t)$ does not increase, (1.3.41) implies

$$\|u\|_{L_q(\Omega,\mu)} \le \int_0^\infty \mu(\mathscr{L}_t)^{1/q}\,\mathrm{d}t \le \sup_{\{g\}} \frac{\mu(g)^{1/q}}{s(\partial g)} \int_0^\infty s(\partial \mathscr{L}_t)\,\mathrm{d}t.$$

Here we used Corollary 1.2.2, according to which almost all sets $\mathscr{L}_t$ are bounded by smooth manifolds. By Theorem 1.2.4, the last integral coincides with $\|\nabla u\|_{L_1(\Omega)}$.

2. Let g be an arbitrary set in $\{g\}$ and let $d(x) = \operatorname{dist}(x, g)$, $g_t = \{x : d(x) < t\}$. Into (1.4.6), we substitute the function $u_\epsilon(x) = \alpha[d(x)]$, where $\alpha(d)$ is a nondecreasing C^∞ function on [0,1], equal to one for $d = 0$ and to zero for $d > \varepsilon$, $\varepsilon > 0$. According to Theorem 1.2.4,

$$\int_\Omega |\nabla u_\varepsilon|\,\mathrm{d}x = \int_0^\varepsilon \alpha'(t) s(\partial g_t)\,\mathrm{d}t.$$

Since $s(\partial g_t) \to s(\partial g)$ as $t \to 0$, we have

$$\int_\Omega |\nabla u_\varepsilon|\,\mathrm{d}x \to s(\partial g). \tag{1.4.11}$$

On the other hand,

$$\|u_\varepsilon\|_{L_q(\Omega,\mu)} \ge \mu(g)^{1/q}. \tag{1.4.12}$$

Combining (1.4.11) and (1.4.12) with (1.4.7), we obtain

$$\mu(g)^{1/q} \le C s(\partial g),$$

which completes the proof. □

From Theorem 1 and the classical isoperimetric inequality

$$m_n(g)^{(n-1)/n} \le n^{-1} v_n^{-1/n} s(\partial g) \tag{1.4.13}$$

(cf. Lyusternik [507], Schmidt [693], Hadwiger [334], and others), it follows that *for all* $u \in \mathscr{D}(\Omega)$

$$\|u\|_{L_{n/(n-1)}} \le n^{-1} v_n^{-1/n} \|\nabla u\|_{L_1}, \tag{1.4.14}$$

with the best constant.

In the case $n > p \ge 1$ we replace u by $|u|^{p(n-1)/(n-p)}$ in (1.4.14) and then estimate the right-hand side by Hölder's inequality. We have

$$\begin{aligned} \|u\|^{p(n-1)(n-p)}_{L_{pn/(n-p)}} &\le \frac{p(n-1)}{n(n-p)} v_n^{-1/n} \big\| |u|^{n(p-1)/(n-p)} \nabla u \big\|_{L_1} \\ &\le \frac{p(n-1)}{n(n-p)} v_n^{-1/n} \|u\|^{n(p-1)/(n-p)}_{L_{pn/(n-p)}} \|\nabla u\|_{L_p}. \end{aligned}$$

Consequently,

$$\|u\|_{Lpn/(n-p)} \le \frac{p(n-1)}{n(n-p)} v_n^{-1/n} \|\nabla u\|_{L_p}.$$

This along with

$$\big|\nabla|\nabla_{l-k}u|\big| \le n^{1/2}|\nabla_{l-k+1}u|, \quad k=1,\dots,l-1,$$

yields

$$\|\nabla_{l-k}u\|_{L_{pn/(n-kp)}} \le \frac{p(n-1)n^{-1/2}}{n-kp} v_n^{-1/n} \|\nabla_{l-k+1}u\|_{L_{pn/(n-(k-1)p)}}, \quad (1.4.15)$$

where $kp < n$. Putting $k = 1, 2, \dots, l$ in (1.4.15) and then multiplying all inequalities obtained, we arrive at the next corollary.

Corollary. *If $n > lp$, $p \ge 1$, then for all $u \in \mathscr{D}$*

$$\|u\|_{L_{pn/(n-lp)}} \le \left(\frac{n-1}{n^{1/2} v_n^{1/n}}\right)^l \frac{\Gamma(n/p - l)}{\Gamma(n/p)} \|\nabla_l u\|_{L_p}. \quad (1.4.16)$$

Thus we obtained the Sobolev inequality for ($p > 1$) and the Gagliardo–Nirenberg inequality ($p = 1$) with an explicit (but not the best possible for $p > 1$, $l \ge 1$ or for $p \ge 1$, $l > 1$) constant. In the case $l = 1$ the best constant is known (2.3.1).

The following extension of Theorem 1 is proved in the same way, and more general facts of a similar nature will be studied in Sect. 2.1.

Theorem 1′. *The best constant C in the inequality*

$$\left(\int_\Omega |u|^q \, \mathrm{d}\mu\right)^{1/q} \le C \|\Phi \nabla u\|_{L_1(\Omega)},$$

where $\mu \ge 0$, $q \ge 1$, $\Phi \in C(\Omega)$, $\Phi \ge 0$, and u is an arbitrary function in $C_0^\infty(\Omega)$, is equal to

$$\sup_{\{g\}} \frac{\mu(g)^{1/q}}{\int_{\partial g} \Phi(x) \, \mathrm{d}s}.$$

Here $\{g\}$ is the same as in Theorem 1.

The next theorem shows that in the case $\Omega = \mathbb{R}^n$ the condition (1.4.6) can be replaced by the equivalent one

$$\sup_{x;\varrho} \varrho^{(1-n)q} \mu\big(B_\varrho(x)\big) < \infty. \quad (1.4.17)$$

Theorem 2. 1. *If (1.4.17) holds, then (1.4.7) holds for all $u \in \mathscr{D}(\mathbb{R}^n)$ with $q \ge 1$ and*

$$C^q \le c^q \sup_{x;\varrho} \varrho^{(1-n)q} \mu\bigl(B_\varrho(x)\bigr), \tag{1.4.18}$$

where c depends only on n.

2. *If* (1.4.7) *holds for all* $u \in \mathscr{D}(\mathbb{R}^n)$, *then*

$$C^q \ge (nv_n)^{-q} \sup_{x;\varrho} \varrho^{(1-n)q} \mu\bigl(B_\varrho(x)\bigr). \tag{1.4.19}$$

Proof. Let $\{B(x_j, \varrho_j)\}$ be the covering of g constructed in Theorem 1.2.1/2. By the obvious inequality

$$\Bigl(\sum_j a_j\Bigr)^{1/q} \le \sum_j a_j^{1/q},$$

where $a_j \ge 0$, $q \ge 1$, we have

$$\begin{aligned}\mu(g) &\le \sum_j \mu\bigl(B(x_j, \varrho_j)\bigr) \le \Bigl[\sum_j \mu\bigl(B(x_j, \varrho_j)\bigr)^{1/q}\Bigr]^q \\ &\le \sup_{x;\varrho} \varrho^{(1-n)q} \mu\bigl(B(x, \varrho)\bigr) \Bigl(\sum_j \varrho_j^{n-1}\Bigr)^q.\end{aligned}$$

This and (1.2.1) imply

$$\mu(g) \le c^q \sup_{x;\varrho} \varrho^{(1-n)q} \mu\bigl(B(x, \varrho)\bigr) s(\partial g),$$

which along with Theorem 1 yields (1.4.7).

The inequality (1.4.19) is an obvious corollary of (1.4.9). The theorem is proved. □

1.4.3 Estimate for the Norm in $L_q(\mathbb{R}^n, \mu)$ by the Integral of the Modulus of the lth Order Gradient

Lemma. *Let* μ *be a measure on* $\mathbb{R}^n$, $n > l$, $1 \le q < (n-l+1)(n-l)^{-1}$ *and* $\tau^{-1} = 1 - n^{-1}(q-1)(n-l)$. *Further, let*

$$(I_1\mu)(x) = \int_{\mathbb{R}^n} |x-y|^{1-n}\, \mathrm{d}\mu(y).$$

Then for all $x \in \mathbb{R}^n$ *and* $\varrho > 0$

$$\varrho^{l-1-n} \|I_1\mu\|_{L_\tau(B(x,\varrho))} \le c \sup_{x\in\mathbb{R}^n, r>0} r^{(l-n)q} \mu\bigl(B(x, r)\bigr).$$

Proof. Without loss of generality we may put $x = 0$. By Minkowski's inequality,

$$\Big(\int_{|x|\le\varrho}\Big(\int_{|y|<2\varrho}\frac{\mathrm{d}\mu(y)}{|x-y|^{n-1}}\Big)^{\tau}\mathrm{d}x\Big)^{1/\tau}$$
$$\le\int_{|y|<2\varrho}\Big(\int_{|x|\le\varrho}\frac{\mathrm{d}x}{|x-y|^{(n-1)\tau}}\Big)^{1/\tau}\mathrm{d}\mu(y). \qquad (1.4.20)$$

Since $(n-1)\tau<n$, we have

$$\int_{|x|\le\varrho}\frac{\mathrm{d}x}{|x-y|^{(n-1)\tau}}\le c\varrho^{n-\tau(n-1)}.$$

Hence the right-hand side in (1.4.20) does not exceed $c\varrho^{n\tau^{-1}-n+1}\mu(B(2\varrho))$. Consequently,

$$\varrho^{l-n-1}\Big(\int_{|x|\le\varrho}\Big(\int_{|y|<2\varrho}\frac{\mathrm{d}\mu(y)}{|x-y|^{n-1}}\Big)^{\tau}\mathrm{d}x\Big)^{1/\tau}\le c\varrho^{(l-n)q}\mu\big(B(2\varrho)\big).$$

On the other hand,

$$\Big(\int_{|x|<\varrho}\Big(\int_{|y|\ge2\varrho}\frac{\mathrm{d}\mu(y)}{|x-y|^{n-1}}\Big)^{\tau}\mathrm{d}x\Big)^{1/\tau}\le c\varrho^{n/\tau}\int_{|y|\ge2\varrho}\frac{\mathrm{d}\mu(y)}{|y|^{n-1}}.$$

The last integral is equal to

$$(n-1)\int_{2\varrho}^{\infty}\mu\big(B(r)\backslash B(2\varrho)\big)r^{-n}\,\mathrm{d}r,$$

and therefore it is majorized by

$$c\varrho^{q(n-l)-n+1}\sup_{0<r<\infty}r^{(l-n)q}\mu\big(B(r)\big).$$

The result follows. □

Theorem. *Let μ be a measure on $\mathbb{R}^n$ and let $l\le n$, $q\ge1$. The inequality*

$$\|u\|_{L_q(\mu)}\le C\|\nabla_l u\|_{L_1},\quad u\in\mathscr{D}, \qquad (1.4.21)$$

holds if and only if

$$\mathscr{K}=\sup_{x\in\mathbb{R}^n,\varrho>0}\varrho^{l-n}\mu\big(B(x,\varrho)\big)^{1/q}<\infty. \qquad (1.4.22)$$

Moreover, $\mathscr{K}$ is equivalent to the best constant C in (1.4.21).

Proof. The estimate $C\ge c\mathscr{K}$ is obvious. We prove the opposite one. In the case $l=n$ it follows from the identity

$$u(x)=\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n}\frac{\partial^n u}{\partial x_1\cdots\partial x_n}\,\mathrm{d}x_1,\dots,\mathrm{d}x_n,\quad u\in\mathscr{D}. \qquad (1.4.23)$$

Let $l < n$. For $l = 1$ the result follows by Theorem 1.4.2/2. First consider the case $l > 1$, $q > n/(n-1)$. By Corollary 1.4.1

$$\|u\|_{L_q(\mu)} \le c\,\mathscr{K}\|\nabla_{l-1}u\|_{L_{n/(n-1)}}.$$

By (1.4.14), we obtain that the right-hand side does not exceed $c\mathscr{K}\|\nabla_l u\|_{L_1}$.

Now let $l > 1$, $q \le n/(n-1)$. We use induction on the number of derivatives. Suppose the assertion holds for derivatives of orders $2, \dots, l-1$. By the integral representation (1.1.10),

$$\int |u|^q \,\mathrm{d}\mu(x) \le c_0 \int \left| \int \frac{(\xi - x)\nabla_\xi |u(\xi)|^q}{|\xi - x|^n} \,\mathrm{d}\xi \right| \le c_0 q \int |\nabla u||u|^{q-1} I_1\mu \,\mathrm{d}\xi.$$

Hence

$$\int |u|^q \,\mathrm{d}\mu(x) \le c\|u\|^{q-1}_{L_{n/(n-1)}} \big\||\nabla u| I_l \mu\big\|_{L_\tau},$$

where $\tau^{-1} = 1-(q-1)(n-l)n^{-1}$. By (1.4.16) the first norm on the right-hand side is majorized by

$$c \sup_{x\in\mathbb{R}^n, \varrho>0} \varrho^{l-1-n}\|I_l\mu\|_{L_\tau(B(x,\varrho))}\|\nabla_l u\|_{L_1},$$

which follows by the induction hypothesis. Since $q \le n(n-1)^{-1}$ then $q < (n-l+1)(n-l)^{-1}$ and we may use the Lemma. Thus, the sufficiency of the condition (1.4.22) as well as the estimate $C \le c\mathscr{K}$ are proved. The necessity of (1.4.22) and the estimate $C \ge c\mathscr{K}$ follow by the insertion of the test function (1.4.5) into (1.4.21). This completes the proof.

1.4.4 Corollaries of Previous Results

The following assertion combines and complements Corollary 1.4.1 and Theorem 1.4.3.

Theorem 1. *Let either $k < l$, $p(l-k) < n$, $1 \le p < q < \infty$ or $l - k = n$, $p = 1 \le q \le \infty$. The best constant in*

$$\|\nabla_k u\|_{L_q(\mu)} \le C\|\nabla_l u\|_{L_p}, \quad u \in \mathscr{D}\big(\mathbb{R}^n\big), \tag{1.4.24}$$

is equivalent to

$$\mathscr{K} = \sup_{x;\varrho} \varrho^{l-k-np^{-1}} \big[\mu\big(B(x;\varrho)\big)\big]^{1/q}.$$

Proof. The estimate $C \le c\,\mathscr{K}$ is proved in Corollary 1.4.1 and in Theorem 1.4.3. Inserting

$$u(y) = (x_1 - y_1)^k \eta\left(\frac{x-y}{\varrho}\right),$$

where $\varrho > 0$ and $\eta \in \mathscr{D}(B_2)$, $\eta = 1$ on B_1, into (1.4.24), we obtain the lower bound for C.

The next assertion is the analog of Theorem 1 for the space V_p^l.

Theorem 2. *Let the conditions of Theorem* 1 *relating the values of* p, q, l, k, *and* n *hold. The best constant in*

$$\|\nabla_k u\|_{L_q(\mu)} \le C\|u\|_{V_p^l}, \quad u \in \mathscr{D},$$

is equivalent to

$$\mathscr{K}_1 = \sup_{x;\varrho\in(0,1)} \varrho^{l-k-np^{-1}} \big[\mu\big(B(x,\varrho)\big)\big]^{1/q}. \tag{1.4.25}$$

Proof. First we derive the upper bound for C. Let the cubes $\mathscr{Q}_j$ form the coordinate net in $\mathbb{R}^n$ with step 1 and let $2\mathscr{Q}_j$ be concentric homothetic cubes with edge length 2. By $\{\eta_j\}$ we denote a partition of unity subordinate to the covering $\{2\mathscr{Q}_j\}$ and such that $|\nabla_m \eta_j| \le c(m)$ for all j. Here $c(m)$ is a positive number and m is an integer. Since the multiplicity of the covering $\{2\mathscr{Q}_j\}$ is finite and depends on n only, it follows that

$$\int |\nabla_k u|^q \,\mathrm{d}\mu \le \int \Big(\sum_j \big|\nabla_k(\eta_j u)\big|\Big)^q \mathrm{d}\mu \le c\sum_j \int \big|\nabla_k(\eta_j u)\big|^q \,\mathrm{d}\mu.$$

Applying Theorem 1 to each summand of the last sum, we obtain

$$\big\|\nabla_k(\eta_j u)\big\|_{L_q(\mu)} \le c, \sup_{x;\varrho} \varrho^{l-k-np^{-1}} \big[\mu\big(2\mathscr{Q}_j \cap B(x,\varrho)\big)\big]^{1/q} \big\|\nabla_l(\eta_j u)\big\|_{L_p}.$$

Consequently,

$$\|\nabla_k u\|_{L_p(\mu)} \le c\,\mathscr{K}_1 \|u\|_{V_p^l},$$

where $\mathscr{K}_1$ is the constant defined by (1.4.25).

The lower bound for C can be obtained in the same way as in Theorem 1.

1.4.5 Generalized Sobolev Theorem

Theorem. *Let Ω be a domain in $\mathbb{R}^n$ with compact closure and let it be the union of a finite number of domains of the class EV_p^l.* (*In particular, according to Sect.* 1.1.9 *and the Stein extension theorem, mentioned in Sect.* 1.1.17, *this assumption holds if Ω has the cone property.*)

Further, let μ be a measure on Ω satisfying

$$\sup_{x\in\mathbb{R}^n,\varrho>0} \varrho^{-s}\mu\big(\Omega\cap B(x,\varrho)\big) < \infty, \tag{1.4.26}$$

where $s > 0$ (for example, if s is an integer, then μ can be s-dimensional Lebesgue measure on $\Omega\cap\mathbb{R}^s$).

Then, for any $u \in C^\infty(\Omega) \cap V_p^l(\Omega)$,

$$\sum_{j=0}^{k} \|\nabla_j u\|_{L_q(\Omega,\mu)} \le C\|u\|_{V_p^l(\Omega)}, \tag{1.4.27}$$

where C *is a constant independent of* u*, and the parameters* q*,* s*,* p*,* l*, and* k *satisfy the inequalities*

(a) $p > 1$, $0 < n - p(l-k) < s \le n$, $q \le sp(n-p(l-k))^{-1}$;
(b) $p = 1$, $0 < n - l + k \le s \le n$, $q \le s(n-l+k)^{-1}$;
(c) $p > 1$, $n = p(l-k)$, $s \le n$, *q is any positive number.*
If either of the conditions holds:
(d) $p > 1$, $n < p(l-k)$;
(e) $p = 1$, $n \le l - k$;
then

$$\sum_{j=0}^{k} \sup_{\Omega} |\nabla_j u| \le C\|u\|_{V_p^l(\Omega)}. \tag{1.4.28}$$

If Ω *belongs to the class* EV_p^l *(for example,* Ω *is in* $C^{0,1}$*), then in case* (d) *the Theorem can be refined as follows.*

(f) *If* $p \ge 1$, $(l-k-1)p < n < (l-k)p$ *and* $\lambda = l - k - n/p$*, then for all* $u \in V_p^l(\Omega) \cap C^\infty(\Omega)$

$$\sup_{x, x+h \in \Omega,\, h \ne 0} \frac{|\nabla_k u(x+h) - \nabla_k u(x)|}{|h|^\lambda} \le C\|u\|_{V_p^l(\Omega)}. \tag{1.4.29}$$

(g) *If* $(l-k-1)p = n$*, then inequality* (1.4.29) *holds for all* $0 < \lambda < 1$ *and* $u \in V_p^l(\Omega) \cap C^\infty(\Omega)$.

Proof. First we note that in cases (c) and (g) the result follows from (e) and (f), respectively, since $V_{p_1}^l(\Omega) \subset V_{p_2}^l(\Omega)$ for $p_1 > p_2$.

It is sufficient to prove (1.4.27) and (1.4.28) for domains of the class EV_p^l. Since for such a domain there exists an extension operator $V_p^l(\Omega) \to V_p^l(\mathbb{R}^n)$, we can limit ourselves to consideration of the case $\Omega = \mathbb{R}^n$. To obtain (1.4.27) in cases (a) and (b) we refer to Theorem 1.4.4/2.

Let (d) hold. It is sufficient to prove (1.4.28) for functions in $V_p^l(\mathbb{R}^n)$ with supports in a ball. Then (1.4.28) results from the integral representation (1.1.10) and Hölder's inequality.

In case (e) the estimate (1.4.28) follows directly from (1.4.23).

Let (f) hold. Clearly, it is sufficient to assume that $k = 0$. Since $\Omega \in EV_p^l$, then, as before, we may put $\Omega = \mathbb{R}^n$. Furthermore, one can assume without loss of generality that $|h| < 1/4$ and that $u(y) = 0$ outside the ball $B(x, 1)$. By (1.1.10)

$$u(x) = \sum_{|\alpha|=l} \int_{\mathbb{R}^n} K_\alpha(x-y) D^\alpha u(y)\, \mathrm{d}y,$$

where $|K_\alpha(z)| \le c|z|^{l-n}$ and

$$\big|K_\alpha(z+h) - K_\alpha(z)\big| \le c|h||z|^{l-1-n} \quad \text{for } |z| \ge 3|h|.$$

Therefore,

$$\begin{aligned}\big|u(x+h) - u(x)\big| & \\ \le c \int_{|x-y|\le 4|h|} & \frac{|\nabla_l u(y)|}{|x-y|^{n-l}}\, \mathrm{d}y \\ + c|h| \int_{|x-y|\ge 4|h|} & \frac{|\nabla_l u(y)|}{|x-y|^{n-l+1}}\, \mathrm{d}y. \end{aligned} \tag{1.4.30}$$

It remains to apply Hölder's inequality to both integrals on the right-hand side. The theorem is proved.

Remark 1. All the relations between n, p, l, k, and λ in cases (d)–(g) of the Theorem are the best possible. This fact can be verified using examples of functions $x_1^k \log|\log|x||$, $|x|^\alpha$.

Remark 2. From the Theorem it follows that $V_p^l(\Omega)$ is continuously embedded into $V_q^k(\Omega)$, $q = np(n-p(l-k))^{-1}$ for $n > p(l-k)$, $p \ge 1$, if Ω is bounded and has the cone property. In the case $n = p(l-k)$ the same holds for any $q < \infty$. Also note that for this critical dimension there is no embedding of $V_p^l(\Omega) \cap C^{k-1,1}(\Omega)$ into $C^k(\Omega)$. The corresponding counterexample is provided by the function

$$u(x) = x_1 \sin\log\log\frac{e}{|x|},$$

defined on the unit ball.

In the cases $p(l-k) > n$ and $p = 1$, $l-k \ge n$ the space $V_p^l(\Omega)$ is embedded into $C^k(\Omega)$.

Remark 3. If $\Omega \in C^{0,1}$, then under the conditions (f) and (g) the space $V_p^l(\Omega)$ is embedded into the space, obtained by the completion of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\sum_{j=0}^{k} \|\nabla_j u\|_{L_\infty(\Omega)} + \sup_{x,y\in\Omega, x\ne y} \frac{|\nabla_k u(x) - \nabla_k u(y)|}{\psi(|x-y|)},$$

where

$$\psi(t) = \begin{cases} t^\lambda & \text{for } \lambda \in (0,1), \\ t(1+|\log t|)^{(p-1)/p} & \text{for } \lambda = 1, \end{cases}$$

(if $\lambda = 1$, the right-hand side in (1.4.30) is majorized by $c|h| \cdot |\log|h||^{\frac{p-1}{p}} \times \|\nabla_l u\|_{L_p}$ with the aid of Hölder's inequality). The exponent $(p-1)/p$ is sharp for $\lambda = 1$, which can be checked by using the trial function $B_{1/4}(0) \ni x \mapsto x_1^{k+1}(-\log|x|)^\nu$ with $\nu \in (0,(p-1)/p)$, $l-k-n/p = 1$.

From conditions (a), (b), and (c) it follows that for integer s the restriction operator

$$C^\infty(\Omega) \cap V_p^l(\Omega) \ni u \to u|_{\mathbb{R}^s \cap \Omega}, \tag{1.4.31}$$

can be uniquely extended to a linear operator $V_p^l(\Omega) \to V_q^k(\mathbb{R}^s \cap \Omega)$.

Using Lemma 1.1.11 we may rewrite (1.4.27) as

$$\sum_{j=0}^{k} \big\| \nabla_j (u - \Pi) \big\|_{L_q(\Omega,\mu)} \le C \|\nabla_l u\|_{L_p(\Omega)},$$

where Π is the polynomial (1.1.12). This enables us to introduce a continuous restriction operator $\dot{L}_p^l \to V_p^k(\mathbb{R}^s \cap \Omega)/\mathscr{P}_{l-1}$ to $\mathbb{R}^s \cap \Omega$ for $\mu = m_s$. Analogously, we may establish that in cases (d) and (e) and (f) and (g) the space $\dot{L}_p^l(\Omega)$ is continuously embedded into $C^k(\bar{\Omega})/\mathscr{P}_{l-1}$ and into $C^{k,\lambda}(\bar{\Omega})/\mathscr{P}_{l-1}$, respectively.

In conclusion we note that the Theorem of the present subsection refines Theorem 1.1.2 on local properties of functions in $L_p^l(\Omega)$, where Ω is an arbitrary open subset of $\mathbb{R}^n$.

1.4.6 Compactness Theorems

The embedding and restriction operators mentioned in Remark 1.4.5/2, which are continuous by Theorem 1.4.5, turn out to be compact for certain values of p, l, q, n, and s. This result will be proved at the end of the present subsection.

Lemma. *Any bounded subset of the space of restrictions of the functions in $V_p^l(\mathbb{R}^n)$ to a bounded domain Ω is relatively compact in $V_p^{l-1}(\Omega)$.*

Proof. It suffices to limit consideration to the case $l = 1$. Let f be a summable nonnegative function on $[0, a+\delta]$, where $a > 0$, $\delta > 0$. Then

$$\int_0^a \mathrm{d}t \int_t^{t+\delta} f(\tau)\,\mathrm{d}\tau \le \delta \int_0^{a+\delta} f(t)\,\mathrm{d}t. \tag{1.4.32}$$

In fact, the integral on the left-hand side is

$$\int_0^a \mathrm{d}t \int_0^{\delta} f(\tau + t)\,\mathrm{d}\tau = \int_0^{\delta} \mathrm{d}\tau \int_\tau^{a+\tau} f(t)\,\mathrm{d}t \le \delta \int_0^{a+\delta} f(t)\,\mathrm{d}t.$$

Now let $u \in C_0^\infty(\mathbb{R}^n)$. Obviously, for all $h \in \mathbb{R}^n$,

$$\int_\Omega \big|u(x+h) - u(x)\big|^p\,\mathrm{d}x \le \int_\Omega \bigg(\int_{\sigma_{x,h}} |\nabla u|\,\mathrm{d}l \bigg)^p \mathrm{d}x,$$

where $\sigma_{x,h} = [x, x+h]$. Hence

$$\int_\Omega \big|u(x+h) - u(x)\big|^p\,\mathrm{d}x \le |h|^{p-1} \int_\Omega \int_{\sigma_{x,h}} |\nabla u|^p\,\mathrm{d}l\,\mathrm{d}x.$$

Applying (1.4.32) with $\delta = |h|$ to the last integral, we obtain

$$\left(\int_{\Omega} \bigl|u(x+h) - u(x)\bigr|^p \, \mathrm{d}x\right)^{1/p} \le |h| \|\nabla u\|_{L_p(\mathbb{R}^n)}.$$

It remains to note that by Riesz's theorem, a set of functions, defined on a bounded open domain Ω, is compact in $L_p(\Omega)$, if it is bounded in $L_p(\Omega)$ and

$$\int_{\Omega} \bigl|u(x+h) - u(x)\bigr|^p \, \mathrm{d}x \to 0,$$

uniformly as $|h| \to 0$, where h is an arbitrary vector in $\mathbb{R}^n$. This completes the proof. □

Theorem 1. *Let a bounded domain $\Omega \subset \mathbb{R}^n$ be the union of a finite number of domains in EV_p^l (for example, Ω has the cone property owing to Lemma 1.1.9/1). Let μ be a nonnegative measure in $\mathbb{R}^n$ with support in Ω. Further, let $k < l$, $p(l-k) < n$, and either $1 \le p < q < \infty$ or $1 = p < q < \infty$.*

Then any subset of the space $C^\infty(\bar{\Omega})$, bounded in $V_p^l(\Omega)$, is relatively compact in the metric

$$\sum_{j=0}^{k} \|\nabla_j u\|_{L_q(\bar{\Omega}, \mu)}, \tag{1.4.33}$$

if and only if

$$\lim_{\varrho \to 0} \sup_{x \in \mathbb{R}^n} \varrho^{q(l-k-n/p)} \mu\bigl(B(x, \varrho)\bigr) = 0. \tag{1.4.34}$$

Proof. Sufficiency. We may assume from the very beginning that $\Omega \in EV_p^l$. Then it suffices to prove that any subset of the space $C^\infty(\mathbb{R}^n) \cap W_p^l(\mathbb{R}^n)$, bounded in $W_p^l(\mathbb{R}^n) = V_p^l(\mathbb{R}^n)$, is relatively compact in the metric (1.4.33).

According to (1.4.34), given any $\varepsilon > 0$, there exists a number δ such that

$$\varrho^{q(1-k-n/p)} \sup_x \mu\bigl(B(x, \varrho)\bigr) < \varepsilon$$

for $\varrho \le \delta$. We construct a covering $\{\mathscr{B}_i\}$ of $\bar{\Omega}$ by balls with diameter $\delta \le 1$, the multiplicity of the covering being bounded by a constant which depends on n. Let μ_i be the restriction of μ to $\mathscr{B}_i$ and let $\{\eta_i\}$ be a partition of unity subordinate to the covering $\{\mathscr{B}_i\}$. Using Theorem 1.4.4/1, we obtain

$$\begin{aligned}\int_{\mathscr{B}_i} \sum_{j=0}^{k} \bigl|\nabla_j(u\eta_i)\bigr|^q \, \mathrm{d}\mu_i &\le c \sup_{\varrho; x} \varrho^{q(l-k-n/p)} \mu_i\bigl(B(x, \varrho)\bigr) \|u\eta_i\|^q_{V_p^l(\mathscr{B}_i)} \\ &\le c\varepsilon \sum_{j=0}^{l} \left(\delta^{p(j-l)} \int_{\mathscr{B}_i} |\nabla_j u|^p \, \mathrm{d}x\right)^{q/p}.\end{aligned}$$

Summing over i, we arrive at

$$\int_{\bar{\Omega}} \sum_{j=0}^{k} |\nabla_j u|^q \, d\mu \le c\varepsilon \|\nabla_l u\|^q_{L_p(\mathbb{R}^n)} + C(\varepsilon)\|u\|^q_{V_p^{l-1}(\bigcup_i \mathscr{B}_i)}.$$

It remains to note that, by the Lemma, any bounded set in $V_p^l(\mathbb{R}^n)$ is compact in $V_p^{l-1}(\bigcup_i \mathscr{B}_i)$.

Necessity. Let us take the origin of Cartesian coordinates to be an arbitrary point $O \in \mathbb{R}^n$. Let η denote a function in $\mathscr{D}(B_{2\varrho})$ which is equal to unity on B_ϱ, $\varrho < 1$, and such that $|\nabla_j \eta| \le c\varrho^{-j}$, $j = 1, 2, \dots$.

Note that μ has no point charges by (1.4.34) and the inequality $p(l-k) < n$. From the relative compactness of the set $\{u \in C^\infty(\bar{\Omega}) : \|u\|_{V_p^l(\mathbb{R}^n)} \le 1\}$ in the metric (1.4.27) it follows that given any $\varepsilon > 0$, any function of this set and any point O we have

$$\int_{B_{2\varrho}} |\nabla_k u|^q \, d\mu \le \varepsilon$$

for some ϱ. Inserting the function

$$u(x) = \frac{x_1^k \eta(x)}{\|x_1^k \eta\|_{V_p^l(\mathbb{R}^n)}},$$

into the last inequality, we obtain

$$(k!)^q \mu(B_\varrho) \le \varepsilon \|x_1^k \eta\|^q_{V_p^l(B_{2\varrho})} \le c\varepsilon \varrho^{q(n/p-l+k)}.$$

The result follows. □

Theorem 2. *Let a bounded domain $\Omega \subset \mathbb{R}^n$ be the union of a finite number of domains in EV_p^l. Then for $l > k \ge 0$, $p \ge 1$ we have*:

(a) *If s is a positive integer and $n > (l-k)p$, then the restriction operator* (1.4.31) *is compact as an operator, mapping $V_p^l(\Omega)$ into $V_q^k(\Omega \cap \mathbb{R}^s)$ for $n - (l-k)p < s \le n$ and $q < sp(n-(l-k)p)^{-1}$.*

(b) *If s is a positive integer and $n = (l-k)p$, then the operator* (1.4.31) *is compact as an operator, mapping $V_p^l(\Omega)$ into $V_q^k(\Omega \cap \mathbb{R}^s)$ for any $q \ge 1$, $s \le n$.*

(c) *If $n < (l-k)p$, then the embedding of $V_p^l(\Omega)$ into the space $C^k(\Omega)$ equipped with the norm*

$$\sum_{j=0}^{k} \sup_{\Omega} |\nabla_j u|$$

is compact.

Proof. Since $V_{p_1}^l(\Omega) \subset V_{p_2}^l(\Omega)$ for $p_1 > p_2$, then (b) follows from (a). In turn, (a) is a corollary of Theorem 1.

To obtain (c) it suffices to prove the compactness of the unit ball in $V_p^l(\mathbb{R}^n)$ with respect to the metric of the space $C^k(G)$, where $(l-k)p > n$ and G is any bounded domain. Let $x \in \bar{G}$ and $\varrho > 0$. By Sobolev's estimate (1.4.28)

$$\sum_{j=0}^{k} \sup_{B(x,1)} |\nabla_j u| \le C\|u\|_{V_p^l(B(x,1))}.$$

Applying a dilation with coefficient ϱ, we obtain

$$\sum_{j=0}^{k} \varrho^j \sup_{B(x,\varrho)} |\nabla_j u| \le c\sum_{i=0}^{l} \varrho^{i-n/p}\|\nabla_i u\|_{L_p(B(x,\varrho))}.$$

Therefore, for $j = 0, \ldots, k$

$$\sup_{B(x,\varrho)} |\nabla_j u| \le c\varrho^{l-j-n/p}\|\nabla_l u\|_{L_p(B(x,\varrho))} + C(\varrho)\|u\|_{V_p^{l-1}(B(x,\varrho))}$$

and thus

$$\sum_{j=0}^{k} \sup_{\bar{G}} |\nabla_j u| \le c\,\varrho^{l-k-n/p}\|u\|_{V_p^l(\mathbb{R}^n)} + C(\varrho)\|u\|_{V_p^{l-1}(G_\varrho)},$$

where G_ϱ is the ϱ neighborhood of $\bar{G}$. Since ϱ is an arbitrarily small number, it follows by the Lemma that the unit ball in $V_p^l(\mathbb{R}^n)$ is compact in $V_p^{l-1}(G_\varrho)$. Thus (c) is proved. □

1.4.7 Multiplicative Inequalities

Most of this subsection is dedicated to a necessary and sufficient condition for the validity of the inequality

$$\|\nabla_k u\|_{L_q(\mu)} \le C\|\nabla_l u\|^{\tau}_{L_p}\|u\|^{1-\tau}_{L_p}. \tag{1.4.35}$$

Lemma. *Let μ be a measure in $\mathbb{R}^n$ with support in $B_\varrho = \{x : |x| < \varrho\}$ and such that*

$$K = \sup_{x;r} r^{-s}\mu\big(B(x,r)\big) < \infty \tag{1.4.36}$$

for some $s \in [0, n]$. Further, let $p \ge 1$, let k and l be integers $k < l$, and let $s > n - p(l-k)$ if $p > 1$, $s \ge n - l + k$ if $p = 1$.

Then, for all $v \in C(\bar{B}_\varrho)$ and for q satisfying the inequalities $l - k - n/p + s/q > 0$, $q \ge p$, we have

$$\|\nabla_k v\|_{L_q(\mu,B_\varrho)} \le cK^{1/q}\varrho^{s/q-n/p-k}\big(\varrho^l\|\nabla_l v\|_{L_p(B_\varrho)} + \|v\|_{L_p(B_\varrho)}\big). \tag{1.4.37}$$

Proof. According to Sect. 1.1.17, any function $w \in C^\infty(\bar{B}_1)$ can be extended to a function $w \in C_0^l(B_2)$ satisfying the inequality

$$\|\nabla_l w\|_{L_p(B_2)} \le c\|w\|_{V_p^l(B_1)}.$$

Since $V_p^l(B_1) = W_p^l(B_1)$ (see Corollary 1.1.11), the last estimate is equivalent to

$$\|\nabla_l w\|_{L_p(B_2)} \le c\big(\|\nabla_l w\|_{L_p(B_1)} + \|w\|_{L_p(B_1)}\big).$$

Thus, applying a dilation, we obtain that the function v, mentioned in the statement of the Lemma, admits an extension $v \in C_0^l(B_{2\varrho})$ such that

$$\|\nabla_l v\|_{L_p(B_{2\varrho})} \le c\big(\|\nabla_l v\|_{L_p(B_\varrho)} + \varrho^{-1}\|v\|_{L_p(B_\varrho)}\big). \tag{1.4.38}$$

Let $(l-k)p < n$, $p > 1$ or $l - k \le n$, $p = 1$. By Theorem 1.4.4/1 we obtain

$$\|\nabla_k v\|_{L_t(\mu, B_{2\varrho})} \le cK^{1/t}\|\nabla_l v\|_{L_p(B_{2\varrho})}, \tag{1.4.39}$$

where $t = ps/(n - p(l-k))$.

In the case $(l-k)p = n$, $p > 1$, we let p_1 denote a number in $[1, p)$, sufficiently close to p. We put $t = p_1 s/(n - p_1(l-k))$. Then, by Corollary 1.4.1,

$$\begin{aligned}\|\nabla_k v\|_{L_t(\mu, B_{2\varrho})} &\le cK^{1/t}\|\nabla_l v\|_{L_{p_1}(B_{2\varrho})}\\ &\le cK^{1/t}\varrho^{n/p_1 - n/p}\|\nabla_l v\|_{L_p(B_{2\varrho})}.\end{aligned} \tag{1.4.40}$$

In the case $(l-k)p > n$, $p \ge 1$ we put $t = \infty$. By Sobolev's theorem

$$\|\nabla_k v\|_{L_t(\mu, B_{2\varrho})} \le c\varrho^{l-k-n/p}\|\nabla_l v\|_{L_p(B_{2\varrho})}. \tag{1.4.41}$$

Combining (1.4.39)–(1.4.41), we obtain

$$\|\nabla_k v\|_{L_t(\mu, B_{2\varrho})} \le c\,K^{1/t}\varrho^{l-k-n/p+s/t}\|\nabla_l v\|_{L_p(B_{2\varrho})}. \tag{1.4.42}$$

By Hölder's inequality,

$$\begin{aligned}\|\nabla_k u\|_{L_q(\mu, B_\varrho)} &\le \big[\mu(B_\varrho)\big]^{1/q-1/t}\|\nabla_k v\|_{L_t(\mu, B_\varrho)}\\ &\le K^{1/q-1/t}\varrho^{s(1/q-1/t)}\|\nabla_k v\|_{L_t(\mu, B_\varrho)},\end{aligned}$$

which along with (1.4.42) gives

$$\|\nabla_k v\|_{L_t(\mu, B_{2\varrho})} \le cK^{1/q}\varrho^{s/q+l-k-n/p}\|\nabla_l v\|_{L_p(B_{2\varrho})}.$$

Using (1.4.38), we complete the proof. □

Theorem. 1. *Let μ be a measure in $\mathbb{R}^n$ which satisfies the condition (1.4.36) for some $s \in [0, n]$. Let $p \ge 1$ and let k, l be integers, $0 \le k \le l - 1$; $s > n - p(l-k)$ if $p > 1$ and $s \ge n - l + k$ if $p = 1$. Then, for all $u \in \mathscr{D}$, the estimate (1.4.35) holds, where $C \le cK^{1/q}$, $n/p - l + k < s/q$, $q \ge p$ and $\tau = (k - s/q + n/p)/l$.*

2. *If (1.4.35) is valid for all $u \in \mathscr{D}$, then $C \ge c\,K^{1/q}$.*

Proof. According to the Lemma, for all $x \in \mathbb{R}^n$ and $\varrho > 0$,

$$\|\nabla_k u\|_{L_q(\mu,B(x,\varrho))} \le c\, K^{1/q} \varrho^{s/q-n/p-k}\big(\varrho^l \|\nabla_l u\|_{L_p(B(x,\varrho))} + \|u\|_{L_p(B(x,\varrho))}\big). \quad (1.4.43)$$

We fix an arbitrary $\varrho_0 > 0$. If the first term on the right-hand side of (1.4.43) exceeds the second for $\varrho = \varrho_0$, then we cover a point $x \in \operatorname{supp}\mu$ by the ball $B(x,\varrho)$. Otherwise we increase ϱ until the first term becomes equal to the second. Then the point x is covered by the ball $B(x,\varrho)$, where

$$\varrho = \|u\|^{1/l}_{L_p(B(x,\varrho))} \|\nabla_l u\|^{-1/l}_{L_p(B(x,\varrho))}.$$

In both cases

$$\|\nabla_k u\|^q_{L_q(\mu,B(x,\varrho))} \le cK\big(\varrho_0^{s-q(n/p-l+k)} \|\nabla_l u\|^q_{L_p(B(x,\varrho))} + \|\nabla_l u\|^{q\tau}_{L_p(B(x,\varrho))} \|u\|^{q(1-\tau)}_{L_p(B(x,\varrho))}\big). \quad (1.4.44)$$

According to Theorem 1.2.1/1, we can select a subcovering $\{\mathscr{B}^{(i)}\}_{i\ge1}$ of finite multiplicity, depending only on n, from the covering $\{B(x,\varrho)\}$ of $\operatorname{supp}\mu$. Summing (1.4.44) over all balls $\mathscr{B}^{(i)}$ and noting that

$$\sum_i a_i^\alpha b_i^\beta \le \Big(\sum_i a_i^{\alpha+\beta}\Big)^{\alpha/(\alpha+\beta)} \Big(\sum_i b_i^{\alpha+\beta}\Big)^{\beta/(\alpha+\beta)} \le \Big(\sum_i a_i\Big)^\alpha \Big(\sum_i b_i\Big)^\beta,$$

where a_i, b_i, α, and β are positive numbers $\alpha+\beta \ge 1$, we arrive at

$$\|\nabla_k v\|^q_{L_q(\mu)} \le cK(\varrho_0^{q(n/p-l+k)} \Big(\sum_i \|\nabla_l u\|^p_{L_p(\mathscr{B}^{(i)})}\Big)^{q/p} + \Big(\sum_i \|\nabla_l u\|^p_{L_p(\mathscr{B}^{(i)})}\Big)^{\tau q/p} \Big(\sum_i \|u\|^p_{L_p(\mathscr{B}^{(i)})}\Big)^{(1-\tau)q/p}.$$

Since the multiplicity of the covering $\{\mathscr{B}^{(i)}\}$ depends only on n, the right-hand side is majorized by

$$cK\big(\varrho_0^{s-q(n/p-l+k)} \|\nabla_l u\|^q_{L_q} + \|\nabla_l u\|^{\tau q}_{L_p} \|u\|^{(1-\tau)q}_{L_p}\big).$$

Passing to the limit as $\varrho_0 \to 0$, we complete the proof of case 1.

To prove case 2 it is sufficient to insert the function $u_\varrho(x) = (y_1 - x_1)^k \times \varphi(\varrho^{-1}(x-y))$, where $\varphi \in \mathscr{D}(B_2)$, $\varphi = 1$ on B_1, into (1.4.35). The result follows. $\square$

Corollary 1. 1. *Let μ be a measure in $\mathbb{R}^n$ such that*

$$K_1 = \sup_{x\in\mathbb{R}^n, r\in(0,1)} r^{-s}\mu\big(B(x,r)\big) < \infty, \tag{1.4.45}$$

for some $s \in [0,n]$. Further let $p \geq 1$, let k and l be integers $0 \leq k \leq l-1$; $s > n - p(l-k)$ if $p > 1$ and $s \geq n - l + k$ if $p = 1$. Then, for all $u \in \mathscr{D}$,

$$\|\nabla_k u\|_{L_q(\mu)} \leq C_1 \|u\|^{\tau}_{V_p^l} \|u\|^{1-\tau}_{L_p}, \tag{1.4.46}$$

where $C_1 \leq cK_1^{1/q}$, $n/p - l + k < s/q$, $q \geq p$, and $\tau = (k - s/p + n/p)/l$.

2. *If (1.4.46) is valid for all $u \in \mathscr{D}$, then $C_1 \geq c\,K_1^{1/q}$.*

Proof. Let $\{\mathscr{Q}^{(i)}\}$ denote a sequence of closed cubes with edge length 1 which forms a coordinate grid in $\mathbb{R}^n$. Let $\mathscr{O}^{(i)}$ be the center of the cube $\mathscr{Q}^{(i)}$, $\mathscr{O}^{(0)} = O$, and let $2\mathscr{Q}^{(i)}$ be the concentric homothetic cube with edge length 2. We put $\eta_i(x) = \eta(x - \mathscr{O}^{(i)})$,where $\eta \in C_0^\infty(2\mathscr{Q}^{(0)})$, $\eta = 1$ on $\mathscr{Q}^{(0)}$.

Applying the Theorem of the present subsection to the function $u\eta_i$ and to the measure $e \to \mu(e \cap \mathscr{Q}^{(i)})$, we obtain

$$\big\|\nabla_k(u\eta_i)\big\|^p_{L_q(\mu)} \leq cK_1^{p/q}\big\|\nabla_l(u\eta_i)\big\|^{p\tau}_{L_p}\|u\eta_1\|^{p(1-\tau)}_{L_p}.$$

Summing over i and using the inequality

$$\Big(\sum a_i\Big)^{p/q} \leq \sum a_i^{p/q},$$

where $a_i \geq 0$, we arrive at (1.4.46).

The second assertion follows by insertion of the function u_ϱ, defined at the end of the proof of the Theorem, into (1.4.46).

The next assertion follows immediately from Corollary 1.

Corollary 2. *Suppose there exists an extension operator which maps $V_p^l(\Omega)$ continuously into $V_p^l(\mathbb{R}^n)$ and $L_p(\Omega)$ into $L_p(\mathbb{R}^n)$ (for instance, Ω is a bounded domain of the class $C^{0,1}$). Further, let μ be a measure in $\bar{\Omega}$ satisfying (1.4.45), where s is a number subject to the same inequalities as in Corollary 1. Then for all $u \in C^l(\Omega)$*

$$\|\nabla_k u\|_{L_q(\mu,\bar{\Omega})} \leq C\|u\|^{\tau}_{V_p^l(\Omega)}\|u\|^{1-\tau}_{L_p(\Omega)}, \tag{1.4.47}$$

where $n/p - l + k < s/q$, $q \geq p \geq 1$ and $\tau = (k - s/q + n/p)/l$.

2. *If for all $u \in C^l(\bar{\Omega})$ the estimate (1.4.47) holds, then the measure μ with support in $\bar{\Omega}$ satisfies (1.4.45).*

1.4.8 Comments to Sect. 1.4

Theorem 1.4.1/2 is due to D.R. Adams [2, 3]. The proof given above is borrowed from the paper by D.R. Adams [3]. The following analog of Corollary 1.4.1 was obtained by Maz'ya and Preobrazhenski [577] and will be proved in Sect. 11.9.

If $1 < p < q$, $lp = n$, *then the best constant* C *in*

$$\|u\|_{L_q(\mu)} \le C\|u\|_{W_p^l}, \quad u \in C_0^\infty$$

is equivalent to

$$\sup_{x\in\mathbb{R}^n, r\in(0,1)} \left(\log\frac{2}{r}\right)^{\frac{p-1}{p}} \big[\mu\big(B_r(x)\big)\big]^{1/q}.$$

For $\mu = m_s$, i.e., for the s-dimensional Lebesgue measure in $\mathbb{R}^s$, inequality (1.4.4) was proved by Sobolev [712] in the case $s = n$ and by Il'in [394] in the case $s < n$. They used the integral representation (1.1.10) and the multidimensional generalization of the following Hardy–Littlwood theorem (cf. Hardy, Littlewood, and Pólya [351]).

If $1 < p < q < \infty$ *and* $\mu = 1 - p^{-1} + q^{-1}$, *then the operator* $|x|^{-\mu} * f$ *with* $f : \mathbb{R}^1 \to \mathbb{R}^1$ *maps* $L_p(\mathbb{R}^1)$ *continuously into* $L_q(\mathbb{R}^1)$.

For one particular case, Lieb [496, 497] found an explicit expression for the norm of the operator $|x|^{-\mu} * f$, $\mu \in (0, n)$, acting on functions of n variables. His result can be written as the inequality

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\right| \le \pi^{\frac{1}{2}} \frac{\Gamma((n-\mu)/2)}{\Gamma(n-\mu/2)} \left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{\frac{\mu-n}{n}} \|f\|_{L_{\frac{2n}{2n-\mu}}} \|g\|_{L_{\frac{2n}{2n-\mu}}}, \quad (1.4.48)$$

with the equality if and only if f and g are proportional to the function $(|x-x_0|^2 + a^2)^{(\mu-2n)/2}$, where $a \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$.

Theorems 1.4.2/1 and 1.4.2/2 are due to the author [543, 548]. Inequality

$$\left(\int \big|u(x)\big|^{n/(n-1)}\,\mathrm{d}x\right)^{(n-1)/n} \le C_n \int \big|\nabla u(x)\big|\,\mathrm{d}x \quad (1.4.49)$$

was proved independently by Gagliardo [299] and Nirenberg [641] using the same method, without discussion of the best value of C_n.

The proof based on the classical isoperimetric inequality (1.4.13) in $\mathbb{R}^n$, which gives the sharp constant (see (1.4.14)), was proposed simultaneously and independently by Federer and Fleming [273] and by Maz'ya [527].

Briefly, the proof by Gagliardo [299] and Nirenberg [640] runs as follows. One notes that

$$\bigl|u(x)\bigr| \le 2^{-1} \int_{\mathbb{R}} \left| \frac{\partial u}{\partial y_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n) \right| \mathrm{d}y_i$$

for $1 \le i \le n$ and for all $u \in C_0^\infty$. This yields

$$\bigl|u(x)\bigr|^{n/(n-1)} \le 2^{n/(1-n)} \biggl(\prod_{1 \le i \le n} \int_{\mathbb{R}} \left| \frac{\partial u}{\partial y_i} \right| \mathrm{d}y_i \biggr)^{1/(n-1)}.$$

Integrating successively with respect to x_1, x_2, and so on, and using the generalized Hölder inequality

$$\left| \int f_1 \cdots f_{n-1} \, \mathrm{d}\mu \right| \le \prod_{1 \le j \le n-1} \|f_j\|_{L_{p_j}(\mu)}$$

with $p_1 = p_2 = \cdots = p_{n-1} = n-1$ after every integration, we arrive at the inequality

$$\|u\|_{L_{n/(n-1)}} \le 2^{-1} \biggl(\prod_{1 \le i \le n} \int_{\mathbb{R}^n} \left| \frac{\partial u}{\partial x_i} \right| \mathrm{d}x \biggr)^{1/n}. \tag{1.4.50}$$

(Note that (1.4.50) is equivalent to the isoperimetric inequality

$$\bigl(m_n(g)\bigr)^{n-1} \le 2^{-1} \prod_{1 \le i \le n} \int_{\partial g} \bigl|\cos(\nu, x_i)\bigr| \, \mathrm{d}s,$$

which can be proved by a straightforward modification of the proof of Theorem 1.4.2/1.)

By the inequality between the geometric and arithmetic means, we obtain the estimate

$$\|u\|_{L_{n/(n-1)}} \le (2n)^{-1} \int_{\mathbb{R}^n} \sum_{1 \le i \le n} \left| \frac{\partial u}{\partial x_i} \right| \mathrm{d}x. \tag{1.4.51}$$

Its optimality is checked by a sequence of mollifications of the characteristic function of the cube $\{x : 0 \le x_i \le 1\}$.

Obviously, (1.4.51) implies (1.4.49) with $C_n = (2n^{1/2})^{-1}$, but this value of C_n is not the best possible.

It is worth mentioning that Gagliardo's paper [299] contains a more general argument based on the same idea which leads to the embedding of $W_p^l(\Omega)$ to $L_q(\Omega_s)$, where Ω_s is an s-dimensional surface situated in $\bar{\Omega}$.

Gromov [325] gave a proof of (1.4.49) where the integration is taken over a normed n-dimensional space X. The value $C_n = n^{-1}$ in Gromov's proof is the best possible provided the unit ball in X has volume 1. This proof is based on the so-called increasing triangular mappings which were apparently introduced to Convex Geometry by Knothe [437], who used them to obtain some generalizations of the geometric Brunn–Minkowski inequality. Such a mapping transports a given probability measure on the Euclidean space to another one, and under mild regularity assumptions, it is defined in a unique way. These mappings have a simple description in terms of conditional probabilities, and were apparently known in Probability Theory before Knothe's work.

In time it became clear that triangular mappings may be used to obtain various geometric and analytic inequalities. Bourgain [137] applied them to prove Khinchin-type (i.e., reverse Hölder) inequalities for polynomials of a bounded degree over high-dimensional convex bodies, with constants that are dimension free.

There is a discussion of this method in Bobkov [111, 112], where triangular mappings were used to study geometric inequalities of dilation type.

Using wavelet decompositions, weak estimates, and interpolation, Cohen, DeVore, Petrushev, and Xu [208] for $n = 2$, and Cohen, Meyer and Oru [209] for $n \geq 2$, obtained the following improvement of (1.4.49):

$$\|u\|_{L_{\frac{n}{n-1}}} \leq C\|\nabla u\|_{L_1}^{(n-1)/n}\|u\|_B^{1/n}, \tag{1.4.52}$$

where B is the distributional Besov space $B^{1-n}_{\infty,\infty}$. One equivalent norm in $B^{1-n}_{\infty,\infty}$,

$$f \to \sup_{t>0} t^{(n-1)/2}\|P_t f\|_{L_\infty},$$

where P_t is the heat semigroup on $\mathbb{R}^n$, was used by Ledoux [485] in his direct semigroup argument leading to (1.4.52).

Another powerful method for proving Sobolev-type inequalities is based upon symmetrization of functions (it will be demonstrated in Sects. 2.3.5 and 2.3.8) was developed in different directions during the last 40 years. In particular, it led to generalizations and refinements of those inequalities for the so-called *rearrangement invariant spaces*: Klimov [426, 427, 430]; Mossino [619]; Kolyada [443, 444], Talenti [742, 743]; Klimov and Panasenko [436]; Edmunds, Kerman, and Pick [253]; Bastero, M. Milman, and Ruiz [76]; M. Milman and Pustylnik [607]; Cianchi [197]; Kerman and Pick [418, 419]; Martin and M. Milman [518–522]; Martin, M. Milman, and Pustylnik [524]; Pick [659, 660]; Cianchi, Kerman, and Pick [205]; and Cianchi and Pick [207], et al.

Using symmetrization methods, Martin and M. Milman [517] showed that for $\alpha < 0$, and $f \in (W^1_1 + W^1_\infty) \cap B^\alpha_{\infty,\infty}$,

$$f^{**}(s) \leq c(n,\alpha)\big(|\nabla f|^{**}(s)\big)^{\frac{|\alpha|}{1+|\alpha|}}\|f\|_{B^\alpha_{\infty,\infty}}^{\frac{1}{1+|\alpha|}},$$

where $h^{**}(t) = \frac{1}{t}\int_0^t h^*(s)\,\mathrm{d}s$. This gives another approach to (1.4.52) and other inequalities of a similar nature.

Although the constant in (1.4.14) is the best possible, it can be improved by restricting the class of admissible functions in this inequality. For example, since for any N-gon $\Omega_N \subset \mathbb{R}^2$ the isoperimetric inequality

$$\big[s(\partial\Omega_N)\big]^2 \geq (4/N)\tan(\pi/N)m_2(\Omega_N)$$

is valid (see [714]) then duplicating the proof of Theorem 1.4.2/1 we obtain the following assertion.

Let u_N be a function on $\mathbb{R}^2$ with compact support, whose graph is a polygon with N sides. Then

$$(4/N)\tan(\pi/N)\int_{\mathbb{R}^2}|u_N|^2\,\mathrm{d}x \le \Big(\int_{\mathbb{R}^2}|\nabla u_N|\,\mathrm{d}x\Big)^2.$$

Lemma 1.4.3 is a special case of a result due to D.R. Adams [2]. Theorem 1.4.3 was proved by the author [551].

Theorem 1.4.5 for $\mu = m_s$ is the classical Sobolev theorem (see Sobolev [712, 713]) with supplements due to Il'in [394], Gagliardo [299], Nirenberg [640], and Morrey [612]. Here we stated this theorem in the form presented by Gagliardo [299].

The continuity of functions in $W_p^1(\Omega)$ for $p > 2$, $n = 2$, was proved by Tonelli [754].

To Remark 1.4.5 we add that if $n = p(l-k)$, $l > k$, $p > 1$, the inequality

$$\int_\Omega \exp\Big(c\frac{|\nabla_k u(x)|}{\|u\|_{V_p^l(\Omega)}}\Big)^{p/(p-1)}\mathrm{d}x \le c_0 \tag{1.4.53}$$

holds with positive constants c and c_0, as shown for the first time by Yudovich in 1961 [809]. (See also Pohozhaev [662] and Trudinger [762]. Concerning the best value of c_0 in inequalities of type (1.4.53) see Comments to Chap. 11.)

The estimate (1.4.32) is contained in the paper by Morrey [612]. Lemma 1.4.6 is the classical lemma due to Rellich [672]. Theorem 1.4.6/2 was proved by Kondrashov [447] for $p > 1$ and by Gagliardo [299] for $p = 1$.

In connection with the estimate (1.4.35) we note that multiplicative inequalities of the form

$$\|\nabla_j u\|_{L_q} \le c\|\nabla_l u\|_{L_p}^{\tau}\|u\|_{L_r}^{1-\tau}$$

and their modifications are well known (see Il'in [393] and Ehrling [257]). Their general form is due to Gagliardo [300] and Nirenberg [640] (see also Solonnikov [717]). The papers by Gagliardo [300] and Nirenberg [640] contain the following theorem.

Theorem 1. *Let Ω be a bounded domain having the cone property and let*

$$\langle\!\langle u\rangle\!\rangle_\sigma = \Big(\int_\Omega |u|^\sigma\,\mathrm{d}x\Big)^{1/\sigma}$$

for $\sigma > 0$. Then

$$\langle\!\langle \nabla_j u\rangle\!\rangle_q \le c\big(\langle\!\langle \nabla_l u\rangle\!\rangle_p + \langle\!\langle u\rangle\!\rangle_r\big)^\tau \langle\!\langle u\rangle\!\rangle_r^{1-\tau}, \tag{1.4.54}$$

where $p \ge 1$, $1/q = j/n + \tau(1/p - 1/n) + (1-\tau)/r$ for all $\tau \in [j/l, 1]$ unless $1 < p < \infty$ and $l - j - n/p$ is a nonnegative integer when (1.4.54) holds for $\tau \in [j/l, 1)$.

In the paper by Nirenberg [641] the stated result is supplemented by the following assertion.

Theorem 2. *Let* $\sigma < 0$, $s = [-n/\sigma]$, $-\alpha = s + n/\sigma$ *and let*

$$\langle\langle u \rangle\rangle_\sigma = \sup |\nabla_s u| \quad \textit{for } \alpha = 0, \qquad \langle\langle u \rangle\rangle_s = [\nabla_s u]_\alpha \quad \textit{for } \alpha > 0,$$

where

$$[f]_\alpha = \sup_{x \neq y} |x-y|^{-\alpha} \big| f(x) - f(y) \big|.$$

Further, let $1/r = -\beta/n$, $\beta > 0$. *Then* (1.4.54) *holds for* $\beta \le j < l$ *and for all* $\tau \in [(j-\beta)/(l-\beta), 1]$, *except the case mentioned in Theorem* 1.

The proof is reduced to derivation of the inequality

$$\int_{\mathscr{I}} \big|u^{(i)}\big|^q \, \mathrm{d}x \le c \bigg(\int_{\mathscr{I}} \big|u^{(l)}\big|^p \, \mathrm{d}x + [u]_\beta^p \bigg) [u]_\beta^{q-p}$$

for functions of the variable x on a unit interval $\mathscr{I}$.

1.5 More on Extension of Functions in Sobolev Spaces

1.5.1 Survey of Results and Examples of Domains

In Sect. 1.1.17, we introduced the class EV_p^l of domains in $\mathbb{R}^n$ for which there exists a linear continuous extension operator $\mathscr{E} : V_p^l(\Omega) \to V_p^l(\mathbb{R}^n)$. There we noted that the class EV_p^l contains Lipschitz graph domains.

Vodop'yanov, Gol'dshtein, and Latfullin [779] proved that a simply connected plane domain belongs to the class EV_2^1 if and only if its boundary is a quasicircle, i.e., the image of a circle under a quasiconformal mapping of the plane onto itself. By Ahlfors' theorem [30] (see also Rickman [678]) the last condition is equivalent to the inequality

$$|x - z| \le c|x - y|, \quad c = \text{const}, \tag{1.5.1}$$

where x, y are arbitrary points of $\partial\Omega$ and z is an arbitrary point on that subarc of $\partial\Omega$ which joins x and y and has the smaller diameter.

We give an example of a quasicircle of infinite length.

Example 1. Let Q be the square $\{(x_1, x_2); 0 < x_i < 1, i = 1, 2\}$. We divide the sides of the square Q into three equal parts and construct the squares Q_{i_1}, $i_1 = 1, \dots, 4$, $Q_{i_1} \cap Q = \varnothing$, on the middle segments. Proceeding in the same manner with each Q_{i_1}, we obtain the squares Q_{i_1,i_2}, $i_2 = 1, \dots, 4$ with edge length 3^{-2}. Repeating the procedure, we construct a sequence of squares

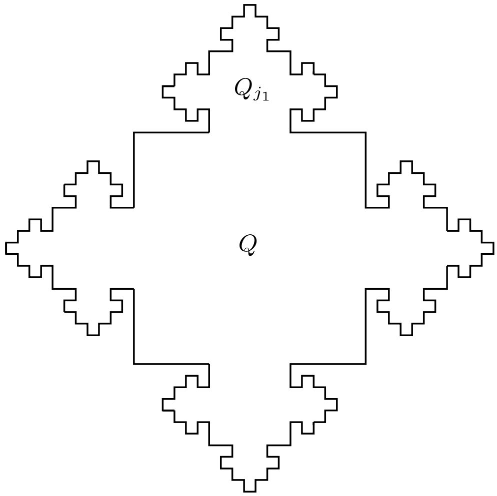

Fig. 6.

$\{Q_{i_1,i_2,\dots,i_k}\}$ $(k = 1, 2, \dots; i_k = 1, \dots, 4)$, whose union with Q is denoted by Ω (see Fig. 6). Clearly,

$$m_1(\partial\Omega) = 4\sum_{k=1}^{\infty} 3^{k-1}(2/3)^k = \infty.$$

Let $x, y \in \partial\Omega$. It suffices to consider the case $x \in \partial Q_{i_1,\dots,i_k}$ and $y \in \partial Q_{j_1,\dots,j_m}$ where $i_1 = j_1, \dots, i_l = j_l$, $i_{l+1} \neq j_{l+1}$. Then $|x - y| \geq c_1 3^{-l}$ and any point z in (1.5.1) satisfies the inequality $|x - y| \leq c_2 3^{-l}$. Thus, $\partial\Omega$ is a quasicircle.

A domain in $\mathbb{R}^2$ which is bounded by a quasicircle belongs to the class EV_p^l for all $p \in [1, \infty)$, $l = 1, 2, \dots$ (cf. Gol'dshtein and Vodop'yanov [320] for $l = 1$, Jones [404] for $l \geq 1$). In the just-mentioned paper by Jones, a class of n-dimensional domains in EV_p^l is described. It is larger than $C^{0,1}$ and coincides with the class of quasidisks for $n = 2$. Gol'dshtein [313] showed that the simultaneous inclusion of a plane simply connected domain Ω and the domain $\mathbb{R}^2\backslash\Omega$ in EV_p^l implies that $\partial\Omega$ is a quasicircle.

Is the last property true under the single condition $\Omega \in EV_p^1$ for some $p \neq 2$? In other words, are quasidisks the only plane simply connected domains contained in EV_p^l, $p \neq 2$? This question is discussed in the present subsection.

We give two examples which speak in favor of an affirmative answer. The first shows that "cusps" directed onto the exterior of a domain do not allow us to construct an extension operator.

Example 2. Let $\Omega = \{(x_1, x_2) : 0 < x_1 < 1, 0 < x_2 < x_1^{\alpha}\}$ where $\alpha > 1$. Suppose $\Omega \in EV_p^l$. Then $V_p^l(\Omega) \subset V_q^{l-1}(\Omega)$ for $1 \leq p < 2$, $q = 2p/(2 - p)$; $V_2^l(\Omega) \subset V_q^{l-1}(\Omega)$ for any $q < \infty$ and $V_q^l(\Omega) \subset C^{l-1,1-2/p}(\bar{\Omega})$ for $p > 2$.

Let $u(x) = x_1^{l-\beta}$. If $\beta < (\alpha+1)/p$, then $u \in V_p^l(\Omega)$. Under the additional condition that β is close to $(\alpha+1)/p$ the function u does not belong to $V_q^{l-1}(\Omega)$ ($p<2, q=2p/(2-p)$ or $p=2$, q is a large number) and does not belong to $C^{l-1,1-1/p}(\bar{\Omega})$ ($p>2$). Thus, $\Omega \notin EV_p^l$.

The following example excludes domains with inward cusps at the boundary from EV_p^l, $p>1$. It shows, incidentally, that the union of two domains in EV_p^l is not always in the same class.

Example 3. Let Ω be the domain considered above. We shall prove that $\mathbb{R}^2 \backslash \bar{\Omega} \notin EV_p^l$. We introduce polar coordinates (r,θ) with origin $x=0$ so that the ray $\theta=0$ is directed along the halfaxis $x_1>0$, $x_2=0$. We put $u(x) = r^{l-\beta}\psi(\theta)\eta(x)$. Here, β satisfies the inequality $\beta<2/p$ and is close to $2/p$; $\eta \in C_0^\infty(\mathbb{R}^2)$, $\eta(x)=1$ for $r<1$ and ψ is a smooth function on $(0,2\pi]$, $\psi(\theta)=1$ for small values of $\theta>0$ and $\psi(\theta)=0$ for $\theta\in[\pi,2\pi]$. Let $v\in V_p^l(\mathbb{R}^2)$ be an extension of $u\in V_p^l(\mathbb{R}^2\backslash\bar{\Omega})$. Since for small positive values of x_1

$$\frac{\partial^{l-1}v}{\partial x_1^{l-1}}(x_1,x_1^\alpha) \geq c x_1^{1-\beta}, \qquad \frac{\partial^{l-1}v}{\partial x_1^{l-1}}(x_1,0)=0,$$

it follows that

$$\begin{aligned}\int_\Omega \left|\frac{\partial^l v}{\partial x_1^{l-1}\partial x_2}\right|^p \mathrm{d}x &\geq \int_0^\delta \left(\int_0^{x_1^\alpha}\left|\frac{\partial^l v}{\partial x_1^{l-1}\partial x_2}\right| \mathrm{d}x_2\right)^p \frac{\mathrm{d}x_1}{x_1^{\alpha(p-1)}}\\ &\geq \int_0^\delta \left|\frac{\partial^{l-1}v}{\partial x_1^{l-1}}(x_1,x_1^\alpha)\right|^p \frac{\mathrm{d}x_1}{x_1^{\alpha(p-1)}}\\ &\geq c\int_0^\delta x_1^{1-p\beta-(\alpha-1)(p-1)}\,\mathrm{d}x_1=\infty\end{aligned}$$

if $p>1$. The latter contradicts the inclusion $v\in EV_p^l(\Omega)$. Thus $\mathbb{R}^2\backslash\bar{\Omega}\notin EV_p^l$ for $p>1$. Nevertheless, we shall show that $\mathbb{R}^2\backslash\bar{\Omega}\in EV_1^1$. Let $u\in V_1^1(\mathbb{R}^2\backslash\bar{\Omega})$. Suppose for a moment that $u=0$ for $x_1>1/2$.

We put $u^-(x)=u(x_1,-x_2)$ and $u^+(x)=u(x_1,2x_1^\alpha-x_2)$ for $x\in\Omega$. It is clear that

$$\left\|u^-\right\|_{V_1^1(\Omega)}+\left\|u^+\right\|_{V_1^1(\Omega)}\leq c\|u\|_{V_1^1(\mathbb{R}^2\backslash\bar{\Omega})}.$$

The function v, defined in $\mathbb{R}^2$ by

$$v(x)=\begin{cases}u(x), & x\notin\Omega,\\ u^-(x)+x_2x_1^{-\alpha}(u^+(x)-u^-(x)), & x\in\Omega,\end{cases}$$

is absolutely continuous on almost all straight lines parallel to coordinate axes. Also,

$$\|\nabla v\|_{L_1(\Omega)}\leq\left\|\nabla u^-\right\|_{L_1(\Omega)}+\left\|\nabla u^+\right\|_{L_1(\Omega)}+c\left\|x_1^{-\alpha}\left(u^+-u^-\right)\right\|_{L_1(\Omega)}.$$

Since

$$\big|u^+(x) - u^-(x)\big| \le \big|u^+(x) - u\big(x_1, x_1^\alpha\big)\big| + \big|u\big(x_1, x_1^\alpha\big) - u(x_1, 0)\big| \\ + \big|u(x_1, 0) - u^-(x)\big|,$$

we have

$$\begin{aligned}
&\big\|x_1^{-\alpha}\big(u^+ - u^-\big)\big\|_{L_1(\Omega)} \\
&\quad\le \left\|x_1^{-\alpha}\int_1^{x_1^\alpha}\big|u_t^+(x_1, t)\big|\,\mathrm{d}t\right\|_{L_1(\Omega)} + \big\|x_1^{-\alpha}\big(u\big(x_1, x_1^\alpha\big) - u(x_1, 0)\big)\big\|_{L_1(\Omega)} \\
&\qquad + \left\|x_1^{-\alpha}\int_0^{x_1^\alpha}\big|u_t^-(x_1, t)\big|\,\mathrm{d}t\right\|_{L_1(\Omega)} \\
&\quad\le \big\|u^+\big\|_{V_1^1(\Omega)} + \big\|u^-\big\|_{V_1^1(\Omega)} + \int_0^1 \big|u\big(x_1, x_1^\alpha\big) - u(x_1, 0)\big|\,\mathrm{d}x_1.
\end{aligned}$$

Clearly, the last integral does not exceed $c\|u\|_{V_1^1(\mathbb{R}^2\backslash\bar{\Omega})}$. We put $\mathscr{E}_0 u = v$. Thus we have

$$\|\mathscr{E}_0 u\|_{V_1^1(\mathbb{R}^2)} \le c\|u\|_{V_1^1(\mathbb{R}^2\backslash\bar{\Omega})}.$$

In the general case we introduce a truncating function $\eta \in C^\infty(\mathbb{R}^1)$ equal to unity on $(-\infty, 1/3]$ and to zero on $[1/2, +\infty)$. Further, let $\Omega_1 = \Omega \cap \{1/3 < x_1 < 1\}$. The required extension operator $\mathscr{E} : V_1^1(\mathbb{R}^2\backslash\bar{\Omega}) \to V_1^1(\mathbb{R}^2)$ is defined by

$$\mathscr{E}u = \mathscr{E}_0(\eta u) + \mathscr{E}_1\big((1-\eta)u\big),$$

where $\mathscr{E}_1 : V_1^1(\mathbb{R}^2\backslash\bar{\Omega}_1) \to V_1^1(\mathbb{R}^2)$ is a linear continuous extension operator.

In general, for $p \in [1, \infty)$, $l = 1, 2, \dots$, Poborchi and the author proved [576, Chap. 5] the existence of a linear bounded extension operator mapping $V_p^l(\mathbb{R}^2\backslash\bar{\Omega})$ into the space $V_p^l(\mathbb{R}^2, \sigma)$ with the weighted norm

$$\left(\int_{\mathbb{R}^2}\sum_{s=0}^{l}|\nabla_s u|^p\sigma\,\mathrm{d}x\right)^{1/p}, \quad 1 \le p < \infty,$$

where σ is a function which is equal to unity outside Ω and coincides with $x_1^{(\alpha-1)(lp-1)}$ on Ω. Moreover, if there exists an extension operator: $V_p^l(\mathbb{R}^2\backslash\bar{\Omega}) \to V_p^l(\mathbb{R}^2;\sigma)$ and the weight σ is nonnegative, depends only on x_1 on Ω and increases then

$$\sigma(x) \le c\,x_1^{(\alpha-1)(lp-1)}, \quad c = \mathrm{const},$$

for $x \in \Omega$ and for small enough x_1.

1.5.2 Domains in EV_p^1 which Are Not Quasidisks

The examples in Sect. 1.5.1 suggest that the class of Jordan curves that bound domains in EV_p^l consists of quasicircles only. However, we shall show that this conjecture is false.

Theorem. *There exists a domain $\Omega \subset \mathbb{R}^2$ with compact closure and Jordan boundary such that*:

(α) *$\partial\Omega$ is not a quasicircle.*

(β) *$\partial\Omega$ is of finite length and Lipschitz in the neighborhood of all but one of its points.*

(γ) *Ω belongs to EV_p^1 for $p \in [1, 2)$.*

(δ) *$\mathbb{R}^2 \backslash \bar{\Omega}$ belongs to EV_p^1 for $p > 2$.*

(From the aforementioned theorem by Gol'dshtein [313] and from the conditions (α), (γ), and (δ) we obtain in addition that $\Omega \notin EV_p^1$ for $p \geq 2$ and $\mathbb{R}^2\backslash\bar{\Omega} \notin EV_p^1$ for $p \in [1, 2]$.)

Before we prove this theorem we recall a well-known inequality that will be used later.

Lemma 1. *Let Ω be a sector defined in polar coordinates by the inequalities $0 < \theta < \alpha$ and $0 < r < a$. Let $u \in W_p^1(\Omega)$, $u|_{r=a} = 0$ for $p > 2$. Then*

$$\left\| \frac{u}{r} \right\|_{L_p(\Omega)} \leq \frac{p}{|2-p|} \|\nabla u\|_{L_p(\Omega)}. \tag{1.5.2}$$

This estimate is an immediate corollary of the following particular case of Hardy's inequality:

$$\int_0^a |u|^p r^{1-p}\, \mathrm{d}r \leq \frac{p^p}{|2-p|^p} \int_0^a |u'|^p r\, \mathrm{d}r$$

(cf. Sect. 1.3).

Proof of Theorem. Figure 7 presents a domain Ω satisfying the conditions (α)–(δ). The corresponding upper and lower "teeth" come close so rapidly that (1.5.1) does not hold. So $\partial\Omega$ is a quasicircle. The "teeth" almost do not change their form and decrease in geometric progression, so (β) holds.

Now we verify (γ). Let G be the difference of the rectangle $R = \{-1/3 < x_1 < 1,\ 0 < x_2 < 1/3\}$ and the union T of the sequence of isosceles right triangles $\{t_k\}_{k\geq 0}$ (cf. Fig. 8). The hypotenuse of t_k is the segment $[2^{-k-1}, 2^{-k}]$.

Lemma 2. *There exists a linear continuous extension operator $\mathscr{E}_1 : V_p^1(G) \to V_p^1(R)$, $1 \leq p < 2$, such that $\mathscr{E}_1 u = 0$ almost everywhere on the interval $x_2 = 0$, $0 < x_1 < 1$.*

Proof. Since "the saw" $\{x \in \partial T, x_2 > 0\}$ is a curve of the class $C^{0,1}$ there exists a linear continuous extension operator $V_p^1(G) \to V_p^1(R)$. Let v be

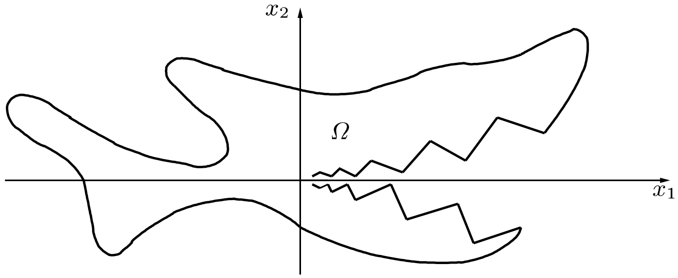

Fig. 7.

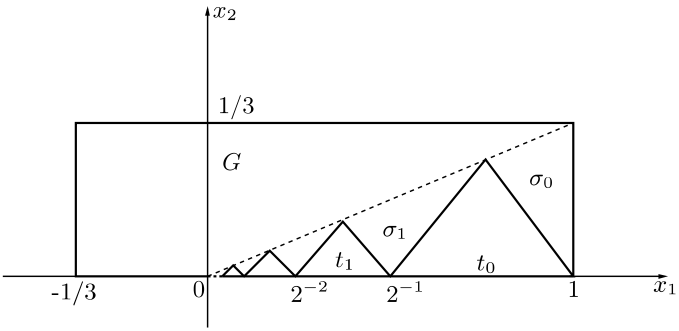

Fig. 8.

an extension of $u \in V_p^1(G)$. We introduce a truncating function θ which is equal to unity on G and to zero almost everywhere on the interval $x_2 = 0$, $0 < x_1 < 1$. Namely, we put $\theta = \theta_k$ on the triangle t_k, where

$$\theta_k = \begin{cases} \frac{4}{\pi} \arctan \frac{x_2}{x_1 - 2^{-k-1}} & \text{for } 2^{-k-1} < x_1 < 3 \cdot 2^{-k-2}, \\ \frac{4}{\pi} \arctan \frac{x_2}{2^{-k} - x_1} & \text{for } 3 \cdot 2^{-k-2} < x_1 < 2^{-k}. \end{cases}$$

Clearly,

$$|\nabla \theta_k| \le c\big(d_k^{-1} + d_{k+1}^{-1}\big) \tag{1.5.3}$$

on t_k, where $d_k(x)$ is the distance from x to the point $x_2 = 0$, $x_1 = 2^{-k}$.

The required extension of u is θv. To prove this we need to verify the inequality

$$\big\|\nabla(\theta v)\big\|_{L_p(T)} \le c\|\nabla v\|_{V_p^1(R)}. \tag{1.5.4}$$

We have

$$\big\|\nabla(\theta v)\big\|_{L_p(T)} \le \|\nabla v\|_{L_p(T)} + \|v \nabla \theta\|_{L_p(T)}.$$

By (1.5.3),

$$\|v\nabla\theta\|^p_{L_p(T)} \le c \sum_{k \ge 0} \big(\big\|d_k^{-1} v\big\|^p_{L_p(t_k^+)} + \big\|d_{k+1}^{-1} v\big\|^p_{L_p(t_k^-)}\big),$$

where t_k^+ and t_k^- are the right and left halves of t_k. Since $1 \le p < 2$, then

$$\big\| d_k^{-1} v \big\|_{L_p(t_k^+)} \le c\big(\|\nabla v\|_{L_p(t_k^+)} + 2^k \|v\|_{L_p(t_k^+)} \big).$$

The same estimate holds for $\|d_{k+1}^{-1} v\|_{L_p(t_k^-)}$. Consequently,

$$\|v\nabla\theta\|^p_{L_p(T)} \le c\bigg(\|\nabla v\|^p_{L_p(T)} + \bigg\| \frac{v}{|x|} \bigg\|^p_{L_p(T)} \bigg).$$

Applying Lemma 1, we obtain

$$\bigg\| \frac{v}{|x|} \bigg\|_{L_p(R)} \le c\|v\|_{V_p^1(R)}.$$

Thus, (1.5.4) as well as Lemma 1 are proved. □

Clearly, the domain $\Omega_+ = \{x \in \Omega : x_2 > 0\}$ (cf. Fig. 8) can be mapped onto the domain G in Lemma 2 by a quasi-isometric mapping. Therefore, any function in $V_p^1(\Omega_+)$ has a norm-preserving extension onto the upper-halfplane that vanishes on the halfaxis Ox_1. Applying the same reasoning to $\Omega_- = \Omega\backslash\bar{\Omega}_+$, we conclude that (γ) holds.

Now we verify (δ). Let $S = \{x : 0 < x_1 < 1, 0 < x_2 < x_1/3\}$. Let σ_k, $k = 0, 1, \dots$, denote the components of the set $S\backslash T$ (cf. Fig. 8), and let γ denote the union $\gamma_k \cup \gamma_{k+1}$ of legs of the triangle σ_k. Further, let $\tilde{V}_p^1(T)$ be the space of functions $u \in V_p^1(T)$ that satisfy the following condition. The limit values of u out of the triangles t_k and t_{k+1} coincide in their common vertex for $k = 0, 1, \dots$. We equip $\tilde{V}_p^1(T)$ with the norm of $V_p^1(T)$.

Clearly, (δ) follows immediately from the next lemma.

Lemma 3. *There exists a linear continuous extension operator* $\mathscr{E}_2$: $\tilde{V}_p^1(T) \to V_p^1(S)$ *with* $p \in (2, \infty)$.

Proof. Consider the rectangle $Q = \{(x, y) : 0 < x < a, 0 < y < b\}$ with vertices O, $A = (a, 0)$, $B = (0, b)$, and $C = (a, b)$. Let the function $w \in V_p^l$, $p > 2$, be defined on the triangle OAC and let $w(0) = 0$.

We shall show that there exists a linear extension operator $w \to f \in V_p^l(Q)$ such that $f(0, y) = 0$ for $y \in (0, b)$ and

$$\|f\|_{L_\infty(Q)} \le \|w\|_{L_\infty(OAC)},$$
$$\|\nabla f\|_{L_p(Q)} \le c\|\nabla w\|_{L_p(OAC)}.$$

Here, c is a constant which depends only on a/b and p. Clearly, we may assume that Q is a square. We construct an even extension of w across the diagonal OC to the triangle OBC. By η we denote a smooth function of the polar angle such that $\eta(\theta) = 1$ for $\theta < \pi/4$ and $\eta(\pi/2) = 0$. Since $w(0) = 0$, it follows by Lemma 1 that

$$\|r^{-1} w\|_{L_p(Q)} \le c\|\nabla w\|_{L_p(Q)},$$

where r is the distance to the point O. Hence $f = \eta w$ is the required extension.

Using the described procedure, we can construct an extension v_k of $u \in \tilde{V}_p^1(T)$ to the triangle σ_k such that $v_k(2^{-k}, y) = u(2^{-k}, 0)$ and

$$\|v_k\|_{L_\infty(\sigma_k)} \le \|u\|_{L\infty(t_k \cup t_{k+1})},$$
$$\|\nabla v_k\|_{L_p(\sigma_k)} \le c, \|\nabla u\|_{L_p(t_k \cup t_{k+1})},$$

where $k = 1, 2, \dots$. For $k = 0$ we obtain an extension v_0 of u to the triangle σ_0 satisfying similar inequalities, where $t_k \cup t_{k+1}$ is replaced by t_0.

We define an extension of u to S by $v = u$ on T, $v = v_k$ on σ_k. Clearly,

$$\|\nabla v\|_{L_p(S)} + \|v\|_{L_\infty(S)} \le c\big(\|\nabla u\|_{L_p(T)} + \|u\|_{L_\infty(T)}\big). \tag{1.5.5}$$

From the integral representation (1.1.8) it follows that

$$\operatorname*{osc}_{t_k} u \le c \int_{t_k} \big|\nabla u(y)\big| \frac{\mathrm{d}y}{|x-y|}.$$

Consequently,

$$\operatorname*{osc}_{t_k} u \le c 2^{-k(1-2/p)} \|\nabla u\|_{L_p(t_k)}.$$

Thus the right-hand side in (1.5.5) is equivalent to the norm in $\tilde{V}_p^1(T)$. The lemma is proved. □

1.5.3 Extension with Zero Boundary Data

Let G and Ω be bounded domains in $\mathbb{R}^n$, $\Omega \in EV_p^l$. Let $\mathring{V}_p^l(G)$ denote the completion of $\mathscr{D}(G)$ with respect to the norm in $V_p^l(G)$. If $\bar{\Omega} \subset G$ then, multiplying the operator $\mathscr{E} : V_p^l(G) \to V_p^l(\mathbb{R}^n)$ by a truncating function $\eta \in \mathscr{D}(G)$, $\eta = 1$ on Ω, we obviously obtain a linear continuous operator $\mathring{\mathscr{E}} : V_p^l(\Omega) \to \mathring{V}_p^l(G)$. If $\Omega \subset G$ and the boundaries ∂G, $\partial \Omega$ have a nonempty intersection, then proving the existence of $\mathring{\mathscr{E}}$ becomes a nontrivial problem. Making no attempt at a detailed study, we shall illustrate possibilities arising here by an example borrowed from the paper by Havin and the author [568]. In that paper, the above-formulated problem appeared in connection with certain problems of approximation in the mean by analytic functions.

Let Ω and G be plane domains such that $\Omega \in EV_p^1$ and $\Omega \subset G$, and let the origin be the only common point of intersection of the disk $B_R = \{\varrho \mathrm{e}^{\mathrm{i}\theta} : 0 \le \varrho < R\}$ with $G \backslash \bar{\Omega}$ (see Fig. 9). If R is sufficiently small, then $B_R \cap (G \backslash \bar{\Omega})$ is the union of two disjoint domains ω_1 and ω_2. We assume that the intersection of any circle $|z| = \varrho$, $\varrho \in (0, R)$, with each domain ω_j $(j = 1, 2)$ is a single arc. Let this arc be given by the equation $z = \varrho \mathrm{e}^{\mathrm{i}\theta}$ with $\theta \in (\alpha_j(\varrho), \beta_j(\varrho))$, where α_j and β_j are functions satisfying a Lipschitz condition on $[0, R]$, and let $\varrho \mathrm{e}^{\mathrm{i}\alpha_j(\varrho)} \in \partial\Omega$, $\varrho \mathrm{e}^{\mathrm{i}\beta_j(\varrho)} \in \partial G$. Further, let $\partial_j(\varrho) = \beta_j(\varrho) - \alpha_j(\varrho)$, $l_j(\varrho) = \varrho \delta_j(\varrho)$.

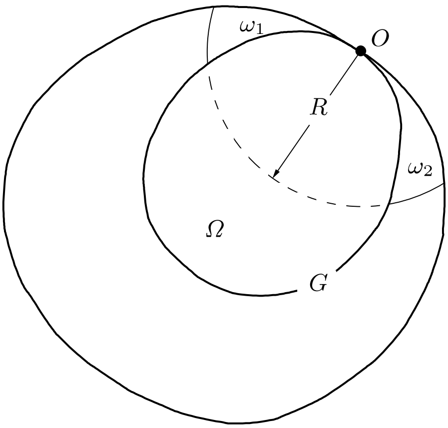

Fig. 9.

Theorem. *The following properties are equivalent.*
(1) *The function* $u \in L_p^1(\Omega)$ *can be extended to a function in* $\overset{\circ}{V}{}_p^1(G)$.

(2)
$$\int_0^R \frac{|u(\varrho e^{i\alpha_j(\varrho)})|^p}{[l_j(\varrho)]^{p-1}}\, d\varrho < \infty. \tag{1.5.6}$$

(*Here* $u(\varrho e^{i\alpha_j(\varrho)})$ *is the boundary value of* u *at the point* $\varrho e^{i\alpha_j(\varrho)} \in \partial\Omega$. *This boundary value exists almost everywhere on* $\partial\Omega$.)

Proof. Since $\Omega \in EV_p^1$, then to prove that (2) implies (1) we may assume that u has already been extended to a function in $V_p^1(B)$, where B is a disk containing $\bar{G}$.

Let η satisfy a Lipschitz condition on the exterior of $|z| = R$, $\eta = 0$ on $\mathbb{R}^2 \backslash G$, $\eta = 1$ on Ω and $\eta(\varrho e^{i\theta}) = 1 - (\theta - \alpha_j(\varrho))/\delta_j(\varrho)$ for $\varrho e^{i\theta} \in \omega_j$, $j = 1, 2$. Clearly, for $\theta \in (\alpha_j(\varrho), \beta_j(\varrho))$,

$$\begin{aligned} \big|u\big(\varrho e^{i\theta}\big) - u\big(\varrho e^{i\alpha_j(\varrho)}\big)\big| &\le \int_{\alpha_j(\varrho)}^{\beta_j(\varrho)} \left|\frac{\partial u}{\partial\theta}\big(\varrho e^{i\theta}\big)\right| d\theta \\ &\le \varrho \int_{\alpha_j(\varrho)}^{\beta_j(\varrho)} \big|(\nabla u)\big(\varrho e^{i\theta}\big)\big|\, d\theta \\ &\le \varrho\big[\delta_j(\varrho)\big]^{(p-1)/p} \bigg(\int_{\alpha_j(\varrho)}^{\beta_j(\varrho)} \big|(\nabla u)\big(\varrho e^{i\theta}\big)\big|^p\, d\theta\bigg)^{1/p}. \end{aligned} \tag{1.5.7}$$

Thus

$$\int_{\omega_j} \frac{|u(\varrho e^{i\theta})|^p}{[l_j(\varrho)]^p}\varrho\, d\varrho\, d\theta \le c\bigg(\|\nabla u\|^p_{L_p(\omega_j)} + \int_0^R \frac{|u(\varrho e^{i\alpha_j(\varrho)})|^p}{[l_j(\varrho)]^{p-1}}\, d\varrho\bigg).$$

We can easily deduce that the preceding inequality implies $u\eta \in \mathring{V}_p^1(G)$.

If $u \in \mathring{V}_p^1(G)$ then by (1.5.7)

$$\int_0^R \frac{|u(\varrho e^{i\alpha_j(\varrho)})|^p}{[l_j(\varrho)]^{p-1}}\, d\varrho \le \|\nabla u\|^p_{L_p(\omega_j)}.$$

Since $l_j(\varrho) \le 2\pi\varrho$, it follows that the condition (1.5.6) with $p \ge 2$ cannot be valid for all $u \in V_p^1(\Omega)$ and hence the operator $\mathring{\mathscr{E}}$ does not exist. The same holds for $1 \le p < 2$ provided $l_j(\varrho) = O(\varrho^{1+\varepsilon})$, $\varepsilon > 0$. In fact, the function $u \in V_p^1(\Omega)$, defined near 0 by the equality $u(\varrho e^{i\theta}) = \varrho^{1+\delta-2/p}$ with $0 < \delta < \varepsilon(p-1)/p$ does not satisfy (1.5.6).

Now let $1 \le p < 2$ and $l_j(\varrho) \ge c\,\varrho$, $c > 0$. Using an estimate similar to (1.5.7), we arrive at

$$\int_0^R \big|u\big(\varrho e^{i\alpha_j(\varrho)}\big)\big|^p \frac{d\varrho}{\varrho^{p-1}} \le c\big(\|\nabla u\|^p_{L_p(\omega_j)} + \big\|\varrho^{-1}u\big\|^p_{L_p(\omega_j)}\big),$$

which together with Hardy's inequality (1.5.2) shows that (1.5.6) is valid for all $u \in V_p^1(\Omega)$. Consequently, for $p \in [1, 2)$ and $l_j(\varrho) \ge c\,\varrho$ the operator $\mathring{\mathscr{E}}$ exists. □

1.5.4 Comments to Sect. 1.5

P. Jones [404] introduced a class of so-called (ε, δ) domains and showed that these domains belong to EV_p^l for $p \in [1, \infty]$ and $l = 1, 2, \dots$. For $\varepsilon \in (0, \infty)$, $\delta \in (0, \infty]$, $\Omega \subset \mathbb{R}^n$ is an (ε, δ) domain if any points $x, y \in \Omega$ with $|x-y| < \delta$ can be joined by a rectifiable arc $\gamma \subset \Omega$ such that

$$\ell(\gamma) \le |x-y|/\varepsilon, \qquad \operatorname{dist}(z, \partial\Omega) \ge \varepsilon\,|x-z||y-z|/|x-y|,$$

where $\ell(\gamma)$ is the length of γ and $z \in \gamma$ an arbitrary point. It should be noted that Jones' bounded extension operator $V_p^l(\Omega) \to V_p^l(\mathbb{R}^n)$ depends on l (while that of Stein does not). Any domain in $C^{0,1}$ with compact closure is an (ε, δ) domain for some ε, δ and for $n = 2$ the class of simply connected (ε, δ) domains coincides with the class of quasidisks [404]. It is interesting to observe that multidimensional domains with isolated inward cusps satisfy Jones' theorem and hence lie in EV_p^l for all $p \ge 1$ and $l = 1, 2, \dots$ (cf. Example 1.5.1/3). Fain [267] and Shvartsman [698] extended Jones' theorem to anisotropic Sobolev spaces, and Chua [188] extended the result of Jones to weighted Sobolev spaces. We also mention here the paper by Rogers [680], where a bounded extension operator $V_p^l(\Omega) \to V_p^l(\mathbb{R}^n)$, independent of l, p was constructed for an (ε, δ) domain.

Romanov [681] showed that the role of the critical value 2 in Theorem 1.5.2 is related to the particular domain dealt with in this theorem. He constructed a planar fractal domain Ω such that $\Omega \in EV_p^1$ for $p \in [1, q)$ and $\Omega \notin EV_p^1$

for $p \geq q$. In [803], S. Yang constructed an example of a homeomorphism of $\mathbb{R}^n$ such that the image of $\mathbb{R}^n_+$ is in EV^1_p for all $p > 1$, but does not satisfy P. Jones' condition [404]. In particular, it is not a quasidisk if $n = 2$.

P. Shvartsman [700] described a class of EW^l_p domains $\Omega \subset \mathbb{R}^n$ whenever $p > n$. Suppose that there exist constants C and θ such that the following condition is satisfied: for every $x, y \in \Omega$ such that $|x - y| \leq \theta$, there exists a rectifiable curve $\gamma \subset \Omega$ joining x to y such that

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{\frac{1-n}{p-1}}\, \mathrm{d}s(z) \leq C|x-y|^{\frac{p-n}{p-1}}. \tag{1.5.8}$$

Then Ω is an EW^l_q domain for every $l \geq 1$ and every $q \geq p$.

For $l = 1$ and $q > p$ this result was proved by Koskela [454].

Buckley and Koskela [148] showed that if a finitely connected bounded domain $\Omega \subset \mathbb{R}^2$ is a EW^1_p domain for some $p > 2$, then there exists a constant $C > 0$ such that *for every* $x,\ y \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ satisfying inequality (1.5.8) (with $n = 2$). Combining this result with Shvartsman's theorem [700] one obtains the following fact. Let $2 < p < \infty$ and let Ω be a finitely connected bounded planar domain. Then Ω is an EW^1_p domain if and only if for every $x, y \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ joining x to y such that

$$\int_\gamma \operatorname{dist}(z, \partial\Omega)^{\frac{1}{1-p}}\, \mathrm{d}s(z) \leq C|x-y|^{\frac{p-2}{p-1}}, \tag{1.5.9}$$

with some $C > 0$.

Here we mention some results by Poborchi and the author [575] on the extension of Sobolev functions from cusp domains. A typical domain with the vertex of an outward cusp on the boundary is

$$\Omega = \big\{x = (y, z) \in \mathbb{R}^n : z \in (0, 1),\ |y| < \varphi(z)\big\}, \quad n \geq 2,$$

where φ is an increasing Lipschitz continuous function on $[0, 1]$ such that $\varphi(0) = \lim_{z\to 0} \varphi'(z) = 0$. Cusp domains are generally not in EV^l_p, but it is possible to extend the elements of $V^l_p(\Omega)$ to some weighted Sobolev space on $\mathbb{R}^n$.

Let σ be a bounded nonnegative measurable function on $\mathbb{R}^n$, which is separated away from zero on the exterior of any ball centered at the origin. By $V^l_{p,\sigma}(\mathbb{R}^n)$ we mean the weighted Sobolev space with norm

$$\|u\|_{V^l_{p,\sigma}(\mathbb{R}^n)} = \sum_{k=0}^{l} \|\sigma\nabla_k u\|_{L_p(\mathbb{R}^n)}.$$

Clearly this weighted space is wider than $V^l_p(\mathbb{R}^n)$. The following assertion gives precise conditions on the weight σ.

Let Ω have an outward cusp as above. In order that there exist a linear continuous extension operator $V_p^l(\Omega) \to V_{p,\sigma}^l(\mathbb{R}^n)$ it is sufficient and if $\sigma(x)$ depends only on $|x|$ and is nondecreasing in the vicinity of the origin, then it is also necessary that

$$\sigma(x) \leq \begin{cases} c(\varphi(|x|)/|x|))^{\min\{l,(n-1)/p\}} & \text{if } lp \neq n-1, \\ c(\frac{\varphi(|x|)}{|x|})^l |\log(\frac{\varphi(|x|)}{|x|})|^{(1-p)/p} & \text{if } lp = n-1, \end{cases}$$

in a neighborhood of the origin, where c is a positive constant independent of x.

Let $p \in (1, \infty)$ and $l = 1, 2, \ldots$. We now give sharp conditions on the exponent $q \in [1, p)$ such that there is a linear continuous extension operator $V_p^l(\Omega) \to V_q^l(\mathbb{R}^n)$ for the same Ω. This extension operator exists if and only if

$$\int_0^1 \left(\frac{t^\beta}{\varphi(t)}\right)^{n/(\beta-1)} \frac{\mathrm{d}t}{t} < \infty,$$

where

$$1/q - 1/p = l(\beta - 1)/\big(\beta(n-1)+1\big) \quad \text{for } lq < n-1$$

and

$$1/q - 1/p = (n-1)(\beta-1)/np \quad \text{for } lq > n-1.$$

In the case $lq = n-1$ the factor $|\log(\varphi(t)/t)|^\gamma$ should be included into the integrand above with

$$\gamma = (1-1/q)/(1/p - 1/q), \qquad \beta = (np - q)/\big(q(n-1)\big).$$

Example. Power cusp. Let $\varphi(z) = c\, z^\lambda$, $\lambda > 1$. A linear continuous extension operator $V_p^l(\Omega) \to V_q^l(\mathbb{R}^n)$ for $q < p$ exists in the following cases.

$1°$ $lq < n-1$ and

$$q^{-1} > p^{-1} + l(\lambda - 1)/\big(1 + \lambda(n-1)\big).$$

$2°$ $lq = n-1$ and either the same inequality as in $1°$ holds or

$$q^{-1} = p^{-1} + l(\lambda-1)/\big(1+\lambda(n-1)\big) \quad \text{with } 2q^{-1} < 1 + p^{-1}.$$

$3°$ $lq > n-1$ and $q^{-1} > p^{-1}(1 + (\lambda-1)(n-1)/n)$.

Various results on extensions of functions in $W_p^l(\Omega)$, with deterioration of the class can be found in Burenkov's survey [156]. Finally we note that properties of extension domains for functions with bounded variation are discussed in Chap. 9 of the present book.

1.6 Inequalities for Functions with Zero Incomplete Cauchy Data

Consider the following question. Are any restrictions on Ω necessary for Sobolev's inequalities to be valid for functions in $W_p^l(\Omega)$ that vanish at $\partial\Omega$ along with their derivatives up to the order $k-1$ for some $k \le l$? To be precise, we mean functions in the space $V_p^l(\Omega) \cap \mathring{W}_p^k(\Omega)$.

Obviously, no requirements on Ω are needed for $k = l$. In the following, we show that the same is true if k satisfies the inequality $2k \ge l$. This result cannot be refined, for in the case $2k < l$ some conditions must be imposed.

The validity of Sobolev's inequalities for $l \le 2k$ follows from the integral representation for differentiable functions in an arbitrary bounded domain, which is derived in Sect. 1.6.2. The necessity of requirements on $\partial\Omega$ in the case $l > 2k$ is shown by an example (cf. Sect. 1.6.4).

1.6.1 Integral Representation for Functions of One Independent Variable

Lemma. *Let k and l be integers $1 < l \le 2k$ and let $z \in W_1^l(a,b) \cap \mathring{V}_1^k(a,b)$. Then*

$$z^{(l-1)}(t) = \int_a^b \mathscr{K}(t,\tau) z^{(l)}(\tau)\,\mathrm{d}\tau, \tag{1.6.1}$$

where

$$\mathscr{K}(t,\tau) = \begin{cases} \Pi_{2[l/2]-1}\left(\frac{2\tau-a-b}{b-a}\right) & \text{for } t > \tau, \\ \Pi_{2[l/2]-1}\left(\frac{a+b-2\tau}{b-a}\right) & \text{for } t < \tau, \end{cases}$$

and Π_{2i-1} is a polynomial of degree $2i-1$, defined by

$$\Pi_{2i-1}(s) + \Pi_{2i-1}(-s) = 1$$

and the boundary conditions

$$\Pi_{2i-1}(-1) = \cdots = \Pi_{2i-1}^{(i-1)}(-1) = 0.$$

Proof. Let l be even, $l = 2q$. Consider the boundary value problem

$$y^{(2q)}(x) = f(x), \qquad y^{(j)}(\pm 1) = 0, \quad j = 0, 1, \ldots, q-1, \tag{1.6.2}$$

on the interval $[-1,1]$. Let $g(x,s)$ denote the Green function of this problem. Clearly, $g(-x,-s) = g(x,s)$ and $g(x,s)$ is a polynomial of degree $2q-1$ in x and s for $x > s$ and $x < s$. First, let $x > s$. The derivative $\partial^{2q-1}g(x,s)/\partial x^{2q-1}$ does not depend on x; it is a polynomial of degree $2q-1$ in s and satisfies the boundary conditions (1.6.2) at the point $s = -1$. We denote this polynomial by $\Pi_{2q-1}(s)$.

In the case $x > s$, i.e., $-x < -s$, we have

$$\frac{\partial^{2q-1}}{\partial x^{2q-1}} g(x,s) = \frac{\partial^{2q-1}}{\partial x^{2q-1}} \big[g(-x,-s)\big] = -\Pi_{2q-1}(-s).$$

Hence, differentiating the equality

$$y(x) = \int_{-1}^{1} g(x,s) y^{(2q)}(s)\,\mathrm{d}s,$$

we obtain

$$y^{(2q-1)}(x) = \int_{-1}^{x} \Pi_{2q-1}(s) y^{(2q)}(s)\,\mathrm{d}s - \int_{x}^{1} \Pi_{2q-1}(-s) y^{(2q)}(s)\,\mathrm{d}s.$$

Passing to the variables t, τ via

$$x = \frac{2t-a-b}{b-a}, \qquad s = \frac{2\tau - a - b}{b-a},$$

and setting $z(t) = y(x)$, $z(\tau) = y(s)$, we arrive at

$$\begin{aligned} z^{(2q-1)}(t) = {} & \int_{a}^{t} \Pi_{2q-1}\left(\frac{2\tau-a-b}{b-a}\right) z^{(2q)}(\tau)\,\mathrm{d}\tau \\ & - \int_{t}^{b} \Pi_{2q-1}\left(\frac{a+b-2\tau}{b-a}\right) z^{(2q)}(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{1.6.3}$$

The lemma is proved for $l = 2q$. In the case $l = 2q+1$ we express $(z')^{(2q-1)}$ in terms of $(z')^{(2q)}$ by the formula just derived. This concludes the proof of the lemma. □

Remark. For $i = 1, 2, 3$, the polynomials Π_{2i-1} are

$$\begin{aligned} &\Pi_1(s) = \frac{1}{2}(s+1), \qquad \Pi_3(s) = \frac{1}{4}(2-s)(s+1)^2, \\ &\Pi_5(s) = \frac{1}{16}(3s^2 - 9s + 8)(s+1)^3. \end{aligned}$$

1.6.2 Integral Representation for Functions of Several Variables with Zero Incomplete Cauchy Data

The basic result of this section is contained in the following theorem.

Theorem. *Let l be a positive integer, $l \le 2k$, and let $u \in L_1^l(\Omega) \cap \mathring{V}_1^k(\Omega)$. Then for almost all $x \in \Omega$*

$$D^\gamma u(x) = \sum_{\{\beta : |\beta| = l\}} \int_\Omega K_{\beta,\gamma}(x,y) D^\beta u(y) \frac{\mathrm{d}y}{|x-y|^{n-1}}. \tag{1.6.4}$$

Here, γ is an arbitrary multi-index of order $l-1$ and $K_{\beta,\gamma}$ is a measurable function on $\Omega\times\Omega$ such that $|K_{\beta,\gamma}(x,y)|\le c$, where c is a constant depending only on n, l, k.

Proof. First suppose that u is infinitely differentiable on an open set ω, $\bar\omega\subset\Omega$. Let L be an arbitrary ray, drawn from the point x; let θ be a unit vector with origin at x and directed along L, $y=x+\tau\theta$, $\tau\in\mathbb{R}^1$. Further, let $\pi(x,\theta)$ be the first point of intersection of L with $\partial\Omega$. We also introduce the notation

$$b(x,\theta)=\bigl|\pi(x,\theta)-x\bigr|,\qquad a(x,\theta)=-b(x,-\theta).$$

Since $u\in \mathring{V}_1^k(\Omega)$ and $\nabla_l u\in L_1(\Omega)$, the function

$$\bigl[a(x,\theta),b(x,\theta)\bigr]\ni\tau\to z(\tau)=u(x+\tau\theta),$$

satisfies the conditions of Lemma 1.6.1 for almost all θ in the $(n-1)$-dimensional unit sphere S^{n-1}. So, from (1.6.3) it follows that

$$z^{(l-1)}(0)=\int_{a(x,\theta)}^0 \varPi_{2[l/2]-1}\biggl(\frac{2\tau-a(x,\theta)-b(x,\theta)}{b(x,\theta)-a(x,\theta)}\biggr)z^{(l)}(\tau)\,\mathrm{d}\tau$$
$$-\int_0^{b(x,\theta)} \varPi_{2[l/2]-1}\biggl(\frac{a(x,\theta)+b(x,\theta)-2\tau}{b(x,\theta)-a(x,\theta)}\biggr)z^{(l)}(\tau)\,\mathrm{d}\tau.\quad (1.6.5)$$

We note that

$$z^{(l-1)}(0)=\sum_{\{\nu:|\nu|=l-1\}}\frac{(l-1)!}{\nu!}\theta^\nu D^\nu u(x).$$

Let γ be any multi-index of order $l-1$ and let $\{P_\gamma(\theta)\}$ be the system of all homogeneous polynomials of degree $l-1$ in the variables $\theta_1,\dots,\theta_n$ such that

$$\int_{S^{n-1}}P_\gamma(\theta)\theta^\nu\,\mathrm{d}s_\theta=\delta_{\gamma\nu}.$$

Before proceeding to further transformations we note that the function $(x,\theta)\to|\pi(x,\theta)-x|$ can be considered as the limit of a sequence of measurable functions on $\Omega\times S^{n-1}$ if Ω is approximated by an increasing nested sequence of polyhedrons. Hence, a and b are measurable functions.

Let $r=|y-x|$, i.e., $\tau=r$, if $\tau>0$ and $\tau=-r$, if $\tau<0$. We multiply (1.6.5) by $P_\gamma(\theta)$ and integrate over S^{n-1}. Then

$$\frac{(l-1)!}{\gamma!}D^\gamma u(x)$$
$$=-\int_{S^{n-1}}P_\gamma(\theta)\,\mathrm{d}s_\theta\int_0^{b(x,\theta)}\varPi_{2[l/2]-1}\biggl(\frac{a(x,\theta)+b(x,\theta)-2r}{b(x,\theta)-a(x,\theta)}\biggr)\frac{\partial^l u(r,\theta)}{\partial r^l}\,\mathrm{d}r$$
$$+\int_{S^{n-1}}P_\gamma(\theta)\,\mathrm{d}s_\theta\int_0^{-a(x,\theta)}(-1)^l\varPi_{2[l/2]-1}\biggl(-\frac{2r+a(x,\theta)+b(x,\theta)}{b(x,\theta)-a(x,\theta)}\biggr)\frac{\partial^l u(r,-\theta)}{\partial r^l}\,\mathrm{d}r.$$

Replacing θ by $-\theta$ in the second term and noting that $a(x,\theta) = -b(x,-\theta)$ and $P_\gamma(\theta) = (-1)^{l-1}P_\gamma(-\theta)$, we find that the first and the second terms are equal, i.e., $D^\gamma u(x)$ is equal to

$$\frac{-2\gamma!}{(l-1)!}\int_{S^{n-1}} P_\gamma(\theta)\,\mathrm{d}s_\theta$$

$$\times \int_0^{b(x,\theta)} \Pi_{2[l/2]-1}\left(\frac{a(x,\theta)+b(x,\theta)-2r}{b(x,\theta)-a(x,\theta)}\right)\frac{\partial^l u(r,\theta)}{\partial r^l}\,\mathrm{d}r$$

$$= -2l\gamma! \int_{\Omega(x)} P_\gamma(\theta)\Pi_{2[l/2]-1}\left(\frac{a(x,\theta)+b(x,\theta)-2r}{b(x,\theta)-a(x,\theta)}\right)$$

$$\times \sum_{\{\beta:|\beta|=l\}} \frac{\theta^\beta}{\beta!}D^\beta u(y)\frac{\mathrm{d}y}{r^{n-1}},$$

where $\Omega(x) = \{y \in \Omega : \theta \in S^{n-1}, r < b(x,\theta)\}$. Let $K_{\beta,\gamma}(x,\cdot)$ denote the function

$$y \to \frac{-2\,l\gamma!}{\beta!}P_\gamma(\theta)\Pi_{2[l/2]-1}\left(\frac{b(x,\theta)-b(x,-\theta)-2r}{b(x,\theta)+b(x,-\theta)}\right)\theta^\beta$$

extended to $\Omega\backslash\Omega(x)$ by zero. Then (1.6.4) follows for $x \in \omega$.

Now we remove the assumption that u is smooth on ω. Let u satisfy the condition of the theorem. Clearly, u can be approximated in the seminorm $\|\nabla_l u\|_{L(\Omega)}$ by the functions that are smooth on $\bar{\omega}$ and which coincide with u near $\partial\Omega$. This and the continuity of the integral operator with the kernel $|x-y|^{1-n}K_{\beta,\gamma}(x,y)$, mapping $L(\Omega)$ into $L(\omega)$, imply (1.6.4) for almost all $x \in \omega$. Since ω is arbitrary, the theorem is proved. □

1.6.3 Embedding Theorems for Functions with Zero Incomplete Cauchy Data

Now we proceed to applications of Theorem 1.6.2.

Theorem 1. *Let m, l, k be integers, $0 \le m < l \le 2k$. Let $p \ge 1$ and let $u \in W_p^l(\Omega) \cap \mathring{V}_1^k(\Omega)$ for $n < p(l-m)$. Then $u \in C^m(\Omega)$, $\nabla_m u \in L_\infty(\Omega)$ and*

$$\|\nabla_m u\|_{L_\infty(\Omega)} \le c d^{l-m-n/p}\|\nabla_l u\|_{L_p(\Omega)}, \tag{1.6.6}$$

where d is the diameter of Ω.

The embedding operator of $W_p^l(\Omega) \cap \mathring{V}_1^k(\Omega)$ into $V_\infty^m(\Omega)$ is compact.

Proof. It is sufficient to assume $d = 1$. Iterating (1.6.4), we obtain the following representation for $D^\gamma u$, $|\gamma| = m < l$,

$$D^\gamma u(x) = \int_\Omega \sum_{\{\beta:|\beta|=l\}} Q_{\beta\gamma}(x,y) D^\beta u(y)\,\mathrm{d}y, \tag{1.6.7}$$

where $Q_{\beta\gamma} = O(r^{l-m-n}+1)$, if $n \neq l-m$ and $Q_{\beta\gamma} = O(\log 2r^{-1})$, if $n = l-m$.

Applying Hölder's inequality to (1.6.7), we obtain (1.6.6). Since any function in $V_p^l(\Omega)$ can be approximated in the $V_p^l(\Omega)$ norm by functions that coincide with u near $\partial\Omega$ and are smooth in $\bar\omega$, we see that $\nabla_m u$ is continuous in Ω. Here, as in the proof of Theorem 1.6.2, ω is an arbitrary open set, $\bar\omega \subset \Omega$. Thus, (1.6.6) is derived.

We can construct a covering $\{\mathscr{B}^{(i)}\}$ of $\mathbb{R}^n$ by balls with diameter δ, with a multiplicity not exceeding some constant which depends only on n. Let $\Omega_i = \Omega \cap \mathscr{B}^{(i)}$ and let $\{\eta_i\}$ be a smooth partition of unity subordinate to $\{\mathscr{B}^{(i)}\}$ such that $\nabla_j \eta_i = O(\delta^{-j})$. By (1.6.6),

$$\begin{aligned} \|\nabla_m u\|_{L_\infty(\Omega)} &\le c \max_i \big\|\nabla_m(\eta_i u)\big\|_{L_\infty(\Omega_i)} \\ &\le c\,\delta^{l-m-n/p} \|\nabla_l u\|_{L_p(\Omega)} + c(\delta)\|u\|_{V_p^{l-p}(\Omega)}. \end{aligned}$$

It remains to note that, by virtue of (1.6.7), any bounded subspace of $V_p^l(\Omega) \cap \mathring{V}_1^k(\Omega)$ is compact in $V_p^{l-1}(\Omega)$. The theorem is proved. □

Further applications of the integral representation (1.6.4) are connected with the results of Sect. 1.4.

Let m and l be integers, $0 \le m < l$, $p > 1$ and let μ be a measure in Ω such that

$$\mu\big(B_\varrho(x) \cap \Omega\big) \le K\varrho^s, \quad K = \text{const},\ 0 < s \le n, \tag{1.6.8}$$

for any ball $B_\varrho(x)$.

Let ω be an open set, $\bar\omega \subset \Omega$, $n > p(l-m) > n-s$ and $q = ps(n-p(l-m))^{-1}$. Further, let $L_q(\omega,\mu)$ be the space of functions on ω which are integrable with respect to μ and $L_q(\Omega,\mu,\mathrm{loc}) = \bigcap_\omega L_q(\omega,\mu)$. By Theorem 1.4.5 there exists a unique linear mapping $\gamma : V_p^{l-m}(\Omega) \to L_q(\Omega,\mu,\mathrm{loc})$ such that:

(i) if $v \in V_p^{l-m}(\Omega)$ and v is smooth on $\bar\omega$, then $\gamma v = v$ on $\bar\omega$;

(ii) the operator $\gamma : V_p^{l-m}(\Omega) \to L_q(\omega,\mu)$ is continuous for an arbitrary set ω.

Theorem 2. *Let m, l, k be integers, $0 \le m < l \le 2k$, $p > 1$, $s > n - p(l-m) > 0$ and let μ be a measure in Ω, satisfying (1.6.8). Then, for any $u \in L_p^l(\Omega) \cap \mathring{V}_1^k(\Omega)$,*

$$\big\|\gamma(\nabla_m u)\big\|_{L_q(\Omega,\mu)} \le c\,K^{1/q}\|\nabla_l u\|_{L_p(\Omega)}, \tag{1.6.9}$$

where $q = ps(n-p(l-m))^{-1}$.

The proof follows immediately from the integral representation (1.6.7) and Theorem 1.4.5.

Theorem 3. *Let m, l, k, p, q, and s be the same as in Theorem 2 and let μ be a measure in Ω satisfying the condition*

$$\lim_{\varrho\to 0}\sup_{x\in\mathbb{R}^n}\varrho^{-s}\mu\big(B_\varrho(x)\cap\Omega\big)=0. \tag{1.6.10}$$

Then the operator $\gamma\nabla_m : W_p^l(\Omega)\cap \overset{\circ}{V}{}_1^k(\Omega)\to L_q(\Omega,\mu)$ is compact.

Proof. Given any $\epsilon>0$ there exists a number $\delta>0$ such that

$$\varrho^{-s}\sup_x\mu\big(B_\varrho(x)\cap\Omega\big)<\epsilon$$

for $\varrho\le\delta$. We shall use the notation η_i, Ω_i, introduced in the proof of Theorem 1. Clearly,

$$\int_\Omega|\gamma\nabla_m u|^q\,\mathrm{d}\mu\le c\sum_i\int_{\Omega_i}\big|\gamma\nabla_m(\eta_i u)\big|^q\,\mathrm{d}\mu.$$

This inequality and Theorem 2 imply

$$\|\gamma\nabla_m u\|_{L_q(\Omega,\mu)}\le c\varepsilon\|\nabla_l u\|_{L_p(\Omega)}+C(\varepsilon)\|u\|_{V_p^{l-1}(\Omega)}.$$

It remains to use the fact that a unit ball in $W_p^l(\Omega)\cap\overset{\circ}{V}{}_1^k(\Omega)$ is compact in $V_p^{l-1}(\Omega)$. □

Remark 1. From Corollary 11.8 and Theorem 11.9.1/4 it follows that Theorems 2 and 3 remain valid for $n=p(l-m)$ if the conditions (1.6.8) and (1.6.10) are replaced by

$$\mu\big(B_\varrho(x)\cap\Omega\big)\le K|\log\varrho|^{q(1-p)/p},\quad 0<\varrho<\frac{1}{2},$$
$$\lim_{\varrho\to 0}\sup_{x\in\mathbb{R}^n}|\log\varrho|^{q(p-1)/p}\mu\big(B_\varrho(x)\cap\Omega\big)=0,$$

where $q>p>1$.

Remark 2. Sometimes assertions similar to Theorems 1, 2, and 3 can be refined via the replacement of $\|\nabla_l u\|_{L_p(\Omega)}$ by $\|(-\Delta)^{l/2}u\|_{L_p(\Omega)}$, where Δ is the Laplace operator. Namely, for any bounded domain Ω and any function $u\in\overset{\circ}{V}{}_2^1(\Omega)$ such that $\Delta u\in L_p(\Omega)$, $2p>n$, we have

$$\|u\|_{L_\infty(\Omega)}\le c(\operatorname{diam}\Omega)^{2-n/p}\|\Delta u\|_{L_p(\Omega)}.$$

This inequality results from an obvious estimate for the Green function of the Dirichlet problem for the Laplace operator, which in turn follows from the maximum principle.

Analogous estimates can be derived from pointwise estimates for the Green function $G_m(x,s)$ of the Dirichlet problem for the m-harmonic operator in an n-dimensional domain (see Maz'ya [550, 558, 559]). For instance, for $n = 5,6,7$, $m = 2$ or $n = 2m+1$, $2m+2$, $m > 2$, we have

$$\big|G_m(x,s)\big| \le c|x-s|^{2m-n}, \quad c = c(m,n).$$

This along with Theorem 1.4.1/2 implies

$$\|\gamma u\|_{L_q(\Omega,\mu)} \le CK^{1/q}\big\|\Delta^m u\big\|_{L_p(\Omega)},$$

where $u \in \mathring{V}_2^m(\Omega)$; $n > 2mp$, $p > 1$ and μ is a measure in Ω satisfying (1.6.8).

1.6.4 Necessity of the Condition $l \le 2k$

Here we show that the condition $l \le 2k$ cannot be weakened in the theorems of the preceding section. We present an example of a domain $\Omega \subset \mathbb{R}^n$ for which $V_p^l(\Omega) \cap \mathring{V}_p^k(\Omega)$, $l < 2k$, is not embedded into $L_\infty(\Omega)$ for $pl > n > 2pk$ and is not embedded into $L_{pn/(n-pl)}(\Omega)$ for $n > pl$.

Consider the function

$$v_\delta(x) = \big(1-\delta^{-2}|x|^2\big)^k,$$

in the ball $B_\delta(0)$. Since v_δ vanishes on $\partial B_\delta(0)$ along with its derivatives up to order $k-1$, then $|\nabla_m v_\delta| \le c\delta^{-k}\varepsilon^{k-m}$ for $k \ge m$ in an ε neighborhood of $\partial B_\delta(0)$. It is also clear that $|\nabla_m v_\delta| = O(\delta^{-m})$ in $B_\delta(0)$ and that $|\nabla_m v_\delta| = 0$ for $m \ge 2k+1$.

We denote by P and Q the lower and upper points at which the axis $0x_n$ intersects $\partial B_\delta(0)$, and construct the balls $B_\varepsilon(P)$, $B_\varepsilon(Q)$, $\varepsilon < \delta/2$. Let η be a smooth function on $\mathbb{R}^n$ vanishing on $B_{1/2}(0)$ and equal to unity on $\mathbb{R}^n \backslash B_1(0)$. On $B_\delta(0)$ we introduce the function

$$w(x) = v_\delta(x)\eta\big(\varepsilon^{-1}(x-P)\big)\eta\big(\varepsilon^{-1}(x-Q)\big),$$

and estimate its derivatives. On the exterior of the balls $B_\varepsilon(P)$, $B_\varepsilon(Q)$ we have

$$|\nabla_j w| = 0 \quad \text{for } j > 2k, \qquad |\nabla_j w| = O\big(\delta^{-j}\big) \quad \text{for } j \le 2k.$$

Also,

$$|\nabla_j w| \le c \sum_{m=0}^{\min\{j,2k\}} \varepsilon^{m-j}|\nabla_m v_\delta|$$

on $B_\varepsilon(P) \cup B_\varepsilon(Q)$. Hence

$$|\nabla_j w| \le c\delta^{-k}\varepsilon^{k-j}, \quad j = 0,1,\dots,l,$$

on $B_\varepsilon(P) \cup B_\varepsilon(Q)$. This implies

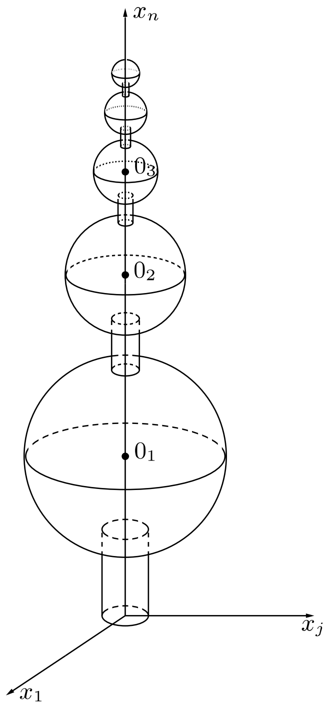

Fig. 10.

$$\|\nabla_j w\|^p_{L_p(B_\delta(0))} \le c\delta^{-pk}\varepsilon^{p(k-j)+n} \quad \text{for } j > 2k.$$

Similarly, since $n > pk$, we obtain that

$$\|\nabla_j w\|^p_{L_p(B_\delta(0))} \le c\big(\delta^{n-pj} + \delta^{-pk}\varepsilon^{p(k-j)+n}\big) \le c_1\delta^{n-pj} \quad \text{for } j \le 2k.$$

Therefore

$$\|w\|^p_{V_p^l(B_\delta(0))} \le c\big(\delta^{n-2pk} + \delta^{-pk}\varepsilon^{n-p(l-k)}\big).$$

We set $\varepsilon = \delta^\alpha$, where α is a number satisfying the inequalities

$$\frac{pk}{n-p(l-k)} < \alpha < \frac{n-pk}{n-p(l-k)}, \quad \alpha > 1.$$

Then

$$\|w\|^p_{V_p^l(B_\delta(0))} \le c\,\delta^\beta,$$

where $\beta = \alpha(n-p(l-k)) - pk > 0$.

Consider the domain (Fig. 10) which is the union of balls $\mathscr{B}_i$ with radii δ_i and centers O_i, joined by cylindrical necks $\mathscr{C}_i$ of arbitrary height and with cross-section diameter $\varepsilon_1 = \delta_i^\alpha$. In each ball $\mathscr{B}_i$ we specify w_i as described previously and extend w_i by zero to $\Omega\backslash\mathscr{B}_i$. Then we put

$$u(x) = \sum_{i=1}^{\infty} h_i w_i(x), \quad x \in \Omega, \tag{1.6.11}$$

where $\{h_i\}$ is the sequence of numbers such that

$$\sum_{i=1}^{\infty} |h_i|^p \delta_i^{\beta} < \infty. \tag{1.6.12}$$

This condition means that $u \in V_p^l(\Omega)$. The partial sums of the series (1.6.11) are functions in $\mathring{V}_p^k(\Omega)$, and so $u \in \mathring{V}_p^k(\Omega)$. Since $w_i = 1$ in the center of $\mathscr{B}_i$, we have

$$\|u\|_{L_\infty(\Omega)} \geq \sup_i |h_i|.$$

Clearly, the series (1.6.12) can converge as $h_i \to \infty$. Therefore, $V_p^l(\Omega) \cap \mathring{V}_p^k(\Omega)$ is not embedded into $L_\infty(\Omega)$.

In the case $n > pl$ we put $|h_i|^p = \delta_i^{lp-n}$. Then

$$\|u\|^q_{L_q(\Omega)} \geq c \sum_{i=1}^{\infty} |h_i|^q \delta_i^n = c \sum_{i=1}^{\infty} 1,$$

with $q = pn/(n - lp)$. On the other hand,

$$\|u\|^p_{V_p^l(\Omega)} \leq c \sum_{i=1}^{\infty} \delta_i^{\gamma},$$

where $\gamma = (\alpha - 1)(n - p(l - k)) > 0$. So, if $\{\delta_i\}$ is a decreasing geometric progression, then $u \in V_p^l(\Omega) \cap \mathring{V}_p^k(\Omega)$, whereas $u \notin L_q(\Omega)$.

The restrictions on Ω under which the Sobolev theorems hold for the space $V_p^l(\Omega) \cap \mathring{V}_p^k(\Omega)$, $2k < l$, will be considered in Sect. 7.6.6.

1.6.5 Comments to Sect. 1.6

The content of the present section is borrowed from the author's paper [553].

1.7 Density of Bounded Functions in Sobolev Spaces

1.7.1 Lemma on Approximation of Functions in $L_p^1(\Omega)$

Lemma. *If $v \in L_p^1(\Omega)$ then the sequence of functions*

$$v^{(m)}(x) = \begin{cases} \min\{v(x), m\} & \text{if } v(x) \geq 0, \\ \max\{v(x), -m\} & \text{if } v(x) \leq 0, \end{cases}$$

$(m = 1, 2, \dots,)$ converges to v in $L_p^1(\Omega)$.

The same is true for the sequence

$$v_{(m)}(x) = \begin{cases} v(x) - m^{-1} & \text{if } v(x) \geq m^{-1}, \\ 0 & \text{if } |v(x)| < m^{-1}, \\ v(x) + m^{-1} & \text{if } v(x) \leq -m^{-1}. \end{cases}$$

Proof. Since functions in $L^1_p(\Omega)$ are absolutely continuous on almost all lines parallel to coordinate axes (Theorem 1.1.3/1), we have, almost everywhere in Ω,

$$\nabla v^{(m)} = \chi^{(m)} \nabla v,$$

where $\chi^{(m)}$ is the characteristic function of the set $\{x : |v(x)| < m^{-1}\}$. Therefore

$$\int_\Omega \left|\nabla (v - v^{(m)})\right|^p \mathrm{d}x = \int_\Omega |\nabla v|^p \big(1 - \chi^{(m)}\big)^p \, \mathrm{d}x.$$

The convergence to zero of the last integral follows from the monotone convergence theorem.

The proof for the sequence $v_{(m)}$ is similar. □

1.7.2 Functions with Bounded Gradients Are Not Always Dense in $L^1_p(\Omega)$

Lemma 1.7.1 says that the set of bounded functions is dense in $L^1_p(\Omega)$ whenever $p \in [1, \infty)$. The following example shows that this set cannot be generally replaced by $L^1_\infty(\Omega)$.

Example. Let $p \in (2, \infty)$ and $\{a_i\}_{i\geq 1}, \{\varepsilon_i\}_{i \geq 1}$ be two sequences of positive numbers satisfying

$$a_1 + \varepsilon_1 < 1, \qquad a_{i+1} + \varepsilon_{i+1} < a_i, \quad i \geq 1, \qquad \lim a_i = 0,$$

and

$$\sum_{i \geq 1} a_i^{1-p} \varepsilon_i < \infty. \tag{1.7.1}$$

The planar domain Ω is the union of the square $\Omega_1 = (-1, 0) \times (0, 1)$, the triangle

$$\Omega_2 = \big\{(x, y) \in \mathbb{R}^2 : x \in (0, 1),\ y \in (0, x)\big\},$$

and the passages

$$\big\{(x, y) : y \in (a_i, a_i + \varepsilon_i),\ 0 \leq x \leq y\big\}, \quad i = 1, 2, \ldots$$

(see Fig. 11). Let u be defined on Ω by $u(x, y) = i - 1$ if $(x, y) \in \Omega_i$, $i = 1, 2$, and $u(x, y) = x/y$ if $(x, y) \in \Omega \setminus (\Omega_1 \cup \Omega_2)$.

Then $u \in L^1_p(\Omega)$ because of (1.7.1) (in fact $u \in C(\Omega) \cap L_\infty(\Omega) \cap L^1_p(\Omega)$). We shall show that u cannot be approximated by functions in $L^1_\infty(\Omega)$ in the metric of the space $L^1_p(\Omega)$.

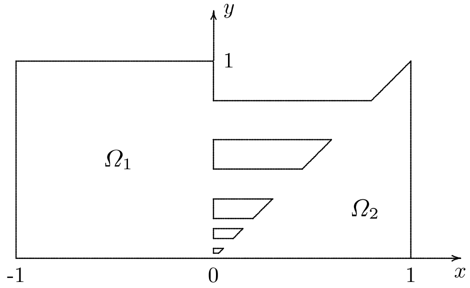

Fig. 11.

Let $v \in L^1_\infty(\Omega)$ be an arbitrary function. This function coincides with a function in $C(\Omega)$ a.e. on Ω. Moreover, by Sobolev's embedding theorem, there is a constant $K > 0$, independent of u and v, such that

$$\|v\|_{L_\infty(\Omega_1)} + \|v - 1\|_{L_\infty(\Omega_2)} \le K\|v - u\|_{L^1_p(\Omega)}. \tag{1.7.2}$$

Using the absolute continuity of v on almost all line segments $[(0, y), (y, y)]$ with

$$y \in \bigcup_{i \ge 1} (a_i, a_i + \varepsilon_i), \tag{1.7.3}$$

we obtain

$$\big|v(y, y) - v(0, y)\big| = \left| \int_0^y \frac{\partial v}{\partial x}(x, y)\, \mathrm{d}x \right| \le y\|\nabla v\|_{L_\infty(\Omega)}.$$

Thus

$$\big|v(y, y) - v(0, y)\big| \le 1/2$$

for sufficiently small y satisfying (1.7.3). Hence the left-hand side of (1.7.2) is not less than $1/2$, and the quantity $\|u - v\|_{L^1_p(\Omega)}$ cannot be less than $(2K)^{-1}$.

1.7.3 A Planar Bounded Domain for Which $L^2_1(\Omega) \cap L_\infty(\Omega)$ Is Not Dense in $L^2_1(\Omega)$

According to Lemma 1.7.1, the subspace of bounded functions is dense in $L^1_p(\Omega)$ for an arbitrary domain if $p \in [1, \infty)$. It turns out that this property cannot be generally extended to Sobolev spaces of higher orders. In this section we give an example of a bounded domain $\Omega \subset \mathbb{R}^2$ and a function $f \in L^2_2(\Omega)$ such that f does not belong to the closure of $L^2_1(\Omega) \cap L_q(\Omega)$ in the norm of $L^2_1(\Omega)$ with arbitrary $q > 0$. In particular, this implies that $L^2_p(\Omega) \cap L_q(\Omega)$ is not dense in $L^2_p(\Omega)$ for $p \le 2$.

First we establish an auxiliary assertion. Below we identify functions in L^2_1 with their continuous representatives.

Lemma. *Let G be a planar subdomain of the disk B_R, starshaped with respect to the disk B_r. Then the following estimate holds for all $f \in L_1^2(G)$:*

$$\big|f(z_1) - f(z_2)\big| \le c\big(\|\nabla_2 f\|_{L_1(G)} + |z_1 - z_2| r^{-1-2/q} \|f\|_{L_q(G)}\big), \qquad (1.7.4)$$

where $z_1, z_2 \in \bar{G}$, and the constant c depends only on q and the ratio R/r.

Proof. It will suffice to consider the case $r = 1$ and then use a similarity transformation. In view of Lemma 1.1.11 there is a linear function ℓ such that

$$\|f - \ell\|_{L_1(G)} \le c\,\|\nabla_2 f\|_{L_1(G)}.$$

Hence by the Sobolev embedding $L_1^2(G) \subset C(\bar{G})$ we obtain two estimates

$$\big|(f-\ell)(z_1) - (f-\ell)(z_2)\big| \le c\|\nabla_2 f\|_{L_1(G)},$$
$$\|\ell\|_{L_{\min\{1,q\}}(G)} \le c\big(\|\nabla_2 f\|_{L_1(G)} + \|f\|_{L_q(G)}\big).$$

Now (1.7.4) follows from the obvious inequality

$$\big|\ell(z_1) - \ell(z_2)\big| \le c|z_1 - z_2|\|\ell\|_{L_{\min\{1,q\}}(G)}.$$

This concludes the proof of the lemma. □

We turn to the required example. Let $\{\delta_i\}_{i\ge 0}$, $\{h_i\}_{i\ge 0}$ be two sequences of positive numbers satisfying $\delta_i < 2^{-i-2}$ and

$$h_i \le \exp\big(-(1+i)^2/\delta_i^2\big), \quad i \ge 0, \qquad (1.7.5)$$

$$\lim_{i\to\infty} \delta_i 2^{ib} = 0 \quad \text{for all } b > 0. \qquad (1.7.6)$$

Next, let $\{\varDelta_i\}_{i\ge 0}$ be the sequence of open isosceles right triangles with hypotenuses of length 2^{1-i}, placed on the lines $y = H_i$, where

$$H_j = 2^{1-j} + \sum_{s\ge j} (h_s - \delta_s), \quad j = 0, 1, \ldots .$$

We assume that all vertices of right angles lie on the axis Oy under the hypotenuses. Let $\varGamma_i$ denote the intersection of $\partial\varDelta_i$ with the halfplane $y \ge H_{i+1} + h_i$. Clearly the distance between $\varGamma_i$ and $\varGamma_{i+1}$ is h_i. By $\varOmega$ we mean the complement of $\bigcup_{i\ge 0} \varGamma_i$ to the rectangle $\{(x, y) \in \mathbb{R}^2 : |x| < 1, 0 < y < H_0\}$, (see Figs. 12 and 13).

Let $\eta \in C_0^\infty(-1, 1)$ and $\eta(t) = 1$ for $|t| \le 1/2$. We now define f on $\varOmega$ as follows. For any strip

$$\varPi_i = \big\{(x, y) \in \varOmega : H_{i+1} < y < H_i\big\}, \quad i \ge 0,$$

f is given on $\varPi_i \cap \varOmega$ by

$$f(x, y) = \operatorname{sign} x \bigg(1 + \frac{(|x| - \delta_i)}{\delta_i \log h_i} \log\big(|x| - \delta_i + h_i\big)\eta\big(2^{i+1}x\big)\bigg)$$

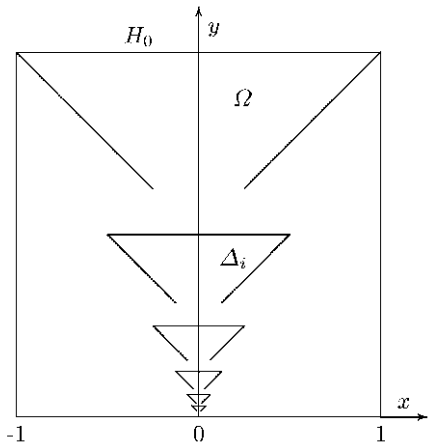

Fig. 12.

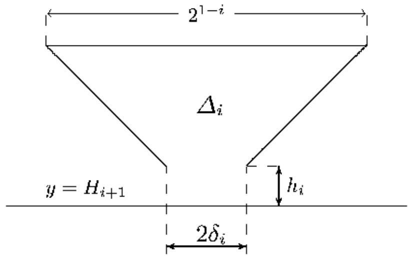

Fig. 13.

for $(x, y) \notin \Delta_i, |x| > \delta_i$, and $f(x, y) = x/\delta_i$ for the remaining points of Π_i. Clearly $\Pi_i \cap \operatorname{supp}(\nabla_2 f)$ is placed in the set

$$\{(x, y) \in \Pi_i \setminus \Delta_i : \delta_i \leq |x| \leq 2^{-i-1}\}.$$

Furthermore, the following estimate holds for $(x, y) \in \Pi_i$

$$|\nabla_2 f| \leq c \frac{\max\{2^i(i+1), (|x| - \delta_i + h_i)^{-1}\}}{\delta_i |\log h_i|}$$

(here and in the following in this section c is a positive constant independent of i). Hence

$$\|\nabla_2 f\|^2_{L_2(\Omega)} = \sum_{i \geq 0} \|\nabla_2 f\|^2_{L_2(\Pi_i)} \leq c \sum_{i \geq 0} \frac{|\log h_i| + (i+1)^2}{(\delta_i \log h_i)^2},$$

which is dominated by $c \sum_{i \geq 1} i^{-2}$ due to (1.7.5).

We now prove by contradiction that f does not belong to the closure of $L_1^2(\Omega) \cap L_q(\Omega)$ in the norm of $L_1^2(\Omega)$. Let

$$\Omega_0 = \Omega \setminus \Big(\bigcup_{i \geq 1} \bar{\Delta}_i\Big).$$

Since Ω_0 has the cone property, Sobolev's theorem says that the space $L_1^2(\Omega_0)$ is embedded into $C(\Omega_0) \cap L_\infty(\Omega_0)$, and

$$\|u\|_{L_\infty(\Omega_0)} \leq K\|u\|_{L_1^2(\Omega)}, \quad K = \text{const} > 0, \tag{1.7.7}$$

for all $u \in L_1^2(\Omega)$. Suppose that there exists a function $g \in L_1^2(\Omega) \cap L_q(\Omega)$ subject to

$$\|f - g\|_{L_1^2(\Omega)} \leq (2K)^{-1}.$$

Put $A_i^{\mp} = (\mp\delta_i, H_{i+1} + h_i)$, $i \geq 0$. Since $f(A_i^-) = -1$, $f(A_i^+) = 1$, the estimate

$$\big|g\big(A_i^+\big) - g\big(A_i^-\big)\big| \geq 1$$

is valid in view of (1.7.7) On the other hand, an application of the above lemma gives

$$c\big|g\big(A_i^+\big) - g\big(A_i^-\big)\big| \leq \|\nabla_2 g\|_{L_1(\Delta_i \cap \Pi_i)} + 2^{i+1+2i/q}\delta_i \, \|g\|_{L_q(\Delta_i \cap \Pi_i)}.$$

This inequality in conjunction with (1.7.6) implies that

$$\lim_{i \to \infty} \big|g\big(A_i^+\big) - g\big(A_i^-\big)\big| = 0,$$

and we arrive at the required contradiction.

1.7.4 Density of Bounded Functions in $L_p^2(\Omega)$ for Paraboloids in $\mathbb{R}^n$

We have seen in the preceding section that bounded domains with a non-smooth boundary may fail to have the property that the set $L_p^2(\Omega) \cap L_\infty(\Omega)$ is dense in $L_p^2(\Omega)$. It turns out that for unbounded domains this property may fail even when Ω is very simple with smooth boundary.

Let f be a function in $C^1([0,\infty))$ such that f is positive on $(0,\infty)$, $|f'(t)| \leq$ const and $f(t) \to \infty$ as $t \to \infty$. If $f(0) = f'(0) = 0$, we impose an additional condition $f'(t) > 0$ in the vicinity of $t = 0$. Let

$$\Omega = \big\{(x,t) \in \mathbb{R}^n : 0 < t < \infty, \ |x| < f(t)\big\}$$

(see Fig. 14). We call this domain an n-dimensional paraboloid.

In this section c denotes various positive constants depending only on p, n, and f. The following assertion gives a necessary and sufficient condition for the space $L_p^2(\Omega) \cap L_\infty(\Omega)$ to be dense in $L_p^2(\Omega)$.

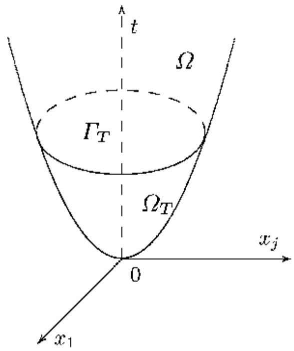

Fig. 14.

Theorem. *The space* $L^2_p(\Omega) \cap L_\infty(\Omega)$ *is dense in* $L^2_p(\Omega), 1 < p < \infty$, *if and only if*

$$\int_1^\infty f(t)^{\frac{1-n}{p-1}}\, \mathrm{d}t = \infty. \tag{1.7.8}$$

(*Clearly* (1.7.8) *holds for* $p > n$.)

The proof of this theorem will be given at the end of the section. First we establish three auxiliary assertions.

Lemma 1. *Let* $u \in L^2_p(\Omega) \cap L_\infty(\Omega)$,

$$\Gamma_T = \big\{(x,t) \in \Omega : t = T\big\},$$
$$\Omega_T = \big\{(x,t) : 0 < t < T,\ |x| < f(t)\big\},$$

and

$$w(T) = f(T)^{1-n} \int_{\Gamma_T} \frac{\partial u}{\partial t}\, \mathrm{d}x.$$

Then for all $T > 0$

$$\big|w(T)\big| \le c\bigg(\int_T^\infty f(t)^{\frac{1-n}{p-1}}\, \mathrm{d}t\bigg)^{(p-1)/p} \bigg(\int_{\Omega\setminus\Omega_T} |\nabla_2 u|^p\, \mathrm{d}x\, \mathrm{d}t\bigg)^{1/p}. \tag{1.7.9}$$

Proof. We may assume that

$$\int_1^\infty f(t)^{\frac{1-n}{p-1}}\, \mathrm{d}t < \infty. \tag{1.7.10}$$

Since

$$w(T) = \int_{|\xi|<1} \frac{\partial u}{\partial t}\big(f(t)\xi, t\big)\big|_{t=T}\, \mathrm{d}\xi,$$

we have for $S > T$

$$
\begin{aligned}
&w(S) - w(T) \\
&\quad = \int_T^S \mathrm{d}t \int_{|\xi|<1} \frac{\partial}{\partial t}\big(u_t\big(f(t)\xi, t\big)\big)\,\mathrm{d}\xi \\
&\quad = \int_T^S \mathrm{d}t \int_{|\xi|<1} \left(\frac{\partial^2 u}{\partial t^2}\big(f(t)\xi, t\big) + f'(t)\xi\left(\nabla_x \frac{\partial u}{\partial t}\right)\big(f(t)\xi, t\big)\right)\mathrm{d}\xi.
\end{aligned}
$$

By Hölder's inequality

$$
\begin{aligned}
&\big|w(T) - w(S)\big| \\
&\quad \le c\left(\int_T^S f(t)^{\frac{1-n}{p-1}}\,\mathrm{d}t\right)^{(p-1)/p}\left(\int_T^S \mathrm{d}t \int_{|x|<f(t)} \big|(\nabla_2 u)(x,t)\big|^p\,\mathrm{d}x\right)^{1/p}.
\end{aligned}
$$

Thus, the limit $d = \lim_{t\to\infty} w(t)$ exists and it suffices to deduce $d = 0$ to prove (1.7.9). Consider the function $\bar{u}$ on $(0, \infty)$ defined by

$$
\bar{u}(t) = f(t)^{1-n}\int_{|x|<f(t)} u(x,t)\,\mathrm{d}x = \int_{|\xi|<1} u\big(f(t)\xi, t\big)\,\mathrm{d}\xi.
$$

Putting $v(\xi, t) = u(f(t)\xi, t)$, we find

$$
\frac{\mathrm{d}\bar{u}}{\mathrm{d}t} = w(t) + \frac{f'(t)}{f(t)}\int_{|\xi|<1} \xi(\nabla_\xi v)(\xi, t)\,\mathrm{d}\xi. \tag{1.7.11}
$$

Furthermore,

$$
\int_{|\xi|<1} \xi \nabla_\xi v\,\mathrm{d}\xi = (1-n)\int_{|\xi|<1} v\,\mathrm{d}\xi + \int_{|\xi|=1} v\,\mathrm{d}s_\xi,
$$

where $\mathrm{d}s_\xi$ is the area element on the sphere $|\xi| = 1$. Since v is bounded, the second term on the right in (1.7.11) tends to zero as $t \to \infty$. So we have $\lim_{t\to\infty} \bar{u}'(t) = d$ and $d = 0$ as long as $\bar{u} \in L_\infty(0, \infty)$. This completes the proof of Lemma 1. □

Lemma 2. *Let* (1.7.8) *hold and let*

$$
F(\varphi) = \int_0^\infty \big(|\varphi'|^p + f^p|\varphi''|^p\big) f^{n-1}\,\mathrm{d}t. \tag{1.7.12}
$$

Then $\inf F(\varphi) = 0$*, where the infimum is taken over the set*

$$
\big\{\varphi : \varphi \in C^\infty(\mathbb{R}^1),\ \varphi(t) = 1 \text{ for } t \le 1,\ \varphi(t) = 0 \text{ for large positive } t\big\}. \tag{1.7.13}
$$

Proof. First let $p > n$. Since $f(t) \le ct$ for large t, it will suffice to prove the equality

$$
\inf \int_1^\infty \big(|\varphi'|^p + t^p|\varphi''|^p\big) t^{n-1}\,\mathrm{d}t = 0. \tag{1.7.14}
$$

Consider the function $\eta \in C^\infty(\mathbb{R}^1), \eta(t) = 1$ for $t \le 1, \eta(t) = 0$ for $t > 2$. By setting $\varphi(t) = \eta(t/N), N = 1, 2, \dots,$ we obtain (1.7.14).

Let $p \le n$. Since

$$f|\varphi''| \le c\big(|\varphi'| + \big|\big(f^{\frac{n-1}{p-1}}\varphi'\big)'\big|\big),$$

the functional (1.7.12) is majorized by

$$c\int_1^\infty \big(\big|f^{\frac{n-1}{p-1}}\varphi'\big|^p + \big|f^{\frac{n-1}{p-1}}\big(f^{\frac{n-1}{p-1}}\varphi'\big)'\big|^p\big)\frac{\mathrm{d}t}{f^{\frac{n-1}{p-1}}}, \tag{1.7.15}$$

on the set (1.7.13). After passing to the new variable $\tau = \tau(t)$ given by

$$\tau = \int_1^t f(\lambda)^{\frac{1-n}{p-1}}\,\mathrm{d}\lambda,$$

the integral in (1.7.15) takes the form

$$G(\psi) = \int_0^\infty \big(|\psi'|^p + |\psi''|^p\big)\,\mathrm{d}\tau, \quad \psi(\tau) = \varphi(t).$$

It remains to observe that $G(\psi_N) \to 0$, where $\psi_N(\tau) = \eta(\tau/\tau(N)), N = 1, 2, \dots,$ and η is the function introduced previously. This concludes the proof of the lemma. □

The proof of the preceding lemma enables us to state the following assertion.

Corollary. *If* (1.7.8) *holds, then there exists a minimizing sequence* $\{\varphi_N\}_{N\ge1}$ *for functional* (1.7.12) *such that* $\varphi_N \in C^\infty(\mathbb{R}^1), \varphi_N(t) = 1$ *for* $t \le N, \varphi_N(t) = 0$ *for large positive* t.

Lemma 3. *Let* $u \in L_p^2(\Omega)$. *For any* $\varepsilon > 0$ *there exist a linear function* ℓ *and a function* $v \in L_p^2(\Omega)$ *such that* $v(x,t) = \ell(x,t)$ *for large* t *and*

$$\|u - v\|_{L_p^2(\Omega)} < \varepsilon.$$

Proof. Let

$$G_N = \big\{(x,t) \in \Omega : N < t < N + f(N)\big\}.$$

By Lemma 1.1.11, there is a linear function ℓ_N satisfying

$$\big\|\nabla(u - \ell_N)\big\|_{L_p(G_N)} + f(N)^{-1}\|u - \ell_N\|_{L_p(G_N)} \le c\,f(N)\|\nabla_2 u\|_{L_p(G_N)}. \tag{1.7.16}$$

We put

$$\eta_N(t) = \eta\big(1 + (t - N)/f(N)\big), \qquad v_N = (u - \ell_N)\eta_N + \ell_N,$$

where η is the function introduced in the proof of Lemma 2. Clearly $v_N = u$ for $t \le N$ and $v_N = \ell_N$ for $t \ge N + f(N)$. Next,

$$\begin{aligned} c\big\|\nabla_2(u-v_N)\big\|_{L_p(\Omega)} &\le \big\|(1-\eta_N)\nabla_2 u\big\|_{L_p(\Omega)} + \big\||\nabla(u-\ell_N)||\nabla\eta_N|\big\|_{L_p(\Omega)} \\ &\quad + \big\|(u-\ell_N)|\nabla_2\eta_N|\big\|_{L_p(\Omega)}. \end{aligned}$$

Since

$$|\nabla\eta_N| \le cf(N)^{-1}, \qquad \|\nabla_2\eta_N\| \le cf(N)^{-2},$$

and in view of (1.7.16), it follows that

$$\big\|\nabla_2(u-v_N)\big\|_{L_p(\Omega)} \le c\|\nabla_2 u\|_{L_p(\Omega\setminus\Omega_N)}.$$

The last norm tends to zero as $N\to\infty$, and we can set $v=v_N$ for sufficiently large N. This establishes Lemma 3.

Proof of Theorem. Let (1.7.10) hold. We check that $L_p^2(\Omega)\cap L_\infty(\Omega)$ is not dense in $L_p^2(\Omega)$. Let $u_\nu\in L_p^2(\Omega)\cap L_\infty(\Omega)$ and $u_\nu\to t$ in the norm of $L_p^2(\Omega)$. By Lemma 1

$$\left|\int_{\Gamma_T}\frac{\partial u_\nu}{\partial t}\,\mathrm{d}x\right| \le c(n,p,f,T)\|\nabla_2 u_\nu\|_{L_p(\Omega)} = o(1) \quad \text{as } \nu\to\infty. \tag{1.7.17}$$

We have $u_\nu\to t$ in $L_p^2(\Omega_2\setminus\bar\Omega_1)$. Since $\Omega_2\setminus\bar\Omega_1\in C^{0,1}$, the spaces L_p^2 and V_p^2 coincide for this domain. Thus, by Sobolev's theorem,

$$\int_{\Gamma_T}\frac{\partial u_\nu}{\partial t}\,\mathrm{d}x \to m_{n-1}\Gamma_T \quad \text{as } \nu\to\infty$$

for almost all $T\in(1,2)$. However, this contradicts (1.7.17).

Suppose (1.7.8) is valid. It will be shown that an arbitrary function $u\in L_p^2(\Omega)$ can be approximated by functions in $L_p^2(\Omega)\cap L_\infty(\Omega)$. According to Lemma 3, it is sufficient to assume that $u=w+\ell$, where ℓ is a linear function and $w(x,t)=0$ for large t. Since any domain Ω_T is of class C, Theorem 1.1.6/2 applies, and w can be approximated in $L_p^2(\Omega)$ by functions in $C^\infty(\bar\Omega)$ with bounded supports. It remains to approximate the functions $t,x_1,\dots,x_{n-1}$.

Let $\{\varphi_N\}$ be the sequence from the Corollary preceding Lemma 3. We set

$$v_N(x,t) = N + \int_N^t \varphi_N(s)\,\mathrm{d}s \quad \text{for } t>N$$

and $v_N(x,t)=t$ for $t\le N$. Clearly $v_N\in L_p^2(\Omega)\cap L_\infty(\Omega)$ and

$$\big\|\nabla_2(t-v_N)\big\|_{L_p(\Omega)}^p = c\int_0^\infty \big|\varphi_N'(t)\big|^p f(t)^{n-1}\,\mathrm{d}t = o(1) \quad \text{as } N\to\infty.$$

To approximate $x_j, 1\le j\le n-1$, we put $w_N(x,t)=x_j\varphi_N(t)$. Then

$$\big\|\nabla_2(x_j-w_N)\big\|_{L_p(\Omega)}^p \le c\int_0^\infty \big(|\varphi_N'|^p + f^p|\varphi_N''|^p\big)f^{n-1}\,\mathrm{d}t,$$

and a reference to the Corollary completes the proof. □

1.7.5 Comments to Sect. 1.7

Lemma 1.7.1 is due to Deny and Lions [234]. In connection with the contents of Sect. 1.7.2, we need two definitions. Let $C_b^\infty(\Omega)$ denote the set of functions in $C^\infty(\Omega)$ with bounded gradients of all orders. A bounded domain $\Omega \subset \mathbb{R}^n$ has the interior segment property if to every $x \in \partial\Omega$ there correspond a number $r > 0$ and a nonzero vector $y \in \mathbb{R}^n$ such that $z + ty \in \Omega$ provided $0 \le t \le 1$ and $z \in \Omega \cap B_r(x)$. (Clearly domains of class C have the interior segment property.)

If $\Omega \subset \mathbb{R}^2$ is a bounded domain that is either starshaped with respect to a point or satisfies the interior segment condition, then $C_b^\infty(\Omega)$ is dense in $V_p^l(\Omega)$ for $p \in [1, \infty)$ and $l = 1, 2, \dots$. This theorem is due to Smith, Stanoyevitch, and Stegenga [707]. In particular, it gives sufficient conditions for the space $V_p^l(\Omega) \cap L_\infty(\Omega)$ to be dense in $V_p^l(\Omega)$, $\Omega \subset \mathbb{R}^2$. The just-mentioned result fails for multidimensional domains.

Example 1.7.2, borrowed from Sect. 2.2 of the book by Maz'ya and Poborchi [576], is analogous to Example 7.1 in the paper by Smith, Stanoyevitch, and Stegenga [707], which was constructed to show that the space $C_b^\infty(\Omega)$ is not always dense in $W_p^1(\Omega)$ (see also O'Farrell [643]).

The contents of Sects. 1.7.3 and 1.7.4 are taken from the paper by Maz'ya and Netrusov [572].

1.8 Maximal Algebra in $W_p^l(\Omega)$

1.8.1 Main Result

Let A be a subset of a Banach function space. The set A is called an algebra with respect to multiplication if there is a constant $c > 0$ such that the inclusions $u \in A, v \in A$ imply $uv \in A$ and $\|uv\| \le c\|u\|\|v\|$.

Note that the space $W_p^l = W_p^l(\mathbb{R}^n)$ is not an algebra when $lp \le n, p > 1$ or $l < n, p = 1$. Indeed, if W_p^l were an algebra, the following inequalities would occur:

$$\left\|u^N\right\|_{L_p}^{1/N} \le \left\|u^N\right\|_{W_p^l}^{1/N} \le c\,\|u\|_{W_p^l},$$

where u is an arbitrary function in W_p^l and $N = 1, 2, \dots$. By letting $N \to \infty$, we can obtain

$$\|u\|_{L_\infty} \le c\|u\|_{W_p^l}.$$

Clearly, the last inequality is not true for the values p, l mentioned earlier. Thus, W_p^l is generally not an algebra with respect to pointwise multiplication. However, it is possible to describe the maximal algebra contained in W_p^l. Namely, the following assertion holds.

Theorem. *The subspace $W_p^l \cap L_\infty$ with the norm $\|\cdot\|_{W_p^l} + \|\cdot\|_{L_\infty}$ is the maximal algebra contained in W_p^l. In particular, the space W_p^l is an algebra if $lp > n$, $p > 1$ or if $l \ge n, p = 1$.*

We need an auxiliary multiplicative inequality to prove this theorem.

Lemma 1. *Let $1 \le p \le \infty$ and $l \ge 2$ be an integer. For any $u \in W_p^l \cap L_\infty$ and any $j = 1, \dots, l-1$, the estimate*

$$\|\nabla_j u\|_{L_{lp/j}} \le c \|u\|_{L_\infty}^{1-j/l} \|u\|_{W_p^l}^{j/l} \tag{1.8.1}$$

holds with c independent of u.

We shall deduce Lemma 1 with the aid of another auxiliary inequality.

Lemma 2. *If $u \in C_0^\infty(\mathbb{R}^n), p \in [1, \infty), 1 \le q \le \infty$, and $2r^{-1} = p^{-1} + q^{-1}$, then*

$$\|\nabla u\|_{L_r} \le c \|\nabla_2 u\|_{L_p}^{1/2} \|u\|_{L_q}^{1/2},$$

where c is a positive constant depending only on n.

Proof. It is sufficient to assume $p > 1$ and $q < \infty$. Then the extreme cases $p = 1$ or $q = \infty$ follow by letting p tend to 1 or q tend to ∞.

We remark that the required inequality is a consequence of the one-dimensional estimate

$$\int |u'|^r \, \mathrm{d}x \le c_0^r \Big(\int |u''|^p \, \mathrm{d}x \Big)^{\frac{r}{2p}} \Big(\int |u|^q \, \mathrm{d}x \Big)^{\frac{r}{2q}} \tag{1.8.2}$$

with $2r^{-1} = p^{-1} + q^{-1}$ and c_0 an absolute constant (the integration is taken over $\mathbb{R}^1$). Once this estimate has been established, the result follows by integrating with respect to the other variables and by applying Hölder's inequality.

Note that for any $u \in C_0^\infty(\mathbb{R}^1)$

$$2c_1 \|u'\|_{L_r(i)} \le |i|^{1+r^{-1}-p^{-1}} \|u''\|_{L_p(i)} + |i|^{p^{-1}-r^{-1}-1} \|u\|_{L_q(i)}, \tag{1.8.3}$$

where c_1 is an absolute positive constant, i an interval, and $|i|$ its length. The last estimate follows from (1.1.18) and the Hölder inequality.

Inequality (1.8.2) is a consequence of the estimate

$$c_1 \|u'\|_{L_r(\varDelta)} \le \|u''\|_{L_p}^{1/2} \|u\|_{L_q}^{1/2}, \tag{1.8.4}$$

where $\varDelta$ is an arbitrary interval in $\mathbb{R}^1$ of finite length. The last is verified as follows.

Fix a positive integer k and introduce the closed interval i of length $|\varDelta|/k$ with the same left point as $\varDelta$. Let us consider inequality (1.8.3) for this interval i. If the first summand on the right of the inequality is greater than the second, we put $i_1 = i$. In this case

$$c_1^r \int_{i_1} |u'|^r \, \mathrm{d}x \le \Big(\frac{|\varDelta|}{k} \Big)^{1+r-r/p} \Big(\int_{\mathbb{R}^1} |u''|^p \, \mathrm{d}x \Big)^{\frac{r}{p}}. \tag{1.8.5}$$

Suppose the first term on the right of (1.8.3) is less than the second. We then increase the interval i leaving the left end point fixed until these two terms are equal (clearly, the equality must take place for some i with $|i| < \infty$ because $1 + r^{-1} - p^{-1} > 0$). Let i_1 denote the resulting interval. Then

$$c_1^r \int_{i_1} |u'|^r \, dx \le \Big(\int_{i_1} |u''|^p \, dx \Big)^{\frac{r}{2p}} \Big(\int_{i_1} |u|^q \, dx \Big)^{\frac{r}{2q}}. \qquad (1.8.6)$$

Putting the end point of i_1 to be the initial point of the next interval, repeat this process with the same k. We stop it when the closed finite intervals $i_1, i_2, \dots$ (each of length at least $|\Delta|/k$) form a covering of the interval Δ. Note that the covering $\{i_1, i_2, \dots\}$ contains at most k elements, each i_s supporting estimates (1.8.5) or (1.8.6) (with i_1 replaced by i_s). Adding these estimates and applying Hölder's inequality, one arrives at

$$c_1^r \int_\Delta |u'|^r \, dx \le k \Big(\frac{|\Delta|}{k} \Big)^{1+r-r/p} \Big(\int |u''|^p \, dx \Big)^{\frac{r}{p}} + \Big(\int |u''|^p \, dx \Big)^{\frac{r}{2p}} \Big(\int |u|^q \, dx \Big)^{\frac{r}{2q}}.$$

Now (1.8.4) follows by letting $k \to \infty$ because $1 + r - r/p > 1$. The proof of Lemma 2 is complete. □

Proof of Lemma 1. It is sufficient to assume $p < \infty$. First let $u \in C_0^\infty(\mathbb{R}^n)$. If $a_j = \|\nabla_j u\|_{pl/j}$, Lemma 2 implies $a_j^2 \le c a_{j-1} a_{j+1}$ for $j = 1, \dots, l-1$. By induction on l, we obtain $a_j \le c a_0^{1-j/l} a_l^{j/l}$, and Lemma 1 is established for smooth functions u with compact support.

Turning to the general case $u \in W_p^l \cap L_\infty$, we first assume that $\operatorname{supp} u$ is bounded and consider a mollification u_h of u with radius h. Since $\|u_h\|_\infty \le c\|u\|_\infty$ and by Lemma 1 applied to u_h we obtain

$$\|\nabla_j u_h\|_{L_{lp/j}} \le c \, \|u_h\|_{W_p^l}^{j/l} \, \|u\|_{L_\infty}^{1-j/l}.$$

Now passage to the limit as $h \to 0$ gives the required inequality for u with bounded support.

To conclude the proof of the lemma, we remove the assumption on the boundedness of $\operatorname{supp} u$. To this end we introduce a cutoff function $\eta \in C_0^\infty(B_2)$ such that $0 \le \eta \le 1$ and $\eta|_{B_1} = 1$. Let $\eta_k(x) = \eta(x/k), k = 1, 2, \dots$. An application of Lemma 1 to the function $u\eta_k$ yields

$$\|\nabla_j u\|_{L_{lp/j}(B_k)} \le c \|u\eta_k\|_{W_p^l}^{j/l} \|u\|_{L_\infty}^{1-j/l}.$$

It remains to pass to the limit as $k \to \infty$ to establish Lemma 1 in the general case. □

We now give a proof of the theorem stated previously.

Proof of Theorem. Let A be an algebra contained in W_p^l.

As we have already seen at the beginning of the section $A \subset L_\infty \cap W_p^l$.

To show that the space $A = L_\infty \cap W_p^l$ is an algebra, we let $u, v \in A$ be arbitrary. Then

$$\big\|\nabla_l(uv)\big\|_{L_p} \le c\sum_{k=0}^{l}\big\||\nabla_k u|\cdot|\nabla_{l-k}v|\big\|_{L_p}$$
$$\le c\sum_{k=0}^{l}\|\nabla_k u\|_{L_{lp/k}}\|\nabla_{l-k}v\|_{L_{lp/(l-k)}}$$

by Hölder's inequality. It follows from Lemma 1 that

$$\big\|\nabla_l(uv)\big\|_{L_p} \le c\sum_{k=0}^{l}\|u\|_{L_\infty}^{(l-k)/l}\|\nabla_l u\|_{L_p}^{k/l}\|v\|_{L_\infty}^{k/l}\|\nabla_l v\|_{L_p}^{(l-k)/l},$$

and hence

$$\big\|\nabla_l(uv)\big\|_{L_p} \le c\big(\|u\|_{L_\infty}\|\nabla_l v\|_{L_p} + \|v\|_{L_\infty}\|\nabla_l u\|_{L_p}\big), \tag{1.8.7}$$

which does not exceed $c\|u\|_A\|v\|_A$. Thus, A is an algebra and hence the space $W_p^l \cap L_\infty$ is the maximal algebra contained in W_p^l.

If $lp > n, p > 1$ or $l \ge n, p = 1$, then $W_p^l \subset L_\infty$ by Sobolev's embedding theorem, and the space W_p^l is an algebra for these p and l. This concludes the proof. □

Let Ω be a domain in $\mathbb{R}^n$. We may ask whether the space $W_p^l(\Omega)\cap L_\infty(\Omega)$ is an algebra with respect to pointwise multiplication. Clearly, for $l = 1$ the answer is affirmative. Since Stein's extension operator from a domain $\Omega \in C^{0,1}$ is continuous as an operator

$$W_p^l(\Omega)\cap L_\infty(\Omega) \to W_p^l\big(\mathbb{R}^n\big)\cap L_\infty\big(\mathbb{R}^n\big),$$

the above question has the affirmative answer for finite sums of domains in $C^{0,1}$. For example, Ω can be a bounded domain having the cone property. However, it turns out that the space $W_p^l(\Omega)\cap L_\infty(\Omega)$ is generally not an algebra.

1.8.2 The Space $W_2^2(\Omega)\cap L_\infty(\Omega)$ Is Not Always a Banach Algebra

Here we give an example of a bounded planar domain Ω such that $W_2^2(\Omega)\cap L_\infty(\Omega)$ is not an algebra.

Let Ω be the union of the rectangle $P = \{(x, y) : x \in (0, 2), y \in (0, 1)\}$, the squares P_k with edge length 2^{-k} and the passages S_k of height 2^{-k} and of width $2^{-\alpha k}$, $k = 1, 2, \dots$, $\alpha > 1$ (see Figs. 15 and 16). Define $u = 0$ on P,

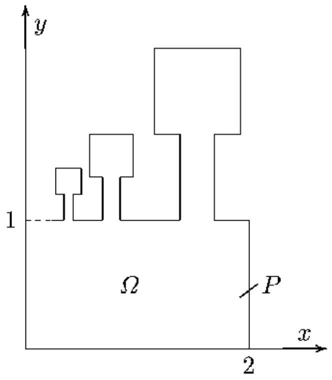

Fig. 15.

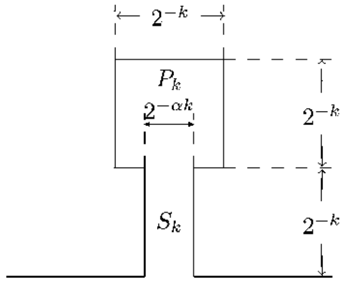

Fig. 16.

$u(x,y) = 2^{3k/2}(y-1)^2$ on S_k, $k = 1, 2, \dots$, and $u(x,y) = 2^{k/2}(2(y-1) - 2^{-k})$ on P_k for $k \geq 1$. Straightforward calculations show that

$$\|\nabla_2 u\|^2_{L_2(S_k)} = 2^{2+(2-\alpha)k}, \quad |u| \leq 3, \qquad \big\|\nabla_2\big(u^2\big)\big\|_{L_2(P_k)} = 8.$$

Thus, if $\alpha > 2$, then $u \in W_2^2(\Omega) \cap L_\infty(\Omega)$, but $u^2 \notin W_2^2(\Omega)$.

1.8.3 Comments to Sect. 1.8

A general form of the inequality obtained in Lemma 1.8.1/1 is due to Gagliardo [300] and Nirenberg [640]. The proof of Lemma 1.8.1/2 follows the paper by Nirenberg [640] where it was also shown that $L_\infty \cap W_p^l$ is an algebra. A counterexample in Sect. 1.8.2 is taken from the paper by Maz'ya and Netrusov [572].

2

Inequalities for Functions Vanishing at the Boundary

The present chapter deals with the necessary and sufficient conditions for the validity of certain estimates for the norm $\|u\|_{L_q(\Omega,\mu)}$, where $u \in \mathscr{D}(\Omega)$ and μ is a measure in Ω. Here we consider inequalities with the integral

$$\int_\Omega \big[\Phi(x, \nabla u)\big]^p \,\mathrm{d}x,$$

on the right-hand side. The function $\Phi(x, \xi)$, defined for $x \in \Omega$ and $\xi \in \mathbb{R}^n$, is positive homogeneous of degree one in ξ. The conditions are stated in terms of isoperimetric (for $p = 1$ in Sect. 2.1) and isocapacitary (for $p \geq 1$, in Sects. 2.2–2.4) inequalities. For example, we give a complete answer to the question of validity of the inequality

$$\|u\|_{L_q(\Omega,\mu)} \leq C\Big(\int_\Omega \big[\Phi(x, \nabla u)\big]^p \,\mathrm{d}x\Big)^{1/p},$$

both for $q \geq p \geq 1$ and $0 < q < p$, $p \geq 1$. In particular, in the first case there hold sharp inequalities for the best constant C

$$\beta^{1/p} \leq C \leq p(p-1)^{(1-p)/p}\beta^{1/p},$$

where

$$\beta = \sup_{F\subset\Omega} \frac{\mu(F)^{p/q}}{(p,\Phi)\text{-cap}(F,\Omega)},$$

with the so-called (p,Φ)-capacity of a compact subset of Ω in the denominator. Actually, this is a special case of a more general assertion concerning Birnbaum–Orlicz spaces.

Among other definitive results we obtain criteria for the validity of multiplicative inequalities of the form

$$\|u\|_{L_p(\Omega,\mu)} \leq C\big\|\Phi(\cdot, \nabla u)\big\|^{\delta}_{L_p(\Omega)}\|u\|^{1-\delta}_{L_r(\Omega,\nu)}$$

V. Maz'ya, *Sobolev Spaces*,
Grundlehren der mathematischen Wissenschaften 342,
DOI 10.1007/978-3-642-15564-2_2,

as well as the necessary and sufficient conditions for compactness of related embedding operators.

In Sect. 2.5 we give applications of the results in Sect. 2.4 to the spectral theory of the multidimensional Schrödinger operator with a nonpositive potential. Here the necessary and sufficient conditions ensuring the positivity and semiboundedness of this operator, discreteness, and finiteness of its negative spectrum are obtained.

Certain properties of quadratic forms of the type

$$\int_{\mathbb{R}^n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,\mathrm{d}x$$

are studied in Sects. 2.6.1 and 2.6.2. Finally, Sects. 2.7 and 2.8 are devoted to sharp constants in some multidimensional inequalities of the Hardy type.

2.1 Conditions for Validity of Integral Inequalities (the Case $p = 1$)

2.1.1 Criterion Formulated in Terms of Arbitrary Admissible Sets

A bounded open set $g \subset \mathbb{R}^n$ will be called admissible if $\bar{g} \subset \Omega$ and ∂g is a C^∞ manifold. In Chaps. 5–7 this definition will be replaced by a broader one.

Let $\mathscr{N}(x)$ denote the unit normal to the boundary of the admissible set g at a point x that is directed toward the interior of g. Let $\Phi(x,\xi)$ be a continuous function on $\Omega \times \mathbb{R}^n$ that is nonnegative and positive homogeneous of the first degree with respect to ξ. We introduce the weighted area of ∂g

$$\sigma(\partial g) = \int_{\partial g} \Phi\big(x, \mathscr{N}(x)\big)\,\mathrm{d}s(x). \tag{2.1.1}$$

Let μ and ν be measures in Ω and $\omega_n = s(\partial B_1)$.

The following theorem contains *a necessary and sufficient condition for the validity of the multiplicative inequality*:

$$\|u\|_{L_q(\Omega,\mu)} \le C\big\|\Phi(\cdot,\nabla u)\big\|^{\delta}_{L_1(\Omega)}\|u\|^{1-\delta}_{L_r(\Omega,\nu)} \tag{2.1.2}$$

for all $u \in \mathscr{D}(\Omega)$. This result will be proved using the same arguments as in Theorem 1.4.2/1.

Theorem. 1. *If for all admissible sets*

$$\mu(g)^{1/q} \le \alpha\sigma(\partial g)^{\delta}\nu(g)^{(1-\delta)/r}, \tag{2.1.3}$$

where $\alpha = \mathrm{const} > 0$, $\delta \in [0,1]$, r, $q > 0$, $\delta + (1-\delta)r^{-1} \ge q^{-1}$, *then* (2.1.2) *holds for all* $u \in \mathscr{D}(\Omega)$ *with* $C \le \alpha r^{\delta}(r\delta + 1 - \delta)^{-\delta-(1-\delta)/r}$.

2. *If* (2.1.2) *holds for all* $u \in \mathscr{D}(\Omega)$ *with* $q > 0$, $\delta \in [0,1]$, *then* (2.1.3) *holds for all admissible sets* g *and* $\alpha \le C$.

Proof. 1. First we note that by Theorem 1.2.4

$$\int_\Omega \Phi(x, \nabla u)\,\mathrm{d}x = \int_{\{x:|\nabla u|>0\}} \Phi\Big(x, \frac{\nabla u}{|\nabla u|}\Big)|\nabla u|\,\mathrm{d}x$$
$$= \int_0^\infty \mathrm{d}t \int_{\mathscr{E}_t} \Phi\Big(x, \frac{\nabla u}{|\nabla u|}\Big)\,\mathrm{d}s = \int_0^\infty \sigma(\partial \mathscr{L}_t)\,\mathrm{d}t. \qquad (2.1.4)$$

Here we used the fact that $|\nabla u| \ne 0$ on $\mathscr{E}_t = \{x : |u(x)| = t\}$ for almost all t and that for such t the sets $\mathscr{L}_t = \{x : |u(x)| > t\}$ are bounded by C^∞ manifolds. By Lemma 1.2.3

$$\|u\|_{L_q(\Omega,\mu)} = \Big(\int_0^\infty \mu(\mathscr{L}_t)\,\mathrm{d}(t^q)\Big)^{1/q}.$$

Since $\mu(\mathscr{L}_t)$ is a nonincreasing function, then, applying (1.3.41), we obtain

$$\|u\|_{L_q(\Omega,\mu)} \le \Big(\int_0^\infty \mu(\mathscr{L}_t)^{\gamma/q}\,\mathrm{d}(t^\gamma)\Big)^{1/\gamma},$$

where $\gamma = r(r\delta + 1 - \delta)^{-1}$, $\gamma \le q$. Using the fact that the sets $\mathscr{L}_t$ are admissible for almost all t, from (2.1.3) we obtain

$$\|u\|_{L_q(\Omega,\mu)} \le \gamma^{1/\gamma}\alpha\Big(\int_0^\infty \sigma(\partial\mathscr{L}_t)^{\gamma\delta}\nu(\mathscr{L}_t)^{\gamma(1-\delta)/r}t^{\gamma-1}\,\mathrm{d}t\Big)^{1/\gamma}.$$

Since $\gamma\delta + \gamma(1-\delta)/r = 1$, then by Hölder's inequality

$$\|u\|_{L_q(\Omega,\mu)} \le \gamma^{1/\gamma}\alpha\Big(\int_0^\infty \sigma(\partial\mathscr{L}_t)\,\mathrm{d}t\Big)^{\delta}\Big(\int_0^\infty \nu(\mathscr{L}_t)t^{r-1}\,\mathrm{d}t\Big)^{(1-\delta)/r},$$

which by virtue of (2.1.4) and Lemma 1.2.3 is equivalent to (2.1.2).

2. Let g be any admissible subset of Ω and let $\mathrm{d}(x) = \mathrm{dist}(x, \mathbb{R}^n \backslash g)$, $g_t = \{x \in \Omega, \mathrm{d}(x) > t\}$. Let α denote a nondecreasing function, infinitely differentiable on $[0, \infty)$, equal to unity for $d \ge 2\varepsilon$ and to zero for $\mathrm{d} \le \varepsilon$, where ε is a sufficiently small positive number. Then we substitute $u_\varepsilon(x) = \alpha[\mathrm{d}(x)]$ into (2.1.2).

By Theorem 1.2.4,

$$\int_\Omega \Phi(x, \nabla u_\varepsilon)\,\mathrm{d}x = \int_0^{2\varepsilon} \alpha'(t)\int_{\partial g_t} \Phi\big(x, \mathscr{N}(x)\big)\,\mathrm{d}s(x),$$

where $\mathscr{N}(x)$ is the normal at $x \in \partial g_t$ directed toward the interior of g_t. Since

$$\int_{\partial g_t} \Phi\big(x, \mathscr{N}(x)\big)\,\mathrm{d}s(x) \xrightarrow{t\to 0} \sigma(\partial g),$$

we obtain

$$\int_{\Omega} \Phi(x, \nabla u_\varepsilon)\,\mathrm{d}x \xrightarrow{\varepsilon\to 0} \sigma(\partial g).$$

Let K be a compactum in g such that $\operatorname{dist}(K, \partial g) > 2\varepsilon$. Then $u_\varepsilon(x) = 1$ on K and

$$\|u_\varepsilon\|_{L_q(\Omega,\mu)} \geq \mu(K)^{1/q}.$$

Using $0 \leq u_\varepsilon(x) \leq 1$ and $\operatorname{supp} u_\varepsilon \subset g$, we see that

$$\|u_\varepsilon\|_{L_r(\Omega,\nu)} \leq \nu(g)^{1/r}.$$

Now from (2.1.2) we obtain

$$\mu(g)^{1/q} = \sup_{K\subset g} \mu(K)^{1/q} \leq C\sigma(\partial g)^{\delta}\nu(g)^{(1-\delta)/r}.$$

The result follows. $\square$

2.1.2 Criterion Formulated in Terms of Balls for $\Omega = \mathbb{R}^n$

In the case $\Phi(x,\xi) = |\xi|$, $\Omega = \mathbb{R}^n$, $\nu = m_n$ it follows from (2.1.2) that for all balls $B_\varrho(x)$

$$\mu\big(B_\varrho(x)\big)^{1/q} \leq A\varrho^{\delta(n-1)+(1-\delta)n/r}. \tag{2.1.5}$$

With minor modification in the proof of Theorem 1.4.2/2 we arrive at the converse assertion.

Theorem. *If* (2.1.5) *holds with* $\delta \in [0,1]$; q, $r > 0$, $\delta + (1-\delta)/r \geq 1/q$ *for all balls* $B_\varrho(x)$, *then*

$$\|u\|_{L_q(\mu)} \leq C\big\|\Phi(\cdot, \nabla u)\big\|_{L_1}^{\delta}\|u\|_{L_r}^{(1-\delta)} \tag{2.1.6}$$

holds for all $u \in \mathscr{D}(\mathbb{R}^n)$ *with* $C \leq cA$.

Proof. As already shown in the proof of Theorem 1.2.1/2, for any bounded open set g with a smooth boundary there exists a sequence $\{B_{\varrho_i}(x_i)\}_{i\geq 1}$ of disjoint balls with the properties

$$(\alpha)\quad g \subset \bigcup_{i\geq 1} B_{3\varrho_i}(x_i),$$

$$(\beta)\quad 2m_n\big(q \cap B_{\varrho_i}(x_i)\big) = v_n\varrho_i^n,$$

$$(\gamma)\quad s(\partial g) \geq c\sum_{i\geq 1}\varrho_i^{n-1}.$$

From (2.1.5) it follows that

$$\mu(g) \leq \sum_{i\geq 1}\mu\big(B_{3\varrho_i}(x_i)\big) \leq A^q\sum_{i\geq 1}(3\varrho_i)^{q[\delta(n-1)+(1-\delta)n/r]}. \tag{2.1.7}$$

Since $q\delta + (1-\delta)n/r \geq 1$, it follows from (2.1.7) that

$$\mu(g) \leq cA^q \Big(\sum_{i\geq 1} \varrho_i^{q\frac{\delta(n-1)+(1-\delta)n/r}{q\delta+(1-\delta)n/r}} \Big)^{q\delta+(1-\delta)n/r},$$

which by Hölder's inequality does not exceed

$$cA^q \Big(\sum_{i\geq 1} \varrho_i^{n-1} \Big)^{q\delta} \Big(\sum_{i\geq 1} \varrho_i^{n} \Big)^{(1-\delta)q/r}.$$

To conclude the proof it remains to apply Theorem 2.1.1. □

2.1.3 Inequality Involving the Norms in $L_q(\Omega, \mu)$ and $L_r(\Omega, \nu)$ (Case $p = 1$)

The next theorem is proved analogously to Theorem 2.1.1.

Theorem. 1. *If for all admissible sets $g \subset \Omega$*

$$\mu(g)^{1/q} \leq \alpha\sigma(\partial g) + \beta\nu(g)^{1/r}, \tag{2.1.8}$$

where $\alpha \geq 0$, $\beta \geq 0$, $q \geq 1 \geq r$, then

$$\|u\|_{L_q(\Omega,\mu)} \leq \alpha\big\|\Phi(x, \nabla u)\big\|_{L(\Omega)} + \beta\|u\|_{L_r(\Omega,\nu)} \tag{2.1.9}$$

holds for all $u \in \mathscr{D}(\Omega)$.

2. *If* (2.1.9) *holds for all $u \in \mathscr{D}(\Omega)$, then* (2.1.8) *holds for all admissible sets g.*

2.1.4 Case $q \in (0, 1)$

Here we deal with the inequality

$$\|u\|_{L_q(\Omega,\mu)} \leq C\big\|\Phi(\cdot, \nabla u)\big\|_{L_1(\Omega)} \tag{2.1.10}$$

for $u \in C_0^\infty(\Omega)$. As a particular case of (2.1.9), we obtain from Theorem 2.1.3 that (2.1.10) *holds with $q \geq 1$ if and only if for all admissible sets g*

$$\mu(g)^{1/q} \leq \alpha\sigma(\partial g) \tag{2.1.11}$$

and α is the best value of C.

We shall show that (2.1.10) can be completely characterized also for $q \in (0, 1)$. Let us start with the basic properties of the so-called nonincreasing rearrangement of a function.

Let u be a function in Ω measurable with respect to the measure μ. We associate with u its nonincreasing rearrangement u_μ^* on $(0, \infty)$, which is introduced by

$$u^*_\mu(t) = \inf\{s > 0 : \mu(\mathscr{L}_s) \le t\}, \tag{2.1.12}$$

where $\mathscr{L}_s = \{x \in \Omega : |u(x)| > s\}$.

Clearly u^*_μ is nonnegative and nonincreasing on $(0, \infty)$. We also have $u^*_\mu(t) = 0$ for $t \ge \mu(\Omega)$. Furthermore, it follows from the definition of u^* that

$$u^*_\mu\big(\mu(\mathscr{L}_s)\big) \le s \tag{2.1.13}$$

and

$$\mu(\mathscr{L}_{u^*(t)}) \le t, \tag{2.1.14}$$

the last because the function $s \to \mu(\mathscr{L}_s)$ is continuous from the right.

The nonincreasing rearrangement of a function has the following important property.

Lemma 1. *If* $q \in (0, \infty)$, *then*

$$\int_\Omega \big|u(x)\big|^q \, \mathrm{d}\mu = \int_0^\infty \big(u^*_\mu(t)\big)^q \, \mathrm{d}t.$$

Proof. The required equality is a consequence of the formula

$$\int_\Omega \big|u(x)\big|^q \, \mathrm{d}\mu = \int_0^\infty \mu(\mathscr{L}_t) \, \mathrm{d}\big(t^q\big)$$

and the identity

$$m_1\big(\mathscr{L}^*_s\big) = \mu(\mathscr{L}_s), \quad s \in (0, \infty) \tag{2.1.15}$$

in which $\mathscr{L}^*_s = \{t > 0 : u^*_\mu(t) > s\}$. To check (2.1.15) we first note that

$$m_1\big(\mathscr{L}^*_s\big) = \sup\{t > 0 : u^*_\mu(t) > s\} \tag{2.1.16}$$

by the monotonicity of u^*_μ. Hence, (2.1.13) yields

$$m_1\big(\mathscr{L}^*_s\big) \le \mu(\mathscr{L}_s).$$

For the inverse inequality, let $\varepsilon > 0$ and $t = m_1(\mathscr{L}^*_s) + \varepsilon$. Then (2.1.16) implies $u^*_\mu(t) \le s$ and therefore

$$m_1\big(\mathscr{L}^*_s\big) \le \mu(\mathscr{L}_{u^*_\mu(t)}) \le t$$

by (2.1.14). Thus $\mu(\mathscr{L}_s) \le m_1(\mathscr{L}^*_s)$ and (2.1.15) follows.

Definition. Let $\mathscr{C}(\varrho)$ denote the infimum $\sigma(\partial g)$ for all admissible sets such that $\mu(g) \ge \varrho$, where $\sigma(\partial g)$ is the weighted area defined by (2.1.1).

Theorem. *Let* Ω *be a domain in* $\mathbb{R}^n$ *and* $0 < q < 1$.

(i) (*Sufficiency*) *If*

$$D := \int_0^{\mu(\Omega)} \left(\frac{s^{1/q}}{\mathscr{C}(s)} \right)^{\frac{q}{1-q}} \frac{\mathrm{d}s}{s} < \infty, \tag{2.1.17}$$

then (2.1.10) *holds for all* $u \in C^\infty(\Omega)$. *The constant* C *satisfies the inequality* $C \le c_1(q) D^{(1-q)/q}$.

(ii) (*Necessity*) *If there is a constant* $C > 0$ *such that* (2.1.10) *holds for all* $u \in C^\infty(\Omega)$, *then* (2.1.17) *holds and* $C \ge c_2(q) D^{(1-q)/q}$.

Proof. (Sufficiency) Note that (2.1.17) implies $\mu(\Omega) < \infty$ and that $\mathscr{C}$ is a positive function. By monotonicity of $\mu(\mathscr{L}_t)$, one obtains

$$\begin{aligned} \int_\Omega |u|^q \,\mathrm{d}\mu &= \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \mu(\mathscr{L}_t) \,\mathrm{d}\big(t^q\big) \\ &\le \sum_{j=-\infty}^{\infty} \mu_j \big(2^{q(j+1)} - 2^{qj}\big), \end{aligned}$$

where $\mu_j = \mu(\mathscr{L}_{2^j})$. We claim that the estimate

$$\sum_{j=r}^{m} \mu_j \big(2^{q(j+1)} - 2^{qj}\big) \le c D^{1-q} \big\|\Phi(\cdot, \nabla u)\big\|^q_{L_q(\Omega)} \tag{2.1.18}$$

is true for any integers r, m, and $r < m$. Once (2.1.18) has been proved, (2.1.17) follows by letting $m \to \infty$ and $r \to -\infty$ in (2.1.18). Clearly, the sum on the left in (2.1.18) is not greater than

$$\mu_m 2^{q(m+1)} + \sum_{j=1+r}^{m} (\mu_{j-1} - \mu_j) 2^{jq}. \tag{2.1.19}$$

Let $S_{r,m}$ denote the sum over $1 + r \le j \le m$. Hölder's inequality implies

$$S_{r,m} \le \left[\sum_{j=1+r}^{m} 2^j \mathscr{C}(\mu_{j-1}) \right]^q \left\{ \sum_{j=1+r}^{m} \frac{(\mu_{j-1} - \mu_j)^{1/(1-q)}}{\mathscr{C}(\mu_{j-1})^{1/(1-q)}} \right\}^{1-q}. \tag{2.1.20}$$

We have

$$(\mu_{j-1} - \mu_j)^{1/(1-q)} \le \mu_{j-1}^{1/(1-q)} - \mu_j^{1/(1-q)}.$$

Hence, by the monotonicity of $\mathscr{C}$, the sum in curly braces is dominated by

$$\sum_{j=1+r}^{m} \int_{\mu_j}^{\mu_{j-1}} \mathscr{C}(t)^{q/(q-1)} \,\mathrm{d}\big(t^{1/(1-q)}\big),$$

which does not exceed $D/(1 - q)$. By (2.1.4) the sum in square brackets in (2.1.20) is not greater than

$$2\sum_{j=-\infty}^{\infty}\int_{\mathscr{L}_{2j-1}\setminus\mathscr{L}_{2j}}\Phi(x,\nabla u)\,\mathrm{d}x.$$

Thus

$$\sum_{j=1+r}^{m}(\mu_{j-1}-\mu_j)2^{qj}\le cD^{1-q}\big\|\Phi(\cdot,\nabla u)\big\|^q_{L_1(\Omega)}.$$

To conclude the proof of (2.1.18), we show that the first term in (2.1.19) is also dominated by the right part of (2.1.18). Indeed, if $\mu_m>0$, then

$$\begin{aligned}\mu_m 2^{mq}&\le\big(2^m\mathscr{C}(\mu_m)\big)^q\Big(\big(\mu_m/\mathscr{C}(\mu_m)\big)^{q/(1-q)}\mu_m\Big)^{1-q}\\&\le c\big\|\Phi(\cdot,\nabla u)\big\|^q_{L_1(\Omega)}\bigg(\int_0^{\mu_m}\bigg(\frac{t}{\mathscr{C}(t)}\bigg)^{q/(1-q)}\mathrm{d}t\bigg)^{1-q}.\end{aligned}$$

The sufficiency of (2.1.17) follows.

We turn to the necessity of (2.1.17). We shall use the following two auxiliary assertions.

Lemma 2. *Let $u\in C_0^{0,1}(\Omega)$. There exists a sequence $\{u_\nu\}_{\nu\ge1}$ of functions $u_\nu\in\mathscr{D}(\Omega)$ such that*

$$\int_\Omega\Phi\big(x,\nabla\big(u_\nu(x)-u(x)\big)\big)\,\mathrm{d}x\to0\quad\textit{as }\nu\to\infty.\tag{2.1.21}$$

Proof. Let $u_\nu=\mathcal{M}_{\nu^{-1}}u$, where $\mathcal{M}_\varepsilon$ stands for a mollification with radius ε. Let U be a neighborhood of supp u, $\bar U\subset\Omega$.

Clearly, supp u_ν is situated in U for all sufficiently large ν. Since $\Phi\in C(\Omega\times S^{n-1})$ and $u\in C_0^{(0,1)}(\Omega)$, it follows that

$$\Phi\big(x,\nabla\big(u_\nu(x)-u(x)\big)\big)=\Phi\bigg(x,\frac{\nabla(u_\nu(x)-u(x))}{|\nabla(u_\nu(x)-u(x))|}\bigg)\big|\nabla\big(u_\nu(x)-u(x)\big)\big|,$$

if $\nabla u_\nu(x)\ne\nabla u(x)$. Therefore, the left-hand side in (2.1.21) does not exceed

$$\max_{\bar U\times S^{n-1}}\Phi\int_\Omega\big|\nabla\big(u_\nu(x)-u(x)\big)\big|\,\mathrm{d}x\to0\quad\text{as }\nu\to\infty.$$

The proof is complete. □

Lemma 3. *Let $\{v_1,\dots,v_N\}$ be a finite collection of functions in the space $C(\Omega)\cap L_p^1(\Omega)$, $p\in[1,\infty)$. Then, for $x\in\Omega$, the function*

$$x\mapsto v(x)=\max\big\{v_1(x),\dots,v_N(x)\big\}$$

belongs to the same space and

$$\big\|\Phi(\cdot, \nabla v)\big\|_{L_1(\Omega)} \le \sum_{i=1}^{N} \big\|\Phi(\cdot, \nabla v_i)\big\|_{L_1(\Omega)}. \tag{2.1.22}$$

Proof. An induction argument reduces consideration to the case $N = 2$. Here

$$v(x) = \max\big\{v_1(x), v_2(x)\big\}.$$

The left-hand side in (2.1.22) is equal to

$$\int_{v_1 \ge v_2} \Phi(x, \nabla v_1)\,\mathrm{d}x + \int_{v_1 < v_2} \Phi(x, \nabla v_2)\,\mathrm{d}x,$$

which implies (2.1.22) for $N = 2$. □

Continuation of the proof of Theorem. (Necessity) First we remark that the claim implies $\mu(\Omega) < \infty$ and that $\mathscr{C}(t) > 0$ for all $t \in (0, \mu(\Omega)]$. Let j be any integer satisfying $2^j \le \mu(\Omega)$. Then there exists a subset g_j of Ω such that

$$\mu(g_j) \ge 2^j, \quad \text{and} \quad \sigma(\Omega \cap g_j) \le 2\mathscr{C}\big(2^j\big).$$

By the definition of $\mathscr{C}$ and by (2.1.4) there is a function $u_j \in C^\infty(\Omega)$ subject to $u_j \ge 1$ on g_j, $u_j = 0$ on $\partial\Omega$ and

$$\int_\Omega \Phi(x, \nabla u_j)\,\mathrm{d}x \le 4\mathscr{C}\big(2^j\big).$$

Let s be the integer for which $2^s < \mu(\Omega) < 2^{s+1}$. For any integer $r < s$, we introduce the Lipschitz function

$$f_{r,s}(x) = \max_{r \le j \le s} \beta_j u_j(x), \quad x \in \Omega,$$

where

$$\beta_j = \big(2^j / \mathscr{C}\big(2^j\big)\big)^{1/(1-q)}.$$

By Lemmas 2 and 3

$$\big\|\Phi(\cdot, \nabla f_{r,s})\big\|_{L_1(\Omega)} \le c \sum_{j=r}^{s} \beta_j \big\|\Phi(\cdot, \nabla u_j)\big\|_{L_1(\Omega)},$$

and one obtains the following upper bound:

$$\big\|\Phi(\cdot, \nabla f_{r,s})\big\|_{L_1(\Omega)} \le c \sum_{j=r}^{s} \beta_j \mathscr{C}\big(2^j\big). \tag{2.1.23}$$

We now derive a lower bound for the norm of $f_{r,s}$ in $L_q(\Omega, \mu)$. Since $f_{r,s}(x) \ge \beta_j$ for $x \in g_j$, $r \le j \le s$, and $\mu(g_j) \ge 2^j$, the inequality

$$\mu\big(\big\{x \in \Omega : \big|f_{r,s}(x)\big| > r\big\}\big) < 2^j$$

implies $r \ge \beta_j$. Hence

$$f^*_{r,s}(t) \ge \beta_j \quad \text{for } t \in \big(0, 2^j\big),\ r \le j \le s,$$

where $f^*_{r,s}$ is the nonincreasing rearrangement of $f_{r,s}$. Then

$$\int_0^{\mu(\Omega)} \big(f^*_{r,s}(t)\big)^q \,\mathrm{d}t \ge \sum_{j=r}^{s} \int_{2^{j-1}}^{2^j} \big(f^*_{r,s}\big)^q \,\mathrm{d}t \ge \sum_{j=r}^{s} \beta_j^q 2^{j-1},$$

which implies

$$\|f_{r,s}\|^q_{L_q(\Omega,\mu)} \ge \sum_{j=r}^{s} \beta_j^q 2^{j-1}. \tag{2.1.24}$$

Next we note that by Lemma 2 if inequality (2.1.10) holds for all $u \in C_0^\infty(\Omega)$, then it holds for all Lipschitz u with compact supports in Ω. In particular,

$$\|f_{r,s}\|_{L_q(\Omega,\mu)} \le C \big\|\Phi(\cdot, \nabla f_{r,s})\big\|_{L_1(\Omega)}.$$

Now (2.1.23) and (2.1.24) in combination with the last inequality give

$$C \ge c\frac{(\sum_{j=r}^{s} \beta_j^q 2^j)^{1/q}}{\sum_{j=r}^{s} \beta_j (2^j)} = c\Bigg(\sum_{j=r}^{s} \frac{2^{j/(1-q)}}{(\mathscr{C}(2^j))^{q/(1-q)}}\Bigg)^{(1-q)/q}.$$

By letting $r \to -\infty$ and by the monotonicity of $\mathscr{C}$, we obtain

$$C \ge c\Bigg(\sum_{j=-\infty}^{s} \bigg(\frac{2^j}{\mathscr{C}(t)}\bigg)^{\frac{q}{1-q}} 2^j\Bigg)^{\frac{1-q}{q}} \ge c\Bigg(\int_0^{\mu(\Omega)} \bigg(\frac{t}{\mathscr{C}(t)}\bigg)^{\frac{q}{1-q}} \mathrm{d}t\Bigg)^{\frac{1-q}{q}}.$$

This completes the proof of the Theorem. □

2.1.5 Inequality (2.1.10) Containing Particular Measures

We give two examples that illustrate applications of the inequality (2.1.10).

Example 1. Let $\Omega = \mathbb{R}^n$, $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n, x_n = 0\}$, $\mu(A) = m_{n-1}(A \cap \mathbb{R}^{n-1})$, where A is any Borel subset of $\mathbb{R}^n$. Obviously,

$$\mu(g) \le \frac{1}{2} s(\partial g)$$

and hence

$$\|u\|_{L_1(\mathbb{R}^{n-1})} \le \frac{1}{2} \|\nabla u\|_{L_1(\mathbb{R}^n)}$$

for all $u \in \mathscr{D}(\mathbb{R}^n)$.

Example 2. Let A be any Borel subset of $\mathbb{R}^n$ with $m_n(A) < \infty$ and let

$$\mu(A) = \int_A |x|^{-\alpha}\,\mathrm{d}x,$$

where $\alpha \in [0, 1]$. Further, let B_r be a ball centered at the origin, whose n-dimensional measure equals $m_n(A)$. In other words,

$$r = \left(\frac{n}{\omega_n} m_n(A)\right)^{1/n}.$$

Obviously,

$$\int_A |x|^{-\alpha}\,\mathrm{d}x \le \int_{A\cap B_r} |x|^{-\alpha}\,\mathrm{d}x + r^{-\alpha} m_n(B_r\backslash A) \le \int_{B_r} |x|^{-\alpha}\,\mathrm{d}x.$$

So

$$\mu(A)^{(n-1)/(n-\alpha)} \le (n-\alpha)^{(1-n)/(n-\alpha)} \omega_n^{\alpha(n-1)/n(n-\alpha)} \big[n m_n(A)\big]^{(n-1)/n}.$$

Let g be any admissible set in $\mathbb{R}^n$. By virtue of the isoperimetric inequality

$$\big[n m_n(g)\big]^{(n-1)/n} \le \omega_n^{-1/n} s(\partial g),$$

we have

$$\mu(g)^{(n-1)/(n-\alpha)} \le (n-\alpha)^{(1-n)/(n-\alpha)} \omega_n^{(\alpha-1)/(n-\alpha)} s(\partial g).$$

This inequality becomes an equality if g is a ball. Therefore

$$\sup_{\{g\}} \frac{\mu(g)^{(n-1)/(n-\alpha)}}{s(\partial g)} = (n-\alpha)^{(1-n)/(n-\alpha)} \omega_n^{(\alpha-1)/(n-\alpha)}$$

and *for all* $u \in \mathscr{D}(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} \big|u(x)\big|^{(n-\alpha)/(n-1)} |x|^{-\alpha}\,\mathrm{d}x\right)^{(n-1)/(n-\alpha)} \le (n-\alpha)^{(1-n)/(n-\alpha)} \omega_n^{(\alpha-1)/(n-\alpha)} \|\nabla u\|_{L_1(\mathbb{R}^n)} \qquad (2.1.25)$$

with the best possible constant.

2.1.6 Power Weight Norm of the Gradient on the Right-Hand Side

In this subsection we denote by $z = (x, y)$ and $\zeta = (\xi, \eta)$ points in $\mathbb{R}^{n+m}$ with $x, \xi \in \mathbb{R}^n$, $y, \eta \in \mathbb{R}^m$, $m, n > 0$. Further, let $B_r^{(d)}(q)$ be the d-dimensional ball with center $q \in \mathbb{R}^d$.

Lemma 1. *Let g be an open subset of $\mathbb{R}^{n+m}$ with compact closure and smooth boundary ∂g, which satisfies*

$$\int_{B_r^{(n+m)}(z)\cap g} |\eta|^\alpha \, d\zeta \Big/ \int_{B_r^{(n+m)}(z)} |\eta|^\alpha \, d\zeta = \frac{1}{2}, \tag{2.1.26}$$

where $\alpha > -m$ for $m > 1$ and $0 \geq \alpha > -1$ for $m = 1$. Then

$$\int_{B_r^{(n+m)}(z)\cap \partial g} |\eta|^\alpha \, ds(\zeta) \geq c r^{n+m-1}\big(r + |y|\big)^\alpha, \tag{2.1.27}$$

where s is the $(n+m-1)$-dimensional area.

The proof is based on the next lemma.

Lemma 2. *Let $\alpha > -m$ for $m > 1$ and $0 \geq \alpha > -1$ for $m = 1$. Then for any $v \in C^\infty(\overline{B_r^{(n+m)}})$ there exists a constant V such that*

$$\int_{B_r^{(n+m)}} \big|v(\zeta) - V\big||\eta|^\alpha \, d\zeta \leq cr \int_{B_r^{(n+m)}} \big|\nabla v(\zeta)\big||\eta|^\alpha \, d\zeta. \tag{2.1.28}$$

Proof. It suffices to derive (2.1.28) for $r = 1$. We put $B_1^{(n+m)} = B$ and $B_1^{(m)} \times B_1^{(n)} = Q$. Let $R(\zeta)$ denote the distance of a point $\zeta \in \partial Q$ from the origin, i.e., $R(\zeta) = (1+|\zeta|^2)^{1/2}$ for $|\eta| = 1$, $|\xi| < 1$ and $R(\zeta) = (1+|\eta|^2)^{1/2}$ for $|\zeta| = 1$, $|\eta| < 1$. Taking into account that B is the quasi-isometric image of Q under the mapping $\zeta \to \zeta/R(\zeta)$, we may deduce (2.1.28) from the inequality

$$\int_Q \big|v(\zeta) - V\big||\eta|^\alpha \, d\zeta \leq c \int_Q \big|\nabla v(\zeta)\big||\eta|^\alpha \, d\zeta, \tag{2.1.29}$$

which will be established now. Since $(m+\alpha)|\eta|^\alpha = \operatorname{div}(|\eta|^\alpha \eta)$, then, after integration by parts in the left-hand side of (2.1.29), we find that it does not exceed

$$(m+\alpha)^{-\alpha}\bigg(\int_Q |\nabla v||\eta|^{\alpha+1} \, d\xi + \int_{B_1^{(n)}} d\xi \int_{\partial B_1^{(m)}} \big|v(\zeta) - V\big| \, ds(\eta)\bigg). \tag{2.1.30}$$

For the sake of brevity we put $T = B_1^{(n)} \times (B_1^{(m)} \backslash B_{1/2}^{(m)})$. Let $m > 1$. The second summand in (2.1.30) is not greater than

$$c\int_T |\nabla v| \, d\zeta + c\int_T |v - V| \, d\zeta.$$

By Lemma 1.1.11, the last assertion and (2.1.30) imply (2.1.29), where V is the mean value of v in T. (Here it is essential that T is a domain for $m > 1$.)

If $m = 1$ then T has two components $T_+ = B_1^{(n)} \times (1/2, 1)$ and $T_- = B_1^{(n)} \times (-1, -1/2)$. Using the same argument as in the case $m > 1$, we obtain

$$\int_{B_1^{(n)}} \bigl|v(\xi,\pm 1) - V_\pm\bigr|\,\mathrm{d}\xi \le c\int_{T_\pm} \bigl|\nabla v(\zeta)\bigr|\,\mathrm{d}\zeta \le c\int_Q \bigl|\nabla v(\zeta)\bigr||\eta|^\alpha\,\mathrm{d}\zeta,$$

where $V_\pm$ are the mean values of v in $T_\pm$. It remains to note that

$$|V_+ - V_-| \le c\int_{B_1^{(n)}} \mathrm{d}\xi \int_{-1}^{1} \left|\frac{\partial v}{\partial \eta}\right| \mathrm{d}\eta \le c\int_Q \bigl|\nabla v(\zeta)\bigr||\eta|^\alpha\,\mathrm{d}\zeta,$$

provided $\alpha \le 0$. So for $m = 1$ we also have (2.1.29) with V replaced by V_+ or V_-. This concludes the proof of the lemma. □

Proof of Lemma 1. For the sake of brevity let $B = B_r^{(n+m)}(z)$. In (2.1.28) we replace v by a mollification of the characteristic function χ_ϱ of the set g with radius ϱ. Then the left-hand side in (2.1.28) is bounded from below by the sum

$$|1 - V|\int_{e_1} |\eta|^\alpha\,\mathrm{d}\zeta + |V|\int_{e_0} |\eta|^\alpha\,\mathrm{d}\zeta,$$

where $e_i = \{\zeta \in B : \chi_\varrho(\zeta) = i\}$, $i = 0, 1$.

Let ε be a sufficiently small positive number. By (2.1.26)

$$\left(\frac{1}{2} - \varepsilon\right)\bigl(|1 - V| + |V|\bigr)\int_B |\eta|^\alpha\,\mathrm{d}\zeta \le cr\int_B |\eta|^\alpha\bigl|\nabla\chi_\varrho(\zeta)\bigr|\,\mathrm{d}\zeta$$

for sufficiently small values of ϱ. Consequently,

$$\frac{1}{2}\int_B |\eta|^\alpha\,\mathrm{d}\zeta \le cr\limsup_{\varrho\to+0}\int_B |\eta|^\alpha\bigl|\nabla\chi_\varrho(\zeta)\bigr|\,\mathrm{d}\zeta = cr\int_{B\cap\partial g} |\eta|^\alpha\,\mathrm{d}s(\zeta).$$

It remains to note that

$$\int_B |\eta|^\alpha\,\mathrm{d}\zeta \ge cr^{m+n}\bigl(r + |y|\bigr)^\alpha.$$

The lemma is proved. □

Remark 1. Lemma 1 fails for $m = 1$, $\alpha > 0$. In fact, let $g = \{\zeta \in \mathbb{R}^{n+1} : \eta > \varepsilon \text{ or } 0 > \eta > -\varepsilon\}$, where $\varepsilon = \text{const} > 0$. Obviously, (2.1.26) holds for this g. However,

$$\int_{B_r^{(n+1)}\cap\partial g} |\eta|^\alpha\,\mathrm{d}s(\zeta) \le c\varepsilon^\alpha,$$

which contradicts (2.1.27).

Theorem 1. *Let ν be a measure in $\mathbb{R}^{n+m}$, $q \ge 1$, $\alpha > -m$. The best constant in*

$$\|u\|_{L_q(\mathbb{R}^{n+m},\nu)} \le C\int_{\mathbb{R}^{n+m}} |y|^\alpha|\nabla_z u|\,\mathrm{d}z, \quad u \in C_0^\infty\bigl(\mathbb{R}^{n+m}\bigr), \tag{2.1.31}$$

is equivalent to

$$K = \sup_{z;\varrho} \big(\varrho + |y|\big)^{-\alpha} \varrho^{1-n-m} \big[\nu\big(B_\varrho^{(n+m)}(z)\big)\big]^{1/q}. \tag{2.1.32}$$

Proof. 1. First, let $m > 1$ or $0 \geq \alpha > -1$, $m = 1$. According to Theorem 2.1.3

$$C = \sup_g \frac{[\nu(g)]^{1/q}}{\int_{\partial g} |y|^\alpha \, \mathrm{d}s(z)},$$

where g is an arbitrary subset of $\mathbb{R}^{n+m}$ with a compact closure and smooth boundary. We show that for each g there exists a covering by a sequence of balls $B_{\varrho_i}^{n+m}(z_i)$, $i = 1, 2, \dots$, such that

$$\sum_i \varrho_i^{n+m-1} \big(\varrho_i + |y_i|\big)^\alpha \leq c \int_{\partial g} |y|^\alpha \, \mathrm{d}s(z).$$

Each point $z \in g$ is the center of a ball $B_r^{(n+m)}(z)$ for which (2.1.26) is valid. In fact, the ratio in the left-hand side of (2.1.26) is a continuous function in r that equals unity for small values of r and converges to zero as $r \to \infty$. By Theorem 1.2.1 there exists a sequence of disjoint balls $B_{r_i}^{(n+m)}(z_i)$ such that

$$g \subset \bigcup_{i=1}^{\infty} B_{3r_i}^{(n+m)}(z_i).$$

According to Lemma 1,

$$\int_{B_{r_i}^{(n+m)}(z_i) \cap \partial g} |y|^\alpha \, \mathrm{d}s(z) \geq c r_i^{n+m-1} \big(r_i + |y_i|\big)^\alpha.$$

Thus $\{B_{3r_i}^{(n+m)}(z_i)\}_{i \geq 1}$ is the required covering.

Obviously,

$$\begin{aligned} \nu(g) &\leq \sum_i \nu\big(B_{3r_i}^{(n+m)}(z_i)\big) \leq \Big(\sum_i \big[\nu\big(B_{3r_i}^{(n+m)}(z_i)\big)\big]^{1/q}\Big)^q \\ &\leq cK^q \Big(\sum_i r_i^{n+m-1} \big(r_i + |y_i|\big)^\alpha\Big)^q \leq \Big(cK \int_{\partial g} |y|^\alpha \, \mathrm{d}s(z)\Big)^q. \end{aligned}$$

Therefore $C \leq cK$ for $m > 1$ and for $m = 1$, $0 \geq \alpha > -1$.

2. Now let $m = 1$, $\alpha > 0$. We construct a covering of the set $\{\zeta : \eta = 0\}$ by balls $\mathscr{B}_j$ with radii ϱ_j, equal to the distance of $\mathscr{B}_j$ from the hyperplane $\{\zeta : \xi = 0\}$. We assume that this covering has finite multiplicity. By $\{\varphi_j\}$ we denote a partition of unity subordinate to $\{\mathscr{B}_j\}$ and such that $|\nabla \varphi_j| \leq c/\varrho_j$ (see Stein [724], Chap. VI, §1.). Using the present theorem for the case $\alpha = 0$, which has already been considered (or equivalently, using Theorem 1.4.2/2), we arrive at

$$\|\varphi_j u\|_{L_q(\mathbb{R}^{n+1},\nu)} \le c \sup_{\varrho;z} \varrho^{-n} \big[\nu_j\big(B_\varrho^{(n+1)}(z)\big)\big]^{1/q} \big\|\nabla(\varphi_j u)\big\|_{L_1(\mathbb{R}^{n+1})},$$

where ν_j is the restriction of ν to $\mathscr{B}_j$. It is clear that

$$\sup_{\varrho;z} \varrho^{-n} \big[\nu_j\big(B_\varrho^{(n+1)}(z)\big)\big]^{1/q} \le c \sup_{\varrho\le\varrho_j, z\in\mathscr{B}_j} \varrho^{-n} \big[\nu\big(B_\varrho^{(n+1)}(z)\big)\big]^{1/q}.$$

Therefore,

$$\|\varphi_j u\|_{L_q(\mathbb{R}^{n+1},\nu)} \le c \sup_{\varrho\le\varrho_j, z\in\mathscr{B}_j} (\varrho+\varrho_j)^{-\alpha} \varrho^{-n} \big[\nu\big(B_\varrho^{(n+1)}(z)\big)\big]^{1/q} \int_{\mathbb{R}^{n+1}} \big|\nabla(\varphi_j u)\big| |\eta|^\alpha \,\mathrm{d}\zeta.$$

Summing over j and using (2.1.29), we obtain

$$\|u\|_{L_q(\mathbb{R}^{n+1},\nu)} \le cK \bigg(\int_{\mathbb{R}^{n+1}} |\nabla u| |\eta|^\alpha \,\mathrm{d}\zeta + \int_{\mathbb{R}^{n+1}} |u| |\eta|^{\alpha-1} \,\mathrm{d}\zeta \bigg).$$

Since

$$\int_{\mathbb{R}^{n+1}} |u| |\eta|^{\alpha-1} \,\mathrm{d}\zeta \le \alpha^{-1} \int_{\mathbb{R}^{n+1}} |\nabla u| |\eta|^\alpha \,\mathrm{d}\zeta$$

for $\alpha > 0$, then $C \le cK$ for $m = 1$, $\alpha > 0$.

3. To prove the reverse estimate, in (2.1.31) we put $U(\zeta) = \varphi(\varrho^{-1}(\zeta - z))$, where $\varphi \in C_0^\infty(B_2^{(n+m)})$, $\varphi = 1$ on $B_1^{(n+m)}$. Since

$$\int_{B_{2\varrho}^{(n+m)}(z)} |\eta|^\alpha |\nabla_\zeta u| \,\mathrm{d}\zeta \le c\varrho^{-1} \int_{B_{2\varrho}^{(n+m)}(z)} |\eta|^\alpha \,\mathrm{d}\zeta \le c\varrho^{n+m-1} \big(\varrho + |y|\big)^\alpha,$$

the result follows. □

Corollary. *Let ν be a measure in $\mathbb{R}^n$, $q \ge 1$, $\alpha > -m$. Then the best constant in (2.1.31) is equivalent to*

$$\sup_{x\in\mathbb{R}^n, \varrho>0} \varrho^{1-m-n-\alpha} \big[\nu\big(B_\varrho^{(n)}(x)\big)\big]^{1/q}.$$

For the proof it suffices to note that K, defined in (2.1.32), is equivalent to the preceding supremum if supp $\nu \subset \mathbb{R}^n$.

Remark 2. The part of the proof of Theorem 1 for the case $m = 1$, $\alpha > 0$ is also suitable for $m > 1$, $\alpha > 1 - m$ since for these values of α and for all $u \in C_0^\infty(\mathbb{R}^{n+m})$ we have

$$\int_{\mathbb{R}^{n+m}} |u| |\eta|^{\alpha-1} \,\mathrm{d}\zeta \le (\alpha + m - 1)^{-1} \int_{\mathbb{R}^{n+m}} |\nabla u| |\eta|^\alpha \,\mathrm{d}\zeta. \tag{2.1.33}$$

This implies that *the best constant C in (2.1.31) is equivalent to*

$$K_1 = \sup_{z\in\mathbb{R}^{n+m};\varrho<|y|/2} |y|^{-\alpha}\varrho^{1-n-m}\big[\nu\big(B_\varrho^{(n+m)}(z)\big)\big]^{1/q}$$

for $m \geq 1$, $\alpha > 1-m$.

Since (2.1.33) is also valid for $\alpha < 1-m$ with the coefficient $(1-m-\alpha)^{-1}$ if u vanishes near the subspace $\eta = 0$, then following the arguments of the second and third parts of the proof of Theorem 1 with obvious changes, we arrive at the next theorem.

Theorem 2. *Let* ν *be a measure in* $\{\zeta \in \mathbb{R}^{n+m} : \eta \neq 0\}$, $q \geq 1$, $\alpha < 1-m$. *Then the best constant in* (2.1.31), *where* $u \in C_0^\infty(\{\zeta : \eta \neq 0\})$, *is equivalent to* K_1.

2.1.7 Inequalities of Hardy–Sobolev Type as Corollaries of Theorem 2.1.6/1

Here we derive certain inequalities for weighted norms which often occur in applications. Particular cases of them are the Hardy inequality

$$\big\||x|^{-l}u\big\|_{L_p(\mathbb{R}^n)} \leq c\|\nabla_l u\|_{L_p(\mathbb{R}^n)}$$

and the Sobolev inequality

$$\|u\|_{L_{pn/(n-lp)}(\mathbb{R}^n)} \leq c\|\nabla_l u\|_{L_p(\mathbb{R}^n)},$$

where $lp < n$ and $u \in \mathscr{D}(\mathbb{R}^n)$. We retain the notation introduced in Sect. 2.1.5.

Corollary 1. *Let*

$$1 \leq q \leq (m+n)/(m+n-1), \qquad \beta = \alpha - 1 + \frac{q-1}{q}(m+n) > -\frac{m}{q}.$$

Then

$$\big\||y|^\beta u\big\|_{L_q(\mathbb{R}^{n+m})} \leq c\big\||y|^\alpha \nabla u\big\|_{L_1(\mathbb{R}^{n+m})} \tag{2.1.34}$$

for $u \in \mathscr{D}(\mathbb{R}^{n+m})$.

Proof. According to Theorem 2.1.6/1 it suffices to establish the uniform boundedness of the value

$$\big(\varrho+|y|\big)^{-\alpha}\varrho^{1-n-m}\bigg(\int_{|z-\zeta|<\varrho} |\eta|^{\beta q}\,\mathrm{d}\zeta\bigg)^{1/q}$$

with respect to ϱ and z. Obviously, it does not exceed

$$c\big(\varrho+|y|\big)^{-\alpha}\varrho^{1-n-m+n/q}\bigg(\int_{|\eta-y|<\varrho} |\eta|^{\beta q}\,\mathrm{d}\eta\bigg)^{1/q}.$$

This value is not greater than $c|y|^{\beta-\alpha}\varrho^{1-(m+n)(q-1)/q}$ for $\varrho \le c|y|$ and $c\varrho^{\beta-\alpha+1-(m+n)(q-1)/q}$ for $\varrho > c|y|$. The result follows. □

In (2.1.34) let us replace q^{-1}, α, and β by $1-p^{-1}+q^{-1}$, $\alpha+(1-p)^{-1}q\beta$, and$((1-p^{-1})q+1)\beta$, respectively, and u by $|u|^s$ with $s=(p-1)qp^{-1}+1$. Then applying Hölder's inequality with exponents p and $p/(p-1)$ to its right-hand side we obtain the following assertion.

Corollary 2. *Let $m+n>p\ge 1$, $p\le q\le p(n+m)(n+m-p)^{-1}$, and $\beta=\alpha-1+(n+m)(1/p-1/q)>-m/q$. Then*

$$\big\||y|^{\beta}u\big\|_{L_q(\mathbb{R}^{n+m})} \le c\big\||y|^{\alpha}\nabla u\big\|_{L_p(\mathbb{R}^{n+m})} \tag{2.1.35}$$

for all $u\in\mathscr{D}(\mathbb{R}^{n+m})$.

For $p=2$, $\alpha=1-m/2$, $n>0$, the substitution of $u(z)=|y|^{-\alpha}v(z)$ into (2.1.35) leads to the next corollary.

Corollary 3. *Let $m+n>2$, $2<q\le 2(n+m)/(n+m-2)$, and $\gamma=-1+(n+m)(2^{-1}-q^{-1})$. Then*

$$\big\||y|^{\gamma}v\big\|^2_{L_q(\mathbb{R}^{n+m})} \le c\bigg(\int_{\mathbb{R}^{n+m}}(\nabla v)^2\,\mathrm{d}z-\frac{(m-2)^2}{4}\int_{\mathbb{R}^{n+m}}\frac{v^2}{|y|^2}\,\mathrm{d}z\bigg) \tag{2.1.36}$$

for all $v\in\mathscr{D}(\mathbb{R}^{n+m})$, subject to the condition $v(x,0)=0$ in the case $m=1$.

In particular, the exponent γ vanishes for $q=2(m+n)/(m+n-2)$ and we obtain

$$c\|v\|^2_{L_{\frac{2(m+n)}{m+n-2}}(\mathbb{R}^{n+m})}+\frac{(m-2)^2}{4}\int_{\mathbb{R}^{n+m}}\frac{v^2}{|y|^2}\,\mathrm{d}z\le\int_{\mathbb{R}^{n+m}}(\nabla v)^2\,\mathrm{d}z, \tag{2.1.37}$$

which is a refinement of both the Sobolev and the Hardy inequalities, the latter having the best constant.

To conclude this subsection we present a generalization of (2.1.35) for derivatives of arbitrary integer order l.

Corollary 4. *Let $m+n>lp$, $1\le p\le q\le p(m+n-lp)^{-1}(m+n)$, and $\beta=\alpha-l+(m+n)(p^{-1}-q^{-1})>-mq^{-1}$. Then*

$$\big\||y|^{\beta}u|\big\|_{L_q(\mathbb{R}^{n+m})} \le c\big\||y|^{\alpha}\nabla_l u\big\|_{L_p(\mathbb{R}^{n+m})} \tag{2.1.38}$$

for $u\in\mathscr{D}(\mathbb{R}^{n+m})$.

Proof. Let $p_j=p(n+m)(n+m-p(l-j))^{-1}$. Successively applying the inequalities

$$\big\||y|^{\beta}u\big\|_{L_q(\mathbb{R}^{n+m})}\le c\big\||y|^{\alpha}\nabla u\big\|_{L_{p_1}(\mathbb{R}^{n+m})},$$

$$\big\||y|^{\alpha}\nabla_j u\big\|_{L_{p_j}(\mathbb{R}^{n+m})}\le c\big\||y|^{\alpha}\nabla_{j+1}u\big\|_{L_{p_{j+1}}(\mathbb{R}^{n+m})},\quad 1\le j<l,$$

which follow from (2.1.35), we arrive at (2.1.38). □

Inequality (2.1.38) and its particular cases (2.1.34) and (2.1.35) obviously fail for $\alpha = l+nq^{-1}-(m+n)p^{-1}$. Nevertheless for this critical α we can obtain similar inequalities that are also invariant under similarity transformations in $\mathbb{R}^{n+m}$ by changing the weight function on the left-hand side.

2.1.8 Comments to Sect. 2.1

The results of Sects. 2.1.1–2.1.3 and 2.1.5 are borrowed from the author's paper [543] (see also [552]).

Properties of the weighted area minimizing function $\mathscr{C}$ introduced in Definition 2.1.4 were studied under the assumption that $\Phi(x,\xi)$ does not depend on x and is convex. In particular, the sharp generalized isoperimetric inequality

$$\int_{\partial g} \Phi\big(\mathscr{N}(x)\big)\,\mathrm{d}s(x) \geq n\varkappa_n^{1/n} m_n(g) \tag{2.1.39}$$

holds for all admissible sets $g \subset \mathbb{R}^n$. Here $\varkappa_n$ is the volume of the set $\{\xi \in \mathbb{R}^n : \Psi(\xi) \leq 1\}$ with

$$\Psi(\xi) = \sup_{x\neq 0} \frac{(x,\xi)_{\mathbb{R}^n}}{\Phi(x)}$$

(see Busemann [158] and Burago, Zalgaller [151]). The surfaces minimizing the integrals of the form

$$\int_{\partial g} \Phi\big(\mathscr{N}(x)\big)\,\mathrm{d}s(x)$$

over all sets g with a fixed volume, called Wulff shapes, appeared in 1901 (see Wulff [798]). The Wulff shape is called the crystal of the function Φ, which in its turn is called crystalline if its crystal is polyhedral (see, in particular, J.E. Taylor [745] for a theory of crystalline integrands as well as the bibliography).

The sharp constant

$$C_\alpha = \left(\frac{\alpha+1}{\alpha+2}\right)^{\frac{\alpha+1}{\alpha+2}} \left(2\int_0^\pi (\sin t)^\alpha\,\mathrm{d}t\right)^{-\frac{1}{\alpha+2}}, \quad \alpha \geq 0,$$

in the weighted isoperimetric inequality

$$m_2(g)^{\frac{\alpha+1}{\alpha+2}} \leq C_\alpha \int_{\partial g} \big(\mathscr{N}_1^2 + |x|^{2\alpha}\mathscr{N}_2^2\big)^{1/2}\,\mathrm{d}s,$$

where $(\mathscr{N}_1, \mathscr{N}_2) = \mathscr{N}$ was found by Monti and Morbidelli [611], which is equivalent to the sharp integral inequality

$$\|u\|_{L_{\frac{\alpha+2}{\alpha+1}}(\mathbb{R}^2)} \leq C_\alpha \int_{\mathbb{R}^2} \left(\left(\frac{\partial u}{\partial x}\right)^2 + |x|^{2\alpha}\left(\frac{\partial u}{\partial y}\right)^2\right)^{1/2} \mathrm{d}x$$

for all $u \in C_0^\infty(\mathbb{R}^2)$.

Theorem 2.1.4 is borrowed from Maz'ya [560], its proof being a modification of that in Maz'ya and Netrusov [572] relating to the case $p > 1$. For the contents of Sect. 2.1.6 see Sect. 2.1.5 in the author's book [556].

Obviously, in Theorem 2.1.6/1 the role of $|y|$ can be played by the distance to the m-dimensional Lipschitz manifold F supporting the measure ν. Horiuchi [383] proved the sufficiency in Theorem 2.1.6/1 for an absolutely continuous measure ν and for a more general class of sets F depending on the behavior as $\varepsilon \to 0$ of the n-dimensional Lebesgue measure of the tubular neighborhood of F, $\{z \in \mathbb{R}^{n+m} : \operatorname{dist}(z, F) < \varepsilon\}$.

The contents of Sect. 2.1.7 were published in [556], Sect. 2.1.6, for the first time. Estimates similar to (2.1.38) are generally well known (except, probably, for certain values of the parameters p, q, l, and α) but they were established by other methods (see Il'in [395]). The multiplicative inequality

$$\big\| |x|^\gamma u \big\|_{L_r(\mathbb{R}^n)} \le C \big\| |x|^\alpha |\nabla u| \big\|^a_{L_p(\mathbb{R}^n)} \big\| |x|^\beta u \big\|^{1-a}_{L_q(\mathbb{R}^n)}$$

was studied in detail by Caffarelli, Kohn, and Nirenberg [162]. Lin [498] has generalized their results to include derivatives of any order.

The inequality (2.1.36) was proved by Maz'ya [556], Sect. 2.1.6. Tertikas and Tintarev [749] (see also Tintarev and Fieseler [753], Sect. 5.6, as well as Benguria, Frank, and Loss [83]) studied the existence and nonexistence of optimizers in (2.1.37) and found sharp constants in some cases. In one particular instance of (2.1.36), the sharp value of c will be given in Sect. 2.7.1. In [277], Filippas, Maz'ya, and Tertikas showed that for any convex domains $\Omega \subset \mathbb{R}^n$ the inequality

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{4} \int \frac{u^2}{d^2} \,\mathrm{d}x \ge c(\Omega) \Big(\int_\Omega |u|^{\frac{2n}{n-2}} \,\mathrm{d}x \Big)^{\frac{n-2}{n}}$$

holds where $u \in C_0^\infty(\Omega)$ and $d = \operatorname{dist}(x, \partial\Omega)$. See Comments to Sect. 2.7 for other contributions to this area.

2.2 (p, Φ)-Capacity

2.2.1 Definition and Properties of the (p, Φ)-Capacity

Let e be a compactum in $\Omega \subset \mathbb{R}^n$ and let Φ be the function specified in Sect. 2.1.1. The number

$$\inf \Big\{ \int_\Omega \big[\Phi(x, \nabla u)\big]^p \,\mathrm{d}x : u \in \mathfrak{N}(e, \Omega) \Big\},$$

where $p \ge 1$, is called *the (p, Φ)-capacity of e relative to Ω* and is denoted by (p, Φ)-cap(e, Ω). Here

$$\mathfrak{N}(e,\Omega) = \{u \in \mathscr{D}(\Omega) : u \geq 1 \text{ on } e\}.$$

If $\Omega = \mathbb{R}^n$, we omit Ω in the notations (p,Ω)-cap(e,Ω), $\mathfrak{N}(e,\Omega)$, and so on.

In the case $\Phi(x,\xi) = |\xi|$, we shall speak of *the p-capacity of a compactum e relative to Ω* and we shall use the notation $\mathrm{cap}_p(e,\Omega)$.

We present several properties of the (p,Φ)-capacity.

(i) *For compact sets $K \subset \Omega$, $F \subset \Omega$, the inclusion $K \subset F$ implies*

$$(p,\Phi)\text{-cap}(K,\Omega) \leq (p,\Phi)\text{-cap}(F,\Omega).$$

This is an obvious consequence of the definition of capacity. From the same definition it follows that the (p,Φ)-capacity of F relative to Ω does not increase under extension of Ω.

(ii) *The equality*

$$(p,\Phi)\text{-cap}(e,\Omega) = \inf\left\{\int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x : u \in \mathfrak{P}(e,\Omega)\right\}, \tag{2.2.1}$$

where $\mathfrak{P}(e,\Omega) = \{u : u \in \mathscr{D}(\Omega), u = 1$ *in a neighborhood of* $e, 0 \leq u \leq 1$ *in* $\mathbb{R}^n\}$ *is valid.*

Proof. Since $\mathfrak{N}(e,\Omega) \subset \mathfrak{P}(e,\Omega)$ it is sufficient to estimate (p,Φ)-cap(e,Ω) from below. Let $\varepsilon \in (0,1)$ and let $f \in \mathfrak{N}(e,\Omega)$ be such that

$$\int_\Omega \big[\Phi(x,\nabla f)\big]^p \,\mathrm{d}x \leq (p,\Phi)\text{-cap}(e,\Omega) + \varepsilon.$$

Let $\{\lambda_m(t)\}_{m\geq 1}$ denote a sequence of functions in $C^\infty(\mathbb{R}^1)$ satisfying the conditions $0 \leq \lambda'_m(t) \leq 1 + m^{-1}$, $\lambda_m(t) = 0$ in a neighborhood of $(-\infty, 0]$ and $\lambda_m(t) = 1$ in a neighborhood of $[1,\infty)$, $0 \leq \lambda_m(t) \leq 1$ for all t. Since $\lambda_m(f(x)) \in \mathfrak{P}(e,\Omega)$, then

$$\inf\left\{\int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x : u \in \mathfrak{P}(e,\Omega)\right\} \leq \int_\Omega \big[\lambda'_m\big(f(x)\big)\big]^p \big[\Phi\big(x,\nabla f(x)\big)\big]^p \,\mathrm{d}x.$$

Passing to the limit as $m \to \infty$, we obtain

$$\inf\left\{\int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x : u \in \mathfrak{P}(e,\Omega)\right\}$$
$$\leq \int_\Omega \big[\Phi(x,\nabla f)\big]^p \,\mathrm{d}x \leq (p,\Phi)\text{-cap}(e,\Omega) + \varepsilon.$$

□

(iii) *For any compactum $e \subset \Omega$ and $\varepsilon > 0$ there exists a neighborhood G such that*

$$(p,\Phi)\text{-cap}(K,\Omega) \leq (p,\Phi)\text{-cap}(e,\Omega) + \varepsilon$$

for all compact sets K, $e \subset K \subset G$.

Proof. From (2.2.1) it follows that there exists a $u \in \mathfrak{P}(e,\Omega)$ such that

$$\int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x \le (p,\Phi)\text{-cap}(e,\Omega) + \varepsilon.$$

Let G denote a neighborhood of e in which $u = 1$. It remains to note that

$$(p,\Phi)\text{-cap}(K,\Omega) \le \int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x$$

for any compactum K, $e \subset K \subset G$. □

The next property is proved analogously.

(iv) *For any compactum $e \subset \Omega$ and any $\varepsilon > 0$ there exists an open set ω, $\bar\omega \subset \Omega$, such that*

$$(p,\Phi)\text{-cap}(e,\omega) \le (p,\Phi)\text{-cap}(e,\Omega) + \varepsilon.$$

(v) *The Choquet inequality*

$$\begin{aligned}(p,\Phi)&\text{-cap}(K\cup F,\Omega) + (p,\Phi)\text{-cap}(K\cap F,\Omega)\\ &\le (p,\Phi)\text{-cap}(K,\Omega) + (p,\Phi)\text{-cap}(F,\Omega)\end{aligned}$$

holds for any compact sets K, $F \subset \Omega$.

Proof. Let u and v be arbitrary functions in $\mathfrak{P}(K,\Omega)$ and $\mathfrak{P}(F,\Omega)$, respectively. We put $\varphi = \max(u,v)$, $\psi = \min(u,v)$. Obviously, φ and ψ have compact supports and satisfy the Lipschitz condition in Ω, $\varphi = 1$ in the neighborhood of $K \cup F$ and $\psi = 1$ in a neighborhood of $K \cap F$. Since the set $\{x : u(x) \ne v(x)\}$ is the union of open sets on which either $u > v$ or $u < v$, and since $\nabla u(x) = \nabla v(x)$ almost everywhere on $\{x : u(x) = v(x)\}$, then

$$\begin{aligned}\int_\Omega &\big[\Phi(x,\nabla\varphi)\big]^p \,\mathrm{d}x + \int_\Omega \big[\Phi(x,\nabla\psi)\big]^p \,\mathrm{d}x\\ &= \int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x + \int_\Omega \big[\Phi(x,\nabla v)\big]^p \,\mathrm{d}x.\end{aligned}$$

Hence, having noted that mollifications of the functions φ and ψ belong to $\mathfrak{P}(K \cup F,\Omega)$ and $\mathfrak{P}(K \cap F,\Omega)$, respectively, we obtain the required inequality. □

A function of compact sets that satisfies conditions (i), (iii), and (v) is called a *Choquet capacity*.

Let E be an arbitrary subset of Ω. The number $(p,\Phi)\text{-cap}(E,\Omega) = \sup_{\{K\}}(p,\Phi)\text{-cap}(K,\Omega)$, where $\{K\}$ is a collection of compact sets contained in E, is called the (p,Φ) *capacity of E relative to Ω*. The number

$$\inf_{\{G\}}(p,\Phi)\text{-cap}(G,\Omega),$$

where $\{G\}$ is the collection of all open subsets of Ω containing E, is called the *outer capacity* (p,Φ)-$\overline{\text{cap}}(E,\Omega)$ of $E \subset \Omega$. A set E is called (p,Φ) *capacitable* if

$$(p,\Phi)\text{-cap}(E,\Omega) = (p,\Phi)\text{-}\overline{\text{cap}}(E,\Omega).$$

From these definitions it follows that any open subset of Ω is (p,Φ) capacitable. If e is a compactum in Ω, then by property (iii), given $\varepsilon > 0$, there exists an open set G such that

$$(p,\Phi)\text{-cap}(G,\Omega) \le (p,\Phi)\text{-cap}(e,\Omega) + \varepsilon.$$

Consequently, all compact subsets of Ω are (p,Φ) capacitable.

From the general theory of Choquet capacities it follows that analytic sets, and in particular, Borel sets are (p,Φ) capacitable (see Choquet [186]).

2.2.2 Expression for the (p,Φ)-Capacity Containing an Integral over Level Surfaces

Lemma 1. *For any compactum $F \subset \Omega$ the (p,Φ)-capacity (for $p > 1$) can be defined by*

$$(p,\Phi)\text{-cap}(F,\Omega) = \inf_{u \in \mathfrak{N}(F,\Omega)} \left\{ \int_0^1 \frac{\mathrm{d}\tau}{\left(\int_{\mathscr{E}_\tau} [\Phi(x,\nabla u)]^p \frac{\mathrm{d}s}{|\nabla u|}\right)^{1/(p-1)}} \right\}^{1-p}, \quad (2.2.2)$$

where $\mathscr{E}_t = \{x : |u(x)| = t\}$.

We introduce the following notation: Λ is the set of nondecreasing functions $\lambda \in C^\infty(\mathbb{R}^1)$, which satisfy the conditions $\lambda(t) = 0$ for $t \le 0$, $\lambda(t) = 1$ for $t \ge 1$, supp $\lambda' \subset (0,1)$; Λ_1 is the set of nondecreasing functions that are absolutely continuous on $\mathbb{R}^1$ and satisfy the conditions $\lambda(t) = 0$ for $t \le 0$, $\lambda(t) = 1$ for $t \ge 1$, $\lambda'(t)$ is bounded.

To prove Lemma 1 we shall use the following auxiliary assertion.

Lemma 2. *Let g be a nonnegative function that is integrable on $[0,1]$. Then*

$$\inf_{\lambda \in \Lambda} \int_0^1 (\lambda')^p g \,\mathrm{d}t = \left(\int_0^1 \frac{\mathrm{d}t}{g^{1/(p-1)}} \right)^{1-p}. \quad (2.2.3)$$

Proof. First we note that by Hölder's inequality

$$1 = \int_0^1 \lambda' \,\mathrm{d}t \le \left(\int_0^1 (\lambda')^p g \,\mathrm{d}t \right)^{1/p} \left(\int_0^1 \frac{\mathrm{d}t}{g^{1/(p-1)}} \right)^{1-1/p},$$

and hence the left-hand side of (2.2.3) is not smaller than the right.

Let $\lambda \in \Lambda_1$, $\zeta_\nu(t) = \lambda'(t)$ for $t \in [\nu^{-1}, 1-\nu^{-1}]$, supp $\zeta_\nu \subset [\nu^{-1}, 1-\nu^{-1}]$, $\nu = 1, 2, \dots$. We set

$$\eta_\nu(t) = \zeta_\nu(t)\biggl(\int_0^1 \zeta_\nu \, \mathrm{d}\tau\biggr)^{-1}.$$

Since the sequence η_ν converges to λ' on (0,1) and is bounded, it follows by Lebesgue's theorem that

$$\int_0^1 \eta_\nu^p g \, \mathrm{d}\tau \to \int_0^1 (\lambda')^p g \, \mathrm{d}\tau.$$

Mollifying η_ν, we obtain the sequence $\{\gamma_\nu\}$, $\gamma_\nu \in C^\infty(\mathbb{R}^1)$, $\operatorname{supp} \gamma_\nu \subset (0,1)$,

$$\int_0^1 \gamma_\nu \, \mathrm{d}\tau = 1, \qquad \int_0^1 \gamma_\nu^p g \, \mathrm{d}\tau \to \int_0^1 (\lambda')^p g \, \mathrm{d}\tau.$$

Setting

$$\lambda_\nu(t) = \int_0^t \gamma_\nu \, \mathrm{d}\tau,$$

we obtain a sequence of functions in Λ such that

$$\int_0^1 (\lambda_\nu')^p g \, \mathrm{d}\tau \to \int_0^1 (\lambda')^p g \, \mathrm{d}\tau.$$

Hence,

$$\inf_{\Lambda} \int_0^1 (\lambda')^p g \, \mathrm{d}\tau = \inf_{\Lambda_1} \int_0^1 (\lambda')^p g \, \mathrm{d}\tau. \tag{2.2.4}$$

Let

$$M_\varepsilon = \bigl\{t : g(t) \geq \varepsilon\bigr\}, \qquad \lambda_0(t) = \int_0^t \eta \, \mathrm{d}\tau,$$

where $\eta(t) = 0$ on $\mathbb{R}^1 \backslash M_\varepsilon$ and

$$\eta(t) = g(t)^{1/(1-p)}\biggl(\int_{M_\varepsilon} g^{1/(1-p)} \, \mathrm{d}\tau\biggr)^{-1} \quad \text{for } t \in M_\varepsilon.$$

Obviously, $\lambda_0 \in \Lambda_1$, and

$$\int_0^1 (\lambda_0')^p g \, \mathrm{d}\tau = \biggl(\int_{M_\varepsilon} g^{1/(1-p)} \, \mathrm{d}\tau\biggr)^{1-p}.$$

By (2.2.4) the left-hand side of (2.2.3) does not exceed

$$\biggl(\int_{M_\varepsilon} g^{1/(1-p)} \, \mathrm{d}\tau\biggr)^{1-p}.$$

We complete the proof by passing to the limit as $\varepsilon \to 0$. □

Proof of Lemma 1. Let $u \in \mathfrak{N}(F, \Omega)$, $\lambda \in \Lambda$. From the definition of capacity and Theorem 1.2.4 we obtain

$$(p,\Phi)\text{-cap}(F,\Omega) = \int_\Omega \big[\lambda'(u)\Phi(x,\nabla u)\big]^p \,\mathrm{d}x = \int_0^1 (\lambda')^p g \,\mathrm{d}t,$$

where

$$g(t) = \int_{\mathscr{E}_t} \big[\Phi(x,\nabla u)\big]^p \frac{\mathrm{d}s}{|\nabla u|}. \tag{2.2.5}$$

By Lemma 2

$$(p,\Phi)\text{-cap}(F,\Omega) \le \left(\int_0^1 g^{1/(1-p)}\,\mathrm{d}\tau\right)^{1-p}.$$

To prove the opposite inequality it is enough to note that

$$\int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x \ge \int_0^1 g\,\mathrm{d}\tau \ge \left(\int_0^1 g^{1/(1-p)}\,\mathrm{d}\tau\right)^{1-p}.$$

The lemma is proved. □

Recalling the property (2.2.1) of the (p,Φ)-capacity, note that, in passing, we have proved here also the following lemma.

Lemma 3. *For any compactum $F \subset \Omega$ the (p,Φ)-capacity $(p > 1)$ can be defined as*

$$(p,\Phi)\text{-cap}(F,\Omega) = \inf_{u\in\mathfrak{P}(F,\Omega)} \left\{ \int_0^1 \frac{\mathrm{d}t}{\left(\int_{\mathscr{E}_t} [\Phi(x,\nabla u)]^p \frac{\mathrm{d}s}{|\nabla u|}\right)^{1/(p-1)}} \right\}^{1-p}.$$

2.2.3 Lower Estimates for the (p,Φ)-Capacity

Lemma. *For any $u \in \mathscr{D}(\Omega)$ and almost all $t \ge 0$,*

$$\big[\sigma(\partial\mathscr{L}_t)\big]^{p/(p-1)} \le \left[-\frac{\mathrm{d}}{\mathrm{d}t} m_n(\mathscr{L}_t)\right] \left(\int_{\partial\mathscr{L}_t} \big[\Phi(x,\nabla u)\big]^p \frac{\mathrm{d}s}{|\nabla u|}\right)^{1/(p-1)}, \tag{2.2.6}$$

where, as usual, $\mathscr{L}_t = \{x \in \Omega : |u(x)| > t\}$.

Proof. By Hölder's inequality, for almost all t and T, $t < T$,

$$\left(\int_{\mathscr{L}_t\setminus\mathscr{L}_T} |u|^{p-1}\Phi(x,\nabla u)\,\mathrm{d}x\right)^{p/(p-1)} \le \int_{\mathscr{L}_t\setminus\mathscr{L}_T} |u|^p\,\mathrm{d}x \left(\int_{\mathscr{L}_t\setminus\mathscr{L}_T} \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x\right)^{1/(p-1)}.$$

Using Theorem 1.2.4, we obtain

$$\left(\int_t^T \tau^{p-1}\sigma(\partial\mathscr{L}_\tau)\,\mathrm{d}\tau\right)^{p/(p-1)} \leq \int_{\mathscr{L}_t\setminus\mathscr{L}_T} |u|^p\,\mathrm{d}x \left(\int_t^T \mathrm{d}\tau \int_{\mathscr{E}_\tau} \big[\Phi(x,\nabla u)\big]^p \frac{\mathrm{d}s}{|\nabla u|}\right)^{1/(p-1)}.$$

We divide both sides of the preceding inequality by $(T-t)^{p/(p-1)}$ and estimate the first factor on the right-hand side

$$\left(\frac{1}{T-t}\int_t^T \tau^{p-1}\sigma(\partial\mathscr{L}_\tau)\,\mathrm{d}\tau\right)^{p/(p-1)} \leq T^p\frac{m_n(\mathscr{L}_t\setminus\mathscr{L}_T)}{T-t}\left(\frac{1}{T-t}\int_t^T \mathrm{d}\tau \int_{\partial\mathscr{L}_\tau} \big[\Phi(x,\nabla u)\big]^p \frac{\mathrm{d}s}{|\nabla u|}\right)^{1/(p-1)}.$$

Passing to the limit as $T \to t$, we obtain (2.2.6) for almost all $t > 0$. The lemma is proved. □

From Lemma 2.2.2/3 and from the Lemma of the present subsection we immediately obtain the following corollary.

Corollary 1. *The inequality*

$$(p,\Phi)\text{-cap}(F,\Omega) \geq \inf_{u\in\mathfrak{P}(F,\Omega)}\left\{-\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}m_n(\mathscr{L}_\tau)\frac{\mathrm{d}\tau}{[\sigma(\partial\mathscr{L}_\tau)]^{p/(p-1)}}\right\}^{1-p} \tag{2.2.7}$$

holds.

Definition. In what follows we use the function $\mathscr{C}$ introduced in Definition 2.1.4 assuming $\mu = m_n$, that is, $\mathscr{C}$ stands for the infimum $\sigma(\partial g)$ for all admissible sets such that $m_n(g) \geq \varrho$. Then from (2.2.7) we obtain the next corollary, containing the so-called *isocapacitary inequalities*.

Corollary 2. *The inequality*

$$(p,\Phi)\text{-cap}(F,\Omega) \geq \left(\int_{m_n(F)}^{m_n(\Omega)} \frac{\mathrm{d}\varrho}{[\mathscr{C}(\varrho)]^{p/(p-1)}}\right)^{1-p} \tag{2.2.8}$$

is valid.

By virtue of the classical isoperimetric inequality

$$s(\partial g) \geq n^{(n-1)/n}\omega_n^{1/n}\big[m_n(g)\big]^{(n-1)/n}, \tag{2.2.9}$$

in the case $\Phi(x,\xi) = |\xi|$ we have

$$\mathscr{C}(\varrho) = n^{(n-1)/n}\omega_n^{1/n}\varrho^{(n-1)/n}.$$

Therefore,

$$\operatorname{cap}_p(F,\Omega) \geq \omega_n^{p/n} n^{(n-p)/n} \left| \frac{p-n}{p-1} \right|^{p-1} \Big| m_n(\Omega)^{(p-n)/n(p-1)} - m_n(F)^{(p-n)/n(p-1)} \Big|^{1-p} \tag{2.2.10}$$

for $p \neq n$ and

$$\operatorname{cap}_p(F,\Omega) \geq n^{n-1}\omega_n \left(\log \frac{m_n(\Omega)}{m_n(F)} \right)^{1-n} \tag{2.2.11}$$

for $p = n$.

In particular, for $n > p$,

$$\operatorname{cap}_p(F) \geq \omega_n^{p/n} n^{(n-p)/n} \left(\frac{n-p}{p-1} \right)^{p-1} m_n(F)^{(n-p)/n}. \tag{2.2.12}$$

2.2.4 *p*-Capacity of a Ball

We show that the estimates (2.2.10) and (2.2.11) become equalities if Ω and F are concentric balls of radii R and r, $R > r$, i.e.,

$$\operatorname{cap}_p(F,\Omega) = \omega_n \left(\frac{|n-p|}{p-1} \right)^{p-1} \left| R^{(p-n)/(p-1)} - r^{(p-n)/(p-1)} \right|^{1-p} \tag{2.2.13}$$

for $n \neq p$ and

$$\operatorname{cap}_n(F,\Omega) = \omega_n \left(\log \frac{R}{r} \right)^{1-n} \tag{2.2.14}$$

for $n = p$.

Let the centers of the balls Ω and F coincide with the origin O of spherical coordinates (ϱ,ω), $|\omega| = 1$. Obviously,

$$\begin{aligned} \operatorname{cap}_p(F,\Omega) &\geq \inf_{u \in \mathfrak{N}(F,\Omega)} \int_{\partial B_1} \mathrm{d}\omega \int_r^R \left| \frac{\partial u}{\partial \varrho} \right|^p \varrho^{n-1}\, \mathrm{d}\varrho \\ &\geq \int_{\partial B_1} \mathrm{d}\omega \inf_{u \in \mathfrak{N}(F,\Omega)} \int_r^R \left| \frac{\partial u}{\partial \varrho} \right|^p \varrho^{n-1}\, \mathrm{d}\varrho. \end{aligned}$$

The inner integral attains its infimum at the function

$$[r,R] \in \varrho \to v(\varrho) = \begin{cases} \frac{R^{(p-n)/(p-1)} - \varrho^{(p-n)/(p-1)}}{R^{(p-n)/(p-1)} - r^{(p-n)/(p-1)}} & \text{for } p \neq n, \\ \frac{\log(\varrho R^{-1})}{\log(rR^{-1})} & \text{for } p = n. \end{cases}$$

This implies the required lower estimates for the p-capacity. The substitution of $v(\varrho)$ into the integral $\int_\Omega |\nabla u|^p\, \mathrm{d}x$ leads to (2.2.13) and (2.2.14).

In particular, the p-capacity of the n-dimensional ball B_r relative to $\mathbb{R}^n$ is equal to $\omega_n(\frac{n-p}{p-1})^{p-1} r^{n-p}$ for $n > p$ and to zero for $n \leq p$. Since the p-capacity is a monotone set function, then for any compactum $p\text{-cap}(F,\mathbb{R}^n) = 0$, if

$n \le p$. In the case $p \le n$ the capacity of a point relative to any open set Ω, containing this point, equals zero. If $p > n$, then the p-capacity of the center of the ball B_R relative to B_R equals $\omega_n(\frac{p-n}{p-1})^{p-1} R^{n-p}$. Therefore, in the last case, the p-capacity of any compactum relative to any bounded open set that contains this compactum is positive.

2.2.5 (p, Φ)-Capacity for $p = 1$

Lemma. *For any compactum $F \subset \Omega$*

$$(1, \Phi)\text{-cap}(F, \Omega) = \inf \sigma(\partial g),$$

where the infimum is taken over all admissible sets g in Ω containing F.

Proof. Let $u \in \mathfrak{N}(F, \Omega)$. Applying Theorem 1.2.4, we obtain

$$\int_\Omega \Phi(x, \nabla u)\, \mathrm{d}x = \int_0^1 \sigma(\partial \mathscr{L}_t)\, \mathrm{d}t \ge \inf_{g \supset F} \sigma(\partial g).$$

This implies the lower estimate for the capacity.

Let g be an admissible set containing F. The function $u_\varepsilon(x) = \alpha(\mathrm{d}(x))$ defined in the proof of the second part of Theorem 2.1.1 belongs to $\mathfrak{N}(F, \Omega)$ for sufficiently small $\varepsilon > 0$. So

$$(1, \Phi)\text{-cap}(F, \Omega) \le \int_\Omega \Phi(x, \nabla u_\varepsilon)\, \mathrm{d}x.$$

In the proof of the second part of Theorem 2.1.1, it was shown that the preceding integral converges to $\sigma(\partial g)$, which yields the required upper estimate for the capacity. The lemma is proved. □

2.2.6 The Measure m_{n-1} and 2-Capacity

Lemma. *If $B_\varrho^{(n-1)}$ is an $(n-1)$-dimensional ball in $\mathbb{R}^n$, $n > 2$, then*

$$\mathrm{cap}_2\big(B_\varrho^{(n-1)}, \mathbb{R}^n\big) = \frac{\omega_n}{c_n} \varrho^{n-2}, \tag{2.2.15}$$

where $c_3 = \frac{\pi}{3}$, $c_4 = 1$, and $c_n = (n-4)!!/(n-3)!!$ for odd $n \ge 5$ and $c_n = \frac{\pi}{2}(n-4)!!/(n-3)!!$ for even $n \ge 6$.

Proof. We introduce ellipsoidal coordinates in $\mathbb{R}^n$: $x_1 = \varrho \sinh\psi \cos\theta_1$, $x_j = \varrho \cosh\psi \sin\theta_1, \dots, \sin\theta_{j-1} \cos\theta_j, j = 2, \dots, n-1$, $x_n = \varrho \cosh\psi \sin\theta_1, \dots, \sin\theta_{n-1}$. A standard calculation leads to the formulas

$$\mathrm{d}x = \varrho^n \big(\cosh^2\psi - \sin^2\theta_1\big)(\cosh\psi)^{n-2}\, \mathrm{d}\psi\, \mathrm{d}\omega,$$

$$(\nabla u)^2 = \varrho^{-2} \left(\frac{\partial u}{\partial \psi}\right)^2 \big(\cosh^2\psi - \sin^2\theta_1\big)^{-1} + \cdots,$$

where $d\omega$ is the surface element of the unit ball in $\mathbb{R}^n$ and the dots denote a positive quadratic form of all first derivatives of u except $\partial u/\partial\psi$. The equation of the ball $B_\varrho^{(n-1)}$ in the new coordinates is $\psi = 0$. Therefore

$$\operatorname{cap}_2\big(B_\varrho^{(n-1)}, \mathbb{R}^n\big) \geq \varrho^{n-2} \int_{|\omega|=1} \bigg(\inf_{\{u\}} \int_0^\infty \Big(\frac{\partial u}{\partial \psi}\Big)^2 (\cosh\psi)^{n-2}\, d\psi \bigg)\, d\omega,$$

where $\{u\}$ is a set of smooth functions on $[0,\infty)$ with compact supports. The infimum on the right-hand side is equal to

$$\bigg(\int_0^\infty \frac{d\psi}{(\cosh\psi)^{n-2}} \bigg)^{-1} = c_n^{-1}.$$

This value is attained at the function

$$v = \int_\psi^\infty \frac{d\tau}{(\cosh\tau)^{n-2}} \bigg(\int_0^\infty \frac{d\tau}{(\cosh\tau)^{n-2}} \bigg)^{-1},$$

which equals unity on $B_\varrho^{(n-1)}$ and decreases sufficiently rapidly at infinity. Substituting v into the Dirichlet integral, we obtain

$$\operatorname{cap}_2\big(B_\varrho^{(n-1)}, \mathbb{R}^n\big) \leq \omega_n \varrho^{n-2} \int_0^\infty \Big(\frac{\partial v}{\partial \psi}\Big)^2 (\cosh\psi)^{n-2}\, d\psi = \frac{\omega_n}{c_n} \varrho^{n-2}.$$

This proves the lemma. □

We now recall the definition of the *symmetrization of a compact set K in $\mathbb{R}^n$ relative to the $(n-s)$-dimensional subspace* $\mathbb{R}^{n-s}$.

Let any point $x \in \mathbb{R}^n$ be denoted by (y, z), where $y \in \mathbb{R}^{n-s}$, $z \in \mathbb{R}^s$. The image K^* of the compact set K under symmetrization relative to the subspace $z = 0$ is defined by the following conditions:

1. The set K^* is symmetric relative to $z = 0$.

2. Any s-dimensional subspace, parallel to the subspace $y = 0$ and crossing either K or K^* also intersects the other one and the Lebesgue measures of both cross sections are equal.

3. The intersection of K^* with any s-dimensional subspace, which is parallel to the subspace $y = 0$, is a ball in $\mathbb{R}^s$ centered at the hyperplane $z = 0$.

Below we follow Pólya and Szegö [666] who established that the 2-capacity does not increase under the symmetrization relative to $\mathbb{R}^{n-1}$. Let π be an $(n-1)$-dimensional hyperplane and let $\mathrm{Pr}_\pi \mathscr{F}$ be the projection of $\mathscr{F}$ onto π. We choose π so that $m_{n-1}(\mathrm{Pr}_\pi\, \mathscr{F})$ attains its maximum value. We symmetrize $\mathscr{F}$ relative to π and obtain a compactum that is also symmetrized relative to a straight line perpendicular to π. So we obtain a body whose capacity does not exceed 2-cap $\mathscr{F}$ and whose intersection with π is an $(n-1)$-dimensional ball with volume $m_{n-1}(\mathrm{Pr}_\pi \mathscr{F})$. Thus the $(n-1)$-dimensional ball has the largest area of orthogonal projections onto an $(n-1)$-dimensional plane among all compacta with fixed 2-capacity.

This and the Lemma imply the isocapacitary inequality

$$\begin{aligned}&\left[m_{n-1}\left(\mathscr{F}\cap\mathbb{R}^{n-1}\right)\right]^{(n-2)/(n-1)}\\&\quad\le\left(\frac{\omega_{n-1}}{n-1}\right)^{(n-2)/(n-1)}\frac{c_n}{\omega_n}\operatorname{cap}_2\left(\mathscr{F},\mathbb{R}^n\right),\end{aligned} \tag{2.2.16}$$

where c_n is the constant defined in the Lemma.

2.2.7 Comments to Sect. 2.2

The capacity generated by the integral

$$\int_\Omega f(x, u, \nabla u)\,\mathrm{d}x$$

was introduced by Choquet [186] where it served as an illustration of general capacity theory. Here the presentation follows the author's paper [543].

Lemma 2.2.2/1 for $p = 2$, $\Phi(x, \xi) = |\xi|$ is the so-called Dirichlet principle with prescribed level surfaces verified in the book by Pólya and Szegö [666] under rigid assumptions on level surfaces of the function u. As for the general case, their proof can be viewed as a convincing heuristic argument. The same book also contains isocapacitary inequalities, which are special cases of (2.2.10) and (2.2.11).

Lemma 2.2.3, leading to lower estimates for the capacity, was published for $\Phi(x, \xi) = |\xi|$ in 1969 by the author [538] and later by Talenti in [741], p. 709.

Properties of symmetrization are studied in the books by Pólya and Szegö [666] and by Hadwiger [334] et al. See, for instance, the book by Hayman [357] where the circular symmetrization and the symmetrization with respect to a straight line in $\mathbb{R}^2$ are considered. Nevertheless, Hayman's proofs can be easily generalized to the n-dimensional case. Lemma 2.2.5 is a straightforward generalization of a similar assertion due to Fleming [281] on 1-capacity.

In the early 1960s the p-capacity was used by the author to obtain the necessary and sufficient conditions for the validity of continuity and compactness properties of Sobolev-type embedding operators [527, 528, 530, 531].

Afterward various generalizations of the p-capacity proved to be useful in the theory of function spaces and nonlinear elliptic equations. A Muckenhoupt A_p-weighted capacity was studied in detail by Heinonen, Kilpeläinen, and Martio [375] and Nieminen [636] et al. A general capacity theory for monotone operators

$$W_p^1 \ni u \to -\operatorname{div}\bigl(a(x, Du)\bigr) \in \bigl(W_p^1\bigr)^*$$

was developed by Dal Maso and Skrypnik [220], whose results were extended to pseudomonotone operators by Casado-Díaz [174]. Biroli [104] studied the p-capacity related to the norm

$$\left(\sum_{i=1}^{m}\int_{\Omega}|X_i u|^p\,\mathrm{d}x+\int_{\Omega}|u|^p\,\mathrm{d}x\right)^{1/p},$$

where X_i are vector fields subject to Hörmander's condition: They and their commutators up to some order span at every point all $\mathbb{R}^n$. Properties of a capacity generated by the Sobolev space $W^1_{p(\cdot)}$ with the variable exponent $p : \mathbb{R}^n \to (1,\infty)$ were investigated by Harjulehto, Hästö, Koskenoja, and Varonen [353]. Another relevant area of research is the p-capacity on metric spaces with a measure (see, for instance, Kinnunen and Martio [425] and Gol'dshtein, Troyanov [317]), in particular, on the Carnot group and Heisenberg group (see Heinonen and Holopainen [374]).

A generalization of the inequality (2.2.8) was obtained by E. Milman [603] in a more general framework of measure metric spaces for the case $\Phi(x,\xi) = |\xi|$. Similarly to Sect. 2.3.8, if we introduce the p-capacity minimizing function

$$\nu_p(t) = \inf \operatorname{cap}_p(F,\Omega),$$

where the infimum is taken over compacta $F \subset \Omega$ with $m_n(F) \geq t$, then for any $p_1 \geq p_0 \geq 1$

$$\frac{1}{\nu_{p_1}(t)} \leq \frac{(\frac{q_0}{q_1}-1)^{p_1/q_0}}{(1-\frac{q_1}{q_0})^{p_1/q_1}}\left(\int_t^{m_n(\Omega)}\frac{\mathrm{d}s}{(s-t)^{q_1/q_0}\nu_{p_0}(s)^{q_1/p_0}}\right)^{p_1/q_1}, \tag{2.2.17}$$

where $q_i = p_i/(p_i-1)$ denote the corresponding conjugate exponents. Clearly (2.2.17) coincides with (2.2.8) when $p_0 = 1$ by Lemma 2.2.5.

Under an appropriate curvature lower bound on the underlying space, it was also shown in [603, 605] that (2.2.17) may, in fact, be reversed, to within numeric constants. An application of this fact was given by E. Milman [604] to the stability of the first positive eigenvalue of the Neumann Laplacian on convex domains, with respect to perturbation of the domain.

2.3 Conditions for Validity of Integral Inequalities (the Case $p > 1$)

2.3.1 The (p,Φ)-Capacitary Inequality

Let $u \in \mathscr{D}(\Omega)$ and let g be the function defined by (2.2.5) with $p > 1$. Further, let

$$T \overset{\text{def}}{=} \sup\bigl\{t > 0 : (p,\Phi)\text{-cap}(\mathscr{N}_t,\Omega) > 0\bigr\} > 0, \tag{2.3.1}$$

where $\mathscr{N}_t = \{x \in \Omega : |u(x)| \geq t\}$. From (2.3.1) it follows that

$$\psi(t) \overset{\text{def}}{=} \int_0^t \frac{\mathrm{d}\tau}{[g(\tau)]^{1/(p-1)}} \leq \infty \tag{2.3.2}$$

for $0 < t < T$. In fact, let

$$v(x) = t^{-2}\big[u(x)\big]^2.$$

Since $v \in \mathfrak{N}(\mathscr{N}_t, \Omega)$, then from Lemma 2.2.2/1 and (2.3.1) we obtain

$$\int_0^1 \bigg(\int_{\{x:v(x)=\tau\}} \big[\Phi(x,\nabla v)\big]^p \frac{\mathrm{d}s}{|\nabla v|}\bigg)^{1/(1-p)} \mathrm{d}\tau$$
$$\le \big[(p,\Phi)\text{-cap}(\mathscr{N}_t, \Omega)\big]^{1/(1-p)} < \infty,$$

and it remains to note that

$$\int_0^1 \frac{\mathrm{d}\tau}{g(\tau)^{1/(p-1)}} = \int_0^1 \bigg(\int_{\{x:v(x)=\tau\}} \big[\Phi(x,\nabla v)\big]^p \frac{\mathrm{d}s}{|\nabla v|}\bigg)^{1/(p-1)} \mathrm{d}\tau.$$

Since by Theorem 1.2.4

$$\int_0^\infty g(\tau)\,\mathrm{d}\tau = \int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x < \infty,$$

it follows that $g(t) < \infty$ for almost all $t > 0$ and the function $\psi(t)$ is strictly monotonic. Consequently, on the interval $[0, \psi(T))$ the function $t(\psi)$, which is the inverse of $\psi(t)$, exists.

Lemma. *Let u be a function in $\mathscr{D}(\Omega)$ satisfying condition (2.3.1). Then the function $t(\psi)$ is absolutely continuous on any segment $[0, \psi(T-\delta)]$, where $\delta \in (0, T)$, and*

$$\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x \ge \int_0^{\psi(t)} \big[t'(\psi)\big]^p\,\mathrm{d}\psi. \tag{2.3.3}$$

If $T = \max|u|$, then we may write the equality sign in (2.3.3).

Proof. Let $0 = \psi_0 < \psi_1 < \cdots < \psi_m = \psi(T-\delta)$ be an arbitrary partition of the segment $[0, \psi(T-\delta)]$. By Hölder's inequality,

$$\frac{[t(\psi_{k+1}) - t(\psi_k)]^p}{(\psi_{k+1}-\psi_k)^{p-1}} = \frac{[t(\psi_{k+1}) - t(\psi_k)]^p}{[\int_{t(\psi_k)}^{t(\psi_{k+1})} g(\tau)^{1/(1-p)}\,\mathrm{d}\tau]^{p-1}} \le \int_{t(\psi_k)}^{t(\psi_{k+1})} g(\tau)\,\mathrm{d}\tau,$$

and consequently,

$$\sum_{k=0}^{m-1} \frac{[t(\psi_{k+1}) - t(\psi_k)]^p}{(\psi_{k+1}-\psi_k)^{p-1}} \le \sum_{k=0}^{m-1} \int_{t(\psi_k)}^{t(\psi_{k+1})} g(\tau)\,\mathrm{d}\tau$$
$$= \int_0^{T-\delta} g(\tau)\,\mathrm{d}\tau \le \int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x. \tag{2.3.4}$$

The last inequality follows from Theorem 1.2.4. By (2.3.4) and F. Riesz's theorem (see Natanson [627]), the function $t(\psi)$ is absolutely continuous and its derivative belongs to $L_p(0, \psi(T-\delta))$. By Theorem 1.2.4,

$$\int_\Omega \big[\varPhi(x,\nabla u)\big]^p\,\mathrm{d}x \ge \lim_{\delta\to+0}\int_0^{T-\delta} g(\tau)\,\mathrm{d}\tau. \tag{2.3.5}$$

Since $t(\psi)$ is a monotonic absolutely continuous function, we can make the change of variable $\tau = t(\psi)$ in the last integral. Then

$$\int_0^{T-\delta} g(\tau)\,\mathrm{d}\tau = \int_0^{\psi(T-\delta)} t'(\psi)g(\psi)\,\mathrm{d}\psi = \int_0^{\psi(T-\delta)} \big[t'(\psi)\big]^p\,\mathrm{d}\psi,$$

which, along with (2.3.5), completes the proof. □

Theorem. (Capacitary Inequality) *Let $u \in \mathscr{D}(\Omega)$. Then for $p \ge 1$,*

$$\int_0^\infty (p,\varPhi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\big(t^p\big) \le \frac{p^p}{(p-1)^{p-1}}\int_\Omega \big[\varPhi(x,\nabla u)\big]^p\,\mathrm{d}x. \tag{2.3.6}$$

For $p = 1$ the coefficient in front of the integral on the right-hand side of (2.3.6) is equal to one. The constant $p^p(p-1)^{1-p}$ is optimal.

Proof. To prove (2.3.6) it is sufficient to assume that the number T, defined in (2.3.1), is positive. Since by Lemma 2.2.5

$$(1,\varPhi)\text{-cap}(\mathscr{N}_t,\Omega) \le \sigma(\partial\mathscr{L}_t)$$

for almost all $t > 0$, we see that (2.3.6) follows from (2.1.4) for $p = 1$.

Consider the case $p > 1$. Let $\psi(t)$ be a function defined by (2.3.2) and let $t(\psi)$ be the inverse of $\psi(t)$. We make the change of variable

$$\begin{aligned}\int_0^\infty (p,\varPhi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\big(t^p\big) &= \int_0^T (p,\varPhi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\big(t^p\big)\\ &= \int_0^{\psi(T)} (p,\varPhi)\text{-cap}(\mathscr{N}_{t(\psi)},\Omega)\,\mathrm{d}\big(t(\psi)^p\big).\end{aligned}$$

Setting $v = t^{-2}u^2$, $\xi = t^{-2}\tau^2$ in (2.3.2), we obtain

$$\psi(t) = \int_0^1 \left(\int_{\{x:v(x)=\xi\}} \big[\varPhi(x,\nabla v)\big]^p \frac{\mathrm{d}s}{|\nabla v|}\right)^{1/(1-p)} \mathrm{d}\xi. \tag{2.3.7}$$

Since $v \in \mathfrak{N}(\mathscr{N}_t,\Omega)$, then by Lemma 2.2.2/1 the right-hand side of (2.3.7) does not exceed

$$\big[(p,\varPhi)\text{-cap}(\mathscr{N}_{t(\psi)},\Omega)\big]^{1/(1-p)}.$$

Consequently,

$$\int_0^\infty (p,\varPhi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\big(t^p\big) \le \int_0^{\psi(T)} \frac{\mathrm{d}[t(\psi)]^p}{\psi^{p-1}} = p\int_0^{\psi(T)} \left[\frac{t(\psi)}{\psi}\right]^{p-1} t'(\psi)\,\mathrm{d}\psi.$$

Applying the Hölder inequality and the Hardy inequality

$$\int_0^{\psi(T)} \frac{[t(\psi)]^p}{\psi^p}\,\mathrm{d}\psi \le \left(\frac{p}{p-1}\right)^p \int_0^{\psi(T)} \bigl[t'(\psi)\bigr]^p\,\mathrm{d}\psi, \tag{2.3.8}$$

we arrive at

$$\int_0^{\infty} (p,\Phi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\bigl(t^p\bigr) \le \frac{p^p}{(p-1)^{p-1}} \int_0^{\psi(T)} \bigl[t'(\psi)\bigr]^p\,\mathrm{d}\psi,$$

which together with Lemma 2.3.1 yields (2.3.6).

To show that the constant factor in the right-hand side of (2.3.6) is sharp, it suffices to put $\Phi(x,y) = |y|$ and $u(x) = f(|x|)$. Then (2.2.13) and (2.3.6) imply

$$\frac{|n-p|^p}{(p-1)^{p-1}} \int_0^{\infty} \frac{|f(r)|^p}{r^p} r^{n-1}\,\mathrm{d}r \le \frac{p^p}{(p-1)^{p-1}} \int_0^{\infty} \bigl|f'(r)\bigr|^p r^{n-1}\,\mathrm{d}r,$$

which is a particular case of the sharp Hardy inequality (1.3.1). □

Remark 1. The inequality

$$\int_0^{\infty} (p,\Phi)\text{-cap}(\mathscr{N}_t,\Omega)\,\mathrm{d}\bigl(t^p\bigr) \le C \int_{\Omega} \bigl[\Phi(x,\nabla u)\bigr]^p\,\mathrm{d}x, \tag{2.3.9}$$

with a cruder constant than in (2.3.6) can be proved more simply in the following way. By the monotonicity of capacity, the integral in the left-hand side does not exceed

$$\varXi \stackrel{\text{def}}{=} \bigl(2^p - 1\bigr) \sum_{j=-\infty}^{+\infty} 2^{pj} (p,\Phi)\text{-cap}(\mathscr{N}_{2^j},\Omega).$$

Let $\lambda_\varepsilon \in C^\infty(\mathbb{R}^1)$, $\lambda_\varepsilon(t) = 1$ for $t \ge 1$, $\lambda_\varepsilon(t) = 0$ for $t \le 0$, $0 \le \lambda'_\varepsilon(t) \le 1+\varepsilon$, and let

$$u_j(x) = \lambda_\varepsilon\bigl(2^{1-j}\bigl|u(x)\bigr| - 1\bigr).$$

Since $u_j \in \mathfrak{N}(\mathscr{N}_{2^j},\Omega)$, we have

$$\begin{aligned}
\varXi &\le 2^{p-1} \sum_{j=-\infty}^{\infty} 2^{pj} \int_{\mathscr{N}_{2^{j-1}}\setminus\mathscr{N}_{2^j}} \bigl[\Phi(x,\nabla u_j)\bigr]^p\,\mathrm{d}x \\
&\le 2^{2p-1} \sum_{j=-\infty}^{\infty} \int_{\mathscr{N}_{2^{j-1}}\setminus\mathscr{N}_{2^j}} \bigl[\lambda'_\varepsilon\bigl(2^{1-j}|u| - 1\bigr)\bigr]^p \bigl[\Phi(x,\nabla u)\bigr]^p\,\mathrm{d}x \\
&\le (1+\varepsilon)^p 2^{2p-1} \int_{\Omega} \bigl[\Phi(x,\nabla u)\bigr]^p\,\mathrm{d}x.
\end{aligned}$$

Letting ε tend to zero, we obtain (2.3.9) with the constant $C = 2^{2p-1}$, which completes the proof.

Remark 2. In fact, the inequality just obtained is equivalent to the following one stronger than (2.3.9) (modulo the best constant)

$$\int_0^\infty (p,\phi)\text{-cap}(\mathscr{N}_{2t},\mathscr{L}_t)\,\mathrm{d}\big(t^p\big) \le c\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x. \tag{2.3.10}$$

Such *conductor inequalities* will be considered in Chap. 3.

2.3.2 Capacity Minimizing Function and Its Applications

Definition. Let $\nu_p(t)$ denote

$$\inf (p,\Phi)\text{-cap}(\bar{g},\Omega),$$

where the infimum is taken over all admissible sets g with

$$\mu(g) \ge t.$$

Note that by Lemma 2.2.5

$$\nu_1(t) = \mathscr{C}(t),$$

where $\mathscr{C}$ is the weighted area minimizing function introduced in Definition 2.1.4.

The following application of the capacity minimizing function ν_p is immediately deduced from the capacitary inequality (2.3.6)

$$\int_0^\infty \nu_p\big(\mu(\mathscr{N}_t)\big)\,\mathrm{d}\big(t^p\big) \le C\int \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x, \tag{2.3.11}$$

where $C \ge p^p(p-1)^{1-p}$. Conversely, minimizing the integral in the right-hand side over $\mathfrak{P}(F,\Omega)$, we see that (2.3.11) gives the isocapacitary inequality

$$\nu_p\big(\mu(F)\big) \le C(p,\Phi)\text{-cap}(F,\Omega).$$

If for instance, $\mu = m_n$, then (2.2.8) leads to the estimate

$$\int_0^\infty \bigg(\int_{m_n(\mathscr{N}_t)}^{m_n(\Omega)} \frac{\mathrm{d}\rho}{[\mathscr{C}(\rho)]^{p/(p-1)}}\bigg)^{1-p} \mathrm{d}\big(t^p\big) \le \frac{p^p}{(p-1)^{p-1}}\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x. \tag{2.3.12}$$

In particular, being set into (2.3.6) with $p = n$ and $\Phi(x,\xi) = |\xi|$, the isocapacitary inequality (2.2.11) implies

$$\int_0^\infty \bigg(\log\frac{m_n(\Omega)}{m_n(\mathscr{N}_t)}\bigg)^{1-n} \mathrm{d}\big(t^n\big) \le \frac{n}{(n-1)^n\omega_n}\int_\Omega |\nabla u|^n\,\mathrm{d}x, \tag{2.3.13}$$

for all $u \in C_0^\infty(\Omega)$, where $C = n(n-1)^{1-n}\omega_n^{-1}$.

Clearly, the inequality (2.3.11) and its special cases (2.3.12) and (2.3.13) can be written in terms of the nonincreasing rearrangement u^*_μ of u introduced by (2.1.12):

$$\int_0^{m_n(\Omega)} \big[u^*_\mu(s)\big]^p \,\mathrm{d}\nu_p(s) \le C \int_\Omega \big[\Phi(x, \nabla u)\big]^p \,\mathrm{d}x,$$

where C is the same as in (2.3.11). In particular, (2.3.13) takes the form

$$\int_0^{m_n(\Omega)} \big[u^*_{m_n}(s)\big]^n \frac{\mathrm{d}s}{(\log \frac{m_n(\Omega)}{s})^n} \le C \int_\Omega |\nabla u|^n \,\mathrm{d}x \tag{2.3.14}$$

for all $u \in C_0^\infty(\Omega)$.

2.3.3 Estimate for a Norm in a Birnbaum–Orlicz Space

We recall the definition of a Birnbaum–Orlicz space (see Birnbaum and Orlicz [103], Krasnosel'skiĭ and Rutickiĭ [463], and Rao and Ren [671]).

On the axis $-\infty < u < \infty$, let the function $M(u)$ admit the representation

$$M(u) = \int_0^{|u|} \varphi(t) \,\mathrm{d}t,$$

where $\varphi(t)$ is a nondecreasing function, positive for $t > 0$, and continuous from the right for $t \ge 0$, satisfying the conditions $\varphi(0) = 0$, $\varphi \to \infty$ as $t \to \infty$. Such functions M are sometimes called Young functions. Further, let

$$\psi(s) = \sup\big\{t : \varphi(t) \le s\big\},$$

be the right inverse of $\varphi(t)$. The function

$$P(u) = \int_0^{|u|} \psi(s) \,\mathrm{d}s$$

is called the complementary function to $M(u)$.

Let $\mathscr{L}_M(\Omega, \mu)$ denote the space of μ-measurable functions for which

$$\|u\|_{\mathscr{L}_M(\Omega,\mu)} = \sup\bigg\{\bigg|\int_\Omega uv \,\mathrm{d}\mu\bigg| : \int_\Omega P(v) \,\mathrm{d}\mu \le 1\bigg\} < \infty.$$

In particular, if $M(u) = q^{-1}|u|^q$, $q > 1$, then $P(u) = (q')^{-1}|u|^{q'}$, $q' = q(q-1)^{-1}$ and

$$\|u\|_{\mathscr{L}_M(\Omega,\mu)} = (q')^{1/q'} \|u\|_{L_q(\Omega,\mu)}.$$

The norm in $\mathscr{L}_M(\Omega, \mu)$ of the characteristic function χ_E of the set E is

$$\|\chi_E\|_{\mathscr{L}_M(\Omega,\mu)} = \mu(E) P^{-1}\Big(\frac{1}{\mu(E)}\Big),$$

where P^{-1} is the inverse of the restriction of P to $[0,\infty)$.

In fact, if $v = P^{-1}(1/\mu(E))\chi_E$, then

$$\int_\Omega P(v)\,\mathrm{d}\mu = 1,$$

and the definition of the norm in $\mathscr{L}_M(\Omega,\mu)$ implies

$$\|\chi_E\|_{\mathscr{L}_M(\Omega,\mu)} \geq \int_\Omega \chi_E v\,\mathrm{d}\mu = \mu(E)P^{-1}\big(1/\mu(E)\big).$$

On the other hand, by Jensen's inequality,

$$\int_\Omega \chi_E v\,\mathrm{d}\mu \leq \mu(E)P^{-1}\bigg(\frac{1}{\mu(E)}\int_E P(v)\,\mathrm{d}\mu\bigg),$$

and if we assume that

$$\int_\Omega P(v)\,\mathrm{d}\mu \leq 1,$$

then the definition of the norm in $\mathscr{L}_M(\Omega,\mu)$ yields

$$\|\chi_E\|_{\mathscr{L}_M(\Omega,\mu)} \leq \mu(E)P^{-1}\big(1/\mu(E)\big).$$

Although formally $M(t) = |t|$ does not satisfy the definition of the Birnbaum–Orlicz space, all the subsequent results concerning $\mathscr{L}_M(\Omega,\mu)$ include this case provided we put $P^{-1}(t) = 1$. Then we have $\mathscr{L}_M(\Omega,\mu) = L_1(\Omega,\mu)$.

Theorem. 1. *If there exists a constant β such that for any compactum $F \subset \Omega$*

$$\mu(F)P^{-1}\big(1/\mu(F)\big) \leq \beta(p,\Phi)\text{-cap}(F,\Omega) \tag{2.3.15}$$

with $p \geq 1$, then for all $u \in \mathscr{D}(\Omega)$,

$$\big\||u|^p\big\|_{\mathscr{L}_M(\Omega,\mu)} \leq C\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x, \tag{2.3.16}$$

where $C \leq p^p(p-1)^{1-p}\beta$.

2. *If (2.3.16) is valid for any $u \in \mathscr{D}(\Omega)$, then (2.3.15) holds for all compacta $F \subset \Omega$ with $\beta \leq C$.*

Proof. 1. From Lemma 1.2.3 and the definition of the norm in $\mathscr{L}_M(\Omega,\mu)$ we obtain

$$\begin{aligned}\big\||u|^p\big\|_{\mathscr{L}_M(\Omega,\mu)} &= \sup\bigg\{\int_0^\infty\int_{\mathscr{N}_\tau} v\,\mathrm{d}\mu\,\mathrm{d}\big(\tau^p\big) : \int_\Omega P(v)\,\mathrm{d}\mu \leq 1\bigg\}\\ &\leq \int_0^\infty \sup\bigg\{\int_\Omega \chi_{\mathscr{N}_\tau} v\,\mathrm{d}\mu : \int_\Omega P(v)\,\mathrm{d}\mu \leq 1\bigg\}\,\mathrm{d}\big(\tau^p\big)\\ &= \int_0^\infty \|\chi_{\mathscr{N}_\tau}\|_{\mathscr{L}_m(\Omega,\mu)}\,\mathrm{d}\big(\tau^p\big).\end{aligned}$$

Consequently,

$$\big\| |u|^p \big\|_{\mathscr{L}_M(\Omega,\mu)} \le \int_0^\infty \mu(\mathscr{N}_\tau) P^{-1}\big(1/\mu(\mathscr{N}_\tau)\big)\, \mathrm{d}\big(\tau^p\big).$$

Using (2.3.15) and Theorem 2.3.1, we obtain

$$\begin{aligned}\big\| |u|^p \big\|_{\mathscr{L}_M(\Omega,\mu)} &\le \beta \int_0^\infty (p,\Phi)\text{-cap}(\mathscr{N}_\tau,\Omega)\, \mathrm{d}\big(\tau^p\big)\\ &\le \frac{p^p\beta}{(p-1)^{p-1}} \int_\Omega \big[\Phi(x,\nabla u)\big]^p\, \mathrm{d}x.\end{aligned}$$

2. Let u be any function in $\mathfrak{N}(F,\Omega)$. By (2.3.16),

$$\|\chi_F\|_{\mathscr{L}_M(\Omega,\mu)} \le C \int_\Omega \big[\Phi(x,\nabla u)\big]^p\, \mathrm{d}x.$$

Minimizing the right-hand side over the set $\mathfrak{N}(F,\Omega)$, we obtain (2.3.15). The theorem is proved. □

Remark 1. Obviously the isocapacitary inequality (2.3.15) can be written in terms of the capacity minimizing function ν_p as follows:

$$sP\big(s^{-1}\big) \le \beta\, \nu_p(s).$$

Remark 2. Let $\Phi(x,y)$ be a function satisfying the conditions stated in Sect. 2.1.1 and let the function $\Psi(x,u,y) : \Omega \times \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$, satisfy:

(i) the Caratheodory conditions: i.e., Ψ is measurable in x for all x, y, and continuous in x and y for almost all x.

(ii) The inequality

$$\Psi(x,u,y) \ge \big[\Phi(x,y)\big]^p$$

holds.

(iii) For all $u \in \mathscr{D}(\Omega)$

$$\liminf_{\lambda\to+\infty} \lambda^{-p} \int_\Omega \Psi(x,\lambda u,\lambda\nabla u)\, \mathrm{d}x \le K \int_\Omega \big[\Phi(x,\nabla u)\big]^p\, \mathrm{d}x.$$

Then (2.3.16) in the Theorem can be replaced by the following more general estimate:

$$\big\| |u|^p \big\|_{\mathscr{L}_M(\Omega,\mu)} \le C \int_\Omega \Psi(x,u,\nabla u)\, \mathrm{d}x. \tag{2.3.17}$$

As an illustration, note that Theorem 2.1.1 shows the equivalence of the inequality

$$\|u\|_{L_q(\mu)} \le \int_\Omega \sqrt{1+(\nabla u)^2}\, \mathrm{d}x,$$

where $u \in \mathscr{D}(\Omega)$ and $q \ge 1$, and the isoperimetric inequality $\mu(g)^{1/q} \le \sigma(\partial g)$.

Here, to prove the necessity of (2.3.15) for (2.3.17), we must set $u = \lambda v$, where $v \in \mathfrak{N}(F,\Omega)$, in (2.3.17) and then pass to the limit as $\lambda \to \infty$. An analogous remark can be made regarding Theorems 2.1.1, 2.1.2, and others.

2.3.4 Sobolev Type Inequality as Corollary of Theorem 2.3.3

Theorem 2.3.3 contains the following assertion, which is of interest in itself.

Corollary. 1. *If there exists a constant β such that for any compactum $F \subset \Omega$*

$$\mu(F)^{\alpha p} \le \beta(p, \Phi)\text{-cap}(F, \Omega), \tag{2.3.18}$$

where $p \ge 1$, $\alpha > 0$, $\alpha p \le 1$, then for all $u \in \mathscr{D}(\Omega)$

$$\|u\|^p_{L_q(\Omega,\mu)} \le C \int_\Omega \big[\Phi(x, \nabla u)\big]^p \, \mathrm{d}x, \tag{2.3.19}$$

where $q = \alpha^{-1}$ and $C \le p^p(p-1)^{1-p}\beta$.

2. *If (2.3.19) holds for any $u \in \mathscr{D}(\Omega)$ and if the constant C does not depend on u, then (2.3.18) is valid for all compacta $F \subset \Omega$ with $\alpha = q^{-1}$ and $\beta \le C$.*

Remark. Obviously, the isocapacitary inequality (2.3.18) is equivalent to the weak-type integral inequality

$$\sup_{t>0}\big(t\mu(\mathscr{L}_t)^{1/q}\big) \le C^{1/p}\big\|\Phi(\cdot, \nabla u)\big\|_{L_p(\Omega)} \tag{2.3.20}$$

with $\mathscr{L}_t = \{x : |u(x)| > t\}$ and this, along with the Corollary, can be interpreted as the equivalence of the weak and the strong Sobolev-type estimates (2.3.20) and (2.3.19).

2.3.5 Best Constant in the Sobolev Inequality ($p > 1$)

From the previous corollary and the isoperimetric inequality (2.2.12) we obtain the Sobolev ($p > 1$)-Gagliardo ($p = 1$) inequality

$$\|u\|_{L_{pn/(n-p)}} \le C\|\nabla u\|_{L_p}, \tag{2.3.21}$$

where $n > p \ge 1$, $u \in \mathscr{D}(\mathbb{R}^n)$ and

$$C = p(n-p)^{(1-p)/p}\omega_n^{-1/n} n^{(p-n)/pn}.$$

The value of the constant C in (2.3.21) is sharp only for $p = 1$ (cf. Theorem 1.4.2/1). To obtain the best constant one can proceed in the following way.

By Lemma 2.3.1

$$\int_0^{\psi(\max|u|)} \big[t'(\psi)\big]^p \, \mathrm{d}\psi = \int_{\mathbb{R}^n} |\nabla u|^p \, \mathrm{d}x.$$

Putting

$$\psi = \frac{p-1}{\omega_n^{1/(p-1)}(n-p)} r^{(n-p)/(1-p)}, \qquad t(\psi) = \gamma(r),$$

and assuming $t(\psi) = \text{const}$ for $\psi \geq \psi(\max|u|)$, we obtain

$$\omega_n \int_0^\infty \big|\gamma'(r)\big|^p r^{n-1}\,\mathrm{d}r = \int_{\mathbb{R}^n} |\nabla u|^p\,\mathrm{d}x.$$

Furthermore, by Lemma 1.2.3,

$$\int_{\mathbb{R}^n} |u|^{pn/(n-p)}\,\mathrm{d}x = \int_0^{\max|u|} m_n(\mathscr{N}_t)\,\mathrm{d}\big(t^{pn/(n-p)}\big).$$

The definition of the function $\psi(t)$, Lemma 2.2.3, and the isoperimetric inequality (2.2.9) imply

$$\psi(t) \leq \omega_n^{1/(1-p)} \frac{p-1}{n-p}\left[\frac{n}{\omega_n} m_n(\mathscr{N}_t)\right]^{(n-p)/n(1-p)}.$$

Consequently,

$$m_n(\mathscr{N}_{t(\psi)}) \leq \omega_n n^{-1} r^n$$

and

$$\int_{\mathbb{R}^n} |u|^{pn/(n-p)}\,\mathrm{d}x \leq \frac{\omega_n}{n}\int_0^\infty r^n\,\mathrm{d}\big[\gamma(r)^{pn/(n-p)}\big].$$

Since

$$\int_0^\infty \big|\gamma'(r)\big|^p r^{n-1}\,\mathrm{d}r < \infty,$$

it follows that $\gamma(r)r^{(n-p)/p} \to 0$ as $r \to \infty$. After integration by parts, we obtain

$$\int_{\mathbb{R}^n} |u|^{pn/(n-p)}\,\mathrm{d}x \leq \omega_n \int_0^\infty \big[\gamma(r)\big]^{pn/(n-p)} r^{n-1}\,\mathrm{d}r.$$

Thus,

$$\sup_{u\in\mathscr{D}} \frac{\|u\|_{L_{pn/(n-p)}}}{\|\nabla u\|_{L_p}} = \omega_n^{-1/n} \sup_{\{\gamma\}} \frac{\big(\int_0^\infty [\gamma(r)]^{pn/(n-p)} r^{n-1}\,\mathrm{d}r\big)^{(n-p)/pn}}{\big(\int_0^\infty |\gamma'(r)|^p r^{n-1}\,\mathrm{d}r\big)^{1/p}}, \tag{2.3.22}$$

where $\{\gamma\}$ is the set of all nonincreasing nonnegative functions on $[0,\infty)$ such that $\gamma(r)r^{(n-p)/p} \to 0$ as $r \to \infty$. Thus, we reduced the question of the best constant in (2.3.21) to a one-dimensional variational problem that was solved explicitly by Bliss [109] as early as 1930 by classical methods of the calculus of variations. Paradoxically, the best constant in the Sobolev inequality had been obtained earlier than the inequality itself appeared. The sharp upper bound in (2.3.22) is attained at any function of the form

$$\gamma(r) = \big(a + br^{p/(p-1)}\big)^{1-n/p}, \quad a, b = \text{const} > 0,$$

and equals

$$n^{-1/p}\left(\frac{p-1}{n-p}\right)^{(p-1)/p}\left[\frac{p-1}{p}B\left(\frac{n}{p},\frac{n(p-1)}{p}\right)\right]^{-1/n}.$$

Finally, the sharp constant in (2.3.21) is given by

$$C=\pi^{-1/2}n^{-1/p}\left(\frac{p-1}{n-p}\right)^{(p-1)/p}\left\{\frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right\}^{1/n}, \qquad (2.3.23)$$

and the equality sign can be written in (2.3.21) if

$$u(x)=\left[a+b|x|^{p/(p-1)}\right]^{1-n/p}, \qquad (2.3.24)$$

where a and b are positive constants (although u does not belong to $\mathscr{D}$ it can be approximated by functions in $\mathscr{D}$ in the norm $\|\nabla u\|_{L_p(\mathbb{R}^n)}$).

2.3.6 Multiplicative Inequality (the Case $p \geq 1$)

The following theorem describes conditions for the equivalence of the generalized Sobolev-type inequality (2.3.19) and a multiplicative integral inequality.

We denote by β the best constant in the isocapacitary inequality (2.3.18).

Theorem. 1. *For any compactum $F \subset \Omega$ let the inequality (2.3.18) hold with $p \geq 1$, $\alpha > 0$. Further, let q be a positive number satisfying one of the conditions* (i) $q \leq q^* = \alpha^{-1}$, *for* $\alpha p \leq 1$, *or* (ii) $q < q^* = \alpha^{-1}$, *for* $\alpha p > 1$.

Then the inequality

$$\|u\|_{L_q(\Omega,\mu)} \leq C\left(\int_\Omega \left[\Phi(x,\nabla u)\right]^p \mathrm{d}x\right)^{(1-\varkappa)/p}\|u\|^{\varkappa}_{L_r(\Omega,\mu)} \qquad (2.3.25)$$

holds for any $u \in \mathscr{D}(\Omega)$, where $r \in (0,q)$, $\varkappa = r(q^*-q)/q(q^*-r)$, $C \leq c\beta^{(1-\varkappa)/p}$.

2. *Let $p \geq 1$, $0 < q^* < \infty$, $r \in (0,q^*]$ and for some $q \in (0,q^*]$ and any $u \in \mathscr{D}(\Omega)$ let the inequality (2.3.25) hold with $\varkappa = r(q^*-q)/q(q^*-r)$ and a constant C independent of u.*

Then (2.3.18) holds for all compacta $F \subset \Omega$ with $\alpha = (q^*)^{-1}$ and $\beta \leq cC^{p/(1-\varkappa)}$.

Proof. 1. Let $\alpha p \leq 1$. By Hölder's inequality,

$$\int_\Omega |u|^q\,\mathrm{d}\mu = \int_\Omega |u|^{q^*(q-r)/(q^*-r)}|u|^{r(q^*-q)/(q^*-r)}\,\mathrm{d}\mu$$
$$\leq \left(\int_\Omega |u|^{q^*}\,\mathrm{d}\mu\right)^{(q-r)/(q^*-r)}\left(\int_\Omega |u|^r\,\mathrm{d}\mu\right)^{(q^*-q)/(q^*-r)},$$

or equivalently,

$$\|u\|_{L_q(\Omega,\mu)} \le \|u\|^{1-\varkappa}_{L_{q^*}(\Omega,\mu)} \|u\|^{\varkappa}_{L_r(\Omega,\mu)}.$$

Estimating the first factor by (2.3.19), we obtain (2.3.25) for $\alpha p \le 1$. Let $\alpha p > 1$. By Lemma 1.2.3,

$$\int_\Omega |u|^q \,\mathrm{d}\mu = q \int_0^\infty \mu(\mathscr{N}_t) t^{q-1} \,\mathrm{d}t.$$

To the last integral we apply inequality (1.3.42), where $x = t^q$, $f(x) = \mu(\mathscr{N}_t)$, $b = p(q^*)^{-1} > 1$, $a > 1$ is an arbitrary number, $\lambda = a(q-r)q^{-1}$, $\mu = p(q^* - q)/q^* q$

$$\int_0^\infty \mu(\mathscr{N}_t) t^{q-1} \,\mathrm{d}t \le c\Big(\int_0^\infty \big[\mu(\mathscr{N}_t)\big]^a t^{ar-1} \,\mathrm{d}t\Big)^{(q^*-q)/a(q^*-r)}$$
$$\times \Big(\int_0^\infty \big[\mu(\mathscr{N}_t)\big]^{p/q^*} t^{p-1} \,\mathrm{d}t\Big)^{q^*(q-r)/p(q^*-r)}.$$

Since $a > 1$ and $\mu(\mathscr{N}_t)$ does not increase, we can apply (1.3.41) to the first factor in the following way:

$$\int_0^\infty \big[\mu(\mathscr{N}_t)\big]^a t^{ar-1} \,\mathrm{d}t \le c\Big(\int_0^\infty \mu(\mathscr{N}_t) t^{r-1} \,\mathrm{d}t\Big)^a.$$

Thus,

$$\|u\|_{L_q(\Omega,\mu)} \le c\Big(\int_0^\infty \big[\mu(\mathscr{N}_t)\big]^{p/q^*} t^{p-1} \,\mathrm{d}t\Big)^{(1-\varkappa)/p} \|u\|^{\varkappa}_{L_r(\Omega,\mu)}.$$

From condition (2.3.18) and Theorem 2.3.1 we obtain

$$\int_0^\infty \big[\mu(\mathscr{N}_t)\big]^{p/q^*} t^{p-1} \,\mathrm{d}t \le c\beta \int_\Omega \big[\varPhi(x, \nabla u)\big]^p \,\mathrm{d}x.$$

The proof of the first part of the theorem is complete.

2. Let G be a bounded open set $\bar{G} \subset \Omega$. We fix a number $\delta > 0$ and we put

$$\beta_\delta = \sup \frac{\mu(F)^{p\alpha}}{(p,\varPhi)\text{-cap}(F,G)}$$

on the set of all compacta F in G satisfying the condition $(p,\varPhi)\text{-cap}(F,G) \ge \delta$. (If $(p,\varPhi)\text{-cap}(F,G) = 0$ for any compactum $F \subset G$, then the substitution of an arbitrary $u \in \mathfrak{N}(F,G)$ into (2.3.25) immediately leads to $\mu = 0$.) Obviously,

$$\beta_\delta \le \delta^{-1}\mu(G)^{p\alpha} < \infty.$$

Let v be an arbitrary function in $\mathfrak{N}(F,G)$ and let $\gamma = \max(pr^{-1}, q^* r^{-1})$. We substitute the function $u = v^\gamma$ into (2.3.25). Then

$$\mu(F)^{1/q} \le cC\bigg(\int_\Omega v^{p(\gamma-1)}\big[\Phi(x,\nabla v)\big]^p\,\mathrm{d}x\bigg)^{(1-\varkappa)/p}\big\|v^\gamma\big\|^{\varkappa}_{L_r(\Omega,\mu)}. \tag{2.3.26}$$

Let $\psi(t)$ be the function defined in (2.3.2), where u is replaced by v. In our case $T=\max v=1$. Clearly,

$$\int_G v^{\gamma r}\,\mathrm{d}\mu = \int_0^\infty \mu(\mathscr{N}_t)\,\mathrm{d}(t^{\gamma r}) = \int_0^1 \mu(\mathscr{N}_t)\big[\psi(t)\big]^{q^*/p'}\frac{\mathrm{d}(t^{\gamma r})}{[\psi(t)]^{q^*/p'}},$$

where $\mathscr{N}_t=\{x\in G: v(x)\ge t\}$. Since $\mathscr{N}_t\supset F$, we have by Lemma 2.2.2/3

$$\mu(\mathscr{N}_t)\psi(t)^{q^*/p'} \le \frac{\mu(\mathscr{N}_t)}{[(p,\Phi)\text{-cap}(\mathscr{N}_t,G)]^{q^*/p}} \le \beta_\delta^{q^*/p}.$$

Hence

$$\int_G v^{\gamma r} \le \beta_\delta^{q^*/p}\int_0^1\big[\psi(t)\big]^{-q^*/p'}\,\mathrm{d}\big(t^{\gamma r}\big).$$

Since $[\psi(t)]^{-q^*/p'}$ is a nonincreasing function, from (1.3.41) we obtain

$$\begin{aligned}\int_G v^{\gamma r}\,\mathrm{d}\mu &\le \beta_\delta^{q*/p}\bigg(\int_0^1\big[\psi(t)\big]^{q^*(1-p)/\gamma r}\,\mathrm{d}\big(t^p\big)\bigg)^{\gamma r/p}\\ &\le \beta_\delta^{q^*/p}\psi(1)^{(\gamma r-q^*)/p'}\bigg(\int_0^1\frac{\mathrm{d}(t^p)}{[\psi(t)]^{p-1}}\bigg)^{\gamma r/p}.\end{aligned}$$

Setting $t=t(\psi)$ in the last integral and applying the inequality (2.3.8) and Lemma 2.3.1, we obtain

$$\int_0^{\psi(1)}\frac{\mathrm{d}[t(\psi)]^p}{\psi^{p-1}} \le c\int_0^{\psi(1)}\big[t'(\psi)\big]^p\,\mathrm{d}\psi = c\int_G\big[\Phi(x,\nabla v)\big]^p\,\mathrm{d}x.$$

Thus,

$$\begin{aligned}\|v^\gamma\|_{L_r(\Omega,\mu)} &\le c\beta_\delta^{q^*/pr}\psi(1)^{(\gamma r-q^*)/rp'}\bigg(\int_G\big[\Phi(x,\nabla v)\big]^p\,\mathrm{d}x\bigg)^{\gamma/p}\\ &\le c\beta_\delta^{q^*/pr}\big[(p,\Phi)\text{-cap}(F,G)\big]^{(q^*-\gamma r)/rp}\bigg(\int_G\big[\Phi(x,\nabla v)\big]^p\,\mathrm{d}x\bigg)^{\gamma/p}.\end{aligned} \tag{2.3.27}$$

The last inequality follows from the estimate

$$\big[\psi(1)\big]^{p-1} \le \big[(p,\Phi)\text{-cap}(F,G)\big]^{-1}$$

(see Lemma 2.2.2/3). Since $0\le v\le 1$ and $\gamma\ge 1$, from (2.3.26) and (2.3.27) it follows that

$$\mu(F)^{1/q} \le cC\beta_\delta^{q^*\varkappa/pr}\big[(p,\Phi)\text{-cap}(F,G)\big]^{\varkappa(q^*-\gamma r)/rp} \times \Big(\int_G \big[\Phi(x,\nabla v)\big]^p\,\mathrm{d}x\Big)^{[1+\varkappa(\gamma-1)]/p}.$$

Minimizing

$$\int_G \big[\Phi(x,\nabla v)\big]^p\,\mathrm{d}x$$

on the set $\mathfrak{P}(F,G)$, we obtain

$$\mu(F)^{1/q} \le cC\beta_\delta^{q^*\varkappa/pr}\big[(p,\Phi)\text{-cap}(F,G)\big]^{1/p+\varkappa(q^*-r)/pr} = cC\beta_\delta^{q^*\varkappa/pr}\big[(p,\Phi)\text{-cap}(F,G)\big]^{q^*/qp}.$$

Hence

$$\mu(F)^{p/q^*} \le cC^{qp/q^*}\beta_\delta^{(q^*-q)/(q^*-r)}(p,\Phi)\text{-cap}(F,G).$$

Consequently,

$$\beta_\delta \le cC^{pq(q^*-r)/q^*(q-r)} = cC^{p/(1-\varkappa)}.$$

Since β_δ is majorized by a constant that depends neither on δ nor G, using the property (iv) of the (p,Φ)-capacity we obtain $\beta \le cC^{p/(1-\varkappa)}$. The theorem is proved. □

Remark. The theorem just proved shows, in particular, the equivalence of the multiplicative inequality (2.3.25) and the Sobolev-type inequality (2.3.19).

2.3.7 Estimate for the Norm in $L_q(\Omega,\mu)$ with $q < p$ (First Necessary and Sufficient Condition)

A characterization of (2.3.19) with $q \ge p$ was stated in Corollary 2.3.4. Now we obtain a condition for the validity of (2.3.19), which is sufficient if $p > q > 0$ and necessary if $p > q \ge 1$.

Definition. Let $S = \{g_j\}_{j=-\infty}^{\infty}$ be any sequence of admissible subsets of Ω with $\bar g_i \subset g_{i+1}$. We put $\mu_i = \mu(g_i)$, $\gamma_i = (p,\Phi)\text{-cap}(\bar g_i, g_{i+1})$, and

$$\varkappa = \sup_{\{S\}}\Big[\sum_{i=-\infty}^{\infty}\Big(\frac{\mu_i^{p/q}}{\gamma_i}\Big)^{q/(p-q)}\Big]^{(p-q)/q}. \tag{2.3.28}$$

(The terms of the form $0/0$ are considered to be zeros.)

Theorem. (i) *If $\varkappa < \infty$, then*

$$\|u\|^p_{L_q(\Omega,\mu)} \le C\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x, \tag{2.3.29}$$

where $u \in \mathscr{D}(\Omega)$ and $p > q > 0$, $C \le c\varkappa$.

(ii) *If there exists a constant* C *such that* (2.3.29) *holds for all* $u \in \mathscr{D}(\Omega)$ *with* $p > q \geq 1$, *then* $\varkappa \leq cC$.

Proof. (i) Let $t_j = 2^{-j} + \varepsilon_j$, $j = 0, \pm 1, \pm 2, \ldots$, where ε_j is a decreasing sequence of positive numbers satisfying $\varepsilon_j 2^j \to 0$ as $j \to \pm\infty$. We assume further that the sets $\mathscr{L}_{t_j}$ are admissible. Obviously,

$$\|u\|^q_{L_q(\Omega,\mu)} = \sum_{j=-\infty}^{\infty} \int_{t_j}^{t_{j-1}} \mu(\mathscr{L}_t)\, \mathrm{d}\big(t^q\big) \leq c \sum_{j=-\infty}^{\infty} 2^{-qj} \mu(\mathscr{L}_{t_j}).$$

Let $g_j = \mathscr{L}_{t_j}$. We rewrite the last sum as

$$c \sum_{j=-\infty}^{\infty} \left(\frac{\mu_j^{p/q}}{\gamma_j}\right)^{q/p} \big(2^{-pj}\gamma_j\big)^{q/p}$$

and apply Hölder's inequality. Then

$$\|u\|^q_{L_q(\Omega,\mu)} \leq c\varkappa^{q/p} \left(\sum_{j=-\infty}^{\infty} 2^{-pj}\gamma_j\right)^{q/p}.$$

Let $\lambda_\varepsilon \in C^\infty(\mathbb{R}^1)$, $\lambda_\varepsilon(t) = 1$ for $t \geq 1$, $\lambda_\varepsilon(t) = 0$ for $t \leq 0$, $0 \leq \lambda'_\varepsilon(t) \leq 1 + \varepsilon$, $(\varepsilon > 0)$ and let

$$u_j(x) = \lambda_\varepsilon\left[\frac{|u(x)| - t_{j+1}}{t_j - t_{j+1}}\right].$$

Since $u_j \in \mathfrak{N}(\bar{g}_j, g_{j+1})$, it follows that

$$\begin{aligned}\sum_{j=-\infty}^{\infty} 2^{-pj}\gamma_j &\leq c \sum_{j=-\infty}^{\infty} (t_j - t_{j+1})^p \int_{g_{j+1}\setminus g_j} \big[\Phi(x, \nabla u_j)\big]^p\, \mathrm{d}x \\ &= c \sum_{j=-\infty}^{\infty} \int_{g_{j+1}\setminus g_j} \left[\lambda'_\varepsilon\left(\frac{u - t_{j+1}}{t_j - t_{j+1}}\right)\right]^p \big[\Phi(x, \nabla u)\big]^p\, \mathrm{d}x.\end{aligned}$$

Letting ε tend to zero, we obtain

$$\sum_{j=-\infty}^{\infty} 2^{-pj}\gamma_j \leq c \int_\Omega \big[\Phi(x, \nabla u)\big]^p\, \mathrm{d}x. \tag{2.3.30}$$

(ii) We introduce the sequence

$$S = \{g_j\}_{j=-\infty}^{\infty}$$

and put $\tau_{N+1} = 0$ and

$$\tau_k = \sum_{j=k}^{N} \left(\frac{\mu_j}{\gamma_j}\right)^{1/(p-q)}$$

for $k = -N, -N+1, \dots, 0, \dots, N-1, N$. By u_k we denote an arbitrary function in $\mathfrak{P}(\bar{g}_k, g_{k+1})$ and define the function

$$u_k = (\tau_k - \tau_{k+1})u_k + \tau_{k+1} \quad \text{on } g_{k+1}\backslash g_k,$$

$$u = \tau_{-N} \quad \text{on } g_{-N}, \qquad u = 0 \quad \text{on } \Omega\backslash g_{N+1}.$$

Since $u \in \mathscr{D}(\Omega)$, it satisfies (2.3.29). Obviously,

$$\int_\Omega u^q \,\mathrm{d}\mu = v\int_0^\infty \mu(\mathscr{L}_t)\,\mathrm{d}\big(t^q\big)$$
$$= \sum_{k=-N}^{N} \int_{\tau_{k+1}}^{\tau_k} \mu(\mathscr{L}_t)\,\mathrm{d}\big(t^q\big) \geq \sum_{k=-N}^{N} \mu_k\big(\tau_k^q - \tau_{k+1}^q\big).$$

Therefore, (2.3.29) and the inequality $(\tau_k - \tau_{k+1})^q \leq (\tau_k^q - \tau_{k+1}^q)$ implies

$$\left[\sum_{k=-N}^{N} \mu_k(\tau_k - \tau_{k+1})^q\right]^{p/q} \leq C \sum_{k=-N}^{N} \int_{g_{k+1}\backslash g_k} \big[\Phi(x, \nabla u_k)\big]^p \,\mathrm{d}x$$
$$= C \sum_{k=-N}^{N} (\tau_k - \tau_{k+1})^p \int_{g_{k+1}\backslash g_k} \big[\Phi(x, \nabla u_k)\big]^p \,\mathrm{d}x.$$

Since u_k is an auxiliary function in $\mathfrak{P}(\bar{g}_k, g_{k+1})$, it follows by minimizing the last sum that

$$\left[\sum_{k=-N}^{N} \mu(\tau_k - \tau_{k+1})^q\right]^{p/q} \leq C \sum_{k=-N}^{N} (\tau_k - \tau_{k+1})^p \gamma_k.$$

Putting here

$$\tau_k - \tau_{k+1} = \mu_k^{1/(p-q)} \gamma_k^{1/(q-p)},$$

we arrive at the result

$$\left|\sum_{k=-N}^{N} \big(\mu_k^{p/q}\gamma_k\big)^{q/(p-q)}\right|^{(p-q)/q} \leq C.$$

□

2.3.8 Estimate for the Norm in $L_q(\Omega, \mu)$ with $q < p$ (Second Necessary and Sufficient Condition)

Lemma. *Let g_1, g_2, and g_3 be admissible subsets of Ω such that $\bar{g}_1 \subset g_2$, $\bar{g}_2 \subset g_3$. We set*

$$\gamma_{ij} = (p, \Phi)\text{-cap}(\bar{g}_i, g_j),$$

where $i < j$. Then

$$\gamma_{12}^{-1/(p-1)} + \gamma_{23}^{-1/(p-1)} \le \gamma_{13}^{-1/(p-1)}.$$

Proof. Let ε be any positive number. We choose functions $u_k \in \mathfrak{P}(\bar{g}_k, g_{k+1})$, $k = 1, 2$, so that

$$\gamma_{k,k+1}^{-1/(p-1)} \le \int_0^1 \left[\int_{\mathscr{E}_\tau^k} \left[\Phi(x, \nabla u_k)\right]^p \frac{\mathrm{d}s}{|\nabla u_k|} \right]^{-1/(p-1)} \mathrm{d}\tau + \varepsilon,$$

where $\mathscr{E}_\tau^k = \{x : u_k(x) = \tau\}$. We put $u(x) = \frac{1}{2} u_2(x)$ for $x \in g_3 \backslash g_2$ and $u(x) = (u_1(x) + 1)/2$ for $x \in g_2$. Then

$$\int_0^1 \left[\int_{\mathscr{E}_\tau^1} \left[\Phi(x, \nabla u_1)\right]^p \frac{\mathrm{d}s}{|\nabla u_1|} \right]^{1/(1-p)} \mathrm{d}\tau = \int_{1/2}^1 \left[\int_{\mathscr{E}_\tau} \left[\Phi(x, \nabla u)\right]^p \frac{\mathrm{d}s}{|\nabla u|} \right]^{1/(1-p)} \mathrm{d}\tau,$$

$$\int_0^1 \left[\int_{\mathscr{E}_\tau^2} \left[\Phi(x, \nabla u_2)\right]^p \frac{\mathrm{d}s}{|\nabla u_2|} \right]^{1/(1-p)} \mathrm{d}\tau = \int_0^{1/2} \left[\int_{\mathscr{E}_\tau} \left[\Phi(x, \nabla u)\right]^p \frac{\mathrm{d}s}{|\nabla u|} \right]^{1/(1-p)} \mathrm{d}\tau,$$

where $\mathscr{E}_\tau = \{x : u(x) = \tau\}$. Therefore,

$$\gamma_{12}^{1/(1-p)} + \gamma_{23}^{1/(1-p)} \le \int_0^1 \left(\int_{\mathscr{E}_\tau} \left[\Phi(x, \nabla u)\right]^p \frac{\mathrm{d}s}{|\nabla u|} \right)^{1/(1-p)} \mathrm{d}\tau + 2\varepsilon.$$

Since $u \in \mathfrak{P}(\bar{g}_1, g_3)$, by Lemma 2.2.2/3 the right-hand side of the last inequality does not exceed $\gamma_{13}^{1/(1-p)} + 2\varepsilon$. The lemma is proved. □

Let ν_p be the capacity minimizing function introduced in Definition 2.3.2. It can be easily checked that condition (2.3.15) is equivalent to

$$\beta \nu_p(t) \ge t P^{-1}(1/t)$$

and condition (2.3.18) to

$$\beta \nu_p(t) \ge t^{\alpha p}.$$

The theorem of the present subsection yields the following necessary and sufficient condition for the validity of (2.3.29) with $p > q > 0$:

$$K = \int_0^{\mu(\Omega)} \left[\frac{\tau}{\nu_p(\tau)} \right]^{q/(p-q)} \mathrm{d}\tau < \infty. \tag{2.3.31}$$

Theorem. *Let $p > q > 0$, $p > 1$.*
1. *If $K < \infty$, then (2.3.29) holds for all $u \in \mathscr{D}(\Omega)$ with $C \le cK^{(p-q)/q}$.*
2. *If (2.3.29) holds with $C < \infty$, then (2.3.31) is valid with $K^{(p-q)/q} \le cC$.*

Proof. 1. By Theorem 2.3.7 it suffices to prove the inequality

$$\sup_{\{S\}} \sum_{j=-\infty}^{\infty} \left(\frac{\mu_j^{p/q}}{\gamma_j} \right)^{q/(p-q)} \le \frac{p}{p-q} \int_0^{\mu(\Omega)} \left[\frac{\tau}{\nu_p(\tau)} \right]^{q/(p-q)} \mathrm{d}\tau, \tag{2.3.32}$$

where the notation of Sect. 2.3.7 is retained.

Let the integral in the right-hand side converge, let N be a positive integer, and let $\Gamma_j = (p, \Phi)\text{-cap}(\bar{g}_j, g_{N+1})$ for $j \le N$, $\Gamma_{N+1} = \infty$. By the Lemma,

$$\gamma_j^{1/(1-p)} \le \Gamma_j^{1/(1-p)} - \Gamma_{j+1}^{1/(1-p)}, \quad j \le N.$$

Since $q(p-1)/(p-q) \ge 1$, then

$$|a-b|^{q(p-1)/(p-q)} \le \left| a^{q(p-1)/(p-q)} - b^{q(p-1)/(p-q)} \right|$$

and hence

$$\gamma_j^{-q/(p-q)} \le \Gamma_j^{-q/(p-q)} - \Gamma_{j+1}^{-q/(p-q)}.$$

This implies

$$\begin{aligned} \sigma_N &\stackrel{\text{def}}{=} \sum_{j=-N}^{N} \left(\frac{\mu_j^{p/q}}{\gamma_j} \right)^{q/(p-q)} \le \sum_{j=-N}^{N} \mu_j^{p/(p-q)} \left(\Gamma_j^{-q/(p-q)} - \Gamma_{j+1}^{-q/(p-q)} \right) \\ &\le \sum_{j=-N+1}^{N} \left(\mu_j^{p/(p-q)} - \mu_{j-1}^{p/(p-q)} \right) \Gamma_j^{-q/(p-q)} + \mu_{-N}^{p/(p-q)} \Gamma_{-N}^{-q/(p-q)}. \end{aligned}$$

It is clear that $\Gamma_j \ge (p, \Phi)\text{-cap}(\bar{g}_j, \Omega) \ge \nu_p(\mu_j)$. Since the function ν_p does not decrease then

$$\mu_{-N}^{p/(p-q)} \left[\nu_p(\mu_{-N}) \right]^{q/(p-q)} \le \int_0^{\mu_{-N}} \frac{\mathrm{d}\left(\tau^{p/(p-q)}\right)}{[\nu_p(\tau)]^{q/(p-q)}}.$$

Similarly,

$$\left(\mu_j^{p/(p-q)} - \mu_{j-1}^{p/(p-q)} \right) \left[\nu_p(\mu_j) \right]^{q/(p-q)} \le \int_{\mu_{j-1}}^{\mu_j} \frac{\mathrm{d}\left(\tau^{p/(p-q)}\right)}{[\nu_p(\tau)]^{q/(p-q)}}.$$

Consequently,

$$\sigma_N \le \int_0^{\mu_N} \left[\nu_p(\tau) \right]^{q/(q-p)} \mathrm{d}\left(\tau^{p/(p-q)} \right).$$

The result follows.

2. With a μ-measurable function f we connect its nonincreasing rearrangement

$$f_\mu^*(t) = \inf\left\{ s : \mu\left\{ x \in \Omega : f(x) \ge s \right\} \le t \right\}. \tag{2.3.33}$$

By Lemma 2.1.4/1

$$\|f\|_{L_q(\Omega,\mu)} = \left(\int_0^{\mu(\Omega)} \big(f_\mu^*(t)\big)^q \,\mathrm{d}t\right)^{1/q}, \quad 0 < q < \infty. \tag{2.3.34}$$

We note that the inequality (2.3.29) implies that $\mu(\Omega) < \infty$ and that $\nu_p(t) > 0$ for all $t \in (0, \mu(\Omega)]$. Let l be any integer satisfying $2^l \le \mu(\Omega)$. We introduce an admissible subset g_l of Ω such that

$$\mu(\bar{g}_l) \ge 2^l, \qquad (p,\Phi)\text{-cap}(g_l,\Omega) \le 2\nu_p\big(2^l\big).$$

By u_l we denote a function in $\mathfrak{P}(\bar{g}_l,\Omega)$ satisfying

$$\int_\Omega \big[\Phi(x,\nabla u_l)\big]^p \,\mathrm{d}x \le 4\nu_p\big(2^l\big). \tag{2.3.35}$$

Let s be the integer for which $2^s \le \mu(\Omega) < 2^{s+1}$. We define the function in $C_0^{0,1}(\Omega)$

$$f_{r,s}(x) = \sup_{r\le l\le s} \beta_l u_l(x), \quad x \in \Omega,$$

where $r < s$ and the values β_l are defined by

$$\beta_l = \left(\frac{2^l}{\nu_p(2^l)}\right)^{1/(p-q)}.$$

By Lemma 2.1.4/2 and by Lemma 2.1.4/3 with Φ^p instead of Φ we have

$$\int_\Omega \big[\Phi(x,\nabla f_{r,s})\big]^p \,\mathrm{d}x \le \sum_{l=r}^s \beta_l^p \int_\Omega \big[\Phi(x,\nabla u_l)\big]^p \,\mathrm{d}x.$$

By (2.3.35) the right-hand side is majorized by

$$c\sum_{l=r}^s \beta_l^p \nu_p\big(2^l\big).$$

Now we obtain a lower estimate for the norm of $f_{r,s}$ in $L_q(\Omega,\mu)$. Since $f_{r,s}(x) \ge \beta_l$ on the set g_l, $r \le l \le s$, and $\mu(\bar{g}_l) \ge 2^l$, the inequality

$$\mu\big(\big\{x \in \Omega : \big|f_{r,s}(x)\big| > \tau\big\}\big) < 2^l$$

implies $\tau \ge \beta_l$. Hence

$$f_{r,s}^*(t) \ge \beta_l \quad \text{for } t \in \big(0, 2^l\big), r \le l \le s. \tag{2.3.36}$$

By (2.3.34) and (2.3.36)

$$\|f_{r,s}\|^q_{L_q(\Omega,\mu)} = \int_0^{\mu(\Omega)} \big((f_{r,s})_\mu^*(t)\big)^q \,\mathrm{d}t \ge c\sum_{l=r}^s \big((f_{r,s})_\mu^*(2^l)\big)^q 2^l \ge c\sum_{l=r}^s \beta_l^q 2^l.$$

Therefore,

$$B := \sup \frac{\|u\|_{L_q(\Omega,\mu)}}{(\int_\Omega [\Phi(x,\nabla u)]^p \,\mathrm{d}x)^{1/p}} \geq c \frac{(\sum_{l=r}^s \beta_l^q 2^l)^{1/q}}{(\sum_{l=r}^s \beta_l^p \nu_p(2^l))^{1/p}}$$
$$= c\left(\sum_{l=r}^s \frac{2^{lp/(p-q)}}{\nu_p(2^l)^{q/(p-q)}}\right)^{1/q-1/p}.$$

With $r \to -\infty$ we obtain

$$B \geq \left(\sum_{l=-\infty}^s \frac{2^{lp/(p-q)}}{\nu_p(2^l)^{q/(p-q)}}\right)^{1/q-1/p}$$
$$\geq c\left(\int_0^{2^{s-1}} \sum_{l=-\infty}^s \frac{t^{p/(p-q)}}{(\nu_p(t))^{q/(p-q)}} \frac{\mathrm{d}t}{t}\right)^{1/q-1/p}.$$

Hence by monotonicity of ν_p we obtain

$$B \geq c\left(\int_0^{\mu(\Omega)} \frac{t^{p/(p-q)}}{(\nu_p(t))^{q/(p-q)}} \frac{\mathrm{d}t}{t}\right)^{1/q-1/p}.$$

The proof is complete. □

We give a sufficient condition for inequality (2.3.29) with $\mu = m_n$ formulated in terms of the weighted isoperimetric function $\mathscr{C}$ introduced in Definition 2.2.3.

Corollary. *If $p > q > 0$, $p > 1$, and*

$$\int_0^{m_n(\Omega)} \left(\int_t^{m_n(\Omega)} \frac{\mathrm{d}\varrho}{(\mathscr{C}(\varrho))^{p/(p-1)}}\right)^{\frac{q(p-1)}{p-q}} t^{\frac{q}{p-q}} \,\mathrm{d}t < \infty,$$

then (2.3.29) with $\mu = m_n$ and any $u \in \mathscr{D}(\Omega)$ holds.

Proof. The result follows directly from the last Theorem and Corollary 2.7.2. □

2.3.9 Inequality with the Norms in $L_q(\Omega, \mu)$ and $L_r(\Omega, \nu)$ (the Case $p \geq 1$)

The next theorem gives conditions for the validity of the inequality

$$\|u\|^p_{L_q(\Omega,\mu)} \leq C\left(\int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x + \|u\|^p_{L_r(\Omega,\nu)}\right) \tag{2.3.37}$$

for all $u \in \mathscr{D}(\Omega)$ with $q \geq p \geq r$, $p > 1$ (compare with Theorem 2.1.3).

Theorem. *Inequality* (2.3.37) *holds if and only if*

$$\mu(g)^{p/q} \le cC\big[(p,\Phi)\text{-cap}(\bar g,\mathscr{G}) + \big[\nu(\mathscr{G})\big]^{p/r}\big] \tag{2.3.38}$$

for all admissible sets g and $\mathscr{G}$ with $\bar g \subset \mathscr{G}$.

Proof. Sufficiency. By Lemma 1.2.3 and inequality (1.3.41),

$$\|u\|^p_{L_q(\Omega,\mu)} = \Big[\int_0^\infty \mu(\mathscr{L}_t)\,\mathrm{d}\big(t^q\big)\Big]^{p/q}$$
$$\le \int_0^\infty \big[\mu(\mathscr{L}_t)\big]^{p/q}\,\mathrm{d}\big(t^p\big) \le c\sum_{j=-\infty}^{\infty} 2^{-pj}\mu(g_j)^{p/q},$$

where $g_j = \mathscr{L}_{t_j}$ and $\{t_j\}$ is the sequence of levels defined in the proof of part (i) of Theorem 2.3.7. We set $\gamma_j = (p,\Phi)\text{-cap}(\bar g_j, g_{j+1})$ and using the condition (2.3.38), we arrive at the inequality:

$$\|u\|^p_{L_q(\Omega,\mu)} \le cC\Big[\sum_{j=-\infty}^{\infty} 2^{-pj}\gamma_j + \sum_{j=-\infty}^{\infty} 2^{-pj}\nu(g_j)^{p/r}\Big]. \tag{2.3.39}$$

We can estimate the first sum on the right-hand side of this inequality by means of (2.3.30). The second sum does not exceed

$$c\int_0^\infty \big[\nu(\mathscr{L}_t)\big]^{p/r}\,\mathrm{d}\big(t^p\big) \le c\Big(\int_0^\infty \nu(\mathscr{L}_t)\,\mathrm{d}\big(t^r\big)\Big)^{p/r} = c\|u\|^p_{L_r(\Omega,\nu)}.$$

Necessity. Let g and $\mathscr{G}$ be admissible and let $\bar g \subset \mathscr{G}$. We substitute any function $u \in \mathfrak{P}(\bar g,\mathscr{G})$ into (2.3.37). Then

$$\mu(g)^{p/q} \le C\Big[\int_\Omega \big[\Phi(x,\nabla u)\big]^p\,\mathrm{d}x + \nu(\mathscr{G})^{p/r}\Big].$$

Minimizing the first term on the right of the set $\mathfrak{P}(\bar g, G)$, we obtain (2.3.38). □

Remark. Obviously, a sufficient condition for the validity of (2.3.37) is the inequality

$$\mu(g)^{p/q} \le C_1\big[(p,\Phi)\text{-cap}(g,\Omega) + \nu(g)^{p/r}\big], \tag{2.3.40}$$

which is simpler than (2.3.38). In contrast to (2.3.37) it contains only one set g. However, as the following example shows, the last condition is not necessary.

Let $\Omega = \mathbb{R}^3$, $q = p = r = 2$, $\Phi(x,y) = |y|$, and let the measures μ and ν be defined as follows:

$$\mu(A) = \sum_{k=0}^{\infty} s(A \cap \partial B_{2^k}),$$
$$\nu(A) = \sum_{k=0}^{\infty} s(A \cap \partial B_{2^k+1}),$$

where A is any Borel subset of $\mathbb{R}^3$ and s is a two-dimensional Hausdorff measure. The condition (2.3.40) is not fulfilled for these measures and the 2-capacity. Indeed, for the sets $g_k = B_{2^k+1} \backslash \bar{B}_{2^k-1}$, $k = 2, 3, \ldots$, we have $\mu(g_k) = \pi 4^{k+1}$, $\nu(g_k) = 0$, 2-cap$(g_k, \mathbb{R}^3) = 4\pi(2^k + 1)$.

We shall show that (2.3.37) is true. Let $u \in \mathscr{D}(\mathbb{R}^3)$ and let (ϱ, ω) be spherical coordinates with center O. Obviously,

$$\left[u\left(2^k, \omega\right)\right]^2 \le 2 \int_{2^k}^{2^k+1} \left(\frac{\partial u}{\partial \varrho}(\varrho, \omega)\right)^2 \mathrm{d}\varrho + 2\left[u\left(2^k + 1, \omega\right)\right]^2.$$

Hence

$$4^k \int_{\partial B_1} \left[u\left(2^k, \omega\right)\right]^2 \mathrm{d}\omega \le 2 \int_{B_{2^k+1} \backslash B_{2^k}} \left(\frac{\partial u}{\partial \varrho}\right)^2 \mathrm{d}x + 2 \cdot 4^k \int_{\partial B_1} \left[u\left(2^k + 1, \omega\right)\right]^2 \mathrm{d}\omega.$$

Summing over k, we obtain

$$\int_{\mathbb{R}^3} u^2 \, \mathrm{d}\mu \le c\left(\int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^3} u^2 \, \mathrm{d}\nu\right).$$

The proof is complete.

2.3.10 Estimate with a Charge σ on the Left-Hand Side

The following assertion yields a condition close in a certain sense to being necessary and sufficient for the validity of the inequality

$$\int_{\Omega} |u|^p \, \mathrm{d}\sigma \le c \int_{\Omega} \left[\Phi(x, \nabla u)\right]^p \mathrm{d}x, \quad u \in \mathscr{D}(\Omega), \tag{2.3.41}$$

where σ is an arbitrary charge in Ω, not a nonnegative measure as in Theorem 2.3.6. (Theorem 2.1.3 contains a stronger result for $p = 1$.)

Theorem. *Let σ^+ and σ^- be the positive and negative parts of the charge σ.*

1. *If for some $\varepsilon \in (0, 1)$ and for all admissible sets g and $\mathscr{G}$ with $\bar{g} \subset \mathscr{G}$ we have the inequality*

$$\sigma^-(g) \le C_\varepsilon(p, \Phi)\text{-cap}(\bar{g}, G) + (1 - \varepsilon)\sigma^-(G), \tag{2.3.42}$$

where $C_\varepsilon = \text{const}$, then (2.3.41) is valid with $C \le cC_\varepsilon$.

2. *If for all $u \in \mathscr{D}(\Omega)$ inequality (2.3.41) holds, then*

$$\sigma^+(g) \le C(p, \Phi)\text{-cap}(\bar{g}, \mathscr{G}) + \sigma^-(\mathscr{G}) \tag{2.3.43}$$

for all admissible sets g and $\mathscr{G}$, $\bar{g} \subset \mathscr{G}$.

Proof. Let $\delta = (1 - \varepsilon)^{-1/2p}$ and $g_j = \mathscr{L}_{\delta^j}$, $j = 0, \pm 1, \ldots$. By Lemma 1.2.3,

$$\|u\|^p_{L_p(\Omega,\sigma^+)} = \int_0^\infty \sigma^+(\mathscr{L}_t)\,\mathrm{d}\big(t^p\big) = \sum_{j=-\infty}^{\infty} \int_{\delta^j}^{\delta^{j+1}} \sigma^+(\mathscr{L}_t)\,\mathrm{d}\big(t^p\big)$$
$$\leq \sum_{j=-\infty}^{\infty} \sigma^+(\mathscr{L}_{\delta^j})\big(\delta^{(j+1)p} - \delta^{jp}\big).$$

This and (2.3.42) imply

$$\|u\|^p_{L_p(\Omega,\sigma^+)} \leq C_\varepsilon \sum_{j=-\infty}^{\infty} (p,\Phi)\text{-cap}(\bar{\mathscr{L}}_{\delta^j}, \mathscr{L}_{\delta^{j-1}})\big(\delta^{(j+1)p} - \delta^{jp}\big)$$
$$+ (1-\varepsilon) \sum_{j=-\infty}^{\infty} \sigma^-(\mathscr{L}_{\delta^{j-1}})\big(\delta^{(j+1)p} - \delta^{jp}\big). \tag{2.3.44}$$

Using the same arguments as in the derivation of (2.3.30), we obtain that the first sum in (2.3.44) does not exceed

$$\frac{(\delta^p - 1)\delta^p}{(\delta - 1)^p} \int_\Omega \big[\Phi(x, \nabla u)\big]^p\,\mathrm{d}x.$$

Since $\sigma^-(\mathscr{L}_t)$ is a nondecreasing function, then

$$\big(\delta^{(j-1)p} - \delta^{(j-2)p}\big)\sigma^-(\mathscr{L}_{\delta^{j-1}}) \leq \int_{\delta^{j-2}}^{\delta^{j-1}} \sigma^-(\mathscr{L}_t)\,\mathrm{d}\big(t^p\big)$$

and hence

$$\sum_{j=-\infty}^{\infty} \sigma^-(\mathscr{L}_{\delta^{j-1}})\big(\delta^{(j+1)p} - \delta^{jp}\big) \leq \delta^{2p} \int_0^\infty \sigma^-(\mathscr{L}_t)\,\mathrm{d}\big(t^p\big).$$

Thus

$$\|u\|^p_{L_p(\Omega,\sigma^+)} \leq C_\varepsilon \frac{(\delta^p - 1)\delta^p}{(\delta - 1)^p} \int_\Omega \big[\Phi(x, \nabla u)\big]^p\,\mathrm{d}x + \delta^{2p}(1-\varepsilon)\|u\|^p_{L_p(\Omega,\sigma^+)}.$$

It remains to note that $\delta^{2p}(1-\varepsilon) = 1$.

2. The proof of the second part of the theorem is the same as that of necessity in Theorem 2.3.9. The theorem is proved. □

2.3.11 Multiplicative Inequality with the Norms in $L_q(\Omega, \mu)$ and $L_r(\Omega, \nu)$ (Case $p \geq 1$)

The following assertion gives a necessary and sufficient condition for the validity of the multiplicative inequality

$$\|u\|^p_{L_q(\Omega,\mu)} \leq C\left\{\int_\Omega \big[\Phi(x, \nabla u)\big]^p\,\mathrm{d}x\right\}^\delta \|u\|^{p(1-\delta)}_{L_r(\Omega,\nu)} \tag{2.3.45}$$

for $p \geq 1$ (cf. Theorem 2.1.1).

Theorem. 1. *Let g and $\mathscr{G}$ be any admissible sets such that $\bar{g} \subset \mathscr{G}$. If a constant α exists such that*

$$\mu(g)^{p/q} \leq \alpha\big[(p,\varPhi)\text{-cap}(\bar{g},\mathscr{G})\big]^{\delta}\nu(\mathscr{G})^{(1-\delta)p/r}, \tag{2.3.46}$$

then (2.3.45) holds for all functions $u \in \mathscr{D}(\varOmega)$ with $C \leq c\alpha$, $1/q \leq (1-\delta)/r + \delta/p$, $r, q > 0$.

2. *If (2.3.45) is true for all $u \in \mathscr{D}(\varOmega)$, then (2.3.46) holds for all admissible sets g and $\mathscr{G}$ such that $\bar{g} \subset \mathscr{G}$. The constant α in (2.3.46) satisfies $\alpha \leq C$.*

Proof. 1. By Lemma 1.2.3 and inequality (1.3.41),

$$\|u\|_{L_q(\varOmega,\mu)} = \bigg[\int_0^\infty \mu(\mathscr{L}_\tau)\,\mathrm{d}\big(\tau^q\big)\bigg]^{1/q} \leq \gamma^{1/\gamma}\bigg[\int_0^\infty \mu(\mathscr{L}_\tau)^{\gamma/q}\tau^{\gamma-1}\,\mathrm{d}\tau\bigg]^{1/\gamma},$$

where $\gamma = pr[p(1-\delta)+\delta r]^{-1}$, $\gamma \leq q$. Consequently,

$$\begin{aligned}\|u\|^p_{L_q(\varOmega,\mu)} &\leq c\bigg[\sum_{j=-\infty}^{\infty} 2^{-\gamma j}\mu(g_j)^{\gamma/q}\bigg]^{p/\gamma}\\ &\leq c\alpha\bigg\{\sum_{j=-\infty}^{\infty} 2^{-\gamma j}\big[(p,\varPhi)\text{-cap}(\bar{g}_j,g_{j+1})\big]^{\delta\gamma/p}\nu(g_{j+1})^{(1-\delta)\gamma/r}\bigg\}^{p/\gamma},\end{aligned}$$

where $g_j = \mathscr{L}_{t_j}$ and $\{t_j\}$ is the sequence of levels defined in the proof of the first part of Theorem 2.3.7. Hence,

$$\begin{aligned}\|u\|^p_{L_q(\varOmega,\mu)} \leq c\alpha&\bigg[\sum_{j=-\infty}^{\infty} 2^{-pj}(p,\varPhi)\text{-cap}(\bar{g}_j,g_{j+1})\bigg]^{\delta}\\ &\times\bigg[\sum_{j=-\infty}^{\infty} 2^{-rj}\nu(g_{j+1})\bigg]^{(1-\delta)p/r}.\end{aligned} \tag{2.3.47}$$

By (2.3.30),

$$\sum_{j=-\infty}^{\infty} 2^{-pj}(p,\varPhi)\text{-cap}(\bar{g}_j,g_{j+1}) \leq c\int_\varOmega \big[\varPhi(x,\nabla u)\big]^p\,\mathrm{d}x.$$

Obviously, the second sum in (2.3.47) does not exceed $c\|u\|^r_{L_r(\varOmega,\nu)}$. Thus (2.3.45) follows.

2. Let g and $\mathscr{G}$ be admissible sets with $\bar{g} \subset \mathscr{G}$. We substitute any function $u \in \mathfrak{P}(\bar{g},\mathscr{G})$ into (2.3.45). Then

$$\mu(g)^{p/q} \leq C\bigg[\int_\varOmega \big[\varPhi(x,\nabla u)\big]^p\,\mathrm{d}x\bigg]^{\delta}\nu(\mathscr{G})^{(1-\delta)p/r},$$

which yields (2.3.46). The theorem is proved. □

2.3.12 On Nash and Moser Multiplicative Inequalities

An important role in Nash's classical work on local regularity of solutions to second-order parabolic equations in divergence form with measurable bounded coefficients [625] is played by the multiplicative inequality

$$\left(\int_{\mathbb{R}^n} u^2 \, dx\right)^{1+2/n} \le C \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \left(\int_{\mathbb{R}^n} |u| \, dx\right)^{4/n}, \quad u \in C_0^\infty. \tag{2.3.48}$$

Another inequality of a similar nature

$$\|u\|_{L_{2+4/n}}^{1+2/n} \le c\|u\|_{L_2}^{2/n} \|\nabla u\|_{L_2}, \quad u \in C_0^\infty, \tag{2.3.49}$$

was used by Moser in his proof of the Harnack inequality for solutions of second-order elliptic equations with measurable bounded coefficients in divergence form [617].

These two inequalities are contained as very particular cases in the Gagliardo–Nirenberg inequality for all $u \in C_0^\infty(\mathbb{R}^n)$

$$\|\nabla_j u\|_{L_q} \le c\|\nabla_l u\|_{L_p}^\alpha \|u\|_{L_r}^{1-\alpha}, \tag{2.3.50}$$

where $1 \le p, r \le \infty$, $0 \le j < l$, $j/l \le \alpha, \le 1$, and

$$\frac{1}{q} = \frac{j}{n} + \alpha\left(\frac{1}{p} - \frac{l}{n}\right) + \frac{1-\alpha}{r}.$$

If $1 < p < \infty$ and $l - j - n/p$ is a nonnegative integer then (2.3.50) holds only for $\alpha \in [j/n, 1)$ (see Gagliardo [299] and Nirenberg [640]).

If $n > 2$, the Nash and Moser inequalities follow directly by the Hölder inequality from the Sobolev inequality

$$\|u\|_{L_{2n/(n-2)}} \le c\|\nabla u\|_{L_2}, \quad u \in C_0^\infty. \tag{2.3.51}$$

We know by the second part of Theorem 2.3.6 that conversely, (2.3.48) and (2.3.49) imply (2.3.51). The just-mentioned theorem does not contain (2.3.48) and (2.3.49) for $n = 2$, which formally corresponds to the exceptional case $\alpha = 0$. However, we show here that both (2.3.48) and (2.3.49) with $n = 2$, and even the more general inequality

$$\int_{\mathbb{R}^n} |u|^q \, dx \le c\left(\int_{\mathbb{R}^n} |\nabla u|^n \, dx\right)^{\frac{q-r}{n}} \int_{\mathbb{R}^n} |u|^r \, dx, \tag{2.3.52}$$

where $q \ge r > 0$ can be deduced from Theorem 2.3.11. In fact, by this theorem, (2.3.52) holds if and only if

$$m_n(g) \le \text{const } \left(\text{cap}_n(\bar{g}, G)\right)^{\frac{q-r}{n}} m_n(G), \tag{2.3.53}$$

where g and G are arbitrary bounded open sets with smooth boundaries, $\bar{g} \subset G$, and cap_n is the n capacity of $\bar{g}$ with respect to G. By the isocapacitary inequality (2.2.11),

$$\mathrm{cap}_n(\bar{g}, G) \geq n^{n-1}\omega_n \left(\log \frac{m_n(G)}{m_n(g)}\right)^{1-n}.$$

Hence (2.3.53) is a consequence of the boundedness of the function

$$(0,1) \ni x \to x\left(\log \frac{1}{x}\right)^{\frac{(q-r)(n-1)}{n}},$$

which, in its turn, implies the multiplicative inequality (2.3.52). □

The original proof of (2.3.52) (see Nirenberg [640], p. 129) is as follows. First, one notes that it suffices to obtain (2.3.52) for large q. Then (2.3.52) results by putting $|u|^{q(1-n)/n}$ instead of u into (1.4.49) and using an appropriate Hölder's inequality.

Extensions of Nash's inequality (2.3.48) to weighted inequalities with indefinite weights on the left-hand side were obtained by Maz'ya and Verbitsky [594] with simultaneously necessary and sufficient conditions on the weights.

2.3.13 Comments to Sect. 2.3

The basic results of Sects. 2.3.1–2.3.4 were obtained by the author in [531, 534] for $p = 2$, $\Phi(x,\xi) = |\xi|$, $M(u) = |u|$, and in [543] for the general case. Some of these results were repeated by Stredulinsky [729]. We shall return to capacitary inequalities similar to (2.3.6) in Chaps. 3 and 11. The inequality (2.3.14) can be found also in Brezis and Wainger [146] and Hansson [348].

Regarding the criterion in Sect. 2.3.3, see Comments to Sect. 2.4, where other optimal embeddings of Birnbaum–Orlicz–Sobolev spaces into C and Birnbaum–Orlicz spaces are discussed.

Inequality (2.3.21) is (up to a constant) the Sobolev ($p > 1$)-Gagliardo–Nirenberg ($p = 1$) inequality. The best constant for the case $p = 1$ (see (1.4.14)) was found independently by Federer and Fleming [273] and by the author [527] using the same method.

The best constant for $p > 1$, presented in Sect. 2.3.5 was obtained by Aubin [55] and Talenti [740] (the case $n = 3$, $p = 2$ was considered earlier by Rosen [682]), whose proofs are a combination of symmetrization and the one-dimensional Bliss inequality [109] (see Sect. 2.3.5). The uniqueness of the Bliss optimizer was proved by Gidas, Ni, and Nirenberg [307].

A different approach leading to the best constant in the Sobolev inequality, which is based on the geometric Brunn–Minkowski–Lyusternik inequality, was proposed in Bobkov and Ledoux [118].

The extremals exhibited in (2.3.24) of the Sobolev inequality (2.3.21) in the whole of $\mathbb{R}^n$, with sharp constant C, are the only ones—see Cordero-Erausquin, Nazaret, and Villani [212] who used the mass transportation techniques referred to in Comments to Sect. 1.4. Strengthened, quantitative versions of this inequality are also available. They involve a remainder term depending on the distance of the function u from the family of extremals. The first result in this connection was established by Bianchi and Egnell [96] for $p = 2$. The case when $p = 1$ was considered in Cianchi [199] and sharpened in Fusco, Maggi, and Pratelli [296] as far as the exponent in the remainder term is concerned. The general case when $1 < p < n$ is the object of Cianchi, Fusco, Maggi, and Pratelli [204]. Related results for $p > n$ are contained in Cianchi [202].

In [811], Zhang proved an improvement of the inequality (1.4.14), called the L_1 affine Sobolev inequality,

$$\int_{S^{n-1}} \|\nabla_u f\|_{L_1}^{-n} \, \mathrm{d}s_u \le n\left(\frac{\omega_n}{2\omega_{n-1}}\right)^n \|f\|_{L_{\frac{n}{n-1}}}^{-n}, \tag{2.3.54}$$

where $\nabla_u f$ is the partial derivative of f in direction u, $\mathrm{d}s_u$ is the surface measure on S^{n-1} and the constant factor on the right-hand side is sharp. Modifications of (2.3.54) for the L_p-gradient norm with $p > 1$ and for the Lorentz and Birnbaum–Orlicz settings are due to Zhang [811]; Lutwak, D. Yang, and Zhang [510]; Haberl and Schuster [333]; Werner and Ye [794]; and Cianchi, Lutwak, D. Yang, and Zhang [206].

A Sobolev-type trace inequality, which attracted much attention, is the following trace inequality:

$$\|f\|_{L_{\frac{p(n-1)}{n-p}}(\partial\mathbb{R}^n_+)} \le \mathscr{K}_{n,p}\|\nabla f\|_{L_p(\mathbb{R}^n_+)}, \tag{2.3.55}$$

where $n > p > 1$. In the case $p = 2$, Beckner [78] and Escobar [259], using different approaches, found the best value of $\mathscr{K}_{n,2}$. Xiao [799] generalized their result to the inequality

$$\|f\|_{L_{\frac{2(n-1)}{n-1-2\alpha}}(\partial\mathbb{R}^n_+)} \le C(n,\alpha)\int_{\mathbb{R}^n_+} \big|\nabla f(x)\big|^2 x_n^{1-2\alpha} \, \mathrm{d}x, \tag{2.3.56}$$

showing that

$$C(n,\alpha) = \left(\frac{2^{1-4\alpha}}{\pi^\alpha \Gamma(2(1-\alpha))}\right)\left(\frac{\Gamma((n-1-2\alpha)/2)}{\Gamma((n-1+2\alpha)/2)}\right)\left(\frac{\Gamma(n-1)}{\Gamma((n-1)/2)}\right)^{\frac{2\alpha}{n-1}}.$$

An idea in [799] is that it suffices to prove (2.3.56) for solutions of the Euler equation

$$\operatorname{div}\big(x_n^{1-2\alpha}\nabla u\big) = 0 \quad \text{on } \mathbb{R}^n_+.$$

Then by the Fourier transform with respect to $x' = (x_1, \ldots, x_{n-1})$ the integral on the right-hand side of (2.3.56) takes the form

$$\text{const} \left\| (-\varDelta_{x'})^{\alpha/2} u \right\|^2_{L_2(\mathbb{R}^{n-1})},$$

and the reference to the Lieb formula (1.4.48) gives the above value of $C(n, \alpha)$.

More recently, Nazaret proved, by using the mass transportation method mentioned in Comments to Sect. 1.4, that the only minimizer in (2.3.55) has the form

$$\text{const} \left((x_n + \lambda)^2 + |x|^2 \right)^{\frac{p-n}{2(p-1)}},$$

where $\lambda = \text{const} > 0$. Sharp Sobolev-type inequalities proved to be crucial in the study of partial differential equations and nowadays there is extensive literature dealing with them. To the works mentioned earlier we add the papers: Gidas, Ni, and Nirenberg [307]; Lieb [496]; Lions [501]; Han [335]; Beckner [78, 79]; Adimurthi and Yadava [29]; Hebey and Vaugon [362, 363]; Hebey [359]; Druet and Hebey [243]; Lieb and Loss [497]; Del Pino and Dolbeault [231]; Bonder, Rossi, and Ferreira [125]; Biezuner [99]; Ghoussoub and Kang [306]; Dem'yanov and A. Nazarov [233]; Bonder and Saintier [126]; et al.

The study of minimizers in the theory of Sobolev spaces based on the so-called concentration compactness is one of the topics in the book by Tintarev and Fieseler [753] where relevant historical information can be found as well.

The material of Sects. 2.3.6–2.3.11 is due to the author [543]. The sufficiency in Theorem 2.3.8 can be found in Maz'ya [543] and the necessity is due to Maz'ya and Netrusov [572].

The equivalence of the Nash and Moser inequalities (2.3.48) and (2.3.49) for $n > 2$ and Sobolev's inequality (2.3.51) is an obvious consequence of Theorem 2.3.6, which was proved by the author [543] (see also [552], Satz 4.3). This equivalence was rediscovered in the 1990s by Bakry, Coulhon, Ledoux, and Saloff-Coste [64] (see also Sect. 3.2 in the book by Saloff-Coste [687]) and by Delin [232]. The best constant in (2.3.48) was found by Carlen and Loss [167]:

$$C = 2n^{-1+2/n}(1 + n/2)^{1+n/2} z_n^{-1} \omega_n^{-2/n},$$

where z_n is the smallest positive root of the equation

$$(1 + n/2) J_{(n-2)/2}(z) + z J'_{(n-2)/2}(z) = 0.$$

The existence of the optimizer is proved in Tintarev and Fieseler [753], 10.3.

2.4 Continuity and Compactness of Embedding Operators of $\mathring{L}^1_p(\Omega)$ and $\mathring{W}^1_p(\Omega)$ into Birnbaum–Orlicz Spaces

Let $\mathring{L}^l_p(\Omega)$ and $\mathring{W}^l_p(\Omega)$ be completions of $\mathscr{D}(\Omega)$ with respect to the norms $\|\nabla_l u\|_{L_p(\Omega)}$ and $\|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}$.

Let μ be a measure in Ω. By $\mathscr{L}_M(\Omega, \mu)$ we denote the Birnbaum–Orlicz space generated by a convex function M, and by P we mean the complementary function of M (see Sect. 2.3.3).

The present section deals with some consequences of Theorem 2.3.3, containing the necessary and sufficient conditions for boundedness and compactness of embedding operators which map $\mathring{L}_p^1(\Omega)$ and $\mathring{W}_p^1(\Omega)$ into the space $\mathscr{L}_{p,M}(\Omega, \mu)$ with the norm $\||u|^p\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)}$, where μ is a measure in Ω. In the case $p = 2$, $M(t) = |t|$ these results will be used in Sect. 2.5 in the study of the Dirichlet problem for the Schrödinger operator.

2.4.1 Conditions for Boundedness of Embedding Operators

With each compactum $F \subset \Omega$ we associate the number

$$\pi_{p,M}(F, \Omega) = \begin{cases} \frac{\mu(F)P^{-1}(1/\mu(F))}{\operatorname{cap}_p(F,\Omega)} & \text{for } \operatorname{cap}_p(F, \Omega) > 0, \\ 0 & \text{for } \operatorname{cap}_p(F, \Omega) = 0. \end{cases}$$

In the case $p = 2$, $M(t) = |t|$, we shall use the notation $\pi(F, \Omega)$ instead of $\pi_{p,M}(F, \Omega)$.

The following assertion is a particular case of Theorem 2.3.3.

Theorem 1. 1. *Suppose that*

$$\pi_{p,M}(F, \Omega) \le \beta$$

for any compactum $F \subset \Omega$. Then, for all $u \in \mathscr{D}(\Omega)$,

$$\big\||u|^p\big\|_{\mathscr{L}_M(\Omega,\mu)} \le C \int_\Omega |\nabla u|^p \, \mathrm{d}x, \tag{2.4.1}$$

where $C \le p^p(p-1)^{1-p}\beta$.

2. *If (2.4.1) is valid for all $u \in \mathscr{D}(\Omega)$, then $\pi_{p,M}(F, \Omega) \le C$ for all compacta $F \subset \Omega$.*

Using this assertion we shall prove the following theorem.

Theorem 2. *The inequality*

$$\big\||u|\big\|^p_{\mathscr{L}_M(\Omega,\mu)} \le C \int_\Omega \big(|\nabla u|^p + |u|^p\big) \, \mathrm{d}x, \tag{2.4.2}$$

where $p < n$ is valid for all $u \in \mathscr{D}(\Omega)$ if and only if, for some $\delta > 0$,

$$\sup\big\{\pi_{p,M}(F, \Omega) : F \subset \Omega,\ \operatorname{diam}(F) \le \delta\big\} < \infty, \tag{2.4.3}$$

where, as usual, F is a compact subset of Ω.

Proof. Sufficiency. We construct a cubic grid in $\mathbb{R}^n$ with edge length $c\delta$, where c is a sufficiently small number depending only on n. With each cube $\mathscr{Q}_i$ of the grid we associate a concentric cube $2\mathscr{Q}_i$ with double the edge length and with faces parallel to those of $\mathscr{Q}_i$. We denote an arbitrary function in

$\mathscr{D}(\Omega)$ by u. Let η_i be an infinitely differentiable function in $\mathbb{R}^n$ that is equal to unity in $\mathscr{Q}_i$, to zero outside $2\mathscr{Q}_i$, and such that $|\nabla\eta_i| \le c_0/\delta$.

By Theorem 1,

$$\begin{aligned}
&\big\||u\eta_i|^p\big\|_{\mathscr{L}_M(\Omega,\mu)} \\
&\quad\le c\sup\left\{\frac{\mu(F)P^{-1}(1/\mu(F))}{\operatorname{cap}_p(F, 2\mathscr{Q}_i\cap\Omega)} : F\subset 2\mathscr{Q}_i\cap\Omega\right\}\int_{2\mathscr{Q}_i\cap\Omega}\big|\nabla(u\eta_i)\big|^p\,\mathrm{d}x \\
&\quad\le c\sup\big\{\pi_{p,M}(F,\Omega) : F\subset\Omega, \operatorname{diam}(F)\le\delta\big\}\int_{2\mathscr{Q}_i\cap\Omega}\big|\nabla(u\eta_i)\big|^p\,\mathrm{d}x.
\end{aligned}$$

Summing over i and noting that

$$\big\||u|^p\big\|_{\mathscr{L}_M(\Omega,\mu)} \le \bigg\|\sum_i |u\eta_i|^p\bigg\|_{\mathscr{L}_M(\Omega,\mu)} \le \sum_i\big\||u\eta_i|^p\big\|_{\mathscr{L}_M(\Omega,\mu)},$$

we obtain the required inequality

$$\begin{aligned}
\big\||u|^p\big\|_{L_M(\Omega,\mu)} &\le c\sup\big\{\pi_{p,M}(F,\Omega) : F\subset\Omega,\ \operatorname{diam}(F)\le\delta\big\} \\
&\quad\times\int_\Omega\big(|\nabla u|^p + \delta^{-p}|u|^p\big)\,\mathrm{d}x. \qquad (2.4.4)
\end{aligned}$$

Necessity. Let F be any compactum in Ω and let $\operatorname{diam}(F)\le\delta<1$. We include F inside two open concentric balls B and $2B$ with diameters δ and 2δ, respectively. Then we substitute an arbitrary $u\in\mathfrak{P}(F, 2B\cap\Omega)$ into (2.4.2).

Since $u = 1$ on F, then by (2.4.2)

$$\|\chi_F\|_{\mathscr{L}_M(\Omega,\mu)} \le C\bigg(\int_{2B}|\nabla u|^p\,\mathrm{d}x + \int_{2B}|u|^p\,\mathrm{d}x\bigg).$$

Consequently,

$$\mu(F)P^{-1}\big(1/\mu(F)\big) \le C\big(1 + c\delta^p\big)\int_{2B}|\nabla u|^p\,\mathrm{d}x.$$

Minimizing the last integral over the set $\mathfrak{P}(F, 2B\cap\Omega)$ we obtain

$$\mu(F)P^{-1}\big(1/\mu(F)\big) \le C\big(1 + c\delta^p\big)\operatorname{cap}_p(F, 2B\cap\Omega).$$

It remains to note that since $p<n$, it follows that

$$\operatorname{cap}_p(F, 2B\cap\Omega) \le c\operatorname{cap}_p(F,\Omega), \qquad (2.4.5)$$

where c depends only on n and p.

In fact, if $u\in\mathfrak{N}(F,\Omega)$ and $\eta\in\mathscr{D}(2B)$, $\eta = 1$ on B, $|\nabla\eta|\le c\delta$, then $u\eta\in\mathfrak{N}(F,\Omega\cap 2B)$ and hence

$$\begin{aligned}\operatorname{cap}_p(F, 2B\cap\Omega) &\le \int_{\Omega\cap 2B} \big|\nabla(u\eta)\big|^p\,\mathrm{d}x\\ &\le c\Big(\int_{2B}|\nabla u|^p\,\mathrm{d}x + \delta^{-p}\int_{2B}|u|^p\,\mathrm{d}x\Big)\\ &\le c\Big(\int_{\Omega}|\nabla u|^p\,\mathrm{d}x + \|u\|^p_{L_{pn/(n-p)}(\Omega)}\Big).\end{aligned}$$

This and the Sobolev theorem imply (2.4.5). The theorem is proved. □

2.4.2 Criteria for Compactness

The following two theorems give the necessary and sufficient conditions for the compactness of embedding operators that map $\mathring{L}_p^1(\Omega)$ and $\mathring{W}_p^1(\Omega)$ into $\mathscr{L}_{p,M}(\Omega,\mu)$.

Theorem 1. *The conditions*

$$\lim_{\delta\to 0}\sup\big\{\pi_{p,M}(F,\Omega) : F\subset\Omega,\ \operatorname{diam}(F)\le\delta\big\}=0 \tag{2.4.6}$$

and

$$\lim_{\varrho\to\infty}\sup\big\{\pi_{p,M}(F,\Omega) : F\subset\Omega\backslash B_\varrho\big\}=0 \tag{2.4.7}$$

are necessary and sufficient for any set of functions in $\mathscr{D}(\Omega)$, bounded in $\mathring{L}_p^1(\Omega)$ $(p<n)$, to be relatively compact in $\mathscr{L}_{p,M}(\Omega,\mu)$.

Theorem 2. *The condition* (2.4.6) *and*

$$\lim_{\varrho\to\infty}\sup\{\pi_{p,M}(F,\Omega) : F\subset\Omega\backslash B_\varrho,\ \operatorname{diam}(F)\le 1\}=0 \tag{2.4.8}$$

are necessary and sufficient for any set of functions in $\mathscr{D}(\Omega)$, bounded in $\mathring{W}_p^1(\Omega)$ $(p<n)$, to be relatively compact in $\mathscr{L}_{p,M}(\Omega,\mu)$.

To prove Theorems 1 and 2 we start with the following auxiliary assertion.

Lemma. *Let $\mu^{(\varrho)}$ be the restriction of μ to the ball B_ϱ. For an arbitrary set, bounded in $\mathring{L}_p^1(\Omega)$ or in $\mathring{W}_p^1(\Omega)$, $p<n$, to be relatively compact in $\mathscr{L}_{p,M}(\Omega,\mu^{(\varrho)})$ for all $\varrho>0$, it is necessary and sufficient that*

$$\lim_{\delta\to 0}\sup\big\{\pi_{p,M}(F,\Omega) : F\subset B_\varrho\cap\Omega,\ \operatorname{diam}(F)\le\delta\big\}=0, \tag{2.4.9}$$

for any $\varrho>0$.

Proof. Sufficiency. Since capacity does not increase under the extension of Ω, we see that for any compactum $F\subset B_\varrho\cap\Omega$,

$$\pi_{p,M}(F, B_\varrho\cap\Omega)\le\pi_{p,M}(F,\Omega).$$

This along with (2.4.9) implies

$$\lim_{\delta\to 0}\sup\bigl\{\pi_{p,M}(F,B_\varrho\cap\Omega): F\subset B_\varrho\cap\Omega,\ \operatorname{diam}(F)\le\delta\bigr\}=0$$

for all $\varrho>0$. This equality, together with (2.4.4), where the role of Ω is played by $B_{2\varrho}\cap\Omega$, yields

$$\bigl\||u|^p\bigr\|_{\mathscr{L}_M(\Omega,\mu^{(2\varrho)})}\le\varepsilon\int_{B_{2\varrho}\cap\Omega}|\nabla u|^p\,\mathrm{d}x+C_1(\varepsilon)\int_{B_{2\varrho}\cap\Omega}|u|^p\,\mathrm{d}x$$

for any $\varepsilon>0$ and for all $u\in\mathscr{D}(B_{2\varrho}\cap\Omega)$. Replacing u by $u\eta$, where η is a truncating function, equal to unity on B_ϱ and to zero outside $B_{2\varrho}$, we obtain

$$\bigl\||u|^p\bigr\|_{L_M(\Omega,\mu^{(\varrho)})}\le\varepsilon\int_{\Omega}|\nabla u|^p\,\mathrm{d}x+C_2(\varepsilon)\int_{B_{2\varrho}\cap\Omega}|u|^p\,\mathrm{d}x. \tag{2.4.10}$$

It remains to note that in the case $p<n$ any set, bounded in $\mathring{L}_p^1(\Omega)$ (and a fortiori in $\mathring{W}_p^1(\Omega)$), is compact in $L_p(B_\varrho\cap\Omega)$ for any $\varrho>0$. The sufficiency of (2.4.8) is proved.

Necessity. Let $F\subset B_\varrho\cap\Omega$ be a compactum and let $\operatorname{diam}(F)\le\delta<1$. We include F inside concentric balls B and $2B$ with radii δ and 2δ, respectively. By u we denote an arbitrary function in $\mathfrak{P}(F,2B\cap\Omega)$. Since any set of functions in $\mathscr{D}(\Omega)$, bounded in $\mathring{W}_p^1(\Omega)$, is relatively compact in $\mathscr{L}_{p,M}(\Omega,\mu^{(\varrho)})$, then for all $v\in\mathscr{D}(\Omega)$

$$\bigl\|\chi_B|v|^p\bigr\|_{\mathscr{L}_M(\Omega,\mu^{(\varrho)})}\le\varepsilon(\delta)\int_\Omega\bigl(|\nabla v|^p+|v|^p\bigr)\,\mathrm{d}x,$$

where χ_B is the characteristic function of B and $\varepsilon(\delta)\to 0$ as $\delta\to 0$. To prove this inequality we must note that Theorem 2.4.1/2, applied to the measure $\mu^{(\varrho)}$, implies $\mu^{(\varrho)}(2B)\to 0$ as $\delta\to 0$. Since u equals zero outside $2B\cap\Omega$ we have

$$\int_\Omega|u|^p\,\mathrm{d}x\le c\delta^p\int_\Omega|\nabla u|^p\,\mathrm{d}x.$$

Therefore,

$$\mu(F)P^{-1}\bigl(1/\mu(F)\bigr)\le\bigl(1+c\delta^p\bigr)\varepsilon(\delta)\int_{2B}|\nabla u|^p\,\mathrm{d}x.$$

Minimizing the last integral over $\mathfrak{P}(F,2B\cap\Omega)$ and using (2.4.5), we arrive at

$$\pi_{p,M}(F,\Omega)\le\bigl(1+c\delta^p\bigr)\varepsilon(\delta).$$

The necessity of (2.4.9) follows. The lemma is proved. □

Proof of Theorem 1. Sufficiency. Let $\zeta\in C^\infty(\mathbb{R}^n)$, $0\le\zeta\le 1$, $|\nabla\zeta|\le c\varrho^{-1}$, $\zeta=0$ in a neighborhood of $B_{\varrho/2}$, $\zeta=1$ outside B_ϱ. It is clear that

$$\big\| |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \big\| (1-\zeta)^p |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} + \big\| \zeta^p |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)}$$
$$\le \big\| |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu^{(\varrho)})} + \big\| |\zeta u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)}. \tag{2.4.11}$$

By the first part of Theorem 2.4.1/1, applied to the set $\Omega\backslash\bar{B}_{\varrho/2}$, by (2.4.7) and the inequality

$$\pi_{p,M}(F, \Omega\backslash\bar{B}_{\varrho/2}) \le \pi_{p,M}(F,\Omega),$$

given any ε, there exists a number $\varrho > 0$ such that

$$\big\| |\zeta u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \varepsilon \big\| \nabla(\zeta u) \big\|_{L_p(\Omega)}.$$

Since $|\nabla\zeta| \le c\varrho^{-1} \le c|x|^{-1}$ and

$$\big\| |x|^{-1} u \big\|_{L_p(\Omega)} \le c \|\nabla u\|_{L_p(\Omega)},$$

we have

$$\big\| |\zeta u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le c\varepsilon \|\nabla u\|_{L_p(\Omega)}.$$

The last inequality along with (2.4.11) implies

$$\big\| |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \big\| |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu^{(\varrho)})} + c\varepsilon \|\nabla u\|_{L_p(\Omega)}. \tag{2.4.12}$$

Obviously, (2.4.6) implies (2.4.9). Therefore, the lemma guarantees that any set of functions in $\mathscr{D}(\Omega)$, bounded in $\mathring{L}^1_p(\Omega)$, is compact in $\mathscr{L}_{p,M}(\Omega,\mu^{(\varrho)})$. This together with (2.4.12) completes the proof of the first part of the theorem.

Necessity. Let F be a compactum in Ω with $\operatorname{diam}(F) \le \delta < 1$. Duplicating the proof of necessity in the Lemma and replacing $\mu^{(\varrho)}$ there by μ, we arrive at the inequality $\pi_{p,M}(F,\Omega) \le (1 + c\delta^p)\varepsilon(\delta)$ and hence at (2.4.6).

Now let $F \subset \Omega\backslash\bar{B}_\varrho$. Using the compactness in $\mathscr{L}_{p,M}(\Omega,\mu)$ of any set of functions in $\mathscr{D}(\Omega)$, which are bounded in $\mathring{L}^1_p(\Omega)$, we obtain

$$\big\| \chi_{\Omega\backslash B_\varrho} |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \varepsilon_\varrho \|\nabla u\|_{L_p(\Omega)},$$

where $\varepsilon_\varrho \to 0$ as $\varrho \to 0$ and u is an arbitrary function in $\mathscr{D}(\Omega)$. In particular, the last inequality holds for any $u \in \mathfrak{P}(F,\Omega)$ and therefore

$$\mu(F) P^{-1}\big(1/\mu(F)\big) \le \varepsilon^p_\varrho \|\nabla u\|^p_{L_p(\Omega)}.$$

Minimizing the right-hand side over the set $\mathfrak{P}(F,\Omega)$, we arrive at (2.4.7). The theorem is proved. □

Proof of Theorem 2. We shall use the same notation as in the proof of Theorem 1.

Sufficiency. From (2.4.4), where $\delta = 1$ and Ω is replaced by $\Omega\backslash\bar{B}_{\varrho/2}$, together with (2.4.8), it follows that given any $\varepsilon > 0$, there exists a $\varrho > 0$ such that

$$\big\| |\zeta u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \varepsilon \big(\big\| \nabla(u\zeta) \big\|_{L_p(\Omega)} + \|\zeta u\|_{L_p(\Omega)} \big).$$

This together with (2.4.11) yields

$$\big\| |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu)} \le \big\| |u|^p \big\|^{1/p}_{\mathscr{L}_M(\Omega,\mu^{(\varrho)})} + c\varepsilon \|u\|_{W^1_p(\Omega)}.$$

The remainder of the proof is the same as the proof of sufficiency in the preceding theorem.

Necessity. The condition (2.4.6) can be derived in the same way as in the proof of necessity in Theorem 1.

Let $F \subset \Omega \backslash \bar{B}_\varrho$, $\varrho > 8$, $\operatorname{diam}(F) \le 1$. From the compactness in $\mathscr{L}_{p,M}(\Omega,\mu)$ of any set of functions in $\mathscr{D}(\Omega)$, bounded in $\mathring{W}^1_p(\Omega)$, it follows that

$$\big\| \chi_{\Omega \backslash B_{\varrho/2}} |u|^p \big\|_{\mathscr{L}_M(\Omega,\mu)} \le \varepsilon_\varrho \|u\|^p_{W^1_p(\Omega)},$$

where $\varepsilon_\varrho \to 0$ as $\varrho \to \infty$ and u is an arbitrary function in $\mathscr{D}(\Omega)$. We include F inside concentric balls B and $2B$ with radii 1 and 2 and let u denote any function in $\mathfrak{P}(F, 2B \cap \Omega)$. Using the same argument as in the proof of necessity in the Lemma we arrive at

$$\pi_{p,M}(F,\Omega) \le (1+c)\varepsilon_\varrho,$$

which is equivalent to (2.4.8). The theorem is proved. □

Remark. Let us compare (2.4.6) and (2.4.9). Clearly, (2.4.9) results from (2.4.6). The following example shows that the converse assertion is not valid. Consider a sequence of unit balls $\mathscr{B}^{(\nu)}$ $(\nu = 1, 2, \dots)$, with dist $(\mathscr{B}^{(\nu)}, \mathscr{B}^{(\mu)}) \ge 1$ for $\mu \ne \nu$. Let $\Omega = \mathbb{R}^n$ and

$$\mu(F) = \int_F p(x)\,\mathrm{d}x,$$

where

$$p(x) = \begin{cases} \varrho^{-2+\nu^{-1}} & \text{for } x \in \mathscr{B}^{(\nu)}, \\ 0 & \text{for } x \notin \bigcup_{\nu=1}^\infty \mathscr{B}^{(\nu)}. \end{cases}$$

Here ϱ is the distance of x from the center of $\mathscr{B}^{(\nu)}$.

We shall show that the measure μ satisfies the condition (2.4.9) with $p = 2$, $M(t) = t$. First of all we note that for any compactum $F \subset \mathscr{B}^{(\nu)}$

$$\mu(F) = \int_F \varrho^{-2+1/\nu}\,\mathrm{d}x \le \int_{\partial B_1} \int_0^{r(F)} \varrho^{n-3+1/\nu}\,\mathrm{d}\varrho\,\mathrm{d}\omega,$$

where

$$r(F) = \left[\frac{n}{\omega} m_n(F) \right]^{1/n}.$$

To estimate $\operatorname{cap}(F)$, i.e., $\operatorname{cap}_2(F,\mathbb{R}^n)$, we apply the isoperimetric inequality (2.2.12)

$$\omega_n^{-1}(n-2)^{-1}\operatorname{cap}(F) \geq \left[\frac{n}{\omega_n} m_n(F)\right]^{(n-2)/n} = \left[r(F)\right]^{n-2}.$$

Now

$$\pi\left(F, \mathbb{R}^n\right) \leq \frac{r(F)^{1/\nu}}{(n-2)(n-2+1/\nu)},$$

and (2.4.9) follows.

If F is the ball $\{x : \varrho \leq \delta\}$, we have

$$\pi\left(F, \mathbb{R}^n\right) = \frac{\delta^{1/\nu}}{(n-2)(n-2+1/\nu)}.$$

Consequently,

$$\sup\left\{\pi\left(F, \mathbb{R}^n\right) : F \subset \mathbb{R}^n, \operatorname{diam}(F) \leq 2\delta\right\} \\ \geq \lim_{\nu\to\infty} \frac{\delta^{1/\nu}}{(n-2)(n-2+1/\nu)} = (n-2)^{-2}$$

and (2.4.6) is not valid.

2.4.3 Comments to Sect. 2.4

The material of this section is borrowed from Sect. 2.5 of the author's book [552]. Sharp embeddings of Birnbaum–Orlicz–Sobolev spaces of order one into the space $L_\infty(\Omega)$ will be considered in Chap. 7 of the present book (see also Maz'ya [528, 545]).

An optimal Sobolev embedding theorem in Birnbaum–Orlicz spaces was established by Cianchi in [194], and in alternative equivalent form, in [195]. A basic version of this result states that if Ω is an open set in $\mathbb{R}^n$ with finite measure, M is any Young function, and M_n is the Young function given by

$$M_n(t) = M\left(H^{-1}(t)\right) \quad \text{for } t > 0,$$

where

$$H(s) = \left(\int_0^s \left(\frac{t}{M(t)}\right)^{\frac{1}{n-1}} \mathrm{d}t\right)^{\frac{n-1}{n}} \quad \text{for } s \geq 0,$$

then there exists a constant C, depending on n, such that

$$\|u\|_{\mathscr{L}_{M_n}(\Omega)} \leq C\|\nabla u\|_{\mathscr{L}_M(\Omega)}$$

for every weakly differentiable function u vanishing, in the appropriate sense, on $\partial\Omega$. Moreover, the Birnbaum–Orlicz space $\mathscr{L}_{M_n}(\Omega)$ is optimal. Analogous results for functions that need not vanish on $\partial\Omega$, and for domains Ω with infinite measure [194]. The case of higher-order derivatives was dealt with by Cianchi in [200].

Some necessary and sufficient conditions for embeddings of Sobolev-type spaces into Birnbaum–Orlicz spaces will be treated in Chap. 11 of this book.

Analog of certain results in the present section were obtained by Klimov [431–435] for the so-called *ideal function spaces*, for which the multiplication by any function α with $|\alpha(x)| \le 1$ a.e. is contractive.

A few words on the so-called *logarithmic Sobolev inequalities*. Let μ be a measure in Ω, $\mu(\Omega) = 1$, $p \ge 1$ and let ν_p be the capacity minimizing function generated by μ (see Definition 2.3.2). The inequality

$$\exp\biggl(-\int_\Omega \log^+ \frac{1}{|u|}\,\mathrm{d}\mu\biggr) \le 4\|\nabla u\|_{L_p(\Omega)} \exp\biggl(-\frac{1}{p}\int_0^1 \log \nu_p(s)\,\mathrm{d}s\biggr) \tag{2.4.13}$$

for all $u \in \mathring{L}_p^1(\Omega)$ was proved in 1968 by Maz'ya and Havin [568]. It shows, in particular, that

$$\int_0^1 \nu_p(s)\,\mathrm{d}s = +\infty$$

implies

$$\int_0^1 \log^+ \frac{1}{|u|}\,\mathrm{d}\mu = +\infty$$

for all $u \in L_p^1(\Omega)$. This fact allows for certain applications of (2.4.13) to complex function theory [568] (see also Sect. 14.3 of the present book for another logarithmic inequality of a similar nature).

Inequality (2.4.13) is completely different from the logarithmic Sobolev inequality obtained in 1978 by Weissler [793],

$$\exp\biggl(\frac{4}{n}\int_{\mathbb{R}^n} |u|^2 \log|u|\,\mathrm{d}x\biggr) \le \frac{2}{\pi e n}\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}x,$$

where $\|u\|_{L_2(\mathbb{R}^n)} = 1$, which is equivalent (see Beckner and Pearson [81]) to the well-known Gross inequality of 1975 [327],

$$\int_{\mathbb{R}^n} u^2 \log\biggl(u^2\Big/\int_{\mathbb{R}^n} u^2\,\mathrm{d}\mu\biggr)\,\mathrm{d}\mu \le C\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}\mu, \tag{2.4.14}$$

where

$$\mathrm{d}\mu = (2\pi)^{-n/2}\exp\bigl(-|x|^2/2\bigr)\,\mathrm{d}x.$$

Various extensions, proofs, and applications of (2.4.14) were the subject of many studies: R.A. Adams [24]; Stroock and Zegarlinski [730]; Holley and Stroock [381]; Davies [222]; Zegarlinski [810]; Beckner [77, 80]; Gross [328]; Aida, Masuda, and Shigekawa [32]; Aida and Stroock [33]; Bakry [63]; Bakry, Ledoux, and Qian [65]; Chen [185]; F.-Y. Wang [787–790]; Bodineau and Helffer [121]; Bobkov and Götze [114]; Ledoux [483–485]; Yosida [808]; Guionnet and Zegarlinski [330]; Xiao [799]; Lugiewicz and Zegarlinski [509]; Otto and Reznikoff [655]; Inglis and Papageorgiou [398]; Cianchi and Pick [207]; Martin and M. Milman [521]; et al.

2.5 Structure of the Negative Spectrum of the Multidimensional Schrödinger Operator

In this section we show how the method and results of Sect. 2.4 can be applied to the spectral theory of the Schrödinger operator.

2.5.1 Preliminaries and Notation

We start with some definitions from the theory of quadratic forms in a Hilbert space H. Let $\mathscr{L}$ be a dense linear subset of H and let $S[u,u]$ be a quadratic form defined on $\mathscr{L}$. If there exists a constant γ such that for all $u \in \mathscr{L}$

$$S[u,u] \geq \gamma \|u\|_H^2, \tag{2.5.1}$$

then the form S is called semibounded from below. The largest constant γ in (2.5.1) is called the greatest lower bound of the form S and is denoted by $\gamma(S)$. If $\gamma(S) > 0$, then S is called positive definite. For such a form the set $\mathscr{L}$ is a pre-Hilbert space with the inner product $S[u,u]$. If $\mathscr{L}$ is a Hilbert space the form S is called closed. If any Cauchy sequence in the metric $S[u,u]^{1/2}$ that converges to zero in H also converges to zero in the metric $S[u,u]^{1/2}$, then S is said to be closable. Completing $\mathscr{L}$ and extending S by continuity onto the completion $\bar{\mathscr{L}}$, we obtain the closure $\bar{S}$ of the form S.

Now, suppose that the form S is only semibounded from below. We do not assume $\gamma(S) > 0$. Then for any $c > -\gamma(S)$ the form

$$S[u,u] + c[u,u] \tag{2.5.2}$$

is positive definite. By definition, S is closable if the form (2.5.2) is closable for some, and therefore for any, $c > \gamma(S)$. The form $\overline{S + cE} - cE$ is called the closure $\bar{S}$ of S.

It is well known and can be easily checked that a semibounded closable form generates a unique self-adjoint operator $\tilde{S}$, for which

$$(\tilde{S}u, u) = S[u,u] \quad \text{for all } u \in \mathscr{L}.$$

Let Ω be an open subset of $\mathbb{R}^n$, $n > 2$, and let h be a positive number. We shall consider the quadratic form

$$S_h[u,u] = h \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \int |u|^2 \, \mathrm{d}\mu(x)$$

defined on $\mathscr{D}(\Omega)$.

We shall study the operator $\tilde{S}_h$ generated by the form $S_h[u,u]$ under the condition that the latter is closable. If the measure μ is absolutely continuous with respect to the Lebesgue measure m_n and the derivative $p = \mathrm{d}\mu/\mathrm{d}m_n$ is locally square integrable, then the operator $\tilde{S}_h$ is the Friedrichs extension of the Schrödinger operator $-h\Delta - p(x)$.

In this section, when speaking of capacity, we mean the 2-capacity and use the notation cap.

Before we proceed to the study of the operator $\tilde{S}_h$ we formulate two lemmas on estimates for capacity that will be used later. For the proofs of these lemmas see the end of the section.

Lemma 1. *Let F be a compactum in $\Omega \cap B_r$. Then for $R > r$*

$$\operatorname{cap}(F, B_r \cap \Omega) \leq \begin{cases} (1 + \frac{2r}{R-r} \log \frac{Re^{1/2}}{r}) \operatorname{cap}(F, \Omega) & \textit{for } n = 3, \\ (1 + \frac{2}{n-3} \frac{r}{R-r}) \operatorname{cap}(F, \Omega) & \textit{for } n > 3. \end{cases}$$

Lemma 2. *Let F be a compactum in $\Omega \backslash \bar{B}_R$. Then for $r < R$*

$$\operatorname{cap}(F, \Omega \backslash \bar{B}_r) \leq \left(1 + \frac{1}{n-2} \frac{r}{R-r}\right) \operatorname{cap}(F, \Omega).$$

All the facts concerning the operator $\tilde{S}_h$ will be formulated in terms of the function

$$\pi(F, \Omega) = \begin{cases} \frac{\mu(F)}{\operatorname{cap}(F,\Omega)} & \text{for } \operatorname{cap}(F, \Omega) > 0, \\ 0 & \text{for } \operatorname{cap}(F, \Omega) = 0, \end{cases}$$

which is a particular case of the function $\pi_{p,M}(F, \Omega)$, introduced in Sect. 2.4, for $M(t) = |t|$, $p = 2$.

2.5.2 Positivity of the Form $S_1[u, u]$

The following assertion is a particular case of Theorem 2.4.1/1.

Theorem. 1. *If for any compactum $F \subset \Omega$*

$$\pi(F, \Omega) \leq \beta, \tag{2.5.3}$$

then for all $u \in \mathscr{D}(\Omega)$

$$\int_\Omega |u|^2 \mu(\mathrm{d}x) \leq C \int_\Omega |\nabla u|^2 \, \mathrm{d}x, \tag{2.5.4}$$

where $C \leq 4\beta$.

2. *If* (2.5.4) *holds for all $u \in \mathscr{D}(\Omega)$, then for any compactum $F \subset \Omega$*

$$\pi(F, \Omega) \leq C. \tag{2.5.5}$$

Corollary. *If*

$$\sup_{F \subset \Omega} \pi(F, \Omega) < \frac{1}{4},$$

then the form $S_1[u,u]$ *is positive, closable in* $L_2(\Omega)$*, and hence it generates a self-adjoint positive operator* $\tilde{S}_1$ *in* $L_2(\Omega)$*.*

Proof. The positiveness of $S_1[u,u]$ follows from the Theorem. Moreover, inequality (2.5.4) implies

$$S_1[u,u] \geq \Big[1 - 4 \sup_{F\subset\Omega} \pi(F,\Omega)\Big] \int_\Omega |\nabla u|^2 \, \mathrm{d}x. \tag{2.5.6}$$

Let $\{u_\nu\}_{\nu\geq 1}$, $u_\nu \in \mathscr{D}(\Omega)$, be a Cauchy sequence in the metric $S_1[u,u]^{1/2}$ and let u_ν converge to zero in $L_2(\Omega)$. Then by (2.5.6), u_ν converges to zero in $\mathring{L}_2^1(\Omega)$ and it is a Cauchy sequence in $L_2(\Omega,\mu)$. Since

$$\int_\Omega |u_\nu|^2 \, \mathrm{d}\mu \leq 4 \sup_{F\subset\Omega} \pi(F,\Omega) \int_\Omega |\nabla u_\nu|^2 \, \mathrm{d}x,$$

then $u_\nu \to 0$ in $L_2(\Omega,\mu)$. Thus, $S_1[u_\nu,u_\nu] \to 0$ and therefore the form $S_1[u,u]$ is closable in $L_2(\Omega)$. The corollary is proved. □

We note that close necessary and sufficient conditions for the validity of the inequality

$$\int_\Omega |u|^2 \, \mathrm{d}\sigma \leq C \int_\Omega |\nabla u|^2 \, \mathrm{d}x, \quad u \in \mathscr{D}(\Omega),$$

where σ is an arbitrary charge in Ω, are contained in Theorem 2.3.10 for $\Phi(x,y) = |y|$, $p = 2$. The conditions in question coincide for $\sigma \geq 0$. They become the condition $\sup\{\pi(F,\Omega) : F \subset \Omega\} < \infty$, which follows from the Theorem.

2.5.3 Semiboundedness of the Schrödinger Operator

Theorem. 1. *If*

$$\lim_{\delta\to 0} \sup\big\{\pi(F,\Omega) : F \subset \Omega,\ \operatorname{diam}(F) \leq \delta\big\} < \frac{1}{4}, \tag{2.5.7}$$

then the form $S_1[u,u]$ *is semibounded from below and closable in* $L_2(\Omega)$*.*

2. *If the form* $S_1[u,u]$ *is semibounded from below in* $L_2(\Omega)$*, then*

$$\lim_{\delta\to 0} \sup\big\{\pi(F,\Omega) : F \subset \Omega,\ \operatorname{diam}(F) \leq \delta\big\} \leq 1. \tag{2.5.8}$$

Proof. 1. If $\varPi$ is a sufficiently large integer, then there exists $\delta > 0$ such that

$$\sup\big\{\pi(F,\Omega) : F \subset \Omega,\ \operatorname{diam}(F) \leq \delta\big\} \leq \frac{1}{4}\left(\frac{\varPi - 1}{\varPi + 2}\right)^n. \tag{2.5.9}$$

We construct a cubic grid in $\mathbb{R}^n$ with edge length $H = \delta/(\Pi + 2)\sqrt{n}$. We include each cube $\mathscr{Q}_i$ of the grid inside concentric cubes $\mathscr{Q}_i^{(1)}$ and $\mathscr{Q}_i^{(2)}$ with faces parallel to those of $\mathscr{Q}_i$. Let the edge lengths of $\mathscr{Q}_i^{(1)}$ and $\mathscr{Q}_i^{(2)}$ be $(\Pi + 1)H$ and $(\Pi + 2)H$, respectively. Since $\operatorname{diam}(\mathscr{Q}_i^{(2)}) = \delta$ then for any compactum $F \subset \mathscr{Q}_i^{(2)} \cap \Omega$

$$\pi\big(F, \Omega \cap \mathscr{Q}_i^{(2)}\big) \leq \pi(F, \Omega) \leq \frac{1}{4}\left(\frac{\Pi - 1}{\Pi + 2}\right)^n. \tag{2.5.10}$$

Let u denote an arbitrary function in $\mathscr{D}(\Omega)$ and let η denote an infinitely differentiable function on $\mathbb{R}^n$ which is equal to unity in $\mathscr{Q}_i^{(1)}$ and to zero outside $\mathscr{Q}_i^{(2)}$. By (2.5.10) and Theorem 2.5.2 we have

$$\int_{\mathscr{Q}_i^{(2)}} |u\eta|^2 \,\mathrm{d}\mu \leq \left(\frac{\Pi - 1}{\Pi + 2}\right)^n \int_{\mathscr{Q}_i^{(2)}} \big|\nabla(u\eta)\big|^2 \,\mathrm{d}x.$$

This implies

$$\int_{\mathscr{Q}_i^{(1)}} |u|^2 \,\mathrm{d}\mu \leq \left(\frac{\Pi - 1}{\Pi + 2}\right)^n \int_{\mathscr{Q}_i^{(2)}} \left(|\nabla u|^2 + \frac{c_1}{H^2}|u|^2\right) \mathrm{d}x.$$

Summing over i and noting that the multiplicity of the covering $\{\mathscr{Q}_i^{(2)}\}$ does not exceed $(\Pi + 2)^n$ and that of $\{\mathscr{Q}_i^{(2)}\}$ is not less than Π^n, we obtain

$$\int_\Omega |u|^2 \,\mathrm{d}\mu \leq \big(1 - \Pi^{-n}\big) \int_\Omega \left(|\nabla u|^2 + c\frac{\Pi^2}{\delta^2}|u|^2\right) \mathrm{d}x. \tag{2.5.11}$$

Thus, the form $S_1[u, u]$ is semibounded. Moreover, if K is a sufficiently large constant, then

$$S_1[u, u] + K \int_\Omega |u|^2 \,\mathrm{d}x \geq \varepsilon \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \quad \varepsilon > 0.$$

Further, using the same argument as in the proof of Corollary 2.5.2, we can easily deduce that the form $S_1[u, u]$ is closable in $L_2(\Omega)$.

2. Let F be an arbitrary compactum in Ω with $\operatorname{diam}(F) \leq \delta < 1$. We enclose F in a ball B with radius δ and construct the concentric ball B' with radius $\delta^{1/2}$.

We denote an arbitrary function in $\mathfrak{P}(F, B' \cap \Omega)$ by u. In virtue of the semiboundedness of the form $S_1[u, u]$ there exists a constant K such that

$$\int_{B'} u^2 \,\mathrm{d}\mu \leq \int_{B'} (\nabla u)^2 \,\mathrm{d}x + K \int_{B'} u^2 \,\mathrm{d}x.$$

Obviously, the right-hand side of this inequality does not exceed

$$\bigl(1 + K\lambda^{-1}\delta\bigr) \int_{B' \cap \Omega} (\nabla u)^2 \, \mathrm{d}x,$$

where λ is the first eigenvalue of the Dirichlet problem for the Laplace operator in the unit ball.

Minimizing the Dirichlet integral and taking into account that $u = 1$ on F, we obtain

$$\mu(F) \le \bigl(1 + K\lambda^{-1}\delta\bigr) \operatorname{cap}(F, B' \cap \Omega).$$

By Lemma 2.5.1,

$$\operatorname{cap}(F, B' \cap \Omega) \le \bigl(1 + o(1)\bigr) \operatorname{cap}(F, \Omega),$$

where $o(1) \to 0$ as $\delta \to 0$. Hence

$$\sup\bigl\{\pi(F, \Omega) : F \subset \Omega, \operatorname{diam}(F) \le \delta\bigr\} \le 1 + o(1).$$

It remains to pass to the limit as $\delta \to 0$. The theorem is proved. $\square$

The two assertions stated in the following are obvious corollaries of the Theorem. The second is a special case of Theorem 2.4.1/2.

Corollary 1. *The condition*

$$\lim_{\delta \to 0} \sup\bigl\{\pi(F, \Omega) : F \subset \Omega, \operatorname{diam}(F) \le \delta\bigr\} = 0 \tag{2.5.12}$$

is necessary and sufficient for the semiboundedness of the form $S_h[u, u]$ in $L_2(\Omega)$ for all $h > 0$.

Corollary 2. *The inequality*

$$\int_{\Omega} |u|^2 \, \mathrm{d}\mu \le C \int_{\Omega} \bigl(|\nabla u|^2 + |u|^2\bigr) \, \mathrm{d}x,$$

where u is an arbitrary function in $\mathscr{D}(\Omega)$ and C is a constant independent of u, is valid if and only if

$$\sup\bigl\{\pi(F, \Omega) : F \subset \Omega, \operatorname{diam}(F) \le \delta\bigr\} < \infty \tag{2.5.13}$$

for some $\delta > 0$.

We shall give an example that illustrates an application of Theorem 2.5.2 and the theorem of the present subsection to the Schrödinger operator generated by a singular measure.

Example. Let M be a plane Borel subset of $\mathbb{R}^3$. We define the measure $\mu(F) = m_2(F \cap M)$ for any compactum $F \subset \mathbb{R}^3$. (In the sense of distribution theory the potential $p(x)$ is equal to the Dirac δ function concentrated on the plane set M.) Then

$$\pi\big(F,\mathbb{R}^3\big)=\frac{m_2(F\cap M)}{\operatorname{cap}(F)}\le\frac{m_2(F\cap M)}{\operatorname{cap}(F\cap M)}.$$

Since

$$\operatorname{cap}(F\cap M)\ge 8\pi^{-1/2}\big[m_2(F\cap M)\big]^{1/2}$$

(cf. (2.2.16)), we have

$$\pi\big(F,\mathbb{R}^3\big)\le\frac{\pi^{1/2}}{8}\big[m_2(F\cap M)\big]^{1/2}. \tag{2.5.14}$$

By Theorem 2.5.2 the form

$$S_1[u,u]=\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x-\int_M|u|^2m_2(\mathrm{d}x)$$

is positive if $m_2(M)\le 4\pi^{-1}$. Using Corollary 1, from (2.5.14) we obtain that the form $S_h[u,u]$ is semibounded and closable in $L_2(\mathbb{R}^3)$ for all $h>0$ for any plane set M.

2.5.4 Discreteness of the Negative Spectrum

Let ϱ be a fixed positive number and let $\mu^{(\varrho)}$ be the restriction of a measure μ to the ball $B_\varrho=\{x:|x|<\varrho\}$. Further, let $\mu_\varrho=\mu-\mu^{(\varrho)}$.

To exclude the influence of singularities of the measure μ, which are located at a finite distance, we shall assume that any subset of $\mathscr{D}(\Omega)$, bounded in $\mathring{W}{}_2^1(\Omega)$ (or in $\mathring{L}{}_2^1(\Omega)$), is compact in $L_2(\mu^{(\varrho)})$. In Lemma 2.4.2 it is shown that this condition is equivalent to

$$\lim_{\delta\to 0}\sup\big\{\pi(F,\Omega):F\subset B_\varrho\cap\Omega,\ \operatorname{diam}(F)\le\delta\big\}=0 \tag{2.5.15}$$

for any $\varrho>0$.

Now we formulate two well-known general assertions that will be used in the following.

Lemma 1. (Friedrichs [292]). *Let $A[u,u]$ be a closed quadratic form in a Hilbert space H with the domain $D[A]$, $\gamma(A)$ being its positive greatest lower bound. Further, let $B[u,u]$ be a real form, compact in $D[A]$. Then the form $A-B$ is semibounded from below in H and closed in $D[A]$, and its spectrum is discrete to the left of $\gamma(A)$.*

Lemma 2. (Glazman [309]). *For the negative spectrum of a self-adjoint operator A to be infinite it is necessary and sufficient that there exists a linear manifold of infinite dimension on which $(Au,u)<0$.*

Now we proceed to the study of conditions for the spectrum of the Schrödinger operator to be discrete.

Theorem. *Let the condition (2.5.15) hold.*

1. *If*

$$\lim_{\delta\to\infty}\lim_{\varrho\to\infty}\sup\bigl\{\pi(F,\Omega): F\subset\Omega\backslash B_\varrho,\ \operatorname{diam}(F)\le\delta\bigr\}<\frac{1}{4}, \tag{2.5.16}$$

then the form $S_1[u,u]$ is semibounded from below closable in $L_2(\Omega)$, and the negative spectrum of the operator $\tilde{S}_1$ is discrete.

2. *If the form $S_1[u,u]$ is semibounded from below and closable in $L_2(\Omega)$, and the negative spectrum of the operator $\tilde{S}_1$ is discrete, then*

$$\lim_{\delta\to\infty}\lim_{\varrho\to\infty}\sup\bigl\{\pi(F,\Omega): F\subset\Omega\backslash B_\varrho,\ \operatorname{diam}(F)\le\delta\bigr\}\le 1. \tag{2.5.17}$$

Proof. 1. We show that the form $S_1[u,u]$ is semibounded from below and closable in $L_2(\Omega)$, and that for any positive γ the spectrum of the operator $\tilde{S}_1+2\gamma I$ is discrete to the left of γ. This will yield the first part of the theorem.

By (2.5.16), there exists a sufficiently large integer Π such that

$$\lim_{\varrho\to\infty}\sup\bigl\{\pi(F,\Omega): F\subset\Omega\backslash B_\varrho,\ \operatorname{diam}(F)\le\delta\bigr\}\le\frac{1}{4}\left(\frac{\Pi-2}{\Pi+2}\right)^n$$

for all $\delta>0$.

Given any δ, we can find a sufficiently large number $\varrho=\varrho(\delta)$ so that

$$\sup\bigl\{\pi(F,\Omega): F\subset\Omega\backslash B_\varrho,\operatorname{diam}(F)\le\delta\bigr\}\le\frac{1}{4}\left(\frac{\Pi-1}{\Pi+2}\right)^n.$$

Hence

$$\sup\left\{\frac{\mu_{(\varrho)}(F)}{\operatorname{cap}(F,\Omega)}: F\subset\Omega,\operatorname{diam}(F)\le\delta\right\}\le\frac{1}{4}\left(\frac{\Pi-1}{\Pi+2}\right)^n.$$

If we replace $\mu_{(\varrho)}$ here by μ, then we obtain the condition (2.5.9), which was used in the first part of Theorem 2.5.3 for the proof of inequality (2.5.11). We rewrite that inequality, replacing μ by $\mu_{(\varrho)}$:

$$\int_\Omega |u|^2\,\mathrm{d}\mu_{(\varrho)}\le\bigl(1-\Pi^{-n}\bigr)\int_\Omega\left(|\nabla u|^2+c\frac{\Pi^2}{\delta^2}|u|^2\right)\mathrm{d}x. \tag{2.5.18}$$

Let γ denote an arbitrary positive number. We specify $\delta>0$ by the equality $c\Pi^2(1-\Pi^{-n})\delta^{-2}=\gamma$ and find ϱ corresponding to δ. Then

$$\int_\Omega |u|^2\,\mathrm{d}\mu_{(\varrho)}\le\bigl(1-\Pi^{-n}\bigr)\int_\Omega|\nabla u|^2\,\mathrm{d}x+\gamma\int_\Omega|u|^2\,\mathrm{d}x.$$

Hence the form

$$A[u,u]=\int_\Omega|\nabla u|^2-\int_\Omega|u|^2\,\mathrm{d}\mu_{(\varrho)}+2\gamma\int_\Omega|u|^2\,\mathrm{d}x,$$

majorizes

$$\Pi^{-n} \int_{\Omega} |\nabla u|^2 \, \mathrm{d}x + \gamma \int_{\Omega} |u|^2 \, \mathrm{d}x.$$

This means that the form $A[u,u]$ has a positive lower bound γ and is closable in $L_2(\Omega)$. Let $\bar{A}[u,u]$ denote the closure of the form $A[u,u]$. Clearly, the domain of the form $\bar{A}[u,u]$ coincides with $\mathring{W}_2^1(\Omega)$.

By (2.5.15) and Corollary 2.5.3/2, the form

$$B[u,u] = \int_{\Omega} |u|^2 \, \mathrm{d}\mu^{(\varrho)}$$

is continuous in $W_2^1(\Omega)$ and is closable in $\mathring{W}_2^1(\Omega)$. Lemma 2.4.2 ensures the compactness of the form $\bar{B}[u,u]$ in $\mathring{W}_2^1(\Omega)$. It remains to apply Lemma 1 to $\bar{A}[u,u]$ and $\bar{B}[u,u]$.

2. Suppose that

$$\lim_{\varrho \to \infty} \sup\bigl\{\pi(F,\Omega) : F \subset \Omega \backslash B_\varrho,\ \operatorname{diam} \le \delta\bigr\} > 1+\alpha, \quad \alpha > 0,$$

for some δ. Then there exists a sequence of compacta F_ν with $\operatorname{diam}(F_\nu) \le \delta$, which tends to infinity and satisfies

$$\mu(F_\nu) > (1+\alpha) \operatorname{cap}(F_\nu, \Omega). \tag{2.5.19}$$

We include F_ν in a ball $B_\delta^{(\nu)}$ with radius δ. Let $B_\varrho^{(\nu)}$ denote a concentric ball with a sufficiently large radius ϱ that will be specified later. Without loss of generality, we may obviously assume that the balls $B_\varrho^{(\nu)}$ are disjoint.

By Lemma 2.5.1/1,

$$\operatorname{cap}\bigl(F_\nu, B_\varrho^{(\nu)} \cap \Omega\bigr) \le \bigl(1+\varepsilon(\varrho)\bigr) \operatorname{cap}(F_\nu, \Omega),$$

where $\varepsilon(\varrho) \to 0$ as $\varrho \to \infty$. This and (2.5.19) imply

$$\mu(F_\nu) > K \operatorname{cap}\bigl(F_\nu, B_\varrho^{(\nu)} \cap \Omega\bigr), \tag{2.5.20}$$

where

$$K = \frac{1+\alpha}{1+\varepsilon(\varrho)}.$$

Let ϱ be chosen so that the constant K exceeds 1. By (2.5.20) there exists a function u_ν in $\mathfrak{P}(F_\nu, B_\varrho^{(\nu)} \cap \Omega)$ such that

$$\int_{B_\varrho^{(\nu)}} u_\nu^2 \, \mathrm{d}\mu > K \int_{B_\varrho^{(\nu)}} (\nabla u_\nu)^2 \, \mathrm{d}x.$$

Hence

$$S_1[u_\nu, u_\nu] < -(K-1)\frac{\lambda}{\varrho^2} \int_{\Omega} u_\nu^2 \, \mathrm{d}x,$$

where λ is the first eigenvalue of the Dirichlet problem for the Laplace operator in the unit ball.

Now, Lemma 2 implies that the spectrum of the operator $\tilde{S}_1$ has a limit point to the left of $-(K-1)\lambda\varrho^{-2}$. So we arrive at a contradiction. $\square$

2.5.5 Discreteness of the Negative Spectrum of the Operator $\tilde{S}_h$ for all h

The following assertion contains a necessary and sufficient condition for the discreteness of the negative spectrum of the operator $\tilde{S}_h$ for all $h > 0$. We note that although the measure μ in Theorem 2.5.4 is supposed to have no strong singularities at a finite distance (condition (2.5.17)), the corresponding criterion for the family of all operators $\{\tilde{S}_h\}_{h>0}$ is obtained for an arbitrary nonnegative measure.

Corollary. *The conditions*

$$\lim_{\delta\to 0} \sup\big\{\pi(F,\Omega) : F \subset \Omega,\ \operatorname{diam}(F) \le \delta\big\} = 0 \tag{2.5.21}$$

and

$$\lim_{\varrho\to\infty} \sup\big\{\pi(F,\Omega) : F \subset \Omega\backslash B_\varrho, \operatorname{diam}(F) \le 1\big\} = 0 \tag{2.5.22}$$

are necessary and sufficient for the semiboundedness of the form $S_h[u,u]$ in $L_2(\Omega)$ and for the discreteness of the negative spectrum of the operator $\tilde{S}_h$ for all $h > 0$.

We also note that the semiboundedness of the form $S_h[u,u]$ for all $h > 0$ implies that $S_h[u,u]$ is closable in $L_2(\Omega)$ for all $h > 0$.

Proof. Sufficiency. We introduce the notation

$$l(\delta) = \lim_{\varrho\to\infty} \sup\big\{\pi(F,\Omega) : F \subset \Omega\backslash B_\varrho,\ \operatorname{diam}(F) \le \delta\big\}.$$

First we note that (2.5.21) implies (2.5.15). Therefore, according to Theorem 2.5.4, the condition $l(\delta) \equiv 0$, combined with (2.5.21), is sufficient for the semiboundedness of the form $S_h[u,u]$ and for the discreteness of the negative spectrum of the operator $\tilde{S}_h$ for all $h > 0$.

To prove the sufficiency of the conditions $l(1) = 0$ and (2.5.21) we represent an arbitrary compactum F with $\operatorname{diam}(F) \le \delta'$, $\delta' > \delta$, as the union $\bigcup_{\nu=1}^{N} F_\nu$, where $\operatorname{diam}(F_\nu) \le \delta$ and N depends only on δ'/δ and n. Since $\operatorname{cap}(F,\Omega)$ is a nondecreasing function of F, then

$$\frac{\mu(F)}{\operatorname{cap}(F,\Omega)} \le \sum_{\nu=1}^{N} \frac{\mu(F_\nu)}{\operatorname{cap}(F_\nu,\Omega)}.$$

This and the monotonicity of $l(\delta)$ immediately imply $l(\delta) \le l(\delta') \le N l(\delta)$, which proves the equivalence of the conditions $l(\delta) \equiv 0$ and $l(1) = 0$.

Necessity. If the form $S_h[u,u]$ is semibounded for all $h > 0$, then by Corollary 2.5.3/1 the condition (2.5.21) holds together with (2.5.15). Under (2.5.15) Theorem 2.5.4 implies the necessity of $l(\delta) \equiv 0$ which is equivalent to $l(1) = 0$. The corollary is proved. $\square$

2.5.6 Finiteness of the Negative Spectrum

Theorem. *Suppose that the condition* (2.5.15) *holds.*

1. *If*

$$\lim_{\varrho\to\infty} \sup\big\{\pi(F,\Omega) : F \subset \Omega\backslash B_\varrho\big\} < \frac{1}{4}, \tag{2.5.23}$$

then the form $S_1[u,u]$ *is semibounded from below and closable in* $L_2(\Omega)$, *and the negative spectrum of the operator* $\tilde{S}_1$ *is finite.*

2. *If the form* $S_1[u,u]$ *is semibounded from below and closable in* $L_2(\Omega)$, *and the negative spectrum of the operator* $\tilde{S}_1$ *is finite, then*

$$\lim_{\varrho\to\infty} \sup\big\{\pi(F,\Omega) : F \subset \Omega\backslash B_\varrho\big\} \le 1. \tag{2.5.24}$$

Proof. 1. Since for any compactum $F \subset \Omega$

$$\frac{\mu(F)}{\operatorname{cap}(F,\Omega)} \le \frac{\mu(F\backslash B_\varrho)}{\operatorname{cap}(F\backslash B_\varrho,\Omega)} + \frac{\mu(F\cap \bar{B}_\varrho)}{\operatorname{cap}(F\cap \bar{B}_\varrho,\Omega)}, \tag{2.5.25}$$

conditions (2.5.15) and (2.5.23) imply

$$\lim_{\delta\to 0}\big\{\pi(F,\Omega) : F\subset\Omega, \operatorname{diam}(F)\le\delta\big\} < \frac{1}{4}.$$

According to the last inequality and Theorem 2.5.3, the form $S_1[u,u]$ is semibounded and closable in $L_2(\Omega)$. From (2.5.11) it follows that the metric

$$C\int_\Omega |u|^2\,\mathrm{d}x + S_1[u,u]$$

is equivalent to the metric of the space $\mathring{W}{}_2^1(\Omega)$ for C large enough.

Turning to condition (2.5.23), we note that there exists a positive constant α such that

$$\sup\big\{\pi(F,\Omega) : F\subset\Omega, F\subset\Omega\backslash B_{\varrho_0}\big\} < \frac{1}{4} - \alpha$$

for sufficiently large ϱ_0. Hence

$$\sup\left\{\frac{\mu_{(\varrho_0)}(F)}{\operatorname{cap}(F,\Omega)} : F\subset\Omega\right\} < \frac{1}{4} - \alpha,$$

and by Theorem 2.5.2 the form

$$(1-4\alpha)\int_\Omega |\nabla u|^2\,\mathrm{d}x - \int_\Omega |u|^2 \mu_{(\varrho_0)}(\mathrm{d}x)$$

is positive. Therefore for any $u\in\mathscr{D}(\Omega)$

$$S_1[u,u] \geq 4\alpha \int_\Omega |\nabla u|^2 \, dx - \int_\Omega |u|^2 \mu^{(\varrho_0)}(dx).$$

We estimate the right-hand side by inequality (2.4.10) with $\varepsilon = 2\alpha$, $p = 2$, $M(t) = |t|$:

$$S_1[u,u] \geq 2\alpha \int_\Omega |\nabla u|^2 \, dx - K \int_{B_{2\varrho_0} \cap \Omega} |u|^2 \, dx. \tag{2.5.26}$$

Passing to the closure of the form $S_1[u,u]$, we obtain (2.5.26) for all $u \in \mathring{W}_2^1(\Omega)$.

Since any set, bounded in $\mathring{L}_2^1(\Omega)$, is compact in the metric

$$\left(\int_{B_\varrho \cap \Omega} |u|^2 \, dx \right)^{1/2}$$

for any $\varrho > 0$, the form

$$2\alpha \int_\Omega |u|^2 \, dx - K \int_{B_{2\varrho_0} \cap \Omega} |u|^2 \, dx,$$

is nonnegative up to a finite-dimensional manifold. Taking (2.5.26) into account, we may say the same for the form $S_1[u,u]$. Now the result follows from Lemma 2.5.4/2.

2. Suppose

$$\lim_{\varrho \to \infty} \sup\bigl\{\pi(F,\Omega) : F \subset \Omega \backslash B_\varrho \bigr\} > 1 + \alpha,$$

where $\alpha > 0$.

Let $\{\varrho_k\}_{k \geq 1}$ denote an increasing sequence of positive numbers such that

$$\varrho_k \varrho_{k+1}^{-1} \xrightarrow{k \to \infty} 0. \tag{2.5.27}$$

We construct the subsequence $\{\varrho_{k_\nu}\}_{\nu \geq 1}$, defined as follows: Let $k_1 = 1$. We find a compactum F_1, contained in $\Omega \backslash \bar{B}_{\varrho_{k_1}}$, such that $\pi(F_1, \Omega) > 1 + \alpha$. Further we select k_2 to be so large that F_1 is contained in $B_{\varrho_{k_2}}$. Let F_2 denote a compactum in $\Omega \backslash B_{\varrho_{k_2+1}}$ such that $\pi(F_2, \Omega) > 1 + \alpha$. If numbers $k_1, \dots, k_\nu$ and compacta $F_1, \dots, F_\nu$ have already been chosen, then $k_{\nu+1}$ is defined by the condition $F_\nu \subset B_{\varrho_{k_{\nu+1}}}$. The set $F_{\nu+1} \subset \Omega \backslash B_{\varrho_{k_{\nu+1}+1}}$ must be chosen to satisfy the inequality

$$\pi(F_{\nu+1}, \Omega) > 1 + \alpha.$$

Thus we obtained a sequence of compacta $F_\nu \subset \Omega$ with F_ν in the spherical layer $B_{\varrho_{k_{\nu+1}}} \backslash \bar{B}_{\varrho_{k_\nu+1}}$ and subject to the condition

$$\mu(F_\nu) > (1 + \alpha) \operatorname{cap}(F_\nu, \Omega). \tag{2.5.28}$$

We introduce the notation $R_\nu = B_{\varrho_{k_{\nu+1}+1}} \backslash \bar{B}_{\varrho_{k_\nu}}$. By Lemma 2.5.1/2, where $r = \varrho_{k_\nu}$ and $R = \varrho_{k_\nu+1}$,

$$\operatorname{cap}(F_\nu, \Omega \cap R_\nu) \le \left(1 + (n-2)^{-1}\frac{\varrho_{k_\nu}}{\varrho_{k_\nu+1} - \varrho_{k_\nu}}\right)\operatorname{cap}(F_\nu, \Omega \cap B_{\varrho_{k_{\nu+1}+1}}),$$

which along with condition (2.5.27) implies

$$\operatorname{cap}(F_\nu, \Omega \cap R_\nu) \le \big[1 + o(1)\big]\operatorname{cap}(F_\nu, \Omega \cap B_{\varrho_{k_{\nu+1}+1}}). \tag{2.5.29}$$

From Lemma 2.5.1/1 with $r = \varrho_{k_\nu+1}$ and $R = \varrho_{k_{\nu+1}+1}$ it follows that

$$\operatorname{cap}(F_\nu, \Omega \cap B_{\varrho_{k_{\nu+1}+1}}) \le \big[1 + o(1)\big]\operatorname{cap}(F_\nu, \Omega).$$

According to (2.5.29),

$$\operatorname{cap}(F_\nu, \Omega \cap R_\nu) \le \big[1 + o(1)\big]\operatorname{cap}(F_\nu, \Omega).$$

Hence by (2.5.28), for sufficiently large ν,

$$\mu(F_\nu) > (1 + \alpha')\operatorname{cap}(F_\nu, \Omega \cap R_\nu),$$

where α' is a positive constant.

Now we can find a sequence of functions $u_\nu \in \mathfrak{P}(F_\nu, \Omega \cap R_\nu)$ such that

$$\int_{R_\nu \cap \Omega} u_\nu^2 \mu(\mathrm{d}x) > (1 + \alpha')\int_{R_\nu \cap \Omega} (\nabla u_\nu)^2\,\mathrm{d}x,$$

which yields the inequality $S_1[u_\nu, u_\nu] < 0$. It remains to note that the supports of the functions u_ν are disjoint and therefore the last inequality holds for all linear combinations of u_ν. This and Lemma 2.5.4/2 imply that the negative spectrum of the operator S_1 is infinite. The theorem is proved. □

2.5.7 Infiniteness and Finiteness of the Negative Spectrum of the Operator $\tilde{S}_h$ for all h

We shall find criteria for the infiniteness and for the finiteness of the negative spectrum of the operator $\tilde{S}_h$ for all h. We underline that here, as in the proof of the discreteness criterion in Corollary 2.5.5, we obtain the necessary and sufficient conditions without additional assumptions on the measure μ.

Corollary 1. *Conditions* (2.5.21) *and*

$$\sup\big\{\pi(F, \Omega) : F \subset \Omega\big\} = \infty \tag{2.5.30}$$

are necessary and sufficient for the semiboundedness of the form $S_h[u, u]$ in $L_2(\Omega)$ and for the infiniteness of the spectrum of the operator $\tilde{S}_h$ for all $h > 0$.

Proof. By Corollary 2.5.3/1, (2.5.21) is equivalent to the semiboundedness of the form $S_h[u, u]$ for all $h > 0$.

We must prove that the criterion

$$\lim_{\varrho\to\infty} \sup\{\pi(F,\Omega) : F \subset \Omega\backslash B_\varrho\} = \infty, \tag{2.5.31}$$

which follows from Theorem 2.5.6, is equivalent to (2.5.30). Obviously, (2.5.30) is a consequence of (2.5.31). Assume that the condition (2.5.30) is valid. Taking into account (2.5.21), we obtain

$$\sup\{\pi(F,\Omega) : F \subset B_\varrho \cap \Omega\} < \infty$$

for any ϱ. On the other hand, (2.5.30) implies

$$\lim_{\varrho\to\infty} \sup\{\pi(F,\Omega) : F \subset B_\varrho \cap \Omega\} = \infty.$$

We choose a sequence $\varrho_\nu \to \infty$ such that

$$\sup\{\pi(F,\Omega) : F \subset B_{\varrho_{\nu+1}} \cap \Omega\} > 2\sup\{\pi(F,\Omega) : F \subset B_{\varrho_\nu} \cap \Omega\}.$$

From this and inequality (2.5.25) we obtain

$$\sup\{\pi(F,\Omega) : F \subset R_{\varrho_\nu,\varrho_{\nu+1}} \cap \Omega\} \geq \sup\{\pi(F,\Omega) : F \subset B_{\varrho_\nu} \cap \Omega\},$$

where $R_{\varrho,\varrho'} = B_{\varrho'}\backslash \bar{B}_\varrho$. Hence

$$\sup\{\pi(F,\Omega) : F \subset R_{\varrho_\nu,\varrho_{\nu+1}} \cap \Omega\} \xrightarrow{\nu\to\infty} \infty,$$

and the result follows. □

Corollary 2. *Conditions* (2.5.21) *and*

$$\lim_{\varrho\to\infty} \sup\{\pi(F,\Omega) : F \subset \Omega\backslash B_\varrho\} = 0 \tag{2.5.32}$$

are necessary and sufficient for the semiboundedness of $\tilde{S}_h$ and for the finiteness of the negative spectrum of $\tilde{S}_h$ for all $h > 0$.

The necessity and sufficiency of conditions (2.5.21) and (2.5.32) immediately follow from Theorem 2.5.6.

2.5.8 Proofs of Lemmas 2.5.1/1 and 2.5.1/2

The following facts are well known (cf. Landkof [477]). For $n \geq 3$ and for any open set $\Omega \subset \mathbb{R}^n$ there exists a unique Green function $G(x, y)$ of the Dirichlet problem for the Laplace operator.

Let μ be a nonnegative measure in Ω. Let V^μ denote the Green potential of the measure μ, i.e.,

$$V^\mu(x) = \int_\Omega G(x, y)\mu(\mathrm{d}y).$$

Obviously, V^{μ} is a harmonic function outside the support of the measure μ. There exists a unique capacitary distribution of a compactum F with respect to Ω, i.e., a measure μ_F, supported on F, such that $V^{\mu_F}(x) \le 1$ in Ω and

$$\mu_F(F) = (n-2)^{-1}\omega_n^{-1}\,\mathrm{cap}(F,\Omega).$$

The potential V^{μ_F} is called the capacitary potential of F relative to Ω. If F is the closure of an open set with C^∞-smooth boundary, then V^{μ_F} is a smooth function in $\overline{\Omega\backslash F}$, equal to unity on F, and continuous in Ω.

Proof of Lemma 2.5.1/1. Using the continuity of the capacity from the right, we can easily reduce the proof for an arbitrary compactum to the consideration of a compactum $F \subset B_r \cap \Omega$, which is the closure of an open set with a C^∞-smooth boundary.

Let V^{μ_F} denote the capacitary potential of F relative to Ω and let η denote a continuous piecewise linear function, equal to unity on $[0,r]$, and to zero outside $[0,R]$.

The function $u(x) = \eta(|x|)V^{\mu_F}(x)$ can be approximated in $\mathring{L}_2^1(\Omega \cap B_R)$ by functions in $\mathfrak{N}(F, B_R \cap \Omega)$. Hence

$$\mathrm{cap}(F, B_R \cap \Omega) \le \int_{B_R \cap \Omega} |\nabla u|^2\,\mathrm{d}x. \tag{2.5.33}$$

We extend V^{μ_F} to be zero outside Ω. It is readily checked that

$$\int_{B_R\cap\Omega} |\nabla u|^2\,\mathrm{d}x = \int_{B_R} \left|\nabla V^{\mu_F}\right|^2 \eta^2\,\mathrm{d}x + A + B, \tag{2.5.34}$$

where

$$A = \frac{1}{R-r}\int_{\partial B_r} \left(V^{\mu_F}\right)^2 s(\mathrm{d}x),$$
$$B = \frac{n-1}{(R-r)^2}\int_{B_R\backslash B_r} \left(V^{\mu_F}\right)^2 \frac{R-|x|}{|x|}\,\mathrm{d}x.$$

Obviously,

$$\int_{\Omega} \left|\nabla V^{\mu_F}\right|^2 \eta^2\,\mathrm{d}x \le \mathrm{cap}(F,\Omega). \tag{2.5.35}$$

Now we note that

$$\int_{\partial B_\varrho} V^{\mu_F} s(\mathrm{d}x) = \int_F \mu_F(\mathrm{d}y) \int_{\partial B_\varrho} G(x,y)s(\mathrm{d}x) \le \int_F \mu_F(\mathrm{d}y)\int_{\partial B_\varrho} \frac{s(\mathrm{d}x)}{|x-y|^{n-2}}.$$

The integral over ∂B_ϱ is a single-layer potential and it is equal to a constant on ∂B_ϱ. Hence, for $y \in B_\varrho$,

$$\int_{\partial B_\varrho} \frac{s(\mathrm{d}x)}{|x-y|^{n-2}} = \omega_n \varrho.$$

Thus

$$\int_{\partial B_\varrho} V^{\mu_F} s(\mathrm{d}x) \le (n-2)^{-1} \varrho \,\mathrm{cap}(F, \Omega). \tag{2.5.36}$$

The following inequality is a direct consequence of the maximum principle

$$V^{\mu_F}(x) \le \frac{r^{n-2}}{|x|^{n-2}} \quad \text{for } |x| \ge r.$$

Now, the bound for A is

$$A \le (R-r)^{-1} \int_{\partial B_r} V^{\mu_F} s(\mathrm{d}x) \le \frac{r}{(n-2)(R-r)} \mathrm{cap}(F, \Omega). \tag{2.5.37}$$

We introduce spherical coordinates (ϱ, ω) in the integral B. Then

$$B = \frac{n-1}{(R-r)^2} \int_r^R \varrho^{n-2}(R-\varrho)\,\mathrm{d}\varrho \int_{\partial B_\varrho} \big(V^{\mu_F}\big)^2 \omega(\mathrm{d}x).$$

Hence

$$B \le \frac{(n-1)r^{n-2}}{R-r} \int_r^R \mathrm{d}\varrho \int_{\partial B_\varrho} V^{\mu_F} \omega(\mathrm{d}x).$$

Using (2.5.36), we obtain

$$B \le \frac{n-1}{n-2}\, \frac{r^{n-2}}{R-r} \int_r^R \varrho^{2-n}\,\mathrm{d}\varrho \,\mathrm{cap}(F, \Omega),$$

which along with (2.5.33)–(2.5.35) and (2.5.37) gives the final result. □

Proof of Lemma 2.5.1/2. The general case can be easily reduced to the consideration of a compactum $F \subset \Omega \backslash \bar{B}_R$, which is the closure of an open set with a smooth boundary. Let V^{μ_F} denote the capacitary potential of F relative to Ω, extended by zero outside Ω.

The function

$$u(x) = \begin{cases} V^{\mu_F}(x) & \text{for } x \in \Omega \backslash B_R, \\ \frac{R(|x|-r)}{|x|(R-r)} V^{\mu_F}(x) & \text{for } x \in \Omega \cap (B_R \backslash B_r), \\ 0 & \text{for } x \in \Omega \cap B_r, \end{cases}$$

can be approximated in $\mathring{L}_2^1(\Omega \backslash \bar{B}_r)$ by the functions in $\mathfrak{N}(F, \Omega \backslash \bar{B}_r)$. Therefore,

$$\mathrm{cap}(F, \Omega \backslash \bar{B}_r) \le \int_{\Omega \backslash \bar{B}_r} (\nabla u)^2 \,\mathrm{d}x.$$

This implies

$$\mathrm{cap}(F, \Omega \backslash \bar{B}_r) \le \int_\Omega \big(\nabla V^{\mu_F}\big)^2 \,\mathrm{d}x + \frac{r}{R(R-r)} \int_{\partial B_R} (V^{\mu_F})^2 s(\mathrm{d}x). \tag{2.5.38}$$

Since $V^{\mu_F} \le 1$ and

$$\int_{\partial B_R} V^{\mu_F} s(\mathrm{d}x) \le (n-2)^{-1} R \,\mathrm{cap}(F, \Omega)$$

(cf. (2.5.36)), it follows that

$$\frac{r}{R(R-r)} \int_{\partial B_r} \left(V^{\mu_F}\right)^2 s(\mathrm{d}x) \le \frac{r}{R-r} (n-2)^{-1} \,\mathrm{cap}(F, \Omega),$$

which together with (2.5.38) completes the proof. □

2.5.9 Comments to Sect. 2.5

The presentation follows the author's paper [534] and the main results were announced in Maz'ya [531]. A number of results on the spectrum of the Schrödinger operator are presented in the monograph by Glazman [309] who used the so-called splitting method. Birman [100, 101] established some important results in the perturbation theory of quadratic forms in Hilbert spaces. In particular, he proved that the discreteness (the finiteness) of the negative spectrum of the operator $S_h = -h\Delta - p(x)$ in $\mathbb{R}^n$ for $p(x) \ge 0$ and for all $h > 0$ is equivalent to the compactness of the embedding of $W_2^1(\mathbb{R}^n)(\mathring{L}_2^1(\mathbb{R}^n))$ into the space with the norm

$$\left(\int_{\mathbb{R}^n} |u|^2 p(x)\,\mathrm{d}x\right)^{1/2}.$$

Using such criteria, Birman derived the necessary or sufficient conditions for the discreteness, finiteness, or infiniteness of the negative spectrum of S_h for all $h > 0$. The statement of these conditions makes no use of the capacity. The results of Birman's paper [101] were developed in the author's paper [534] the content of which is followed here.

The theorems of Sect. 2.5 turned out to be useful in the study of the asymptotic behavior of eigenvalues of the Dirichlet problem for the Schrödinger operator. Rozenblum [683] considered the operator $H = -\Delta + q(x)$ in $\mathbb{R}^n$ with $q = q_+ - q_-$, where $q_- \in L_{n/2,\mathrm{loc}}, n \ge 3$. We state one of his results. Let a cubic grid be constructed in $\mathbb{R}^n$ with d as the edge length of each cube and let $F(d)$ be the union of those cubes $\mathscr{Q}$ of the grid that satisfies the condition

$$\sup\left\{\frac{\int_E q_-(x)\,\mathrm{d}x}{\mathrm{cap}(E)} : E \subset 2\mathscr{Q}\right\} > \gamma,$$

where $2\mathscr{Q}$ is the concentric homothetic cube having edge length $2d$, $\gamma = \gamma(n)$ is a large enough number.

Then, for $\lambda > 0$, the number $\mathscr{N}(-\lambda, H)$ of eigenvalues of H that are less than $-\lambda$ satisfies the inequality

$$\mathscr{N}(-\lambda, H) \le c_1 \int_{F(c_2\lambda^{1/2})} \big(c_3\lambda - q(x)\big)_+^{n/2}\,\mathrm{d}x,$$

where c_1, c_2, and c_3 are certain constants depending only on n.

In the case $\Omega = \mathbb{R}^n$ by Theorem 2.5.2, the inequality (2.5.4) with $\Omega = \mathbb{R}^n$ holds if and only if

$$\sup_F \frac{\mu(F)}{\operatorname{cap}(F)} < \infty,$$

where F is an arbitrary compact set in $\mathbb{R}^n$. For the same case, other criteria for the validity of (2.5.4) are known. The following one is due to Kerman and Sawyer [420] (see Theorem 11.5/1 of the present book):

For every open ball B in $\mathbb{R}^n$,

$$\int_B \int_B \frac{\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)}{|x-y|^{n-2}} \le c\mu(B).$$

Another two criteria for (2.5.4) were obtained by Maz'ya and Verbitsky [591]:

(i) *The pointwise inequality*

$$I_1(I_1\mu)^2(x) \le cI_1(\mu)(x) < \infty \quad \text{a.e.}$$

holds, where I_1 stands for the Riesz potential of order 1, *i.e.*, $I_1\mu = |x|^{1-n} \star \mu$.

(ii) *For every compact set $F \subset \mathbb{R}^n$,*

$$\int_F (I_1\mu)^2\,\mathrm{d}x \le c\operatorname{cap}(F).$$

One more condition necessary and sufficient for (2.5.4) was found by Verbitsky [775]:

For every dyadic cube P in $\mathbb{R}^n$,

$$\sum_{Q\subset P} \left[\frac{\mu(Q)}{|Q|^{1-1/n}}\right]^2 |Q| \le c\mu(P),$$

where the sum is taken over all dyadic cubes Q contained in P and c does not depend on P.

We now state the main result of the paper [592] by the author and Verbitsky, characterizing arbitrary complex-valued distributions V subject to the inequality

$$\left|\int_{\mathbb{R}^n} |u|^2 V\,\mathrm{d}x\right| \le c\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}x \quad \text{for all } u \in \mathscr{D}. \tag{2.5.39}$$

This characterization reduces the case of distributional potentials V to that of nonnegative absolutely continuous weights. (Cf. Sect. 1.3.4, where similar statements are established for functions of one variable.)

Theorem. *Let $V \in \mathscr{D}'$, $n > 2$. Then the inequality (2.5.39) holds, if and only if there is a vector field $\mathbf{\Gamma} \in L_2(\mathbb{R}^n, \mathrm{loc})$ such that $V = \operatorname{div} \mathbf{\Gamma}$ and*

$$\int_{\mathbb{R}^n} \left|u(x)\right|^2 \left|\mathbf{\Gamma}(x)\right|^2 \mathrm{d}x \le C \int_{\mathbb{R}^n} \left|\nabla u(x)\right|^2 \mathrm{d}x$$

for all $u \in \mathscr{D}$. The vector field $\mathbf{\Gamma}$ can be chosen in the form $\mathbf{\Gamma} = \nabla \Delta^{-1} V$.

2.6 Properties of Sobolev Spaces Generated by Quadratic Forms with Variable Coefficients

2.6.1 Degenerate Quadratic Form

In the preceding sections of the present chapter we showed that rather general inequalities, containing the integral $\int_\Omega [\Phi(x, \nabla u)]^p \, \mathrm{d}x$, are equivalent to isocapacitary inequalities that relate (p, Φ)-capacity and measures. Although such criteria are of primary interest, we should note that their verification in particular cases is often difficult. Even for rather simple quadratic forms

$$\left[\Phi(x, \xi)\right]^2 = \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j,$$

the estimates for the corresponding capacities by measures are unknown.

Thus, the general necessary and sufficient conditions obtained in the present chapter cannot diminish the value of straightforward methods of investigation of integral inequalities without using capacity. In the present section this will be illustrated, using as an example the quadratic form

$$\left[\Phi(x, \xi)\right]^2 = \left(|x_n| + |x'|^2\right) \xi_n^2 + |\xi'|^2,$$

where $x' = (x_1, \dots, x_{n-1})$, $\xi' = (\xi_1, \dots, \xi_{n-1})$.

By Corollary 2.3.4, the inequality

$$\int_{\mathbb{R}^{n-1}} \left[u(x', 0)\right]^2 \mathrm{d}x' \le c \int_{\mathbb{R}^n} \left[\Phi(x, \nabla u)\right]^2 \mathrm{d}x \tag{2.6.1}$$

holds for all $u \in \mathscr{D}(\mathbb{R}^n)$ if and only if

$$m_{n-1}\left(\{x \in g, \ x_n = 0\}\right) \le c (2, \Phi)\text{-cap}(g)$$

for any admissible set g. A straightforward proof of the preceding isoperimetric inequality is unknown to the author. Nevertheless, the estimate (2.6.1) is true and will be proved in the sequel.

Theorem 1. *Let*

$$\big[\varPhi(x,\nabla u)\big]^2=\big(|x_n|+|x'|^2\big)(\partial u/\partial x_n)^2+\sum_{i=1}^{n-1}(\partial u/\partial x_i)^2.$$

Then (2.6.1) is valid for all $u\in\mathscr{D}(\mathbb{R}^n)$.

Proof. Let the integral in the right-hand side of (2.6.1) be denoted by $Q(u)$. For any $\delta\in(0,1/2)$ we have

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}\big|u(x',0)\big|^2\,\mathrm{d}x'&\le 2\int_{\mathbb{R}^n}\frac{(|x_n|+|x'|^2)^{1/2}}{|x_n|^{(1-\delta)/2}|x'|^{\delta}}\left|u\frac{\partial u}{\partial x_n}\right|\mathrm{d}x\\&\le 2\big[Q(u)\big]^{1/2}\bigg(\int_{\mathbb{R}^n}|x_n|^{\delta-1}|x'|^{-2\delta}|u|^2\,\mathrm{d}x\bigg)^{1/2}.\end{aligned}\tag{2.6.2}$$

To give a bound for the last integral we use the following well-known generalization of the Hardy–Littlewood inequality:

$$\int_{\mathbb{R}^{n-1}}\bigg(\int_{\mathbb{R}^{n-1}}\frac{f(y)\,\mathrm{d}y}{|x'-y|^{n-1-\delta}}\bigg)^2\frac{\mathrm{d}x'}{|x'|^{2\delta}}\le c\int_{\mathbb{R}^{n-1}}\big[f(y)\big]^2\,\mathrm{d}y.\tag{2.6.3}$$

(For the proof of this estimate see Lizorkin [505]. It can also be derived as a corollary to Theorem 1.4.1/2.) Since the convolution with the kernel $|x'|^{\delta+1-n}$ corresponds to the multiplication by $|\xi'|^{-\delta}$ of the Fourier transform, (2.6.3) can be written as

$$\int_{\mathbb{R}^{n-1}}|u|^2|x'|^{-2\delta}\,\mathrm{d}x'\le c\int_{\mathbb{R}^{n-1}}\big[(-\varDelta_{x'})^{\delta/2}u\big]^2\,\mathrm{d}x',$$

where $(-\varDelta_{x'})^{\delta/2}$ is the fractional power of the Laplace operator. Now we find that the right-hand side in (2.6.2) does not exceed

$$c\bigg(Q(u)+\int_{\mathbb{R}^n}|x_n|^{\delta-1}\big[(-\varDelta_{x'})^{\delta/2}u\big]^2\,\mathrm{d}x\bigg).\tag{2.6.4}$$

From the almost obvious estimate

$$\int_0^\infty g^2t^{\delta-1}\,\mathrm{d}t\le c\bigg(\int_0^\infty(g')^2t\,\mathrm{d}t+\int_0^\infty g^2\,\mathrm{d}t\bigg),$$

it follows that

$$\begin{aligned}&|\xi'|^{2\delta}\int_{\mathbb{R}^n}\big|(F_{x'\to\xi'}u)(\xi',x_n)\big|^2|x_n|^{\delta-1}\,\mathrm{d}x_n\\&\quad\le c\bigg(\int_{\mathbb{R}^1}\bigg|\bigg(F_{x'\to\xi'}\frac{\partial u}{\partial x_n}\bigg)(\xi',x_n)\bigg|^2|x_n|\,\mathrm{d}x_n\\&\qquad+|\xi'|^2\int_{\mathbb{R}^1}\big|(F_{x'\to\xi'}u)(\xi',x_n)\big|^2\,\mathrm{d}x_n\bigg),\end{aligned}$$

where $F_{x'\to\xi'}$ is the Fourier transform in $\mathbb{R}^{n-1}$. So the second integral in (2.6.4) does not exceed

$$c\int_{\mathbb{R}^n}\big(|x_n|(\partial u/\partial x_n)^2+(\nabla_{x'}u)^2\big)\,\mathrm{d}x.$$

The result follows. □

The next assertion shows that Theorem 1 is exact in a certain sense.

Theorem 2. *The space of restrictions to* $\mathbb{R}^{n-1}=\{x\in\mathbb{R}^n : x_n=0\}$ *of functions in the set* $\{u\in\mathscr{D}(\mathbb{R}^n) : Q(u)+\|u\|^2_{L_2(\mathbb{R}^n)}\le 1\}$ *is not relatively compact in* $L_2(B_1^{(n-1)})$*, where* $B_\varrho^{(n-1)}=\{x'\in\mathbb{R}^{n-1} : |x'|<\varrho\}$.

Proof. Let φ denote a function in $C_0^\infty(B_1^{(n-1)})$ such that $\varphi(y)=\varphi(-y)$, $\|\varphi\|_{L_2(\mathbb{R}^{n-1})}=1$ and introduce the sequence $\{\varphi_m\}_{m=1}^\infty$ defined by $\varphi_m(y)=m^{(n-1)/2}\varphi(my)$. Since this sequence is normalized and weakly convergent to zero in $L_2(B_1^{(n-1)})$, it contains no subsequences converging in $L_2(B_1^{(n-1)})$. Further, let $\{v_m\}_{m=1}^\infty$ be the sequence of functions in $\mathbb{R}^n$ defined by

$$v_m(x)=F^{-1}_{\eta'\to x'}\exp\big\{-\langle\eta\rangle^2|x_n|\big\}F_{x'\to\eta'}\varphi_m,$$

where $\eta\in\mathbb{R}^{n-1}$, $\langle\eta\rangle=(|\eta|^2+1)^{1/2}$.

Consider the quadratic form

$$T(u)=\int_{\mathbb{R}^n}\left[\big(|x_n|+|x'|^2\big)\left|\frac{\partial u}{\partial x_n}\right|^2+|\nabla_{x'}u|^2+|u|^2\right]\mathrm{d}x.$$

It is clear that

$$T(u)=(2\pi)^{1-n}\int_{\mathbb{R}^n}\left(|x_n|\left|\frac{\partial Fu}{\partial t}\right|^2+\left|\frac{\partial}{\partial t}\nabla_\eta Fu\right|^2+\langle\eta\rangle^2|Fu|^2\right)\mathrm{d}\eta\,\mathrm{d}x_n.$$

Differentiating the function $T(v_m)$, we obtain from the last equality that $T(v_m)$ does not exceed

$$c\int_{\mathbb{R}^n}\big[\big(1+\langle\eta\rangle^2|x_n|+\langle\eta\rangle^4|x_n|^3\big)\langle\eta\rangle^2|F\varphi_m|^2+\langle\eta\rangle^4|\nabla F\varphi_m|^2\big]$$
$$\times\exp\big(-2\langle\eta\rangle^2|x_n|\big)\,\mathrm{d}\eta\,\mathrm{d}x_n.$$

Thus we obtain

$$\begin{aligned}T(v_m)&\le c\int_{\mathbb{R}^{n-1}}\big(\langle\eta\rangle^2|\nabla F\varphi_m|^2+|F\varphi_m|^2\big)\,\mathrm{d}\eta\\&=c_1\left(\sum_{i=1}^{n-1}\|x_i\varphi_m\|^2_{W_2^1(\mathbb{R}^{n-1})}+\|\varphi_m\|^2_{L_2(\mathbb{R}^{n-1})}\right)\le\text{const}.\end{aligned}$$

Let $\psi\in C_0^\infty(B_2^{(n-1)})$, $\psi=1$ on $B_1^{(n-1)}$. It is clear that $(v_m\psi)|_{\mathbb{R}^{n-1}}=\varphi_m$ and $T(v_m\psi)\le\text{const}$. The sequence $\{v_m\psi/(T(v_m\psi))^{1/2}\}_{m=1}^\infty$ is the required counterexample. The theorem is proved. □

2.6.2 Completion in the Metric of a Generalized Dirichlet Integral

Consider the quadratic form

$$S[u,u] = \int_{\mathbb{R}^n} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + u^2 \right) \mathrm{d}x,$$

where $\|a_{ij}(x)\|_{i,j=1}^n$ is a uniformly positive definite matrix, whose elements $a_{ij}(x)$ are smooth real functions.

Let the completion of $C_0^{0,1}$ with respect to the norm $(S[u,u])^{1/2}$ be denoted by $\overset{\circ}{H}(S)$. Further, we introduce the space $H(S)$ obtained as the completion with respect to this norm of the set of functions in $C^{0,1}$ with the finite integral $S[u,u]$.

If the elements of the matrix $\|a_{ij}\|_{i,j=1}^n$ are bounded functions, then $\overset{\circ}{H}(S) = \overset{\circ}{W}{}_2^1$, $H(S) = W_2^1$ and both spaces, obviously, coincide. It is also known that $\overset{\circ}{H}(S) = H(S)$ if the functions a_{ij} do not grow too rapidly at infinity. Here we consider the problem of the coincidence of $\overset{\circ}{H}(S)$ and $H(S)$ in the general case.

Definition. Let $E \subset \mathbb{R}^n$. In the present subsection the set E is said to have finite $H(S)$ capacity if there exists a function $u \in C^{0,1} \cap H(S)$ that is equal to 1 on E.

Theorem 1. *The spaces $\overset{\circ}{H}(S)$ and $H(S)$ coincide if and only if, for an arbitrary domain G with finite $H(S)$ capacity, there exists a sequence of functions $\{\varphi_m\}_{m\geq 1}$ in $C_0^{0,1}$ that converges in measure to unity on G and is such that*

$$\lim_{m\to\infty} \int_G a_{ij}(x) \frac{\partial \varphi_m}{\partial x_i} \frac{\partial \varphi_m}{\partial x_j} \,\mathrm{d}x = 0. \tag{2.6.5}$$

Before we proceed to the proof, we note that if G is a bounded domain then the sequence $\{\varphi_m\}_{m\geq 1}$ always exists. We can put $\varphi_m = \varphi$ where $\varphi \in C_0^{0,1}$, $\varphi = 1$ on G.

Proof. Sufficiency. We show that any function $u \in C^{0,1} \cap H(S)$ can be approximated in $H(S)$ by functions in $\overset{\circ}{H}(S)$. Without loss of generality we may assume that $u \geq 0$.

First, we note that if $t > 0$ then $\mathscr{L}_t = \{x : u(x) > t\}$ is a set of finite $H(S)$ capacity. In fact, the function $v(x) = t^{-1} \min\{u(x), t\}$ equals unity on $\mathscr{L}_t$, satisfies a Lipschitz condition, and $S[v,v] \leq t^{-2} S[u,u] < \infty$.

From the Lebesgue theorem it follows that the sequences $\min\{u, m\}$, $(u - m^{-1})_+$, $m = 1, 2, \dots$, converge to u in $H(S)$ (see Sect. 5.1.2). So we may assume from the very beginning that u is bounded and vanishes on the exterior of a bounded set G of finite $H(S)$ capacity.

We denote the complements of the set G by G_j and then define the sequence

$$u^{(m)}(x) = \begin{cases} u(x) & \text{for } x \in \bigcup_{j\le m} G_j, \\ 0 & \text{for } x \in \mathbb{R}^n \backslash \bigcup_{j\le m} G_j, \end{cases}$$

$j = 1, 2, \dots$. It is clear that $u^{(m)} \to u$ in $H(S)$ as $m \to \infty$. Since each $u^{(m)}$ vanishes on the exterior of a finite number of domains, we may assume without loss of generality that G is a domain.

Let $\{\varphi_m\}$ be the sequence of functions specified for the domain G in the statement of the theorem. Replacing $\{\varphi_m\}$ by the sequence $\{\psi_m\}$, defined by $|\psi_m| = \min\{2, |\varphi_m|\}$, sgn $\psi_m =$ sgn φ_m, we obtain a bounded sequence with the same properties. Obviously, $\psi_m u \in \mathring{H}(S)$ and $\psi_m u \to u$ in L_2. Moreover,

$$\int_{\mathbb{R}^n} a_{ij} \frac{\partial}{\partial x_i}(u - u\psi_m) \frac{\partial}{\partial x_j}(u - u\psi_m)\, \mathrm{d}x$$
$$\le 2\int_G (1-\psi_m)^2 a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\, \mathrm{d}x + 2\int_G u^2 a_{ij} \frac{\partial \psi_m}{\partial x_i}\frac{\partial \psi_m}{\partial x_j}\, \mathrm{d}x. \qquad (2.6.6)$$

Since the sequence

$$(1-\psi_m)^2 \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j},$$

converges to zero in G with respect to the measure m_n and is majorized by the integrable function

$$9\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j},$$

the first integral on the right in (2.6.6) converges to zero. The convergence to zero of the second integral follows from the boundedness of u and equality (2.6.5). Thus $u\psi_m \to u$ in $H(S)$. The required approximation is constructed.

Necessity. Let G be an arbitrary domain in $\mathbb{R}^n$ with finite $H(S)$ capacity. Let u denote a function in $C^{0,1} \cap H(S)$, which is equal to unity on G. Since $H(S)$ and $\mathring{H}(S)$ coincide u can be approximated in $H(S)$ by the sequence $\{\varphi_m\}_{m\ge 1}$ contained in $C_0^{0,1}$. Noting that $u = 1$ on G and $\varphi_m \to u$ in $L_2(G)$, we obtain that $\varphi_m \to 1$ in G in measure. Furthermore,

$$\int_G a_{ij} \frac{\partial \varphi_m}{\partial x_i}\frac{\partial \varphi_m}{\partial x_j}\, \mathrm{d}x = \int_G a_{ij} \frac{\partial}{\partial x_i}(u-\varphi_m)\frac{\partial}{\partial x_j}(u-\varphi_m)\, \mathrm{d}x \xrightarrow{m\to\infty} 0.$$

So the theorem is proved. □

Although the above result is not very descriptive, it facilitates verification of concrete conditions for coincidence or noncoincidence of $H(S)$ and $\mathring{H}(S)$. We now present some of them.

Theorem 2. (cf. Maz'ya [536]). *The spaces $H(S)$ and $\mathring{H}(S)$ coincide provided $n = 1$ or $n = 2$.*

Proof. Taking into account Theorem 1 and the discussion that follows its statement, we arrive at the equality $H(S) = \overset{\circ}{H}(S)$ if we show that any domain G with finite $H(S)$ capacity is bounded. The case $n = 1$ is obvious. Let $n = 2$ and $u \in C^{0,1} \cap H(S)$, $u = 1$ on G.

Let O and P denote arbitrary points in G and let the axis Ox_2 be directed from O to P. Then

$$S[u,u] \geq c \int_0^{|P|} \mathrm{d}x_2 \int_{\mathbb{R}^1} \left(\left(\frac{\partial u}{\partial x_1} \right)^2 + u^2 \right) \mathrm{d}x_1 \geq c_1 \int_0^{|P|} \max_{x_1} \big[u(x_1, x_2) \big]^2 \, \mathrm{d}x_2.$$

Taking into account that G is a domain and $u = 1$ on G we arrive at

$$\max_{x_1} \big[u(x_1, x_2) \big]^2 \geq 1.$$

Therefore $\operatorname{diam}(G) \leq cS[u,u]$, which completes the proof. □

The following assertion shows that for $n \geq 3$ the form $S[u,u]$ must be subjected to certain conditions by necessity. The result is due to Uraltseva [769]. Our proof, though different, is based on the same idea.

Theorem 3. *Let $n > 2$. Then there exists a form $S[u,u]$ for which $H(S) \neq \overset{\circ}{H}(S)$.*

Proof. 1. Consider the domain $G = \{x : 0 < x_n < \infty, |x'| < f(x_n)\}$ where $x' = (x_1, \dots, x_{n-1})$ and f is a positive decreasing function in $C^\infty[0,\infty)$, $f(0) < 1$. For $x \notin G$ we put $a_{ij}(x) = \delta_i^j$.

For arbitrary functions a_{ij} on G, for any $u \in C^{0,1}$, $u = 1$ on G, we have $S[u,u] = \|u\|^2_{W_2^1}$. This implies that G is a domain with finite $H(S)$ capacity if and only if $\operatorname{cap}(G) < \infty$ (here, as before, cap is the Wiener capacity, i.e., 2-cap). Clearly,

$$\begin{aligned} \operatorname{cap}(G) &\leq \sum_{j=0}^{\infty} \operatorname{cap}\big(\{x \in G : j \leq x_n \leq j+1\}\big) \\ &\leq \sum_{j=0}^{\infty} \operatorname{cap}\big(\{x : |x'| \leq f(j), j \leq x_n \leq j+1\}\big). \end{aligned}$$

This and the well-known estimates for the capacity of the cylinder (cf. Landkof [477] or Proposition 13.1.3/1 of the present book) yield

$$\operatorname{cap}(G) \leq c \sum_{j=0}^{\infty} \big[f(j) \big]^{n-3} \quad \text{for } n > 3,$$

$$\operatorname{cap}(G) \leq c \sum_{j=0}^{\infty} \big| \log f(j) \big|^{-1} \quad \text{for } n = 3.$$

Therefore G is a domain with finite $H(S)$ capacity provided

$$\int_0^\infty \big[f(t)\big]^{n-3}\,\mathrm{d}t < \infty \quad \text{for } n > 3,$$
$$\int_0^\infty \big|\log f(t)\big|^{-1}\,\mathrm{d}t < \infty \quad \text{for } n = 3.$$

2. In the interior of G we define the quadratic form $a_{ij}(x)\xi_i\xi_j$ by

$$a_{ij}(x)\xi_i\xi_j = \xi^2 + \left(\frac{g(x_n)}{f(x_n)}\right)^{n-1}\eta(x)\left(f'(x_n)\sum_{i=1}^{n-1} x_i\xi_i + \xi_n\right)^2,$$

where $\eta \in C_0^\infty(G)$, $0 \le \eta \le 1$, $\eta(x) = 1$ on the set $\{x : 1 < x_n < \infty, |x'| < \frac{1}{2}f(x_n)\}$, and g is an arbitrary positive function on $[0,\infty)$ satisfying the condition

$$\int_0^\infty \big[g(t)\big]^{1-n}\,\mathrm{d}t < \infty.$$

Using the change of variable $x_n = y_n$, $x_i = f(y_n)y_i$, $1 \le i \le n-1$, we map G onto the cylinder $\{y : 0 < y_n < \infty, |y'| < 1\}$. Obviously,

$$\int_G \left(\frac{g(x_n)}{f(x_n)}\right)^{n-1}\eta(x)\left(f'(x_n)\sum_{i=1}^{n-1} x_i\frac{\partial\varphi}{\partial x_i} + \frac{\partial\varphi}{\partial x_n}\right)^2\mathrm{d}x$$
$$\ge \int_C \big[g(y_n)\big]^{n-1}\left(\frac{\partial\varphi}{\partial y_n}\right)^2\mathrm{d}y,$$

where $C = \{y : 1 < y_n < \infty, |y'| < \frac{1}{2}\}$. Applying the Cauchy inequality to the last integral we obtain

$$\int_1^\infty \big[g(t)\big]^{1-n}\,\mathrm{d}t \int_G a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,\mathrm{d}x \ge \int_{|y'|\le 1/2}\left(\int_1^\infty \left|\frac{\partial\varphi}{\partial y_n}\right|\mathrm{d}y_n\right)^2\mathrm{d}y'.$$

If $\varphi \in C_0^{0,1}$ then the right-hand side exceeds

$$\int_{|y'|<1/2}\max_{1<y_n<\infty}\big[\varphi(y',y_n)\big]^2\,\mathrm{d}y' \ge \int_{C_1}\varphi^2\,\mathrm{d}y,$$

where $C_1 = \{y \in C : y_n < 2\}$. Passing to the variables $x_1,\dots,x_n$ on the right, we arrive at

$$\int_1^\infty \big[g(t)\big]^{1-n}\,\mathrm{d}t \int_G a_{ij}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,\mathrm{d}x \ge \int_{G_1}\varphi^2\frac{\mathrm{d}x}{[f(x_n)]^{n-1}},$$

where $G_1 = \{x : |x'| < \frac{1}{2}f(x_n), 1 < x_n < 2\}$. Thus, for any sequence $\{\varphi_m\}_{m\ge1}$ of functions in $C_0^{0,1}$ converging in measure to unity in G we have

$$\liminf_{m\to\infty}\int_G a_{ij}\partial\varphi_m/\partial x_i \cdot \partial\varphi_m/\partial x_j\,\mathrm{d}x > 0.$$

To conclude the proof, it remains to make use of Theorem 1. □

Theorem 3 has an interesting application to the problem of the self-adjointness of an elliptic operator in $L_2(\mathbb{R}^n)$, $n \geq 3$ (cf. Uraltseva [769]). Let the operator

$$u \to S_0 u = -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + u$$

be defined on C_0^∞. If $\|a_{ij}\|_{i,j=1}^n$ is the matrix constructed in Theorem 3, then $H(S) = \mathring{H}(S)$ and hence there exists a function $w \in H(S)$, which does not vanish identically and is orthogonal to any $v \in C_0^\infty$ in $H(S)$, i.e.,

$$0 = \int_{\mathbb{R}^n}\left(a_{ij}\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} + wv\right)\mathrm{d}x = \int_{\mathbb{R}^n} wS_0 v\,\mathrm{d}x.$$

Therefore the range of the closure $\bar{S}_0$ does not coincide with L_2. If $\bar{S}_0$ is self-adjoint then $w \in \mathrm{Dom}(\bar{S}_0)$ and $\bar{S}_0 w = 0$. This obviously implies $w = 0$. We arrived at a contradiction, which means that $\bar{S}_0$ is not self-adjoint. Thus, the condition of the uniform positive definiteness of the matrix $\|a_{ij}\|_{i,j=1}^n$ alone is insufficient for the self-adjointness of $\bar{S}_0$.

2.6.3 Comments to Sect. 2.6

The results of Sect. 2.6.1 are due to the author [556], Sect. 2.6. We note that the proof of Theorem 2.6.1/2 implies nondiscreteness of the spectrum of the Steklov problem

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a(x)u = 0 \quad \text{in } \Omega,$$
$$\sum_{i,j=1}^n a_{ij}\cos(\nu, x_j)\frac{\partial u}{\partial x_i} = \lambda u \quad \text{on } \partial\Omega,$$

under the condition that $\partial\Omega$ is characteristic at least at one point. Here ν is a normal to $\partial\Omega$ and the matrix $\|a_{ij}\|_{i,j=1}^n$ is nonnegative $a(x) > 0$. The coefficients a_{ij}, a, and the surface $\partial\Omega$ are assumed to be smooth.

In conclusion, we note that the topic of Sect. 2.6.2 was also considered in the paper by S. Laptev [482] who studied the form

$$S[u,u] = \int_{\mathbb{R}^n}\big(\alpha(x)(\nabla u)^2 + u^2\big)\,\mathrm{d}x,$$

where $\alpha(x) \geq \mathrm{const} > 0$. He presented an example of a function α for which $H(S) \neq \mathring{H}(S)$ and showed that $H(S)$ and $\mathring{H}(S)$ coincide in each of the following three cases: (i) α is a nondecreasing function in $|x|$, (ii) $\alpha(x) = O(|x|^2+1)$, and (iii) $n = 3$ and α depends only on $|x|$.

2.7 Dilation Invariant Sharp Hardy's Inequalities

2.7.1 Hardy's Inequality with Sharp Sobolev Remainder Term

Here we find the best value of C for a particular case of the inequality

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 \,\mathrm{d}x \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{|v|^2}{x_n^2} \,\mathrm{d}x + C \|x_n^\gamma v\|^2_{L_q(\mathbb{R}^n_+)}, \tag{2.7.1}$$

which is equivalent to (2.1.36) with $m = 1$.

Theorem. *For all $u \in C^\infty(\overline{\mathbb{R}^n_+})$, $u = 0$ on $\mathbb{R}^{n-1}$, the sharp inequality*

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 \,\mathrm{d}x \geq \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{|u|^2}{x_n^2} \,\mathrm{d}x + \frac{\pi^{n/(n+1)}(n^2-1)}{4(\Gamma(\frac{n}{2}+1))^{2/(n+1)}} \left\| x_n^{-1/(n+1)} u \right\|^2_{L_{\frac{2(n+1)}{n-1}}(\mathbb{R}^n_+)} \tag{2.7.2}$$

holds.

Proof. We start with the Sobolev inequality

$$\int_{\mathbb{R}^{n+1}} |\nabla w|^2 \,\mathrm{d}z \geq \mathcal{S}_{n+1} \|w\|^2_{L_{\frac{2(n+1)}{n-1}}(\mathbb{R}^{n+1})} \tag{2.7.3}$$

with the best constant

$$\mathcal{S}_{n+1} = \frac{\pi^{(n+2)/(n+1)}(n^2-1)}{4^{n/(n+1)}(\Gamma(\frac{n}{2}+1))^{2/(n+1)}} \tag{2.7.4}$$

(see (2.3.23)).

Let us introduce the cylindrical coordinates (r, φ, x'), where $r \geq 0$, $\varphi \in [0, 2\pi)$, and $x' \in \mathbb{R}^{n-1}$. Assuming that w does not depend on φ, we write (2.7.3) in the form

$$2\pi \int_{\mathbb{R}^{n-1}} \int_0^\infty \left(\left| \frac{\partial w}{\partial r} \right|^2 + |\nabla_{x'} w|^2 \right) r \,\mathrm{d}r \,\mathrm{d}x' \geq (2\pi)^{(n-1)/(n+1)} \mathcal{S}_{n+1} \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty |w|^{2(n+1)/(n-1)} r \,\mathrm{d}r \,\mathrm{d}x' \right)^{(n-1)/(n+1)}.$$

Replacing r by x_n, we obtain

$$\int_{\mathbb{R}^n_+} |\nabla w|^2 x_n \,\mathrm{d}x \geq (2\pi)^{-2/(n+1)} \mathcal{S}_{n+1} \left(\int_{\mathbb{R}^n_+} |w|^{2(n+1)/(n-1)} x_n \,\mathrm{d}x \right)^{(n-1)/(n+1)}.$$

It remains to set here $w = x_n^{1/2} v$ and to use (2.7.4). □

2.7.2 Two-Weight Hardy's Inequalities

As usual, here and elsewhere $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_n > 0\}$ and $C_0^\infty(\mathbb{R}^n_+)$ and $C_0^\infty(\overline{\mathbb{R}^n_+})$ stand for the spaces of infinitely differentiable functions with compact support in $\mathbb{R}^n_+$ and $\overline{\mathbb{R}^n_+}$, respectively.

Theorem 1. *The inequality*

$$\int_{\mathbb{R}^n_+} \frac{|u(x)|^p}{(x_{n-1}^2 + x_n^2)^{1/2}} \le (2p)^p \int_{\mathbb{R}^n_+} x_n^{p-1} \big|\nabla u(x)\big|^p \,\mathrm{d}x \tag{2.7.5}$$

holds for all $u \in C_0^\infty(\overline{\mathbb{R}^n_+})$.

Proof. We put $\varrho^2 = x_{n-1}^2 + x_n^2$ and denote the integrals on the left- and right-hand sides by $\mathscr{I}$ and $\mathscr{J}$, respectively. Integrating by parts, we obtain

$$\mathscr{I} = -p \int_{\mathbb{R}^n} x_n \varrho^{-1} |u|^{p-1} \operatorname{sgn} u \frac{\partial u}{\partial x_n} \,\mathrm{d}x + \int_{\mathbb{R}^n} x_n^2 \varrho^{-3} |u|^p \,\mathrm{d}x.$$

We denote two summands in the right-hand side by $\mathscr{I}_1$ and $\mathscr{I}_2$. Clearly, by Hölder's inequality we have $|\mathscr{I}_1| \le {}^{(p-1)/p} \mathscr{J}^{1/p}$. To obtain a bound for $\mathscr{I}_2$ we introduce cylindrical coordinates (z, ϱ, θ) with $z \in \mathbb{R}^{n-2}$, $x_{n-1} + i x_n = \varrho \exp(i\theta)$. Then

$$\mathscr{I}_2 = -p \int_{\mathbb{R}^{n-2}} \mathrm{d}z \int_0^\pi \sin^2\theta \,\mathrm{d}\theta \int_0^\infty |u|^{p-1} \operatorname{sgn} u \frac{\partial u}{\partial \varrho} \varrho \,\mathrm{d}\varrho \le \varrho \mathscr{I}^{(p-1)/p} \mathscr{J}^{1/p}.$$

Thus $\mathscr{I} \le 2p \mathscr{I}^{(p-1)/p} \mathscr{J}^{1/p}$ and (2.7.5) follows. □

In this section we are concerned with generalizations of the inequality

$$\int_{\mathbb{R}^n_+} x_n |\nabla u|^2 \,\mathrm{d}x \ge \Lambda \int_{\mathbb{R}^n_+} \frac{|u|^2}{(x_{n-1}^2 + x_n^2)^{1/2}} \,\mathrm{d}x, \quad u \in C_0^\infty(\overline{\mathbb{R}^n_+}). \tag{2.7.6}$$

By substituting $u(x) = x_n^{-1/2} v(x)$ into (2.7.6), one arrives at the improved Hardy inequality

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 \,\mathrm{d}x - \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{|v|^2 \,\mathrm{d}x}{x_n^2} \ge \Lambda \int_{\mathbb{R}^n_+} \frac{|v|^2 \,\mathrm{d}x}{x_n (x_{n-1}^2 + x_n^2)^{1/2}} \tag{2.7.7}$$

for all $v \in C_0^\infty(\mathbb{R}^n_+)$.

More generally, replacing u by $x_n^{-1/2} v(x)$ in the next theorem, we find a condition on the function q that is necessary and sufficient for the inequality

$$\int_{\mathbb{R}^n_+} |\nabla v|^2 \,\mathrm{d}x - \frac{1}{4} \int_{\mathbb{R}^n_+} \frac{|v|^2 \,\mathrm{d}x}{x_n^2} \\ \ge C \int_{\mathbb{R}^n_+} q\Big(\frac{x_n}{(x_{n-1}^2 + x_n^2)^{1/2}} \Big) \frac{|v|^2 \,\mathrm{d}x}{x_n (x_{n-1}^2 + x_n^2)^{1/2}}, \tag{2.7.8}$$

where v is an arbitrary function in $C_0^\infty(\mathbb{R}^n_+)$. This condition implies, in particular, that the right-hand side of (2.7.7) can be replaced by

$$C \int_{\mathbb{R}^n_+} \frac{|v|^2 \, \mathrm{d}x}{x_n^2 (1 - \log \frac{x_n}{(x_{n-1}^2 + x_n^2)^{1/2}})^2}.$$

Theorem 2. (i) *Let q denote a locally integrable nonnegative function on $(0,1)$. The best constant in the inequality*

$$\int_{\mathbb{R}^n_+} x_n |\nabla u|^2 \, \mathrm{d}x \geq C \int_{\mathbb{R}^n_+} q\left(\frac{x_n}{(x_{n-1}^2 + x_n^2)^{1/2}} \right) \frac{|u|^2}{(x_{n-1}^2 + x_n^2)^{1/2}} \, \mathrm{d}x, \tag{2.7.9}$$

for all $u \in C_0^\infty(\overline{\mathbb{R}^n_+})$, is given by

$$\lambda := \inf \frac{\int_0^{\pi/2} (|y'(\varphi)|^2 + \frac{1}{4} |y(\varphi)|^2) \sin \varphi \, \mathrm{d}\varphi}{\int_0^{\pi/2} |y(\varphi)|^2 q(\sin \varphi) \, \mathrm{d}\varphi}, \tag{2.7.10}$$

where the infimum is taken over all smooth functions on $[0, \pi/2]$.

(ii) *Inequalities (2.7.9) and (2.7.8) with a positive C hold if and only if*

$$\sup_{t \in (0,1)} (1 - \log t) \int_0^t q(\tau) \, \mathrm{d}\tau < \infty. \tag{2.7.11}$$

Moreover,

$$\lambda \sim \left(\sup_{t \in (0,1)} (1 - \log t) \int_0^t q(\tau) \, \mathrm{d}\tau \right)^{-1}, \tag{2.7.12}$$

where $a \sim b$ means that $c_1 a \leq b \leq c_2 a$ with absolute positive constants c_1 and c_2.

Proof. (i) Let $U \in C_0^\infty(\overline{\mathbb{R}^2_+})$, $\zeta \in C_0^\infty(\mathbb{R}^{n-2})$, $x' = (x_1, \ldots, x_{n-2})$, and let $N = \text{const} > 0$. Putting

$$u(x) = N^{(2-n)/2} \zeta(N^{-1} x') U(x_{n-1}, x_n)$$

into (2.7.9) and passing to the limit as $N \to \infty$, we see that (2.7.9) is equivalent to the inequality

$$\begin{aligned} &\int_{\mathbb{R}^2_+} x_2 (|U_{x_1}|^2 + |U_{x_2}|^2) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &\qquad \geq C \int_{\mathbb{R}^2_+} q\left(\frac{x_2}{(x_1^2 + x_2^2)^{1/2}} \right) \frac{|U|^2 \, \mathrm{d}x_1 \, \mathrm{d}x_2}{(x_1^2 + x_2^2)^{1/2}}, \end{aligned} \tag{2.7.13}$$

where $U \in C_0^\infty(\overline{\mathbb{R}^2_+})$. Let (ρ, φ) be the polar coordinates of the point $(x_1, x_2) \in \mathbb{R}^2_+$. Then (2.7.13) can be written as

$$\int_0^\infty \int_0^\pi \big(|U_\rho|^2 + \rho^{-2}|U_\varphi|^2\big) \sin\varphi \, \mathrm{d}\varphi \, \rho^2 \, \mathrm{d}\rho \geq C \int_0^\infty \int_0^\pi |U|^2 q(\sin\varphi) \, \mathrm{d}\varphi \, \mathrm{d}\rho.$$

By the substitution

$$U(\rho,\varphi) = \rho^{-1/2} v(\rho,\varphi)$$

the left-hand side becomes

$$\int_0^\infty \int_0^\pi \left(|\rho v_\rho|^2 + |v_\varphi|^2 + \frac{1}{4}|v|^2\right) \sin\varphi \, \mathrm{d}\varphi \, \frac{\mathrm{d}\rho}{\rho} - \mathrm{Re} \int_0^\pi \int_0^\infty \overline{v} v_\rho \, \mathrm{d}\rho \sin\varphi \, \mathrm{d}\varphi. \tag{2.7.14}$$

Since $v(0) = 0$, the second term in (2.7.14) vanishes. Therefore, (2.7.13) can be written in the form

$$\int_0^\infty \int_0^\pi \left(|\rho v_\rho|^2 + |v_\varphi|^2 + \frac{1}{4}|v|^2\right) \sin\varphi \, \mathrm{d}\varphi \, \frac{\mathrm{d}\rho}{\rho} \geq C \int_0^\infty \int_0^\pi |v|^2 q(\sin\varphi) \, \mathrm{d}\varphi \, \frac{\mathrm{d}\rho}{\rho}. \tag{2.7.15}$$

Now, the definition (2.7.10) of λ shows that (2.7.9) holds with $C = \lambda$.

To show the optimality of this value of C, put $t = \log\rho$ and $v(\rho,\varphi) = w(t,\varphi)$. Then (2.7.9) is equivalent to

$$\int_{\mathbb{R}^1} \int_0^\pi \left(|w_t|^2 + |w_\varphi|^2 + \frac{1}{4}|w|^2\right) \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}t \geq C \int_{\mathbb{R}^1} \int_0^\pi |w|^2 q(\sin\varphi) \, \mathrm{d}\varphi \, \mathrm{d}t. \tag{2.7.16}$$

Applying the Fourier transform $w(t,\varphi) \to \hat{w}(s,\varphi)$, we obtain

$$\int_{\mathbb{R}^1} \int_0^\pi \left(|\hat{w}_\varphi|^2 + \left(|s|^2 + \frac{1}{4}\right)|\hat{w}|^2\right) \sin\varphi \, \mathrm{d}\varphi \, \mathrm{d}s \geq C \int_{\mathbb{R}^1} \int_0^\pi |\hat{w}|^2 q(\sin\varphi) \, \mathrm{d}\varphi \, \mathrm{d}s. \tag{2.7.17}$$

Putting here

$$\hat{w}(s,\varphi) = \varepsilon^{-1/2} \eta(s/\varepsilon) y(\varphi),$$

where $\eta \in C_0^\infty(\mathbb{R}^1)$, $\|\eta\|_{L^2(\mathbb{R}^1)} = 1$, and y is a function on $C^\infty([0,\pi])$, and passing to the limit as $\varepsilon \to 0$, we arrive at the estimate

$$\int_0^\pi \left(\big|y'(\varphi)\big|^2 + \frac{1}{4}\big|y(\varphi)\big|^2\right) \sin\varphi \, \mathrm{d}\varphi \geq C \int_0^\pi \big|y(\varphi)\big|^2 q(\sin\varphi) \, \mathrm{d}\varphi, \tag{2.7.18}$$

where π can be changed for $\pi/2$ by symmetry. This together with (2.7.10) implies $\Lambda \leq \lambda$. The proof of (i) is complete.

(ii) Introducing the new variable $\xi = \log\cot\frac{\varphi}{2}$, we write (2.7.10) as

$$\lambda = \inf_z \frac{\int_0^\infty (|z'(\xi)|^2 + \frac{|z(\xi)|^2}{4(\cosh\xi)^2})\,\mathrm{d}\xi}{\int_0^\infty |z(\xi)|^2 q(\frac{1}{\cosh\xi})\frac{\mathrm{d}\xi}{\cosh\xi}}. \tag{2.7.19}$$

Since

$$\big|z(0)\big|^2 \le 2\int_0^1 \big(\big|z'(\xi)\big|^2 + \big|z(\xi)\big|^2\big)\,\mathrm{d}\xi$$

and

$$\begin{aligned}
&\int_0^\infty \big|z(\xi)\big|^2 \frac{e^{2\xi}}{(1+e^{2\xi})^2}\,\mathrm{d}\xi \\
&\quad\le 2\int_0^\infty \big|z(\xi)-z(0)\big|^2\frac{\mathrm{d}\xi}{\xi^2} + 2\big|z(0)\big|^2\int_0^\infty \frac{e^{2\xi}}{(1+e^{2\xi})^2}\,\mathrm{d}\xi \\
&\quad\le 8\int_0^\infty \big|z'(\xi)\big|^2\,\mathrm{d}\xi + \big|z(0)\big|^2
\end{aligned}$$

it follows from (2.7.19) that

$$\lambda \sim \inf_z \frac{\int_0^\infty |z'(\xi)|^2\,\mathrm{d}\xi + |z(0)|^2}{\int_0^\infty |z(\xi)|^2 q(\frac{1}{\cosh\xi})\frac{\mathrm{d}\xi}{\cosh\xi}}. \tag{2.7.20}$$

Setting $z(\xi) = 1$ and $z(\xi) = \min\{\eta^{-1}\xi, 1\}$ for all positive ξ and fixed $\eta > 0$ into the ratio of quadratic forms in (2.7.20), we deduce that

$$\lambda \le \min\bigg\{\bigg(\int_0^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{\mathrm{d}\xi}{\cosh\xi}\bigg)^{-1}, \bigg(\sup_{\eta>0}\eta\int_\eta^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{\mathrm{d}\xi}{\cosh\xi}\bigg)^{-1}\bigg\}.$$

Hence,

$$\lambda \le c\bigg(\sup_{t\in(0,1)}(1-\log t)\int_0^t q(\tau)\,\mathrm{d}\tau\bigg)^{-1}.$$

To obtain the converse estimate, note that

$$\begin{aligned}
&\int_0^\infty \big|z(\xi)\big|^2 q\Big(\frac{1}{\cosh\xi}\Big)\frac{\mathrm{d}\xi}{\cosh\xi} \\
&\quad\le 2\big|z(0)\big|^2\int_0^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{\mathrm{d}\xi}{\cosh\xi} + 2\int_0^\infty \big|z(\xi)-z(0)\big|^2 q\Big(\frac{1}{\cosh\xi}\Big)\frac{\mathrm{d}\xi}{\cosh\xi}.
\end{aligned}$$

The second term in the right-hand side is dominated by

$$8\sup_{\eta>0}\bigg(\eta\int_\eta^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{\mathrm{d}\xi}{\cosh\xi}\bigg)\int_0^\infty \big|z'(\xi)\big|^2\,\mathrm{d}\xi$$

(see Sect. 1.3.2). Therefore,

$$\int_0^\infty \big|z(\xi)\big|^2 q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}$$
$$\le 8\max\Big\{\int_0^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi},\ \sup_{\eta>0}\eta\int_\eta^\infty q\Big(\frac{1}{\cosh\sigma}\Big)\frac{d\sigma}{\cosh\sigma}\Big\}$$
$$\times\Big(\int_0^\infty \big|z'(\xi)\big|^2 d\xi+\big|z(0)\big|^2\Big),$$

which together with (2.7.20) leads to the lower estimate

$$\lambda\ge\min\Big\{\Big(\int_0^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}\Big)^{-1},\Big(\sup_{\eta>0}\eta\int_\eta^\infty q\Big(\frac{1}{\cosh\xi}\Big)\frac{d\xi}{\cosh\xi}\Big)^{-1}\Big\}.$$

Hence,

$$\lambda\ge c\Big(\sup_{t\in(0,1)}(1-\log t)\int_0^t q(\tau)\,d\tau\Big)^{-1}.$$

The proof of (ii) is complete. □

Since (2.7.11) holds for $q(t)=t^{-1}(1-\log t)^{-2}$, Theorem 2(ii) leads to the following assertion.

Corollary 1. *There exists an absolute constant $C>0$ such that the inequality*

$$\int_{\mathbb{R}^n_+}|\nabla v|^2\,dx-\frac14\int_{\mathbb{R}^n_+}\frac{|v|^2\,dx}{x_n^2}\ge C\int_{\mathbb{R}^n_+}\frac{|v|^2\,dx}{x_n^2\big(1-\log\frac{x_n}{(x_{n-1}^2+x_n^2)^{1/2}}\big)^2} \tag{2.7.21}$$

holds for all $v\in C_0^\infty(\mathbb{R}^n_+)$. The best value of C is equal to

$$\lambda:=\inf\frac{\int_0^\pi[|y'(\varphi)|^2+\frac14|y(\varphi)|^2]\sin\varphi\,d\varphi}{\int_0^\pi|y(\varphi)|^2(\sin\varphi)^{-1}(1-\log\sin\varphi)^{-2}\,d\varphi}, \tag{2.7.22}$$

where the infimum is taken over all smooth functions on $[0,\pi/2]$. By numerical approximation, $\lambda=0.16,\dots$.

A particular case of Theorem 2 corresponding to $q=1$ is the following assertion.

Corollary 2. *The sharp value of Λ in (2.7.6) and (2.7.7) is equal to*

$$\lambda:=\inf\frac{\int_0^\pi[|y'(\varphi)|^2+\frac14|y(\varphi)|^2]\sin\varphi\,d\varphi}{\int_0^\pi|y(\varphi)|^2\,d\varphi}, \tag{2.7.23}$$

where the infimum is taken over all smooth functions on $[0,\pi]$. By numerical approximation, $\lambda=0.1564,\dots$.

Remark 1. Let us consider the Friedrichs extension $\tilde{\mathcal{L}}$ of the operator

$$\mathcal{L} : z \to -\big((\sin\varphi)z'\big)' + \frac{\sin\varphi}{4} z, \tag{2.7.24}$$

defined on smooth functions on $[0, \pi]$. It is a simple exercise to show that the energy space of $\tilde{\mathcal{L}}$ is compactly embedded into $L^2(0, \pi)$. Hence, the spectrum of $\tilde{\mathcal{L}}$ is discrete and λ defined by (2.7.23) is the smallest eigenvalue of $\tilde{\mathcal{L}}$.

Remark 2. The argument used in the proof of Theorem 2(i) with obvious changes enables one to obtain the following more general fact. Let P and Q be measurable nonnegative functions in $\mathbb{R}^n$, positive homogeneous of degrees 2μ and $2\mu - 2$, respectively. The sharp value of C in

$$\int_{\mathbb{R}^n} P(x)|\nabla u|^2 \,\mathrm{d}x \geq C \int_{\mathbb{R}^n} Q(x)|u|^2 \,\mathrm{d}x, \quad u \in C_0^\infty\big(\mathbb{R}^n\big), \tag{2.7.25}$$

is equal to

$$\lambda := \inf \frac{\int_{S^{n-1}} P(\omega)(|\nabla_\omega Y|^2 + (\mu - 1 + \frac{n}{2})^2 |Y|^2) \,\mathrm{d}s_\omega}{\int_{S^{n-1}} Q(\omega)|Y|^2 \,\mathrm{d}s_\omega},$$

where the infimum is taken over all smooth functions on the unit sphere S^{n-1}.

2.7.3 Comments to Sect. 2.7

The material of this subsection is borrowed from Maz'ya and Shaposhnikova [587]. In Sect. 2.7.1 we are concerned with the inequality (2.7.1) which is a special case of (2.1.36). Another inequality of a similar nature, whose generalizations are dealt with in Sect. 2.7.2, is (2.7.7). It is equivalent to (2.7.6) and was obtained in 1972 by the author, proving to be useful in the study of the generic case of degeneration in the oblique derivative problem for second-order elliptic differential operators [541].

Without the second term on the right-hand sides of (2.7.1) and (2.7.7), these inequalities reduce to the classical Hardy inequality with the sharp constant 1/4. An interesting feature of (2.7.1) and (2.7.7) is their dilation invariance. The value $\Lambda = 1/16$ in (2.7.7) obtained in Maz'ya [541] is not the best possible. Tidblom replaced it by 1/8 in [752]. As a corollary of Theorem 2.7.2/2, we find an expression for the optimal value of Λ (see Corollary 2.7.2/2).

Sharp constants in Hardy-type inequalities as well as variants, extensions, and refinements of (2.7.1) and (2.7.7), usually called Hardy's inequalities with remainder term, became a theme of many subsequent studies (Davies [225, 226]; Brezis and Marcus [142]; Brezis and Vázquez [145]; Matskewich and Sobolevskii [526]; Sobolevskii [715]; Davies and Hinz [227]; Marcus, Mizel, and Pinchover [516]; Laptev and Weidl [481]; Weidl [792]; Yafaev [801]; Brezis, Marcus, and Shafrir [143]; Vázquez and Zuazua [773]; Eilertsen [256];

Adimurthi [26]; Adimurthi, Chaudhuri, and Ramaswamy [27]; Filippas and Tertikas [278]; M. Hoffman-Ostenhof, T. Hoffman-Ostenhof, and Laptev [380]; Barbatis, Filippas, and Tertikas [73, 74]; Balinsky [67]; Barbatis, Filippas, and Tertikas [72]; Chaudhuri [178]; Z.-Q. Wang and Meijun [791]; Balinsky, Laptev, and A. Sobolev [68]; Dávila and Dupaigne [228]; Dolbeault, Esteban, Loss, and Vega [237]; Filippas, Maz'ya, and Tertikas [275–277]; Gazzola, Grunau, and Mitidieri [303]; Tidblom [751, 752]; Colin [211]; Edmunds and Hurri-Syrjänen [252]; Galaktionov [301, 302]; Samko [689]; Yaotian and Zhihui [804]; Adimurthi, Grossi, and Santra [28]; Alvino, Ferone, and Trombetti [42]; Barbatis [70, 71]; Brandolini, Chiacchio, and Trombetti [140]; Dou, Niu, and Yuan [241]; Evans and Lewis [264]; Tertikas and Tintarev [749]; Tertikas and Zographopoulos [750]; Benguria, Frank, and Loss [83]; Bosi, Dolbeault, and Esteban [127]; Frank and Seiringer [289, 290]; Frank, Lieb, and Seiringer [288]; A. Laptev and A. Sobolev [480]; Cianchi and Ferone [203]; Kombe and Özaydin [446]; Filippas, Tertikas, and Tidblom [279]; Pinchover and Tintarev [661]; Avkhadiev and Laptev [58] et al.).

2.8 Sharp Hardy–Leray Inequality for Axisymmetric Divergence-Free Fields

2.8.1 Statement of Results

Let $\mathbf{u}$ denote a $C_0^\infty(\mathbb{R}^n)$ vector field in $\mathbb{R}^n$. The following n-dimensional generalization of the one-dimensional Hardy inequality,

$$\int_{\mathbb{R}^n} \frac{|\mathbf{u}|^2}{|x|^2}\,\mathrm{d}x \le \frac{4}{(n-2)^2}\int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2\,\mathrm{d}x \tag{2.8.1}$$

appears for $n = 3$ in the pioneering paper by Leray on the Navier–Stokes equations [487]. The constant factor on the right-hand side is sharp. Since one frequently deals with divergence-free fields in hydrodynamics, it is natural to ask whether this restriction can improve the constant in (2.8.1).

We show in the present section that this is the case indeed if $n > 2$ and the vector field $\mathbf{u}$ is axisymmetric by proving that the aforementioned constant can be replaced by the (smaller) optimal value

$$\frac{4}{(n-2)^2}\left(1 - \frac{8}{(n+2)^2}\right), \tag{2.8.2}$$

which, in particular, evaluates to $68/25$ in three dimensions. This result is a special case of a more general one concerning a divergence-free improvement of the multidimensional sharp Hardy inequality

$$\int_{\mathbb{R}^n} |x|^{2\gamma-2}|\mathbf{u}|^2\,\mathrm{d}x \le \frac{4}{(2\gamma+n-2)^2}\int_{\mathbb{R}^n} |x|^{2\gamma}|\nabla \mathbf{u}|^2\,\mathrm{d}x. \tag{2.8.3}$$

Let $\boldsymbol{\phi}$ be a point on the $(n-2)$-dimensional unit sphere S^{n-2} with spherical coordinates $\{\theta_j\}_{j=1,\dots,n-3}$ and ϕ, where $\theta_j \in (0,\pi)$ and $\varphi \in [0,2\pi)$. A point $x \in \mathbb{R}^n$ is represented as a triple $(\rho, \theta, \boldsymbol{\phi})$, where $\rho > 0$ and $\theta \in [0,\pi]$. Correspondingly, we write $\mathbf{u} = (u_\rho, u_\theta, \mathbf{u}_{\boldsymbol{\phi}})$ with $\mathbf{u}_{\boldsymbol{\phi}} = (u_{\theta_{n-3}}, \dots, u_{\theta_1}, u_{\boldsymbol{\phi}})$.

The *condition of axial symmetry* means that $\mathbf{u}$ depends only on ρ and θ.

For higher dimensions, our result is as follows.

Theorem 1. *Let $\gamma \neq 1 - n/2$, $n > 2$, and let $\mathbf{u}$ be an axisymmetric divergence-free vector field in $C_0^\infty(\mathbb{R}^n)$. We assume that $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ for $\gamma < 1 - n/2$. Then*

$$\int_{\mathbb{R}^n} |x|^{2\gamma-2}|\mathbf{u}|^2\,\mathrm{d}x \le C_{n,\gamma} \int_{\mathbb{R}^n} |x|^{2\gamma}|\nabla \mathbf{u}|^2\,\mathrm{d}x \tag{2.8.4}$$

with the best value of $C_{n,\gamma}$ given by

$$C_{n,\gamma} = \frac{4}{(2\gamma+n-2)^2}\left(1 - \frac{2}{n+1+(\gamma-n/2)^2}\right), \tag{2.8.5}$$

for $\gamma \le 1$, and by

$$C_{n,\gamma}^{-1} = \left(\frac{n}{2}+\gamma-1\right)^2 + \min\left\{n-1,\ 2+\min_{x\ge 0}\left(x + \frac{4(n-1)(\gamma-1)}{x+n-1+(\gamma-n/2)^2}\right)\right\} \tag{2.8.6}$$

for $\gamma > 1$.

The two minima in (2.8.6) can be calculated in closed form, but their expressions for arbitrary dimensions turn out to be unwieldy and we omit them.

However, the formula for $C_{3,\gamma}$ is simple.

Corollary. *For $n = 3$ inequality (2.8.4) holds with the best constant*

$$C_{3,\gamma} = \begin{cases} \frac{4}{(2\gamma+1)^2}\cdot\frac{2+(\gamma-3/2)^2}{4+(\gamma-3/2)^2} & \text{for } \gamma \le 1, \\ \frac{4}{8+(1+2\gamma)^2} & \text{for } \gamma > 1. \end{cases} \tag{2.8.7}$$

For $n = 2$, we obtain the sharp constant in (2.8.4) without axial symmetry of the vector field.

Theorem 2. *Let $\gamma \neq 0$, $n = 2$, and let $\mathbf{u}$ be a divergence-free vector field in $C_0^\infty(\mathbb{R}^2)$. We assume that $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ for $\gamma < 0$. Then inequality (2.8.4) holds with the best constant*

$$C_{2,\gamma} = \begin{cases} \gamma^{-2}\frac{1+(1-\gamma)^2}{3+(1-\gamma)^2} & \text{for } \gamma \in [-\sqrt{3}-1, \sqrt{3}-1], \\ (\gamma^2+1)^{-1} & \text{otherwise.} \end{cases} \tag{2.8.8}$$

2.8.2 Proof of Theorem 1

In the spherical coordinates introduced previously, we have

$$\operatorname{div}\mathbf{u} = \rho^{1-n}\frac{\partial}{\partial\rho}\big(\rho^{n-1}u_\rho\big) + \rho^{-1}(\sin\theta)^{2-n}\frac{\partial}{\partial\theta}\big((\sin\theta)^{n-2}u_\theta\big)$$
$$+\sum_{k=1}^{n-3}(\rho\sin\theta\sin\theta_{n-3}\cdots\sin\theta_{k+1})^{-1}(\sin\theta_k)^{-k}\frac{\partial}{\partial\theta_k}\big((\sin\theta_k)^k u_{\theta_k}\big)$$
$$+(\rho\sin\theta\sin\theta_{n-3}\cdots\sin\theta_1)^{-1}\frac{\partial u_\varphi}{\partial\varphi}. \tag{2.8.9}$$

Since the components u_φ and u_{θ_k}, $k=1,\dots,n-3$, depend only on ρ and θ, (2.8.9) becomes

$$\operatorname{div}\mathbf{u} = \rho^{1-n}\frac{\partial}{\partial\rho}\big(\rho^{n-1}u_\rho(\rho,\theta)\big) + \rho^{-1}(\sin\theta)^{2-n}\frac{\partial}{\partial\theta}\big((\sin\theta)^{n-2}u_\theta(\rho,\theta)\big)$$
$$+\sum_{k=1}^{n-3}k(\sin\theta_{n-3}\cdots\sin\theta_{k+1})^{-1}\cot\theta_k\frac{u_{\theta_k}(\rho,\theta)}{\rho\sin\theta}. \tag{2.8.10}$$

By the linear independence of the functions

$$1,\quad (\sin\theta_{n-3}\cdots\sin\theta_{k+1})^{-1}\cot\theta_k,\quad k=1,\dots,n-3,$$

the divergence-free condition is equivalent to the collection of $n-2$ identities

$$\rho\frac{\partial u_\rho}{\partial\rho}+(n-1)u_\rho+\left(\frac{\partial}{\partial\theta}+(n-2)\cot\theta\right)u_\theta=0, \tag{2.8.11}$$
$$u_{\theta_k}=0,\quad k=1,\dots,n-3. \tag{2.8.12}$$

If the right-hand side of (2.8.4) diverges, there is nothing to prove. Otherwise, the matrix $\nabla\mathbf{u}$ is $O(|x|^m)$, with $m>-\gamma-n/2$, as $x\to 0$. Since $\mathbf{u}(\mathbf{0})=\mathbf{0}$, we have $\mathbf{u}(x)=O(|x|^{m+1})$ ensuring the convergence of the integral on the left-hand side of (2.8.4). We introduce the vector field

$$\mathbf{v}(x)=\mathbf{u}(x)|x|^{\gamma-1+n/2}. \tag{2.8.13}$$

The inequality (2.8.4) becomes

$$\left(\frac{1}{C_{n,\gamma}}-\left(\frac{n}{2}+\gamma-1\right)^2\right)\int_{\mathbb{R}^n}\frac{|\mathbf{v}|^2}{|x|^n}\,\mathrm{d}x\le\int_{\mathbb{R}^n}\frac{|\nabla\mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x. \tag{2.8.14}$$

The condition $\operatorname{div}\mathbf{u}=0$ is equivalent to

$$\rho\operatorname{div}\mathbf{v}=\left(\frac{n-2}{2}+\gamma\right)v_\rho. \tag{2.8.15}$$

To simplify the exposition, we assume first that $\mathbf{v}_{\boldsymbol{\varphi}} = \mathbf{0}$. Now, (2.8.15) can be written as

$$\rho \frac{\partial v_\rho}{\partial \rho} + \left(\frac{n}{2} - \gamma\right) v_\rho + \mathcal{D} v_\theta = 0, \tag{2.8.16}$$

where

$$\mathcal{D} := \frac{\partial}{\partial \theta} + (n-2)\cot\theta. \tag{2.8.17}$$

Note that $\mathcal{D}$ is the adjoint of $-\partial/\partial\theta$ with respect to the scalar product

$$\int_0^\pi f(\theta)\overline{g(\theta)}(\sin\theta)^{n-2}\,\mathrm{d}\theta.$$

A straightforward, though lengthy calculation yields

$$\begin{aligned}\rho^2|\nabla \mathbf{v}|^2 = \rho^2\left(\frac{\partial v_\rho}{\partial \rho}\right)^2 + \rho^2\left(\frac{\partial v_\theta}{\partial \rho}\right)^2 + \left(\frac{\partial v_\rho}{\partial \theta}\right)^2 + \left(\frac{\partial v_\theta}{\partial \theta}\right)^2 \\ + v_\theta^2 + (n-1)v_\rho^2 + (n-2)(\cot\theta)^2 v_\theta^2 + 2\left(v_\rho \mathcal{D} v_\theta - v_\theta \frac{\partial v_\rho}{\partial \theta}\right).\end{aligned} \tag{2.8.18}$$

Hence

$$\begin{aligned}\rho^2 \int_{S^{n-1}} |\nabla \mathbf{v}|^2\,\mathrm{d}s = \int_{S^{n-1}} \Bigg\{\rho^2\left(\frac{\partial v_\rho}{\partial \rho}\right)^2 + \left(\frac{\partial v_\theta}{\partial \theta}\right)^2 + \rho^2\left(\frac{\partial v_\theta}{\partial \rho}\right)^2 + \left(\frac{\partial v_\rho}{\partial \theta}\right)^2 \\ + v_\theta^2 + (n-1)v_\rho^2 + (n-2)(\cot\theta)^2 v_\theta^2 + 4 v_\rho \mathcal{D} v_\theta\Bigg\}\,\mathrm{d}s.\end{aligned} \tag{2.8.19}$$

Changing the variable ρ to $t = \log\rho$ and applying the Fourier transform with respect to t,

$$\mathbf{v}(t,\theta) \mapsto \mathbf{w}(\lambda,\theta),$$

we derive

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x = \int_{\mathbb{R}}\int_{S^{n-1}} \Bigg\{ \big(l^2 + n - 1\big)|w_\rho|^2 + \big(l^2 - n + 3\big)|w_\theta|^2 \\ + \left|\frac{\partial w_\rho}{\partial \theta}\right|^2 + \left|\frac{\partial w_\theta}{\partial \theta}\right|^2 + (n-2)(\sin\theta)^{-2}|w_\theta|^2 \\ + 4\Re(\overline{w}_\rho \mathcal{D} w_\theta)\Bigg\}\,\mathrm{d}s\,\mathrm{d}\lambda\end{aligned} \tag{2.8.20}$$

and

$$\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n}\,\mathrm{d}x = \int_{\mathbb{R}}\int_{S^{n-1}} |\mathbf{w}|^2\,\mathrm{d}s\,\mathrm{d}\lambda. \tag{2.8.21}$$

From (2.8.15), we obtain

$$w_\rho = -\frac{\mathcal{D}w_\theta}{i\lambda + n/2 - \gamma}, \tag{2.8.22}$$

which implies

$$|w_\rho|^2 = \frac{|\mathcal{D}w_\theta|^2}{\lambda^2 + (n/2-\gamma)^2} \tag{2.8.23}$$

and

$$\Re(\overline{w}_\rho \mathcal{D}w_\theta) = -\frac{(n/2-\gamma)|\mathcal{D}w_\theta|^2}{\lambda^2 + (n/2-\gamma)^2}. \tag{2.8.24}$$

Introducing this into (2.8.20), we arrive at the identity

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x = \int_0^\infty \int_{S^{n-1}} \Big\{ &\big(\lambda^2+n-1\big)\frac{|\mathcal{D}w_\theta|^2}{\lambda^2+(n/2-\gamma)^2} \\ &+ \big(\lambda^2-n+3\big)|w_\theta|^2 + \left|\frac{\partial w_\theta}{\partial \theta}\right|^2 + (n-2)(\sin\theta)^{-2}|w_\theta|^2 \\ &+ \frac{1}{\lambda^2+(n/2-\gamma)^2}\left|\frac{\partial}{\partial\theta}\mathcal{D}w_\theta\right|^2 \\ &- 4\Big(\frac{n}{2}-\gamma\Big)\frac{|\mathcal{D}w_\theta|^2}{\lambda^2+(n/2-\gamma)^2}\Big\}\,\mathrm{d}s\,\mathrm{d}\lambda.\end{aligned}$$

We simplify the right-hand side to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x = \int_0^\infty \int_{S^{n-1}} \Big\{ &\Big(\frac{-n-1+\lambda^2+4\gamma}{\lambda^2+(n/2-\gamma)^2}+1\Big)|\mathcal{D}w_\theta|^2 \\ &+ \big(\lambda^2-n+3\big)|w_\theta|^2 + \frac{1}{\lambda^2+(n/2-\gamma)^2}\left|\frac{\partial}{\partial\theta}\mathcal{D}w_\theta\right|^2\Big\}\,\mathrm{d}s\,\mathrm{d}\lambda.\end{aligned} \tag{2.8.25}$$

On the other hand, by (2.8.21) and (2.8.22)

$$\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x = \int_0^\infty \int_{S^{n-1}} \Big(\frac{|\mathcal{D}w_\theta|^2}{\lambda^2+(n/2-\gamma)^2} + |w_\theta|^2\Big)\,\mathrm{d}s\,\mathrm{d}\lambda. \tag{2.8.26}$$

Defining the self-adjoint operator

$$T := -\frac{\partial}{\partial\theta}\mathcal{D}, \tag{2.8.27}$$

or equivalently,

$$T = -\delta_\theta + \frac{n-2}{(\sin\theta)^2}, \tag{2.8.28}$$

where δ_θ is the θ part of the Laplace–Beltrami operator on S^{n-1}, we write (2.8.25) and (2.8.26) as

$$\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x = \int_{\mathbb{R}} \int_{S^{n-1}} Q(\lambda, w_\theta)\,\mathrm{d}s\,\mathrm{d}\lambda \tag{2.8.29}$$

and

$$\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n}\,\mathrm{d}x = \int_{\mathbb{R}} \int_{S^{n-1}} q(\lambda, w_\theta)\,\mathrm{d}s\,\mathrm{d}\lambda, \tag{2.8.30}$$

respectively, where Q and q are sesquilinear forms in w_θ, defined by

$$Q(\lambda, w_\theta) = \left(\frac{-n-1+\lambda^2+4\gamma}{\lambda^2+(n/2-\gamma)^2}+1\right)Tw_\theta\cdot\overline{w_\theta} + \left(\lambda^2-n+3\right)|w_\theta|^2 + \frac{1}{\lambda^2+(n/2-\gamma)^2}|Tw_\theta|^2$$

and

$$q(\lambda, w_\theta) = \frac{Tw_\theta\cdot\overline{w_\theta}}{\lambda^2+(n/2-\gamma)^2} + |w_\theta|^2. \tag{2.8.31}$$

The eigenvalues of T are $\alpha_\nu = \nu(\nu+n-2)$, $\nu\in\mathbb{Z}^+$. Representing w_θ as an expansion in eigenfunctions of T, we find

$$\inf_{w_\theta} \frac{\int_{\mathbb{R}}\int_{S^{n-1}} Q(\lambda, w_\theta)\,\mathrm{d}s\,\mathrm{d}\lambda}{\int_{\mathbb{R}}\int_{S^{n-1}} q(\lambda, w_\theta)\,\mathrm{d}s\,\mathrm{d}\lambda} = \inf_{\lambda\in\mathbb{R}}\inf_{\nu\in\mathbb{N}} \frac{(\frac{-n-1+\lambda^2+4\gamma}{\lambda^2+(n/2-\gamma)^2}+1)\alpha_\nu + \lambda^2 - n + 3 + \frac{\alpha_\nu^2}{\lambda^2+(n/2-\gamma)^2}}{\frac{\alpha_\nu}{\lambda^2+(n/2-\gamma)^2+1}}. \tag{2.8.32}$$

Thus our minimization problem reduces to finding

$$\inf_{x\ge 0}\inf_{\nu\in\mathbb{N}} f(x, \alpha_\nu, \gamma), \tag{2.8.33}$$

where

$$f(x,\alpha_\nu,\gamma) = x - n + 3 + \alpha_\nu\left(1 - \frac{16(1-\gamma)}{4x+4\alpha_\nu+(n-2\gamma)^2}\right). \tag{2.8.34}$$

Since $\gamma \le 1$, it is clear that f is increasing in x, so the value (2.8.33) is equal to

$$\inf_{\nu\in\mathbb{N}} f(0,\alpha_\nu,\gamma) = \inf_{\nu\in\mathbb{N}}\left(3 - n + \alpha_\nu\left(1 - \frac{16(1-\gamma)}{4\alpha_\nu+(n-2\gamma)^2}\right)\right). \tag{2.8.35}$$

We have

$$\frac{\partial}{\partial\alpha_\nu} f(0,\alpha_\nu,\gamma) = 1 - \frac{16(1-\gamma)(n-2\gamma)}{(4\alpha_\nu+(n-2\gamma)^2)^2}. \tag{2.8.36}$$

Noting that

$$4\alpha_\nu + (n-2\gamma)^2 \ge 4(n-1) + (n-2\gamma)^2 \ge 4\sqrt{n-1}(n-2\gamma), \tag{2.8.37}$$

we see that

$$\frac{\partial}{\partial\alpha_\nu} f(0,\alpha_\nu,\gamma) \ge 1 - \frac{1-\gamma}{(n-1)(n-2\gamma)} > 0. \tag{2.8.38}$$

Thus the minimum of $f(0,\alpha_\nu,\gamma)$ is attained at $\alpha_1 = n-1$ and equals

$$3-n+(n-1)\left(1-\frac{16(1-\gamma)}{4(n-1)+(n-2\gamma)^2}\right)=\frac{2(\gamma-1+n/2)^2}{n-1+(\gamma-n/2)^2}. \quad (2.8.39)$$

This completes the proof for the case $\mathbf{v}_{\boldsymbol{\varphi}} = \mathbf{0}$.

If we drop the assumption $\mathbf{v}_{\boldsymbol{\varphi}} = \mathbf{0}$, then, to the integrand on the right-hand side of (2.8.19), we should add the terms

$$\rho^2\left(\frac{\partial v_\varphi}{\partial \rho}\right)^2+\left(\frac{\partial v_\varphi}{\partial \theta}\right)^2+(\sin\theta\sin\theta_{n-3}\cdots\sin\theta_1)^{-2}v_\varphi^2. \quad (2.8.40)$$

The expression in (2.8.40) equals

$$\rho^2\big|\nabla\big(v_\varphi \mathrm{e}^{\mathrm{i}\varphi}\big)\big|^2. \quad (2.8.41)$$

As a result, the right-hand side of (2.8.29) is augmented by

$$\int_{\mathbb{R}}\int_{S^{n-1}} R(\lambda,w_\varphi)\,\mathrm{d}s\,\mathrm{d}\lambda, \quad (2.8.42)$$

where

$$R(\lambda,w_\varphi)=\lambda^2|w_\varphi|^2+\big|\nabla_\omega\big(w_\varphi \mathrm{e}^{\mathrm{i}\varphi}\big)\big|^2 \quad (2.8.43)$$

with $\omega = (\theta,\theta_{n-3},\dots,\varphi)$. Hence,

$$\inf_{\mathbf{v}}\frac{\int_{\mathbb{R}^n}\frac{|\nabla\mathbf{v}|^2}{|x|^{n-2}}\,\mathrm{d}x}{\int_{\mathbb{R}^n}\frac{|\mathbf{v}|^2}{|x|^n}\,\mathrm{d}x}=\inf_{w_\theta,w_\varphi}\frac{\int_{\mathbb{R}}\int_{S^{n-1}}(Q(\lambda,w_\theta)+R(\lambda,w_\varphi))\,\mathrm{d}s\,\mathrm{d}\lambda}{\int_{\mathbb{R}}\int_{S^{n-1}}(q(\lambda,w_\theta)+|w_\varphi|^2)\,\mathrm{d}s\,\mathrm{d}\lambda}. \quad (2.8.44)$$

Using the fact that w_θ and w_φ are independent, the right-hand side is the minimum of (2.8.32) and

$$\inf_{w_\varphi}\frac{\int_{\mathbb{R}}\int_{S^{n-1}}R(\lambda,w_\varphi)\,\mathrm{d}s\,\mathrm{d}\lambda}{\int_{\mathbb{R}}\int_{S^{n-1}}|w_\varphi|^2\,\mathrm{d}s\,\mathrm{d}\lambda}. \quad (2.8.45)$$

Since $w_\varphi \mathrm{e}^{\mathrm{i}\varphi}$ is orthogonal to one on S^{n-1}, we have

$$\int_{S^{n-1}}\big|\nabla_\omega\big(w_\varphi \mathrm{e}^{\mathrm{i}\varphi}\big)\big|^2\,\mathrm{d}s \geq (n-1)\int_{S^{n-1}}|w_\varphi|^2\,\mathrm{d}s. \quad (2.8.46)$$

Hence the infimum in (2.8.45) is at most $n-1$, which exceeds the value in (2.8.39). The result follows for $\gamma \leq 1$.

For $\gamma > 1$ the proof is similar. Differentiation of f in α_ν gives

$$1+\frac{16(\gamma-1)((n-2\gamma)^2+4x)}{(4x+4\alpha_\nu+(n-2\gamma)^2)^2}, \quad (2.8.47)$$

which is positive. Hence the role of the value (2.8.39) is played by the smallest value of $f(\cdot,n-1,\gamma)$ on $\mathbb{R}^+$. Therefore,

$$\inf_{\mathbf{v}} \frac{\int_{\mathbb{R}^n} \frac{|\nabla \mathbf{v}|^2}{|x|^{n-2}}\, dx}{\int_{\mathbb{R}^n} \frac{|\mathbf{v}|^2}{|x|^n}\, dx} = 2 + \min_{x\geq 0}\Big(x + \frac{4(n-1)(\gamma-1)}{x+n-1+(\gamma-n/2)^2}\Big). \tag{2.8.48}$$

The proof is complete. □

Proof of Corollary 1. We need to consider only $\gamma > 1$. It follows directly from (2.8.6) that

$$C_{3,\gamma}^{-1} = \Big(\frac{3}{2} + \gamma - 1\Big)^2 + 2,$$

which gives the result.

Remark. Using (2.8.22), we see that a minimizing sequence $\{\mathbf{v}_k\}_{k\geq 1}$, which shows the sharpness of inequality (2.8.4) with the constant (2.8.5), can be obtained by taking $\mathbf{v}_k = (v_{\rho,k}, v_{\theta,k}, \mathbf{0})$ with the Fourier transform $\mathbf{w}_k = (w_{\rho,k}, w_{\theta,k}, \mathbf{0})$ chosen as follows:

$$w_{\theta,k}(\lambda,\theta) = h_k(\lambda)\sin\theta, \qquad w_{\rho,k}(\lambda,\theta) = \frac{1-n}{i\lambda + n/2 - \gamma} h_k(\lambda)\cos\theta. \tag{2.8.49}$$

The sequence $\{|h_k|^2\}_{k\geq 1}$ converges in distributions to the delta function at $\lambda = 0$. The minimizing sequence that gives the value (2.8.7) of $C_{3,\gamma}$ is

$$w_{\theta,k}(\lambda,0) = 0, \quad w_{\rho,k}(\lambda,\theta) = 0, \quad \text{and} \quad w_{\phi,k}(\lambda,\theta) = h_k(\lambda)\sin\theta,$$

where $\{|h_k|^2\}_{k\geq 1}$ is as previously.

2.8.3 Proof of Theorem 2

The calculations are similar but simpler than those in the previous section. We start with the substitution $\mathbf{v}(x) = \mathbf{u}(x)|x|^{2\gamma}$ and write (2.8.14) in the form

$$\frac{1}{C_{2,\gamma}} = \gamma^2 + \inf_{\mathbf{v}} \frac{\int_{\mathbb{R}^2} |\nabla \mathbf{v}|^2\, dx}{\int_{\mathbb{R}^2} |\mathbf{v}|^2 |x|^{-2}\, dx}. \tag{2.8.50}$$

In polar coordinates ρ and φ, with $\varphi \in [0, 2\pi)$, we have

$$\int_{\mathbb{R}^2} |\nabla \mathbf{v}|^2\, dx = \int_{\mathbb{R}^2} \big\{|\nabla v_\rho|^2 + |\nabla v_\varphi|^2 + \rho^{-2}\big(v_\rho^2 + v_\varphi^2 - 4v_\rho(\partial_\varphi v_\varphi)\big)\big\}\, dx. \tag{2.8.51}$$

Changing the variable ρ to $t = \log\rho$ and applying the Fourier transform $\mathbf{v}(\rho,\varphi) \to \mathbf{w}(\lambda,\varphi)$, we obtain that the right-hand side is

$$\int_{\mathbb{R}} \int_0^{2\pi} \big\{(\lambda^2+1)\big(|w_\rho|^2 + |w_\varphi|^2\big) + |\partial_\varphi w_\varphi|^2 + |\partial_\varphi w_\rho|^2 - 4(\partial_\varphi w_\varphi)\overline{w_\rho}\big\}\, d\varphi\, d\lambda. \tag{2.8.52}$$

The divergence-free condition for u becomes

$$w_\rho = -\frac{\partial_\varphi w_\varphi}{i\lambda + 1 - \gamma}, \tag{2.8.53}$$

which yields

$$\int_{\mathbb{R}^2} |\nabla \mathbf{v}|^2 \, \mathrm{d}x = \int_{\mathbb{R}} \int_0^{2\pi} \Big\{ \Big(\frac{\lambda^2 + 4\gamma - 3}{\lambda^2 + (1-\gamma)^2} + 1 \Big) |\partial_\varphi w_\varphi|^2 + \frac{|\partial_\varphi^2 w_\varphi|^2}{\lambda^2 + (1-\gamma)^2} + \big(\lambda^2 + 1\big) |w_\varphi|^2 \Big\} \, \mathrm{d}\varphi \, \mathrm{d}\lambda. \tag{2.8.54}$$

Analogously,

$$\int_{\mathbb{R}^2} |\mathbf{v}|^2 |x|^{-2} \, \mathrm{d}x = \int_{\mathbb{R}} \int_0^{2\pi} \big(|w_\rho|^2 + |w_\varphi|^2 \big) \, \mathrm{d}\varphi \, \mathrm{d}\lambda = \int_{\mathbb{R}} \int_0^{2\pi} \Big(\frac{|\partial_\varphi w_\varphi|^2}{\lambda^2 + (1-\gamma)^2} + |w_\varphi|^2 \Big) \, \mathrm{d}\varphi \, \mathrm{d}\lambda. \tag{2.8.55}$$

Therefore, by (2.8.50)

$$\frac{1}{C_{2,\gamma}} = \gamma^2 + \inf_{x \geq 0} \inf_{\nu \in \mathbb{N} \cup 0} f(x, \nu, \gamma), \tag{2.8.56}$$

where

$$f(x, \nu, \gamma) = x + 1 + \nu \Big(1 - \frac{4(1-\gamma)}{x + \nu + (1-\gamma)^2} \Big). \tag{2.8.57}$$

Let first $\gamma \leq 1$. Then f is increasing in x, which implies $f(x, \nu, \gamma) \geq f(0, \nu, \gamma)$. Since the derivative

$$\frac{\partial}{\partial \nu} f(0, \nu, \gamma) = 1 - \frac{4(1-\gamma)^3}{(\nu + (1-\gamma)^2)^2}, \tag{2.8.58}$$

is positive for $\nu \geq 2$, we need to compare only the values $f(0, 0, \gamma)$, $f(0, 1, \gamma)$, and $f(0, 2, \gamma)$. An elementary calculation shows that both $f(0, 0, \gamma)$ and $f(0, 2, \gamma)$ exceed $f(0, 1, \gamma)$ for $\gamma \notin (-1 - \sqrt{3}, -1 + \sqrt{3})$.

Let now $\gamma > 1$. We have

$$\frac{\partial}{\partial \nu} f(x, \nu, \gamma) = 1 + \frac{4(\gamma - 1)(x + (1-\gamma)^2)}{(x + \nu + (1 - \gamma^2))^2} > 0 \tag{2.8.59}$$

and therefore $f(x, \nu, \gamma) \geq f(x, 0, \gamma) = x + 1 \geq 1$. The proof of Theorem 2 is complete. □

Remark. Minimizing sequences that give $C_{2,\gamma}$ in (2.8.8) can be chosen as follows:

$$w_{\rho,k}(\lambda, \varphi) = 0, \qquad w_{\varphi,k}(\lambda, \varphi) = h_k(\lambda),$$

for $\gamma \notin (-1 - \sqrt{3}, -1 + \sqrt{3})$, and

$$w_{\rho,k} = \frac{h_k(\lambda)\sin(\varphi - \varphi_0)}{i\lambda + 1 - \gamma}, \qquad w_{\varphi,k} = h_k(\lambda)\cos(\varphi - \varphi_0),$$

when $\gamma \in (-1-\sqrt{3}, -1+\sqrt{3})$, for any constant φ_0. Here $\{|h_k|^2\}_{k\geq 1}$ converges in distributions to the delta function at 0.

Corollary. *Let $\gamma \neq 0$. Denote by ψ a real-valued scalar function in $C_0^\infty(\mathbb{R}^2)$ and assume, in addition, that $\nabla\psi(\mathbf{0}) = \mathbf{0}$ if $\gamma < 0$. Then the sharp inequality*

$$\int_{\mathbb{R}^2} |\nabla\psi|^2 |x|^{2(\gamma-1)}\, \mathrm{d}x \leq C_{2,\gamma} \int_{\mathbb{R}^2} \big(\psi^2_{x_1x_1} + 2\psi^2_{x_1x_2} + \psi^2_{x_2x_2}\big)|x|^{2\gamma}\, \mathrm{d}x \quad (2.8.60)$$

holds with $C_{2,\gamma}$ given in (2.8.8).

Indeed, for $n = 2$, inequality (2.8.4) becomes (2.8.60) if ψ is interpreted as a stream function of the vector field $\mathbf{u}$, i.e., $\mathbf{u} = \nabla \times \psi$.

2.8.4 Comments to Sect. 2.8

The results of this section are borrowed from the paper by Costin and Maz'ya [214]. In [715], Sobolevskii stated that the sharp constant in Hardy's inequality for arbitrary solenoidal vector functions in a convex domain coincides with the same constant in the classical one-dimensional case.

3

Conductor and Capacitary Inequalities with Applications to Sobolev-Type Embeddings

3.1 Introduction

Let Ω be an open set in $\mathbb{R}^n$ and let μ and ν be locally finite, nonzero Borel measures on Ω. We also use the following notation: l is a positive integer, $1 \le p \le \infty$, $q > 0$, $\mathrm{d}x$ is an element of the Lebesgue measure m_n on $\mathbb{R}^n$, and f is an arbitrary function in $C_0^\infty(\Omega)$, i.e., an infinitely differentiable function with compact support in Ω. By $\mathcal{L}_t$ we mean the set $\{x \in \Omega : |f(x)| > t\}$, where $t > 0$. We shall use the equivalence relation $a \sim b$ to denote that the ratio a/b admits upper and lower bounds by positive constants depending only on n, l, p, and q.

This chapter is a continuation of the previous one. We shall discuss the conductor inequality

$$\int_0^\infty \mathrm{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\, \mathrm{d}\big(t^p\big) \le c(a,p) \int_\Omega |\operatorname{grad} f|^p \, \mathrm{d}x, \tag{3.1.1}$$

where $a = \mathrm{const} > 1$ and cap_p is the p-capacity of $\overline{\mathcal{L}_{at}}$ with respect to $\mathcal{L}_t$, or in other words, the p conductivity of the conductor $\mathcal{L}\backslash\overline{\mathcal{L}_{at}}$. (This inequality was only mentioned in Sect. 2.3.1.)

By monotonicity of cap_p the *conductor inequality* (3.1.1) implies the *capacitary inequality*

$$\int_0^\infty \mathrm{cap}_p(\overline{\mathcal{L}_t}, \Omega)\, \mathrm{d}\big(t^p\big) \le C(p) \int_\Omega |\operatorname{grad} f|^p \, \mathrm{d}x, \tag{3.1.2}$$

which was also proved in Theorem 2.3.1 with the best constant

$$C(p) = p^p (p-1)^{1-p}. \tag{3.1.3}$$

Note that the left-hand side in (3.1.2) can be zero for all $f \in C_0^\infty(\Omega)$. (This happens if and only if either $p > n$ and $\Omega = \mathbb{R}^n$, or $p = n$ and the complement of Ω has zero n capacity.) At the same time, the left-hand side in

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(3.1.1) is always positive if $f \neq 0$. The layer cake texture of the left-hand side in the conductor inequality (3.1.1) admits significant corollaries that cannot be directly deduced from the inequality (3.1.2). For instance, as a straightforward consequence of (3.1.1) and the *isocapacitary inequality* (2.2.11), one deduces

$$\int_0^\infty \frac{\mathrm{d}(t^n)}{(\log \frac{m_n(\mathcal{L}_t)}{m_n(\mathcal{L}_{at})})^{n-1}} \leq c(a) \int_\Omega |\operatorname{grad} f|^n \, \mathrm{d}x, \tag{3.1.4}$$

where $n > 1$ and $a > 1$. Note that (3.1.4) is stronger than the well-known inequality

$$\int_0^\infty \frac{\mathrm{d}(t^n)}{(\log \frac{m_n(\Omega)}{m_n(\mathcal{L}_t)})^{n-1}} \leq c \int_\Omega |\operatorname{grad} f|^n \, \mathrm{d}x, \tag{3.1.5}$$

(see Maz'ya [543], Hansson [348], and Brezis and Wainger [146]) which is informative only if the volume of Ω is finite.

In the case $p \neq n$ and $p > 1$, another straightforward consequence of (3.1.1) with a similar flavor is the following improvement of the classical Sobolev inequality:

$$\int_0^\infty \left| m_n(\mathcal{L}_t)^{\frac{p-n}{n(p-1)}} - m_n(\mathcal{L}_{at})^{\frac{p-n}{n(p-1)}} \right|^{1-p} \mathrm{d}\big(t^p\big) \leq c(p,a) \int_\Omega |\operatorname{grad} f|^p \, \mathrm{d}x. \tag{3.1.6}$$

An immediate application of the conductor inequality (3.1.1) that seems to be unattainable with the help of the capacitary inequality (3.1.2) is a necessary and sufficient condition for the two-measure Sobolev-type inequality

$$\left(\int_\Omega |f|^q \, \mathrm{d}\mu \right)^{1/q} \leq C \left(\int_\Omega \big|\Phi(x, \operatorname{grad} f)\big|^p \, \mathrm{d}x + \int_\Omega |f|^p \, \mathrm{d}\nu \right)^{1/p}, \tag{3.1.7}$$

where $q \geq p$, μ, and ν are locally finite Radon measures on Ω, and the function $\Omega \times \mathbb{R}^n \ni (x, y) \to \Phi(x, y)$ is positively homogeneous in y of degree 1 (see Theorem 2.3.9). The just-mentioned characterization is formulated in terms of the conductivity generated by the integral

$$\int_\Omega \big|\Phi(x, \operatorname{grad} f)\big|^p \, \mathrm{d}x. \tag{3.1.8}$$

In the one-dimensional case and when $\Phi(x, \operatorname{grad} f) = |f'|$, the conductivity is calculated explicitly (see Lemma 2.2.2/2) and this characterization takes the following simple form:

$$\mu\big(\sigma_d(x)\big)^{p/q} \leq \operatorname{const}\big(\tau^{1-p} + \nu\big(\sigma_{d+\tau}(x)\big)\big), \tag{3.1.9}$$

where $\sigma_d(x)$ denotes any open interval $(x-d, x+d)$, such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

In the present chapter, we deal mostly with applications of conductor inequalities to two-measure Sobolev-type embeddings that seem to be unattainable with the help of capacitary inequalities. In particular, we sometimes assume that $n = 1$ and we study inequalities of the type

$$\Big(\int_\Omega |f|^q \, d\mu\Big)^{1/q} \le C\Big(\int_\Omega \big|f^{(l)}\big|^p \, dx + \int_\Omega |f|^p \, d\nu\Big)^{1/p}, \tag{3.1.10}$$

and their analogs involving a fractional Sobolev norm.

We conclude the Introduction with a brief outline of the contents of the chapter. In Sect. 3.2 we discuss the inequalities (3.1.4) and (3.1.5). A proof of (3.1.1) is given in Sect. 3.3. In Sects. 3.4 and 3.5 we characterize the inequality (3.1.7) and give a criterion for its multiplicative analog. A necessary and sufficient condition for the compactness and two-sided estimates of the essential norm of the embedding operator associated with (3.1.7) are obtained in Sect. 3.6.

In Sect. 3.7 we give a necessary and sufficient condition for the inequality

$$\Big(\int_\Omega |f|^q \, d\mu\Big)^{1/q} \le C\Big(\int_\Omega \big|f''(x)\big|^p \, dx + \int_\Omega |f|^p \, d\nu\Big)^{1/p} \tag{3.1.11}$$

with $1 < p \le q < \infty$, restricted to nonnegative functions $f \in C_0^\infty(\Omega)$. This is the estimate

$$\mu\big(\sigma_d(x)\big)^{p/q} \le \text{const}\big(\tau^{1-2p} + \nu\big(\sigma_{d+\tau}(x)\big)\big), \tag{3.1.12}$$

valid for all intervals $\overline{\sigma_{d+\tau}(x)} \subset \Omega$. A simple example shows that (3.1.12) does not guarantee (3.1.11) for all $f \in C_0^\infty(\Omega)$. We also give counterexamples showing that the necessary condition for (3.1.10)

$$\mu\big(\sigma_d(x)\big)^{p/q} \le \text{const}\big(\tau^{1-lp} + \nu\big(\sigma_{d+\tau}(x)\big)\big) \tag{3.1.13}$$

is not sufficient if $l \ge 3$.

Section 3.8 is dedicated to multidimensional (p,l)-conductivity inequalities for fractional Sobolev L_p norms of order l in $(0,1)$ and $(1,2)$. The section is concluded with the necessary and sufficient conditions for two-measure multidimensional inequalities of type (3.1.7) involving fractional norms.

3.2 Comparison of Inequalities (3.1.4) and (3.1.5)

Inequalities (3.1.4) and (3.1.5) follow directly from (3.1.1) combined with the isocapacitary inequality (2.2.11).

Let us compare the integrals on the left-hand sides of (3.1.4) and (3.1.5):

$$\int_0^\infty \frac{d(t^n)}{(\log \frac{m_n(\mathcal{L}_t)}{m_n(\mathcal{L}_{at})})^{n-1}}, \tag{3.2.1}$$

and

$$\int_0^\infty \frac{d(t^n)}{(\log \frac{m_n(\Omega)}{m_n(\mathcal{L}_t)})^{n-1}}, \tag{3.2.2}$$

where $m_n(\Omega) < \infty$. Clearly, the first of them exceeds the second. However, the convergence of the second integral does not imply the convergence of the first. In fact, let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, $\Omega = B_2$ and

$$f(x) = \begin{cases} 5 - |x| & \text{for } 0 \le |x| < 1, \\ 2 - |x| & \text{for } 1 \le |x| < 2. \end{cases} \tag{3.2.3}$$

We have

$$\mathcal{L}_t = \begin{cases} B_{2-t} & \text{for } 0 \le t < 1, \\ B_1 & \text{for } 1 \le t \le 4, \\ B_{5-t} & \text{for } 4 < t \le 5. \end{cases}$$

Let $1 < a < 4$. Then both sets $\mathcal{L}_t$ and $\mathcal{L}_{at}$ for $1 < t < 4a^{-1}$ coincide with the ball B_1 which makes (3.2.1) divergent whereas integral (3.2.2) is finite. Furthermore, integral (3.2.1) is convergent for $a \ge 4$.

Therefore, inequality (3.1.4) is strictly better than (3.1.5), even for domains Ω of finite volume. We see also that the convergence of integral (3.2.1) for a bounded function f may depend on the value of a.

The same argument shows that inequality (3.1.6) with any $f \in C_0^\infty(\Omega)$ and $1 < p < n$, i.e.,

$$\int_0^\infty \frac{\mathrm{d}(t^p)}{\Big(\frac{1}{m_n(\mathcal{L}_{at})^{\frac{n-p}{n(p-1)}}} - \frac{1}{m_n(\mathcal{L}_t)^{\frac{n-p}{n(p-1)}}}\Big)^{p-1}} \le c \int_\Omega |\operatorname{grad} f|^p \,\mathrm{d}x, \tag{3.2.4}$$

improves the Lorentz space $L_{\frac{np}{n-p},p}(\Omega)$ inequality

$$\int_0^\infty \big(m_n(\mathcal{L}_t)\big)^{\frac{n-p}{n}} \,\mathrm{d}\big(t^p\big) \le c \int_\Omega |\operatorname{grad} f|^p \,\mathrm{d}x,$$

which results from the capacitary inequality (2.3.6) and is stronger, in its turn, than the Sobolev estimate for the norm $\|f\|_{L_{\frac{pn}{n-p}}}$.

In conclusion, we add that the convergence of the integral on the left-hand side of (3.2.4) may depend on the choice of a, as shown by the function (3.2.3).

3.3 Conductor Inequality (3.1.1)

Let g and G denote arbitrary bounded open sets in $\mathbb{R}^n$ subject to $\bar{g} \subset G, \bar{G} \subset \Omega$. We introduce the p conductivity of the conductor $G \backslash \bar{g}$ (in other terms, the relative p-capacity of the set $\bar{g}$ with respect to G) as

$$\operatorname{cap}_p(\bar{g}, G) = \inf\Big\{ \int_\Omega \big|\operatorname{grad} \varphi(x)\big|^p \,\mathrm{d}x : \varphi \in C_0^\infty(G), 0 \le \varphi \le 1 \text{ on } G \text{ and } \varphi = 1 \text{ on a neighborhood of } \bar{g} \Big\}. \tag{3.3.1}$$

This infimum does not change if the class of admissible functions φ is enlarged to

$$\{\varphi \in C^\infty(\Omega) : \varphi \geq 1 \text{ on } g,\ \varphi \leq 0 \text{ on } \Omega \backslash G\} \tag{3.3.2}$$

(see Sect. 2.2.1).

Now we derive a generalization of the conductor inequality (3.1.1).

Proposition. *For all $f \in C_0^\infty(\Omega)$ and for an arbitrary $a > 1$ inequality* (3.1.1) *holds with*

$$c(a,p) = \frac{p \log a}{(a-1)^p}.$$

Proof. We show first that the function $t \to \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)$ is measurable. Let us introduce the open set $\mathcal{S} := \{t > 0 : |\operatorname{grad} f| > 0 \text{ on } \partial\mathcal{L}_t\}$ whose complement has zero one-dimensional Lebesgue measure by the Morse theorem. Let $t_0 \in \mathcal{S}$. Given an arbitrary $\varepsilon > 0$, there exists a function $\varphi \in C_0^\infty(\mathcal{L}_{t_0})$, $\varphi = 1$ on a neighborhood $\overline{\mathcal{L}_{at_0}}$, and such that

$$\|\operatorname{grad} \varphi\|_{L_p}^p \leq \operatorname{cap}_p(\overline{\mathcal{L}_{at_0}}, \mathcal{L}_{t_0}) + \varepsilon.$$

Since $t_0 \in \mathcal{S}$ we deduce from (3.3.1) that for all sufficiently small $\delta > 0$

$$\|\operatorname{grad} \varphi\|_{L_p}^p \geq \operatorname{cap}_p(\overline{\mathcal{L}_{a(t_0-\delta)}}, \mathcal{L}_{t_0+\delta}).$$

Therefore,

$$\operatorname{cap}_p(\overline{\mathcal{L}_{a(t_0\pm\delta)}}, \mathcal{L}_{t_0\pm\delta}) \leq \operatorname{cap}_p(\overline{\mathcal{L}_{at_0}}, \mathcal{L}_{t_0}) + \varepsilon,$$

which means that the function $t \to \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)$ is upper semicontinuous on $\mathcal{S}$. The measurability of this function follows.

Let φ denote a locally integrable function on $(0, \infty)$ such that there exist the limits $\varphi(0)$ and $\varphi(\infty)$. Then there holds the identity

$$\int_0^\infty \big(\varphi(t) - \varphi(at)\big) \frac{\mathrm{d}t}{t} = \big(\varphi(0) - \varphi(\infty)\big) \log a. \tag{3.3.3}$$

Setting here

$$\varphi(t) := \int_{\mathcal{L}_t} |\operatorname{grad} f|^p \, \mathrm{d}x,$$

we obtain

$$\int_\Omega |\operatorname{grad} f|^p \, \mathrm{d}x \geq \frac{1}{\log a} \int_0^\infty \int_{\mathcal{L}_t \backslash \mathcal{L}_{at}} |\operatorname{grad} f|^p \, \mathrm{d}x \frac{\mathrm{d}t}{t}.$$

By (3.3.1) the right-hand side exceeds

$$\frac{(a-1)^p}{p \log a} \int_0^\infty \operatorname{cap}_p(\overline{\mathscr{L}_{at}}, \mathscr{L}_t) \, \mathrm{d}\big(t^p\big),$$

and (3.1.1) follows. □

3.4 Applications of the Conductor Inequality (3.1.1)

The following lemma, which can be easily obtained from (3.1.1), is a particular case of Theorem 2.3.9.

Lemma 1. *Let* $1 \le p \le q$*. The inequality*

$$\Big(\int_\Omega |f|^q \,\mathrm{d}\mu\Big)^{1/q} \le C\Big(\int_\Omega |\operatorname{grad} f|^p \,\mathrm{d}x + \int_\Omega |f|^p \,\mathrm{d}\nu\Big)^{1/p} \tag{3.4.1}$$

holds for all $f \in C_0^\infty(\Omega)$ *if and only if there exists a constant* $K > 0$ *such that for all open bounded sets* g *and* G*, subject to* $\bar g \subset G, \bar G \subset \Omega$*, the inequality*

$$\mu(g)^{1/q} \le K\big(\operatorname{cap}_p(\bar g, G) + \nu(G)\big)^{1/p} \tag{3.4.2}$$

holds.

From this lemma, we shall deduce a sufficient condition for (3.4.1) which does not involve the p conductivity.

Corollary. *Let* $n < p \le q$*. If for all bounded open sets* g *and* G *in* $\mathbb{R}^n$ *such that* $\bar g \subset G$*,* $\bar G \subset \Omega$*, we have*

$$\mu(g)^{1/q} \le K\big(\operatorname{dist}(\partial g, \partial G)^{n-p} + \nu(G)\big)^{1/p}, \tag{3.4.3}$$

then (3.4.1) *holds for all* $f \in C_0^\infty(\Omega)$*.*

Proof. Let φ be an arbitrary admissible function in (3.3.1). By Sobolev's theorem 1.1.10/1, for all $y \in g$ and $z \in \Omega \backslash G$

$$1 \le \big(\varphi(y) - \varphi(z)\big)^p \le c|y-z|^{n-p} \int_\Omega \big|\operatorname{grad} \varphi(x)\big|^p \,\mathrm{d}x,$$

which implies

$$\big(\operatorname{dist}(\partial g, \partial G)\big)^{n-p} \le c \operatorname{cap}_p(\bar g, G).$$

It remains to refer to Lemma 1. □

Let us see how criterion (3.1.9) follows from Lemma 1.

Theorem 1. *Let* $n = 1$ *and* $1 \le p \le q < \infty$*. The inequality* (3.4.1) *holds for all* $f \in C_0^\infty(\Omega)$ *if and only if condition* (3.1.9) *is satisfied.*

Proof. Let $g_0 = (a, b), G_0 = (A, B)$, and $A < a < b < B$. It is an easy exercise to show that

$$\operatorname{cap}_p(\bar g_0, G_0) = (a-A)^{1-p} + (B-b)^{1-p}. \tag{3.4.4}$$

(For the proof of a more general formula for a weighted p conductivity see Lemma 2.2.2/2.) Hence, by setting $g = \sigma_d(x)$ and $G = \sigma_{d+\tau}(x)$ into (3.4.2), we obtain

$$\mu\big(\sigma_d(x)\big)^{1/q} \le K\big(2\tau^{1-p} + \nu\big(\sigma_{d+\tau}(x)\big)\big)^{1/p},$$

which implies the necessity of (3.1.9). To prove the sufficiency we need to obtain (3.4.2) for all admissible sets g and G. Let G be the union of nonoverlapping intervals G_i and let $g_i = G_i \cap g$. Denote by h_i the smallest interval containing g_i and by τ_i the minimal distance from h_i to $\mathbb{R}\backslash G_i$. By definition (3.3.1) in the one-dimensional case, we have

$$\mathrm{cap}_p(\bar{g}_i, G_i) = \mathrm{cap}_p(\bar{h}_i, G_i)$$

and

$$\mathrm{cap}_p(\bar{g}, G) = \sum_i \mathrm{cap}_p(\bar{g}_i, G_i).$$

Hence, and by (3.4.4) applied to the intervals h_i and G_i,

$$\mathrm{cap}_p(\bar{g}, G) \ge \sum_i \tau_i^{1-p}. \tag{3.4.5}$$

Using (3.1.9), we obtain

$$\mu(g_i)^{1/q} \le \mu(h_i)^{1/q} \le A\big(\tau_i^{1-p} + \nu(G_i)\big)^{1/q},$$

where A is a positive constant independent of g and G. Since $q \ge p$, we have

$$\mu(g)^{p/q} \le \sum_i \mu(g_i)^{p/q},$$

which, together with (3.4.5), implies

$$\mu(g)^{p/q} \le A^p \sum_i \big(\tau_i^{1-p} + \nu(G_i)\big) \le A^p \big(\mathrm{cap}_p(\bar{g}, G) + \nu(G)\big).$$

The result follows from Lemma 1. □

Remark 1. A possible modification of Theorem 3.4 concerns the Birnbaum–Orlicz space $L_M(\mu)$, where M is an arbitrary convex function on $(0, \infty)$, $M(+0) = 0$. Let P denote the complementary convex function to M. One can easily deduce from Theorems 2.3.3 (compare with Theorem 3.4) that the condition

$$\mu\big(\sigma_d(x)\big) P^{-1}\left(\frac{1}{\mu(\sigma_d(x))}\right) \le \mathrm{const}\ \big(\tau^{1-p} + \nu\big(\sigma_{d+\tau}(x)\big)\big)^{1/p}$$

is necessary and sufficient for the inequality

$$\int_0^\infty \mu(\mathcal{L}_\tau) P^{-1}\left(\frac{1}{\mu(\mathcal{L}_\tau)}\right) \mathrm{d}\big(\tau^p\big) \le c\left(\int_\Omega |f'|^p \,\mathrm{d}x + \int_\Omega |f|^p \,\mathrm{d}\nu\right)$$

as well as for the inequality

$$\big\||u|^p\big\|_{L_M(\mu)} \le c\Big(\int_\Omega |f'|^p\,\mathrm{d}x + \int_\Omega |f|^p\,\mathrm{d}\nu\Big).$$

It is well known that the weight w in the integral

$$\int_\Omega \big|f'(x)\big|^p w(x)\,\mathrm{d}x$$

can be removed by the change of the variable x

$$\xi = \int \frac{\mathrm{d}x}{w(x)^{1/(p-1)}}.$$

Therefore, Theorem 3.4 leads to a criterion for the three-weight inequality

$$\Big(\int_\Omega |f|^q\,\mathrm{d}\mu\Big)^{1/q} \le \Big(\int_\Omega |f'|^p\,\mathrm{d}\lambda + \int_\Omega |f|^p\,\mathrm{d}\nu\Big)^{1/p},$$

where λ is a nonnegative measure. Note that the singular part of λ does not influence the validity of the last inequality (compare with Muckenhoupt [620] and Sect. 1.3.2).

Remark 2. Let $n = 1$. With $p \in (1,\infty)$ and the measure ν, we associate a function $\mathcal{R}$ of an interval $\sigma_d(x)$ by the equality

$$\mathcal{R}\big(\sigma_d(x)\big) = \sup\big\{\tau : \tau^{1-p} > \nu\big(\sigma_{d+\tau}(x)\big)\big\} \tag{3.4.6}$$

with $\overline{\sigma_{d+\tau}(x)} \subset \Omega$ as everywhere. Clearly,

$$\mathcal{R}\big(\sigma_d(x)\big)^{1-p} \le \inf\big\{\tau : \tau^{1-p} + \nu\big(\sigma_{d+\tau}(x)\big)\big\} \le 2\mathcal{R}\big(\sigma_d(x)\big)^{1-p}, \tag{3.4.7}$$

which shows that criterion (3.1.9) can be written as

$$\sup_{\overline{\sigma_d(x)} \subset \Omega} \mathcal{R}\big(\sigma_d(x)\big)^{(p-1)/p} \mu\big(\sigma_d(x)\big)^{1/q} < \infty.$$

Remark 3. According to Theorem 2.1.3, inequality (3.4.1) with $p = 1, q \ge 1$ is equivalent to the inequality

$$\mu(g)^{1/q} \le C\big(2 + \nu(g)\big),$$

where g is an arbitrary interval and C is the same constant as in (3.4.1).

Similarly to (3.4.1) with $n = 1$, we can characterize the inequality

$$\Big(\int_\Omega |f|^q\,\mathrm{d}\mu\Big)^{1/q} \le C\Big(\int_\Omega |f'|^p\,\mathrm{d}x\Big)^{\delta/p}\Big(\int_\Omega |f|^r\,\mathrm{d}\nu\Big)^{(1-\delta)/r} \tag{3.4.8}$$

by using the following particular case of Theorem 2.3.11.

Lemma 2. *Let $n \geq 1$, $p \geq 1$, and $\delta \in [0,1]$. If the inequality*

$$\Big(\int_\Omega |f|^q \,\mathrm{d}\mu\Big)^{1/q} \leq C\Big(\int_\Omega |\operatorname{grad} f|^p \,\mathrm{d}x\Big)^{\delta/p}\Big(\int_\Omega |f|^r \,\mathrm{d}\nu\Big)^{(1-\delta)/r}, \tag{3.4.9}$$

is valid for all $f \in C_0^\infty(\Omega)$ and some positive r and q, then there exists a constant α such that for all open bounded subsets g and *G of Ω such that $\bar{g} \subset G$, $\bar{G} \subset \Omega$, there holds the inequality*

$$\mu(g)^{p/q} \leq \alpha \ \operatorname{cap}_p(\bar{g}, G)^\delta \nu(G)^{(1-\delta)p/r}. \tag{3.4.10}$$

If (3.4.10) holds for all g and G as above, then (3.4.9) is valid for all functions $f \in C_0^\infty(\Omega)$ with $1/q \leq (1-\delta)/r + \delta/p$.

Arguing as in the proof of Theorem 3.4, we arrive at the following criterion for (3.4.8).

Theorem 2. *Let $n = 1$, $p \geq 1$, and $\delta \in [0,1]$. If the inequality (3.4.8) holds for all $f \in C_0^\infty(\Omega)$ and some positive r* and *q, then there exists a constant $\beta > 0$ such that*

$$\mu\big(\sigma_d(x)\big)^{1/q} \leq \frac{\beta}{\tau^{\delta(p-1)/p}} \nu\big(\sigma_{d+\tau}(x)\big)^{(1-\delta)/r} \tag{3.4.11}$$

for all $x \in \Omega, d > 0$ and $\tau > 0$ such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$. Conversely, if (3.4.11) is true for some positive r and q such that $1/q \leq (1-\delta)/r + \delta/p$, then (3.4.8) holds.

Note that for $p = 1$ condition (3.4.11) is simplified:

$$\mu\big(\sigma_d(x)\big)^{r/q(1-\delta)} \leq \text{const } \nu\big(\sigma_d(x)\big).$$

For the particular case $\mu = \nu$, inequality (3.4.8) admits the following simpler characterization that results from Theorem 2.3.6.

Theorem 3. 1. *Let $n = 1$ and let for all $x \in \Omega$, $d > 0$ and $\tau > 0$ such that $\sigma_{d+\tau}(x) \subset \Omega$, there holds*

$$\mu\big(\sigma_d(x)\big)^\alpha \leq \text{const } \tau^{(1-p)/p}, \tag{3.4.12}$$

where $p \geq 1$ and $\alpha > 0$. Furthermore, let q be a positive number satisfying one of the conditions (i) *$q \leq \alpha^{-1}$ if $\alpha p \leq 1$ or* (ii) *$q < \alpha^{-1}$ if $\alpha p > 1$. Then the inequality*

$$\Big(\int_\Omega |f|^q \,\mathrm{d}\mu\Big)^{1/q} \leq C\Big(\int_\Omega |f'|^p \,\mathrm{d}x\Big)^{\delta/p}\Big(\int_\Omega |f|^r \,\mathrm{d}\mu\Big)^{(1-\delta)/r} \tag{3.4.13}$$

with $r \in (0,q)$ and $\delta = (q-r)/(1-\alpha r)q$ is valid for any function $f \in C_0^\infty(\Omega)$.

2. *Conversely, let* $p \geq 1$*,* $\alpha > 0$ *and* $r \in (0, \alpha^{-1}]$*. Furthermore, let the inequality* (3.4.13) *with* $\delta = (q-r)/(1-\alpha r)q$ *hold for any function* $f \in C_0^\infty(\Omega)$*. Then* (3.4.12) *holds for all* x *and* d *such that* $\overline{\sigma_{d+\tau}(x)} \subset \Omega$*.*

Remark 4. By Theorem 3 the multiplicative inequality (3.4.13) is equivalent to

$$\Big(\int_\Omega |f|^q \, \mathrm{d}\mu\Big)^{1/q} \leq C\Big(\int_\Omega |f'|^p \, \mathrm{d}x\Big)^{1/p}$$

with $p \leq q$.

The next assertion concerning an arbitrary charge λ (not a nonnegative measure as elsewhere) follows directly from Theorem 2.3.8.

Theorem 4. *Let* λ^+ *and* λ^- *denote the positive and negative parts of the charge* λ*, respectively.*

(i) *If for a certain* $\varepsilon \in (0, 1)$ *and for any* $x \in \Omega$*,* $d > 0$*,* $\tau > 0$*, such that* $\overline{\sigma_{d+\tau}(x)} \subset \Omega$*, there holds*

$$\lambda^+\big(\sigma_d(x)\big) \leq C_\varepsilon \tau^{1-p} + (1-\varepsilon)\lambda^-\big(\sigma_{d+\tau}(x)\big),$$

where $p > 1$*, then for all* $f \in C_0^\infty(\Omega)$

$$\int_\Omega |f|^p \, \mathrm{d}\lambda \leq C \int_\Omega |f'|^p \, \mathrm{d}x. \tag{3.4.14}$$

(ii) *If* (3.4.14) *is true for any* $x \in \Omega$*,* $d > 0$*,* $\tau > 0$ *such that* $\overline{\sigma_{d+\tau}(x)} \subset \Omega$*, then*

$$\lambda^+\big(\sigma_d(x)\big) \leq C\tau^{1-p} + \lambda^-\big(\sigma_{d+\tau}(x)\big). \tag{3.4.15}$$

Example. We show that (3.4.15) is not sufficient for (3.4.14). Let λ^+ and λ^- be the Dirac measures concentrated at the points 0 and 1, respectively. We introduce the sequence of piecewise linear functions $\{\varphi_m\}_{m=1}^\infty$ on $\mathbb{R}$ by

$$\varphi_m(x) = 0 \quad \text{for } |x| > m^{p/(p-1)},$$

$$\varphi_m(0) = 1, \qquad \varphi_m(1) = 1 - m^{-1}.$$

Then

$$\int_{\mathbb{R}} |\varphi_m|^p \, \mathrm{d}\lambda = \frac{p}{m}\big(1 + o(1)\big) \quad \text{and} \quad \int_{\mathbb{R}} |\varphi_m'|^p \, \mathrm{d}x \sim m^{-p} \quad \text{as } m \to \infty,$$

and therefore (3.4.14) fails. However, (3.4.15) holds with $C = 1$. To check this, we need to consider only the case $\lambda^+(\sigma_d(x)) = 1$ and $\lambda^-(\sigma_{d+\tau}(x)) = 0$, when clearly $\tau \leq 1$ and $\tau^{1-p} \geq 1$.

3.5 p-Capacity Depending on ν and Its Applications to a Conductor Inequality and Inequality (3.4.1)

Let $n \geq 1$ and let K denote a compact subset of Ω. We introduce a relative p-capacity of K with respect to Ω, depending on the measure ν, by

$$\operatorname{cap}_p(K, \Omega, \nu) = \inf\left(\| \operatorname{grad} \varphi \|^p_{L_p} + \int_\Omega |\varphi|^p \, d\nu \right), \tag{3.5.1}$$

where the infimum is extended over all functions $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \geq 1$ on K. Arguing as in Sect. 2.3.1, one can show that the infimum in (3.5.1) will be the same if the set of admissible functions is replaced by $\{\varphi \in C_0^\infty(\Omega) : \varphi = 1 \text{ on } K, 0 \leq \varphi \leq 1 \text{ on } \Omega\}$.

Making small changes in the proof of Proposition 3.3, one arrives at the inequality

$$\int_0^\infty \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t, \nu) \, d\big(t^p\big) \leq c(p) \left(\| \operatorname{grad} f \|^p_{L_p} + \int_\Omega |f|^p \, d\mu \right),$$

where $a = \text{const} > 1$ and $f \in C_0^\infty(\Omega)$. By this inequality one can easily obtain the following condition, necessary and sufficient for (3.4.1) with $q \geq p$:

$$\mu(g)^{p/q} \leq \text{const} \operatorname{cap}_p(g, \Omega, \nu) \tag{3.5.2}$$

for all bounded open sets g with $\bar{g} \subset \Omega$.

The next lemma shows directly that (3.5.2) is equivalent to (3.4.2).

Lemma. *There holds the equivalence relation*

$$\operatorname{cap}_p(K, \Omega, \nu) \sim \inf_G \big(\operatorname{cap}_p(K, G) + \nu(G) \big), \tag{3.5.3}$$

where the infimum is taken over all bounded open sets G such that $K \subset G$ and $\bar{G} \subset \Omega$.

Proof. Let $\varepsilon > 0$, $f \in C_0^\infty(\Omega)$, $f = 1$ on K, $0 \leq f \leq 1$ on Ω, and let

$$\operatorname{cap}_p(K, \Omega, \nu) + \varepsilon \geq \| \operatorname{grad} f \|^p_{L_p} + \int_\Omega |f|^p \, d\nu.$$

Then

$$\begin{aligned} &\operatorname{cap}_p(K, \Omega, \nu) + \varepsilon \\ &\quad \geq \sum_{k=0}^\infty 2^{-p(k+1)} \int_{\mathcal{L}_{2^{-k-1}} \setminus \mathcal{L}_{2^{-k}}} \big| \operatorname{grad}\big(2^{k+1} f - 1\big) \big|^p \, dx + \int_0^1 \nu(\mathcal{L}_t) \, d\big(t^p\big) \\ &\quad \geq c \sum_{k=0}^\infty 2^{-pk} \big(\operatorname{cap}_p(\overline{\mathcal{L}_{2^{-k}}}, \mathcal{L}_{2^{-k-1}}) + \nu(\mathcal{L}_{2^{-k-1}}) \big). \end{aligned}$$

Since $\mathrm{cap}_p(\overline{\mathcal{L}_{2^{-k}}}, \mathcal{L}_{2^{-k-1}}) \ge \mathrm{cap}_p(K, \mathcal{L}_{2^{-k-1}})$, it follows that

$$\mathrm{cap}_p(K, \Omega, \nu) + \varepsilon \ge c \inf_G \big(\mathrm{cap}_p(K, G) + \nu(G)\big).$$

The estimate

$$\mathrm{cap}_p(K, \Omega, \nu) \le \mathrm{cap}_p(K, G) + \nu(G)$$

is obvious. The result follows. □

We introduce the capacity minimizing function

$$S_p(t) = \inf \mathrm{cap}_p(g, \Omega, \nu),$$

where the infimum is taken over all bounded open sets g, $\bar{g} \subset \Omega$, satisfying $\mu(g) > t$. By the last lemma,

$$S_p(t) \sim \inf_{g,G} \big(\mathrm{cap}_p(g, G) + \nu(G)\big)$$

with the infimum extended over open sets g and G such that $\bar{g} \subset G$, $\bar{G} \subset \Omega$, and $\mu(g) > t$. Obviously, the condition (3.5.2) is equivalent to

$$\sup \frac{t^{p/q}}{S_p(t)} < \infty.$$

Making trivial changes in the proof of Theorem 2.3.8, we arrive at *the condition, necessary and sufficient for* (3.4.1) *with* $0 < q < p$, $p \ge 1$:

$$\int_0^\infty \left(\frac{t^{p/q}}{S_p(t)}\right)^{q/(p-q)} \frac{\mathrm{d}t}{t} < \infty. \tag{3.5.4}$$

It follows from the proof of Theorem 3.4/1 that in the one-dimensional case there holds the equivalence relation

$$S_p(t) \sim \inf\big\{\tau : \tau^{1-p} + \nu\big(\sigma_{d+\tau}(x)\big)\big\}$$

with the infimum taken over all x, d, and τ such that $\overline{\sigma_{d+\tau}(x)} \subset \Omega$ and

$$\mu\big(\sigma_d(x)\big) > t. \tag{3.5.5}$$

By (3.4.7),

$$S_p(t) \sim \inf \mathcal{R}\big(\sigma_d(x)\big)^{1-p},$$

where the infimum is taken over all intervals $\sigma_d(x)$, $\overline{\sigma_d(x)} \subset \Omega$, satisfying (3.5.5).

3.6 Compactness and Essential Norm

We define the space $\mathring{\mathfrak{W}}^1_p(\nu)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|_{\mathring{\mathfrak{W}}^1_p(\nu)} = \left(\int_\Omega \big|f'(x)\big|^p \,\mathrm{d}x + \int_\Omega \big|f(x)\big|^p \,\mathrm{d}\nu\right)^{1/p}.$$

The condition (3.1.9) is a criterion of boundedness for the embedding operator

$$I_{p,q} : \mathring{\mathfrak{W}}^1_p(\nu) \to L_q(\mu)$$

for $q \geq p \geq 1$.

In this section we establish a compactness criterion for $I_{p,q}$ with $q \geq p \geq 1$ and obtain sharp two-sided estimates for the essential norm of $I_{p,q}$. We recall that the essential norm of a bounded linear operator A acting from X into Y, where X and Y are linear normed spaces, is defined by

$$\operatorname{ess}\|A\| = \inf_T \|A - T\|,$$

with the infimum taken over all compact operators $T : X \to Y$.

Theorem 1. *If $q \geq p \geq 1$, then the operator $I_{p,q}$ is compact if and only if*

$$\lim_{M\to\infty} \sup_{x,\tau,d} \frac{\mu(\sigma_d(x) \setminus [-M,M])^{1/q}}{\{\tau^{1-p} + \nu(\sigma_{d+\tau}(x))\}^{1/p}} = 0. \tag{3.6.1}$$

Proof. Sufficiency. Let μ' stand for the restriction of μ to the segment $[-M, M]$ and let $\mu_M = \mu - \mu'_M$. We define the embedding operators

$$I_M : \mathring{\mathfrak{W}}^1_p(\nu) \to L_q(\mu_M) \quad \text{and} \quad I'_M : \mathring{\mathfrak{W}}^1_p(\nu) \to L_q(\mu'_M),$$

as well as the embedding operators

$$i_M :\ L_q(\mu_M) \to L_q(\mu) \quad \text{and} \quad i'_M : L_q(\mu'_M) \to L_q(\mu).$$

We have

$$I_{p,q} = i_M \circ I_M + i'_M \circ I'_M. \tag{3.6.2}$$

We prove that I'_M is compact. Consider the embedding operators

$$I^C_M : \mathring{\mathfrak{W}}^1_p(\nu) \to C\big([-M,M]\big),$$

$$i^C_M : C\big([-M,M]\big) \to L_q(\mu'_M),$$

where $C([-M, M])$ is the space of continuous functions with the usual norm. Clearly, $I'_M = i^C_M \circ I^C_M$. Since I^C_M is compact for any $M > 0$ by the Arzela theorem, the operator I'_M is compact as well. The condition $\|I_M\| \to 0$ as $M \to \infty$ is equivalent to (3.6.1) owing to Theorem 3.4/1.

Necessity. Let $I_{p,q}$ be compact and let B denote the unit ball in $\mathring{\mathfrak{W}}_p^1(\nu)$. The set $I_{p,q}B$ is relatively compact in $L_q(\mu)$. Therefore, for any $\varepsilon > 0$ there exists a finite ε-net $\{f_j\}_{j=1}^N \subset I_{p,q}B = B$ for the set $I_{p,q}B$. Given any f_j, there exists a number $M_j(\varepsilon)$ such that

$$\int_{|x|>M_j(\varepsilon)} \big|f_j(x)\big|^q \,\mathrm{d}\mu(x) < \varepsilon^q.$$

Let $M(\varepsilon)$ be equal to $\sup_j M_j(\varepsilon)$. Then for any $f \in B$ and for some $i \in \{1, N\}$ we have

$$\begin{aligned}\left(\int_\Omega \big|f(x)\big|^q \,\mathrm{d}\mu_{M(\varepsilon)}(x)\right)^{1/q} &\le \|f - f_j\|_{L_q(\mu)} + \left(\int_\Omega \big|f_j(x)\big|^q \,\mathrm{d}\mu_{M_j(\varepsilon)}(x)\right)^{1/q} \\ &< 2\varepsilon.\end{aligned}$$

Hence inequality (3.1.10) holds, where $l = 1$, $\mu_{M(\varepsilon)}$ and 2ε instead of μ and C. Now (3.6.1) follows from the necessity part in Theorem 3.4/1. □

Theorem 2. *Let $q \ge p \ge 1$ and*

$$E(\mu,\nu) := \lim_{M\to\infty} \sup_{x,\tau,d} \frac{\mu(\sigma_d(x)\setminus[-M,M])^{1/q}}{\{\tau^{1-p} + \nu(\sigma_{d+\tau}(x))\}^{1/p}}.$$

There exist positive constants c_1 and c_2 such that

$$c_1 E(\mu,\nu) \le \operatorname{ess}\|I_{p,q}\| \le c_2 E(\mu,\nu). \tag{3.6.3}$$

Proof. We use the same notation as in the previous theorem. The upper bound in (3.6.3) is a consequence of the sufficiency part in the proof of Theorem 3.4/1.

Let T be any compact operator: $\mathring{\mathfrak{W}}_p^1(\nu) \to L_q(\mu)$ and let ε be any positive number. We choose T to satisfy

$$\operatorname{ess}\|I_{p,q}\| \ge \|I_{p,q} - T\| - \varepsilon. \tag{3.6.4}$$

There exists a positive $M(\varepsilon)$ such that for any $f \in B$

$$\int_\Omega \big|Tf(x)\big|^q \,\mathrm{d}\mu_{M(\varepsilon)}(x) < \varepsilon^q. \tag{3.6.5}$$

We introduce the truncation operator $\tau_M : L_q(\mu) \to L_q(\mu_M)$ by

$$(\tau_M f)(x) = \begin{cases} 0, & |x| < M, \\ f(x), & |x| \ge M. \end{cases}$$

Using (3.6.4) and (3.6.5), we obtain

$$\operatorname{ess}\|I_{p,q}\| \geq \|I_{M(\varepsilon)} - \tau_{M(\varepsilon)} \circ T\| - \varepsilon \geq \|I_{M(\varepsilon)}\| - 2\varepsilon.$$

By Theorem 3.6,

$$\|I_{M(\varepsilon)}\| \geq c \sup_{x,\tau,d} \frac{\mu_{M(\varepsilon)}(\sigma_d(x))^{1/q}}{\{\tau^{1-p} + \nu(\sigma_{d+\tau}(x))\}^{1/p}} \geq c_1 E(\mu, \nu).$$

The result follows. □

3.7 Inequality (3.1.10) with Integer $l \geq 2$

Let us deduce a characterization of inequality (3.1.11) for nonnegative functions.

Theorem. *Let $n = 1$ and $1 < p \leq q < \infty$. The inequality (3.1.11) holds for all $f \in C_0^\infty(\Omega)$ and $f \geq 0$ on Ω if and only if there exists a constant $K > 0$ such that*

$$\mu\big(\sigma_d(x)\big)^{1/q} \leq K\big(\tau^{1-2p} + \nu\big(\sigma_{d+\tau}(x)\big)\big)^{1/p}, \tag{3.7.1}$$

for all $x \in \Omega, d > 0$ and $\tau > 0$ satisfying $\overline{\sigma_{d+\tau}(x)} \subset \Omega$.

Proof. To prove the necessity, we set a function f in (3.1.11), which is subject to $f \in C_0^\infty(\Omega)$, $f = 1$ on $\sigma_d(x)$, $f = 0$ outside $\sigma_{d+\tau}(x)$, and $0 \leq f(x) \leq 1$ on Ω. Then

$$\mu\big(\sigma_d(x)\big)^{1/q} \leq C\bigg(\int_{\sigma_{d+\tau}(x)\backslash\sigma_d(x)} \big|f''(y)\big|^p \,\mathrm{d}y + \nu\big(\sigma_{d+\tau}(x)\big)\bigg)^{1/p}.$$

Clearly, f can be chosen on $\sigma_{d+\tau}(x)\backslash\sigma_d(x)$ so that the integral on the right does not exceed $c(p)r^{1-2p}$. The estimate (3.7.1) follows.

For the proof of sufficiency we need the following lemma.

Lemma. *If $f \in \mathscr{D}(\mathbb{R})$, $f \geq 0$, then $f^{1/2} \in L^1_{2p}(\mathbb{R})$ and*

$$\int_{\mathbb{R}} \frac{|f'|^{2p}}{f^p}\,\mathrm{d}t \leq \left(\frac{2p-1}{p-1}\right)^p \int_{\mathbb{R}} |f''|^p \,\mathrm{d}t, \tag{3.7.2}$$

where the integration is taken over the support of f. The constant factor on the right-hand side is optimal.

Proof. Obviously, for $\varepsilon > 0$,

$$\int_{\mathbb{R}} \frac{|f'|^{2p}}{(f+\varepsilon)^p}\,\mathrm{d}t = \frac{1}{1-p}\int_{\mathbb{R}} |f'|^{2p-2} f'\big((f+\varepsilon)^{1-p}\big)'\,\mathrm{d}t.$$

Integrating by parts, we obtain

$$\int_{\mathbb{R}} \frac{|f'|^{2p}}{(f+\varepsilon)^p}\,\mathrm{d}t = \frac{2p-1}{p-1}\int_{\mathbb{R}} \frac{|f'|^{2(p-1)}}{(f+\varepsilon)^{p-1}} f''\,\mathrm{d}t$$
$$\le \frac{2p-1}{p-1}\Big(\int_{\mathbb{R}} |f''|^p\,\mathrm{d}t\Big)^{1/p}\Big(\int_{\mathbb{R}} \frac{|f'|^{2p}}{(f+\varepsilon)^p}\,\mathrm{d}t\Big)^{(p-1)/p}.$$

Therefore

$$\int_{\mathbb{R}} \frac{|f'|^{2p}}{(f+\varepsilon)^p}\,\mathrm{d}t \le \Big(\frac{2p-1}{p-1}\Big)^p \int_{\mathbb{R}} |f''|^p\,\mathrm{d}t.$$

It remains to pass to the limit as $\varepsilon \to +0$.

Let $f_\varepsilon(x) = \|x\|^{\varepsilon+2-1/p}\eta(x)$, where $\varepsilon > 0$, $\eta \in C_0^\infty(\mathbb{R})$ and $\eta = 1$ near the origin. Putting the family $\{f_\varepsilon\}$ into (3.7.2) we see that the constant factor on the right-hand side of (3.7.2) is sharp. The lemma is proved. □

Let us turn to the proof of sufficiency of (3.7.1). Let $f \in C_0^\infty(\Omega)$ satisfy $\operatorname{supp} f \subset \Omega$ and $f \ge 0$. Then $u = f^{1/2}$ satisfies

$$\Big(\int_a^b |u|^{2q}\,\mathrm{d}\mu\Big)^{1/2q} \le cK^{1/2}\Big(\int_a^b |u'|^{2p}\,\mathrm{d}x + \int_a^b |u|^{2p}\,\mathrm{d}\nu\Big)^{1/2p}$$

by Theorem 3.4/1. This inequality, along with (3.7.2) gives (3.1.11). □

The proof of the following conductor inequality, involving nonnegative functions, is based upon the *smooth level truncation* and the last lemma.

Proposition. *Let* $n \ge 1$, $f \in C_0^\infty(\Omega)$, $f \ge 0$, $a = \mathrm{const} > 1$, *and* $p > 1$. *Then*

$$\int_0^\infty \operatorname{cap}^+_{p,2}(\overline{\mathscr{L}_{at}}, \mathscr{L}_t)\,\mathrm{d}\big(t^p\big) \le c(p,a)\int_\Omega |\operatorname{grad}_2 f|^p\,\mathrm{d}x, \tag{3.7.3}$$

where $\operatorname{grad}_2 = \{\partial^2/\partial x_i \partial x_j\}_{i,j=1}^n$ *and*

$$\operatorname{cap}^+_{p,2}(\bar g, G) = \inf\Big\{\int_G \big|\operatorname{grad}_2 \varphi(x)\big|^p\,\mathrm{d}x : \varphi \in C_0^\infty(G), 1 \ge \varphi \ge 0 \text{ on } G,$$
$$\varphi = 1 \text{ in a neighborhood of } \bar g\Big\}. \tag{3.7.4}$$

(Concerning the measurability of the function $t \to \operatorname{cap}^+_{p,2}(\overline{\mathscr{L}_{at}}, \mathscr{L}_t)$ see the beginning of the proof of Proposition 3.3.)

Proof. Let $H \in C^2(\mathbb{R})$,

$$H(x) = \begin{cases} 0 & \text{for } x < \varepsilon, \\ 1 & \text{for } x > 1-\varepsilon, \end{cases}$$

where ε is an arbitrary number in $(0,1)$. By (3.7.4),

$$\begin{aligned}\operatorname{cap}^+_{p,2}(\overline{\mathscr{L}_{at}}, \mathscr{L}_t) &\leq \int_\Omega \left| \operatorname{grad}_2 \left(H\left(\frac{f(x)-t}{(a-1)t} \right) \right) \right|^p \mathrm{d}x \\ &\leq \frac{c(a)}{t^p} \int_{\mathscr{L}_t \setminus \mathscr{L}_{at}} \left(\frac{|\operatorname{grad} f|^{2p}}{f^p} + |\operatorname{grad}_2 f|^p \right) \mathrm{d}x.\end{aligned}$$

Hence the left-hand side in (3.7.3) is dominated by

$$pc(a) \int_0^\infty \int_{\mathscr{L}_t \setminus \mathscr{L}_{at}} \left(\frac{|\operatorname{grad} f|^{2p}}{f^p} + |\operatorname{grad}_2 f|^p \right) \mathrm{d}x \, \frac{\mathrm{d}t}{t}.$$

Owing to (3.3.3), this can be written as

$$pc(a) \log a \int_\Omega \left(\frac{|\operatorname{grad} f|^{2p}}{f^p} + |\operatorname{grad}_2 f|^p \right) \mathrm{d}x,$$

which does not exceed the right-hand side of inequality (3.7.3) in view of (3.7.2). The result follows. □

Remark. A direct generalization of this proof for derivatives of higher than second order is impossible since there is no analog of the inequality (3.7.2) for higher derivatives. In fact, the example of a function $\mathbb{R}^1 \ni t \to u \in C_0^\infty(\mathbb{R}^1)$, $u \geq 0$, coinciding with t^2 for $|t| < 1$, shows that the finiteness of the norm $\|u^{(l)}\|_{L_p(\mathbb{R}^1)}$ does not imply the finiteness of the integral

$$\int_{\mathbb{R}^1} \left| u^{(j)}(t) \right|^{pl/j} u(t)^{p(j-l)/j} \, \mathrm{d}t$$

for $l > 2$.

Nevertheless, in Sect. 11.2.1 it will be shown that "the smooth truncation" can be used in the proof of a capacitary inequality for $\Omega = \mathbb{R}^n$ and any integer $l > 0$ being applied not to an arbitrary nonnegative function but to a potential with nonnegative density.

Example 1. Let us show that condition (3.7.1) is not sufficient for (3.1.11) with $p = 1$. Let ν be Dirac's measure concentrated at $x = 0$ and let $\mathrm{d}\mu(x) = (1+x^2)^{-1}\,\mathrm{d}x$. Obviously, condition (3.7.1) holds. We construct a sequence of nonnegative functions $\eta_m \in C_0^\infty(\mathbb{R}), m = 1, 2, \ldots$, defined by $\eta_m(x) = \varphi_m(x - m - 1)$, where φ_m is a smooth, nonnegative, even function on $\mathbb{R}$, vanishing for $x \geq m+1$ and such that $\varphi_m(x) = m + 1 - x$ for $1 \leq x \leq m$. Then $\eta_m(0) = 0$,

$$\int_{\mathbb{R}} |\eta_m''| \, \mathrm{d}x = \text{const}, \qquad \int_{\mathbb{R}} \eta_m^q \, \mathrm{d}\mu \to \infty,$$

i.e., inequality (3.1.11) with $p = 1$ fails.

Example 2. We shall check that (3.7.1) does not suffice for (3.1.11) to be valid for all $f \in C_0^\infty(\mathbb{R})$ if $p \geq 1$. Let ν and μ be Dirac's measures concentrated at 0 and 1, respectively. Consider the function $\varphi_0 \in C_0^\infty(\mathbb{R})$ such that $\varphi_0(x) = x$ for $x \in [-1, 1]$. We set $\varphi_m(x) = \varphi_0(x/m)$. Then

$$\Big(\int_{\mathbb{R}} |\varphi_m|^q \,\mathrm{d}\mu\Big)^{1/q} = m^{-1}, \qquad \Big(\int_{\mathbb{R}} |\varphi_m''|^p \,\mathrm{d}x\Big)^{1/p} = cm^{-2+1/p},$$

and (3.1.11) fails for $p > 1$. The case $p = 1$ was treated in Example 1.

Example 3. Now we consider the case of the derivative of order $l \geq 3$ in inequality (3.1.10) for all $f \in C_0^\infty(\Omega)$ such that $f(x) \geq 0$ on Ω. By the obvious relation

$$\inf\Big\{\int_a^b \big|f^{(l)}(x)\big|^p \,\mathrm{d}x : f \in C^\infty[a,b], f(x) \geq 0, f(a) = 0, f(b) = 1\Big\}$$
$$= c_{l,p}(b-a)^{1-lp},$$

we obtain the following necessary condition for (3.1.10)

$$\sup_{x\in\Omega, d>0} \mu\big(\sigma_d(x)\big)^{1/q} \Big(\inf_{\sigma_{d+\tau}(x)\subset\Omega} \big(\tau^{1-pl} + \nu\big(\sigma_{d+\tau}(x)\big)\big)\Big)^{-1/p} < \infty. \qquad (3.7.5)$$

We shall verify that this condition is not sufficient for (3.1.10) when $l \geq 3$ and $p \geq 1$.

Suppose first that $p > 1$. Let ν and μ be Dirac's measures concentrated at 0 and 1, respectively. Then (3.7.5) holds. Let φ_0 be a nonnegative function in $C_0^\infty(\mathbb{R})$ such that $\varphi_0(x) = x^2$ for $|x| \leq 1$. We put $\varphi_m(x) = \varphi_0(x/m)$, $m = 1, 2, \ldots$. Then

$$\int_{\mathbb{R}} |\varphi_m|^q \,\mathrm{d}\mu = m^{-2q}, \qquad \int_{\mathbb{R}} \big|\varphi^{(l)}(x)\big|^p \,\mathrm{d}x = cm^{1-pl},$$

and inequality (3.1.10) fails.

Consider the remaining case $l = 3$, $p = 1$. Let ν be Dirac's measure at O. Then (3.7.1) has the form

$$\sup_x \big(1 + x^2\big)\mu\big((x,\infty)\big)^{1/q} < \text{const}.$$

For $\mathrm{d}\mu(x) = (1 + |x|)^{-2q-1}$ the last condition holds.

We introduce the sequence $\{\Gamma_m(x)\}_{m\geq 1}$ by

$$\Gamma_m(x) = \int_0^x \eta_m(t)\,\mathrm{d}t \quad \text{for } |x| \leq 2m+2,$$

where η_m is the same as in Example 2. For $|x| \geq 2m+2$ we define Γ_m so that $\Gamma_m \geq 0$ and

$$\sup_m \int_{2m+2}^{\infty} \left|\Gamma_m^{(3)}(t)\right| \mathrm{d}t < \infty.$$

We see that

$$\int_{\mathbb{R}} \left|\Gamma_m^{(3)}\right| \mathrm{d}t = \int_{-\infty}^{2m+2} |\varphi_m''| \,\mathrm{d}t + \int_{2m+2}^{\infty} \left|\Gamma_m^{(3)}\right| \mathrm{d}t < \infty,$$

and inequality (3.1.10) with $p = 1$, $l = 3$ does not hold.

3.8 Two-Weight Inequalities Involving Fractional Sobolev Norms

Consider the inequality

$$\left(\int_{\mathbb{R}^n} |f|^q \,\mathrm{d}\mu\right)^{1/q} \le c\left(\langle f\rangle_{p,l}^p + \int_{\mathbb{R}^n} |f|^p \,\mathrm{d}\nu\right)^{1/p}, \tag{3.8.1}$$

where $f \in C_0^\infty(\mathbb{R}^n)$, $p \ge 1$, $0 < l < 1$, and

$$\langle f\rangle_{p,l}^p = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+pl}} \,\mathrm{d}x\,\mathrm{d}y.$$

As is well known (Uspenskiĭ [770]), any smooth extension of f onto $\mathbb{R}^{n+1}$ admits the estimate

$$\langle f\rangle_{p,l}^p \le c\int_{\mathbb{R}^{n+1}} |x_{n+1}|^{p(1-l)-1} |\operatorname{grad} F|^p \,\mathrm{d}x\,\mathrm{d}x_{n+1}, \tag{3.8.2}$$

and there exists a linear extension operator $f \to F \in C^\infty(\mathbb{R}^{n+1})$, where F decays to 0 at infinity and such that

$$\int_{\mathbb{R}^{n+1}} |x_{n+1}|^{p(1-l)-1} |\operatorname{grad} F|^p \,\mathrm{d}x\,\mathrm{d}x_{n+1} \le c\langle f\rangle_{p,l}^p. \tag{3.8.3}$$

The same argument as in Proposition 3.3 leads to a conductor inequality, similar to (3.1.1), for the integral

$$\int_{\mathbb{R}^{n+1}} |x_{n+1}|^{p(1-l)-1} |\operatorname{grad} F|^p \,\mathrm{d}x\,\mathrm{d}x_{n+1}, \tag{3.8.4}$$

with the left-hand side involving the conductivity generated by (3.8.4) (compare with (3.3.1)).

Minimizing (3.8.4) over all extensions of f and using (3.8.2) and (3.8.3), we arrive at the fractional conductor inequality

$$\int_0^\infty \operatorname{cap}_{p,l}(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\,\mathrm{d}\big(t^p\big) \le c(l,p,a)\langle f\rangle_{p,l}^p, \tag{3.8.5}$$

where $a > 1$ and

$$\operatorname{cap}_{p,l}(\bar{g}, G) = \inf \langle \varphi \rangle_{p,l}^{p}, \tag{3.8.6}$$

with the infimum taken over all $\varphi \in C_0^\infty(G)$ subject to $\varphi = 1$ on $\bar{g}$, $\varphi = 0$ outside G, and $1 \geq \varphi \geq 0$ on G. This infimum does not change if one requires $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \geq 1$ on $\bar{g}$, and $\varphi \leq 0$ outside G.

By (3.8.5) we obtain the following criterion for (3.8.1).

Theorem 1. *Let* $1 \leq p \leq q$. Inequality (3.8.1) *holds for all* $f \in C_0^\infty(\mathbb{R}^n)$ *if and only if there exists a constant* K *such that for all open bounded sets* g *and* G *subject to* $\bar{g} \subset G$ *there holds*

$$\mu(g)^{1/q} \leq K \big(\operatorname{cap}_{p,l}(\bar{g}, G) + \nu(G)\big)^{1/p}. \tag{3.8.7}$$

Remark. The last criterion can be simplified for $p = 1$, $q \geq 1$, as follows:

$$\mu(g)^{1/q} \leq K \bigg(\int_g \int_{\mathbb{R}^n \setminus g} \frac{\mathrm{d}x \, \mathrm{d}y}{|x-y|^{n-pl}} + \nu(g) \bigg)$$

for all open bounded sets g. In fact, the necessity results by setting the characteristic function of g into (3.8.1). The sufficiency follows from

$$\begin{aligned} \langle u \rangle_{1,l} &= 2 \int \int_{|u(x)| \leq |u(y)|} \int_{|u(x)|}^{|u(y)|} \mathrm{d}t \, \frac{\mathrm{d}x \, \mathrm{d}y}{|x-y|^{n+l}} \\ &= 2 \int_0^\infty \int_{\mathcal{L}_t} \int_{\mathbb{R}^n \setminus \mathcal{L}_t} \frac{\mathrm{d}x \, \mathrm{d}y}{|x-y|^{n+l}} \, \mathrm{d}t \end{aligned}$$

combined with (3.8.7) where $p = 1$.

We turn to the inequality

$$\bigg(\int_{\mathbb{R}^n} |f|^q \, \mathrm{d}\mu \bigg)^{1/q} \leq c \bigg(\langle \operatorname{grad} f \rangle_{p,1+l}^{p} + \int_{\mathbb{R}^n} |f|^p \, \mathrm{d}\nu \bigg)^{1/p}, \tag{3.8.8}$$

where $f \in C_0^\infty(\mathbb{R}^n)$, $f \geq 0$, and $0 < l < 1$.

Lemma 1. *Let* f *be a nonnegative function on* $\mathbb{R}$ *with absolutely continuous* f'*. Suppose the weight function* w *belongs to the Muckenhoupt class* A_p*,* $p \in (1, \infty)$*. Then*

$$\bigg\| \frac{(f')^2}{f} \bigg\|_{L_p(w \, \mathrm{d}x)} \leq c \|f'' w\|_{L_p(w \, \mathrm{d}x)}. \tag{3.8.9}$$

Proof. Since $\mathcal{M}_{\pm} g \leq 2 \mathcal{M} g$, where $\mathcal{M} g$ is the centered maximal function

$$(\mathcal{M}g)(x) = \sup_{\tau>0} \frac{1}{2\tau} \int_{x-\tau}^{x+\tau} \bigl|g(y)\bigr|\, \mathrm{d}y$$

and $\mathcal{M}_\pm$ are one-sided maximal functions introduced in (1.3.57) and (1.3.58), the result follows from (1.3.59) and the weighted norm inequality for the maximal function due to Muckenhoupt [621]. □

Corollary 1. *Let the conditions in Lemma* 1 *hold and let* $\Phi \in C^2(\mathbb{R})$ *be such that*

$$\sup_{t>0} \bigl|t^{i-1}\Phi^{(i)}\bigr| < \infty, \quad i = 1, 2.$$

Then

$$\bigl\|\bigl(\Phi(f)\bigr)''\bigr\|_{L_p(w\,\mathrm{d}x)} \le c\|f''\|_{L_p(w\,\mathrm{d}x)}.$$

Lemma 2. *Let* $F \in C^\infty(\mathbb{R}^{n+1})$ *and* $F \ge 0$*. Then there exists a positive constant* $c = c(n, p, l)$ *such that*

$$\int_{\mathbb{R}^{n+1}} |x_{n+1}|^{p(1-l)-1} \frac{|\operatorname{grad} F|^{2p}}{F^p}\, \mathrm{d}x\, \mathrm{d}x_{n+1}$$
$$\le c \int_{\mathbb{R}^{n+1}} |x_{n+1}|^{p(1-l)-1} |\operatorname{grad}_2 F|^p\, \mathrm{d}x\, \mathrm{d}x_{n+1}. \tag{3.8.10}$$

Proof. The inequality (3.8.10) with $(\partial F/\partial x_1, \dots, \partial F/\partial x_n)$ instead of $\operatorname{grad} F$ on the left-hand side follows immediately from (3.7.2). To estimate the integral involving only the derivative $\partial F/\partial x_{n+1}$ we need the next inequality for nonnegative functions of one variable:

$$\int_{\mathbb{R}} |t|^{p(1-l)-1} \frac{|f'(t)|^{2p}}{f(t)^p}\, \mathrm{d}t \le c \int_{\mathbb{R}} |t|^{p(1-l)-1} \bigl|f''(t)\bigr|^p\, \mathrm{d}t, \tag{3.8.11}$$

which is a particular case of Lemma 1 since the weight $|t|^{p(1-l)-1}$ belongs to the Muckenhoupt class A_p. □

We state a direct corollary of Lemma 2.

Corollary 2. *Let* F *be the same as in Lemma* 2 *and let* h *be a function in* $C^{1,1}(0,\infty)$ *such that* $C := \sup\{t > 0 : |h'(t)| + |t||h''(t)| < \infty\}$*. Then*

$$\bigl\||x_{n+1}|^{1-l-1/p} \operatorname{grad}_2 h(F)\bigr\|_{L_p(\mathbb{R}^{n+1})} \le cC\bigl\||x_{n+1}|^{1-l-1/p} \operatorname{grad}_2 F\bigr\|_{L_p(\mathbb{R}^{n+1})}.$$

Let $f \in C_0^\infty(\mathbb{R}^n)$, $f \ge 0$. The standard extension operator with nonnegative radial kernel gives a nonnegative extension $F \in C^\infty(\mathbb{R}^{n+1})$ of f satisfying

$$\bigl\||x_{n+1}|^{1-l-1/p} \operatorname{grad}_2 F\bigr\|_{L_p(\mathbb{R}^{n+1})} \le c\langle f\rangle_{p,1+l}.$$

Therefore, arguing as in the proof of Proposition 3.7 and using the last inequality and the trace inequality (3.8.2), we arrive at the conductor inequality

$$\int_0^\infty \operatorname{cap}^+_{p,1+l}(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\,\mathrm{d}\big(t^p\big) \le c(l,p,a)\langle f\rangle^p_{p,1+l}, \tag{3.8.12}$$

where

$$\operatorname{cap}^+_{p,1+l}(\bar g, G) = \inf\big\{\langle\varphi\rangle^p_{p,1+l} : \varphi \in C_0^\infty(G), 1 \ge \varphi \ge 0 \text{ on } G,\\ \text{and } \varphi = 1 \text{ on a neighborhood of } \bar g\big\}.$$

Using (3.8.12), we arrive at the following criterion.

Theorem 2. *Let* $1 \le p \le q$. *The inequality*

$$\Big(\int_{\mathbb{R}^n} |f|^q\,\mathrm{d}\mu\Big)^{1/q} \le c\Big(\langle f\rangle^p_{p,1+l} + \int_{\mathbb{R}^n} |f|^p\,\mathrm{d}\nu\Big)^{1/p}$$

holds for all nonnegative $f \in C_0^\infty(\mathbb{R}^n)$ *if and only if there exists a constant* K *such that*

$$\mu(g)^{1/q} \le K\big(\operatorname{cap}^+_{p,1+l}(\bar g, G) + \nu(G)\big)^{1/p}$$

for all open bounded sets g *and* G *subject to* $\bar g \subset G$.

3.9 Comments to Chap. 3

The material of this chapter is mostly borrowed from the author's paper [562]. (For $p = 2$ inequality (3.1.2) with $C(2) = 4$ was used without explicit formulation already in [531, 534], and [543].)

Capacitary inequality (3.1.2) and its various extensions are of independent interest and have numerous applications to the theory of Sobolev spaces, linear and nonlinear partial differential equations, calculus of variations, theories of Dirichlet forms, and also in Probability Theory, especially in connection with multidimensional concentration phenomena and the problems on the convergence of Markov semigroups, and so on (see Maz'ya [543, 551]; D.R. Adams [5]; Dahlberg [219]; Hansson [348]; Kolsrud [441, 442]; Netrusov [631]; Rao [670]; D.R. Adams and Pierre [18]; Kaimanovich [410]; Maz'ya and Netrusov [572]; Vondraček [783]; D.R. Adams and Hedberg [15]; Amghibech [44]; Aikawa [36]; Verbitsky [774, 775]; Hansson, Maz'ya, and Verbitsky [349]; Grigor'yan [324]; Takeda [739]; Fitzsimmons [280]; Hajłasz [341]; Fukushima and Uemura [294, 295]; Adams and Xiao [20, 21]; Ben Amor [82]; M. Chen [183, 184]; Barthe, Cattiaux, and Roberto [75]; Maz'ya and Verbitsky [594]; Xiao [800]; Cattiaux, Gentil, and Guillin [177]; Phuc and Verbitsky [658]; Bobkov and Zegarlinski [120]; et al.).

It is, perhaps, worth mentioning that the proof of (3.1.1) is so simple and generic that it works in a much more general frame of analysis on manifolds and metric spaces (see Grigor'yan [324], Hajłasz [341], et al.). We shall discuss such generalizations in the next chapter.

Inequality (3.1.10) and its applications were the subject of extensive work. See, for example, books by Maz'ya [556]; Mynbaev and Otelbaev [622]; Opic and Kufner [646]; Davies [224]; Kufner and Persson [469]; Kufner, Maligranda, and Persson [468]; papers by Muckenhoupt [620]; Maz'ya [543]; Otelbaev [653]; Oinarov [644]; Davies [226]; Nasyrova and Stepanov [626]; Chua and Wheeden [190]; Maz'ya and Verbitsky [593]; Stepanov and Ushakova [727]; Prokhorov and Stepanov [667]; and references therein.

Inequality (3.1.1) was generalized to Sobolev–Lorentz spaces by Costea and Maz'ya [213] as follows:

$$\int_0^\infty \operatorname{cap}_{p,q}(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\, \mathrm{d}\big(t^p\big) \le c(a,p,q)\|\nabla f\|^p_{L^{p,q}(\Omega,m_n,\mathbb{R}^n)} \quad \text{when } 1 \le q \le p$$

and

$$\int_0^\infty \operatorname{cap}_{p,q}(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)^{q/p}\, \mathrm{d}\big(t^q\big) \le c(a,p,q)\|\nabla f\|^q_{L^{p,q}(\Omega,m_n,\mathbb{R}^n)} \quad \text{when } p < q < \infty$$

with $\operatorname{cap}_{p,q}$ standing for the capacity generated by the Sobolev–Lorentz norm. In [213] one can find generalizations of other results in Sect. 3.1 to the context of Sobolev–Lorentz spaces.

4

Generalizations for Functions on Manifolds and Topological Spaces

4.1 Introduction

The results and arguments in Chaps. 2 and 3 based upon isoperimetric and isocapacitary inequalities for sets in the Euclidean space can be readily extended to much more general situtations; the present chapter only touches such opportunities.

Applications to estimates of the principal eigenvalue of the Dirichlet Laplacian on a Riemannian manifold are presented in Sects. 4.2 and 4.3. In Sects. 4.4–4.6 we derive some conductor inequalities for functions defined on a locally compact Hausdorff space $\mathcal{X}$. It is worth mentioning that, unlike the Sobolev inequalities, the conductor inequalities do not depend on the dimension of $\mathcal{X}$. Furthermore, with a lower estimate for the p-conductivity by a certain measure on $\mathcal{X}$, one can readily deduce the Sobolev–Lorentz type inequalities involving this measure.

In Sect. 4.4 we are interested in conductor inequalities for the Dirichlet-type integral

$$\int_{\mathcal{X}} \mathcal{F}_p[f], \tag{4.1.1}$$

where $\mathcal{F}_p$ is a measure-valued operator acting on a function f and satisfying locality and contractivity conditions. A prototype of (4.1.1) is the functional

$$\int_{\Omega} \big|\varPhi(x, \operatorname{grad} f)\big|^p \,\mathrm{d}x + \int_{\Omega} |f|^p \,\mathrm{d}\nu, \tag{4.1.2}$$

with a function $y \to \varPhi(x,y)$, positively homogeneous of order 1.

In Theorem 4.4 we obtain the conductor inequality

$$M^{-1}\bigg(\int_0^\infty M\big(t^p \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\big)\frac{\mathrm{d}t}{t}\bigg) \le c(a,p)\int_{\mathcal{X}} \mathcal{F}_p[f]; \tag{4.1.3}$$

here and elsewhere $\mathcal{L}_t = \{x \in \mathcal{X} : |f(x)| > t\}$, M is a positive convex function on $(0,\infty)$, $M(+0) = 0$, and M^{-1} stands for the inverse of M. By cap_p we mean the p-capacity generated by the operator $\mathcal{F}_p$.

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In Sect. 4.5 we derive the conductor inequality

$$\left(\int_0^\infty \left(\operatorname{cap}_{p,\Gamma}(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\right)^{q/p} \mathrm{d}\left(t^q\right)\right)^{p/q} \le c(a,p,q)\langle f\rangle^p_{p,\Gamma}, \tag{4.1.4}$$

where $q \ge p \ge 1$,

$$\langle f\rangle^p_{p,\Gamma} := \left(\int_{\mathcal{X}}\int_{\mathcal{X}} \left|f(x)-f(y)\right|^p \Gamma(\mathrm{d}x \times \mathrm{d}y)\right)^{1/p}, \tag{4.1.5}$$

and $\operatorname{cap}_{p,\Gamma}$ is the relative p-capacity corresponding to the seminorm (4.1.5). We apply (4.1.4) to obtain a necessary and sufficient condition for a two-measure Sobolev inequality involving $\langle f\rangle^p_{p,\Gamma}$.

In Sect. 4.6 we handle variants of the sharp capacitary inequality (2.3.6). We show in Theorem 4.6/2 that a fairly general capacitary inequality is a direct consequence of a one-dimensional inequality for functions with the first derivative in $L_p(0,\infty)$. A corollary of this result is the following inequality with the best constant, complementing (2.3.6):

$$\left(\int_\Omega \operatorname{cap}_p(\overline{\mathcal{L}_t}, \Omega)^{q/p}\, \mathrm{d}\left(t^q\right)\right)^{1/q} \le \left(\frac{\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{q}{q-p}\right)\Gamma\left(p\frac{q-1}{q-p}\right)}\right)^{1/p-1/q} \left(\int_\Omega |\operatorname{grad} f|^p\, \mathrm{d}x\right)^{1/p}, \tag{4.1.6}$$

where $q > p \ge 1$. Combined with the isocapacitary inequality, (4.1.6) with $q = pn/(n-p)$, $n > p$, immediately gives the classical Sobolev estimate

$$\left(\int_\Omega |f|^{\frac{pn}{n-p}}\, \mathrm{d}x\right)^{1-p/n} \le c\int_\Omega |\operatorname{grad} f|^p\, \mathrm{d}x \tag{4.1.7}$$

with the best constant (see Sect. 2.3.5).

Another example of the application of Theorem 4.6/2 is the inequality

$$\sup \int_0^\infty \exp\left(-c\operatorname{cap}_p(\overline{\mathcal{L}_t}, \Omega)^{1/(1-p)}\right) \mathrm{d}\exp\left(ct^{p/(p-1)}\right) < \infty, \tag{4.1.8}$$

where $c = \text{const}$, the supremum is taken over all $f \in C_0^\infty(\Omega)$ subject to $\|\operatorname{grad} f\|_{L_p(\Omega)} \le 1$.

Inequality (4.1.8) with $p = n$ is stronger than the sharp form of the Yudovich inequality [809] due to Moser [618], which immediately follows from (4.1.8) and an isocapacitary inequality. Yudovich's inequality was rediscovered by Pohozhaev [662] and Trudinger [764].

A capacitary improvement of the Faber–Krahn isoperimetric property of the first eigenvalue of the Dirichlet problem for the Laplacian on a Riemannian manifold is given in Sect. 4.7. Finally, in Sect. 4.8 the capacitary inequality

(4.1.6) is used to obtain a sharp constant in the Sobolev-type two-weighted Il'in's inequality

$$\Big(\int_{\mathbb{R}^n} \big|u(x)\big|^{\frac{(n-b)p}{(n-p-a)}} \frac{\mathrm{d}x}{|x|^b}\Big)^{\frac{n-p-a}{n-b}} \le \mathcal{C}_{p,a,b}\Big(\int_{\mathbb{R}^n} \big|\nabla u(x)\big|^p \frac{\mathrm{d}x}{|x|^a}\Big)^{\frac{1}{p}}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

4.2 Integral Inequalities for Functions on Riemannian Manifolds

Until now, and mostly in the sequel, we restrict our attention to Sobolev spaces in subdomains of $\mathbb{R}^n$. However, the methods of proof of many of the previous results do not use in a specific way the properties of the Euclidean space. Let us turn, for instance, to Corollary 2.3.4.

The assumption $\Omega \subset \mathbb{R}^n$ is not essential for its proof: The capacitary inequality (2.3.6) is also valid for functions on a Riemannian manifold, while the rest of the argument has a general character. Let Ω be an open subset of a Riemannian manifold $\mathfrak{R}_n$ and let $\mu = m_n$ be the n-dimensional measure on subsets of $\mathfrak{R}_n$. Then by virtue of Theorem 2.3.3 we have the following two-sided estimate:

$$\begin{aligned}\inf_{\{F\}} \frac{[\mathrm{cap}_p(F,\Omega)]^{1/p}}{[m_n(F)]^{1/q}} &\ge \inf_{u \in C_0^\infty(\Omega)} \frac{\|\operatorname{grad} u\|_{L_p(\Omega)}}{\|u\|_{L_p(\Omega)}} \\ &\ge \frac{(p-1)^{(p-1)/p}}{p} \inf_{\{F\}} \frac{[\mathrm{cap}_p(F,\Omega)]^{1/p}}{[m_n(F)]^{1/q}}, \qquad (4.2.1)\end{aligned}$$

where $\{F\}$ is the family of all compact subsets of Ω. Generally speaking, we preserve the notation introduced in this chapter for the Euclidean case.

Let $\mathscr{C}$ be a function on $(0, m_n(\Omega))$ such that for any open set g with compact closure $\bar{g} \subset \Omega$ and the smooth boundary one has the *isoperimetric inequality*

$$s(\partial g) \ge \mathscr{C}\big(m_n(g)\big). \qquad (4.2.2)$$

Then, in view of (2.2.8), the following isocapacitary inequality holds:

$$\mathrm{cap}_p(F,\Omega) \ge \Big(\int_{m_n(F)}^{m_n(\Omega)} \frac{\mathrm{d}v}{[\mathscr{C}(v)]^{p/(p-1)}}\Big)^{1-p},$$

and consequently,

$$\begin{aligned}&\inf_{u \in C_0^\infty(\Omega)} \frac{\|\operatorname{grad} u\|^p_{L_p(\Omega)}}{\|u\|^p_{L_p(\Omega)}} \\ &\quad\ge \frac{(p-1)^{p-1}p^{-p}}{\sup_{0<t<m_n(\Omega)}\big(t\big(\int_t^{m_n(\Omega)} \frac{\mathrm{d}v}{[\mathscr{C}(v)]^{p/(p-1)}}\big)^{p-1}\big)}. \qquad (4.2.3)\end{aligned}$$

Choosing for $\mathscr{C}$ a power function we derive the estimate

$$\inf_{u\in C_0^\infty(\Omega)} \frac{\|\operatorname{grad} u\|^p_{L_p(\Omega)}}{\|u\|^p_{L_p(\Omega)}} \geq p^{-p}\left(\inf_{\{g\}} \frac{s(\partial g)}{m_n(g)}\right)^p. \tag{4.2.4}$$

In the case $p = 2$ the left-hand side of the last inequality coincides with the first eigenvalue $\Lambda(\Omega)$ of Dirichlet's problem for the Laplace operator on $\mathfrak{R}_n$. Estimates of eigenvalues for the Dirichlet and Neumann problem by geometric characteristics of the manifold have been the subject of many investigations (see Berger, Gauduchon, and Mazet [85]; Yau [805]; Chavel [179]; Donnelly [238, 239]; Escobar [258]; Chung, Grigor'yan, and Yau [191]; Nadirashvili [623]; et al.). A significant role here was played by *Cheeger's inequality* from 1970 [181]

$$\Lambda(\Omega) \geq \frac{1}{4}\left(\inf_{\{g\}} \frac{s(\partial g)}{m_n(g)}\right)^2, \tag{4.2.5}$$

which coincides with (4.2.4) if $p = 2$. The constant $1/4$ in (4.2.5) cannot be improved (Buser [159]).[1] The earlier proof of (4.2.4) shows that Cheeger's inequality is a consequence of the estimate

$$\Lambda(\Omega) \geq \frac{1}{4} \inf_{F\subset\Omega} \frac{\operatorname{cap}(F,\Omega)}{m_n(F)}, \tag{4.2.6}$$

which was established by the author in 1962. Inequality (4.2.3) for $p = 2$ can be written as

$$\frac{1}{\Lambda(\Omega)} \leq 4 \sup_{0<t<m_n(\Omega)} \left(t\int_t^{m_n(\Omega)} \frac{\mathrm{d}v}{[\mathscr{C}(v)]^2}\right). \tag{4.2.7}$$

Specializing the function $\mathscr{C}$ in (4.2.2), one can derive from it strengthenings of known lower bounds for the eigenvalue $\Lambda(\Omega)$.

Let us now turn to an example. Let Ω be a subdomain of a two-dimensional Riemannian manifold $\mathfrak{R}_2$, whose Gauss curvature K does not exceed $-\alpha^2$ on Ω, $\alpha=$ const > 0. Let us show that

$$\frac{1}{\Lambda(\Omega)} \leq \frac{4}{\alpha^2}\big(1 - \big(1 + \alpha^2 m_2(\Omega)/4\big)^{-1/2}\big). \tag{4.2.8}$$

This is a sharpening of the *inequality of McKean* [595], $\mathscr{C}(\Omega) \geq \alpha^2/4$, which follows from (4.2.5). Indeed, as is known, in the hypothesis $K \leq -\alpha^2$ inequality (4.2.2) is fulfilled with $\mathscr{C}(v) = (4\pi v + \alpha^2 v^2)^{1/2}$ (Burago–Zalgaller [151]). Plugging this function into (4.2.7), we find

[1] Cheeger's paper from 1970 is actually about the first eigenvalue of the *Neumann* (and not the Dirichlet) Laplacian, but the Dirichlet case is only simpler, Buser's example shows that the constant $1/4$ in Cheeger's inequality cannot be improved in both the Dirichlet and no-boundary cases.

$$\frac{1}{\Lambda(\Omega)} \le \frac{4}{\alpha^2} \max_{0<t<A}\left(t \log \frac{A(t+1)}{(A+1)t}\right) = \frac{4}{\alpha^2}\Psi^{-1}\left(\frac{A}{A+1}\right),$$

where $A = \alpha^2 m_2(\Omega)/4\pi$, $\Psi(y) = ye^{1-y}$, $0 \le y \le 1$, Ψ^{-1} being the inverse of Ψ. Now (4.2.8) follows from the obvious inequality $\Psi^{-1}(s) \le 1-(1-s)^{1/2}$.

Analogous considerations can be done also in connection with other integral inequalities for functions on a domain on a Riemannian manifold. Then the powers of integrability do not depend only on the amount of irregularity of the boundary, but also on the geometry of the manifold. As before, the properties of the embedding operator are fixed by the constants in the corresponding inequalities between measures and capacities, so the main difficulty consists of the verification of these inequalities.

Recently this theme has been given considerable attention thanks to the applications to physics and geometry. Let us turn to some such results.

As the isoperimetric inequality (1.4.13) is also true for subsets of the Lobachevskiĭ space $\mathscr{H}^n$ of constant negative curvature, the estimate (1.4.14) with an exact constant remains in force for functions on $\mathscr{H}^n$.

For subsets g of a smooth k-dimensional submanifold of $\mathbb{R}^n$ without boundary the following inequality holds:

$$v(g) \le c_k\big(s(\partial g) + Q(g)\big)^{k/(k-1)}, \tag{4.2.9}$$

where v and s are the k-dimensional volume and the $(k-1)$-dimensional area and Q the absolute integral mean curvature (Michael and Simon [600]). As a particular case of Theorem 2.1.3 we find that the isoperimetric inequality (4.2.9) is equivalent to the integral inequality

$$\left(\int_{\mathfrak{R}_n} |u|^{k/(k-1)}\,\mathrm{d}v\right)^{(k-1)/k} \le c_k\left(\int_{\mathfrak{R}_n} |\operatorname{grad} u|\,\mathrm{d}v + \int_{\mathfrak{R}_n} |uH|\,\mathrm{d}v\right), \tag{4.2.10}$$

where H is the mean curvature. If $\mathfrak{R}_n$ is a manifold with boundary $\partial\mathfrak{R}_n$, then one must add to the right-hand side of (4.2.10) the norm of the trace of u in $L_1(\partial\mathfrak{R}_n)$. The constant c_k depends only on k.

Let us remark that an isoperimetric inequality of the type (4.2.9) and the corresponding embedding theorem have likewise been proved for objects generalizing k-dimensional surfaces in $\mathbb{R}^n$ such as currents and varifolds (cf. Burago and Zalgaller [151], Sect. 7.4).

Inequality (4.2.9) (and therefore also its consequence (4.2.10)) can be extended to manifolds $\mathfrak{M}$ smoothly embedded in an n-dimensional Riemannian manifold $\mathfrak{R}_n$ (Hoffmann and Spruck [379]). Then some natural geometrical restrictions arise, which are connected with the volume of $\mathfrak{M}$, the injectivity radius of $\mathfrak{R}_n$, and its sectional curvature.

The question of exact constants in inequalities of Sobolev type for functions on a Riemannian manifold have been studied by Aubin. He proved, in particular, that for any compact two-dimensional Riemannian manifold $\mathfrak{R}_2$ there exists a constant $A(p)$ such that for all $u \in W_p^1(\mathfrak{R}_2)$, $1 \le p \le 2$, one has

$$\|u\|_{L_{2p/(2-p)}(\mathfrak{R}_2)} \le K(2,p)\|\operatorname{grad} u\|_{L_p(\mathfrak{R}_2)} + A(p)\|u\|_{L_p(\mathfrak{R}_2)}, \tag{4.2.11}$$

where $K(n,p)$ is the best constant in (2.3.21), as given in (2.3.23). For manifolds of arbitrary dimension one does not have very complete results. There the necessity follows naturally from attempts to solve special nonlinear elliptic equations on Riemannian manifolds, for example, in connection with the well-known *Yamabe problem* concerning the conformal equivalence of an arbitrary Riemannian metric with a metric of constant scalar curvature. The issue is to find a number λ such that the equation

$$4\frac{n-1}{n-2}\Delta u + Ru = \lambda u^{(n+2)/(n-2)} \tag{4.2.12}$$

has positive solution on a Riemannian manifold (M,g) of dimension $n \ge 3$. Here $R = R(x)$ is the scalar curvature defined by the metric g and λ the constant scalar curvature of the conformal metric. Yamabe's paper [802] devoted to this problem contains an error. It was found by Trudinger [763] who corrected it in the case of nonnegative scalar curvature. The next step was done by Aubin [56]. He stated his results in terms of the best constant of a certain inequality of Sobolev type. The nonlinear spectral problem (4.2.12) can be associated with a variational problem where one looks for the infimum J of the functional

$$\int_M \Big(|\operatorname{grad} u|^2 + \frac{n-2}{4(n-1)}Ru^2\Big)\,\mathrm{d}s/\|u\|^2_{L_{2n/(n-2)}(M)},$$

defined in the space $W_2^1(M)$. Aubin proved that $J \le [K(n,2)]^{-2}$ and that Yamabe's problem is positively solved if this inequality is sharp.

Let us further remark that for the n-dimensional ball S of unit measure Aubin proved the inequality

$$\|u\|^2_{L_{2n/(n-2)}(S)} \le \big[K(n,2)\big]^2\|\operatorname{grad} u\|^2_{L_2(S)} + \|u\|^2_{L_2(S)},$$

with an exact constant for both terms in the right-hand side. In 1986 Gil-Medrano [308] gave a partial solution to Yamabe's problem for compact n-dimensional locally conformally flat manifolds with positive scalar curvature and infinite fundamental group, proving the inequality $J < [K(n,2)]^{-2}$. The complete solution of Yamabe's problem was achieved by Schoen [694]. Escobar [260] contributed by solving this problem for manifolds with a boundary. The work on Yamabe's problem was an important step in the study of nonlinear partial differential equations on Riemannian manifolds.

We mentioned only a few early results to give the flavor of a very large area developed during the last 40 years in geometric analysis. Numerous studies showed the usefulness of isoperimetric inequalities, and to a smaller degree of isocapacitary inequalities, in the theory of Sobolev spaces and differential equations on manifolds, graphs, and Markov chains. Without aiming at completeness, we mention the monographs by Aubin [57]; Varopoulos, Saloff-Coste, and Coulhon [772]; Hebey [360]; Chavel [180]; and Saloff-Coste [687]. Considerable attention was paid to extensions for the so-called sub-Riemannian differential geometry, in particular to the analysis on Carnot and Heisenberg groups (see Capogna, Danielli, Pauls, and Tyson [166]).

4.3 The First Dirichlet–Laplace Eigenvalue and Isoperimetric Constant

In this section we show by a counterexample that the fundamental eigenvalue of the Dirichlet Laplacian is not equivalent to an isoperimetric constant, called, as a rule, Cheeger's constant, in contrast with an isocapacitary constant[2] (see (4.2.1)). This equivalence, even uniform with respect to the dimension, holds for domains whose boundaries have nonnegative mean curvature, as proved recently by E. Milman (oral communication), but as we shall see, it fails even in the class of domains starshaped with respect to a ball.

Let Ω be a subdomain of an n-dimensional Riemannian manifold $\mathfrak{R}_n$ and let $\Lambda(\Omega)$ be the first eigenvalue of the Dirichlet problem for the Laplace operator $-\Delta$ in Ω, or more generally, the upper lower bound of the spectrum of this operator

$$\Lambda(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \frac{\|\nabla u\|^2_{L_2(\Omega)}}{\|u\|^2_{L_2(\Omega)}}. \tag{4.3.1}$$

By Corollary 2.3.4, $\Lambda(\Omega)$ admits the two-sided estimate

$$\frac{1}{4}\Gamma(\Omega) \le \Lambda(\Omega) \le \Gamma(\Omega) \tag{4.3.2}$$

with

$$\Gamma(\Omega) := \inf_{\{F\}} \frac{\operatorname{cap}(F;\Omega)}{m_n(F)}.$$

By Theorem 2.1.3, the set function

$$\gamma(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \frac{\|\nabla u\|_{L_1(\Omega)}}{\|u\|_{L_1(\Omega)}} \tag{4.3.3}$$

admits the geometric representation

$$\gamma(\Omega) = \inf_{\{g\}} \frac{s(\partial g)}{m_n(g)}, \tag{4.3.4}$$

where g is an arbitrary open subset of $\mathfrak{R}_n$ with compact closure $\overline{g}$ in Ω and smooth boundary ∂g, and s is the $(n-1)$-dimensional Hausdorff measure. Obviously, for all $u \in C_0^\infty(\Omega)$,

---

[2] By the equivalence of the set functions a and b, defined on subsets of $\mathbb{R}^n$, we mean here the existence of positive constants c_1 and c_2 depending only on n and such that $c_1 a \le b \le c_2 a$.

$$\gamma(\Omega) \le \frac{\int_\Omega |\nabla(u^2)|\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x} \le 2\frac{\|\nabla u\|_{L_2(\Omega)}}{\|u\|_{L_2(\Omega)}}.$$

Hence

$$\gamma(\Omega)^2 \le 4\Lambda(\Omega), \tag{4.3.5}$$

which shows, together with (4.3.2) and (4.3.3), that

$$\gamma(\Omega)^2 \le 4\Gamma(\Omega). \tag{4.3.6}$$

Note that (4.3.5) is nothing but (4.2.6), and we arrived at it in a much simpler way than in Sect. 4.2. One can ask whether an upper bound for $\Gamma(\Omega)$ formulated in terms of $\gamma(\Omega)$ exists.

By the following counterexample we show that the answer is negative for domains in $\mathbb{R}^n$ starshaped with respect to balls.

Example. Let Ω be a subdomain of the n-dimensional unit ball B, starshaped with respect to a concentric ball $B_\rho = \{x : |x| < \rho\}$. Here we check that the inequality opposite to (4.3.6),

$$\gamma(\Omega)^2 \ge C\Gamma(\Omega), \tag{4.3.7}$$

is impossible with C independent of ρ. Moreover, we shall construct a sequence of domains $\{\Omega_N\}_{n\ge 1}$ situated in B and such that:

(i) Ω_N is starshaped with respect to a ball $B(0,\rho_N)$, where $\rho_N \to 0$,
(ii) $\Gamma(\Omega_N) \to \infty$,
(iii) $\gamma(\Omega_N) \le c$, where c depends only on n.

Let N stand for a sufficiently large integer number. By $\{\omega_j\}_{j=1}^{N^{n-1}}$ we denote a collection of points on the unit sphere S^{n-1} placed uniformly in the sense that the distance from every point ω_j to the set of other points of the collection lies between $c_1 N^{-1}$ and $c_2 N^{-1}$, where c_1 and c_2 are positive constants, depending only on n. Consider a closed rotational cone C_j with the axis $O\omega_j$ and the vertex at the distance $c_0 N^{-1}$ from O, where c_0 is an absolute constant large enough. The opening of C_j will be independent of j and denoted by ε_N. Let $\varepsilon_N = o(N^{\frac{1-n}{n-2}})$. Clearly, the complement of C_j is visible from a sufficiently small ball $B(0,\rho_N)$. Therefore, the domain

$$\Omega_N := B\backslash \bigcup\nolimits_j C_j$$

is starshaped with respect to $B(0,\rho_N)$ (see Fig. 17).

We shall find the limit of $\gamma(\Omega_N)$ as $N\to\infty$ as well as a lower estimate for $\Gamma(\Omega_N)$. Clearly, $\gamma(\Omega_N) \ge \gamma(B) = n$. Furthermore, by (4.3.4),

$$\gamma(\Omega_N) \le \frac{s(\partial\Omega_N)}{m_n(\Omega_N)} = \frac{s(\partial B) + s(\bigcup_j (B\cap\partial C_j))}{m_n(B) - m_n(\bigcup_j (B\cap C_j))}$$
$$\le \frac{\omega_n + c_1\varepsilon_N^{n-1}N^{n-1}}{\omega_n/n - c_2\varepsilon_N^{n-2}N^{n-1}},$$

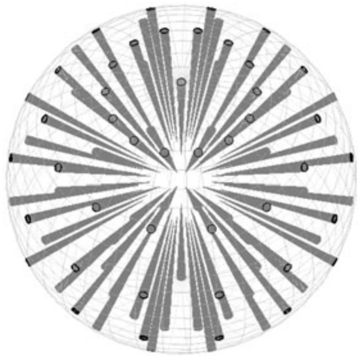

Fig. 17. The boundary of the subdomain Ω_N of the unit ball is the union of many thin conic surfaces and a spherical part.

and therefore,

$$\lim_{N\to\infty} \gamma(\Omega_N) = n.$$

To estimate $\Gamma(\Omega_N)$ from below, we construct a covering of B by the balls $\mathcal{B}_k := B(x_k, 4c_0N^{-1})$, whose multiplicity does not exceed a constant depending only on n. Let $|x_k| \ge c_0N^{-1}$. Theorem 14.1.2, to be proved later, implies

$$cN^n \operatorname{cap}(\mathcal{B}_k\backslash\Omega_N) \int_{\mathcal{B}_k} u^2\,\mathrm{d}x \le \int_{\mathcal{B}_k} |\nabla u|^2\,\mathrm{d}x \tag{4.3.8}$$

for all $u \in C_0^\infty(\Omega_N)$, and the result will stem from a proper lower bound for $\operatorname{cap}(\mathcal{B}_k\backslash\Omega_N)$.

First, let us consider $n = 3$. Clearly, $\mathcal{B}_k\backslash\Omega_N$ contains a right rotational cylinder T_k with height c_0N^{-1} and diameter of the base $\varepsilon_N N^{-1}$. Now, by Proposition 13.1.3/1 to appear in Chap. 13,

$$\operatorname{cap}(T_k) \ge cN^{-1}|\log\varepsilon_N|^{-1}.$$

This estimate in combination with (4.3.8) gives

$$cN^2|\log\varepsilon_N|^{-1} \int_{\mathcal{B}_k} u^2\,\mathrm{d}x \le \int_{\mathcal{B}_k} |\nabla u|^2\,\mathrm{d}x. \tag{4.3.9}$$

Choosing $\varepsilon_N = \exp(-N)$ and summing (4.3.9) over all balls $\mathcal{B}_k$, we obtain $\lambda(\Omega_N) \ge cN$. Hence $\lambda(\Omega_N) \to \infty$ where as $\gamma(\Omega_N) \le c$. Thus, in particular, there is no inequality

$$\left(\inf_{\{g\}} \frac{s(\partial g)}{m_3(g)}\right)^2 \ge C \inf_{\{F\}} \frac{\operatorname{cap}(F;\Omega)}{m_3(F)},$$

and equivalently,

$$\left(\inf_{\{g\}} \frac{s(\partial g)}{m_3(g)}\right)^2 \geq C\Lambda(\Omega)$$

with constant factors C independent of the radius ρ.

For dimensions greater than three, the very end of the argument remains intact, but the estimation of $\mathrm{cap}(\mathcal{B}_k \backslash \Omega_N)$ becomes a bit more complicated and the choice of ε_N will be different.

Let $\alpha\mathcal{B}_k$ stand for the ball concentric with $\mathcal{B}_k$ and dilated with coefficient α. We introduce the set $s_k = \{j : C_j \cap \frac{1}{2}\mathcal{B}_k \neq \varnothing\}$. With every j in s_k we associate a right rotational cylinder T_j coaxial with the cone C_j and situated in $C_j \cap \frac{1}{2}\mathcal{B}_k$. The height of T_j will be equal to $c_0 N^{-1}$ and the diameter of the base equal to $\varepsilon_N |x_k|$. We define a cutoff function η_j, equal to 1 on the $\varepsilon_N|x_k|$-neighborhood of T_j, zero outside the $2\varepsilon_N|x_k|$-neighborhood of T_j, and satisfying the estimate

$$\left|\nabla\eta_j(x)\right| \leq c\delta(x)^{-1},$$

where $\delta(x)$ is the distance from x to the intersection of T_j with the axis of C_j.

By $\mathcal{P}_k$ we denote the equilibrium potential of $\mathcal{B}_k \backslash \Omega_N$. We have

$$\begin{aligned}\sum_{j\in s_k} \mathrm{cap}(C_j \cap \mathcal{B}_k) &\leq \sum_{j\in s_k} \int_{\mathbb{R}^n} \left|\nabla(\mathcal{P}_k \eta_j)\right|^2 \mathrm{d}x \\ &\leq c\biggl(\int_{\mathbb{R}^n} |\nabla\mathcal{P}_k|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} \mathcal{P}_k^2 \delta^{-2}\,\mathrm{d}x\biggr).\end{aligned}$$

Changing the constant c, one can majorize the last integral by the previous one due to Hardy's inequality. Hence,

$$\mathrm{cap}(\mathcal{B}_k \backslash \Omega_N) \geq c\sum_{j\in s_k} \mathrm{cap}(T_j). \tag{4.3.10}$$

By Proposition 13.1.3/1 to be proved in the sequel,

$$\mathrm{cap}(T_j) \geq c\bigl(\varepsilon_N |x_k|\bigr)^{n-3} N^{-1}.$$

Furthermore, it is visible that the number of integers in s_k is between two multiples of $|x_k|^{1-n}$. Now, by (4.3.10)

$$\mathrm{cap}(\mathcal{B}_k \backslash \Omega_N) \geq c|x_k|^{1-n}\bigl(\varepsilon_N |x_k|\bigr)^{n-3} N^{-1}$$

and by (4.3.8)

$$cN^{n-1}|x_k|^{-2}\varepsilon_N^{n-3} \int_{\mathcal{B}_k} u^2 \,\mathrm{d}x \leq \int_{\mathcal{B}_k} |\nabla u|^2 \,\mathrm{d}x. \tag{4.3.11}$$

Since $|x_k| \leq 1$, it follows by summation of (4.3.11) over k that

$$\lambda(\Omega_N) \geq c\varepsilon_N^{n-3} N^{n-1}.$$

Putting, for instance,

$$\varepsilon_N = N^{(1-n)/(n-5/2)},$$

we see that $\Gamma(\Omega_N) \to \infty$, and the desired counterexample is constructed for $n > 3$.

4.4 Conductor Inequalities for a Dirichlet-Type Integral with a Locality Property

Let $\mathcal{X}$ denote a locally compact Hausdorff space and let $C(\mathcal{X})$ stand for the space of continuous real-valued functions given on $\mathcal{X}$. By $C_0(\mathcal{X})$ we denote the set of functions $f \in C(\mathcal{X})$ with compact supports in $\mathcal{X}$.

We introduce an operator $\mathcal{F}_p$ defined on a subset $\operatorname{dom}(\mathcal{F}_p)$ of $C(\mathcal{X})$ and taking values in the cone of nonnegative locally finite Borel measures on $\mathcal{X}$. We suppose that $1 \in \operatorname{dom}(\mathcal{F}_p)$ and $\mathcal{F}_p$ is positively homogeneous of order $p \geq 1$, i.e., for every real α, $f \in \operatorname{dom}(\mathcal{F}_p)$ implies $\alpha f \in \operatorname{dom}(\mathcal{F}_p)$ and

$$\mathcal{F}_p[\alpha f] = |\alpha|^p \mathcal{F}_p[f]. \tag{4.4.1}$$

It is also assumed that $\mathcal{F}_p$ is contractive, that is, $\lambda(f) \in \operatorname{dom}(\mathcal{F}_p)$ and

$$\mathcal{F}_p\big[\lambda(f)\big] \leq \mathcal{F}_p[f], \tag{4.4.2}$$

for all $f \in \operatorname{dom}(\mathcal{F}_p)$, where λ is an arbitrary real-valued Lipschitz function on the line $\mathbb{R}$ such that $|\lambda'| \leq 1$ and $\lambda(0) = 0$. We suppose that the following locality condition holds:

$$f(x) = c \in \mathbb{R} \text{ on a compact set } \mathcal{C} \quad \Longrightarrow \quad \int_{\mathcal{C}} \mathcal{F}_p[f] = \int_{\mathcal{C}} \mathcal{F}_p[c]. \tag{4.4.3}$$

An example of the measure satisfying conditions (4.4.1)–(4.4.3) is given by (4.1.2), where $\nu = 0$ and

$$\Omega \times \mathbb{R}^n \ni (x, z) \to \Phi(x, z) \in \mathbb{R} \tag{4.4.4}$$

is a continuous function, positively homogeneous of degree 1 with respect to z. One can take the space of locally Lipschitz functions on Ω as $\operatorname{dom}(\mathcal{F}_p)$.

Let g and G denote open sets in $\mathcal{X}$ such that the closure $\bar{g}$ is a compact subset of G. We introduce the p-conductivity of the conductor $G\backslash\bar{g}$ (in other terms, the relative p-capacity of the set $\bar{g}$ with respect to G) as

$$\begin{aligned}\operatorname{cap}_p(\bar{g}, G) = \inf\bigg\{ \int_{\mathcal{X}} \mathcal{F}_p[\varphi] : \varphi \in \operatorname{dom}(\mathcal{F}_p), 0 \leq \varphi \leq 1 \text{ on } G \\ \text{and } \varphi = 1 \text{ on a neighborhood of } \bar{g} \bigg\}.\end{aligned} \tag{4.4.5}$$

Using the truncation

$$\lambda(\xi) = \min\bigg\{ \frac{(\xi - \varepsilon)_+}{1 - \varepsilon}, 1 \bigg\},$$

with $\varepsilon \in (0, 1)$ and $\xi \in \mathbb{R}$, we see that the infimum in (4.4.5) does not change if the class of admissible functions φ is enlarged to

$$\{\varphi \in \operatorname{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X}) : \varphi \geq 1 \text{ on } g, \varphi \leq 0 \text{ on } \mathcal{X} \backslash G\} \tag{4.4.6}$$

(compare with Sect. 2.2.1).

Lemma. *Let* $f \in \operatorname{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$, $a = \text{const} > 1$, $\mathcal{L}_t = \{x \in \mathcal{X} : |f(x)| > t\}$. *Then the function* $t \to \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)$ *is upper semicontinuous.*

Proof. It follows from (4.4.3) that

$$\int_{\mathcal{X}} \mathcal{F}_p[f] = \int_{\operatorname{supp} f} \mathcal{F}_p[f] < \infty. \tag{4.4.7}$$

Let $t_0 > 0$ and $\varepsilon > 0$. There exist open sets g and G such that

$$\overline{\mathcal{L}_{at}} \subset g, \qquad \overline{g} \subset G, \qquad \overline{G} \subset \mathcal{L}_t. \tag{4.4.8}$$

It follows from the definition of cap_p that for all compact sets $C \subset g$

$$\operatorname{cap}_p(C, G) \leq \operatorname{cap}_p(\overline{\mathcal{L}_{at_0}}, \mathcal{L}_{t_0}) + \varepsilon \tag{4.4.9}$$

(compare with Sect. 2.2.1). By (4.4.8),

$$\max\{f(x) : x \in \overline{g}\} < at_0 \quad \text{and} \quad \min\{f(x) : x \in \overline{G}\} > t_0.$$

We denote

$$\delta_1 = t_0 - a^{-1} \max\{f(x) : x \in \overline{g}\}$$

and

$$\delta_2 = \min\{f(x) : x \in \overline{G}\} - t_0.$$

Then

$$\overline{\mathcal{L}_{a(t_0-\delta)}} \subset g \quad \text{and} \quad \overline{G} \subset \mathcal{L}_{t_0+\delta}$$

for every $\delta \in (0, \min\{\delta_1, \delta_2\})$. Putting $C = \overline{\mathcal{L}_{a(t_0-\delta)}}$ in (4.4.9) and recalling that cap_p decreases with enlargement of the conductor, we obtain

$$\operatorname{cap}_p(\overline{\mathcal{L}_{a(t_0-\delta)}}, \mathcal{L}_{t_0+\delta}) \leq \operatorname{cap}_p(\overline{\mathcal{L}_{at_0}}, \mathcal{L}_{t_0}) + \varepsilon. \tag{4.4.10}$$

Using the monotonicity of cap_p again, we deduce from (4.4.10) that

$$\operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t) \leq \operatorname{cap}_p(\overline{\mathcal{L}_{at_0}}, \mathcal{L}_{t_0}) + \varepsilon$$

for every t sufficiently close to t_0. In other words, the function $t \to \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)$ is upper semicontinuous. The result follows. □

We prove a general conductor inequality in the integral form for the functional (4.1.1).

Theorem. *Let* M *denote an increasing convex* (*not necessarily strictly convex*) *function given on* $[0, \infty)$, $M(0) = 0$. *Then the inequality* (4.1.3) *holds for all* $f \in \operatorname{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$ *and for an arbitrary* $a > 1$.

Proof. We have

$$\operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t) \le \int_{\mathcal{X}} \mathcal{F}_p[\varphi]$$

for every $\varphi \in \operatorname{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$ satisfying

$$\varphi = 1 \quad \text{on } \overline{\mathcal{L}_{at}}, \quad \varphi = 0 \quad \text{on } \mathcal{X}\backslash\mathcal{L}_t, \quad \text{and} \quad 0 \le \varphi \le 1 \quad \text{on } \mathcal{X}.$$

By the homogeneity of $\mathcal{F}_p$ and by (4.4.3),

$$t^p \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t) \le \int_{\mathcal{L}_t} \mathcal{F}_p[t\varphi].$$

We set here

$$\varphi_(x) = \frac{\Lambda_t(f(x))}{(a-1)t},$$

where

$$\Lambda_t(\xi) = \min\big\{(|\xi| - t)_+, (a-1)t\big\}, \quad \xi \in \mathbb{R}, \tag{4.4.11}$$

with $\xi_+ = (|\xi| + \xi)/2$. By $\Lambda_t = \text{const}$ on $\overline{\mathcal{L}_{at}}$ and by (4.4.3) we have

$$t^p \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t) \le \frac{1}{(a-1)^p} \int_{\mathcal{L}_t\backslash\overline{\mathcal{L}_{at}}} \mathcal{F}_p\big[\Lambda_t(f)\big].$$

Since the mapping $\xi \to \Lambda_t(\xi)$ is contractive and since the function $t \to \int_{\mathcal{L}_t} \mathcal{F}_p[f]$ has, at most, a countable set of discontinuities, it follows that

$$t^p \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t) \le \frac{1}{(a-1)^p} \int_{\mathcal{L}_t\backslash\mathcal{L}_{at}} \mathcal{F}_p[f] + t^p \int_{\mathcal{L}_{at}} \mathcal{F}_p[1], \tag{4.4.12}$$

for almost every $t > 0$. Hence,

$$\begin{aligned}
&\int_0^\infty M\big(t^p \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\big)\frac{\mathrm{d}t}{t} \\
&\quad \le \int_0^\infty M\bigg(\frac{1}{(a-1)^p} \int_{\mathcal{L}_t\backslash\mathcal{L}_{at}} \mathcal{F}_p[f] + t^p \int_{\mathcal{L}_{at}} \mathcal{F}_p[1]\bigg)\frac{\mathrm{d}t}{t} \\
&\quad \le \frac{1}{2}\int_0^\infty M\bigg(\frac{2}{(a-1)^p} \int_{\mathcal{L}_t\backslash\mathcal{L}_{at}} \mathcal{F}_p[f]\bigg)\frac{\mathrm{d}t}{t} \\
&\qquad + \frac{1}{2}\int_0^\infty M\bigg(2t^p \int_{\mathcal{L}_{at}} \mathcal{F}_p[1]\bigg)\frac{\mathrm{d}t}{t}.
\end{aligned} \tag{4.4.13}$$

Let γ denote a locally integrable function on $(0, \infty)$ such that there exist the limits $\gamma(0)$ and $\gamma(\infty)$. Then the identity

$$\int_0^\infty \big(\gamma(t) - \gamma(at)\big)\frac{\mathrm{d}t}{t} = \big(\gamma(0) - \gamma(\infty)\big)\log a, \tag{4.4.14}$$

holds. Setting here

$$\gamma(t) := M\left(\frac{1}{(a-1)^p}\int_{\mathcal{L}_t}\mathcal{F}_p[f]\right),$$

and using the convexity of M, we obtain

$$\begin{aligned}\int_0^\infty & M\left(\frac{2}{(a-1)^p}\int_{\mathcal{L}_t\setminus\mathcal{L}_{at}}\mathcal{F}_p[f]\right)\frac{\mathrm{d}t}{t}\\ &\le \int_0^\infty\left\{M\left(\frac{2}{(a-1)^p}\int_{\mathcal{L}_t}\mathcal{F}_p[f]\right) - M\left(\frac{2}{(a-1)^p}\int_{\mathcal{L}_{at}}\mathcal{F}_p[f]\right)\right\}\frac{\mathrm{d}t}{t}\\ &= \log a, M\left(\frac{2}{(a-1)^p}\int_{\mathcal{X}}\mathcal{F}_p[f]\right). \end{aligned}\tag{4.4.15}$$

By convexity of M,

$$\begin{aligned}\int_0^\infty & M\left(2t^p\int_{\mathcal{L}_{at}}\mathcal{F}_p[1]\right)\frac{\mathrm{d}t}{t}\\ &\le 2\int_0^\infty M'\left(2t^p\int_{\mathcal{L}_{at}}\mathcal{F}_p[1]\right)t^{p-1}\int_{\mathcal{L}_{at}}\mathcal{F}_p[1]\,\mathrm{d}t\\ &\le 2\int_0^\infty M'\left(2p\int_0^t\tau^{p-1}\int_{\mathcal{L}_{a\tau}}\mathcal{F}_p[1]\,\mathrm{d}\tau\right)t^{p-1}\int_{\mathcal{L}_{at}}\mathcal{F}_p[1]\,\mathrm{d}t\\ &= \frac{1}{p}M\left(2p\int_0^\infty\tau^{p-1}\int_{\mathcal{L}_{a\tau}}\mathcal{F}_p[1]\,\mathrm{d}\tau\right). \end{aligned}\tag{4.4.16}$$

Clearly,

$$\int_0^\infty\tau^{p-1}\int_{\mathcal{L}_{a\tau}}\mathcal{F}_p[1]\,\mathrm{d}\tau = \left(a^p-1\right)\int_0^\infty\tau^{p-1}\int_{\mathcal{L}_\tau\setminus\mathcal{L}_{a\tau}}\mathcal{F}_p[1]\,\mathrm{d}\tau. \tag{4.4.17}$$

Using the truncation

$$\lambda(\xi) = \begin{cases}|\xi| & \text{for } |\xi| > a\tau,\\ a\tau & \text{for } |\xi| \le a\tau\end{cases}$$

together with (4.4.2), we deduce from (4.4.17) and (4.4.14) that

$$\begin{aligned}\int_0^\infty\tau^{p-1}\int_{\mathcal{L}_{a\tau}}\mathcal{F}_p[1]\,\mathrm{d}\tau &\le \frac{a^p-1}{a^p}\int_0^\infty\int_{\mathcal{L}_\tau\setminus\mathcal{L}_{a\tau}}\mathcal{F}_p[f]\frac{\mathrm{d}\tau}{\tau}\\ &= \log a\frac{a^p-1}{a^p}\int_{\mathcal{X}}\mathcal{F}_p[f]. \end{aligned}$$

Combining this with (4.4.16), we arrive at

$$\int_0^\infty M\left(2t^p\int_{\mathcal{L}_{at}}\mathcal{F}_p[1]\right)\frac{\mathrm{d}t}{t} \le \frac{1}{p}M\left(2p\log a\frac{a^p-1}{a^p}\int_{\mathcal{X}}\mathcal{F}_p[f]\right).$$

Adding (4.4.15) and the last inequality, we deduce from (4.4.13) that

$$\int_0^\infty M\big(t^p \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\big) \frac{\mathrm{d}t}{t} \\ \le \frac{1}{2} \log a M\bigg(\frac{2}{(a-1)^p} \int_{\mathcal{X}} \mathcal{F}_p[f]\bigg) + \frac{1}{2p} M\bigg(2p \log a \frac{a^p - 1}{a^p} \int_{\mathcal{X}} \mathcal{F}_p[f]\bigg),$$

and (4.1.3) follows. □

Remark 1. Suppose that (4.4.3) is replaced by the following more restrictive locality condition:

$$f(x) = \text{const} \quad \text{on a compact set } \mathcal{C} \quad \Longrightarrow \quad \int_{\mathcal{C}} \mathcal{F}_p[f] = 0, \tag{4.4.18}$$

which holds, for example, if the measure ν in (4.1.2) is zero. Then the above proof becomes simpler. In fact, we can replace (4.4.13) by

$$\int_0^\infty M\big(t^p \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\big) \frac{\mathrm{d}t}{t} \le \int_0^\infty M\bigg(\frac{1}{(a-1)^p} \int_{\mathcal{L}_t \setminus \mathcal{L}_{at}} \mathcal{F}_p[f]\bigg).$$

Estimating the right-hand side by (4.4.15) we obtain the inequality

$$\int_0^\infty M\big(t^p \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\big) \frac{\mathrm{d}t}{t} \le \log a M\bigg(\frac{1}{(a-1)^p} \int_{\mathcal{X}} \mathcal{F}_p[f]\bigg). \tag{4.4.19}$$

The next assertion follows directly from (4.1.3) and (4.4.19) by setting $M(\xi) = \xi^{q/p}$ for $\xi \ge 0$.

Corollary 1. *Let $q \ge p$ and let $\mathcal{F}_p$ satisfy the locality condition (4.4.3). Then for all $f \in \operatorname{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$ and for an arbitrary $a > 1$*

$$\bigg(\int_0^\infty \big(\operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\big)^{q/p} \,\mathrm{d}\big(t^q\big)\bigg)^{1/q} \le C\bigg(\int_{\mathcal{X}} \mathcal{F}_p[f]\bigg)^{1/p}. \tag{4.4.20}$$

If additionally $\mathcal{F}_p$ is subject to (4.4.18), then one can choose

$$C = \frac{(q \log a)^{1/q}}{a-1}.$$

Remark 2. Let $\mathcal{F}_p$ satisfy (4.4.18). Then for every sequence $\{t_k\}_{k=-\infty}^{\infty}$ such that $0 < t_k < t_{k+1}$,

$$t_k \to 0 \quad \text{as } k \to -\infty \quad \text{and} \quad t_k \to \infty \quad \text{as } k \to \infty,$$

the following discrete conductor inequality holds:

$$\sum_{k=-\infty}^{\infty} (t_{k+1} - t_k)^p \operatorname{cap}_p(\overline{\mathcal{L}_{t_{k+1}}}, \mathcal{L}_{t_k}) \le \mathcal{F}_p[f]. \tag{4.4.21}$$

Putting $t_k = a^k$, where $a > 1$, we see that

$$\sum_{k=-\infty}^{\infty} a^{pk} \operatorname{cap}_p(\overline{\mathcal{L}_{a^{k+1}}}, \mathcal{L}_{a^k}) \le (a-1)^{-p} \mathcal{F}_p[f]. \tag{4.4.22}$$

Using Lemma 4.4 and monotonicity properties of the conductivity, we check that (4.4.22) is equivalent to (4.4.20) with $q = p$ modulo the value of the coefficient c.

The capacitary inequality

$$\Big(\int_0^\infty \big(\operatorname{cap}_p(\overline{\mathcal{L}_t}, \mathcal{X})\big)^{q/p} \,\mathrm{d}\big(t^q\big)\Big)^{1/q} \le C\Big(\int_{\mathcal{X}} \mathcal{F}_p[f]\Big)^{1/p} \tag{4.4.23}$$

results directly from (4.4.20).

An immediate consequence of (4.4.23) is the following criterion for the Sobolev-type inequality:

$$\|f\|_{L_q(\mu)} \le C\Big(\int_{\mathcal{X}} \mathcal{F}_p[f]\Big)^{1/p}, \tag{4.4.24}$$

where μ is a locally finite Radon measure on $\mathcal{X}$, $q \ge p$, and f is an arbitrary function in $\operatorname{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$.

Corollary 2. *Inequality* (4.4.24) *holds if and only if*

$$\sup \frac{\mu(g)^{p/q}}{\operatorname{cap}_p(\overline{g}, \mathcal{X})} < \infty. \tag{4.4.25}$$

Proof. The necessity of (4.4.25) is obvious and its sufficiency follows from the well-known and easily checked inequality

$$\Big(\int_0^\infty \mu(\mathcal{L}_t) \,\mathrm{d}\big(t^q\big)\Big)^{1/q} \le \Big(\int_0^\infty \mu(\mathcal{L}_t)^{p/q} \,\mathrm{d}\big(t^p\big)\Big)^{1/p},$$

where $q \ge p \ge 1$ (see Hardy, Littlewood, and Pólya [350]).

4.5 Conductor Inequality for a Dirichlet-Type Integral without Locality Conditions

Here the notations $\mathcal{X}$ and $\operatorname{Lip}_0(\mathcal{X})$ have the same meaning as in Sect. 4.4. Let $\times$ stand for the Cartesian product of sets and let Γ denote a nonnegative symmetric Radon measure on $\mathcal{X}^2 := \mathcal{X} \times \mathcal{X}$, locally finite outside the diagonal $\{(x, y) \in \mathcal{X}^2 : x = y\}$. We shall derive a conductor inequality for the seminorm (4.1.4) where f is an arbitrary function in $C_0(\mathcal{X})$ such that

$$\langle f\rangle_{p,\Gamma} < \infty. \tag{4.5.1}$$

Clearly, the seminorm $\langle\lambda(f)\rangle_{p,\Gamma}$ is contractive, that is,

$$\big\langle\lambda(f)\big\rangle_{p,\Gamma} \le \langle f\rangle_{p,\Gamma},$$

with the same λ as in (4.4.2).

Let, as before, g and G denote open sets in $\mathcal{X}$ such that $\overline{g}$ is a compact subset of G. Similarly to Sect. 4.4, we introduce the conductivity of the conductor $G\backslash\overline{g}$, in other terms, the capacity of $\bar{g}$ with respect to G:

$$\begin{aligned}\operatorname{cap}_{p,\Gamma}(\overline{g}, G) = \inf\{\langle f\rangle^p_{p,\Gamma} : \varphi \in C_0(\mathcal{X}), 0 \le \varphi \le 1 \text{ on } G \\ \text{and } \varphi = 1 \text{ on a neighborhood of } \overline{g}\}.\end{aligned}$$

It is straightforward that this infimum does not change if the class of admissible functions φ is replaced with (4.4.6) (compare with the definition of $\operatorname{cap}_p(\overline{g}, G)$ in Sect. 4.4).

Theorem. *For all $f \in C_0(\mathcal{X})$, for all $q \ge p \ge 1$, and for an arbitrary $a > 1$ the conductor inequality* (4.1.4) *holds.*

Proof. The measurability of the function $t \to \operatorname{cap}_{p,\Gamma}(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)$ is proved word for word as in Lemma 4.4.

Clearly,

$$(a-1)^p t^p \operatorname{cap}_{p,\Gamma}(\overline{\mathcal{L}_{at}}, \mathcal{L}_t) \le \big\langle\Lambda_t(f)\big\rangle^p_{p,\Gamma} \tag{4.5.2}$$

with Λ_t defined by (4.4.11). Let K_t denote the conductor $\mathcal{L}_t\backslash\overline{\mathcal{L}_{at}}$. Since

$$S^2 \subset (S\times T)\cup(T\times S)\cup(S\backslash T)^2$$

for all sets S and T and since Γ is symmetric, it follows that

$$\begin{aligned}&(a-1)^p t^p \operatorname{cap}_{p,\Gamma}(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\\ &\quad\le \left(2\int_{K_t\times\mathcal{X}} + \int_{(\mathcal{X}\backslash K_t)^2}\right)\big|\Lambda_t\big(f(x)\big) - \Lambda_t\big(f(y)\big)\big|^p \Gamma(\mathrm{d}x\times \mathrm{d}y)\\ &\quad\le 2\int_{K_t\times\mathcal{X}}\big|f(x)-f(y)\big|^p\Gamma(\mathrm{d}x\times\mathrm{d}y) + 2(a-1)^p t^p \Gamma\big(\overline{\mathcal{L}_{at}}\times(\mathcal{X}\backslash\mathcal{L}_t)\big).\end{aligned} \tag{4.5.3}$$

By Minkowski's inequality,

$$(a-1)^p\left(\int_0^\infty \big(\operatorname{cap}_{p,\Gamma}(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)\big)^{q/p} t^{q-1}\,\mathrm{d}t\right)^{p/q} \le A + B,$$

where

$$A = 2\left(\int_0^\infty \left(\int_{K_t\times\mathcal{X}}\big|f(x)-f(y)\big|^p \Gamma(\mathrm{d}x\times\mathrm{d}y)\right)^{q/p}\frac{\mathrm{d}t}{t}\right)^{p/q}$$

and

$$B = 2(a-1)^p \Big(\int_0^\infty \Gamma\big(\overline{\mathcal{L}_{at}} \times (\mathcal{X} \backslash \mathcal{L}_t)\big)^{q/p} t^{q-1}\, \mathrm{d}t \Big)^{p/q}.$$

Since $q \geq p$, we have

$$A \leq a \Big(\int_0^\infty \frac{\gamma(t) - \gamma(at)}{t}\, \mathrm{d}t \Big)^{p/q},$$

where

$$\gamma(t) = \Big(\int_{\mathcal{L}_t \times \mathcal{X}} \big|f(x) - f(y)\big|^p \Gamma(\mathrm{d}x \times \mathrm{d}y) \Big)^{p/q}.$$

Using (4.4.14), we obtain

$$A \leq 2(\log a)^{p/q} \langle f \rangle_p^p. \tag{4.5.4}$$

Let us estimate B. Clearly,

$$B = 2(a-1)^p \Big(\int_0^\infty \Big(\int_{\mathcal{X}^2} \varkappa(x, \overline{\mathcal{L}_{at}}) \varkappa(y, \mathcal{X} \backslash \mathcal{L}_t) \Gamma(\mathrm{d}x \times \mathrm{d}y) \Big)^{q/p} t^{q-1}\, \mathrm{d}t \Big)^{p/q},$$

where $\varkappa(\cdot, S)$ is the characteristic function of a set S. By Minkowski's inequality,

$$\begin{aligned} B &\leq 2(a-1)^p \int_{\mathcal{X}^2} \Big(\int_0^\infty \varkappa(x, \overline{\mathcal{L}_{at}}) \varkappa(y, \mathcal{X} \backslash \mathcal{L}_t) t^{q-1}\, \mathrm{d}t \Big)^{p/q} \Gamma(\mathrm{d}x \times \mathrm{d}y) \\ &= \frac{2(a-1)^p}{q^{p/q} a^p} \int_{\mathcal{X}^2} \big(|f(x)|^q - a^q |f(y)|^q \big)_+^{p/q} \Gamma(\mathrm{d}x \times \mathrm{d}y). \end{aligned}$$

Obviously, the inequality $|f(x)| \geq a|f(y)|$ implies

$$\big|f(x)\big|^q - a^q \big|f(y)\big|^q \leq \big|f(x)\big|^q \leq \frac{a^q}{(a-1)^q} \big(\big|f(x)\big| - |f(y)| \big)_+^q.$$

Hence

$$B \leq q^{-p} \langle f \rangle_p^p.$$

Adding this estimate with (4.5.4), we arrive at (4.1.4) with

$$c(a,p,q) = \frac{1 + 2(q \log a)^{p/q}}{(a-1)^p q^{p/q}}.$$

The proof is complete. □

To show the usefulness of inequality (4.1.4), we give a criterion of a two-weight Sobolev-type inequality involving the seminorm $\langle f \rangle_{p,\Gamma}$.

Corollary. *Let $q \geq p \geq 1$, $q \geq r > 0$, and let μ and ν be locally finite nonnegative Radon measures on $\mathcal{X}$. The inequality*

$$\int_{\mathcal{X}} |f|^q \,\mathrm{d}\mu \le C\bigg(\langle f\rangle_{p,\Gamma}^q + \Big(\int_{\mathcal{X}} |f|^r \,\mathrm{d}\nu\Big)^{q/r}\bigg) \tag{4.5.5}$$

holds for every $f \in \mathrm{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$ *if and only if all bounded open sets* g *and* G *in* $\mathcal{X}$ *such that* $\overline{g} \subset G$ *satisfy the inequality*

$$\mu(g) \le Q\big(\mathrm{cap}_{p,\Gamma}(\overline{g}, G)^{q/p} + \nu(G)^{q/r}\big). \tag{4.5.6}$$

The best constants C *and* Q *in* (4.5.5) *and* (4.5.6) *are related by* $Q \le C \le c(p,q)Q$.

Proof. The necessity of (4.5.6) and the estimate $G \le C$ are obtained by putting an arbitrary function $f \in \mathrm{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$ subject to $f = 1$ on g, $f = 0$ on $\mathcal{X}\backslash G$, $0 \le f \le 1$, into (4.5.5).

The sufficiency of (4.5.6) results by the following argument:

$$\begin{aligned}\int_{\mathcal{X}} |f|^q \,\mathrm{d}\mu &= \int_0^\infty \mu(\mathcal{L}_t)\,\mathrm{d}\big(t^q\big)\\ &\le Q\bigg(\int_0^\infty \mathrm{cap}_{p,\Gamma}(\overline{\mathcal{L}_{at}}, \mathcal{L}_t)^{q/p}\,\mathrm{d}\big(t^q\big) + \int_0^\infty \nu(\mathcal{L}_t)^{q/r}\,\mathrm{d}\big(t^q\big)\bigg)\\ &\le Q\bigg(c(a,p,q)^{q/p}\langle f\rangle_{p,\Gamma}^q + \int_{\mathcal{X}} |f|^r\,\mathrm{d}\nu\bigg)^{q/r},\end{aligned}$$

where $c(a,p,q)$ is the same constant as in (4.1.4). The proof is complete. □

Remark. Using the obvious identity

$$\big\langle |f|\big\rangle_{1,\Gamma} = \int_0^\infty \Gamma(\mathcal{L}_t \times \mathcal{X}\backslash\mathcal{L}_t)\,\mathrm{d}t,$$

instead of the conductor inequality (4.1.4), we deduce with the same argument that the inequality

$$\bigg(\int_{\mathcal{X}} |f|^q\,\mathrm{d}\mu\bigg)^{1/q} \le C\bigg(\langle f\rangle_{1,\Gamma} + \int_{\mathcal{X}} |f|\,\mathrm{d}\nu\bigg), \tag{4.5.7}$$

with $q \ge 1$ holds if and only if, for all bounded open sets g,

$$\mu(g)^{1/q} \le C\big(\Gamma\big(g \times (\mathcal{X}\backslash g)\big) + \nu(g)\big),$$

with the same value of C as in (4.5.7).

4.6 Sharp Capacitary Inequalities and Their Applications

Let Ω denote an open set in $\mathbb{R}^n$ and let the function

$$\Omega \times \mathbb{R}^n \ni (x,z) \to \Phi(x,z) \in \mathbb{R}$$

be a continuous function, positively homogeneous of degree 1 with respect to z. Clearly, the measure

$$\mathcal{F}_p[f] := \big|\varPhi\big(x, \operatorname{grad} f(x)\big)\big|^p \,\mathrm{d}x$$

satisfies (4.4.1), (4.4.2), and (4.4.18). Hence, (4.4.20) implies the inequality

$$\left(\int_0^\infty \big(\operatorname{cap}_p(\overline{\mathcal{L}_t}, \mathcal{X})\big)^{q/p} \,\mathrm{d}(t^q)\right)^{1/q} \le C\left(\int_{\mathcal{X}} \big|\varPhi\big(x, \operatorname{grad} f(x)\big)\big|^p \,\mathrm{d}x\right)^{1/p}, \tag{4.6.1}$$

where cap_p is the p-capacity corresponding to the integral (3.1.8), $C = \text{const} > 0$, and f is an arbitrary function in $C_0^\infty(\varOmega)$. The next assertion gives the sharp value of C for $q > p$. In the case $q = p$ the sharp value of C is given by (2.3.6) and is obtained by the same argument.

Theorem 1. *Inequality* (4.6.1) *with* $q > p$ *holds with*

$$C = \left(\frac{\varGamma(\frac{pq}{q-p})}{\varGamma(\frac{q}{q-p})\varGamma(p\frac{q-1}{q-p})}\right)^{1/p-1/q}. \tag{4.6.2}$$

This value of C *is sharp if* $\varPhi(x, y) = |y|$ *and if either* $\varOmega$ *is a ball or* $\varOmega = \mathbb{R}^n$.

Proof. Let

$$\psi(t) = \int_t^\infty \left(\int_{|f(x)|=\tau} \big|\varPhi\big(x, N(x)\big)\big|^p \big|\operatorname{grad} f(x)\big|^{p-1} \,\mathrm{d}s(x)\right)^{1/(1-p)} \mathrm{d}\tau$$

with $\mathrm{d}s$ standing for the surface element and $N(x)$ denoting the normal vector at x directed inward $\mathcal{L}_\tau$. Further, let $t(\psi)$ denote the inverse function of $\psi(t)$. Then

$$\int_\varOmega \big|\varPhi\big(x, \operatorname{grad} f(x)\big)\big|^p \,\mathrm{d}x = \int_0^\infty \big|t'(\psi)\big|^p \,\mathrm{d}\psi \tag{4.6.3}$$

(see Lemma 2.3.1 for more details). By Bliss's inequality [109]

$$\left(\int_0^\infty t(\psi)^q \frac{\mathrm{d}\psi}{\psi^{1+q(p-1)/p}}\right)^{1/q} \le \left(\frac{p}{q(p-1)}\right)^{1/q} C\left(\int_0^\infty \big|t'(\psi)\big|^p \,\mathrm{d}\psi\right)^{1/p} \tag{4.6.4}$$

with C as in (4.6.2), and by (4.6.3) this is equivalent to

$$\left(\int_0^\infty \frac{\mathrm{d}(t(\psi)^q)}{\psi^{q(p-1)/p}}\right)^{1/q} \le C\left(\int_\varOmega \big|\varPhi\big(x, \operatorname{grad} f(x)\big)\big|^p \,\mathrm{d}x\right)^{1/p}.$$

To obtain (4.6.1) with C given by (4.6.2) it remains to note that

$$\operatorname{cap}_p(\overline{\mathcal{L}_t}) \le \frac{1}{\psi(t)^{p-1}} \tag{4.6.5}$$

(see Lemma 2.2.2/1). The constant (4.6.2) is the best possible since (4.6.5) becomes equality for radial functions. □

Following Definition 2.1.4, we introduce the weighted perimeter minimizing function σ on $(0,\infty)$ by

$$\mathscr{C}(m) := \inf \int_{\partial g} \bigl|\Phi\bigl(x, N(x)\bigr)\bigr| \, \mathrm{d}s(x), \tag{4.6.6}$$

where the infimum is extended over all bounded open sets g with smooth boundaries subject to

$$m_n(g) \geq m.$$

According to Corollary 2.2.3/2, the following isocapacitary inequality holds:

$$\operatorname{cap}_p(\overline{g}, G) \geq \biggl(\int_{m_n(g)}^{m_n(G)} \frac{\mathrm{d}m}{\mathscr{C}(m)^{p'}}\biggr)^{1-p}. \tag{4.6.7}$$

Therefore, (4.6.1) leads to

Corollary 1. *For, all $f \in C_0^\infty(\Omega)$,*

$$\biggl(\int_0^\infty \biggl(\int_{m_n(\mathcal{L}_t)}^{m_n(\Omega)} \frac{\mathrm{d}m}{\mathscr{C}(m)^{p'}}\biggr)^{-q/p'} \mathrm{d}\bigl(t^q\bigr)\biggr)^{1/q} \leq C\biggl(\int_\Omega \bigl|\Phi\bigl(x, \operatorname{grad} f(x)\bigr)\bigr|^p \, \mathrm{d}x\biggr)^{1/p} \tag{4.6.8}$$

with $q > p$ and C defined by (4.6.2). For $p = 1$ the last inequality should be replaced by

$$\biggl(\int_0^\infty \mathscr{C}\bigl(m_n(\mathcal{L}_t)\bigr)^q \, \mathrm{d}\bigl(t^q\bigr)\biggr)^{1/q} \leq \int_\Omega \bigl|\Phi\bigl(x, \operatorname{grad} f(x)\bigr)\bigr| \, \mathrm{d}x \tag{4.6.9}$$

with $q \geq 1$.

In addition, this corollary, combined with the isoperimetric inequality

$$s(\partial g) \geq n^{1/n'} \omega_n^{1/n} m_n(g)^{1/n'},$$

immediately gives the following well-known sharp result.

Corollary 2. (*See* (1.4.14) *for $p = 1$ and Sect.* 2.3.5 *for $p > 1$*) *Let $n > p \geq 1$ and $q = pn(n-p)^{-1}$. Then every $f \in C_0^\infty(\mathbb{R}^n)$ satisfies the Sobolev inequality* (4.1.7) *with the best constant*

$$c = \pi^{-1/2} n^{-1/2} \Bigl(\frac{p-1}{n-p}\Bigr)^{1/p'} \Bigl(\frac{\Gamma(n)\Gamma(1+n/2)}{\Gamma(n/p)\Gamma(1+n-n/p)}\Bigr)^{1/n}.$$

The next assertion resulting from (4.6.3) and (4.6.5) shows that a quite general capacitary inequality is a consequence of a certain inequality for functions of one variable.

Theorem 2. *Let α and β be positive nondecreasing functions on $(0,\infty)$ such that*

$$\sup \int_0^\infty \beta\big(\psi^{1-p}\big)\,\mathrm{d}\big(\alpha\big(t(\psi)\big)\big) < \infty \tag{4.6.10}$$

with the supremum taken over all absolutely continuous functions $[0,\infty) \ni \psi \to t(\psi) \geq 0$ subject to $t(0) = 0$ and

$$\int_0^\infty \big|t'(\psi)\big|^p\,\mathrm{d}\psi \leq 1. \tag{4.6.11}$$

Then

$$\sup \int_0^\infty \beta\big(\mathrm{cap}_p(\overline{\mathcal{L}_t}, \Omega)\big)\,\mathrm{d}\alpha(t) < \infty \tag{4.6.12}$$

with the supremum extended over all f subject to

$$\int_\Omega \big|\Phi\big(x, \mathrm{grad}\, f(x)\big)\big|^p\,\mathrm{d}x \leq 1. \tag{4.6.13}$$

The least upper bounds (4.6.10) and (4.6.12) coincide.

In fact, the above Theorem 1 is a particular case of Theorem 2 corresponding to the choice

$$\alpha(t) = t^q \quad \text{and} \quad \beta(\xi) = \xi^{q/p}.$$

The next result is another consequence of Theorem 2.

Theorem 3. *For every $c > 0$*

$$\sup \int_0^\infty \exp\bigg(\frac{-c}{\mathrm{cap}_p(\overline{\mathcal{L}_t}, \Omega)^{1/(p-1)}}\bigg)\,\mathrm{d}\big(\exp(ct^{p'})\big) < \infty, \tag{4.6.14}$$

where the supremum is taken over all $f \in C_0^\infty(\Omega)$ subject to (4.6.13) and $p' = p/(p-1)$, $p > 1$.

Proof. It follows from a theorem by Jodeit [402] that

$$\sup \int_0^\infty \exp\big(t(\psi)^{p'} - \psi\big)\,\mathrm{d}\psi < \infty \tag{4.6.15}$$

with the supremum taken over all absolutely continuous functions $[0,\infty) \ni \psi \to t(\psi) \geq 0$ subject to $t(0) = 0$ and (4.6.11). Hence, for every $c > 0$,

$$\sup \int_0^\infty \exp\big(ct(\psi)^{p'} - c\psi\big)\,\mathrm{d}\psi < \infty.$$

It remains to refer to Theorem 4.6/2 with

$$\alpha(t) = \exp\bigl(ct^{p'}\bigr) \quad \text{and} \quad \beta(\xi) = \exp\bigl(-c\xi^{1/(1-p)}\bigr).$$

A direct consequence of Theorem 3 and the isocapacitary inequality (2.3.17) is the following result by Moser.

Corollary 3. (Moser [618]) *Let $m_n(\Omega) < \infty$ and let*

$$\{f\} := \bigl\{f \in C_0^\infty(\Omega) : \|\operatorname{grad} f\|_{L_n(\Omega)} \le 1\bigr\}.$$

Then

$$\sup_{\{f\}} \int_\Omega \exp\bigl(n\omega_n^{1/(n-1)} \bigl|f(x)\bigr|^{n'}\bigr)\, \mathrm{d}x < \infty.$$

Proof. The first inequality (2.2.11) can be written as

$$m_n(\overline{g}) \le m_n(G) \exp\bigl(-n\omega_n^{1/(n-1)} \operatorname{cap}_n(\overline{g}, G)^{1/(1-n)}\bigr).$$

Hence, putting $c = n\omega_n^{1/(n-1)}$ and $p = n$ in (4.6.14), we obtain

$$\int_0^\infty m_n(\mathcal{L}_t)\, \mathrm{d} \exp\bigl(n\omega_n^{1/(n-1)} t^{n'}\bigr) < \infty.$$

The result follows. □

One needs no changes in proofs to see that the main results of this section, Theorems 1–3, hold if Ω is an open subset of a Riemannian manifold $\mathfrak{R}_n$ and $\operatorname{grad} f$ is the Riemannian gradient. As an application, we obtain another Moser inequality [618].

Corollary 4. *Let Ω be a proper subdomain of the unit sphere S^2 and let $\{f\}$ be defined as in Corollary* 3. *Then*

$$\sup_{\{f\}} \int_\Omega \exp\bigl(4\pi f^2(\omega)\bigr)\, \mathrm{d}s_\omega < \infty.$$

Proof. By Theorem 3 with $c = 4\pi$ we have the capacitary integral inequality

$$\sup_{\{f\}} \int_0^\infty \exp\left(\frac{-4\pi}{\operatorname{cap}_2(\overline{\mathcal{L}_t}, \Omega)}\right) \mathrm{d}\bigl(\exp\bigl(4\pi t^2\bigr)\bigr) < \infty. \tag{4.6.16}$$

The classical isoperimetric inequality on S^2

$$s(\partial g)^2 \ge m_2(g)\bigl(4\pi - m_2(g)\bigr)$$

(see Rado [668]), combined with (4.6.7) implies the isocapacitary inequality

$$\operatorname{cap}_2(\bar{g}, G) \geq 4\pi\left(\log\frac{m_2(G)(4\pi - m_2(g))}{m_2(g)(4\pi - m_2(G))}\right)^{-1}.$$

Setting here $G = \Omega$, $g = \mathcal{L}_t$, and using (4.6.16), we complete the proof. □

Remark. We can go even further, extending the previous results to the measure-valued operator $\mathcal{F}_p[f]$ in Sect. 4.4 subject to the condition

$$\mathcal{F}_p\big[\lambda(f)\big] = \big|\lambda'(f)\big|^p \mathcal{F}_p[f], \tag{4.6.17}$$

with the same λ as in (4.4.2). In fact, (4.6.17) implies

$$\int_{\mathcal{X}} \mathcal{F}_p[f] = \int_0^\infty \big|t'(\psi)\big|^p \,\mathrm{d}\psi, \tag{4.6.18}$$

where $t(\psi)$ is the inverse of the function

$$\psi(t) = \int_t^\infty \left|\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{F}_p[f](\mathcal{L}_\tau)\right|^{1/(1-p)} \mathrm{d}\tau.$$

Identity (4.6.18) is the core of the proof of the results in the present section.

4.7 Capacitary Improvement of the Faber–Krahn Inequality

By the Faber [265]–Krahn [462] inequality for the Dirichlet–Laplace eigenvalues, the ball has the minimum eigenvalue among all domains in $\mathbb{R}^n$ with the same volume.

Here we obtain a capacitary version of this inequality valid for any open subset of an arbitrary Riemannian manifold. We state and prove the main result of this section.

Theorem 1. *Let $\mathcal{R} > 0$, $u \in C_0^\infty(\Omega)$, and $\mathcal{N}_t = \{x \in \Omega : |u(x)| \geq t\}$. If $n > 2$, then*

$$\left(\frac{j_{(n-2)/2}}{\mathcal{R}}\right)^2 m_n(B_{\mathcal{R}}) \int_0^\infty \left(\frac{\operatorname{cap}(\mathcal{N}_t;\Omega)}{\operatorname{cap}(B_{\mathcal{R}}) + \operatorname{cap}(\mathcal{N}_t;\Omega)}\right)^{\frac{n}{n-2}} \mathrm{d}\big(t^2\big) \leq \|\nabla u\|^2_{L_2(\Omega)}, \tag{4.7.1}$$

where cap *is the 2-capacity and j_ν is the first positive root of the Bessel function J_ν. If $n = 2$, then*

$$\pi j_0^2 \int_0^\infty \exp\left(\frac{-4\pi}{\operatorname{cap}(\mathcal{N}_t;\Omega)}\right) \mathrm{d}\big(t^2\big) \leq \|\nabla u\|^2_{L_2(\Omega)}. \tag{4.7.2}$$

Proof. Let w be an arbitrary absolutely continuous function on $(0,\mathcal{R}]$, such that $w(\mathcal{R})=0$. The inequality

$$\left(\frac{j_{(n-2)/2}}{\mathcal{R}}\right)^2 \int_0^{\mathcal{R}} w(\rho)^2\rho^{n-1}\,\mathrm{d}\rho \le \int_0^{\mathcal{R}} w'(\rho)^2\rho^{n-1}\,\mathrm{d}\rho, \tag{4.7.3}$$

where $n>2$, is equivalent to the fact that the first eigenvalue of the Dirichlet–Laplace operator in the unit ball B equals $j^2_{(n-2)/2}$. Similarly, the inequality

$$\left(\frac{j_0}{\mathcal{R}}\right)^2 \int_0^{\mathcal{R}} w(\rho)^2\rho\,\mathrm{d}\rho \le \int_0^{\mathcal{R}} w'(\rho)^2\rho\,\mathrm{d}\rho, \tag{4.7.4}$$

is associated with $n=2$.

In the case $n>2$, we introduce the new variables

$$\psi = \frac{\rho^{2-n}-\mathcal{R}^{2-n}}{(n-2)\omega_n}, \qquad t(\psi)=w\big(\rho(\psi)\big),$$

and write (4.7.3) in the form

$$\begin{aligned}&\big(\omega_n j_{(n-2)/2}\mathcal{R}^{-1}\big)^2 \int_0^\infty \frac{t(\psi)^2\,\mathrm{d}\psi}{((n-2)\omega_n\psi+\mathcal{R}^{2-n})^{2(n-1)/(n-2)}}\\ &\quad\le \int_0^\infty t'(\psi)^2\,\mathrm{d}\psi.\end{aligned} \tag{4.7.5}$$

Similarly, for $n=2$, putting

$$\psi=(2\pi)^{-1}\log\frac{\mathcal{R}}{\rho}, \qquad t(\psi)=w\big(\rho(\psi)\big),$$

we write (4.7.4) as

$$(2\pi j_0)^2\int_0^\infty t(\psi)^2\exp(-4\pi\psi)\,\mathrm{d}\psi \le \int_0^\infty t'(\psi)^2\,\mathrm{d}\psi. \tag{4.7.6}$$

Note that the function t in (4.7.5) and (4.7.6) is subject to the boundary condition $t(0)=0$. We write (4.7.5) and (4.7.6) as

$$\begin{aligned}&n^{-1}\omega_n\left(\frac{j_{(n-2)/2}}{\mathcal{R}}\right)^2\int_0^\infty \frac{\mathrm{d}t(\psi)^2}{((n-2)\omega_n\psi+\mathcal{R}^{2-n})^{n/(n-2)}}\\ &\quad\le \int_0^\infty t'(\psi)^2\,\mathrm{d}\psi\end{aligned} \tag{4.7.7}$$

and

$$\pi j_0^2\int_0^\infty \exp(-4\pi\psi)\,\mathrm{d}t(\psi)^2 \le \int_0^\infty t'(\psi)^2\,\mathrm{d}\psi. \tag{4.7.8}$$

Now, as in Sect. 2.3.1, we introduce the function

$$\psi(t) = \int_0^t \frac{\mathrm{d}\tau}{\int_{|u|=\tau} |\nabla u| \, \mathrm{d}s} \tag{4.7.9}$$

as well as its inverse $\psi \to t(\psi)$, and replace the integral on the right-hand side of (4.7.7) and (4.7.8) by $\|\nabla u\|^2_{L_2(\Omega)}$. It remains to note that

$$\psi(t) \le \big(\mathrm{cap}(\mathcal{N}_t; \Omega)\big)^{-1} \tag{4.7.10}$$

by Lemma 2.2.2/1.

Let us use the area minimizing function of Ω

$$\lambda(v) = \inf s(\partial g), \tag{4.7.11}$$

where the infimum is extended over all sets g with smooth boundaries and compact closures $\overline{g} \subset \Omega$, subject to the inequality $m_n(g) \ge v$. This function is a particular case of the function $\mathscr{C}$ from Definition 2.1.4, corresponding to $\Phi(x, \xi) = |\xi|$. The function λ appears in the isocapacitary inequality

$$\mathrm{cap}(F; \Omega) \ge \left(\int_{m_n(F)}^{m_n(\Omega)} \frac{\mathrm{d}v}{\lambda(v)^2} \right)^{-1}$$

(see Corollary 2.2.3/2). Therefore, (4.7.1), (4.7.2), and the identity

$$\mathrm{cap}(B_{\mathcal{R}}) = (n-2)\omega_n \mathcal{R}^{n-2},$$

lead to the following Lorentz-type estimates.

Corollary 2. *If $n > 2$ and $\mathcal{R} > 0$, then, for all $u \in C_0^\infty(\Omega)$,*

$$\left(\frac{j_{(n-2)/2}}{\mathcal{R}}\right)^2 m_n(B_{\mathcal{R}}) \int_0^\infty \left(\mathrm{cap}(B_{\mathcal{R}}) \int_{m_n(\mathcal{N}_t)}^{m_n(\Omega)} \frac{\mathrm{d}v}{\lambda(v)^2} + 1 \right)^{\frac{n}{2-n}} \mathrm{d}\big(t^2\big) \le \|\nabla u\|^2_{L_2(\Omega)}. \tag{4.7.12}$$

If $n = 2$, then, for all $u \in C_0^\infty(\Omega)$,

$$\pi j_0^2 \int_0^\infty \exp\left(-4\pi \int_{m_n(\mathcal{N}_t)}^{m_n(\Omega)} \frac{\mathrm{d}v}{\lambda(v)^2} \right) \mathrm{d}\big(t^2\big) \le \|\nabla u\|^2_{L_2(\Omega)}. \tag{4.7.13}$$

Remark. Since

$$\lambda(v) \ge n^{\frac{n-1}{n}} \omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}}, \tag{4.7.14}$$

by the classical isoperimetric inequality for $\mathbb{R}^n$, the estimates (4.7.12) and (4.7.13) imply the Faber–Krahn property

$$\Lambda(\Omega) \ge \left(\frac{j_{(n-2)/2}}{\mathcal{R}}\right)^2$$

for any n-dimensional Euclidean domain Ω with $m_n(\Omega) = n^{-1}\omega_n \mathcal{R}^n$.

Theorem 1 is a very special case of the following general assertion.

Theorem 2. *Let* $\mathbf{M}$ *be a decreasing nonnegative function on* $[0, \infty)$ *and let* $q > 0$ *and* $p \geq 1$*. Suppose that for all absolutely continuous functions* $\psi \to t(\psi)$ *on* $[0, \infty)$*, the inequality*

$$\Big(-\int_0^\infty \big|t(\psi)\big|^q \,\mathrm{d}\mathbf{M}(\psi)\Big)^{1/q} \leq \Big(\int_0^\infty \big|t'(\psi)\big|^p \,\mathrm{d}\psi\Big)^{1/p} \tag{4.7.15}$$

holds. Then, for all $u \in C_0^\infty(\Omega)$*,*

$$\Big(\int_0^\infty \mathbf{M}\big(\big(\mathrm{cap}_p(\mathcal{N}_t;\Omega)\big)^{1/(1-p)}\big)\,\mathrm{d}\big(t^q\big)\Big)^{1/q} \leq \|\nabla u\|_{L_p(\Omega)}, \tag{4.7.16}$$

where cap_p *is the* p*-capacity defined by*

$$\mathrm{cap}_p(F;\Omega) = \inf\Big\{\int_\Omega |\nabla u|^p \,\mathrm{d}x : u \in C_0^\infty(\Omega), u \geq 1 \ \text{on } F\Big\}. \tag{4.7.17}$$

Proof. The role of the function ψ given by (4.7.9) is played in the present proof by

$$\psi(t) = \int_0^t \frac{\mathrm{d}\tau}{(\int_{|u|=\tau} |\nabla u|^{p-1}\,\mathrm{d}s)^{1/(p-1)}}. \tag{4.7.18}$$

We write the left-hand side of (4.7.15) in the form

$$\Big(\int_0^\infty \mathbf{M}(\psi)\,\mathrm{d}\big(t(\psi)\big)^q\Big)^{1/q}$$

and use the monotonicity of $\mathbf{M}$ and the inequality

$$\psi \leq \big(\mathrm{cap}_p(\mathcal{N}_{t(\psi)};\Omega)\big)^{1/(1-p)} \tag{4.7.19}$$

proved in Lemma 2.2.2/1. It remains to apply (4.7.15) and the identity

$$\int_0^\infty \big|f'(\psi)\big|^p \,\mathrm{d}\psi = \int_\Omega |\nabla u|^p \,\mathrm{d}x \tag{4.7.20}$$

found in Lemma 2.3.1. □

Using the area minimizing function λ defined by (4.7.11) and the estimate

$$\mathrm{cap}_p(F;\Omega) \geq \Big(\int_{m_n(F)}^{m_n(\Omega)} \frac{\mathrm{d}v}{\lambda(v)^{p/(p-1)}}\Big)^{1-p} \tag{4.7.21}$$

(see Corollary 2.2.3/2), we obtain the following from Theorem 2.

Corollary 2. *Let μ, p, and q be the same as in Theorem 2 and let (4.7.15) hold. Then*

$$\Big(\int_0^\infty \mathbf{M}\Big(\int_{m_n(\mathcal{N}_t)}^{m_n(\Omega)} \frac{\mathrm{d}v}{\lambda(v)^{p/(p-1)}}\Big)\,\mathrm{d}\big(t^q\big)\Big)^{1/q} \le \|\nabla u\|_{L_p(\Omega)} \tag{4.7.22}$$

for all $u \in C_0^\infty(\Omega)$.

Clearly, (4.7.22) is a generalization of the estimates (4.7.12) and (4.7.13) which were obtained for $p = 2$ with a particular choice of μ. Another obvious remark is that (4.7.15), where $\mathbf{M}$ is defined on the interval $0 < t < m_n(\Omega)$ by

$$\mathbf{M}\Big(\int_t^{m_n(\Omega)} \frac{\mathrm{d}v}{\lambda(v)^{p/(p-1)}}\Big) = \Lambda_{p,q} t$$

with a constant $\Lambda_{p,q}$ depending on $m_n(\Omega)$, implies the inequality

$$\Lambda_{p,q}^{1/q}\|u\|_{L_q(\Omega)} \le \|\nabla u\|_{L_p(\Omega)}, \tag{4.7.23}$$

for all $u \in C_0^\infty(\Omega)$.

4.8 Two-Weight Sobolev Inequality with Sharp Constant

Let the measure μ_b be defined by

$$\mu_b(K) = \int_K \frac{\mathrm{d}x}{|x|^b} \tag{4.8.1}$$

for any compact set K in $\mathbb{R}^n$. In this section we obtain the best constant in the inequality

$$\|u\|_{\mathcal{L}_{\tau,q}(\mu_b)} \le C\Big(\int_{\mathbb{R}^n} \big|\nabla u(x)\big|^p \frac{\mathrm{d}x}{|x|^a}\Big)^{1/p},$$

where the left-hand side is the quasinorm in the Lorentz space $\mathcal{L}_{\tau,q}(\mu_b)$, i.e.,

$$\|u\|_{\mathcal{L}_{\tau,q}(\mu_b)} = \Big(\int_0^\infty \big(\mu_b\{x : \big|u(x)\big| \ge t\}\big)^{q/\tau}\,\mathrm{d}\big(t^q\big)\Big)^{1/q}.$$

As a particular case of this result we obtain the best constant in the Hardy-Sobolev inequality

$$\Big(\int_{\mathbb{R}^n} \big|u(x)\big|^q \frac{\mathrm{d}x}{|x|^b}\Big)^{1/q} \le \mathcal{C}\Big(\int_{\mathbb{R}^n} \big|\nabla u(x)\big|^p \frac{\mathrm{d}x}{|x|^a}\Big)^{1/p}. \tag{4.8.2}$$

Let Ω denote an open set in $\mathbb{R}^n$ and let $p \in [1,\infty)$. By (p,a)-capacity of a compact set $K \subset \Omega$ we mean the set function

$$\operatorname{cap}_{p,a}(K,\Omega) = \inf\Big\{\int_\Omega |\nabla u|^p |x|^{-a}\,\mathrm{d}x : u \in C_0^\infty(\Omega), u \geq 1 \text{ on } K\Big\}.$$

In the case $a = 0$, $\Omega = \mathbb{R}^n$ we write simply $\operatorname{cap}_p(K)$.

The following inequality is a particular case of a more general one obtained in Theorem 4.6/1, where Ω is an open subset of an arbitrary Riemannian manifold and $|\Phi(x, \nabla u(x))|$ plays the role of $|\nabla u(x)||x|^{-a/p}$.

Let $q \geq p \geq 1$ and let Ω be an open set in $\mathbb{R}^n$. Then for an arbitrary $u \in C_0^\infty(\Omega)$,

$$\Big(\int_0^\infty \big(\operatorname{cap}_{p,a}(\overline{\mathcal{L}_t}, \Omega)\big)^{q/p}\,\mathrm{d}\big(t^q\big)\Big)^{1/q} \leq \mathcal{A}_{p,q}\Big(\int_\Omega \big|\nabla u(x)\big|^p |x|^{-a}\,\mathrm{d}x\Big)^{1/p}, \tag{4.8.3}$$

where

$$\mathcal{A}_{p,q} = \left(\frac{\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{q}{q-p})\Gamma(p\frac{q-1}{q-p})}\right)^{1/p-1/q} \tag{4.8.4}$$

for $q > p$ and

$$\mathcal{A}_{p,p} = p(p-1)^{(1-p)/p}. \tag{4.8.5}$$

The sharpness of this constant is checked by a sequence of radial functions in $C_0^\infty(\Omega)$. Moreover, there exists a radial optimizer vanishing at infinity, if $\Omega = \mathbb{R}^n$.

Being combined with the isocapacitary inequality

$$\mu(K)^\gamma \leq \Lambda_{p,\gamma} \operatorname{cap}_{p,a}(K, \Omega), \tag{4.8.6}$$

where μ is a Radon measure in Ω, (4.8.3) implies the estimate

$$\Big(\int_0^\infty \big(\mu(\mathcal{L}_t)\big)^{\gamma q/p}\,\mathrm{d}\big(t^q\big)\Big)^{1/q} \leq \mathcal{A}_{p,q}\Lambda_{p,\gamma}^{1/p}\Big(\int_\Omega \big|\nabla u(x)\big|^p |x|^{-a}\,\mathrm{d}x\Big)^{1/p} \tag{4.8.7}$$

for all $u \in C_0^\infty(\Omega)$.

This estimate of u in the Lorentz space $\mathcal{L}_{p/\gamma,q}(\mu)$ becomes the estimate in $L_q(\mu)$ for $\gamma = p/q$

$$\|u\|_{L_q(\mu)} \leq \mathcal{A}_{p,q}\Lambda_{p,\gamma}^{1/p}\Big(\int_\Omega \big|\nabla u(x)\big|^p |x|^{-a}\,\mathrm{d}x\Big)^{1/p}.$$

In the next assertion we find the best value of $\Lambda_{p,\gamma}$ in (4.8.6) for the measure $\mu = \mu_b$ defined by (4.8.1).

Lemma. *Let*

$$1 \leq p < n, \quad 0 \leq a < n-p, \quad \textit{and} \quad a + p \geq b \geq \frac{an}{n-p}. \tag{4.8.8}$$

Then

$$\left(\int_{\mathbb{R}^n} \frac{\mathrm{d}x}{|x|^b}\right)^{\frac{n-p-a}{n-b}} \le \left(\frac{p-1}{n-p-a}\right)^{p-1} \frac{\omega_n^{\frac{b-p-a}{n-b}}}{(n-b)^{\frac{n-p-a}{n-b}}} \operatorname{cap}_{p,a}(K). \tag{4.8.9}$$

The value of the constant factor in front of the capacity is sharp and the equality in (4.8.9) is attained at any ball centered at the origin.

Proof. Introducing spherical coordinates (r, ω) with $r > 0$ and $\omega \in S^{n-1}$, we have

$$\operatorname{cap}_{p,a}(K) = \inf_{u|_K \ge 1} \int_{S^{n-1}} \int_0^\infty \left(\left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2} |\nabla_\omega u|^2 \right)^{\frac{p}{2}} r^{n-1-a}\, \mathrm{d}r\, \mathrm{d}s_\omega. \tag{4.8.10}$$

Let us put here $r = \rho^{1/\varkappa}$, where

$$\varkappa = \frac{n-p-a}{n-p},$$

and $y = (\rho, \omega)$. The mapping $(r, \omega) \to (\rho, \omega)$ will be denoted by σ. Then (4.8.10) takes the form

$$\operatorname{cap}_{p,a}(K) = \varkappa^{p-1} \inf_v \int_{\mathbb{R}^n} \left(\left|\frac{\partial u}{\partial \rho}\right|^2 + (\varkappa\rho)^{-2} |\nabla_\omega u|^2 \right)^{\frac{p}{2}} \mathrm{d}y, \tag{4.8.11}$$

where the infimum is taken over all $v = u \circ \sigma^{-1}$. Since $0 \le \varkappa \le 1$ owing to the conditions $p < n$, $0 < a < n - p$, and $a \ge 0$, inequality (4.8.11) implies

$$\operatorname{cap}_{p,a}(K) \ge \varkappa^{p-1} \inf_v \int_{\mathbb{R}^n} |\nabla u|^p\, \mathrm{d}y \ge \varkappa^{p-1} \operatorname{cap}_p\big(\sigma(K)\big), \tag{4.8.12}$$

which together with the isocapacitary property of cap_p (see (2.2.12)) leads to the estimate

$$\operatorname{cap}_p\big(\sigma(K)\big) \ge \left(\frac{n-p}{p-1}\right)^{p-1} \omega_n^{\frac{p}{n}} n^{\frac{n-p}{n}} \big(m_n\big(\sigma(K)\big)\big)^{\frac{n-p}{n}}. \tag{4.8.13}$$

Clearly,

$$\mu_b(K) = \frac{1}{\varkappa} \int_{\sigma(K)} \frac{\mathrm{d}y}{|y|^\alpha}$$

with

$$\alpha = n - \frac{n-b}{\varkappa} = \frac{b(n-p) - an}{n-p-a} \ge 0. \tag{4.8.14}$$

Furthermore, one can easily check that

$$\mu_b(K) \le \frac{n^{1-\frac{\alpha}{n}}}{n-b} \omega_n^{\frac{\alpha}{n}} \big(m_n\big(\sigma(K)\big)\big)^{1-\frac{\alpha}{n}} \tag{4.8.15}$$

(see, for instance, Example 2.1.5/2). Combining (4.8.15) with (4.8.13), we find

$$\big(\mu_b(K)\big)^{\frac{n-p-a}{n-b}} \le \Big(\frac{p-1}{n-p}\Big)^{p-1} \frac{\omega_n^{\frac{b-p-a}{n-b}}}{(n-b)^{\frac{n-p-a}{n-b}}}\,\mathrm{cap}_p\big(\sigma(K)\big), \tag{4.8.16}$$

which together with (4.8.12) completes the proof of (4.8.9). □

The main result of this section is as follows.

Theorem. *Let conditions* (4.8.8) *hold and let* $q \ge p$. *Then for all* $u \in C_0^\infty(\mathbb{R}^n)$

$$\bigg(\int_0^\infty \big(\mu_b(\mathcal{L}_t)\big)^{\frac{(n-p-a)q}{(n-b)p}}\, \mathrm{d}\big(t^q\big)\bigg)^{\frac{1}{q}} \le \mathcal{C}_{p,q,a,b}\bigg(\int_{\mathbb{R}^n} \big|\nabla u(x)\big|^p \frac{\mathrm{d}x}{|x|^a}\bigg)^{\frac{1}{p}}, \tag{4.8.17}$$

where

$$\mathcal{C}_{p,q,a,b} = \bigg(\frac{\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{q}{q-p})\Gamma(p\frac{q-1}{q-p})}\bigg)^{\frac{1}{p}-\frac{1}{q}}\Big(\frac{p-1}{n-p-a}\Big)^{1-\frac{1}{p}} \times \bigg(\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}(n-b)^{\frac{n-p-a}{p+a-b}}}\bigg)^{\frac{p+a-b}{(n-b)p}}. \tag{4.8.18}$$

The constant (4.8.18) *is the best possible that can be shown by constructing a radial optimizing sequence in* $C_0^\infty(\mathbb{R}^n)$.

Proof. Inequality (4.8.17) is obtained by the substitution of (4.8.4) and (4.8.9) into (4.8.7). The sharpness of (4.8.18) follows from the sharpness of the constant $\mathcal{A}_{p,q}$ in (4.8.3) and from the fact that the isocapacitary inequality (4.8.9) becomes equality for balls. □

The last theorem contains the best constant in the Il'in inequality (4.8.2) as a particular case $q = (n-b)p/(n-p-a)$. We formulate this as the following assertion.

Corollary. *Let conditions* (4.8.8) *hold. Then for all* $u \in C_0^\infty(\mathbb{R}^n)$

$$\bigg(\int_{\mathbb{R}^n} \big|u(x)\big|^{\frac{(n-b)p}{(n-p-a)}} \frac{\mathrm{d}x}{|x|^b}\bigg)^{\frac{n-p-a}{n-b}} \le \mathcal{C}_{p,a,b}\bigg(\int_{\mathbb{R}^n} \big|\nabla u(x)\big|^p \frac{\mathrm{d}x}{|x|^a}\bigg)^{\frac{1}{p}}, \tag{4.8.19}$$

where

$$\mathcal{C}_{p,a,b} = \Big(\frac{p-1}{n-p-a}\Big)^{1-\frac{1}{p}}\bigg(\frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}(n-b)^{\frac{n-p-a}{p+a-b}}}\bigg)^{\frac{p+a-b}{(n-b)p}} \times \bigg(\frac{\Gamma(\frac{p(n-b)}{p-b})}{\Gamma(\frac{n-b}{p-b})\Gamma(1+\frac{(n-b)(p-1)}{p-b})}\bigg)^{\frac{p-b}{p(n-b)}}.$$

This constant is the best possible, which can be shown by constructing a radial optimizing sequence in $C_0^\infty(\mathbb{R}^n)$.

4.9 Comments to Chap. 4

Sections 4.2, 4.3. We follow the articles by Maz'ya [557] and [564]. In the class of convex domains $\Omega \subset \mathbb{R}^n$, it was shown by B. Klartag and E. Milman (oral communication) that the Faber–Krahn inequality admits a matching converse inequality for some linear image $T(\Omega)$ of Ω (having the same measure). Specifically, there exists a numerical constant $C > 0$, such that if $m_n(\Omega) = n^{-1}\omega_n R^n$, then

$$\Lambda\big(T(\Omega)\big) \le C\big(\log(1+n)\big)^2 \left(\frac{j_{(n-2)/2}}{R}\right)^2.$$

Sections 4.4–4.6. Here we follow the author's article [561]. Various aspects of the theory of Sobolev spaces of the first order on metric spaces with a measure were studied in numerous papers, to name only a few: Korevaar and Schoen [452]; Biroli and Mosco [105]; Hajłasz [338]; Kinnunen and Martio [425]; Cheeger [182]; Franchi, Hajłasz, and Koskela [287]; Kilpeläinen, Kinnunen, and Martio [421]; Shanmugalingam [696]; Koskela [456]; Björn, MacManus, and Shanmugalingam [106]; Heinonen [373]; Gol'dshtein and Troyanov [318]; Ambrosio and Tilli [43]; Ostrovskii [650]; Hajłasz, Koskela, and Tuominen [345]; and Kinnunen and Korte [423].

In connection with the capacitary improvement (4.6.14) of Moser's inequality we note that Hudson and Leckband [387] established the existence of an extremal in Jodeit's inequality (4.6.15).

Section 4.7. Theorem 4.7/1 was proved in Maz'ya [564].

Section 4.8. The material of this section is borrowed from the paper by Maz'ya and Shaposhnikova [587].

Inequality (4.8.2) was first proved by Il'in in 1961 ([395], Theorem 1.4) without discussion of the value of C. For certain values of parameters the best constant $\mathcal{C}$ was found in Chua and Wheeden [190] ($p = 2$), in Maz'ya [543], Sect. 2 ($p = 1$, $a = 0$), in Glazer, Martin, Grosse, and Thirring [310] ($p = 2$, $n = 3$, $a = 0$), Lieb [496] ($p = 2$, $n \ge 3$, $a = 0$), in Chou and Chu [187] ($p = 2$, $q \ge 2$), and in A. Nazarov [628] ($1 < p < n$, $a = 0$), where different methods were used.

5

Integrability of Functions in the Space $L_1^1(\Omega)$

Now we turn to embedding theorems for functions with unrestricted boundary values. The present chapter contains conditions on Ω that are necessary and sufficient for the embedding operator $L_1^1(\Omega) \to L_q(\Omega)$ to be continuous or compact. These criteria are intimately connected with relative isoperimetric inequalities and isoperimetric functions. In Sect. 5.2 we consider the cases $q \geq 1$ and $0 < q < 1$ separately.

In more detail, by a relative isoperimetric inequality we mean the inequality of the form

$$m_n(\mathscr{G})^\alpha \leq \text{const}\, s(\partial_i \mathscr{G}), \quad \alpha > 0, \tag{5.0.1}$$

connecting the volume of "admissible" subsets $\mathscr{G}$ of Ω with the area of the surfaces $\partial_i \mathscr{G} = \Omega \cap \partial \mathscr{G}$. The sets $\mathscr{G}$ should satisfy an additional requirement $m_n(\mathscr{G}) \leq M$, where $M \in (0, m_n(\Omega))$. By Theorem 5.2.3, the above isoperimetric inequality with $\alpha \leq 1$ holds if and only if the embedding operator $L_1^1(\Omega) \to L_{1/\alpha}(\Omega)$ is continuous. The sets Ω satisfying (5.0.1) are said to belong to the class $\mathscr{J}_\alpha$. They can be characterized by the inequality

$$\liminf_{\mu \to +0} \mu^{-\alpha} \lambda_M(\mu) > 0,$$

where λ_M is an isoperimetric (in other words, area minimizing function) of Ω, i.e., the least upper bound of the numbers $s(\partial_i \mathscr{G})$ over all sets $\mathscr{G}$ subject to $\mu \leq m_n(\mathscr{G}) \leq M$. The embedding $L_1^1(\Omega) \to L_{1/\alpha}(\Omega)$ for $\alpha > 1$ proves to be continuous if and only if Ω belongs to the class $\mathcal{H}_\alpha$ defined by the inequality

$$\int_0^M \left(\frac{s^\alpha}{\lambda_M(s)} \right)^{\frac{1}{\alpha - 1}} \frac{\mathrm{d}s}{s} < \infty.$$

To find more visible sufficient conditions on the domain ensuring Sobolev embeddings, we use the so-called subareal mappings in Sect. 5.3. Those are mappings of one domain onto another that do not increase $s(\partial_i \mathscr{G})$ up to a constant factor.

V. Maz'ya, *Sobolev Spaces*, 287
Grundlehren der mathematischen Wissenschaften 342,
DOI 10.1007/978-3-642-15564-2_5,

A particular irregular domain that was considered by Nikodým in 1933 as a counterexample to the Poincaré inequality is discussed in Sect. 5.4.

In Sect. 5.5 we show that the embedding $L_1^1(\Omega) \subset L_q(\Omega)$ is always compact if $0 < q < 1$ and it is compact for $q \geq 1$ if and only if

$$\lim_{\mu \to +0} \mu^{1/q} \lambda_M(\mu) = \infty.$$

It might be natural to expect that a weakening of the assumptions on the domain in embedding theorems of the Sobolev type can be compensated by supplementary assumptions on the boundary behavior of the function. The question of the influence of such assumptions on Sobolev-type inequalities is considered in Sect. 5.6. Let the space $W_{p,r}^1(\Omega, \partial\Omega)$ be the completion of $L_p^1(\Omega) \cap C^\infty(\Omega) \cap C(\bar{\Omega})$ equipped with the norm $(\int_\Omega |\nabla u|^p \,\mathrm{d}x)^{1/p} + (\int_{\partial\Omega} |u|^r \,\mathrm{d}s)^{1/r}$.

We show in Theorem 5.6.3 that $W_{1,r}^1(\Omega, \partial\Omega)$ is continuously embedded into $L_q(\Omega)$ with $q \geq 1$, $r \leq q$, if and only if

$$m_n(\mathscr{G})^{1/q} \leq C\big[s(\partial_i \mathscr{G}) + s(\partial_e \mathscr{G})^{1/r}\big]$$

for every admissible $\mathscr{G} \subset \Omega$, where $\partial_e \mathscr{G} = \partial \mathscr{G} \cap \partial\Omega$. Here the sharp inequality

$$\|u\|_{L_{n/(n-1)}(\Omega)} \leq \frac{[\Gamma(1+n/2)]^{1/n}}{n\sqrt{\pi}} \big(\|\nabla u\|_{L(\Omega)} + \|u\|_{L(\partial\Omega)}\big),$$

valid for an arbitrary bounded set Ω and $u \in W_{1,1}^1(\Omega, \partial\Omega)$, can be found.

5.1 Preliminaries

5.1.1 Notation

In this chapter, as well as in Chaps. 6 and 7, we shall use the symbols introduced in Sect. 1.1.1 and the following notation.

A bounded open subset $\mathscr{G}$ of the set Ω is called admissible if $\Omega \cap \partial\mathscr{G}$ is a manifold of the class C^∞ (this term was understood in a more restrictive sense in Chap. 2).

Let $\bar{E}$ be the closure of the set $E \subset \mathbb{R}^n$ and let ∂E be the boundary of E. Further, let $\mathrm{clos}_\Omega E$ be the closure of E in Ω and let $\partial_i E$ be the inner part of ∂E with respect to Ω, i.e., $\partial_i E = \Omega \cap \partial E$.

We put $\Omega_\varrho = \Omega \cap B_\varrho$, $u^+ = \max\{u, 0\}$, $u_- = u_+ - u$, $\mathscr{E}_t = \{x : |u(x)| = t\}$, $\mathscr{L}_t = \{x : |u(x)| > t\}$, $\mathscr{N}_t = \{x : |u(x)| \geq t\}$.

As before, we shall write

$$\|u\|_{L_q(\Omega)} = \left(\int_\Omega |u|^q \,\mathrm{d}x\right)^{1/q}.$$

This notation also will be used for $q \in (0,1)$ when the right-hand side is a pseudonorm. (We recall that a linear space is called pseudonormed if there is a functional $\|x\| \geq 0$, defined on its elements, which satisfies the conditions 1. if $\|x\| = 0$, then $x = 0$; 2. $\|\alpha x\| = |\alpha|\,\|x\|$, where $\alpha \in \mathbb{R}^1$; 3. if $\|x_m\| \to 0$, $\|y_m\| \to 0$, then $\|x_m + y_m\| \to 0$.) Clearly, in the case $0 < q < 1$ the functional

$$\varrho(u,v) = \int_\Omega |u-v|^q \,\mathrm{d}x$$

satisfies the axioms of a metric.

Let $C^{0,1}(\Omega)$ denote the space of functions that satisfy a Lipschitz condition on any compact subset of Ω.

If Ω is a domain, we endow the space $L_p^l(\Omega)$, $p \geq 1$, $l = 1,2,\dots$ (cf. Sect. 1.1.2) with the norm

$$\|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\omega)},$$

where ω is an open nonempty set with compact closure $\bar{\omega} \subset \Omega$. From (1.1.13) it follows that varying ω leads to an equivalent norm.

Further, let $W_{p,r}^l(\Omega) = L_p^l(\Omega) \cap L_r(\Omega)$ be the space equipped with the norm for $r \geq 1$ and with the psuedonorm for $r \in (0,1)$ as follows:

$$\|u\|_{W_{p,r}^l(\Omega)} = \|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_r(\Omega)}.$$

In accordance with Sect. 1.1.4, $W_{p,p}^l(\Omega) = W_p^l(\Omega)$. By Theorems 1.1.5/1 and 1.1.5/2, the sets $L_p^l(\Omega) \cap C^\infty(\Omega)$ and $W_p^l(\Omega) \cap C^\infty(\Omega)$ are dense in $L_p^l(\Omega)$ and $W_p^l(\Omega)$, respectively.

5.1.2 Lemmas on Approximation of Functions in $W_{p,r}^1(\Omega)$ and $L_p^1(\Omega)$

Lemma 1. *The set of functions in $L_p^1(\Omega) \cap C^\infty(\Omega) \cap L_\infty(\Omega)$ $(p \geq 1)$ with bounded supports is dense in $W_{p,r}^1(\Omega)$ $(\infty > r > 0)$.*

Proof. Let $v \in W_{p,r}^1(\Omega)$. We use the sequences $v^{(m)}$ and $v_{(m)}$ introduced in Lemma 1.7.1. Since $v^{(m)} \to v$ and $v_{(m)} \to v$ in $W_{p,r}^1(\Omega)$, the set of bounded functions $v \in L_p^1(\Omega)$ with $m_n(\operatorname{supp} v) < \infty$ is dense in $W_{p,r}^1(\Omega)$. Suppose v satisfies these conditions. We define the sequence

$$v_m(x) = \eta\big(m^{-1}x\big)v(x), \quad m = 1,2,\dots,$$

where $\eta \in C_0^\infty(B_2)$, $\eta = 1$ on B_1. Obviously,

$$\|v_m - v\|_{W_{p,r}^1(\Omega)} \leq c\|\nabla v\|_{L_p(\Omega\backslash B_m)} + c\,m^{-1}\|v\|_{L_\infty(\Omega)}\big[m_n(\operatorname{supp} v)\big]^{1/p} + \|v\|_{L_r(\Omega\backslash B_m)} \to 0,$$

as $m \to \infty$. To approximate each v_m by smooth functions it is sufficient to use a partition of unity and mollifying operators (cf. the proof of Theorem 1.1.5/1). □

From Lemma 1 we obtain the following corollary.

Corollary. *If Ω is domain with finite volume, then the set of functions in $L_p^1(\Omega) \cap C^\infty(\Omega) \cap L_\infty(\Omega)$ with bounded supports is dense in $L_p^1(\Omega)$.*

Lemma 2. *Let G be an open subset of Ω and let $u \in C^{0,1} \cap L_p^1(\Omega)$, $u = 0$ outside G. Then there exists a sequence of functions in $L_p^1(\Omega) \cap C^\infty(\Omega)$ that also vanish outside G which converges to u in $L_p^1(\Omega)$.*

Proof. Since u can be approximated in $L_p^1(\Omega)$ by the sequence $u_{(m)}$ defined in Lemma 1.7.1, we may assume that $u = 0$ outside some open set $g \subset G$ with $\mathrm{clos}_\Omega\, g \subset G$.

We let $\{\mathscr{B}^{(k)}\}$ denote a locally finite covering of g by open balls $\mathscr{B}^{(k)}$, $\overline{\mathscr{B}^{(k)}} \subset G$ and then we repeat the proof of Theorem 1.1.5/1. The lemma is proved. □

Remark. If we assume that the function u referred to in the statement of Lemma 2 is continuous on $\bar{\Omega}$, we may also assume the functions of an approximating sequence to have the same property (cf. Remark 1.1.5).

If, in addition to the condition of Lemma 2, $u \in L_r(\Omega)$ then the approximating sequence can be taken to be convergent in $W_{p,r}^1(\Omega)$.

Both of these assertions are immediate corollaries of Lemma 2 and are proved similarly.

5.2 Classes of Sets $\mathscr{J}_\alpha$, $\mathscr{H}_\alpha$ and the Embedding $L_1^1(\Omega) \subset L_q(\Omega)$

5.2.1 Classes $\mathscr{J}_\alpha$

Definition. A bounded domain Ω belongs to *the class* $\mathscr{J}_\alpha(\alpha \geq \frac{n-1}{n})$ if there exists a constant $M \in (0, m_n(\Omega))$ such that

$$\mathfrak{A}_\alpha(M) \stackrel{\text{def}}{=} \sup_{\{\mathscr{G}\}} \frac{m_n(\mathscr{G})^\alpha}{s(\partial_i \mathscr{G})} < \infty, \tag{5.2.1}$$

where $\{\mathscr{G}\}$ is a collection of admissible subsets of Ω with $m_n(\mathscr{G}) \leq M$ and s is the $(n-1)$-dimensional area.

The condition (5.2.1) gives a local characterization of the boundary of Ω. We briefly comment on this property. If Ω is a domain with sufficiently smooth boundary then it can be easily seen that the $(n-1)$-dimensional area of the surface $\partial\mathscr{G} \cap \partial\Omega$ is bounded from below (up to a constant factor) by the area of $\partial_i \mathscr{G} = \Omega \cap \partial \mathscr{G}$ for any $\mathscr{G}$ of sufficiently small volume. So by the classical isoperimetric inequality we have the *relative isoperimetric inequality*

$$m_n(\mathscr{G})^{(n-1)/n} \leq \text{const} \cdot s(\partial_i \mathscr{G})$$

and hence Ω belongs to the class $\mathscr{J}_{(n-1)/n}$. If $\partial\Omega$ has cusps directed into Ω, then it is intuitively clear that the last inequality still holds. If a cusp is directed outward from the domain then there exists a sequence of sets $\mathscr{G}_\nu \subset \Omega$ for which

$$\lim_{\nu\to\infty} \frac{m_n(\mathscr{G}_\nu)^{(n-1)/n}}{s(\partial_i \mathscr{G}_\nu)} = \infty.$$

Along with this property the domain may satisfy (5.2.1) for some $\alpha > (n-1)/n$. The exponent α characterizes the degree of sharpness of a cusp.

In what follows we shall see that it is relative (with respect to Ω) isoperimetric inequalities of the type (5.2.1) (and more complicated ones) that determine the order of integrability of functions in Sobolev spaces.

Lemma 1. *Let Ω be an open unit ball and let g be an open subset of Ω such that $\partial_i g$ is a manifold of the class $C^{0,1}$. Then*

$$\min\{m_n(g), m_n(\Omega\backslash g)\} \leq \frac{v_n}{2} v_{n-1}^{n/(1-n)} s(\partial_i g)^{n/(n-1)}. \tag{5.2.2}$$

The constant in (5.2.2) is the best possible. More generally, the minimum value of $s(\partial_i g)$ over all sets g with $m_n(g) = \text{const} < v_n/2$, is attained at the ball which is orthogonal to $\partial\Omega$.

Proof. It is sufficient to assume that $2m_n(g) < v_n$. Applying the spherical symmetrization of g with respect to a ray l that emanates from the center of the ball Ω, we obtain a set $f \subset \Omega$ symmetric with respect to l and such that

$$m_n(f) = m_n(g), \qquad s(\partial_i f) \leq s(\partial_i g).$$

(The spherical symmetrization with respect to a ray is defined similarly to the symmetrization with respect to an $(n-s)$-dimensional subspace introduced in Sect. 2.3.4; we just need to replace the s-dimensional subspaces orthogonal to $\mathbb{R}^{n-s}$ by $(n-1)$-dimensional spheres centered at the origin of the ray.)

Let b be a ball such that $b \cap \partial\Omega = \partial f \cap \partial\Omega$ and $m_n(b\cap\Omega) = m_n(f)$. Since

$$m_n\big[f \cup (b\backslash\Omega)\big] = m_n(b),$$

then by the isoperimetric property of the ball we have

$$s(\Omega \cap \partial b) \leq s(\partial_i f).$$

An elementary calculation shows that the minimum value of $s(\Omega \cap \partial b)$ over all balls with $m_n(\Omega\cap b) = \text{const} < \frac{1}{2} m_n(\Omega)$ is attained at the ball which is orthogonal to $\partial\Omega$.

It can be easily checked that the function

$$\frac{s(\Omega \cap \partial b_\varrho)}{[m_n(\Omega \cap b_\varrho)]^{(n-1)/n}},$$

where b_ϱ is a ball with radius ϱ, orthogonal to Ω and such that $m_n(b_\varrho \cap \Omega) < \frac{1}{2}m_n(\Omega)$, decreases. The proof is complete. □

The next corollary follows immediately from Lemma 1.

Corollary 1. *If Ω is an n-dimensional ball, then $\Omega \in \mathscr{J}_{(n-1)/n}$ and*

$$\mathfrak{A}_{(n-1)/n}\left(\frac{1}{2}m_n(\Omega)\right) = v_{n-1}^{-1}(v_n/2)^{(n-1)/n},$$

where v_s is the volume of the s-dimensional unit ball.

Corollary 2. *A bounded domain starshaped with respect to a ball belongs to the class $\mathscr{J}_{(n-1)/n}$.*

Proof. According to Lemma 1.1.8, the set $\bar{\Omega}$ is the quasi-isometric image of a ball. This along with Lemma 1 implies

$$m_n(g)^{(n-1)/n} \le C(M)s(\partial_i g) \tag{5.2.3}$$

for any constant $M \in (0, m_n(\Omega))$ and for all open sets $g \subset \Omega$ such that $\partial_i g$ is a manifold of the class $C^{0,1}$ and $m_n(g) \le M$. Here $C(M)$ is a constant independent of g. The corollary is proved. □

Remark. The condition (5.2.1) does not hold for $\alpha < (n-1)/n$ since in this case

$$\lim_{\varrho\to 0} \frac{m_n(B_\varrho)^\alpha}{s(\partial B_\varrho)} = c \lim_{\varrho \to 0} \varrho^{1-n+n\alpha} = \infty.$$

We give an example of a domain that does not belong to the class $\mathscr{J}_\alpha$ for any α.

Example. Consider the domain Ω depicted in Fig. 18. For the sequence of subsets Q_m $(m = 2, 4, \dots,)$, we have

$$m_2(Q_m) = m^{-4}, \qquad s(\partial_i Q_m) = m^{-2m},$$
$$\frac{m_2(Q_m)^\alpha}{s(\partial_i Q_m)} = m^{2m-4\alpha} \xrightarrow{m\to\infty} \infty,$$

and hence $\Omega \notin \mathscr{J}_\alpha$.

Lemma 2. *If a bounded open set is the union of a finite number of open sets of the class $\mathscr{J}_\alpha$ then it also belongs to $\mathscr{J}_\alpha$.*

Proof. Let $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_k \in \mathscr{J}_\alpha (k = 1, 2)$, and let $M = \min\{M_1, M_2\}$ where M_1 and M_2 are constants for Ω_1 and Ω_2 in the definition of the class $\mathscr{J}_\alpha$. Let $\mathscr{G}$ denote an admissible subset of Ω with $m_n(\mathscr{G}) \le M$.

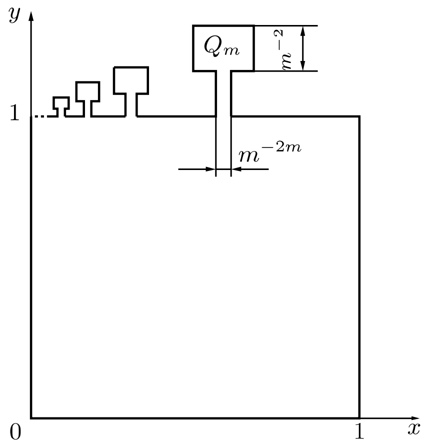

Fig. 18.

Since

$$m_n(\mathscr{G}) \le m_n(\mathscr{G} \cap \Omega_1) + m_n(\mathscr{G} \cap \Omega_2)$$

and $s(\Omega \cap \partial\mathscr{G}) \ge s(\Omega_k \cap \partial\mathscr{G})$, then

$$\frac{m_n(\mathscr{G})^\alpha}{s(\Omega \cap \partial\mathscr{G})} \le c \sum_{k=1}^{2} \frac{m_n(\Omega_k \cap \mathscr{G})^\alpha}{s(\Omega_k \cap \partial\mathscr{G})}. \qquad \square$$

This along with Lemma 1.1.9/1 and Corollary 5.2.1/2 implies the next corollary.

Corollary 3. *A bounded domain having the cone property belongs to the class* $\mathscr{J}_{(n-1)/n}$.

5.2.2 Technical Lemma

The following assertion will be used in Sect. 5.2.3.

Lemma. *Let* $\mathscr{G}$ *be an admissible subset of* Ω *such that* $s(\partial_i\mathscr{G}) < \infty$. *Then there exists a sequence of functions* $\{w_m\}_{m\ge1}$ *with the properties*:

1. $w_m \in C^{0,1}(\Omega)$,
2. $w_m = 0$ in $\Omega\backslash\mathscr{G}$,
3. $w_m \in [0,1]$ in Ω,
4. for any compactum $e \subset \mathscr{G}$ there exists a natural number $m(e)$, such that $w_m(x) = 1$ for $x \in e$ and $m \ge m(e)$,
5. $\lim_{m\to\infty} \sup \int_\Omega |\nabla w_m|\, dx = s(\partial_i\mathscr{G})$.

Proof. Let ω be a bounded open set, $\bar{\omega} \subset \Omega$, $s[(\Omega \backslash \omega) \cap \partial \mathscr{G}] < m^{-1}$. There exists a small positive number $\varepsilon = \varepsilon(m)$ such that for some locally finite (in Ω) covering of $(\Omega\backslash\omega) \cap \partial\mathscr{G}$ by open balls $\mathscr{B}_i$ with radii $r_i < \varepsilon$ the inequality

$$\sum_i r_i^{n-1} < 2m^{-1}, \tag{5.2.4}$$

holds. Obviously, we may assume that each ball $\mathscr{B}_i$ intersects $(\Omega\backslash\omega) \cap \partial\mathscr{G}$.

We introduce the notation $2\mathscr{B}_i$ is the ball with radius $2\,r_i$ concentric with $\mathscr{B}_i; C_1 = \bigcup_i \mathscr{B}_i$, $C_2 = \bigcup_i 2\mathscr{B}_i$; $\varrho(x) = \operatorname{dist}(x, \partial_i\mathscr{G})$.

Let $g = \{x : x \in \mathscr{G}, \varrho(x) < \delta\}$ where $\delta = \delta(m)$ is a small number, $\delta \in (0, \varepsilon)$, such that $g \cap \partial\omega$ is contained in C_1 (this can be achieved since the covering $\{\mathscr{B}_i\}$ is locally finite in Ω).

We construct a function $v(x)$, which is equal to zero in $\Omega\backslash\mathscr{G}$, to $\delta^{-1}\varrho(x)$ for $x \in g \cap \omega$, and to unity on the remaining portion of Ω. This function is discontinuous on the sets $(\Omega\backslash\omega) \cap \partial\mathscr{G}$ and $g \cap \partial\omega$. We eliminate this defect by using a truncating function $\eta(x)$ that is defined as follows.

Let $\eta_i \in C^{0,1}(\mathbb{R}^n)$, $\eta_i = 1$ outside $2\mathscr{B}_i$, $\eta = 0$ in $\mathscr{B}_i$, $0 \le \eta_i \le 1$, $|\nabla\eta_i| \le r_i^{-1}$ in $2\mathscr{B}_i$, and $\eta(x) = \inf_i\{\eta_i(x)\}$.

Obviously, $\eta \in C^{0,1}(\Omega)$, $\eta = 0$ in C_1, $\eta = 1$ outside C_2. Consider the function $w_m = \eta v$, which equals zero in $C_1 \cap \Omega$ and hence vanishes in the neighborhood of the set of discontinuities of v. Clearly,

$$\int_\Omega |\nabla w_m|\,\mathrm{d}x = \int_{\Omega\backslash C_1} \big|\nabla(\eta v)\big|\,\mathrm{d}x \le \int_{\Omega\backslash C_1} |\nabla\eta|\,\mathrm{d}x + \int_{\Omega\backslash C_1} |\nabla v|\,\mathrm{d}x.$$

We note that

$$\int_{\Omega\backslash C_1} |\nabla\eta|\,dx \le \sum_i \int_{\Omega\backslash C_1} |\nabla\eta_i|\,\mathrm{d}x \le \sum_i \int_{2\mathscr{B}_i} |\nabla\eta_i|\,\mathrm{d}x \le c\sum_i r_i^{n-1}.$$

Here and henceforth in this lemma c is a constant that depends only on n. The preceding inequalities along with (5.2.4) imply

$$\int_{\Omega\backslash C_1} |\nabla\eta|\,\mathrm{d}x \le cm^{-1}.$$

Further, since by Theorem 1.2.4

$$\int_{\Omega\backslash C_1} |\nabla v|\,\mathrm{d}x \le \delta^{-1}\int_{g\cap\omega} |\nabla\varrho|\,\mathrm{d}x = \delta^{-1}\int_0^\delta s(\Gamma_\tau)\,\mathrm{d}\tau,$$

where $\Gamma_\tau = \{x \in \omega \cap g : \varrho(x) = \tau\}$, it follows for sufficiently small $\delta = \delta(m)$ that

$$\int_{\Omega\backslash C_1} |\nabla v|\,\mathrm{d}x \le s(\partial_i\mathscr{G}) + cm^{-1}.$$

Finally we have

$$\int_\Omega |\nabla w_m| \,\mathrm{d}x \le s(\partial_i \mathscr{G}) + cm^{-1}.$$

The function w_m is equal to zero outside $\mathscr{G}$ and to unity outside an ε-neighborhood of $\Omega \cap \partial\mathscr{G}$. The lemma is proved. □

5.2.3 Embedding $L_1^1(\Omega) \subset L_q(\Omega)$

Let G be an open subset of Ω and let

$$\mathfrak{A}_G^{(\alpha)} \stackrel{\text{def}}{=} \sup \frac{[m_n(\mathscr{G})]^\alpha}{s(\partial_i \mathscr{G})}, \tag{5.2.5}$$

where the supremum is taken over all admissible sets $\mathscr{G}$ with $\operatorname{clos}_\Omega \mathscr{G} \subset G$.

Lemma 1. 1. *If $\mathfrak{A}_G^{(\alpha)} < \infty$, $\alpha \le 1$, then for all functions $u \in C^{0,1}(\Omega) \cap L_1^1(\Omega)$, equal to zero outside G,*

$$\|u\|_{L_q(\Omega)} \le C \|\nabla u\|_{L_1(\Omega)}, \tag{5.2.6}$$

where $q = \alpha^{-1}$ and $C \le \mathfrak{A}_G^{(\alpha)}$.

2. *If for all functions $u \in C^{0,1}(\Omega) \cap L_1^1(\Omega)$, equal to zero outside G, inequality (5.2.6) holds, then $C \ge \mathfrak{A}_G^{(\alpha)}$.*

Proof. 1. By Lemma 5.1.2/2, it is sufficient to prove (5.2.6) for functions $u \in C^\infty(\Omega) \cap L_1^1(\Omega)$ that are equal to zero outside G.

Since

$$\|u\|^q_{L_q(\Omega)} = \int_0^\infty m_n(\mathscr{N}_t) \,\mathrm{d}\big(t^q\big),$$

(1.3.5) implies

$$\|u\|_{L_q(\Omega)} \le \int_0^\infty m_n(\mathscr{N}_t)^{1/q} \,\mathrm{d}t.$$

Now we note that, for almost all $t > 0$,

$$m_n(\mathscr{N}_t)^{1/q} \le \mathfrak{A}_G^{(\alpha)} s(\mathscr{E}_t).$$

From this inequality and Theorem 1.2.4 we obtain

$$\|u\|_{L_q(\Omega)} \le \mathfrak{A}_G^{(\alpha)} \int_0^\infty s(\mathscr{E}_t) \,\mathrm{d}t = \mathfrak{A}_G^{(\alpha)} \|\nabla u\|_{L_1(\Omega)}.$$

2. Let $\mathscr{G}$ be an admissible subset of G with $\operatorname{clos}_\Omega \mathscr{G} \subset G$. We insert the sequence $\{w_m\}_m \ge 1$ specified in Lemma 5.2.2 into (5.2.6). For any compactum $e \subset \mathscr{G}$ we have

$$\left(\int_e |w_m|^q \,\mathrm{d}x \right)^{1/q} \le Cs(\partial_i \mathscr{G})$$

and hence $m_n(\mathscr{G})^{1/q} \le Cs(\partial_i \mathscr{G})$. The lemma is proved. □

For any domain Ω with finite volume we put

$$\mathfrak{A}_\alpha = \mathfrak{A}_\alpha\Big(\frac{1}{2}m_n(\Omega)\Big).$$

Theorem. *Let Ω be a domain with $m_n(\Omega) < \infty$.*

1. If $\Omega \in \mathscr{J}_\alpha$, where $\alpha \in [(n-1)/n, 1]$, then, for all $u \in L_1^1(\Omega)$,

$$\inf_{c\in\mathbb{R}^1} \|u-c\|_{L_q(\Omega)} \le C\|\nabla u\|_{L_1(\Omega)}, \tag{5.2.7}$$

where $q = \alpha^{-1}$ and $C \le \mathfrak{A}_\alpha$.

2. If for all $u \in L_1^1(\Omega)$ inequality (5.2.7) is true, then $\Omega \in \mathscr{J}_\alpha$ with $\alpha = q^{-1}$ and $2^{(q-1)/q}C \ge \mathfrak{A}_\alpha$ for $q>1$, $C \ge \mathfrak{A}_\alpha$ for $0<q\le 1$.

Proof. 1. By Corollary 5.1.2, it is sufficient to obtain (5.2.7) for functions $u \in L_1^1(\Omega)\cap L_\infty(\Omega)\cap C^\infty(\Omega)$ with bounded supports. Let τ denote a number such that

$$2m_n\big(\{x : u(x) \ge \tau\}\big) \ge m_n(\Omega),$$
$$2m_n\big(\{x : u(x) > \tau\}\big) \le m_n(\Omega).$$

According to Lemma 1,

$$\Big(\int_\Omega (u-\tau)_+^q \,\mathrm{d}x\Big)^{1/q} \le \mathfrak{A}_\alpha \int_{\{x:u(x)>\tau\}} |\nabla u|\,\mathrm{d}x$$

and

$$\Big(\int_\Omega (\tau-u)_+^q \,\mathrm{d}x\Big)^{1/q} \le \mathfrak{A}_\alpha \int_{\{x:u(x)<\tau\}} |\nabla u|\,\mathrm{d}x,$$

which completes the proof of the first part of the theorem.

2. For all $u \in L_1^1(\Omega)$, let inequality (5.2.7) hold and let $\mathscr{G}$ be any admissible subset of Ω with $2m_n(\mathscr{G}) \le m_n(\Omega)$. We insert the sequence $\{w_m\}$ specified in Lemma 5.2.2 into (5.2.7). For any compactum $e \subset \mathscr{G}$ we have

$$\inf_{c\in\mathbb{R}^1}\Big(\int_e |1-c|^q\,\mathrm{d}x + \int_{\Omega\backslash\mathscr{G}} |c|^q\,\mathrm{d}x\Big)^{1/q} \le Cs(\partial_i\mathscr{G})$$

and consequently,

$$\min_c\big(|1-c|^q m_n(\mathscr{G}) + |c|^q m_n(\Omega\backslash\mathscr{G})\big)^{1/q} \le Cs(\partial_i\mathscr{G}).$$

The minimum value of the left-hand side with $q>1$ is attained at

$$c = \frac{m_n(\mathscr{G})^{1/(q-1)}}{m_n(\mathscr{G})^{1/(q-1)} + m_n(\Omega\backslash\mathscr{G})^{1/(q-1)}}.$$

Hence

$$\frac{[m_n(\mathscr{G})m_n(\Omega\backslash\mathscr{G})]^{1/q}}{[m_n(\mathscr{G})^{1/(q-1)} + m_n(\Omega\backslash\mathscr{G})^{1/(q-1)}]^{(q-1)/q}} \le Cs(\partial_i\mathscr{G}).$$

Taking into account the condition $2m_n(\mathscr{G}) \le m_n(\Omega)$, we obtain

$$m_n(\mathscr{G})^{1/q} \le 2^{(q-1)/q} Cs(\partial_i\mathscr{G}).$$

The case $0 < q \le 1$ is treated similarly. The theorem is proved. □

Lemma 2. *Let Ω be a domain with $m_n(\Omega) < \infty$. The space $L_p^1(\Omega)$ is embedded into $L_q(\Omega)$ $(p \ge 1,\ q > 0)$ if and only if*

$$\inf_{c\in\mathbb{R}^1} \|u - c\|_{L_q(\Omega)} \le C\|\nabla u\|_{L_p(\Omega)} \tag{5.2.8}$$

for all $u \in L_p^1(\Omega)$.

Proof. Necessity. Let $\mathscr{Z}$ be the subspace of functions equal to a constant on Ω and let $\dot{W}_{p,q}^1(\Omega)$ be the factor space $W_{p,q}^1(\Omega)/\mathscr{Z}$, equipped with the norm

$$\inf_{c\in\mathscr{Z}} \|u - c\|_{L_q(\Omega)} + \|\nabla u\|_{L_p(\Omega)}.$$

Let $\mathscr{E}$ denote the identity mapping of $\dot{W}_{p,q}^1$ into $\dot{L}_q^1(\Omega)$. This mapping is linear, continuous, and one to one. Since $L_q^1(\Omega) \subset L_q(\Omega)$, we see that $\mathscr{E}$ is surjective. By the Banach theorem (cf. Bourbaki [128], I, 3, 3), $\mathscr{E}$ is an isomorphism and hence (5.2.8) holds.

Sufficiency. Let (5.2.8) be true. We must show that $\mathscr{E}$ is surjective. By (5.2.8), the image of $\dot{W}_{p,q}^1(\Omega)$ is closed in $L_p^1(\Omega)$. So it is sufficient to take into account that, by Corollary 5.1.2, the space $\dot{W}_{p,q}^1(\Omega)$ considered as a subspace of $\dot{L}_p^1(\Omega)$ is dense in $\dot{L}_p^1(\Omega)$. The lemma is proved. □

Theorem and Lemma 2 immediately imply the next corollary.

Corollary. *If Ω is a domain with $m_n(\Omega) < \infty$ then $L_1^1(\Omega)$ is embedded into $L_q(\Omega)$, $q \ge 1$, if and only if*

$$\sup \frac{m_n(\mathscr{G})^{1/q}}{s(\partial_i\mathscr{G})} < \infty,$$

where the supremum is taken over all admissible subsets $\mathscr{G}$ of Ω with $m_n(\mathscr{G}) \le \frac{1}{2}m_n(\Omega)$.

Remark. Since a planar domain Ω bounded by a quasicircle belongs to the class EV_1^1 (cf. Sect. 1.4.8), the embedding $L_1^1(\Omega) \subset L_2(\Omega)$ holds. The last assertion along with Lemma 5.2.1/2 and the just-formulated Corollary implies that the union of a finite number of quasidisks belongs to the class $\mathscr{J}_{1/2}$.

5.2.4 Area Minimizing Function λ_M and Embedding of $L_1^1(\Omega)$ into $L_q(\Omega)$

Definition. Let $M \in (0, m_n(\Omega))$. By $\lambda_M(\mu)$ we denote the greatest lower bound of the numbers $s(\partial_i \mathscr{G})$ considered over all admissible sets $\mathscr{G} \subset \Omega$ that satisfy the condition $\mu \le m_n(\mathscr{G}) \le M$.

Obviously, $\lambda_M(\mu)$ is nondecreasing in μ and nonincreasing in M.

We can give an equivalent definition of the class $\mathscr{J}_\alpha$ in terms of the function λ_M. Namely, $\Omega \in \mathscr{J}_\alpha$ if and only if

$$\liminf_{\mu \to +0} \mu^{-\alpha} \lambda_M(\mu) > 0. \tag{5.2.9}$$

Lemma. *If $M \in (0, m_n(\Omega))$ and Ω is a domain with finite volume, then*

$$\lambda_M(\mu) > 0 \quad \textit{for all } \mu \in (0, M].$$

Proof. Let $0 < \mu \le \min\{M, m_n(\Omega) - M\}$ and let ω be a domain with a smooth boundary such that $\bar{\omega} \subset \Omega$ and $2m_n(\Omega \backslash \omega) < \mu$. If $\mathscr{G}$ is an admissible subset of Ω, $M \ge m_n(\mathscr{G}) \ge \mu$, then obviously,

$$2m_n(\mathscr{G} \cap \omega) \ge \mu \quad \text{and} \quad 2m_n(\omega \backslash \mathscr{G}) \ge m_n(\Omega) - M \ge \mu.$$

Since $\partial\omega$ is a smooth surface, we have

$$\inf_{c \in \mathbb{R}^1} \|u - c\|_{L(\omega)} \le C(\omega) \|\nabla u\|_{L(\omega)}$$

for all $u \in L_1^1(\omega)$. Hence, according to the second part of Theorem 5.2.3,

$$\min\bigl\{m_n(\mathscr{G} \cap \omega), m_n(\omega \backslash \mathscr{G})\bigr\} \le C(\omega) s(\omega \cap \partial \mathscr{G})$$

for any admissible subset $\mathscr{G}$ of Ω. Thus,

$$\mu \le C(\omega) s(\Omega \cap \partial \mathscr{G})$$

and therefore $\lambda_M(\mu) > 0$ for small values of μ. Since $\lambda_M(\mu)$ is nondecreasing in μ, we conclude that $\lambda_M(\mu) > 0$ for all $\mu \in (0, M]$. The lemma is proved. □

It can be easily seen that the condition of the connectedness of Ω as well as the condition $m_n(\Omega) < \infty$ are essential for the validity of the lemma.

From the lemma we immediately obtain the next corollary.

Corollary. *If Ω is a domain of the class $\mathscr{J}_\alpha$ and $m_n(\Omega) < \infty$, then the value*

$$\sup\bigl\{m_n(\mathscr{G})^\alpha / s(\partial_i \mathscr{G}) : \mathscr{G} \textit{ is an admissible subset of } \Omega,\ m_n(\mathscr{G}) \le M\bigr\}$$

is finite for arbitrary constant $M \in (0, m_n(\Omega))$.

In what follows we shall write $\lambda(\mu)$ instead of $\lambda_{m_n(\Omega)/2}(\mu)$. The next criterion is proved exactly in the same way as Theorem 2.1.4.

Theorem. *Let Ω be a domain with $m_n(\Omega) < \infty$.*
1. *If*

$$\mathfrak{B}_\alpha := \int_0^{m_n(\Omega)/2} \left(\frac{s^\alpha}{\lambda(s)} \right)^{\frac{1}{\alpha-1}} \frac{\mathrm{d}s}{s} < \infty, \tag{5.2.10}$$

then (5.2.7) *holds for all $u \in L_1^1(\Omega)$ with $q = 1/\alpha$ and $C \le c\mathfrak{B}_\alpha^{\alpha-1}$.*
2. *If there is a constant C such that* (5.2.7) *holds with $q < 1$ for all $u \in L_1^1(\Omega)$, then* (5.2.10) *holds with $\alpha = q^{-1}$ and $C \ge c\mathfrak{B}_\alpha^{\alpha-1}$.*

It can be easily verified by the previous Lemma that (5.2.10) holds simultaneously with the condition

$$\int_0^M \left(\frac{s^\alpha}{\lambda_M(s)} \right)^{\frac{1}{\alpha-1}} \frac{\mathrm{d}s}{s} < \infty$$

for every M in $(0, m_n(\Omega))$.

We shall say that domains satisfying (5.2.10) belong to the class $\mathcal{H}_\alpha$.

5.2.5 Example of a Domain in $\mathscr{J}_1$

We shall show that the union of Ω of the squares $Q_m = \{(x,y) : 2^{-m-1} \le x \le 3\cdot 2^{-m-2},\ 0 < y < 2^{-m-2}\}$ and the rectangles $R_m = \{(x,y) : 3\cdot 2^{-m-2} < x < 2^{-m}, 0 < y < 1\}$ $(m = 0, 1, \dots)$ (Fig. 19) belongs to the class $\mathscr{J}_1$ and does not belong to $\mathscr{J}_\alpha$ for $\alpha < 1$.

Let T be the triangle $\{(x,y) : 0 < y < x/3,\ 0 < x < 1\}$ contained in Ω. Further let $v = u$ on $\Omega \backslash T$ and $v = 3yx^{-1}u$ on T. Clearly,

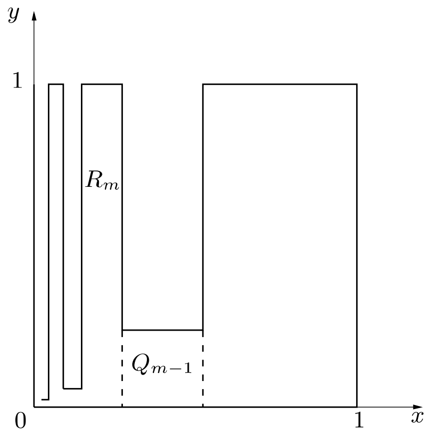

Fig. 19.

$$\|v\|_{L(\Omega)} \le \left\|\frac{\partial v}{\partial y}\right\|_{L(\Omega)} \le \left\|\frac{\partial u}{\partial y}\right\|_{L(\Omega)} + c\left\|\frac{u}{r}\right\|_{L(T)},$$

where $r = (x^2 + y^2)^{1/2}$. Consequently,

$$\|u\|_{L(\Omega)} \le \|\nabla u\|_{L(\Omega)} + c\left\|\frac{u}{r}\right\|_{L(T)}.$$

Since by the obvious inequality

$$\int_0^1 |w(r)|\,dr \le \int_0^1 |w'(r)|r\,dr, \quad w(1) = 0,$$

$L_1^1(\Omega)$ is embedded into the space with the norm $\|r^{-1}u\|_{L(T)}$, it follows that $L_1^1(\Omega) \subset L(\Omega)$. Applying Lemma 5.2.3/2 we obtain $\Omega \in \mathscr{J}_1$. On the other hand, the rectangles $G_m = R_m \cap \{(x, y) : y > 2^{-m-1}\}$ satisfy $\lim m_2(G_m)/s(\partial_i G_m) = 1$. Therefore $c_1\mu \le \lambda(\mu) \le c_2\mu$ for small values of μ. Hence $L_1^1(\Omega)$ is embedded into $L_1(\Omega)$ and is not embedded into $L_q(\Omega)$ with $q > 1$.

5.3 Subareal Mappings and the Classes $\mathscr{J}_\alpha$ and $\mathscr{H}_\alpha$

In this section we introduce and study properties of "subareal" mappings of a domain, i.e., the mappings that do not essentially enlarge the $(n-1)$-dimensional area of surfaces. We shall use these mappings to verify the conditions for concrete domains to be in $\mathscr{J}_\alpha$ and $\mathscr{H}_\alpha$.

5.3.1 Subareal Mappings

Consider a locally quasi-isometric mapping

$$\Omega \ni x \to \xi \in \mathbb{R}^n.$$

Let ξA denote the image of an arbitrary set $A \subset \Omega$ under the mapping ξ. Let ξ'_x be the matrix $(\partial\xi_i/\partial x_k)_{i,k=1}^n$ and let $\det \xi'_x$ be the Jacobian of ξ. The notations x'_ξ and $\det x'_\xi$ have a similar meaning.

Definition. The mapping ξ is called *subareal* if there exists a constant k such that

$$s(\xi\partial_i\mathscr{G}) \le ks(\partial_i\mathscr{G}) \tag{5.3.1}$$

for any admissible subset $\mathscr{G}$ of Ω.

Lemma 1. *The mapping ξ is a subareal if and only if*

$$|\det x'_\xi| \ge k\|x'_\xi\|, \tag{5.3.2}$$

for almost all $x \in \Omega$, where $\|\cdot\|$ is the norm of a matrix.

To prove this lemma we need the following assertion.

Lemma 2. *Let $u \in C^\infty(\Omega) \cap L_1^1(\Omega)$ and let E be any measurable subset of Ω. In order that*

$$\int_E |\nabla_x u| \, dx \geq k \int_{\xi E} |\nabla_\xi u| \, d\xi \tag{5.3.3}$$

with a constant k independent of u and E, it is necessary and sufficient for the mapping ξ to satisfy (5.3.2). Moreover, (5.3.2) follows from the validity of (5.3.3) for any ball in Ω.

Proof. The sufficiency results immediately from (5.3.2) along with

$$\int_{\xi E} |\nabla_\xi u| \, d\xi = \int_E \Big| \sum_{i=1}^n \frac{\partial u}{\partial x_i} \nabla_\xi x_i \Big| |\det \xi'_x| \, dx$$
$$= \int_E |\nabla_x u| \big|(x'_\xi)^* \alpha\big| |\det x'_\xi|^{-1} \, dx,$$

where $\alpha = |\nabla_x u|^{-1} \nabla_x u$ and $(\,)^*$ stands for the transposed matrix.

Necessity. We fix a unit vector α and consider the ball $B_\varrho(x_0) \subset \Omega$. We put $u(x) = \alpha x$. By virtue of (5.3.3)

$$\int_{B_\varrho(x_0)} dx \geq k \int_{\xi B_\varrho(x_0)} |\nabla_\xi u| \, d\xi = k \int_{B_\varrho(x_0)} \big|(x'_\xi)^* \alpha\big| |\det x'_\xi|^{-1} \, dx.$$

Passing to the limit as $\varrho \to 0$ and using the fact that α is arbitrary, we obtain (5.3.3) for almost all $x_0 \in \Omega$. The lemma is proved. □

Proof of Lemma 1. Sufficiency. Let $\mathscr{G}$ be an arbitrary admissible subset of Ω. We choose an arbitrary point $y \in \partial_i \mathscr{G}$. Let $\mathscr{U}_y$ be a neighborhood of y so small that the set $\overline{\mathscr{U}_y \cap \mathscr{G}}$ is represented by the inequality $x_n \leq f(x_1, \dots, x_{n-1})$ in some Cartesian coordinate system with an infinitely differentiable f.

Let δ be a fixed small positive number. Let $\mathscr{V}_\varepsilon$ denote the set of points $x \in \mathscr{U}_y$ defined by the equation $x_n = \varepsilon + f(x_1, \dots, x_{n-1})$ where $\varepsilon \in [0, \delta]$. We put

$$u_\delta(x) = \begin{cases} 1 & \text{for } x_n < f(x_1, \dots, x_{n-1}), \\ 1 - \varepsilon\delta^{-1} & \text{for } x \in \mathscr{V}_\varepsilon, \\ 0 & \text{for } x_n \geq f(x_1, \dots, x_{n-1}) + \delta. \end{cases}$$

Obviously, $u_\delta(x)$ is a Lipschitz function. By (5.3.2) and Lemma 2,

$$\int_{\mathscr{U}_y} |\nabla_x u_\delta| \, dx \geq k \int_{\xi \mathscr{U}_y} |\nabla_\xi u_\delta| \, d\xi.$$

Using Theorem 1.2.4, we can rewrite the last inequality as

$$\delta^{-1}\int_0^\delta s(\mathscr{V}_\varepsilon)\,\mathrm{d}\varepsilon \geq k\delta^{-1}\int_0^\delta s(\xi\mathscr{V}_\varepsilon)\,\mathrm{d}\varepsilon.$$

Since $s(\mathscr{V}_\varepsilon)$ is continuous, we have

$$s(\mathscr{V}_0) \geq k\liminf_{\delta\to 0}\delta^{-1}\int_0^\delta s(\xi\mathscr{V}_\varepsilon)\,\mathrm{d}\varepsilon.$$

Taking into account the lower semicontinuity of the area, we obtain

$$s(\mathscr{V}_0) \geq ks(\xi\mathscr{V}_0).$$

The latter implies (5.3.1) since y is an arbitrary point.

Necessity. Let ξ be a subareal mapping of Ω. Consider an arbitrary $u \in C^\infty(\Omega)\cap L_1^1(\Omega)$. According to Theorem 1.2.2, the level sets of $|u|$ are smooth manifolds for almost all $t > 0$. Let B be an arbitrary ball in Ω. By Theorem 1.2.4,

$$\int_B |\nabla_x u|\,\mathrm{d}x = \int_0^\infty s(B\cap\mathscr{E}_t)\,\mathrm{d}t,$$
$$\int_{\xi B} |\nabla_\xi u|\,\mathrm{d}\xi = \int_0^\infty s(\xi B\cap\xi\mathscr{E}_t)\,\mathrm{d}t.$$

By the definition of subareal mappings, $s(\mathscr{E}_t) \geq ks(\xi\mathscr{E}_t)$. Consequently, (5.3.3) holds. Now it remains to refer to Lemma 2. Lemma 1 is proved. □

5.3.2 Estimate for the Function λ in Terms of Subareal Mappings

The following theorem yields lower bounds for the function λ, introduced at the end of Sect. 5.2.4.

Theorem. *Let Ω be a domain with finite volume for which there exists a subareal mapping onto a bounded domain $\xi\Omega$ starshaped with respect to a ball.*

We put

$$\pi(\mu) = \inf_{\{\mathscr{G}\}}\int_{\mathscr{G}} |\det\xi'_x|\,\mathrm{d}x,$$

where the infimum is taken over all admissible subsets $\mathscr{G}$ of Ω such that

$$\mu \leq m_n(\mathscr{G}) \leq \frac{1}{2}m_n(\Omega).$$

Then there exists a constant Q such that

$$Q\lambda(\mu) \geq \pi(\mu)^{(n-1)/n}. \tag{5.3.4}$$

Proof. Let $\mathscr{G}$ be an admissible subset of Ω with $m_n(\mathscr{G}) < \frac{1}{2}m_n(\Omega)$. We put

$$M = \sup_{\{\mathscr{G}\}} m_n(\xi\mathscr{G}).$$

Since $|\det \xi'_x| \geq \text{const} > 0$ on any compact subset of Ω, it follows that $M < m_n(\xi\Omega)$. Taking into account the fact that $\xi\Omega$ is starshaped with respect to a ball and the fact that $\xi\mathscr{G}$ is a manifold of the class $C^{0,1}$, by (5.2.2) we obtain

$$\big[m_n(\xi\mathscr{G})\big]^{(n-1)/n} \leq Cs\big(\xi(\partial_i\mathscr{G})\big).$$

The latter along with (5.3.1) implies

$$\big[m_n(\xi\mathscr{G})\big]^{(n-1)/n} \leq Cks(\partial_i\mathscr{G}).$$

Thus, if $m_n(\mathscr{G}) \geq \mu$, then

$$\pi(\mu)^{(n-1)/n} \leq \big[m_n(\xi\mathscr{G})\big]^{(n-1)/n} \leq Ck\lambda(\mu).$$

The theorem is proved. □

5.3.3 Estimates for the Function λ for Special Domains

We give some applications of Theorem 5.3.2.

Example 1. Consider the domain

$$\Omega = \left\{x \in \mathbb{R}^n : \left(\sum_{i=1}^{n-1} x_i^2\right)^{1/2} < f(x_n),\ 0 < x_n < a\right\},$$

where f is a nonnegative convex function with $f'(a-0) < \infty$ and $f(0) = 0$ (cf. Fig. 20).

We show that the function $\lambda(\mu)$ specified for Ω satisfies

$$k\big[f\big(\alpha(\mu)\big)\big]^{n-1} \leq \lambda(\mu) \leq \big[f\big(\alpha(\mu)\big)\big]^{n-1}, \tag{5.3.5}$$

where $k \in (0,1)$ and $\alpha(\mu)$ is defined by

$$\mu = v_{n-1}\int_0^{\alpha(\mu)} \big[f(\tau)\big]^{n-1}\,\mathrm{d}\tau.$$

Proof. Consider the domain $G_t = \Omega \cap \{x : 0 < x_n < t\}$, where t is subject to the condition

$$m_n(G_t) = v_{n-1}\int_0^t \big[f(\tau)\big]^{n-1}\,\mathrm{d}\tau \leq \frac{1}{2}m_n(\Omega).$$

Obviously,

$$s(\Omega \cap \partial G_t) = v_{n-1}\big[f(t)\big]^{n-1}.$$

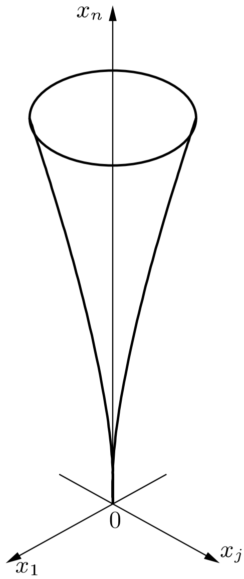

Fig. 20.

Since by the definition of $\lambda(\mu)$, $\lambda[m_n(G_t)] \le s(\Omega \cap \partial G_t)$, the right-hand side of (5.3.5) is proved.

The mapping $x \to \xi = (x_1, \dots, x_{n-1}, f(x_n))$ maps Ω onto the truncated cone

$$\xi\Omega = \left\{ \xi : \sum_{i=1}^{n-1} \xi_i^2 < \xi_n^2,\ 0 < \xi_n < f(a) \right\}.$$

The condition (5.3.2) is equivalent to the boundedness of f'. Hence the mapping $x \to \xi$ is subareal.

It remains to give the lower bound for the function $\pi(\mu)$ defined in Theorem 5.3.2. Since the Jacobian of the mapping $x \to \xi$ is equal to $f'(x_n)$, we must estimate the integral

$$\int_{\mathscr{G}} f'(x_n)\,\mathrm{d}x, \tag{5.3.6}$$

from below, provided $\frac{1}{2} m_n(\Omega) > m_n(\mathscr{G}) \ge \mu$. Since f' is nondecreasing, the integral (5.3.6) attains its minimum value at the set $G_{\alpha(\mu)}$. Therefore,

$$\int_{\mathscr{G}} f'(x_n)\,\mathrm{d}x \ge \int_0^{\alpha(\mu)} f'(c)\,\mathrm{d}c \int_{|x'| \le f(c)} \mathrm{d}x' = \frac{v_{n-1}}{n} \big[f\big(\alpha(\mu)\big)\big]^n.$$

Thus

$$\pi(\mu) \ge n^{-1} v_{n-1} \big[f\big(\alpha(\mu)\big)\big]^n.$$

Applying Theorem 5.3.2 to the last inequality we obtain the left-hand side of (5.3.5). The proof is complete. □

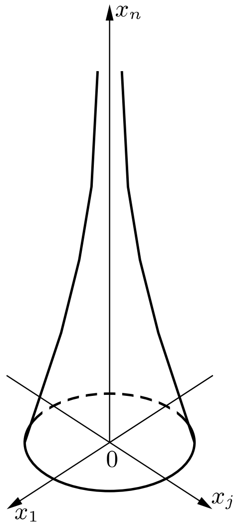

Fig. 21.

For the domain

$$\Omega = \left\{ x : \sum_{i=1}^{n-1} x_i^2 < x_n^{2\beta},\ 0 < x_n < a \right\}, \quad \beta \geq 1,$$

from (5.3.5) it immediately follows that

$$c_1\mu^\alpha \leq \lambda(\mu) \leq c_2\mu^\alpha, \quad \alpha = \frac{\beta(n-1)}{\beta(n-1)+1}, \tag{5.3.7}$$

for small values of μ. Consequently, $\Omega \in \mathscr{J}_{\frac{\beta(n-1)}{\beta(n-1)+1}}$ and $\Omega \notin \mathscr{J}_\alpha$ for $\alpha < \frac{\beta(n-1)}{\beta(n-1)+1}$.

Example 2. Consider the domain

$$\Omega = \left\{ x : 0 < x_n < \infty,\ \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} < f(x_n) \right\},$$

where f is the nonnegative convex function with $f'(+0) > -\infty$ and $f(+\infty) = 0$ (cf. Fig. 21). Suppose $m_n(\Omega) < \infty$, i.e.,

$$\int_0^\infty \big[f(t)\big]^{n-1}\,\mathrm{d}t < \infty.$$

We can show that

$$k\big[f\big(\alpha(\mu)\big)\big]^{n-1} \leq \lambda(\mu) \leq \big[f\big(\alpha(\mu)\big)\big]^{n-1}, \tag{5.3.8}$$

where $k \in (0,1)$ and $\alpha(\mu)$ is defined by

$$\mu = v_{n-1} \int_{\alpha(\mu)}^{\infty} \big[f(t)\big]^{n-1}\, \mathrm{d}t.$$

The proof is the same as in the previous example. The role of the auxiliary subareal mapping is played by

$$x \to \xi = \big(x_1, \dots, x_{n-1}, -f(x_n)\big)$$

onto the truncated cone

$$\Big\{\xi : \sum_{i=1}^{n-1} \xi_i^2 < \xi_n^2,\ 0 > \xi_n > -f(0)\Big\}.$$

By (5.3.8) and Theorem 5.2.4, $\Omega \in \mathcal{H}_\alpha$, $\alpha > 1$, i.e., $L_1^1(\Omega)$ is embedded into $L_q(\Omega)$, $q = 1/\alpha$, if and only if

$$\int_1^{\infty} \frac{f(t)^{(n-1)(1+\frac{\alpha}{\alpha-1})}\, \mathrm{d}t}{(\int_t^{\infty} f(\tau)^{n-1}\, \mathrm{d}\tau)^{\frac{\alpha}{\alpha-1}}} < \infty. \tag{5.3.9}$$

In particular, for the domain

$$\Omega = \big\{x : x_1^2 + \dots + x_{n-1}^2 < (1+x_n)^{-2\beta},\ 0 < x_n < \infty\big\}, \qquad \beta(n-1) > 1,$$

we have

$$c_1 \mu^{\alpha} \le \lambda(\mu) \le c_2 \mu^{\alpha}, \quad \alpha = \frac{\beta(n-1)}{\beta(n-1)-1}, \tag{5.3.10}$$

for small μ, i.e., $\Omega \in \mathscr{J}_{\frac{\beta(n-1)}{\beta(n-1)-1}}$ and $\Omega \notin \mathscr{J}_\alpha$ for $\alpha < \frac{\beta(n-1)}{\beta(n-1)-1}$.

Example 3. Consider the plane spiral domain Ω (cf. Fig. 22) defined in polar coordinates by

$$1 - \varepsilon_1(\theta) > \varrho > 1 - \varepsilon_2(\theta), \quad 0 < \theta < \infty.$$

Here $0 < \varepsilon_2(\theta + 2\pi) < \varepsilon_1(\theta) < \varepsilon_2(\theta) < 1$, ε_1, ε_2 are functions satisfying a uniform Lipschitz condition on $[0, \infty)$ and such that $\varepsilon_2 - \varepsilon_1$ is convex on $[0, \infty)$. Further, we suppose that the area of Ω is finite, i.e.,

$$\int_0^{\infty} \big(\varepsilon_2(\theta) - \varepsilon_1(\theta)\big)\, \mathrm{d}\theta < \infty.$$

Applying Theorem 5.3.2 to the subareal mapping ξ

$$\xi_1 = 1 - \varrho - \frac{1}{2}\big[\varepsilon_1(\theta) + \varepsilon_2(\theta)\big], \qquad \xi_2 = \frac{1}{2}\big[\varepsilon_2(\theta) - \varepsilon_1(\theta)\big],$$

of Ω onto the triangle

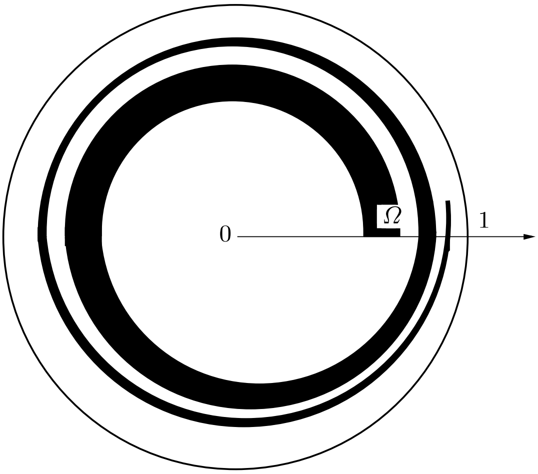

Fig. 22.

$$|\xi_1| < \xi_2, \qquad 0 < \xi_2 < \frac{1}{2}\big[\varepsilon_2(\theta) - \varepsilon_1(\theta)\big],$$

and using the same arguments as in Example 1, we obtain

$$c_1\big[\varepsilon_2\big(\theta(\mu)\big) - \varepsilon_1\big(\theta(\mu)\big)\big] \le \lambda(\mu) \le c_2\big[\varepsilon_2\big(\theta(\mu)\big) - \varepsilon_1\big(\theta(\mu)\big)\big], \qquad (5.3.11)$$

where $\theta(\mu)$ is the function defined by

$$\mu = \iint_{\{\varrho e^{i\theta}\in\Omega:\theta>\theta(\mu)\}} \varrho \,\mathrm{d}\varrho\,\mathrm{d}\theta. \qquad (5.3.12)$$

In particular, for the domain

$$\big\{\varrho e^{i\theta} : 1-(8+\theta)^{1-\beta} > \varrho > 1-(8+\theta)^{1-\beta} - c(8+\theta)^{-\beta},\ 0<\theta<\infty\big\}, \quad (5.3.13)$$

where $0 < c < 2\pi(\beta-1)$, $\beta > 1$, we have $c_1\mu^\alpha \le \lambda(\mu) \le c_2\mu^\alpha$ with $\alpha = \beta/(\beta-1)$ for small μ. Thus, the domain (5.3.13) belongs to $\mathscr{J}_{\beta/(\beta-1)}$.

A more complicated example of a domain in $\mathscr{J}_\alpha$, $\alpha>1$, is considered in the following section.

Remark. Incidentally, if $x \to \xi$ is a subareal mapping of Ω onto a bounded starshaped domain, then, obviously, $L_1^1(\Omega)$ is embedded into the space with the norm

$$\Big(\int_\Omega |u|^{n/(n-1)} |\det \xi'_x| \,\mathrm{d}x\Big)^{(n-1)/n}.$$

In particular, for domains in Examples 1 and 2 we have

$$L_1^1(\Omega) \subset L_{n/(n-1)}\big(\Omega, \big|f'(x_n)\big|\,\mathrm{d}x\big).$$

This is an illustration of the natural idea that the limit exponent $n/(n-1)$ is also preserved for "bad" domains if we consider a weighed Lebesgue measure, degenerating at "bad" boundary points, instead of m_n.

In connection with this remark we note that, although the present chapter deals with the problem of the integrability of functions in $L_1^1(\Omega)$ with respect to Lebesgue measure, the proofs from Sect. 5.2.3 do not change essentially after replacing $L_q(\Omega)$ by the space $L_q(\Omega, \mu)$, where μ is an arbitrary measure in Ω (cf. Chap. 2).

5.4 Two-Sided Estimates for the Function λ for the Domain in Nikodým's Example

In this section we consider the domain Ω specified in Example 1.1.4/1 provided $\varepsilon_m = \delta(2^{-m-1})$, where δ is a Lipschitz function on [0,1] such that $\delta(2t) \sim \delta(t)$. We shall show that $\lambda(\mu) \sim \delta(\mu)$.

Lemma. *If $\mathscr{G}$ is an admissible subset of Ω, $\mathscr{G} \cap C = \varnothing$, then*

$$s(\partial_i \mathscr{G}) \geq k\delta\big(m_2(\mathscr{G})\big), \quad k = const > 0. \tag{5.4.1}$$

Proof. Let $\mathscr{G}_m = (A_m \cup B_m) \cap \mathscr{G}$. Let N denote the smallest number for which

$$s(\partial_i \mathscr{G}_N) \geq \delta\big(2^{-N-1}\big).$$

This and the obvious inequality $m_2(\mathscr{G}_m) \leq 2^{-m-1}$ imply that

$$\delta\left[m_2\bigg(\bigcup_{m \geq N} \mathscr{G}_m\bigg)\right] \leq \delta\big(2^{-N}\big) \leq ks(\partial_i \mathscr{G}). \tag{5.4.2}$$

Since

$$s(\partial_i \mathscr{G}_m) < \delta\big(2^{-m-1}\big)$$

for all $m < N$, then $A_m \cup B_m$ does not contain components of $\partial_i \mathscr{G}$, connecting the polygonal line *abcd* with the segment *ef* (cf. Fig. 23). So for $m < N$

$$2s(\partial_i \mathscr{G}_m) \geq s(\partial_e \mathscr{G}_m),$$

where $\partial_e A = \partial A \backslash \partial_i A$. From this along with the isoperimetric inequality

$$s(\partial \mathscr{G}_m)^2 \geq 4\pi m_2(\mathscr{G}_m)$$

we obtain

$$m_2(\mathscr{G}_m)^{1/2} \leq cs(\partial_i \mathscr{G}_m). \tag{5.4.3}$$

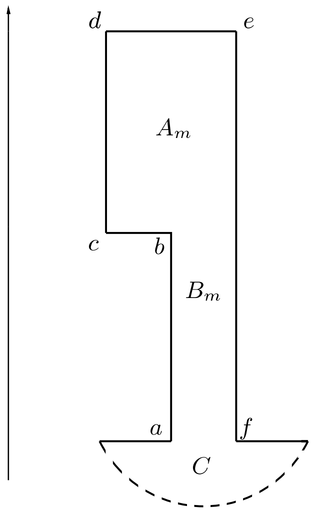

Fig. 23.

Summing (5.4.3) over m and using the inequality

$$\Big(\sum_m a_m\Big)^{1/2} \le \sum_m a_m^{1/2},$$

with positive a_m, we obtain

$$\Big[m_2\Big(\bigcup_{m<N} \mathscr{G}_m\Big)\Big]^{1/2} \le cs(\partial_i \mathscr{G}). \tag{5.4.4}$$

Combining (5.4.2) with (5.4.4), we arrive at the required estimate.

Proposition. *The function* λ *satisfies*

$$k_1\delta(\mu) \le \lambda(\mu) \le k_2\delta(\mu),$$

where k_1 *and* k_2 *are positive constants.*

Proof. From (5.4.1) and Theorem 1.2.4 it follows that

$$k\int_0^\infty \delta\big(m_2(\mathscr{N}_t)\big)\,\mathrm{d}t \le \int_\Omega |\nabla u|\,\mathrm{d}x \tag{5.4.5}$$

for all $u \in C^\infty(\Omega)$ that vanish on C. From Lemma 5.1.2/2 we obtain that this inequality is valid for all $u \in C^{0,1}(\Omega)$, $u = 0$ on C.

Now let u be an arbitrary function in $C^{0,1}(\Omega)$ and let η be a continuous piecewise linear function on [0,1] equal to zero on [0, 1/3] and to unity on (2/3,1). We put

$$u^{(1)}(x,y) = u(x,y)\eta(y),$$
$$u^{(2)}(x,y) = u(x,y)\big(1-\eta(y)\big).$$

Since $|u| \le |u^{(1)}| + |u^{(2)}|$, we have

$$\mathscr{N}_t \subset \mathscr{N}_{t/2}^{(1)} \cup \mathscr{N}_{t/2}^{(2)}$$

and hence

$$m_2(\mathscr{N}_t) \le m_2\big(\mathscr{N}_{t/2}^{(1)}\big) + m_2\big(\mathscr{N}_{t/2}^{(2)}\big).$$

Taking into account the condition $\delta \in C^{0,1}[0,1]$, we obtain

$$\delta\big(m_2(\mathscr{N}_t)\big) \le \delta\big(m_2\big(\mathscr{N}_{t/2}^{(1)}\big)\big) + \delta\big(m_2\big(\mathscr{N}_{t/2}^{(2)}\big)\big).$$

This and (5.4.5) applied to $u^{(1)}$ yield

$$\int_0^\infty \delta\big(m_2(\mathscr{N}_t)\big)\,\mathrm{d}t \le K\bigg(\iint_\Omega \big|\nabla u^{(1)}\big|\,\mathrm{d}x\,\mathrm{d}y + \iint_\Omega \big|u^{(2)}\big|\,\mathrm{d}x\,\mathrm{d}y\bigg).$$

Here the right-hand side does not exceed

$$K\bigg(\int_\Omega |\nabla u|\,\mathrm{d}x\,\mathrm{d}y + \iint_\omega |u|\,\mathrm{d}x\,\mathrm{d}y\bigg), \tag{5.4.6}$$

where $\omega = \Omega \cap \{(x,y) : |y| < 2/3\}$.

We give a bound for the integral over ω in (5.4.6). Let $(x,y) \in \omega$ and let $(x,z) \in C$. Obviously

$$\big|u(x,y)\big| \le \big|u(x,z)\big| + \int \big|\nabla u(x,\bar{y})\big|\,\mathrm{d}\bar{y},$$

where the integration is taken over a vertical segment, contained in Ω and passing through $(x,0)$. After integration over x, y, z we obtain

$$\iint_\omega |u|\,\mathrm{d}x\,\mathrm{d}y \le c\bigg(\iint_C |u|\,\mathrm{d}x\,\mathrm{d}y + \iint_\Omega |\nabla u|\,\mathrm{d}x\,\mathrm{d}y\bigg).$$

Thus

$$\int_0^\infty \delta\big(m_2(\mathscr{N}_t)\big)\,\mathrm{d}t \le K\bigg(\iint_\Omega |\nabla u|\,\mathrm{d}x\,\mathrm{d}y + \iint_C |u|\,\mathrm{d}x\,\mathrm{d}y\bigg).$$

The rectangle C belongs to the class $\mathscr{J}_{1/2}$. Therefore, if $u = 0$ on a subset of Ω with area not less than $\frac{1}{2}m_2(\Omega)$, then

$$\iint_C |u|\,\mathrm{d}x\,\mathrm{d}y \le K \iint_C |\nabla u|\,\mathrm{d}x\,\mathrm{d}y.$$

Consequently,

$$\int_0^\infty \delta\big(m_2(\mathscr{N}_t)\big)\,\mathrm{d}t \leq K \iint_C |\nabla u|\,\mathrm{d}x\,\mathrm{d}y. \tag{5.4.7}$$

Inserting the sequence $\{w_m\}$, constructed in Lemma 5.2.2, into (5.4.7), we obtain (5.4.1) for any admissible set with an area that does not exceed $\frac{1}{2}m_n(\Omega)$. Thus, the lower bound for $\lambda(\mu)$ is obtained.

To derive the estimate $\lambda(\mu) \leq c\delta(\mu)$ it is sufficient to note that the sequence $\{\mathscr{G}_m\}_{m\geq 1}$, where $\mathscr{G}_m$ is the interior of $A_m \cup B_m$, satisfies

$$s(\partial_i \mathscr{G}_m) = \delta\big(2^{-m-1}\big),$$
$$\frac{1}{3}\delta\big(2^{-m-1}\big) \leq m_2(\mathscr{G}_m) \leq \frac{2}{3}\delta\big(2^{-m-1}\big).$$

The proposition is proved. □

Obviously, the Nikodým domain Ω belongs to $\mathscr{J}_\alpha (\alpha \geq 1)$ if and only if

$$\liminf_{t\to+0} t^{-\alpha}\delta(t) > 0. \tag{5.4.8}$$

Similarly, $\Omega \in \mathcal{H}_\alpha$, $\alpha > 1$, i.e., $L_1^1(\Omega) \subset L_{1/\alpha}(\Omega)$ if and only if

$$\int_0^1 \left(\frac{s^\alpha}{\delta(s)}\right)^{\frac{1}{\alpha-1}} \frac{\mathrm{d}s}{s} < \infty.$$

We shall return to the domain considered here in Sect. 6.5.

5.5 Compactness of the Embedding $L_1^1(\Omega) \subset L_q(\Omega)$ $(q \geq 1)$

5.5.1 Class $\mathring{\mathscr{J}}_\alpha$

Definition. The set Ω belongs to *the class* $\mathring{\mathscr{J}}_\alpha$, $\alpha > (n-1)/n$, if

$$\lim_{\mu\to 0} \sup \frac{m_n(\mathscr{G})^\alpha}{s(\partial_i \mathscr{G})} = 0, \tag{5.5.1}$$

where the supremum is taken over all admissible subsets $\mathscr{G}$ of Ω such that $m_n(\mathscr{G}) \leq \mu$.

Remark 1. It is clear that (5.5.1) is equivalent to

$$\lim_{\mu\to 0} \mu^{-\alpha}\lambda_M(\mu) = \infty, \tag{5.5.2}$$

where M is a fixed number in $(0, m_n(\Omega))$.

Remark 2. The exponent α in the definition of the class $\mathring{\mathscr{J}}_\alpha$ exceeds $(n-1)/n$ since in the case $\alpha = (n-1)/n$ we have

$$\big[m_n(B_\varrho)\big]^{(n-1)/n} = \mathrm{const}\cdot s(\partial B_\varrho).$$

5.5.2 Compactness Criterion

Theorem. *Let $m_n(\Omega) < \infty$ and let Ω be a domain. For the compactness of the embedding $L_1^1(\Omega) \subset L_q(\Omega)$, where $n/(n-1) > q \geq 1$, it is necessary and sufficient that Ω belong to $\mathring{\mathscr{J}}_\alpha$ with $\alpha = q^{-1}$.*

Proof. Sufficiency. Let u be an arbitrary function in $L_1^1(\Omega) \cap L_\infty(\Omega) \cap C^\infty(\Omega)$ with bounded support. By Corollary 5.1.2 the set of such functions is dense in $L_1^1(\Omega)$. Obviously,

$$\int_\Omega |u|^q \,\mathrm{d}x \leq c\Big(\int_{\mathscr{N}_\tau} |u|^q \,\mathrm{d}x + \tau^q m_n(\Omega)\Big)$$
$$\leq c_1\Big(\int_\Omega (|u|-\tau)_+^q \,\mathrm{d}x + \tau^q m_n(\Omega)\Big)$$

for $\tau \geq 0$. Let τ be such that

$$m_n(\mathscr{N}_\tau) \geq \mu, \qquad m_n(\mathscr{L}_\tau) \leq \mu,$$

where μ is an arbitrary number in $(0, M]$ and M is the constant in the definition of $\mathscr{J}_\alpha$.

Using Lemma 5.2.3/1, we obtain

$$\int_\Omega (|u|-\tau)_+^q \,\mathrm{d}x \leq \frac{\mu}{[\lambda_M(\mu)]^q}\Big(\int_\Omega |\nabla u| \,\mathrm{d}x\Big)^q. \tag{5.5.3}$$

Let ω be a bounded set with smooth boundary such that

$$\bar{\omega} \subset \Omega, \qquad 2m_n(\Omega\backslash\omega) < \mu.$$

Since $m_n(\mathscr{N}_\tau) \geq \mu$, we have $2m_n(\Omega \cap \mathscr{N}_\tau) \geq \mu$. Consequently,

$$\int_\omega |u|^q \,\mathrm{d}x \geq 2^{-1}\mu\tau^q.$$

Thus,

$$c\|u\|_{L_q(\Omega)} \leq \frac{\mu^{1/q}}{\lambda_M(\mu)} \int_\Omega |\nabla u| \,\mathrm{d}x + \Big[\frac{m_n(\Omega)}{\mu}\Big]^{1/q} \|u\|_{L_q(\omega)}. \tag{5.5.4}$$

Let $\{u_k\}_{k\geq 1}$ be a sequence satisfying

$$\|\nabla u_k\|_{L(\Omega)} + \|u_k\|_{L(\omega)} \leq 1.$$

Since the boundary of ω is smooth, the embedding operator $L_1^1(\omega) \to L_q(\omega)$ is compact and we may suppose that $\{u_k\}_{k\geq 1}$ is a Cauchy sequence in $L_q(\omega)$. By (5.5.4),

$$c\|u_m - u_l\|_{L_q(\Omega)} \leq 2\frac{\mu^{1/q}}{\lambda_M(\mu)} + \left[\frac{m_n(\Omega)}{\mu}\right]^{1/q} \|u_m - u_l\|_{L_q(\omega)}$$

and hence

$$c \limsup_{m,l\to\infty} \|u_m - u_l\|_{L_q(\Omega)} \leq 2\frac{\mu^{1/q}}{\lambda_M(\mu)}.$$

It remains to pass to the limit in the right-hand side as $\mu \to +0$ and take (5.5.2) into account.

Necessity. Let the embedding $L_1^1(\Omega) \subset L_q(\Omega)$ be compact. Then $L_1^1(\Omega) \subset L_1(\Omega)$ and elements of a unit ball in $W_1^1(\Omega)$ have absolutely equicontinuous norms in $L_q(\Omega)$. Hence, for all $u \in L_1^1(\Omega)$

$$\left(\int_{\mathscr{G}} |u|^q \, dx\right)^{1/q} \leq \varepsilon(\mu) \int_\Omega \big(|\nabla u| + |u|\big) \, dx, \tag{5.5.5}$$

where $\mathscr{G}$ is an arbitrary admissible subset of Ω whose measure does not exceed μ and $\varepsilon(\mu)$ converges to zero as $\mu \to +0$.

We insert the sequence $\{w_m\}$ constructed in Lemma 5.2.2 into (5.5.5). Then, for any compactum $K \subset \mathscr{G}$,

$$m_n(K)^{1/q} \leq c\varepsilon(\mu)\big(s(\partial_i \mathscr{G}) + m_n(\mathscr{G})\big),$$

and hence

$$m_n(\mathscr{G})^{1/q} \leq c_1 \varepsilon(\mu) s(\partial_i \mathscr{G}).$$

The theorem is proved. □

Remark. One can prove, in a similar way, that the compactness of the embedding $L_1^1(\Omega)$ into $L_q(\Omega)$, $q < 1$, holds if and only if $\Omega \in \mathcal{H}_{1/q}$, i.e., the embedding $L_1^1(\Omega) \subset L_q(\Omega)$, $q > 1$, is compact and continuous simultaneously.

The role of the inequality (5.5.3) in the proof of the statement should be played by the estimate

$$\int_\Omega \big(|u| - \tau\big)_+^q \, dx \leq c\left(\int_0^M \left(\frac{\mu^{\frac{1}{q}}}{\lambda_M(\mu)}\right)^{\frac{q}{1-q}} \frac{d\mu}{\mu}\right)^{1-q} \left(\int_\Omega |\nabla u| \, dx\right)^q,$$

which can be obtained by obvious changes in the proof of Theorem 2.3.8.

Example. The condition (5.5.1) for the domain

$$\Omega = \big\{x = (x', x_n),\ x' = (x_1, \dots, x_{n-1}) : |x'| < f(x_n),\ 0 < x_n < a\big\}$$

(cf. Example 5.3.3/1) is equivalent to

$$\lim_{x\to 0} \left(\int_0^x \big[f(\tau)\big]^{n-1} \, d\tau\right)^\alpha \big[f(x)\big]^{1-n} = 0. \tag{5.5.6}$$

Since

$$\int_0^x \big[f(t)\big]^{n-1}\,\mathrm{d}\tau \le \big[f(x)\big]^{n-1} x,$$

we see that (5.5.6) holds if

$$\lim_{x\to 0} x^\alpha \big[f(x)\big]^{(n-1)(\alpha-1)} = 0$$

(in particular, (5.5.6) always holds for $\alpha = 1$).

By the last Remark, the condition (5.3.11) is the equivalent to the compactness of the embedding $L_1^1(\Omega)$ into $L_q(\Omega)$, $q > 1$, for the infinite funnel in Example 5.3.3/2.

5.6 Embedding $W_{1,r}^1(\Omega, \partial\Omega) \subset L_q(\Omega)$

5.6.1 Class $\mathscr{K}_{\alpha,\beta}$

Let $r > 0$, $u \in C(\bar{\Omega})$ and

$$\|u\|_{L_r(\partial\Omega)} = \Big(\int_{\partial\Omega} |u|^r\,\mathrm{d}s\Big)^{1/r},$$

where s is the $(n-1)$-dimensional Hausdorff measure. If $r \ge 1$, then $\|u\|_{L_r(\partial\Omega)}$ is a norm and it is a pseudonorm for $r \in (0,1)$ (cf. Sect. 5.1.1).

Definition 1. We denote by $W_{p,r}^1(\Omega, \partial\Omega)$ the completion of the set of functions in $L_p^1(\Omega) \cap C^\infty(\Omega) \cap C(\bar{\Omega})$ with respect to the norm (pseudonorm)

$$\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_r(\partial\Omega)}.$$

In this section we study the conditions for the embedding $W_{1,r}^1(\Omega, \partial\Omega) \subset L_q(\Omega)$. Contrary to Sobolev's theorem for domains of the class $C^{0,1}$, the norm in $L_r(\partial\Omega)$ does not always play the role of a "weak perturbation" for the $L_1(\Omega)$-norm of the gradient in the inequality

$$\|u\|_{L_q(\Omega)} \le c\big(\|\nabla u\|_{L_1(\Omega)} + \|u\|_{L_r(\partial\Omega)}\big). \tag{5.6.1}$$

In the case of a "bad" boundary the exponent q may depend on the order of integrability of the function on the boundary.

In particular, we shall see that functions in $W_{1,r}^1(\Omega, \partial\Omega)$ are integrable with power $q = n/(n-1)$ in Ω, if $r = 1$ and Ω is an arbitrary open set.

Definition 2. An open set Ω belongs to *the class* $\mathscr{K}_{\alpha,\beta}$, if there exists a constant $\mathscr{E}$ such that

$$\big[m_n(g)\big]^\alpha \le \mathscr{E}\big[s(\partial_i g) + s(\partial_e g)^\beta\big] \tag{5.6.2}$$

for any admissible set $g \subset \Omega$ (here $\partial_e g = \partial g \cap \partial\Omega$).

5.6.2 Examples of Sets in $\mathscr{K}_{\alpha,\beta}$

Example 1. An arbitrary open set Ω belongs to the class $\mathscr{K}_{(n-1)/n,1}$ since the condition (5.6.2) for $\alpha = (n-1)/n$, $\beta = 1$ is the classical isoperimetric inequality

$$\big[m_n(g)\big]^{(n-1)/n} \le \frac{[\Gamma(1+n/2)]^{1/n}}{n\sqrt{\pi}} s(\partial g) \tag{5.6.3}$$

(cf. Sect. 9.1.5 and Remark 9.2.2).

Proposition. *If $\Omega \in \mathscr{K}_{\alpha,1}$, where $\alpha \ge (n-1)/n$ then $\Omega \in \mathscr{K}_{\alpha\beta,\beta}$, where β is an arbitrary number in $[(n-1)/n\alpha, 1]$.*

Proof. From (5.6.3) and

$$\big[m_n(g)\big]^{\alpha} \le \mathscr{E} s(\partial g), \tag{5.6.4}$$

it follows that

$$\big[m_n(g)\big]^{\alpha\beta} \le c\mathscr{E}^{\theta} s(\partial g) \quad \text{with } \beta = \theta + (1-\theta)(n-1)/n\alpha$$

for any $\theta \in [0,1]$. If $s(\partial_e g) \le s(\partial_i g)$, then

$$\big[m_n(g)\big]^{\alpha\beta} \le 2c\mathscr{E}^{\theta} s(\partial_i g). \tag{5.6.5}$$

Otherwise, if $s(\partial_e g) > s(\partial_i g)$, then (5.6.4) yields

$$\big[m_n(g)\big]^{\alpha\beta} \le \mathscr{E}^{\beta} \big[s(\partial g)\big]^{\beta}. \tag{5.6.6}$$

Combining (5.6.5) with (5.6.6), we complete the proof.

Example 2. We show that the plane domain

$$\Omega = \big\{(x,y) : 0 < x < \infty,\ 0 < y < (1+x)^{\gamma-1}\big\},$$

where $0 < \gamma \le 1$, belongs to $\mathscr{K}_{1/\gamma,1}$ and by the Proposition, $\Omega \in \mathscr{K}_{\beta/\gamma,\beta}$, where β is an arbitrary number in $[\gamma/2, 1]$.

Let g be an arbitrary admissible subset of Ω. Obviously,

$$s(\partial g) \ge s(\Pr_{Ox} g), \tag{5.6.7}$$

where s is the length and $\Pr_{Ox}$ is the orthogonal projection onto the axis Ox. Since $\gamma \le 1$, we have

$$\begin{aligned}\int_{\Pr_{Ox} g} (1+x)^{\gamma-1}\, \mathrm{d}x &\le \int_0^{s(\Pr_{Ox} g)} (1+x)^{\gamma-1}\, \mathrm{d}x = \gamma^{-1}\big[\big(s(\Pr_{Ox} g)+1\big)^{\gamma} - 1\big] \\ &\le \gamma^{-1}\big[s(\Pr_{Ox} g)\big]^{\gamma}.\end{aligned}$$

Consequently,

$$\Big(\int_{\mathrm{Pr}_{Ox}\, g}(1+x)^{\gamma-1}\,\mathrm{d}x\Big)^{1/\gamma}\le \gamma^{-1/\gamma}s(\partial g). \tag{5.6.8}$$

Taking into account that

$$m_2(g)\le \int_{\mathrm{Pr}_{Ox}\, g}(1+x)^{\gamma-1}\,\mathrm{d}x,$$

from (5.6.8) we obtain

$$m_2(g)^{1/\gamma}\le \gamma^{-1/\gamma}s(\partial g), \tag{5.6.9}$$

which means that $\Omega\in\mathscr{K}_{1/\gamma,1}$.

Example 3. We shall show that any set Ω in $\mathscr{J}_\alpha$ with finite volume belongs to the class $\mathscr{K}_{\alpha,\beta}$, where β is an arbitrary positive number.

For any admissible subset g of Ω satisfying the condition $m_n(g)\le M$ we have

$$m_n(g)^\alpha\le \mathfrak{A}(M)s(\partial_i g). \tag{5.6.10}$$

Let $m_n(g)>M$ and

$$2\alpha_n s(\partial_e g)<M^{(n-1)/n},\quad \text{where } \alpha_n=\frac{[\Gamma(1+n/2)]^{1/n}}{n\sqrt{\pi}}. \tag{5.6.11}$$

By the isoperimetric inequality (5.6.3),

$$M^{(n-1)/n}\le m_n(g)^{(n-1)/n}\le \alpha_n\big(s(\partial_i g)+s(\partial_e g)\big)$$

and hence

$$M^{(n-1)/n}\le 2\alpha_n s(\partial_i g).$$

This implies

$$m_n(g)^\alpha\le m_n(\Omega)^\alpha 2\alpha_n M^{(1-n)/n}s(\partial_i g). \tag{5.6.12}$$

If (5.6.11) is not valid, then

$$m_n(g)^\alpha\le m_n(\Omega)^\alpha(2\alpha_n)^\beta s(\partial_e g)^\beta. \tag{5.6.13}$$

From (5.6.10), (5.6.12), and (5.6.13) it follows that Ω belongs to $\mathscr{K}_{\alpha,\beta}$.

5.6.3 Continuity of the Embedding Operator $W^1_{1,r}(\Omega,\partial\Omega)\to L_q(\Omega)$

Theorem. *If $\Omega\in\mathscr{K}_{\alpha,\beta}$, where $\alpha\le 1$, $\beta\ge\alpha$, then (5.6.1) holds for all $u\in W^1_{1,1/\beta}(\Omega,\partial\Omega)$ with $q=1/\alpha$, $r=1/\beta$.*

Proof. Consider the case $1>\beta\ge\alpha$. Let u be an arbitrary function in $C^\infty(\Omega)\cap C(\bar\Omega)$ with bounded support. From Lemma 1.2.3 and Lemma 1.3.5/1 we obtain that

$$\|u\|_{L_{1/\alpha}(\Omega)} \le \Big(\int_0^\infty m_n(\mathscr{N}_t)^{\alpha/\beta}\, \mathrm{d}\big(t^{1/\beta}\big) \Big)^\beta. \tag{5.6.14}$$

We introduce the set

$$A_t = \big\{ t : s(\mathscr{E}_t) \le s(\bar{\mathscr{N}_t} \cap \partial\Omega)^\beta \big\}$$

and represent the integral above in the form

$$\int_{A_t} + \int_{CA_t},$$

where CA_t is the complement of A_t with respect to the positive halfaxis. We give a bound for the integral

$$I_1 = \int_{A_t} m_n(\mathscr{N}_t)^{\alpha/\beta}\, \mathrm{d}\big(t^{1/\beta}\big).$$

Since $\Omega \in \mathscr{K}_{\alpha,\beta}$, then

$$m_n(\mathscr{N}_t)^\alpha \le 2\mathscr{E} s(\bar{\mathscr{N}_t} \cap \partial\Omega)^\beta$$

for almost all $t \in A_t$ and hence

$$I_1 \le (2\mathscr{E})^{1/\beta} \int_{A_t} s(\bar{\mathscr{N}_t} \cap \partial\Omega)\, \mathrm{d}\big(t^{1/\beta}\big) \le (2\mathscr{E})^{1/\beta} \int_{\partial\Omega} |u|^{1/\beta}\, \mathrm{d}s. \tag{5.6.15}$$

Now we consider the integral

$$I_2 = \int_{CA_t} \big[m_n(\mathscr{N}_t)\big]^{\alpha/\beta}\, \mathrm{d}\big(t^{1/\beta}\big).$$

Obviously,

$$I_2 \le \beta^{-1} \int_0^\infty \big[m_n(\mathscr{N}_t)\big]^\alpha\, \mathrm{d}t \sup_{\tau \in CA_t} \big(\tau \big[m_n(\mathscr{N}_\tau)\big]^\alpha\big)^{(1-\beta)/\beta}.$$

By (5.6.2),

$$\big[m_n(\mathscr{N}_\tau)\big]^\alpha \le 2\mathscr{E} s(\mathscr{E}_\tau)$$

for $\tau \in CA_t$, and hence

$$I_2 \le 2\mathscr{E}\beta^{-1} \int_0^\infty s(\mathscr{E}_t)\, \mathrm{d}t \sup_{\tau>0} \big(\tau \big[m_n(\mathscr{N}_\tau)\big]^\alpha\big)^{(1-\beta)/\beta}.$$

The preceding inequality and Theorem 1.2.4 yield

$$I_2 \le 2\mathscr{E}\beta^{-1} \|\nabla u\|_{L_1(\Omega)} \|u\|^{(1-\beta)/\beta}_{L_{1/\alpha}(\Omega)}. \tag{5.6.16}$$

From (5.6.14)–(5.6.16) it follows that

$$\|u\|_{L_{1/\alpha}(\Omega)} \le c\mathscr{E}\big(\|\nabla u\|^{\beta}_{L_1(\Omega)}\|u\|^{1-\beta}_{L_{1/\alpha}(\Omega)} + \|u\|_{L_{1/\beta}(\partial\Omega)}\big).$$

Now consider the case $\beta \ge 1$. The condition $\Omega \in \mathscr{K}_{\alpha,\beta}$ implies

$$\int_0^\infty \big[m_n(\mathscr{N}_\tau)\big]^\alpha \,\mathrm{d}t \le \mathscr{E}\bigg(\int_0^\infty s(\mathscr{E}_t)\,\mathrm{d}t + \int_0^\infty \big[s(\bar{\mathscr{N}_t} \cap \partial\Omega)\big]^\beta \,\mathrm{d}t\bigg). \qquad (5.6.17)$$

Applying Lemma 1.2.3 and Lemma 1.3.5/1, we obtain

$$\int_\Omega |u|^{1/\alpha}\,\mathrm{d}x \le \bigg(\int_0^\infty \big[m_n(\mathscr{N}_t)\big]^\alpha \,\mathrm{d}t\bigg)^{1/\alpha},$$

$$\bigg(\int_0^\infty \big[s(\bar{\mathscr{N}_t} \cap \partial\Omega)\big]^\beta \,\mathrm{d}t\bigg)^{1/\beta} \le \int_0^\infty s(\bar{\mathscr{N}_t} \cap \partial\Omega)\,\mathrm{d}\big(t^{1/\beta}\big) = \int_{\partial\Omega} |u|^{1/\beta}\,\mathrm{d}s.$$

These estimates together with (5.6.17) lead to

$$\|u\|_{L_{1/\alpha}(\Omega)} \le \mathscr{E}\big(\|\nabla u\|_{L_1(\Omega)} + \|u\|_{L_{1/\beta}(\partial\Omega)}\big). \qquad (5.6.18)$$

This concludes the proof. □

We give an example that shows that the Theorem is not true if $\Omega \in \mathscr{K}_{\alpha,\beta}$, $\alpha > \beta$, and $r = 1/\beta$.

Example. In Example 5.6.2/2 we showed that the domain

$$\Omega = \big\{(x,y) : 0 < x < \infty,\ 0 < y < (1+x)^{2\beta-1}\big\}$$

belongs to the class $\mathscr{K}_{1/2,\beta}$ with $\beta \in (0, 1/2)$. Consider the sequence of functions

$$\Omega \ni (x,y) \to u_m(x,y) = (1+x)^{-\varkappa_m}, \quad \varkappa_m > \beta.$$

Each of these functions obviously belongs to $W^1_{1,1/\beta}(\Omega,\partial\Omega)$ since it can be approximated in the norm of the space mentioned after multiplication by an "expanding" sequence of truncating functions. We have

$$\|u_m\|_{L_2(\Omega)} = 2^{-1/2}(\varkappa_m - \beta)^{-1/2}, \qquad \|\nabla u_m\|_{L(\Omega)} = \varkappa_m/(\varkappa_m - 2\beta + 1),$$

$$\|u_m\|_{L_{1/\beta}(\partial\Omega)} > \beta^\beta(\varkappa_m - \beta)^{-\beta}.$$

For $u = u_m$ and $\varkappa_m \to \beta + 0$ the left-hand side of the inequality

$$\|u\|_{L_2(\Omega)} \le C\big(\|\nabla u\|_{L(\Omega)} + \|u\|_{L_{1/\beta}(\partial\Omega)}\big)$$

grows more rapidly than its right-hand side and hence (5.6.1) is not true for $q = \alpha^{-1}$, $r = \beta^{-1}$.

Remark. A review of the proof of the previous Theorem shows that the theorem also remains valid for $\beta < \alpha$, provided (5.6.1) in its statement is replaced by

$$\|u\|_{L_q(\Omega)} \le C\Big(\|\nabla u\|_{L(\Omega)} + \int_0^\infty \big[s(\bar{\mathscr{N}_t} \cap \partial\Omega)\big]^{1/r}\, dt\Big). \tag{5.6.19}$$

Corollary. *The inequality*

$$\|u\|_{L_{n/(n-1)}(\Omega)} \le \frac{[\Gamma(1+n/2)]^{1/n}}{n\sqrt{\pi}}\big(\|\nabla u\|_{L(\Omega)} + \|u\|_{L(\partial\Omega)}\big), \tag{5.6.20}$$

holds for an arbitrary bounded set Ω and $u \in W^1_{1,1}(\Omega, \partial\Omega)$. The constant in (5.6.20) is the best possible.

The proof follows immediately from the isoperimetric inequality (5.6.14) and (5.6.18). That the constant is exact was already remarked in Sect. 1.4.2, where the inequality (1.4.14) was derived.

5.7 Comments to Chap. 5

A substantial part of the results presented in this chapter were stated in the author's paper [547].

Section 5.2. Lemma 5.2.1/1 was proved by Burago and the author [151]. Other results of this section (except Lemma 5.2.3/2) are due to the author.

The Poincaré-type inequality (5.2.7) with $q = n/(n-1)$ was deduced from the isoperimetric inequality (5.2.1), where $\alpha = (n-1)/n$ and $M = \frac{1}{2}m_n(\Omega)$, simultaneously and independently from the author by Fleming and Rishel [282].

For the further development of the idea concerning the relation between integral and geometrical inequalities see Miranda [608], Burago and Maz'ya [150] (the integrability of traces on $\partial\Omega$ of functions in $BV(\Omega)$, and so on; these results are presented in Chap. 9), Federer [271, 272] (embedding theorems for currents), Klimov [428–430, 433] (embedding theorems involving Birnbaum–Orlicz and rearrangement invariant spaces), Michael and Simon [600], Hoffman and Spruck [379], Aubin [55] (the Sobolev–Gagliardo inequalities for functions on manifolds), Otsuki [654], Martin and M. Milman [523] (embedding theorems involving rearrangement invariant spaces). Lemma 5.2.3/2 was proved by Deny and J.L. Lions [234].

The following optimal result was obtained by Cianchi in [193].

Theorem. *Let $n \ge 2$ and $q \in (0, n/(n-1)]$. Let Ω be the unit ball in $\mathbb{R}^n$. Then*

$$\|u - u_\Omega\|_{L_q(\Omega)} \le \frac{v_n^{1/q}}{2v_{n-1}}\|\nabla u\|_{L_1(\Omega)} \tag{5.7.1}$$

for $u \in L^1_1(\Omega)$, where u_Ω denotes the mean value of u over Ω. The constant $v_n^{1/q}(2v_{n-1})^{-1}$ is sharp.

Although the constant $v_n^{1/q}(2v_{n-1})^{-1}$ is sharp in (5.7.1), it is not achieved. It is actually achieved when u equals the characteristic function of a half-ball in a version of (5.7.1) for functions of bounded variation, where $\|\nabla u\|_{L_1(\Omega)}$ is replaced by the total variation of the distributional gradient of u (see Chap. 9).

The fact that among all subsets of the Euclidean ball of a given measure, the ones minimizing the surface area in the interior of the ball are exactly spherical caps perpendicular to the boundary, is due to Burago and Maz'ya [150] (see also Almgren [40] and Bokowski and Sperner [124]). Cianchi noted in [193] that for the unit ball in $\mathbb{R}^n$ symmetrization arguments as in the proof of Lemma 5.2.1/1 and basic calculus give the formula

$$\lambda_{v_n/2}(s) = B\big(A^{-1}(s)\big) \quad \text{for } s \in [0, v_n/2],$$

where

$$A(\theta) = v_{n-1}\bigg(\int_0^\theta (\sin t)^n \,\mathrm{d}t + (\tan\theta)^n \int_\theta^{\pi/2} (\cos t)^n \,\mathrm{d}t\bigg) \quad \text{for } \theta \in [0, \pi/2]$$

and

$$B(\theta) = (n-1)v_{n-1}(\tan\theta)^{n-1} \int_\theta^{\pi/2} (\cos t)^{n-2} \,\mathrm{d}t \quad \text{for } \theta \in [0, \pi/2].$$

Isoperimetric inequalities in the form (5.2.2) with optimal constants for special convex sets Ω in the plane are discussed in Cianchi [192]. The case when Ω is an n-dimensional convex cone is considered in P.-L. Lions and Pacella [502].

Except for the methods of obtaining Sobolev-type inequalities mentioned in the Comments to Sect. 1.4, there is another powerful approach to reduce many multidimensional geometric and analytic inequalities to specific problems in dimension one, called the localization technique. It was proposed in 1960 by Payne and Weinberger [656] and developed in 1990s by Lovász and Simonovits [508], together with Kannan in [413]. Localization technique was also used by Gromov and V. Milman [326] to recover the isoperimetric theorem on the sphere.

The advantage of localization over triangular mappings is that it often leads to optimal results. For example, localization easily reduces the Gaussian isoperimetric inequality to dimension one, which was demonstrated in Bobkov [110]. More generally, with this technique one may obtain sharp dilation-type inequalities for convex domains and so-called convex probability measures (cf. F. Nazarov, Sodin, and Volberg [629], Bobkov and F. Nazarov [119]). Another interesting line of applications is Sobolev-type inequalities (cf. Kannan, Lov'asz, and Simonovits [413], Bobkov [113]).

Here is a statement from Bobkov [113] that refines one of main results in Kannan, Lov'asz, and Simonovits [413]. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with, for definiteness, volume one. The best constant in the Poincaré-type inequality

$$\int_{\Omega} |u - u_{\Omega}| \, \mathrm{d}x \le A \int_{\Omega} |\nabla u| \, \mathrm{d}x \tag{5.7.2}$$

in the class of all smooth functions u satisfies

$$A \le C \left[\int_{\Omega} |x|^4 \, \mathrm{d}x - \left(\int_{\Omega} |x|^2 \, \mathrm{d}x \right)^2 \right]^{1/4}, \tag{5.7.3}$$

where C is an absolute constant. In the case of the Euclidean ball centered at the origin and of radius of order $\sqrt{n}$, as $n \to \infty$, so that it has volume equal to 1, we have that the right-hand side is of order 1. This follows also from (5.7.1) with $q = 1$. The localization method was further developed in Fradelizi and Guédon [283, 284]. We also mention that the value $2^{-1} \operatorname{diam} \Omega$ of the constant A in (5.7.2) for a convex domain, which is the best constant formulated in terms of diameter, was found by Acosta and Durán [1].

Membership of two-dimensional domains with Hölder continuous boundary in the classes $\mathscr{J}_{\alpha}$, with α depending on the Hölder exponent, was established by Cianchi in [192]. The case of n-dimensional domains with boundaries enjoying more general regularity properties was considered by Labutin in [471]. As Buckley and Koskela showed in [147], bounded planar Jordan domains are in the class $\mathscr{J}_{1/2}$ only if they are Jordan domains. By Bojarski [122] this condition is also sufficient.

Section 5.3. The contents of this section are borrowed from the author's paper [538], where subareal mappings that preserve the space $L_1^1(\Omega)$, were considered in particular. Since then various properties of homeomorphisms of Euclidean domains generating a bounded composition operator of Sobolev spaces were studied. Without aiming at completeness we mention the works by Vodop'yanov and Gol'dshtein [778]; Maz'ya and Shaposhnikova [578, Sect. 9.4], [588]; Holopainen and Rickman [382]; Ukhlov [767]; [768], Gol'dshtein and Gurov [314]; Gol'dshtein, Gurov, and Romanov [315]; Vodop'yanov [777]; Hajłasz [340]; Kauhanen, Koskela, and Malý [417]; Troyanov and Vodop'yanov [761]; Hencl, Koskela, and Malý [377]; and Hencl and Koskela [376], Gol'dshtein and Ukhlov [319].

A different direction is taken by a theory of the so-called Sobolev mappings of Riemannian manifolds and metric spaces. This vast and diverse area, important for applications to geometry, physics, elasticity etc. is outside the scope of the present book.

Section 5.4. The estimates for the relative isoperimetric function λ are presented in Maz'ya [552].

Section 5.5. The results of this section are borrowed from [547].

Section 5.6. The contents of this section are taken from the author's thesis [529]. The inequality (5.6.20) can be found in Maz'ya [538]. This inequality proved to be useful in the theory of the Robin boundary value problem for arbitrary domains developed by Daners [221].

6

Integrability of Functions in the Space $L_p^1(\Omega)$

Here we continue our study of integral inequalities for functions with unrestricted boundary values started in the previous chapter. For the embedding operator $L_p^1(\Omega) \to L_q(\Omega)$, $p \geq 1$, we find necessary and sufficient conditions on Ω ensuring the continuity of this operator (Sects. 6.2–6.4). To get criteria, analogous to those obtained in Sect. 2.2, for the space $L_p^1(\Omega)$, we introduce classes of sets defined with the aid of the so-called p-conductivity, which plays the same role as p-capacity in Chap. 2. Geometrical conditions formulated in terms of the isoperimetric inequalities prove to be only sufficient if $p > 1$.

We give some details. Let G be a bounded open subset of Ω and let F be a relatively closed subset of G. The difference $K = G \backslash F$ is called a conductor and its p-conductivity $c_p(K)$ is defined as $\inf \|\nabla f\|^p_{L_p(\Omega)}$ extended over Lipschitz functions f such that $f \geq 1$ on F and $f \leq 0$ on $\Omega \backslash G$. The infimum of $c_p(K)$ taken over all conductors K with $m_n(F) \geq t$ and $m_n(G) \leq M$, where $M \in (0, m_n(\Omega))$, is called the p-conductivity minimizing function $\nu_{M,p}(t)$. We say that Ω belongs to the class $\mathscr{I}_{p,\alpha}$ if

$$\nu_{M,p}(t) \geq \mathrm{const}\, t^{\alpha p},$$

with $t \in (0, M]$. For $p = 1$ this class coincides with the class $\mathscr{J}_\alpha$ introduced in Chap. 5. By one of the results in the present chapter, Theorem 6.3.3, the inclusion $\Omega \in \mathscr{I}_{p,\alpha}$ implies the inequality of the form

$$\|u\|_{L_q(\Omega)} \leq C\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_s(\Omega)}\big)^{1-\varkappa} \|u\|^{\varkappa}_{L_r(\Omega)},$$

where $q \leq 1/\alpha$ if $p\alpha < 1$ and $q < 1/\alpha$ if $p\alpha > 1$. Conversely, the last multiplicative inequality with $q < 1/\alpha$ ensures the inclusion $\Omega \in \mathscr{I}_{p,\alpha}$ (see Theorem 6.3.3 for other assumptions about p, q, s, and $\varkappa$). By this result and by Theorem 6.4.2 the inequality

$$\|u\|_{L_{1/\alpha}(\Omega)} \leq C\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_s(\Omega)}\big)$$

holds with $s < 1/\alpha$ if and only if $\Omega \in \mathscr{I}_{p,\alpha}$ in the case $\alpha p \leq 1$ and $\Omega \in \mathscr{H}_{p,\alpha}$ in the case $\alpha p > 1$. The class $\mathscr{H}_{p,\alpha}$ is defined by the condition

V. Maz'ya, *Sobolev Spaces*,
Grundlehren der mathematischen Wissenschaften 342,
DOI 10.1007/978-3-642-15564-2_6,

$$\int_0^M \left[\frac{\tau^{\alpha p}}{\nu_{M,p}(\tau)}\right]^{1/(\alpha p-1)} \frac{\mathrm{d}\tau}{\tau} < \infty$$

in Sect. 6.4.2.

As stated in Proposition 6.3.5/1, the p-conductivity minimizing function admits the lower estimate in terms of the area minimizing function

$$\nu_{M,p}(t) \geq \left(\int_t^M \big[\lambda_M(\sigma)\big]^{p/(1-p)}\, \mathrm{d}\sigma\right)^{1-p},$$

which shows, for instance, that $\mathscr{J}_{\alpha+(p-1)/p} \subset \mathscr{I}_{p,\alpha}$. The failure of the converse inclusion is illustrated in Sect. 6.5, where we find equivalent lower and upper estimates for the p-conductivity minimizing function for the Nikodým domain already dealt with in Sect. 5.4.

Some generalizations of the criteria mentioned are discussed in Sect. 6.6 and then, in Sect. 6.7, the necessary and sufficient conditions for integral inequalities involving domains of infinite volume are found. Next, in Sect. 6.8 we characterize the domains for which the embedding $L_p^1(\Omega) \subset L_q(\Omega)$ is compact. Sufficient conditions on Ω for the boundedness and compactness of the embedding operator $L_p^l(\Omega) \to L_q(\Omega)$, $l \geq 1$, following from the previous results are obtained in Sect. 6.9. We present applications to the Neumann and other boundary value problems for strongly elliptic operators giving criteria of their solvability and discreteness of spectrum in Sects. 6.10 and 6.11. Another topic of Sect. 6.11 concerns inequalities containing integrals over the boundary, which extend those in Sect. 5.6 to the case $p > 1$. Here we prove that the refined Friedrichs-type inequality

$$\|u\|_{L_q(\Omega)} \leq C\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_r(\partial\Omega)}\big)$$

holds with $n > p$, $(n-p)r \leq p(n-1)$, $r \geq 1$, and $q = rn/(n-1)$ for an arbitrary open set Ω with finite volume and show that the exponent $q = rn/(n-1)$ is critical provided no regularity assumptions on the domain are made.

6.1 Conductivity

6.1.1 Equivalence of Certain Definitions of Conductivity

Let Ω be an open set in $\mathbb{R}^n$. Let F and G denote bounded closed (in Ω) and open subsets of Ω, respectively, $F \subset G$.

As we mentioned in the introduction to this chapter, the set $K = G\backslash F$ is called a *conductor*. In what follows $U_\Omega(K)$ is the class of functions $f \in C^{0,1}(\Omega)$ with $f(x) \geq 1$ for $x \in F$ and $f(x) \leq 0$ for $x \in \Omega\backslash G$. The value

$$c_p(K) = \inf\left\{\int_\Omega |\nabla f|^p\, \mathrm{d}x : f \in U_\Omega(K)\right\}$$

is called *the p-conductivity of the conductor K*.

Lemma 1. *Let $V_\Omega(K)$ denote the class of functions $\{f \in C^\infty(\Omega) : f(x) = 1$ for $x \in F, f(x) = 0$ for $x \in \Omega\backslash G\}$. Then*

$$c_p(K) = \inf\biggl\{ \int_\Omega |\nabla f|^p \,\mathrm{d}x : f \in V_\Omega(K) \biggr\}.$$

Proof. Since $U_\Omega(K) \supset V_\Omega(K)$, it suffices to obtain only the lower bound for $c_p(K)$. Let $\varepsilon \in (0,1)$ and let $f \in U_\Omega(K)$ be such that

$$\int_\Omega |\nabla f|^p \,\mathrm{d}x \le c_p(K) + \varepsilon.$$

We put $\psi = (1+\varepsilon)^2 f - \varepsilon$ and $\varphi = \min(\psi_+, 1)$. Then

$$\|\nabla \psi\|_{L_p(\Omega)} \le \bigl(c_p(K) + \varepsilon\bigr)^{1/p} (1+\varepsilon)^2. \tag{6.1.1}$$

We introduce some notation: $\Phi_1 = \{x : \varphi = 1\}$, $\Phi_2 = \{x : \varphi = 0\}$, and $Q = \{x : 1 > \varphi(x) > 0\}$. Since $Q = \{x : (1+\varepsilon)^{-1} > f > \varepsilon(1+\varepsilon)^{-2}\}$, we have $\mathrm{clos}_\Omega Q \subset K$.

We construct a locally finite (in Ω) covering of the set $\mathrm{clos}_\Omega\, Q$ by open balls $\mathscr{B}_{0,i}$, $i = 1, 2, \ldots$. Let $\mathfrak{A}_0$ denote the union of these balls. By the inclusion $\mathrm{clos}_\Omega\, Q \subset K$ we can choose the covering to satisfy $\mathrm{clos}_\Omega\, \mathfrak{A}_0 \subset K$. Next we construct locally finite (in Ω) coverings of the sets $\Phi_k \backslash \mathfrak{A}_0$ $(k = 1,2)$ by open balls $\mathscr{B}_{k,i}$ such that the closures in Ω of the sets $\mathfrak{A}_k = \bigcup_i \mathscr{B}_{k,i}$ $(k = 1,2)$ are disjoint with $\mathrm{clos}_\Omega\, Q$. The latter is possible since $(\Phi_k \backslash \mathfrak{A}_0) \cap \mathrm{clos}_\Omega\, Q = \varnothing$. Clearly, $F \subset \mathfrak{A}_1$ and $(\Omega \backslash G) \subset \mathfrak{A}_2$.

Let $\alpha_{k,i} \in \mathscr{D}(\mathscr{B}_{k,i})$, $k = 0,1,2;$ $i = 1, 2, \ldots$, and let

$$\sum_{k=0}^{2} \sum_{i=1}^{\infty} \alpha_{k,i} = 1 \quad \text{in } \Omega.$$

For any $i = 1, 2, \ldots$, we introduce a function $\beta_i \in \mathscr{D}(\mathscr{B}_{0,i})$ such that

$$\bigl\|\nabla(\alpha_{0,i}\varphi - \beta_i)\bigr\|_{L_p(\Omega)} \le \varepsilon^i.$$

Next we put $v_{0,i} = \beta_i$, $v_{1,i} = \alpha_{1,i}$, $v_{2,i} = 0$,

$$u = \sum_{k=0}^{2} \sum_{i=1}^{\infty} v_{k,i}. \tag{6.1.2}$$

The function u is infinitely differentiable in Ω since each point of Ω is contained only in a finite number of balls $\mathscr{B}_{k,i}$ and therefore (6.1.2) has a finite number of summands. Obviously,

$$\bigl\|\nabla(\varphi - \psi)\bigr\|_{L_p(\Omega)} \le \sum_{k=0}^{2} \sum_{i=1}^{\infty} \bigl\|\nabla(\alpha_{k,i}\varphi - v_{k,i})\bigr\|_{L_p(\Omega)}.$$

Since $\varphi = 1$ on $\mathscr{B}_{1,i}$, $v_{1,i} = \alpha_{1,i}$ and $\varphi = 0$ on $\mathscr{B}_{2,i}$, $v_{2,i} = 0$, it follows that

$$\big\|\nabla(\varphi - u)\big\|_{L_p(\Omega)} \le \sum_{i=1}^{\infty} \big\|\nabla(\alpha_{0,i}\varphi - \beta_i)\big\|_{L_p(\Omega)} \le \varepsilon/(1-\varepsilon). \tag{6.1.3}$$

Using $F \subset \mathfrak{A}_1$ and $F \cap \mathfrak{A}_0 = \varnothing$ we have $v_{0,i} = \beta_i = 0$, $v_{2,i} = 0$, $v_{1,i} = \alpha_{1,i}$, $\alpha_{0,i} = 0$, $\alpha_{2,i} = 0$ on F. Therefore

$$u = \sum_{i=1}^{\infty} v_{1,i} = \sum_{k=0}^{2} \sum_{i=1}^{\infty} \alpha_{k,i} = 1 \quad \text{on } F.$$

Also, $v_{k,i} = 0$ on $\Omega \backslash G$ and hence $u = 0$ on the same set. Thus, $u \in V_\Omega(K)$.

Finally, by (6.1.1) and (6.1.3),

$$\|\nabla u\|_{L_p(\Omega)} \le (1+\varepsilon)^2 \big(c_p(K) + \varepsilon\big)^{1/p} + \varepsilon(1-\varepsilon)^{-1},$$

which completes the proof. □

Let $T_\Omega(K) = \{f \in V_\Omega(K) : 0 \le f(x) \le 1 \text{on } K\}$.

Henceforth, the following modification of Lemma 1 will be useful.

Lemma 2. *The equality*

$$c_p(K) = \inf\bigg\{ \int_\Omega |\nabla f|^p \,\mathrm{d}x : f \in T_\Omega(K) \bigg\}$$

is valid.

Proof. Since $T_\Omega(K) \subset V_\Omega(K)$, it suffices to obtain only the lower bound for $c_p(K)$. Let $\varepsilon > 0$, $\lambda_\varepsilon \in C^\infty(-\infty, +\infty)$, $\lambda_\varepsilon(t) = 1$ for $t \ge 1$, $\lambda_\varepsilon(t) = 0$ for $t \le 0$, $0 \le \lambda'_\varepsilon(t) \le 1 + \varepsilon$. Further, let $\varphi \in V_\Omega(K)$. We introduce the function $f = \lambda_\varepsilon(\varphi) \in T_\Omega(K)$. Obviously,

$$\int_\Omega |\nabla f|^p \,\mathrm{d}x = \int_\Omega \big[\lambda'_\varepsilon(\varphi)\big]^p |\nabla \varphi|^p \,\mathrm{d}x \le (1+\varepsilon)^p \int_\Omega |\nabla \varphi|^p \,\mathrm{d}x$$

and consequently,

$$\inf\bigg\{ \int_\Omega |\nabla f|^p \,\mathrm{d}x : f \in T_\Omega(K) \bigg\} \le c_p(K).$$

The result follows.

6.1.2 Some Properties of Conductivity

We shall comment on some simple properties of p-conductivity.

Consider two conductors $K = G\backslash F$ and $K' = G'\backslash F'$ contained in Ω. We say that K' is a part of $K (K' \subset K)$ if $F \subset F' \subset G' \subset G$.

The definition of p-conductivity immediately implies the following proposition.

Proposition 1. *If $K' \subset K$, then*

$$c_p(K) \le c_p(K').$$

Proposition 2. *Given any $\varepsilon > 0$ and any conductor K with finite p-conductivity, we can construct a conductor $K' \subset K$ such that*

$$\varepsilon \ge c_p(K') - c_p(K) \ge 0. \tag{6.1.4}$$

The conductor K' can be chosen so that $\partial_i F'$ and $\partial_i G'$ are C^∞-manifolds.

Proof. The right inequality follows from Proposition 1.

Let a function $f \in U_\Omega(K)$ satisfy

$$c_p(K) + \varepsilon/2 > \int_\Omega |\nabla f|^p \, \mathrm{d}x \tag{6.1.5}$$

and let

$$2\delta = 1 - \left[\frac{\varepsilon + 2c_p(K)}{2\varepsilon + 2c_p(K)}\right]^{1/p}.$$

We may assume that the sets $\{x \in \Omega : f(x) = 1 - \delta\}$ and $\{x \in \Omega : f(x) = \delta\}$ are C^∞-manifolds since otherwise δ can be replaced by an arbitrarily close number having the aforementioned property. We construct the conductor K' as follows: $K' = G'\backslash F'$, where $F' = \{x \in \Omega : f(x) \ge 1 - \delta\}$, $G' = \{x \in \Omega : f(x) > \delta\}$. Then (6.1.5) implies

$$\frac{c_p(K) + \varepsilon/2}{(1 - 2\delta)^p} > \int_\Omega \left|\nabla\left(\frac{f(x) - \delta}{1 - 2\delta}\right)\right|^p \mathrm{d}x.$$

The function $(1 - 2\delta)^{-1}(f - \delta)$ is contained in $U_\Omega(K')$. Hence

$$\frac{c_p(K) + \varepsilon/2}{(1 - 2\delta)^p} \ge c_p(K'),$$

which is equivalent to the left inequality in (6.1.4).

The proposition is proved. □

Proposition 3. *Let $K_1 = G_1\backslash F_1$ and $K_2 = G_2\backslash F_2$ be any conductors in Ω. Then*

$$c_p(K_\cup) + c_p(K_\cap) \le c_p(K_1) + c_p(K_2), \tag{6.1.6}$$

where $K_\cup = (G_1 \cup G_2)\backslash(F_1 \cup F_2)$, $K_\cap = (G_1 \cap G_2)\backslash(F_1 \cap F_2)$.

Proof. Let ε be an arbitrary positive number and let f_1, f_2 be functions in $U_\Omega(K_1)$, $U_\Omega(K_2)$, respectively, satisfying

$$\int_{K_i} |\nabla f_i|^p \, dx < c_p(K_i) + \varepsilon. \tag{6.1.7}$$

We introduce the functions

$$M = \sup\{f_1, f_2\}, \qquad m = \inf\{f_1, f_2\}.$$

It is clear that M and m satisfy the Lipschitz condition in Ω and that $M \geq 1$ on $F_1 \cup F_2$, $M \leq 0$ on $\Omega \backslash (G_1 \cup G_2)$, as well as $m \geq 1$ on $F_1 \cap F_2$, $m \leq 0$ on $\Omega \backslash (G_1 \cap G_2)$. Besides,

$$\int_{K_\cup} |\nabla M|^p \, dx + \int_{K_\cap} |\nabla m|^p \, dx = \sum_{i=1}^{2} \int_{K_i} |\nabla f_i|^p \, dx.$$

This along with (6.1.7) and Lemma 6.1.1/1 implies (6.1.6). □

6.1.3 Dirichlet Principle with Prescribed Level Surfaces and Its Corollaries

The proofs of the following Lemmas 1 and 2 and Corollaries 1 and 2 do not differ from the proofs of similar assertions on the (p, Φ)-capacity in Sects. 2.2.1–2.2.3.

Lemma 1. *For any conductor K in Ω with finite p-conductivity we have*

$$c_p(K) = \inf_{f \in V_\Omega(K)} \left(\int_0^1 \|\nabla f\|^{-1}_{L_{p-1}(\mathscr{E}_\tau)} \, d\tau \right)^{1-p}, \tag{6.1.8}$$

where $\mathscr{E}_\tau = \{x \in \Omega : f(x) = \tau\}$.

Lemma 2. *Let f be in $C^\infty(\Omega) \cap L_p^1(\Omega)$. Then, for almost all t,*

$$\left[s(\mathscr{E}_t)\right]^{p/(p-1)} \|\nabla f\|^{-1}_{L_{p-1}(\mathscr{E}_t)} \leq -\frac{d}{dt} \left[m_n(\mathscr{L}_t)\right], \tag{6.1.9}$$

where $\mathscr{L}_t = \{x : f(x) > t\}$.

Corollary 1. *For any conductor K in Ω the inequality*

$$c_p(K) \geq \inf \left\{ \left(- \int_0^1 \frac{d}{dt} m_n(\mathscr{L}_t) \frac{dt}{[s(\mathscr{E}_t)]^{p/(p-1)}} \right)^{1-p} : f \in V_\Omega(K) \right\} \tag{6.1.10}$$

holds.

Corollary 2. *Let F be an open set closed in Ω and let G, H be bounded open subsets of Ω such that*

$$F \subset G, \qquad \operatorname{clos}_\Omega G \subset H.$$

The conductors

$$K^{(1)} = G \backslash F, \qquad K^{(2)} = H \backslash \operatorname{clos}_\Omega G, \qquad K^{(3)} = H \backslash F,$$

satisfy the inequality

$$\big[c_p\big(K^{(1)}\big)\big]^{-1/(p-1)} + \big[c_p\big(K^{(2)}\big)\big]^{-1/(p-1)} \le \big[c_p\big(K^{(3)}\big)\big]^{-1/(p-1)}. \tag{6.1.11}$$

We present one more property of the p-conductivity, which is proved similarly to Theorem 2.3.1.

Lemma 3. *Let $u \in C^{0,1}(\Omega)$, $u = 0$ on the exterior of an open bounded set $G \subset \Omega$ and let $\mathscr{K}_t$ be the conductor $G \backslash \mathscr{N}_t$. Then, for $p \ge 1$,*

$$\int_0^\infty c_p(\mathscr{K}_t)\, \mathrm{d}\big(t^p\big) \le \frac{p^p}{(p-1)^{p-1}} \int_\Omega |\nabla u|^p \, \mathrm{d}x. \tag{6.1.12}$$

(*For $p = 1$ the coefficient in front of the second integral in* (6.1.12) *is equal to unity.*)

6.2 Multiplicative Inequality for Functions Which Vanish on a Subset of Ω

In this section we find a necessary and sufficient condition for the validity of the inequality

$$\|u\|_{L_q(\Omega)} \le C \|\nabla u\|^{1-\varkappa}_{L_p(\Omega)} \|u\|^{\varkappa}_{L_r(\Omega)} \tag{6.2.1}$$

for all functions that vanish on some subset of Ω.

Let G be an open bounded subset of Ω. For $p > 1$ we put

$$\mathfrak{A}_G^{(p,\alpha)} = \sup_{\{F\}} \frac{[m_n(F)]^\alpha}{[c_p(G \backslash F)]^{1/p}},$$

where $\{F\}$ is the collection of closed (in Ω) subsets of G with $c_p(G \backslash F) > 0$.

Let $\mathfrak{A}_G^{(1,\alpha)}$ denote the value $\mathfrak{A}_G^{(\alpha)}$ introduced in Sect. 5.2.3, i.e.,

$$\mathfrak{A}_G^{(1,\alpha)} = \sup_{\{\mathscr{G}\}} \frac{[m_n(\mathscr{G})]^\alpha}{s(\partial_i \mathscr{G})},$$

where $\{\mathscr{G}\}$ is the collection of admissible subsets of G.

The following assertion is a generalization of Lemma 5.2.3/1 (the case $q^* \ge p = 1$).

Lemma. *Let $p \geq 1$ and let G be an open bounded subset of Ω.*

1. *If $\mathfrak{A}_G^{(p,\alpha)} < \infty$ and the numbers q, α, p satisfy either one of the following conditions:*

$$\text{(i)} \quad q \leq q^* = \alpha^{-1} \quad \text{for } \alpha p \leq 1,$$
$$\text{(ii)} \quad q < q^* = \alpha^{-1} \quad \text{for } \alpha p > 1,$$

then for all functions $u \in C^{0,1}(\Omega)$ which vanish on the exterior of G, the inequality (6.2.1) holds with $r \in (0, q)$, $\varkappa = r(q^* - q)/q(q^* - r)$, and $C \leq c(\mathfrak{A}_G^{(p,\alpha)})^{1-\varkappa}$.

2. *Let $q^* > 0$, $r \in (0, q^*)$ and for some $q \in (0, q^*]$ and for any function $u \in C^{0,1}(\Omega)$ which vanishes on the exterior of G, let the inequality (6.2.1) be valid with $\varkappa = r(q^* - q)/q(q^* - r)$ and with a constant C that is independent of u.*

Then $C \geq c(\mathfrak{A}_G^{(p,\alpha)})^{1-\varkappa}$.

Proof. 1. Duplicating the proof of the first part of Theorem 2.3.6 (for $\mu = m_n$), we arrive at

$$\|u\|_{L_q(\Omega)} \leq c\left(\int_0^\infty \left[m_n(\mathscr{N}_t)\right]^{p\alpha} t^{p-1}\, \mathrm{d}t\right)^{(1-\varkappa)/p} \|u\|^{\varkappa}_{L_r(\Omega)}. \tag{6.2.2}$$

For $p > 1$ this implies

$$\|u\|_{L_q(\Omega)} \leq c\big(\mathfrak{A}_G^{(p,\alpha)}\big)^{1-\varkappa}\left(\int_0^\infty c_p(\mathscr{K}_t) t^{p-1}\, \mathrm{d}t\right)^{(1-\varkappa)/p} \|u\|^{\varkappa}_{L_r(\Omega)},$$

where $\mathscr{K}_t$ is the conductor $G \backslash \mathscr{N}_t$. Now the result follows from Lemma 6.1.3/3.

In the case $p = 1$ inequality (6.2.1) results from (6.2.2) along with Lemma 5.1.2/2 and the formula

$$\int_\Omega |\nabla u|\, \mathrm{d}x = \int_0^\infty s(\mathscr{E}_t)\, \mathrm{d}t \tag{6.2.3}$$

(cf. Theorem 1.2.4).

2. Let $p > 1$. We fix a small positive number $\delta > 0$ and put

$$\beta_\delta = \sup \frac{[m_n(F)]^{p\alpha}}{c_p(G \backslash F)}$$

on the set of all $F \subset G$ with $c_p(G \backslash F) \geq \delta$. (The substitution of an arbitrary function in $T_\Omega(G \backslash F)$ into (6.2.1) implies

$$\big[c_p(G \backslash F)\big]^{1-\varkappa} \geq C^{-p}\big[m_n(F)\big]^{p/q}\big[m_n(G)\big]^{-\varkappa p/r},$$

which means that the collection of sets F, contained in G and satisfying the inequality $c_p(G \backslash F) \geq \delta$, is not empty.) Obviously, $\beta_\delta \leq \delta^{-1} m_n(G)^{p\alpha}$.

Further, we must duplicate the proof of the second part of Theorem 2.3.6 with Lemma 2.2.2/1 replaced by Lemma 6.1.3/1 and with (p,Φ)-cap(F,Ω) replaced by $c_p(G\backslash F)$. Then we arrive at $\beta_\delta \le cC^{p(1-\varkappa)}$, which, since δ is arbitrary, implies $C \ge c(\mathfrak{A}_G^{(p,\alpha)})^{1-\varkappa}$ for $p>1$.

Consider the case $p=1$. Let $\mathscr{G}$ be any admissible subset of G. We insert the sequence of functions $\{w_m\}_{m\ge 1}$ specified in Lemma 5.2.2 into (6.2.1). Then

$$\big[s(\partial_i\mathscr{G})\big]^{1-\varkappa} \ge C^{-1}\big[m_n(e)\big]^{1/q}\big[m_n(\mathscr{G})\big]^{-\varkappa/r},$$

which, since e is arbitrary, yields

$$s(\partial_i\mathscr{G}) \ge C^{-1}\big[m_n(\mathscr{G})\big]^{\alpha}.$$

The lemma is proved. □

Corollary. *If* $p_1>p\ge 1$ *and* $\alpha_1-p_1^{-1}=\alpha-p^{-1}$, *then* $\mathfrak{A}_G^{(p_1,\alpha_1)} \le c\mathfrak{A}_G^{(p,\alpha)}$.

Proof. We put $|u|^{q_1/q}$ with $q_1>q$, $q_1^{-1}-p_1^{-1}=q^{-1}-p^{-1}$ in place of u in (6.2.1). Then

$$\|u\|_{L_{q_1}(\Omega)}^{q_1/q} \le c\bigg(\frac{q_1}{q}\bigg)^{1-\varkappa}\big(\mathfrak{A}_G^{(p,\alpha)}\big)^{1-\varkappa}\big\||u|^{(q_1-q)/q}\nabla u\big\|_{L_p(\Omega)}^{1-\varkappa}\|u\|_{L_{r_1}(\Omega)}^{\varkappa q_1/q}, \quad (6.2.4)$$

where $r_1=rq_1/q$. Applying the Hölder inequality, we obtain

$$\begin{aligned}&\int_\Omega |u|^{p(q_1-q)/q}|\nabla u|^p\,\mathrm{d}x\\ &\quad\le \bigg(\int_\Omega |u|^{(q_1-q)pp_1/(p_1-p)q}\,\mathrm{d}x\bigg)^{(p_1-p)/p_1}\bigg(\int_\Omega |\nabla u|^{p_1}\,\mathrm{d}x\bigg)^{p/p_1}\\ &\quad= \|u\|_{L_{q_1}(\Omega)}^{(p_1-p)q_1/p_1}\|\nabla u\|_{L_{p_1}(\Omega)}^{p}.\end{aligned}$$

The preceding inequality and (6.2.4) imply

$$\|u\|_{L_{q_1}(\Omega)} \le c\big(\mathfrak{A}_G^{(p,\alpha)}\big)^{1-\varkappa_1}\|\nabla u\|_{L_{p_1}(\Omega)}^{1-\varkappa_1}\|u\|_{L_{r_1}(\Omega)}^{\varkappa_1},$$

where $\varkappa_1=r_1(q_1^*-q_1)/q_1(q_1^*-r_1)$, $q_1^*=\alpha_1^{-1}$. Using the second part of the above Lemma we obtain that $\mathfrak{A}_G^{(p_1,\alpha_1)} \le c\mathfrak{A}_G^{(p,\alpha)}$. The corollary is proved. □

6.3 Classes of Sets $\mathscr{I}_{p,\alpha}$

6.3.1 Definition and Simple Properties of $\mathscr{I}_{p,\alpha}$

Definition 1. A domain Ω belongs to *the class* $\mathscr{I}_{p,\alpha}(p\ge 1,\alpha>0)$ if there exists a constant $M\in(0,m_n(\Omega))$ such that

$$\mathfrak{A}_{p,\alpha}(M) \overset{\text{def}}{=} \sup_{\{K\}} \frac{[m_n(F)]^\alpha}{[c_p(K)]^{1/p}} < \infty, \tag{6.3.1}$$

where $\{K\}$ is the collection of conductors $K = G\backslash F$ in Ω with positive p-conductivity and such that $m_n(G) \le M$.

Proposition 6.1.2/2 implies that $m_n(F) = 0$ for any conductor $K = G\backslash F$ with zero p-conductivity provided $\Omega \in \mathscr{I}_{p,\alpha}$.

Remark. The class $\mathscr{I}_{p,\alpha}$ is empty for $\alpha < (n-p)/np$, $n > p$ since in this case $c_p(B_{2r}\backslash \bar{B}_r) = \text{const}\, r^{n-p}$ and therefore

$$\frac{[m_n(B_r)]^\alpha}{[c_p(B_{2r}\backslash \bar{B}_r)]^{1/p}} = \text{const}\, r^{n(\alpha-(n-p)/np)} \to \infty \quad \text{as } r \to 0.$$

Proposition 1. *If a domain Ω is the union of a finite number of domains in $\mathscr{I}_{p,\alpha}$, then Ω is in the same class.*

Proof. Let $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_i \in \mathscr{I}_{p,\alpha}$, $i = 1, 2$. Then there exist constants M_1, M_2 such that

$$\big[m_n(F_i)\big]^\alpha \le \mathfrak{A}^{(i)}_{p,\alpha}(M_i)\big[c_p(K_i)\big]^{1/p}$$

for any conductor $K_i = G_i\backslash F_i$ in Ω_i with $m_n(G_i) \le M_i$.

Put $M = \min\{M_1, M_2\}$ and let K denote a conductor $G\backslash F$ with $m_n(G) \le M$. Further let $G_i = G \cap \Omega_i$, $K_i = G_i\backslash F_i$. If $c_p(K_1) = 0$, then $m_n(F_1) = 0$ and hence $m_n(F) = m_n(F_2)$. Therefore,

$$\big[m_n(F)\big]^\alpha \le \mathfrak{A}^{(2)}_{p,\alpha}(M_2)\big[c_p(K_2)\big]^{1/p} \le \mathfrak{A}^{(2)}_{p,\alpha}(M_2)\big[c_p(K)\big]^{1/p}.$$

In the case $c_p(K_i) > 0$, $i = 1, 2$, we have

$$\frac{[m_n(F)]^\alpha}{[c_p(K)]^{1/p}} \le c\bigg(\frac{[m_n(F_1)]^\alpha}{[c_p(K_1)]^{1/p}} + \frac{[m_n(F_2)]^\alpha}{[c_p(K_2)]^{1/p}}\bigg) \le c\sum_{i=1}^{2} \mathfrak{A}^{(i)}_{p,\alpha}(M_i).$$

The proposition is proved. □

Definition 2. We say that $\mathscr{K}$ is an *admissible conductor* if $\mathscr{K} = \mathscr{G}\backslash \operatorname{clos}_\Omega g$, where $\mathscr{G}$ and g are admissible subsets of Ω (cf. the definition at the beginning of Sect. 5.1.1).

Proposition 2. *We have*

$$\mathfrak{A}_{p,\alpha}(M) = \sup_{\{\mathscr{K}\}} \frac{[m_n(g)]^\alpha}{[c_p(\mathscr{K})]^{1/p}}, \tag{6.3.2}$$

where $\{\mathscr{K}\}$ is the collection of admissible conductors $K = \mathscr{G}\backslash \operatorname{clos}_\Omega g$ with positive p-conductivity and with $m_n(\mathscr{G}) \le M$. (Hence we may restrict ourselves to admissible conductors in the definition of the class $\mathscr{I}_{p,\alpha}$.)

Proof. We prove the inequality

$$\mathfrak{A}_{p,\alpha}(M) \leq \sup_{\{\mathscr{K}\}} \frac{[m_n(g)]^\alpha}{[c_p(\mathscr{K})]^{1/p}}. \tag{6.3.3}$$

The reverse inequality is obvious. Let K be any conductor in Definition 1. Given any $\varepsilon > 0$ we can find an admissible conductor $\mathscr{K} = \mathscr{G} \backslash \operatorname{clos}_\Omega g$, $\mathscr{K} \subset K$, such that $c_p(K) \geq (1-\varepsilon)c_p(\mathscr{K})$ (cf. Proposition 6.1.2/2). It is clear that $m_n(\mathscr{G}) \leq M$ and

$$\frac{[m_n(F)]^\alpha}{[c_p(K)]^{1/p}} \leq \frac{[m_n(g)]^\alpha}{(1-\varepsilon)^{1/p}[c_p(\mathscr{K})]^{1/p}},$$

which immediately implies (6.3.3). □

6.3.2 Identity of the Classes $\mathscr{I}_{1,\alpha}$ and $\mathscr{J}_\alpha$

Lemma. *The classes $\mathscr{I}_{1,\alpha}$ and $\mathscr{J}_\alpha$ coincide and*

$$\mathfrak{A}_{1,\alpha}(M) = \sup_{\{\mathscr{G}\}} \frac{[m_n(\mathscr{G})]^\alpha}{s(\partial_i \mathscr{G})}, \tag{6.3.4}$$

where $\{\mathscr{G}\}$ is the collection of admissible subsets of Ω with $m_n(\mathscr{G}) \leq M$.

Proof. Let $\mathscr{G}$ be an admissible subset of Ω with $m_n(\mathscr{G}) \leq M$ and let $\{w_m\}_{m\geq 1}$ be the sequence of functions specified in Lemma 5.2.2. The properties of $\{w_m\}$ imply $s(\partial_i \mathscr{G}) \geq c_1(\mathscr{G} \backslash e)$ for any compactum e contained in $\mathscr{G}$. If $\Omega \in \mathscr{I}_{1,\alpha}$, then $[m_n(e)]^\alpha \leq \mathfrak{A}_{1,\alpha}(M)\, s(\partial_i \mathscr{G})$ and hence

$$\big[m_n(\mathscr{G})\big]^\alpha = \sup_{e \subset \mathscr{G}} \big[m_n(e)\big]^\alpha \leq \mathfrak{A}_{1,\alpha}(M)\, s(\partial_i \mathscr{G}).$$

Suppose $\Omega \in \mathscr{J}_\alpha$. Let K be an arbitrary conductor $G \backslash F$ with $m_n(G) \leq M$. By (6.2.3) for any $f \in T_\Omega(K)$ we obtain

$$\int_\Omega |\nabla f|\, \mathrm{d}x \geq \inf_{\{\mathscr{G}\}} \big\{ s(\partial_i \mathscr{G}) : G \supset \mathscr{G} \supset F \big\} \geq \inf_{\{\mathscr{G}\}} \frac{s(\partial_i \mathscr{G})}{[m_n(\mathscr{G})]^\alpha} \big[m_n(F)\big]^\alpha. \tag{6.3.5}$$

By Lemma 6.1.1/2 we have

$$\inf \bigg\{ \int |\nabla f|\, \mathrm{d}x : f \in T_\Omega(K) \bigg\} = c_1(K),$$

so (6.3.5) implies

$$\mathfrak{A}_{1,\alpha}(M) \leq \sup_{\{\mathscr{G}\}} \frac{[m_n(\mathscr{G})]^\alpha}{s(\partial_i \mathscr{G})}.$$

The result follows. □

6.3.3 Necessary and Sufficient Condition for the Validity of a Multiplicative Inequality for Functions in $W^1_{p,s}(\Omega)$

Lemmas 6.2 and 6.3.2 imply the following obvious assertion.

Corollary. 1. *Let $\mathfrak{A}_{p,\alpha}(M) < \infty$ for some $M \in (0, m_n(\Omega))$.*

Then for all $u \in C^{0,1}(\Omega)$ with $m_n(\operatorname{supp} u) \le M$ the inequality (6.2.1) holds with $r \in (0,q)$, $\varkappa = r(q^* - q)/q(q^* - r)$, q being the number specified in Lemma 6.2 and $C \le c[\mathfrak{A}_{p,\alpha}(M)]^{1-\varkappa}$.

2. *If (6.2.1) holds for all $u \in C^{0,1}(\Omega)$ with $m_n(\operatorname{supp} u) \le M < m_n(\Omega)$, then $C \ge c[\mathfrak{A}_{p,\alpha}(M)]^{1-\varkappa}$.*

The introduction of the classes $\mathscr{I}_{p,\alpha}$ is justified by the following theorem.

Theorem. 1. *Let $\Omega \in \mathscr{I}_{p,\alpha}$ and let q be a positive number satisfying either one of the conditions:* (i) *either $q \le q^* = \alpha^{-1}$ for $p \le q^*$, or* (ii) *$q < q^*$ for $p > q^*$.*

Then for any $u \in W^1_{p,s}(\Omega)$ we have

$$\|u\|_{L_q(\Omega)} \le \big(C_1 \|\nabla u\|_{L_p(\Omega)} + C_2 \|u\|_{L_s(\Omega)}\big)^{1-\varkappa} \|u\|^{\varkappa}_{L_r(\Omega)}, \tag{6.3.6}$$

where $s < q^*$, $r < q$, $\varkappa = r(q^* - q)/q(q^* - r)$, $C_2 = cM^{(s-q^*)/sq^*}$, and $C_1 \le c\mathfrak{A}_{p,\alpha}(M)$.

2. *Let $q^* > 0$ and for some $q \in (0, q^*]$ and all $u \in W^1_{p,s}(\Omega)$, let (6.3.6) hold with $0 < s < q^*$, $0 < r < q^*$, $\varkappa = r(q^* - q)/q(q^* - r)$.*

Then $\Omega \in \mathscr{I}_{p,\alpha}$ with $\alpha = 1/q^*$. Moreover, if the constant M in the definition of the class $\mathscr{I}_{p,\alpha}$ is specified by $M = c\, C_2^{sq^*/(s-q^*)}$, where c is a small enough positive constant depending only on p, q^*, s, then $C_1 \ge c\, \mathfrak{A}_{p,\alpha}(M)$.

Proof. 1. By Lemma 6.1.1/1, it suffices to obtain (6.3.6) for functions $u \in L^1_p(\Omega) \cap L_\infty(\Omega) \cap C^\infty(\Omega)$ with bounded supports. Let $T = \inf\{t : m_n(\mathscr{N}_t) < M\}$. Clearly, $m_n(\mathscr{L}_t) \le M \le m_n(\mathscr{N}_t)$. Further we note that

$$\int_\Omega |u|^q \,\mathrm{d}x \le \int_{\Omega\setminus\mathscr{N}_T} |u|^q \,\mathrm{d}x + cT^q m_n(\mathscr{N}_T) + c\int_{\mathscr{N}_T} \big(|u| - T\big)^q \,\mathrm{d}x. \tag{6.3.7}$$

To get a bound for the first summand on the right we rewrite it as follows:

$$\int_{\Omega\setminus\mathscr{N}_T} |u|^q \,\mathrm{d}x = \int_{\Omega\setminus\mathscr{N}_T} |u|^{q^*(q-r)/(q^*-r)} |u|^{r(q^*-q)/(q^*-r)} \,\mathrm{d}x,$$

and use the Hölder inequality

$$\int_{\Omega\setminus\mathscr{N}_T} |u|^q \,\mathrm{d}x \le \Big(\int_{\Omega\setminus\mathscr{N}_T} |u|^{q^*} \,\mathrm{d}x\Big)^{(q-r)/(q^*-r)} \Big(\int_{\Omega\setminus\mathscr{N}_T} |u|^r \,\mathrm{d}x\Big)^{(q^*-q)/(q^*-r)}.$$

Since $u < T$ on $\Omega\setminus\mathscr{N}_T$, the right-hand side of the last estimate does not exceed

$$T^{(q^*-s)(q-r)/(q^*-r)}\left(\int_{\Omega\setminus\mathscr{N}_T}|u|^s\,\mathrm{d}x\right)^{(q-r)/(q^*-r)}$$
$$\times\left(\int_{\Omega\setminus\mathscr{N}_T}|u|^r\,dx\right)^{(q^*-q)/(q^*-r)}.$$

This implies

$$\begin{aligned}\int_{\Omega\setminus\mathscr{N}_T}|u|^q\,\mathrm{d}x &\le M^{(s-q^*)(q-r)/s(q^*-r)}\left(\int_{\Omega\setminus\mathscr{N}_T}|u|^r\,\mathrm{d}x\right)^{(q^*-q)/(q^*-r)}\\ &\quad\times\left(\int_{\mathscr{N}_T}|u|^s\,\mathrm{d}x\right)^{(q^*-s)(q-r)/s(q^*-r)}\\ &\quad\times\left(\int_{\Omega\setminus\mathscr{N}_T}|u|^s\,\mathrm{d}x\right)^{(q-r)/(q^*-r)}\\ &\le M^{(s-q^*)(q-r)/s(q^*-r)}\left(\int_{\Omega}|u|^s\,\mathrm{d}x\right)^{q^*(q-r)/s(q^*-r)}\\ &\quad\times\left(\int_{\Omega}|u|^r\,\mathrm{d}x\right)^{(q^*-q)/(q^*-r)}.\end{aligned}\qquad(6.3.8)$$

Next we estimate the second summand on the right in (6.3.7). Since $|u(x)|\ge T$ on $\mathscr{N}_T$ and $m_n(\mathscr{N}_T)\ge M$, we obtain

$$\begin{aligned}T^q m_n(\mathscr{N}_T) &\le M^{(s-q^*)(q-r)/s(q^*-r)}\left(\int_{\mathscr{N}_T}|u|^s\,\mathrm{d}x\right)^{q^*(q-r)/s(q^*-r)}\\ &\quad\times\left(\int_{\mathscr{N}_t}|u|^r\,\mathrm{d}x\right)^{(q^*-q)/(q^*-r)}.\end{aligned}$$

Combining this inequality with (6.3.8) and (6.3.7) we arrive at

$$\|u\|_{L_q(\Omega)}\le c\big\|\big(|u|-T\big)^+\big\|_{L_q(\Omega)}+cM^{(1-\varkappa)(s-q^*)/sq^*}\|u\|^{1-\varkappa}_{L_s(\Omega)}\|u\|^{\varkappa}_{L_r(\Omega)}.$$

It remains to apply the first part of the Corollary to $(|u|-T)^+$.

2. Let M be an arbitrary constant satisfying

$$2c_0M^{1/s-1/q^*}C_2<1,$$

where c_0 is a constant that depends only on s, p, q, q^*, r to be specified at the end of the proof.

Let δ denote a small enough positive number; by K we mean the conductor $G\setminus F$ in Ω with $m_n(G)\le M$, $c_p(K)\ge\delta$. Further we introduce the function $\beta_\delta=\sup[m_n(G)]^{p/q}/c_p(K)$ where the supremum is taken over the above set of conductors. It is clear that $\beta_\delta<\infty$ and

$$\mathfrak{A}_{p,\alpha}(M) = \lim_{\delta\to+0} \beta_\delta^{1/p}. \tag{6.3.9}$$

Since $s < q^*$, for any $u \in T_\Omega(K)$, we have

$$\|u\|_{L_s(\Omega)} \le c\Big(\sup_{0<t<1} \frac{[m_n(\mathscr{N}_t)]^{1/q^*}}{[c_p(G\backslash\mathscr{N}_t)]^{1/p}}\|\nabla u\|_{L_p(\Omega)}\Big)^{q^*(s-t)/s(q^*-t)} \times \|u\|_{L_t(\Omega)}^{r(q^*-s)/s(q^*-t)},$$

where $t < s$ (cf. the proof of the first part of Lemma 6.2).

Now we note that $[m_n(\mathscr{N}_t)]^{p/q^*} \le \beta_\delta c_p(G\backslash\mathscr{N}_t)$ since $c_p(G\backslash\mathscr{N}_t) \ge c_p(K) \ge \delta$. Besides,

$$\|u\|_{L_t(\Omega)} \le M^{1/t-1/s}\|u\|_{L_s(\Omega)}.$$

Consequently,

$$\|u\|_{L_s(\Omega)} \le c\, M^{1/s-1/q}\beta_\delta^{1/p}\|\nabla u\|_{L_p(\Omega)}.$$

Using the preceding estimate, from (6.3.6) we obtain

$$\big[m_n(F)\big]^{1/q} \le \big(C_1 + c\beta_\delta^{1/p}M^{1/s-1/q^*}C_2\big)^{1-\varkappa}\|\nabla u\|_{L_p(\Omega)}^{1-\varkappa}\|u\|_{L_r(\Omega)}^{\varkappa}.$$

Further, we must duplicate the proof of the second part of Theorem 2.3.6 with Lemma 2.2.2/3 replaced by Lemma 6.1.3/1 and with (p,Φ)-cap(F,G) replaced by $c_p(K)$. As a result we obtain

$$\beta_\delta^{1/p} \le c\big(C_1 + \beta_\delta^{1/p}M^{1/s-1/q^*}C_2\big).$$

This inequality and the definition of the constant M imply

$$\beta_\delta(M) \le (2c_0C_1)^p.$$

It remains to make use of (6.3.9). This completes the proof. □

6.3.4 Criterion for the Embedding $W_{p,s}^1(\Omega) \subset L_{q^*}(\Omega)$, $p \le q^*$

An important particular case of Theorem 6.3.3 is the following criterion for the embedding of $W_{p,s}^1(\Omega)$ into $L_{q^*}(\Omega)$.

Corollary. 1. *If $\Omega \in \mathscr{I}_{p,\alpha}$, $q^* \ge p$ and $s < q^*$, then for any $u \in W_{p,s}^1(\Omega)$*

$$\|u\|_{L_{q^*}(\Omega)} \le C_1\|\nabla u\|_{L_p(\Omega)} + c_2\|u\|_{L_s(\Omega)}, \tag{6.3.10}$$

where $C_1 \le c\mathfrak{A}_{p,1/q^*}(M)$ and $C_2 = cM^{(s-q^*)/sq^*}$.

2. *Let for all $u \in W_{p,s}^1(\Omega)$ the inequality (6.3.10) hold, where $0 < s < q^*$. Then $\Omega \in \mathscr{I}_{p,1/q^*}$ and if the constant M in the definition of the class $\mathscr{I}_{p,1/q^*}$ is defined by $M \le cC_2^{q^*/(s-q^*)}$ with sufficiently small $c = c(q^*, s)$, then $C_1 \ge c\mathfrak{A}_{p,1/q^*}(M)$.*

6.3.5 Function $\nu_{M,p}$ and the Relationship of the Classes $\mathscr{I}_{p,\alpha}$ and $\mathscr{J}_\alpha$

Definition. Let $\nu_{M,p}(t)$ be the infimum of $c_p(K)$ taken over the collection of all conductors $K = G\backslash F$ with $m_n(F) \geq t$, $m_n(G) \leq M$.

By an argument similar to that in the proof of Proposition 6.3.1/2 we can prove that we may restrict ourselves to admissible conductors in this definition.

Inequality (6.1.10) immediately implies the following assertion on the connection of $\nu_{M,p}$ with the isoperimetric function λ_M introduced in Sect. 5.2.4.

Proposition 1. *The inequality*

$$\nu_{M,p}(t) \geq \left(\int_t^M \big[\lambda_M(\sigma)\big]^{p/(1-p)}\, \mathrm{d}\sigma\right)^{1-p} \tag{6.3.11}$$

holds.

Obviously, for $p \geq 1$

$$\mathfrak{A}_{p,\alpha}(M) = \sup_{0<t\leq M} t^\alpha \big[\nu_{M,p}(t)\big]^{-1/p}. \tag{6.3.12}$$

Hence $\Omega \in \mathscr{I}_{p,\alpha}$ if and only if

$$\liminf_{t\to+0} t^{-\alpha p}\nu_{M,p}(t) > 0. \tag{6.3.13}$$

We have noted already in Sect. 5.2.4 that $\Omega \in \mathscr{J}_\alpha = \mathscr{I}_{1,\alpha}$ if and only if

$$\liminf_{t\to+0} t^{-\alpha}\lambda_M(t) > 0.$$

Proposition 2. *If $\Omega \in \mathscr{J}_{\alpha+(p-1)/p}$, then $\Omega \in \mathscr{I}_{p,\alpha}$ and*

$$\mathfrak{A}_{p,\alpha}(M) \leq \left(\frac{p-1}{p\,\alpha}\right)^{(p-1)/p} \mathfrak{A}_{1,\alpha+(p-1)/p}(M). \tag{6.3.14}$$

Proof. From (6.3.11) we obtain

$$\begin{aligned}\nu_{M,p}(t) &\geq \big[\mathfrak{A}_{1,\alpha+(p-1)/p}(M)\big]^{-p}\left(\int_t^M \sigma^{-(p\alpha+p-1)/(p-1)}\mathrm{d}\sigma\right)^{1-p}\\ &> \left(\frac{p\alpha}{p-1}\right)^{p-1}\big[\mathfrak{A}_{1,\alpha+(p-1)/p}(M)\big]^{-p} t^{\alpha p}.\end{aligned}$$

The result follows. □

Remark. By Corollary 6.2, the class $\mathscr{I}_{p,\alpha}$ is a part of the class $\mathscr{I}_{p_1,\alpha_1}$ and $\mathfrak{A}_{p_1,\alpha_1}(M) \leq c\,\mathfrak{A}_{p,\alpha}(M)$ provided $p_1 > p \geq 1$ and $\alpha_1 - p_1^{-1} = \alpha - p^{-1}$.

6.3.6 Estimates for the Conductivity Minimizing Function $\nu_{M,p}$ for Certain Domains

We consider some domains for which explicit two-sided estimates for $\nu_{M,p}$ are valid.

Example 1. Let Ω be the domain $\{x : |x'| < f(x_n),\ 0 < x_n < a\}$ considered at the beginning of Sect. 5.3.3.

We show that

$$k^p \left(\int_{\alpha(\mu)}^{\alpha(M)} \big[f(\tau)\big]^{(1-n)/(p-1)} \, d\tau \right)^{1-p}$$
$$\leq \nu_{M,p}(\mu) \leq \left(\int_{\alpha(\mu)}^{\alpha(M)} \big[f(\tau)\big]^{(1-n)/(p-1)} \, d\tau \right)^{1-p}, \tag{6.3.15}$$

where k is the constant in the inequality (6.3.5) that depends on M and the function α is specified by

$$\mu = v_{n-1} \int_0^{\alpha(\mu)} \big[f(\tau)\big]^{n-1} \, d\tau.$$

Consider the conductor $K_{\mu,M} = G_{\alpha(M)} \backslash \operatorname{clos}_\Omega G_{\alpha(\mu)}$, where $G_\alpha = \{x \in \Omega : 0 < x_n < \alpha\}$. Let the function u be defined by $u(x) = 0$ outside $G_{\alpha(M)}$, $u(x) = 1$ on $G_{\alpha(\mu)}$, and

$$u(x) = \int_{x_n}^{\alpha(M)} \frac{d\tau}{[f(\tau)]^{(n-1)/(p-1)}} \left(\int_{\alpha(\mu)}^{\alpha(M)} \frac{d\tau}{[f(\tau)]^{(n-1)/(p-1)}} \right)^{-1}$$

on $G_{\alpha(M)} \backslash G_{\alpha(\mu)}$. Clearly, u is contained in $U_\Omega(K_{\mu,M})$. Hence, inserting u into the integral

$$\int_\Omega |\nabla u|^p \, dx,$$

we obtain

$$c_p(K_{\mu,M}) \leq \left(\int_{\alpha(\mu)}^{\alpha(M)} \big[f(\tau)\big]^{(1-n)/(p-1)} \, d\tau \right)^{1-p}. \tag{6.3.16}$$

Taking into account the definition of $\nu_{M,p}(\mu)$, we arrive at the right inequality (6.3.15).

Substituting the left inequality (5.3.5) into (6.3.11), we obtain the required lower bound for $\nu_{M,p}$.

From (6.3.15) we obtain that $\Omega \in \mathscr{I}_{p,\alpha}$ if and only if

$$\limsup_{x \to +0} \left(\int_0^x \big[f(\tau)\big]^{n-1} \, d\tau \right)^{\alpha p/(p-1)} \int_x^a \big[f(\tau)\big]^{(1-n)/(p-1)} \, d\tau < \infty.$$

In particular, for the domain

$$\Omega^{(\lambda)} = \big\{x : \big(x_1^2 + \cdots + x_{n-1}^2\big)^{1/2} < a x_n^\lambda,\ 0 < x_n < 1\big\} \tag{6.3.17}$$

with $\lambda > (p-1)/(n-1)$, for small t we have

$$ct^{\alpha p} \le \nu_{M,p}(t) \le c' t^{\alpha p},$$

where $\alpha = [\lambda(n-1) + 1 - p]/p[\lambda(n-1)+1]$. Thus the domain (6.3.17) is contained in $\mathscr{I}_{p,\alpha}$ for $p\alpha < 1$.

Example 2. Let Ω be the domain in Example 5.3.3/2. Using the estimates (5.3.8) for λ_M and following the same discussion as in Example 1 we obtain

$$k^p \bigg(\int_{\alpha(M)}^{\alpha(\mu)} \big[f(\tau)\big]^{(1-n)/(p-1)}\, \mathrm{d}\tau\bigg)^{1-p} \le \nu_{M,p}(\mu) \le \bigg(\int_{\alpha(M)}^{\alpha(\mu)} \big[f(\tau)\big]^{(1-n)/(p-1)} d\tau\bigg)^{1-p}. \tag{6.3.18}$$

Consequently, $\Omega \in \mathscr{I}_{p,\alpha}$ if and only if

$$\limsup_{x\to+\infty} \bigg(\int_x^\infty \big[f(\tau)\big]^{n-1}\, \mathrm{d}\tau\bigg)^{\alpha p/(p-1)} \int_0^x \big[f(\tau)\big]^{(1-n)/(p-1)}\, \mathrm{d}\tau < \infty.$$

In particular, for the domain

$$\big\{x : x_1^2 + \cdots + x_{n-1}^2 < (1+x_n)^{-2\beta},\ 0 < x_n < \infty\big\}$$

with $\beta(n-1) > 1$ we have

$$ct^{\alpha p} \le \nu_{M,p}(t) \le c' t^{\alpha p} \quad \text{for small } t,$$

provided $\alpha p = [\beta(n-1) - 1 + p][\beta(n-1)-1]^{-1}$. Here $\alpha p > 1$.

For the domain

$$\big\{x : x_1^2 + \cdots + x_{n-1}^2 < e^{-cx_n},\ 0 < x_n < \infty\big\}, \quad c = \text{const} > 0,$$

by (6.3.18) we obtain

$$c\mu \le \nu_{M,p}(\mu) \le c'\mu,$$

and hence this domain is in $\mathscr{I}_{p,1/p}$.

Example 3. Let Ω be the spiral domain considered in Example 5.3.3/3. Proposition 6.3.5/1 along with the lower bound (5.3.11) for λ yields the following lower bound for $\nu_{M,p}(t)$:

$$\nu_{M,p}(t) \ge C_M^{(1)} \bigg(\int_{\theta(M)}^{\theta(t)} \big[\delta(\theta)\big]^{-1/(p-1)}\, \mathrm{d}\theta\bigg)^{1-p}, \tag{6.3.19}$$

where $\theta(t)$ is specified by (5.3.12) and $\delta = \varepsilon_2 - \varepsilon_1$.

To derive a similar upper bound consider the conductor $G_{\theta(t)} \setminus \operatorname{clos}_\Omega G_{\theta(M)}$, where $G_\theta = \{\varrho e^{i\varphi} \in \Omega : 0 < \varphi < \theta\}$ and the function u defined by

$$u = 0 \quad \text{outside } G_{\theta(t)}, \qquad u = 1 \quad \text{on } G_{\theta(M)},$$

$$u(\theta) = \int_{\theta(M)}^{\theta} \big[\delta(\varphi)\big]^{1/(1-p)} \,\mathrm{d}\varphi \bigg(\int_{\theta(M)}^{\theta(t)} \big[\delta(\varphi)\big]^{1/(1-p)} \,\mathrm{d}\varphi \bigg)^{-1} \quad \text{on } G_{\theta(t)} \setminus \bar{G}_{\theta(M)}.$$

It is clear that

$$\begin{aligned} c_p\big[G_{\theta(t)} \setminus \operatorname{clos}_\Omega\ G_\theta(M)\big] &\le \iint_{G_{\theta(t)} \setminus G_{\theta(M)}} \big|\varrho^{-1} u'(\varphi)\big|^p \varrho \,\mathrm{d}\varrho \,\mathrm{d}\varphi \\ &\le C_M^{(2)} \bigg(\int_{\theta(M)}^{\theta(t)} \big[\delta(\varphi)\big]^{1/(1-p)} \,\mathrm{d}\varphi \bigg)^{1-p}. \end{aligned}$$

This and (6.3.19) imply

$$\begin{aligned} &C_M^{(1)} \bigg(\int_{\theta(M)}^{\theta(t)} \big[\delta(\varphi)\big]^{1/(1-p)} \,\mathrm{d}\varphi \bigg)^{1-p} \\ &\quad \le \nu_{M,p}(t) \le C_M^{(2)} \bigg(\int_{\theta(M)}^{\theta(t)} \big[\delta(\varphi)\big]^{1/(1-p)} \,\mathrm{d}\varphi \bigg)^{1-p}. \end{aligned} \tag{6.3.20}$$

In particular, for the domain

$$\big\{\varrho e^{i\varphi} : 1-(8+\theta)^{1-\beta} > \varrho > 1-(8+\theta)^{1-\beta} - c(8+\theta)^{-\beta},\ 0 < \theta < \infty\big\} \tag{6.3.21}$$

with $0 < c < 2\pi(\beta - 1)$, $\beta > 1$, we have $ct^{\alpha p} \le \nu_{M,p}(t) \le c' t^{\alpha p}$ for small t, where $\alpha = (\beta - 1 + p)/p(\beta - 1)$. Thus (6.3.21) is a bounded domain in the class $\mathscr{I}_{p,\alpha}$ for $p\,\alpha > 1$.

Example 4. We show that the adjoining nonintersecting cylinders

$$G_j = \big\{x : |x_n - \alpha_j| < a_j,\ x_1^2 + \cdots + x_{n-1}^2 < b_j^2\big\}$$

with $\sum_j a_j b_j^{n-1} < \infty$ (cf. Fig. 23) form a domain which is not in $\mathscr{I}_{p,1/p}$ if $\limsup_{j\to\infty} a_j = \infty$. Let

$$F_j = \big\{x \in G_j : |x_n - \alpha_j| \le a_j/2\big\}$$

and let $\eta \in C_0^\infty([0,1])$, $\eta(t) = 1$ for $0 \le t < \frac{1}{2}$. We insert the function $\eta(|x_n - a_j|/a_j)$ into the norm $\|\nabla u\|_{L_p(G_j)}$. Then

$$c_p(G_j \setminus F_j) \le c a_j^{-p} m_n(F_j).$$

Thus

$$\liminf_{\mu\to 0} \mu^{-1} \nu_{M,p}(\mu) = 0.$$

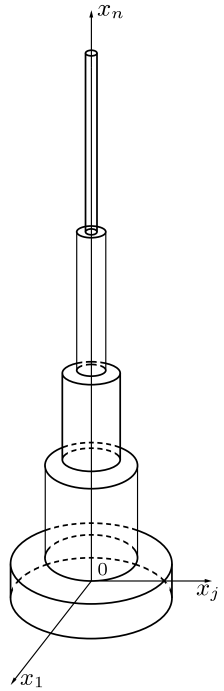

Fig. 23.

6.4 Embedding $W^1_{p,s}(\Omega) \subset L_{q^*}(\Omega)$ for $q^* < p$

6.4.1 Estimate for the Norm in $L_{q^*}(\Omega)$ with $q^* < p$ for Functions which Vanish on a Subset of Ω

If $\Omega \in \mathscr{I}_{p,\alpha}$ with $\alpha p \le 1$, we have by Theorem 6.3.3 that the embedding operator of $W^1_{p,s}(\Omega)$ into $L_{q^*}(\Omega)$ with $q^* = \alpha^{-1}$ is bounded. The following example shows that inequality (6.3.6) with the limit exponent $q = q^* = \alpha^{-1}$ may fail provided $\alpha p > 1$ (or equivalently, provided $p > q^*$).

Example. Consider the plane domain

$$\Omega = \big\{(x_1, x_2) : |x_1| < (1 + x_2)^{-\beta},\ 0 < x_2 < \infty\big\}$$

with $\beta > 1$. In Example 6.3.6/2 it was established that this domain is in the class $\mathscr{I}_{p,\,1/p+1/(\beta-1)}$. We show that inequality (6.3.6) is not valid for

$$q = q^* = p(\beta - 1)/(p + \beta - 1).$$

The sequence of functions $u_m(x_1, x_2) = (1+x_2)^{\gamma_m}$ with $\gamma_m < 1+(\beta-1)/p$, $m = 1, 2, \dots,$ satisfies

$$\|u\|^{q^*}_{L_{q^*}(\Omega)} = 2\big[1 + (\beta-1)/p\big]/\big[\beta - 1 - p(\gamma_m - 1)\big],$$

$$\|\nabla u_m\|^p_{L_p(\Omega)} = 2\gamma_m/\big[\beta - 1 - p(\gamma_m - 1)\big], \qquad \|u_m\|^s_{L_s(\Omega)} = 2/(\beta - 1 - \gamma_m s).$$

Since $s < q^* < p$, the left-hand side in (6.3.6), written for u_m, tends to infinity as $m \to \infty$ more rapidly than the right-hand side. Thus the inclusion $\Omega \in \mathscr{I}_{p,\alpha}$ with $p > q^* = \alpha^{-1}$ is necessary, but not sufficient for the validity of (6.3.6) with the limit exponent $q = q^*$. A necessary and sufficient condition will be given in Theorem 6.4.2.

Let G be a bounded open subset of Ω and let $\nu_G^{(p)}(t)$ denote the infimum of $c_p(K)$ taken over the collection of all conductors $K = G\backslash F$ with $m_n(F) \geq t$. We put

$$\mathfrak{B}_G^{(p,\alpha)} = \left\{\int_0^{m_n(G)} \left[\frac{\tau}{\nu_G^{(p)}(\tau)}\right]^{1/(\alpha p - 1)} \mathrm{d}\tau\right\}^{\alpha - 1/p}, \tag{6.4.1}$$

where $\alpha p > 1$, $p \geq 1$.

Lemma. 1. *Let $\mathfrak{B}_G^{(p,\alpha)} < \infty$. Then the inequality*

$$\|u\|_{L_{q^*}(\Omega)} \leq C\|\nabla u\|_{L_p(\Omega)} \tag{6.4.2}$$

holds for all $u \in C^{0,1}(\Omega)$ which vanish outside G, with $q^* = \alpha^{-1}$, $\alpha p > 1$ and $C \leq c\mathfrak{B}_G^{(p,\alpha)}$.

2. *If (6.4.2) is valid for all $u \in C^{0,1}(\Omega)$ which vanish outside G, with some $q^* \in [1, p)$, then $C \geq c\mathfrak{B}_G^{(p,\alpha)}$.*

Proof differs only by obvious details from that of Theorem 2.3.8.

6.4.2 Class $\mathscr{H}_{p,\alpha}$ and the Embedding $W^1_{p,s}(\Omega) \subset L_{q^*}(\Omega)$ for $0 < q^* < p$

In this subsection we introduce the classes of sets which are adequate for stating a necessary and sufficient condition for the embedding $W^1_{p,s}(\Omega) \subset L_{q^*}(\Omega)$ with $0 < q^* < p$.

Let $\alpha p > 1$. We put

$$\mathfrak{B}_{p,\alpha}(M) = \left(\int_0^M \left[\frac{\tau}{\nu_{M,p}(\tau)}\right]^{1/(\alpha p - 1)} \mathrm{d}\tau\right)^{\alpha - 1/p}, \tag{6.4.3}$$

where M is a constant in $(0, m_n(\Omega))$.

Definition. The set Ω belongs to *the class* $\mathscr{H}_{p,\alpha}$ provided $\mathfrak{B}_{p,\alpha}(M) < \infty$ for some $M \in (0, m_n(\Omega))$.

The next corollary immediately follows from Lemma 6.4.1.

Corollary 1. 1. *Let $\alpha p > 1$ and $\mathfrak{B}_{p,\alpha}(M) < \infty$. Then for all $u \in C^{0,1}(\Omega)$ such that $m_n(\operatorname{supp} u) \leq M$ we have (6.4.2) with $q^* = \alpha^{-1}$ and $C \leq c\mathfrak{B}_{p,\alpha}(M)$.*

2. *If* (6.4.2) *holds for all* $u \in C^{0,1}(\Omega)$ *with* $m_n(\operatorname{supp} u) \le M$ *then* $C \ge c\mathfrak{B}_{p,\alpha}(M)$.

Duplicating with obvious simplifications the proof of Theorem 6.3.3, we obtain the following theorem from Corollary 1.

Theorem. 1. *Let* $\alpha p > 1$ *and* $\Omega \in \mathscr{H}_{p,\alpha}$. *Then*

$$\|u\|_{L_{q^*}(\Omega)} \le C_1 \|\nabla u\|_{L_p(\Omega)} + C_2 \|u\|_{L_s(\Omega)} \tag{6.4.4}$$

for all $u \in W^1_{p,s}(\Omega)$ *with* $q^* = \alpha^{-1}$, $s < q^*$,

$$C_2 = cM^{(s-q^*)/sq^*}, \qquad C_1 \le c\mathfrak{B}_{p,\alpha}(M).$$

2. *Let* (6.4.4) *hold for all* $u \in W^1_{p,s}(\Omega)$ *with* $1 \le q^* < p$, $s < q^*$. *Then* $\Omega \in \mathscr{H}_{p,\alpha}$. *Moreover, if* M *in the definition of* $\mathscr{H}_{p,\alpha}$ *is specified by* $M = cC_2^{sq^*/(s-q^*)}$, *where* c *is a small enough positive constant that depends only on* p, q^*, s, *then* $C_1 \ge c\mathfrak{B}_{p,\alpha}(M)$.

Corollary 2. *If* $p_1 > p \ge 1$ *and* $\alpha_1 - p_1^{-1} = \alpha - p^{-1}$, *then* $\mathfrak{B}^{(p_1,\alpha_1)}_G \le c\mathfrak{B}^{(p,\alpha)}_G$ *for any* G. (*Consequently,* $\mathfrak{B}_{p_1,\alpha_1}(M) \le c\mathfrak{B}_{p,\alpha}(M)$ *and* $\mathscr{H}_{p,\alpha} \subset \mathscr{H}_{p_1,\alpha_1}$.)

This assertion can be proved in the same way as Corollary 6.2.

6.4.3 Embedding $L^1_p(\Omega) \subset L_{q^*}(\Omega)$ for a Domain with Finite Volume

We present a necessary and sufficient condition for the validity of the Poincaré-type inequality

$$\inf_{c \in \mathbb{R}^1} \|u - c\|_{L_{q^*}(\Omega)} \le C \|\nabla u\|_{L_p(\Omega)}, \tag{6.4.5}$$

provided Ω is a domain with $m_n(\Omega) < \infty$.

By Lemma 5.2.3/2 the embedding $L^1_p(\Omega) \subset L_{q^*}(\Omega)$ $(p \ge 1, q^* \ge 1)$ and inequality (6.4.5) are equivalent. The case $m_n(\Omega) = \infty$ will be considered in 6.7.5.

Theorem 1. *Let* Ω *be a domain with finite volume.*

1. *If* $\mathfrak{A}_{p,\alpha}(\frac{1}{2}m_n(\Omega)) < \infty$ *for* $\alpha p \le 1$ *or* $\mathfrak{B}_{p,\alpha}(\frac{1}{2}m_n(\Omega)) < \infty$ *for* $\alpha p > 1$, *then* (6.4.5) *holds for all* $u \in L^1_p(\Omega)$ *with* $q^* = \alpha^{-1}$, $C \le c\mathfrak{A}_{p,\alpha}(\frac{1}{2}m_n(\Omega))$ *for* $\alpha p \le 1$ *and* $C \le c\mathfrak{B}_{p,\alpha}(\frac{1}{2}m_n(\Omega))$ *for* $\alpha p > 1$.

2. *If there exists a constant* C *such that* (6.4.5) *holds for all* $u \in L^1_p(\Omega)$ *with* $q^* \ge 1$, *then* $C \ge c\mathfrak{A}_{p,\alpha}(\frac{1}{2}m_n(\Omega))$ *for* $\alpha^{-1} = q^* \ge p$ *and* $C \ge c\mathfrak{B}_{p,\alpha}(\frac{1}{2}m_n(\Omega))$ *for* $\alpha^{-1} = q^* < p$.

Proof. The proof of *sufficiency* follows the same argument as in the proof of Theorem 5.2.3 except that we must apply Corollary 6.3.3 instead of Lemma 5.2.3/1 for $\alpha p \ge 1$ and Corollary 6.4.2/1 for $\alpha p < 1$.

Necessity. Let u be an arbitrary function in $C^{0,1}(\Omega) \cap L_p^1(\Omega)$ with

$$m_n(\operatorname{supp} u) \le \frac{1}{2} m_n(\Omega). \tag{6.4.6}$$

There exists a number c_0 such that

$$\|u - c_0\|_{L_{q^*}(\Omega)} = \inf_{c \in \mathbb{R}^1} \|u - c\|_{L_{q^*}(\Omega)}. \tag{6.4.7}$$

The latter and (6.4.5) imply

$$\int_{\operatorname{supp} u} |u - c_0|^{q^*} \, dx + |c_0|^{q^*} m_n(\Omega \backslash \operatorname{supp} u) \le C^{q^*} \|\nabla u\|_{L_p(\Omega)}^{q^*}.$$

Therefore

$$\frac{1}{2} m_n(\Omega) |c_0|^{q^*} \le C^{q^*} \|\nabla u\|_{L_p(\Omega)}^{q^*}.$$

Since

$$\|u\|_{L_{q^*}(\Omega)} \le |c_0| \big[m_n(\Omega)\big]^{1/q^*} + \|u - c_0\|_{L_{q^*}(\Omega)},$$

making use of (6.4.5) and (6.4.7), we obtain

$$\|u\|_{L_{q^*}(\Omega)} \le cC \|\nabla u\|_{L_p(\Omega)}$$

for all $u \in C^{0,1}(\Omega)$ satisfying (6.4.6). A reference to Corollaries 6.3.3 and 6.4.2/1 completes the proof. □

Theorem 2. *Let Ω be a domain with finite volume. The space $L_p^1(\Omega)$ is embedded into $L_{q^*}(\Omega)$ if and only if $\Omega \in \mathscr{I}_{p,1/q^*}$ for $p \le q^*$ and $\Omega \in \mathscr{H}_{p,1/q^*}$ for $p > q^* > 0$.*

Proof. The necessity follows from Lemma 5.2.3/2 and Theorem 1. To prove the sufficiency we must show that (6.4.5) is valid provided $\mathfrak{A}_{p,\alpha}(M)$ and $\mathfrak{B}_{p,\alpha}(M)$ are finite for some $M \in (0, \frac{1}{2} m_n(\Omega))$. Let $u \in L_p^1(\Omega) \cap C^{0,1}(\Omega)$ and $T = \inf\{t : m_n(\mathscr{N}_t) \le M\}$. We note that

$$\int_\Omega |u|^{q^*} \, dx \le c \bigg[\int_\Omega \big(|u| - T\big)_+^{q^*} \, dx + T^{q^*} M \bigg].$$

Using Corollaries 6.3.3 and 6.4.2/1, we obtain

$$\|u\|_{L_{q^*}(\Omega)} \le C \|\nabla u\|_{L_p(\Omega)} + cTM^{1/q},$$

where C is a constant independent of u. Let ω denote a bounded subdomain of Ω with smooth boundary such that $m_n(\Omega \backslash \omega) < M/2$. Obviously,

$$\int_\omega |u|^p \, dx \ge \int_{\omega \cap \mathscr{N}_T} |u|^p \, dx \ge \frac{1}{2} T^p M.$$

Therefore

$$\|u\|_{L_{q^*}(\Omega)} \le C\|\nabla u\|_{L_p(\Omega)} + cM^{1/q^*-1/p}\|u\|_{L_p(\omega)}. \tag{6.4.8}$$

Since ω satisfies

$$\inf_{c\in\mathbb{R}^1} \|u-c\|_{L_p(\omega)} \le K\|\nabla u\|_{L_p(\Omega)},$$

then (6.4.8) implies (6.4.5) for Ω. □

6.4.4 Sufficient Condition for Belonging to $\mathscr{H}_{p,\alpha}$

Here we give the following sufficient condition for a set to belong to $\mathscr{H}_{p,\alpha}$:

$$\int_0^M \left[\frac{\tau}{\lambda_M(\tau)}\right]^{p/(\alpha p-1)} \mathrm{d}\tau < \infty. \tag{6.4.9}$$

Proposition. *If $\alpha p > 1$, $\alpha \le 1$, then*

$$\mathfrak{B}_{p,\alpha}(M) \le (p-1)^{(p-1)/p}\left\{\int_0^M \left[\frac{\tau}{\lambda_M(\tau)}\right]^{p/(\alpha p-1)} \mathrm{d}\tau\right\}^{\alpha-1/p}. \tag{6.4.10}$$

Proof. By (6.4.3) and the inequality (6.3.11) we have

$$\left[\mathfrak{B}_{p,\alpha}(M)\right]^{p/(\alpha p-1)} \le \int_0^M \tau^{1/(\alpha p-1)}\left[\int_\tau^M \left[\lambda_M(\sigma)\right]^{p/(1-p)} \mathrm{d}\sigma\right]^{(p-1)/(\alpha p-1)} \mathrm{d}\tau.$$

To find a bound for the right-hand side we apply (1.3.1) in the form

$$\int_0^M \tau^{-r}\left(\int_\tau^M f(\sigma)\,\mathrm{d}\sigma\right)^q \mathrm{d}\tau \le \left(\frac{q}{1-r}\right)^q \int_0^M \tau^{q-r}\left[f(\tau)\right]^q \mathrm{d}\tau, \tag{6.4.11}$$

where $f(\tau) \ge 0$, $r < 1$, $q > 1$. Putting $r = (1-\alpha p)^{-1}$, $q = (p-1)/(\alpha p-1)$ in (6.4.11), we arrive at (6.4.10). □

6.4.5 Necessary Conditions for Belonging to the Classes $\mathscr{I}_{p,\alpha}$ and $\mathscr{H}_{p,\alpha}$

Using definitions of $\mathscr{I}_{p,\alpha}$ and $\mathscr{H}_{p,\alpha}$ we obtain some necessary conditions for Ω to be contained in these classes.

Proposition 1. *Let O be an arbitrary point in $\bar{\Omega}$ and let $s(t)$ be the area of the intersection of Ω with the sphere ∂B_t centered at O.*

If $\Omega \in \mathscr{I}_{p,\alpha}$ then

$$\left(\int_0^r s(t)\,\mathrm{d}t\right)^{\alpha p/(p-1)} \int_r^\varrho \frac{\mathrm{d}t}{[s(t)]^{1/(p-1)}} \le \mathrm{const}$$

for sufficiently small ϱ and for $r < \varrho$.

Proof. If $\Omega \in \mathscr{I}_{p,\alpha}$, then obviously

$$\big[m_n(\Omega_r)\big]^\alpha \le \mathrm{const}\big[c_p(\Omega_\varrho\backslash \operatorname{clos}_\Omega \Omega_r)\big]^{1/p}, \tag{6.4.12}$$

for small enough ϱ and $r < \varrho$. Let $u = 1$ in Ω_r, $u = 0$ outside Ω_ϱ and

$$u(x) = \int_{|x|}^{\varrho} \frac{\mathrm{d}t}{[s(t)]^{1/(p-1)}} \bigg(\int_r^{\varrho} \frac{\mathrm{d}t}{[s(t)]^{1/(p-1)}}\bigg)^{-1}, \quad x \in \Omega_\varrho\backslash\Omega_r.$$

Inserting u into the definition of p-conductivity, we obtain

$$c_p(\Omega_\varrho\backslash \operatorname{clos}_\Omega \Omega_r) \le \bigg(\int_r^{\varrho} \frac{\mathrm{d}t}{[s(t)]^{1/(p-1)}}\bigg)^{1-p}, \tag{6.4.13}$$

which together with (6.4.12) completes the proof. □

The proof of the following assertion is similar.

Proposition 2. *If $m_n(\Omega) < \infty$ and $\Omega \in \mathscr{I}_{p,\alpha}$, then*

$$\bigg(\int_r^{\infty} s(t)\,\mathrm{d}t\bigg)^{\alpha p/(p-1)} \int_\varrho^r \frac{\mathrm{d}t}{[s(t)]^{1/(p-1)}} \le \mathrm{const} \tag{6.4.14}$$

for large enough ϱ and for $r > \varrho$.

Corollary. *If Ω is an unbounded domain with $m_n(\Omega) < \infty$ and $\Omega \in \mathscr{I}_{p,\alpha}$, then $\alpha p \ge 1$.*

Proof. By the Hölder inequality for $r > \varrho$ we have

$$r - \varrho = \int_\varrho^r \big[s(t)\big]^{1/p} \frac{\mathrm{d}t}{[s(t)]^{1/p}} \le \bigg(\int_\varrho^r s(t)\,\mathrm{d}t\bigg)^{1/p} \bigg(\int_\varrho^r \frac{dt}{[s(t)]^{1/(p-1)}}\bigg)^{(p-1)/p}.$$

Therefore, (6.4.14) implies

$$r - \varrho \le \mathrm{const}\, \frac{(\int_\varrho^r s(t)\,\mathrm{d}t)^{1/p}}{(\int_r^\infty s(t)\,\mathrm{d}t)^\alpha}.$$

Let the sequence $\{\varrho_j\}$ be specified by

$$\int_{\varrho_j}^\infty s(t)\,\mathrm{d}t = 2^{-j}.$$

Then

$$\varrho_j - \varrho_{j-1} \le \mathrm{const}\, \frac{(\int_{\varrho_{j-1}}^{\varrho_j} s(t)\,\mathrm{d}t)^{1/p}}{(\int_{\varrho_j}^\infty s(t)\,\mathrm{d}t)^\alpha} = \mathrm{const}\big(2^{\alpha-1/p}\big)^j.$$

If $\alpha p < 1$, then the series

$$\sum_j (\varrho_j - \varrho_{j-1})$$

converges, which contradicts the assumption that Ω is unbounded. The result follows. □

Since the condition $\Omega \in \mathscr{I}_{p,1/q}$ is necessary for the continuity of the embedding operator $L^1_p(\Omega) \to L_q(\Omega)$, it follows by the Corollary that the embedding $L^1_p(\Omega) \subset L_q(\Omega)$ for an unbounded domain Ω with finite volume implies $p \geq q$.

Next we present a necessary condition for Ω to belong to $\mathscr{H}_{p,\alpha}$ stated in terms of the function s introduced in Proposition 1.

Proposition 3. *If $m_n(\Omega) < \infty$ and $\Omega \in \mathscr{H}_{p,\alpha}$, $\alpha p > 1$, then*

$$\int_c^\infty \left(\int_c^\varrho \frac{dt}{[s(t)]^{1/(p-1)}} \right)^{\frac{p-1}{\alpha p-1}} \left(\int_\varrho^\infty s(t)\,\mathrm{d}t \right)^{\frac{1}{\alpha p-1}} s(\varrho)\,\mathrm{d}\varrho < \infty,$$

where c is a positive constant.

The result follows from Definitions 6.3.5 and 6.4.2 combined with (6.4.13).

6.4.6 Examples of Domains in $\mathscr{H}_{p,\alpha}$

Example 1. By Definition 6.4.2 and the two-sided estimate (6.3.18) the domain Ω considered in Example 6.3.6/2 is contained in $\mathscr{H}_{p,\alpha}$ if and only if

$$\int_0^\infty \left[\left(\int_0^x \frac{\mathrm{d}s}{f(s)^{\frac{n-1}{p-1}}} \right)^{p-1} \int_x^\infty \big[f(s)\big]^{n-1}\,\mathrm{d}s \right]^{1/(\alpha p-1)} \big[f(x)\big]^{n-1}\,\mathrm{d}x < \infty. \tag{6.4.15}$$

By Proposition 6.4.4 and the left estimate (6.3.18) we obtain the sufficient condition for $\Omega \subset \mathscr{H}_{p,\alpha}$

$$\int_0^\infty \left(\int_x^\infty \big[f(t)\big]^{n-1}\,\mathrm{d}t \right)^{p/(\alpha p-1)} \big[f(x)\big]^{(n-1)\beta}\,\mathrm{d}x < \infty, \tag{6.4.16}$$

where

$$\beta = \frac{(\alpha-1)p-1}{\alpha p-1}.$$

Assuming that for large x

$$\int_x^\infty \big[f(s)\big]^{n-1}\,\mathrm{d}s \leq cx\big[f(x)\big]^{n-1}, \tag{6.4.17}$$

we see that (6.4.16) follows from the condition

$$\int_0^\infty x^{p/\alpha p-1}\big[f(x)\big]^{n-1}\,\mathrm{d}x < \infty. \tag{6.4.18}$$

On the other hand, the inequality

$$f(2x) \geq cf(x) \quad \text{for large } x \tag{6.4.19}$$

and the monotonicity of f show that (6.4.15) implies (6.4.18). Hence, (6.4.18) is equivalent to the inclusion $\Omega \subset \mathscr{H}_{p,\alpha}$ under the additional requirements (6.4.17) and (6.4.19).

Example 2. Consider the spiral Ω in Examples 5.3.3/3 and 6.3.6/3. We assume in addition that

$$\delta(2\theta) \geq c\delta(\theta), \qquad \int_\theta^\infty \delta(\varphi)\, \mathrm{d}\varphi \leq c\theta\delta(\theta)$$

for large θ. Then, using the same argument as in Example 1 we can show that $\Omega \in \mathscr{H}_{p,\alpha}$ if and only if

$$\int_0^\infty \delta(\varphi)\varphi^{p/(\alpha p-1)}\, \mathrm{d}\varphi < \infty. \tag{6.4.20}$$

A more complicated example of a domain contained in $\mathscr{H}_{p,\alpha}$ will be given in Sect. 6.5.

6.4.7 Other Descriptions of the Classes $\mathscr{I}_{p,\alpha}$ and $\mathscr{H}_{p,\alpha}$

We show that the class of conductors $K = G\backslash F$ used in Definition 6.3.1/1 of $\mathscr{I}_{p,\alpha}$ can be reduced.

Theorem 1. *Let Ω be bounded domain and let ω be a fixed open set with $\bar\omega \subset \Omega$. The domain Ω belongs to $\mathscr{I}_{p,\alpha}$ $(p \geq 1,\ \alpha \geq p^{-1} - n^{-1})$ if and only if*

$$\sup_{\{F\}} \frac{[m_n(F)]^\alpha}{[c_p(G\backslash F)]^{1/p}} < \infty, \tag{6.4.21}$$

where $G = \Omega\backslash\bar\omega$ and $\{F\}$ is the collection of closed (in Ω) subsets of G.

Proof. Let $S(\omega)$ denote the left-hand side in (6.4.21). The *necessity* of (6.4.21) is obvious since

$$S(\omega) \leq \mathfrak{A}_{p,\alpha}\bigl(m_n(\Omega\backslash\bar\omega)\bigr).$$

We prove the *sufficiency*. Let D be a domain with smooth boundary and such that $\bar\omega \subset D \subset \bar D \subset \Omega$. Further, let η be a smooth function, $\eta = 1$ on $\Omega\backslash D$, $\eta = 0$ on ω and $0 \leq \eta \leq 1$.

For any $u \in L_p^1(\Omega)$ we have

$$\int_0^\infty \bigl[m_n\bigl(\{x : (\eta|u|)(x) \geq t\}\bigr)\bigr]^{\alpha p}\, \mathrm{d}(t^p) \leq \bigl[S(\omega)\bigr]^p \int_0^\infty c_p\bigl(K_t^{(1)}\bigr)\, \mathrm{d}(t^p),$$

where $K_t^{(1)}$ is the conductor $(\Omega\backslash\bar{\omega})\backslash\{x \in \Omega : (\eta|u|)(x) \geq t\}$. The preceding inequality and Lemma 6.1.3/3 imply

$$\int_0^\infty \big[m_n\big(\{x : (\eta|u|)(x) \geq t\}\big)\big]^{\alpha p}\,\mathrm{d}\big(t^p\big)$$
$$\leq c\big[S(\omega)\big]^p \int_\Omega \big|\nabla\big(\eta|u|\big)\big|^p\,\mathrm{d}x$$
$$+ c\int_0^\infty m_n\big(\{x : \big(1-\eta(x)\big)\big|u(x)\big| \geq t\}\big)\,\mathrm{d}\big(t^p\big). \tag{6.4.22}$$

Since $\operatorname{supp}(1-\eta)|u| \subset \bar{D}$ and D has a smooth boundary, we have

$$\big[m_n\big(\{x : \big(1-\eta(x)\big)\big|u(x)\big| \geq t\}\big)\big]^\alpha \leq \mathrm{const}\, c_p\big(K_t^{(2)}\big),$$

where $K_t^{(2)}$ is the conductor $D\backslash\{x : (1-\eta(x))|u(x)| \geq t\}$. Hence from (6.4.22) and Lemma 6.1.3/3 we obtain

$$\int_0^\infty \big[m_n\big(\{x : \big|u(x)\big| \geq t\}\big)\big]^{\alpha p}\,\mathrm{d}\big(t^p\big)$$
$$\leq \mathrm{const}\bigg(\int_\Omega \big|\nabla\big(\eta|u|\big)\big|^p\,\mathrm{d}x + \int_\Omega \big|\nabla\big((1-\eta)|u|\big)\big|^p\,\mathrm{d}x\bigg)$$
$$\leq \mathrm{const}\bigg(\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_D |u|^p\,\mathrm{d}x\bigg).$$

Let $M = \frac{1}{2}m_n(D)$ and $m_n(\operatorname{supp} u) \leq M$. Then

$$\int_D |u|^p\,\mathrm{d}x \leq \mathrm{const}\int_D |\nabla u|^p\,\mathrm{d}x,$$

and thus

$$\int_0^\infty \big[m_n\big(\{x : \big|u(x)\big| \geq t\}\big)\big]^{\alpha p}\,\mathrm{d}(t^p) \leq \mathrm{const}\int_\Omega |\nabla u|^p\,\mathrm{d}x. \tag{6.4.23}$$

Consider the conductor $K^* = G^*\backslash F^*$ in Ω subject to the condition $m_n(G^*) \leq M$. We insert $u \in T_\Omega(K^*)$ into (6.4.23). Then we arrive at

$$\big[m_n\big(F^*\big)\big]^{\alpha p} \leq \mathrm{const}\, c_p\big(K^*\big),$$

which is equivalent to the inclusion $\Omega \in \mathscr{I}_{p,\alpha}$. □

Remark. The assertion we just proved remains valid for $p = 1$ provided $S(\omega)$ designates the supremum of $[m_n(g)]^\alpha/s(\partial_i g)$ taken over all admissible sets g contained in Ω with $\operatorname{clos}_\Omega g \subset \Omega\backslash\bar{\omega}$.

Following the same argument as in the proof of Theorem 1, we can establish that the inclusion $\Omega \in \mathscr{H}_{p,\alpha}$ is equivalent to the finiteness of $\mathfrak{B}_G^{(p,\alpha)}$ where $G = \Omega\backslash\bar{\omega}$ (cf. Sect. 6.4.1).

We can give a different description of the class $\mathscr{I}_{p,\alpha}$ for $\alpha p < 1$ replacing G by $\Omega_\varrho(x) = \Omega \cap B_\varrho(x)$, $x \in \partial\Omega$. Namely, we introduce the function

$$[0,1] \ni \varrho \to a_{p,\alpha}(\varrho) = \sup_{x \in \partial\Omega} \sup_{\{F\}} \frac{[m_n(F)]^\alpha}{[c_p(\Omega_\varrho(x) \backslash F)]^{1/p}},$$

where $\{F\}$ is the collection of closed (in Ω) subsets of $\Omega_\varrho(x)$.

Theorem 2. *Let Ω be a bounded domain and let $\alpha p < 1$. Then $\Omega \in \mathscr{I}_{p,\alpha}$ if and only if $a_{p,\alpha}(\varrho) < \infty$ for some ϱ.*

Proof. Necessity does not require proof. Let $a_{p,\alpha}(\varrho) < \infty$. We construct a finite covering of $\bar{\Omega}$ by open balls with radius ϱ and take a partition of unity $\{\eta_i\}$ subordinate to it. By Lemma 6.2

$$\|u\,\eta_i\|_{L_{q^*}(\Omega)} \le c a_{p,\alpha}(\varrho) \big\|\nabla(u\eta_i)\big\|_{L_p(\Omega)},$$

where $q^* = \alpha^{-1}$. Summing over i, we conclude that $W_p^1(\Omega) \subset L_{q^*}(\Omega)$. Consequently, by Theorem 6.3.3 (part 2)) $\Omega \in \mathscr{I}_{p,\alpha}$. $\square$

6.4.8 Integral Inequalities for Domains with Power Cusps

Let us consider in more detail the domain

$$\Omega^{(\lambda)} = \big\{x = (x', x_n) : |x'| < a x_n^\lambda,\ 0 < x_n < 1\big\},$$

presented in Example 6.3.6/1. There we showed that $\Omega^{(\lambda)} \in \mathscr{I}_{p,\alpha}$ provided $n > p$, $\lambda > (p-1)/(n-1)$, and $\alpha = [\lambda(n-1)+1-p]/p[\lambda(n-1)+1]$. By Theorem 6.4.3/2, the latter is equivalent to the inclusion $L_p^1(\Omega^{(\lambda)}) \subset L_{q^*}(\Omega^{(\lambda)})$ with $q^* = p[\lambda(n-1)+1]/[\lambda(n-1)+1-p]$. Although here the exponent q^* is the best possible, it is natural to try to refine the stated result by using spaces with weighted norms (cf. Remark 5.3.3).

Let

$$\|u\|_{L_r(\sigma,\Omega^{(\lambda)})} = \bigg(\int_{\Omega^{(\lambda)}} \big|u(x)\big|^r x_n^{r\sigma}\,\mathrm{d}x\bigg)^{1/r}.$$

The coordinate transformation $\varkappa : x \to \xi$ defined by $\xi_i = x_i$, $1 \le i \le n-1$, $\xi_n = x_n^\lambda$, maps $\Omega^{(\lambda)}$ onto $\Omega^{(1)}$. Since $\Omega^{(1)}$ is a domain of the class $C^{0,1}$, it easily follows from Corollary 2.1.7/2 that

$$\|v\|_{L_q(\beta,\Omega^{(1)})} \le c\big(\|\nabla v\|_{L_p(\alpha,\Omega^{(1)})} + \|v\|_{L_1(\varkappa(\omega))}\big),$$

where $n > p \ge 1$, $p \le q \le pn/(n-p)$, $\beta = \alpha - 1 + n(p^{-1} - q^{-1}) > -n/q$ and ω is nonempty domain, $\bar{\omega} \subset \Omega^{(\lambda)}$. Returning to the variable x, we obtain

$$\begin{aligned}&\|u\|_{L_q(\lambda\beta+(\lambda-1)/q,\Omega^{(\lambda)})}\\ &\quad\le c\big(\|\nabla_{x'} u\|_{L_p(\alpha\lambda+(\lambda-1)/p,\Omega^{(\lambda)})} + \|\partial u/\partial x_n\|_{L_p(\alpha\lambda-(\lambda-1)(p-1)/p,\Omega^{(\lambda)})}\\ &\qquad + \|u\|_{L_1(\omega)}\big),\end{aligned}$$

where $\nabla_{x'} = (\partial/\partial x_1, \dots, \partial x_{n-1})$. Putting $\alpha = (\lambda-1)(p-1)/\lambda p$, we have

$$\|u\|_{L_q(\lambda\beta+(\lambda-1)/q,\Omega^{(\lambda)})} \le c\big(\|\nabla_{x'}u\|_{L_p(\lambda-1,\Omega^{(\lambda)})} + \|\partial u/\partial x_n\|_{L_p(\Omega^{(\lambda)})} + \|u\|_{L_1(\omega)}\big). \quad (6.4.24)$$

Choosing q to eliminate the weight on the left-hand side, we arrive at

$$\|u\|_{L_{q^*}(\Omega^{(\lambda)})} \le c\big(\|\nabla_{x'}u\|_{L_p(\lambda-1,\Omega^{(\lambda)})} + \|\partial u/\partial x_n\|_{L_p(\Omega^{(\lambda)})} + \|u\|_{L_1(\omega)}\big),$$

where, as before, $q^* = p[\lambda(n-1)+1]/[\lambda(n-1)+1-p]$. Since $\lambda > 1$, the preceding result is better than the embedding $L^1_p(\Omega^{(\lambda)}) \subset L_{q^*}(\Omega^{(\lambda)})$.

For $\lambda > 1$ we can take q to be the limit exponent $pn/(n-p)$ in Sobolev's theorem. Then (6.4.24) becomes

$$\|u\|_{L_{pn/(n-p)}((\lambda-1)(n-1)/n,\Omega^{(\lambda)})} \le c\big(\|\nabla_{x'}u\|_{L_p(\lambda-1,\Omega^{(\lambda)})} + \|\partial u/\partial x_n\|_{L_p(\Omega^{(\lambda)})} + \|u\|_{L_1(\omega)}\big),$$

which, in particular, guarantees the inclusion $L^1_p(\Omega) \subset L_{pn/(n-p)}((\lambda-1)(n-1)/n, \Omega^{(\lambda)})$. We can readily check by the example of the function x_n^τ with $\tau = 1+\varepsilon-[\lambda(n-1)+1]/p$ (ε is a small positive number) that the power exponent of the weight is exact.

In conclusion we remark that one can obtain an integral representation similar to (1.1.8) for the domain $\Omega^{(\lambda)}$.

By (1.1.8), for any $v \in C^1(\Omega^{(1)}) \cap L^1_1(\Omega^{(1)})$ we have

$$v(\xi) = \int_{\Omega^{(1)}} \varphi(\eta)v(\eta)\,\mathrm{d}\eta + \sum_{i=1}^n \int_{\Omega^{(1)}} \frac{f_i(\xi,\eta)}{|\xi-\eta|^{n-1}} \frac{\partial v}{\partial \eta_i}(\eta)\,\mathrm{d}\eta,$$

where $\varphi \in \mathscr{D}(\Omega^{(1)})$ and $f_i \in L_\infty(\Omega^{(1)} \times \Omega^{(1)})$. Therefore the function $u = v \circ \varkappa$ has the integral representation

$$u(x) = \int_{\Omega^{(\lambda)}} \Phi(y)u(y)\,\mathrm{d}y + \sum_{i=1}^{n-1} \int_{\Omega^{(\lambda)}} \frac{F_i(x,y)y_n^{\lambda-1}}{(|x'-y'|^2+(x_n^\lambda-y_n^\lambda)^2)^{(n-1)/2}} \frac{\partial u}{\partial y_i}(y)\,\mathrm{d}y + \int_{\Omega^{(\lambda)}} \frac{F_n(x,y)}{(|x'-y'|^2+(x_n^\lambda-y_n^\lambda)^2)^{(n-1)/2}} \frac{\partial u}{\partial y_n}(y)\,\mathrm{d}y, \quad (6.4.25)$$

where $\Phi \in \mathscr{D}(\Omega^{(\lambda)})$ and $F_i \in L_\infty(\Omega^{(\lambda)} \times \Omega^{(\lambda)})$. It is easily seen that we can take u to be an arbitrary function in $C^1(\overline{\Omega^{(\lambda)}})$. Since by Theorem 1.1.6/1 the space $C^1(\bar{\Omega}^{(\lambda)})$ is dense in $L^1_1(\Omega^{(\lambda)})$ we conclude that (6.4.25) is valid for all $u \in L^1_1(\Omega^{(\lambda)})$.

6.5 More on the Nikodým Example

In this section we consider the domain described in Example 1.1.4/1 with $\varepsilon_m = \delta(2^{-m-1})$ where δ is a nondecreasing function such that $2\delta(t) < t$. Here we show in particular that *the convergence of the integral*

$$\int_0^1 \left[\frac{t}{\delta(t)}\right]^{1/(\alpha p-1)} \mathrm{d}t \tag{6.5.1}$$

is necessary and sufficient for Ω to belong to $\mathscr{H}_{p,\alpha}$, for $\alpha p > 1$.

Lemma. (The Inverse Minkowski Inequality) *If g_m are nonnegative measurable functions on* $[0,1]$, *then*

$$\left\{\int_0^1 \frac{\mathrm{d}t}{[\sum_m g_m(t)]^{1/(p-1)}}\right\}^{1-p} \geq \sum_m \left\{\int_0^1 \frac{\mathrm{d}t}{[g_m(t)]^{1/(p-1)}}\right\}^{1-p}. \tag{6.5.2}$$

Proof. Let λ be an absolutely continuous nondecreasing function on [0,1] with $\lambda(0) = 0$ and $\lambda(1) = 1$. We put $g(t) = \sum_m g_m(t)$. Then

$$\int_0^1 (\lambda')^p g \,\mathrm{d}t = \sum_m \int_0^1 (\lambda')^p g_m \,\mathrm{d}t.$$

By Hölder's inequality

$$\sum_m \int_0^1 (\lambda')^p g_m \,\mathrm{d}t \geq \sum_m \frac{(\int_0^1 \lambda' \,\mathrm{d}t)^p}{(\int_0^1 g_m^{1/(1-p)} \,\mathrm{d}t)^{p-1}} = \sum_m \left(\int_0^1 \frac{\mathrm{d}t}{g_m^{1/(p-1)}}\right)^{1-p}.$$

Finally, by Lemma 2.2.2/2,

$$\inf_\lambda \int_0^1 (\lambda')^p g \,\mathrm{d}t = \left\{\int_0^1 \frac{dt}{g^{1/(p-1)}}\right\}^{1-p}.$$

Hence (6.5.2) holds. □

Next we prove that $\Omega \in \mathscr{H}_{p,\alpha}$ provided (6.5.1) is finite. Consider an arbitrary nonnegative function $u \in L_p^1(\Omega)$ that is infinitely differentiable in $A_m \cup B_m$ for any m and that vanishes in the rectangle C.

We fix an arbitrary number m and note that each level set $\mathscr{E}_t^{(m)} = \{(x,y) : u(x,y) = t\} \cap (A_m \cup B_m)$ consists of a finite number of smooth homeomorphic images of a circle and simple arcs with end points on $\partial(A_m \cup B_m)\backslash\{y = 2/3\}$ for almost all levels $t \in (0,\infty)$ (cf. Corollary 1.2.2). Henceforth we shall always consider only such levels.

If t satisfies

$$s(\mathscr{E}_t^{(m)}) \geq 2^{-m-3}, \tag{6.5.3}$$

then

$$m_2\left(\mathscr{N}_t^{(m)}\right) \le \frac{1}{3}\left[1 + 2^{m+1}\delta\left(2^{-m-1}\right)\right]2^{-m-1} < s\left(\mathscr{E}_t^{(m)}\right), \tag{6.5.4}$$

where $\mathscr{N}_t^{(m)} = \{(x,y) : u \ge t\} \cap (A_m \cup B_m)$. Let $\mathfrak{B}_m$ denote the collection of all levels t for which (6.5.3) holds. We show that for $t \notin \mathfrak{B}_m$ one of the following three cases occurs:

$$s\left(\mathscr{E}_t^{(m)}\right) \ge \delta\left(2^{-m-1}\right), \tag{6.5.5}$$

$$m_2\left(\mathscr{N}_t^{(m)}\right) \le k\delta\left(2^{-m-1}\right), \tag{6.5.6}$$

where k is a constant depending on integral (6.5.1);

$$s\left(\mathscr{E}_t^{(m)}\right) \ge \delta\left(2^{-m-1}\right), \tag{6.5.7}$$

$$m_2\left(\mathscr{L}_t^{(m)}\right) \le k\delta\left(2^{-m-1}\right), \tag{6.5.8}$$

where $\mathscr{L}_t^{(m)} = (A_m \cup B_m)\backslash\mathscr{N}_t^{(m)}$;

$$s\left(\mathscr{E}_t^{(m)}\right) < \delta\left(2^{-m-1}\right), \tag{6.5.9}$$

$$m_2\left(\mathscr{N}_t^{(m)}\right) \le ks\left(\mathscr{E}_t^{(m)}\right). \tag{6.5.10}$$

We note that there are no components of $\mathscr{E}_t^{(m)}$ joining cd and ef since $t \notin \mathfrak{B}_m$ (cf. Fig. 24). Besides, the set $\mathscr{E}_t^{(m)}$ $(t > 0)$ is disjoint with the line $y = \frac{2}{3}$ because $u = 0$ in C.

(a) Suppose that (6.5.5) is valid. Let $\tilde{\mathscr{E}}_t^{(m)}$ denote the upper component of $\mathscr{E}_t^{(m)}$ which joins the polygonal line abc with fe. The set $\tilde{\mathscr{E}}_t^{(m)}$ divides $A_m \cup B_m$

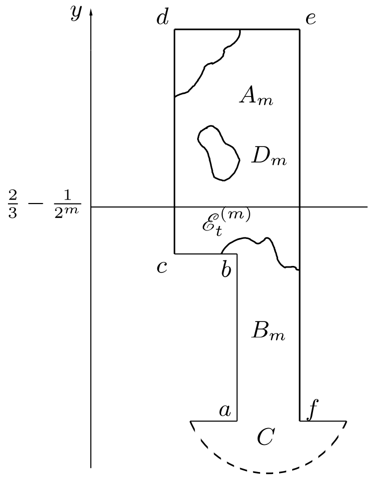

Fig. 24.

into components; the component containing de will be denoted by D_m. Since $t \notin \mathfrak{B}_m$ it follows that $\tilde{\mathscr{E}}_t^{(m)}$ is placed below the line $y = \frac{2}{3} + 2^{-m}$ and hence

$$m_2\big(\mathscr{N}_t^{(m)} \backslash D_m\big) \le \left(2^{-2m-1} + \frac{1}{3}\delta\big(2^{-m-1}\big)\right),$$
$$m_2\big(\mathscr{L}_t^{(m)} \backslash D_m\big) \le \left(2^{-2m-1} + \frac{1}{3}\delta\big(2^{-m-1}\big)\right).$$

Taking into account that δ increases and $2\delta(t) < t$, we obtain

$$\int_0^{2^{-m-1}} \left[\frac{t}{\delta(t)}\right]^\gamma \mathrm{d}t \ge 2^{\gamma-1} \int_0^{2^{-m-1}} \frac{t\,\mathrm{d}t}{\delta(t)} \ge 2^{\gamma-2} \frac{2^{-2(m+1)}}{\delta(2^{-m-1})} \tag{6.5.11}$$

with $\gamma = (\alpha p - 1)^{-1}$. Consequently,

$$m_2\big(\mathscr{N}_t^{(m)} \backslash D_m\big) \le k_1 \delta\big(2^{-m-1}\big), \qquad m_2\big(\mathscr{L}_t^{(m)} \backslash D_m\big) \le k_1 \delta\big(2^{-m-1}\big), \tag{6.5.12}$$

where

$$k_1 = \frac{1}{3} + 2^{\gamma-1} \int_0^1 \big(t \backslash \delta(t)\big)^\gamma \,\mathrm{d}t.$$

The set $\mathscr{E}_t^{(m)} \backslash D_m$ divides $\Omega \backslash D_m$ into components. Let $\tilde{D}_m$ denote one of them with the boundary containing $\tilde{\mathscr{E}}_t^{(m)}$ and the end points of the segment de. Suppose $\tilde{D}_m \subset \mathscr{L}_t^{(m)}$. We estimate $m_2(\mathscr{N}_t^{(m)} \cap D_m)$. First we note that $\mathscr{N}_t^{(m)} \cap D_m$ is bounded by the components of $\mathscr{E}_t^{(m)}$ which (with the exception of $\tilde{\mathscr{E}}_t^{(m)}$) are either closed in $A_m \cup B_m$ or join points of the polygonal lines $abcde$ or def. This implies

$$s\big(\partial(A_m \cup B_m) \cap \overline{\mathscr{N}_t^{(m)}} \cap D_m\big) \le 2s\big(\mathscr{E}_t^{(m)} \cap D_m\big).$$

Using the isoperimetric inequality, we obtain

$$\big[m_2\big(\mathscr{N}_t^{(m)} \cap D_m\big)\big]^{1/2} \le \frac{3}{2\sqrt{\pi}} s\big(\mathscr{E}_t^{(m)} \cap D_m\big). \tag{6.5.13}$$

Since $t \notin \mathfrak{B}_m$, the latter and (6.5.11) yield

$$m_2\big(\mathscr{N}_t^{(m)} \cap D_m\big) \le \frac{9}{4\pi} 2^{-2(m+1)} \le \frac{9 \cdot 2^{2-\gamma}}{4\pi} \delta\big(2^{-m-1}\big) \int_0^1 \left(\frac{t}{\delta(t)}\right)^\gamma \mathrm{d}t,$$

which together with (6.5.12) leads to (6.5.6).

(b) Inequality (6.5.8) can be derived in the same way provided that $\tilde{D}_m \subset \mathscr{N}_t^{(m)}$.

(c) Suppose that (6.5.9) holds. Then $\mathscr{E}_t^{(m)}$ does not contain components which join $abcd$ and ef. Following the same argument as in the derivation of (6.5.13), we obtain

$$\big[m_2\big(\mathscr{N}_t^{(m)}\big)\big]^{1/2} \le \frac{3}{2\sqrt{\pi}} s\big(\mathscr{E}_t^{(m)}\big).$$

The last inequality and (6.5.9) imply (6.5.10).

Thus, one of the following cases is possible: either (6.5.10) is valid, or (6.5.5) and (6.5.6), or (6.5.7) and (6.5.8). Let $\mathfrak{B}'_m$ denote the set of levels t for which (6.5.10) is valid. Let $\mathfrak{B}''_m$ and $\mathfrak{B}'''_m$ be the sets of levels satisfying (6.5.5), (6.5.6) or (6.5.7), (6.5.8), respectively.

Let $\psi_m(t)$ be defined by

$$\psi_m(t) = \int_0^t \bigg(\int_{\mathscr{E}_t^{(m)}} |\nabla u|^{p-1}\,\mathrm{d}s\bigg)^{1/(1-p)}.$$

We have

$$\psi_m(t) \le -\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\big[m_2\big(\mathscr{N}_\tau^{(m)}\big)\big]\frac{\mathrm{d}\tau}{[s(\mathscr{E}_\tau^{(m)})]^{p/(p-1)}}$$

(cf. Corollary 6.1.3/1). We express the integral on the right-hand side as the sum

$$\int_{\mathfrak{B}'_m} + \int_{\mathfrak{B}''_m} + \int_{\mathfrak{B}'''_m}.$$

By (6.5.10),

$$\int_{\mathfrak{B}'_m} \le \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\big[m_2\big(\mathscr{N}_\tau^{(m)}\big)\big]\frac{\mathrm{d}\tau}{[m_2(\mathscr{N}_\tau^{(m)})]^{p/(p-1)}} \le \frac{p-1}{[m_2(\mathscr{N}_t^{(m)})]^{1/(p-1)}}.$$

Using (6.5.5) and (6.5.6), we obtain

$$\int_{\mathfrak{B}''_m} \le \big[\delta\big(2^{-m-1}\big)\big]^{p/(1-p)} \sup_{\tau\in\mathfrak{B}''_m} m_2\big(\mathscr{N}_\tau^{(m)}\big) \le k\big[\delta\big(2^{-m-1}\big)\big]^{1/(1-p)}.$$

We note that

$$-\frac{\mathrm{d}}{\mathrm{d}\tau} m_2\big(\mathscr{N}_\tau^{(m)}\big) = \frac{\mathrm{d}}{\mathrm{d}\tau} m_2\big(\mathscr{L}_\tau^{(m)}\big),$$

and use (6.5.7) and (6.5.8). Then

$$\begin{aligned}\int_{\mathfrak{B}'''_m} &\le \int_{\mathfrak{B}''_m} \frac{\mathrm{d}}{\mathrm{d}t} m_2\big(\mathscr{L}_\tau^{(m)}\big)\frac{\mathrm{d}\tau}{[s(\mathscr{E}_\tau^{(m)})]^{p/(p-1)}} \sup_{\tau\in\mathfrak{B}'''_m} m_2\big(\mathscr{L}_t^{(m)}\big)\\ &\le k\big[\delta\big(2^{-m-1}\big)\big]^{1/(1-p)}.\end{aligned}$$

Consequently,

$$\psi_m(t) \le 2k\big[\delta\big(2^{-m-1}\big)\big]^{1/(1-p)} + \big[m_2\big(\mathscr{N}_t^{(m)}\big)\big]^{1/(1-p)}. \tag{6.5.14}$$

Let l denote the smallest number for which

$$m_2\big(\mathscr{N}_t^{(l)}\big) \ge (2c)^{1-p}\delta\big(2^{-l-1}\big).$$

Then (6.5.14) implies

$$\psi_l(t) \le 4k\big[\delta\big(2^{-l-1}\big)\big]^{1/(1-p)}. \tag{6.5.15}$$

Next, taking into account that

$$m_2\big(\mathscr{N}_t^{(m)}\big) < \frac{1}{3}\cdot 2^{-m},$$

we obtain

$$\delta\left[\frac{3}{4}m_2\left(\bigcup_{m=l}^{\infty}\mathscr{N}_t^{(m)}\right)\right] \le \delta\big(2^{-l-1}\big),$$

which together with (6.5.15) yields

$$\delta\left[\frac{3}{4}m_2\left(\bigcup_{m=l}^{\infty}\mathscr{N}_t^{(m)}\right)\right] \le \big[4k/\psi_l(t)\big]^{p-1}. \tag{6.5.16}$$

Since

$$m_2\big(\mathscr{N}_t^{(m)}\big) < (2c)^{1-p}\delta\big(2^{-m-1}\big)$$

for $m < l$, by (6.5.14) we have

$$\psi_m(t) \le 2\big[m_2\big(\mathscr{N}_t^{(m)}\big)\big]^{1/(1-p)},$$

for $m < l$. Consequently,

$$m_2\left(\bigcup_{m=1}^{l-1}\mathscr{N}_t^{(m)}\right) \le 2^{p-1}\sum_{m=1}^{l-1}\big[\psi_m(t)\big]^{1-p},$$

and thus

$$\delta\left[m_2\left(\bigcup_{m=1}^{l-1}\mathscr{N}_t^{(m)}\right)\right] \le 2^{p-2}\sum_{m=1}^{l-1}\big[\psi_m(t)\big]^{1-p}, \tag{6.5.17}$$

because $\delta(t) < t/2$. If

$$m_2\left(\bigcup_{m=1}^{l-1}\mathscr{N}_t^{(m)}\right) < \frac{3}{4}m_2\left(\bigcup_{m=l}^{\infty}\mathscr{N}_t^{(m)}\right), \tag{6.5.18}$$

then by (6.5.16)

$$\delta\left[\frac{3}{8}m_2(\mathscr{N}_t)\right] \le \big[4k/\psi_l(t)\big]^{p-1}.$$

Otherwise, if the reverse of (6.5.18) is valid, then by (6.5.17)

$$\delta\left[\frac{3}{8}m_2(\mathscr{N}_t)\right] \le 2^{p-2}\sum_{m=1}^{l-1}\big[\psi_m(t)\big]^{1-p}.$$

Thus, we always have

$$\delta\left[\frac{3}{8}m_2(\mathscr{N}_t)\right] \le k' \sum_{m=1}^{\infty} \left[\psi_m(t)\right]^{1-p}, \tag{6.5.19}$$

where k' is the maximum of $(4k)^{p-1}$ and 2^{p-2}.

Let $\psi(t)$ be defined by

$$\psi(t) = \int_0^t \left(\int_{\mathscr{E}_\tau} |\nabla u|^{p-1} \, \mathrm{d}s \right)^{1/(1-p)} \mathrm{d}\tau.$$

Since by Lemma

$$\left[\psi(t)\right]^{1-p} \ge \sum_{m\ge 1} \left[\psi_m(t)\right]^{1-p},$$

(6.5.19) implies

$$\delta\left[\frac{3}{8}m_2(N_t)\right] \le k' \left[\psi(t)\right]^{1-p}. \tag{6.5.20}$$

Let F be an arbitrary subset of $G = \Omega\backslash\bar{C}$ closed in Ω and let u be an arbitrary function in $V_\Omega(K)$, $K = G\backslash F$. By (6.5.20) we have

$$\delta\left[\frac{3}{8}m_2(F)\right] \le k' \left[\psi(1)\right]^{1-p},$$

which together with Lemma 6.1.3/1 yields

$$\delta\left[\frac{3}{8}m_2(F)\right] \le k' c_p(K).$$

Consequently,

$$\delta\left(\frac{3}{8}t\right) \le k' \nu_G^{(p)}(t), \tag{6.5.21}$$

where $\nu_G^{(p)}$ is the function introduced in Sect. 6.4.1.

Taking into account the convergence of the integral (6.5.1) as well as Lemma 6.4.1, from (6.5.21) we obtain that there exists a constant Q such that

$$\|u\|_{L_q(\Omega)} \le Q \|\nabla u\|_{L_p(\Omega)} \tag{6.5.22}$$

for all $u \in C^{0,1}(\Omega)$ which vanish in C with $q = \alpha^{-1}$.

Now let u be an arbitrary function in $C^{0,1}(\Omega)$ and let η be continuous in Ω, vanishing in C, equal to unity in $\bigcup_{m\ge 1} A_m$ and linear in B_m $(m = 1, 2, \dots)$. Then

$$\|u\|_{L_q(\Omega)} \le \|u\eta\|_{L_q(\Omega\backslash C)} + \|u\|_{L_q(\Omega\backslash \bigcup_{m\ge 1} A_m)}. \tag{6.5.23}$$

Using (6.5.22), we obtain

$$\|u\eta\|_{L_q(\Omega\backslash C)} \le Q_1\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_p(\bigcup_{m\ge1} B_m)}\big).$$

The latter and (6.5.23) imply

$$\|u\|_{L_q(\Omega)} \le Q_2\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega\backslash\cup_{m\ge1} A_m)}\big). \tag{6.5.24}$$

We estimate the second norm on the right in (6.5.24). Let $(x,y) \in \Omega\backslash\cup_{m\ge1}A_m$ and $(x,z) \in C$. Obviously,

$$\big|u(x,y)\big|^p \le Q_3\bigg(\big|u(x,z)\big|^p + \int \big|\nabla u(x,\bar y)\big|^p\, \mathrm{d}\bar y\bigg),$$

where the integral is taken over the vertical segment contained in Ω and passing through the point $(x,0)$. Integrating in x, y and z, we obtain

$$\iint_{\Omega\backslash\bigcup_{m\ge1}A_m} |u|^p\, \mathrm{d}x\, \mathrm{d}y \le Q_4\bigg(\iint_C |u|^p\, \mathrm{d}x\, \mathrm{d}y + \iint_{\Omega\backslash\bigcup_{m\ge1}A_m} |\nabla u|^p\, \mathrm{d}x\, \mathrm{d}y\bigg).$$

Therefore,

$$\|u\|_{L_q(\Omega)} \le Q_5\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_p(C)}\big).$$

Hence

$$\inf_{c\in\mathbb{R}^1} \|u-c\|_{L_q(\Omega)} \le Q_5\big(\|\nabla u\|_{L_p(\Omega)} + \inf_{c\in\mathbb{R}^1} \|u-c\|_{L_p(C)}\big).$$

Using the Poincaré inequality for the rectangle C, we obtain

$$\inf_{c\in\mathbb{R}^1} \|u-c\|_{L_q(\Omega)} \le Q_6\|\nabla u\|_{L_p(\Omega)},$$

which according to Theorem 6.4.3/2 is equivalent to $\Omega \in \mathscr{H}_{p,\alpha}$ with $\alpha = q^{-1}$.

Next we show that the convergence of the integral (6.5.1) is necessary for Ω to be contained in $\mathscr{H}_{p,\alpha}$. Let

$$\int_0^1 \big[t/\delta(t)\big]^\gamma\, \mathrm{d}t = \infty,$$

where $\gamma = q/(p-q)$, $q = \alpha^{-1}$. Then

$$\sum_{m\ge1} \frac{\lambda_m^{\gamma+1}}{\delta(\lambda_m)} = \infty, \tag{6.5.25}$$

where $\lambda_m = 2^{-m-1}$. Consider a continuous function u_m in Ω vanishing in C, linear in B_m and equal to

$$\big[\lambda_m/\delta(\lambda_m)\big]^{(\gamma+1)/p},$$

in A_m, $m \ge 1$. For $v_N = \sum_{1\le m\le N} u_m$ we have

$$\iint_{\Omega} v_N^{p\gamma/(\gamma+1)}\,\mathrm{d}x\,\mathrm{d}y \geq \frac{1}{3}\sum_{m=1}^{N} u_m^{p\gamma/(\gamma+1)}\lambda_m = \sum_{m=1}^{N} \lambda_m^{\gamma+1}\big[\delta(\lambda_m)\big]^{-\gamma}.$$

On the other hand,

$$\iint_{\Omega} |\nabla v_N|^p\,\mathrm{d}x\,\mathrm{d}y = 3^p\sum_{m=1}^{N} u_m^p\delta(\lambda_m) = 3^p\sum_{m=1}^{N} \lambda_m^{\gamma+1}\big[\delta(\lambda_m)\big]^{-\gamma}.$$

If Ω is contained in $\mathscr{H}_{p,\alpha}$, then for all $u \in C^{0,1}(\Omega)$ vanishing in C

$$\|u\|^p_{L_{p\gamma/(\gamma+1)}(\Omega)} \leq Q\iint_{\Omega} |\nabla u|^p\,\mathrm{d}x\,\mathrm{d}y, \tag{6.5.26}$$

where Q does not depend on u. From (6.5.26) we obtain

$$\left(\sum_{m=1}^{N}\frac{\lambda_m^{\gamma+1}}{[\delta(\lambda_m)]^{\gamma}}\right)^{(\gamma+1)/\gamma} \leq Q\sum_{m=1}^{N}\frac{\lambda_m^{\gamma+1}}{[\delta(\lambda_m)]^{\gamma}}.$$

Hence

$$\sum_{m=1}^{N}\frac{\lambda_m^{\gamma+1}}{[\delta(\lambda_m)]^{\gamma}} \leq Q^{\gamma},$$

which contradicts (6.5.25).

Remark 1. From (6.5.2) and Lemma 6.1.3/2 it follows that

$$\int_0^{\infty}\delta\Big(\frac{3}{8}m_2(\mathscr{N}_t)\Big)\,\mathrm{d}\big(t^p\big) \leq Q\iint_{\Omega} |\nabla u|^p\,\mathrm{d}x\,\mathrm{d}y \tag{6.5.27}$$

for all $u \in C^{\infty}(\Omega)$, $u = 0$ in C. Further let $\delta \in C^{0,1}[0,1]$ and $\delta(2t) \leq \mathrm{const}\,\delta(t)$. Then, following the same argument as in the proof of Proposition 5.4, from (6.5.2) we obtain $\nu_p(t) \geq k\delta(t)$. The reverse estimate follows by considering the sequence of conductors $\{G_m\backslash F_m\}$ where G_m is the interior of $A_m \cup B_m$ and $F_m = \mathrm{clos}_{\Omega} A_m$. In fact, for a piecewise linear function u_m which vanishes in C and is equal to 1 in A_m we have

$$c_p(G_m\backslash F_m) \leq \iint_{\Omega} |\nabla u_m|^p\,\mathrm{d}x\,\mathrm{d}y = 3^p\delta\big(2^{-m-1}\big) \leq k\delta\big(m_2(F)\big).$$

Thus, for the Nikodým domain and for any $p \geq 1$,

$$\nu_p(t) \sim \delta(t), \tag{6.5.28}$$

and so $\Omega \in \mathscr{I}_{p,\alpha}$ if and only if

$$\liminf_{t\to+0} t^{-\alpha p}\delta(t) > 0 \tag{6.5.29}$$

(since $\delta(t) \le t$ then $\alpha p \ge 1$). Similarly, $\Omega \in \mathscr{H}_{p,\alpha}$ if and only if

$$\int_0^1 \left[\frac{\tau}{\delta(\tau)}\right]^{1/(\alpha p-1)} \mathrm{d}\tau < \infty.$$

Remark 2. The domain considered in the present section is interesting in the following respect. While the conditions for the domains in the examples of Section 6.3.6 to belong to $\mathscr{J}_{\alpha+1-1/p}$ and $\mathscr{I}_{p,\alpha}$ coincide (i.e., Proposition 6.3.5/2 is exact), the Nikodým domain is simultaneously contained in $\mathscr{J}_{p,\alpha}$ and $\mathscr{I}_{p,\alpha}$ by virtue of (6.5.29). This means that if, for example, $\delta(t) = t^\beta$, $\beta > 1$, then the sufficient conditions for the embedding $L_p^1(\Omega) \subset L_q(\Omega)$ ($p > 1$) being formulated in terms of the function λ, give an incorrect value of the limit exponent $q^* = p/(1 + p(\beta - 1))$.

The actual maximal value of q^*, obtained here by the direct estimate of ν_p is equal to p/β.

6.6 Some Generalizations

The spaces $L_s(\Omega)$, $L_r(\Omega)$, and $L_q(\Omega)$ can be replaced by the spaces $L_s(\Omega, \sigma)$, $L_r(\Omega, \sigma)$, and $L_q(\Omega, \sigma)$ of functions that are integrable of order s, r, q, respectively, with respect to the measure σ in Ω; we just need to replace the Lebesgue measure by σ in the corresponding necessary and sufficient conditions. As an example, we pause for a moment to present a generalization of Lemma 6.2.

Let G be an open bounded subset of Ω. For $p > 1$ we put

$$\mathfrak{A}_{G,\sigma}^{(p,\alpha)} = \sup_{\{F\}} \frac{[\sigma(F)]^\alpha}{[c_p(G\backslash F)]^{1/p}},$$

where $\{F\}$ is the collection of subsets of G which are closed in Ω. Further, let

$$\mathfrak{A}_{G,\sigma}^{(1,\alpha)} = \sup_{\{g\}} \frac{[\sigma(g)]^\alpha}{s(\partial_i g)},$$

where $\{g\}$ is the collection of admissible subsets of G. Thus by definition, $\mathfrak{A}_{G,m_n}^{(p,\alpha)} = \mathfrak{A}_G^{(p,\alpha)}$.

Theorem. *Let $p \ge 1$ and let G be an open bounded subset of Ω.*

1. *If $\mathfrak{A}_{G,\sigma}^{(p,\alpha)} < \infty$ and the numbers q, α, p are related by any of the following conditions:*

(i) $q \le q^* = \alpha^{-1}$ *for* $\alpha p \le 1$, (ii) $q < q^* = \alpha^{-1}$ *for* $\alpha p > 1$, *then, for all* $u \in C^{0,1}(\Omega)$ *vanishing outside* G, *we have*

$$\|u\|_{L_q(\Omega,\sigma)} \le C \|\nabla u\|_{L_p(\Omega)}^{1-\varkappa} \|u\|_{L_r(\Omega,\sigma)}^{\varkappa}, \tag{6.6.1}$$

with $r \in (0, q)$, $\varkappa = r(q^* - q)/(q^* - r)q$, $C \le c[\mathfrak{A}_{G,\sigma}^{(p,\alpha)}]^{1-\varkappa}$.

2. *Let* $q^* > 0$, $r \in (0, q^*)$ *and let* (6.5.1) *hold for some* $q \in (0, q^*)$ *and for any* $u \in C^{0,1}(\Omega)$ *vanishing outside* G *with* $\varkappa = r(q^* - q)/(q^* - r)q$ *and with a constant* C *independent of* u. *Then* $C \geq c\,[\mathfrak{A}^{(p,\alpha)}_{G,\sigma}]^{1-\varkappa}$.

The proof does not differ from that of Lemma 6.2. □

Next we present a sufficient condition for the finiteness of $\mathfrak{A}^{(p,\gamma)}_{G,\sigma}$.

Proposition. *We have*

$$\mathfrak{A}^{(p,\gamma)}_{G,\sigma} \leq c\big(\mathfrak{A}^{(p,\beta)}_{G}\big)^{1-\gamma/\alpha}\big(\mathfrak{A}^{(1,\alpha)}_{G,\sigma}\big)^{\gamma/\alpha}, \tag{6.6.2}$$

where $\gamma = \alpha\beta p(p - 1 + p\beta)^{-1}$.

Proof. Let $K = G\backslash F$ and let u be any function in $V_\Omega(K)$. We put

$$\psi(t) = \int_0^t \bigg(\int_{\mathscr{E}_\tau} |\nabla u|^{p-1}\,\mathrm{d}s\bigg)^{1/(1-p)} \mathrm{d}\tau.$$

The definition of $\mathfrak{A}^{(p,\beta)}_{G}$ and (6.1.8) imply

$$\psi(t) \leq \big[c_p(G\backslash\mathscr{N}_t)\big]^{1/(1-p)} \leq \big(\mathfrak{A}^{(p,\beta)}_{G}\big[m_n(\mathscr{N}_t)\big]^{-\beta}\big)^{p/(p-1)}, \tag{6.6.3}$$

where $t \in [0, 1)$. Thus, if $\mathfrak{A}^{(p,\beta)}_{G} < \infty$ then $\psi(t)$ is finite and hence absolutely continuous on any segment $[0, 1-\varepsilon]$, $\varepsilon > 0$. Let $t(\psi)$ be the inverse of $\psi(t)$. Since the function $\psi \to \sigma(\mathscr{N}_{t(\psi)})$ does not increase, by Lemma 1.3.5/3 we have

$$\psi^{\alpha/\gamma}\big[\sigma(\mathscr{N}_{t(\psi)})\big]^{\alpha p/(p-1)} \leq c\sup_{\psi}\psi^{(p-1)/p\beta}\int_{\psi}^{\psi(1-\varepsilon)}\big[\sigma(\mathscr{N}_{t(\psi)})\big]^{\alpha p/(p-1)}\,\mathrm{d}\psi,$$

for $t(\psi) \in [0, 1-\varepsilon]$. The right-hand side is equal to

$$c\sup_{t\in[0,1-\varepsilon]}\big[\psi(t)\big]^{(p-1)/p\beta}\int_t^{1-\varepsilon}\frac{[\sigma(\mathscr{N}_\tau)]^{\alpha p/(p-1)}\,\mathrm{d}\tau}{\|\nabla u\|_{L_{p-1}(\mathscr{E}_\tau)}}.$$

(The change of variable $\psi = \psi(t)$ is possible since $\psi(t)$ is absolutely continuous on $[0, 1-\varepsilon]$.) The latter, (6.6.3) and Lemma 6.1.3/2 imply

$$\begin{aligned}
&\psi^{\alpha/\gamma}\big[\sigma(\mathscr{N}_{t(\psi)})\big]^{\alpha p/(p-1)} \\
&\quad\leq c\big(\mathfrak{A}^{(p,\beta)}_{G}\big)^{1/\beta}\sup_{t\in[0,1-\varepsilon]}\frac{1}{m_n(\mathscr{N}_t)} \\
&\qquad\times\int_t^{1-\varepsilon}\bigg(\frac{[\sigma(\mathscr{N}_\tau)]^\alpha}{s(\mathscr{E}_\tau)}\bigg)^{p/(p-1)}\bigg(-\frac{\mathrm{d}}{\mathrm{d}\tau}m_n(\mathscr{N}_\tau)\bigg)\,\mathrm{d}\tau.
\end{aligned} \tag{6.6.4}$$

Taking into account that

$$\big[\sigma(\mathscr{N}_\tau)\big]^\alpha \le \mathfrak{A}^{(1,\alpha)}_{G,\sigma}\, s(\mathscr{E}_\tau)$$

for almost all $\tau \in (0,1)$, from (6.6.4) we obtain

$$\begin{aligned}\big[\psi(t)\big]^{\alpha/\gamma}\big[\sigma(\mathscr{N}_t)\big]^{\alpha p/(p-1)} &\le c\big(\mathfrak{A}^{(p,\beta)}_G\big)^{1/\beta}\big(\mathfrak{A}^{(1,\alpha)}_{G,\sigma}\big)^{p/(p-1)} \\ &\quad\times \sup_{t\in[0,1-\varepsilon]} \frac{1}{m_n(\mathscr{N}_t)}\int_t^{1-\varepsilon}\Big(-\frac{\mathrm{d}}{\mathrm{d}\tau}m_n(\mathscr{N}_\tau)\Big)\,\mathrm{d}\tau \\ &\le c\big(\mathfrak{A}^{(p,\beta)}_G\big)^{1/\beta}\big(\mathfrak{A}^{(1,\alpha)}_{G,\sigma}\big)^{p/(p-1)}.\end{aligned}$$

Passing to the limit as $t \to 1-0$ we arrive at

$$\big[\sigma(F)\big]^\gamma \le c\big(\mathfrak{A}^{(p,\beta)}_G\big)^{1-\sigma/\alpha}\big(\mathfrak{A}^{(1,\alpha)}_{G,\sigma}\big)^{\gamma/\alpha}\big[\psi(1)\big]^{(1-p)/p}.$$

Minimizing the right-hand side over $V_\Omega(K)$, we obtain

$$\big[\sigma(F)\big]^\gamma \le c\big(\mathfrak{A}^{(p,\beta)}_G\big)^{1-\gamma/\alpha}\big(\mathfrak{A}^{(1,\alpha)}_{G,\sigma}\big)^{\gamma/\alpha}\big[c_p(G\backslash F)\big]^{1/p}.$$

The result follows. □

Now we give an example of the application of the preceding proposition in a concrete situation.

Example. Let

$$\Omega = \big\{x : |x'| < x_n^\lambda,\ 0 < x_n < \infty\big\},$$

where $\lambda > 1$, $x' = (x_1, \dots, x_{n-1})$ and let $G = \{x \in \Omega : 0 < x_n < 1\}$. Here the role of the measure σ is played by the $(n-1)$-dimensional measure s on the set $\Pi = \{x \in \Omega : x_1 = 0\}$. We show that $\mathfrak{A}^{(p,\gamma)}_{G,s}$ is finite and the value of γ is the best possible provided $\lambda(n-1)+1 > p \ge 1$ and $\gamma = (\lambda(n-1)+1-p)/(\lambda(n-2)+1)$.

First let $p = 1$. It was shown in Example 5.3.3/1 that the mapping

$$x \to \xi = (x_1, \dots, x_{n-1}, x_n^\lambda),$$

of Ω onto the cone $\xi\Omega = \{\xi : |\xi'| < \xi_n,\ 0 < \xi_n < \infty\}$ is subareal. Hence

$$s(\partial_i g) \ge c s(\partial_i \xi g) \tag{6.6.5}$$

for any admissible set g with $\operatorname{clos}_\Omega g \subset G$. Since $\xi\Omega$ is a cone and $\xi\Pi$ is its cross section by a hyperplane, by Theorem 1.4.5 we have

$$\|u\|_{L(\xi\Pi)} \le c\|\nabla_\xi u\|_{L(\xi\Omega)} \tag{6.6.6}$$

for all $u \in L_p^1(\xi\Omega)$ that vanish outside ξG. Therefore,

$$c\, s(\partial_i \xi g) \ge s(\xi\Pi \cap \xi g). \tag{6.6.7}$$

(The latter estimate results from the substitution of the sequence $\{w_m\}$ constructed in Lemma 5.2.2 into (6.6.6).)

It is clear that

$$s(\xi\varPi \cap \xi g) = \lambda \int_{\varPi\cap g} x_n^{\lambda-1}\, \mathrm{d}x_2, \ldots, \mathrm{d}x_n. \tag{6.6.8}$$

Since $\lambda > 1$, the infimum of the integral on the right in (6.6.8), taken over all sets $\varPi\cap g$ with the fixed measure $s(\varPi\cap g)$, is attained at $\{x \in \varPi : 0 < x_n < a\}$, where a is the number defined by

$$v_{n-2}\big(\lambda(n-2)+1\big)^{-1} a^{\lambda(n-2)+1} = s(\varPi \cap g).$$

Therefore,

$$s(\xi\varPi \cap \xi g) \geq c\big[s(\varPi \cap g)\big]^{\lambda(n-1)/(\lambda(n-2)+1)},$$

which together with (6.6.5) and (6.6.7) yields

$$s(\partial_i g) \geq c\big[s(\varPi \cap g)\big]^{\alpha},$$

where $\alpha = \lambda(n-1)/(\lambda(n-2)+1)$. Thus, $\mathfrak{A}^{(1,\alpha)}_{G,s} < \infty$. On the other hand, since $\varOmega \in \mathscr{I}_{p,\beta}$ with $\beta = (\lambda(n-1)+1-p)p/(\lambda(n-1)+1)$ (cf. (6.3.16)), it follows that $\mathfrak{A}^{(\alpha,\beta)}_{G} < \infty$. Now (6.6.2) implies $\mathfrak{A}^{(p,\gamma)}_{G,s} < \infty$ with

$$\begin{aligned}\gamma &= \frac{\lambda(n-1)}{\lambda(n-2)+1}\cdot\frac{\lambda(n-1)+1-p}{p(\lambda(n-1)+1)}p\left(p-1+\frac{\lambda(n-1)+1-p}{\lambda(n-1)+1}\right)^{-1}\\ &= \frac{\lambda(n-1)+1-p}{p(\lambda(n-2)+1)}.\end{aligned}$$

This value of γ is the best possible, which may be verified using the sequence $F_m = \{x \in \varOmega : 0 < x_n < m^{-1}\}$, $m = 1, 2, \ldots$. In fact,

$$c_p(G\backslash F_m) \leq c\left(\int_{m^{-1}}^{1} \tau^{\lambda(1-n)/(p-1)}\, \mathrm{d}\tau\right)^{1-p} \leq cm^{p-1-\lambda(n-1)},$$

(cf. (6.3.16)) and $s(F_m \cap \varPi) = c\, m^{-1-\lambda(n-2)}$. Thus the estimate

$$\big[s(F_m \cap \varPi)\big]^{\gamma} \leq \mathrm{const}\big[c_p(G\backslash F_m)\big]^{1/p}$$

implies $\gamma \geq (\lambda(n-1)+1-p)/p(\lambda(n-2)+1)$.

In conclusion we consider briefly some other generalizations of the previous results.

Following Chap. 2 with a minor modification in the proofs, we can generalize the results of the present chapter and Chap. 5 to encompass functions with the finite integral

$$\int_\Omega \big[\Phi(x,\nabla u)\big]^p \,\mathrm{d}x,$$

and even those satisfying the following more general condition:

$$\int_\Omega \Psi(x,u,\nabla u)\,\mathrm{d}x < \infty$$

(cf. Remark 2.3.3/2).

Another possible generalization, which needs no essential changes in the proofs, is the replacement of $L_q(\Omega)$ by a Birnbaum–Orlicz space (cf. Theorem 2.3.3).

6.7 Inclusion $W^1_{p,r}(\Omega) \subset L_q(\Omega)$ $(r > q)$ for Domains with Infinite Volume

6.7.1 Classes $\overset{\infty}{\mathscr{J}}_\alpha$ and $\overset{\infty}{\mathscr{I}}_{p,\alpha}$

The classes $\mathscr{J}_\alpha$, $\mathscr{I}_{p,\alpha}$ introduced previously characterize local properties of the domain. In the present section we are interested in the structure of domains at infinity.

Consider, for example, the unbounded plane domain

$$\Omega = \big\{(x,y) : 0 < x < \infty,\ |y| < x^{1/2}\big\}.$$

We have

$$\|u\|_{L_2(\Omega)} \le C\big(\|\nabla u\|_{L_1(\Omega)} + \|u\|_{L_r(\Omega)}\big), \tag{6.7.1}$$

where r is an arbitrary positive number that does not exceed 2.

This inequality can be proved in the following way. Let $Q_{m,n}$ be an arbitrary square of the integral coordinate grid. Each of the domains $\Omega_{m,n} = \Omega \cap Q_{m,n}$ can be mapped onto $Q_{0,0}$ by a quasi-isometric mapping so that the Lipschitz constants of the mapping functions are uniformly bounded and the Jacobian determinants are uniformly separated from zero. This and Theorem 1.4.5 imply the sequence of inequalities

$$\int_{\Omega_{m,n}} u^2\,\mathrm{d}x \le c\big(\|\nabla u\|^2_{L_1(\Omega_{m,n})} + \|u\|^2_{L_r(\Omega_{m,n})}\big)$$

with constant c independent of m, n. Summing over m,n and using

$$\Big(\sum_i a_i^{1/\alpha}\Big)^{1/\alpha} \le \sum_i a_i$$

with $\alpha \ge 1$, $a_i > 0$ we arrive at (6.7.1).

The condition $r \le 2$ is essential for the validity of (6.7.1). In fact, for $u(x,y) = (x+1)^{-\gamma}$ with $3r/2 < \gamma < \frac{3}{4}$, $2 < r < 3$, the right-hand side in (6.7.1) is finite whereas $u \notin L_2(\Omega)$.

In a sense the estimate (6.7.1) is unsatisfactory: The norm $L_r(\Omega)$ is not weaker than that in $L_2(\Omega)$ (contrary to the case $m_n(\Omega) < \infty$). If we discuss the rate of decrease of a function at infinity then the finiteness of the norm in $L_r(\Omega)$ $(r < 2)$ is a more restrictive condition than that of the norm in $L_2(\Omega)$.

So we may pose the following question. Let $m_n(\Omega) = \infty$. What is the space $L_q(\Omega)$ containing $W^1_{p,r}(\Omega)$ for large r?

To this end we introduce classes similar to $\mathscr{J}_\alpha$, $\mathscr{I}_{p,\alpha}$.

Definition 1. The set Ω is contained in *the class* $\overset{\infty}{\mathscr{J}}_\alpha$ if there exists a constant $M > 0$ such that

$$\sup_{\{g\}} \frac{[m_n(g)]^\alpha}{s(\partial_i g)} < \infty, \tag{6.7.2}$$

where the supremum is taken over all admissible sets $g \subset \Omega$ with $m_n(g) \ge M$.

We note, for the time being without proof, that the domain Ω inside the parabola, mentioned at the beginning of this section, is in the class $\overset{\infty}{\mathscr{J}}_{1/3}$ and that

$$\|u\|_{L_3(\Omega)} \le C\big(\|\nabla u\|_{L_1(\Omega)} + \|u\|_{L_r(\Omega)}\big) \tag{6.7.3}$$

for any $r \ge 3$. The exponent 3 on the left-hand side of the above inequality cannot be reduced.

Let F be a bounded subset of Ω closed in Ω and let $\Omega_R = \Omega \cap B_R$. By the p-capacity of F relative to Ω we mean the limit of $c_p(\Omega_R \backslash F)$ as $R \to \infty$. We denote it by $p\text{-cap}_\Omega(F)$.

Definition 2. The domain Ω is contained in *the class* $\overset{\infty}{\mathscr{I}}_{p,\alpha}$, if there exists a constant $M > 0$ such that

$$\overset{\infty}{\mathfrak{A}}_{p,\alpha}(M) \overset{\text{def}}{=} \sup_{\{F\}} \frac{[m_n(F)]^\alpha}{[p\text{-cap}_\Omega(F)]^{1/p}} < \infty. \tag{6.7.4}$$

Here the supremum is taken over all F with $m_n(F) \ge M$, $p\text{-cap}_\Omega(F) > 0$.

Similarly to Lemma 6.3.2, we can prove that the classes $\overset{\infty}{\mathscr{I}}_{1,\alpha}$ and $\overset{\infty}{\mathscr{J}}_\alpha$ coincide and that

$$\overset{\infty}{\mathfrak{A}}_{1,\alpha}(M) = \sup_{\{g\}} \frac{[m_n(g)]^\alpha}{s(\partial_i g)},$$

where $\{g\}$ has the same meaning as in (6.7.2).

Proposition 1. *The class* $\overset{\infty}{\mathscr{I}}_{p,\alpha}$ *is empty provided* $\alpha > 1/p - 1/n$.

Proof. Let $\Omega \in \overset{\infty}{\mathscr{I}}_{p,\alpha}$. If ϱ is a large enough positive number such that $m_n(\Omega_\varrho) \ge M$ and $R > \varrho$, then by (6.7.4)

$$\big[m_n(\Omega_\varrho)\big]^\alpha \le K\big[c_p(\Omega_R\backslash \operatorname{clos}_\Omega \Omega_\varrho)\big]^{1/p}, \quad K = \text{const}.$$

Let $u(x) = \eta(|x|)$ where η is a piecewise linear continuous function which vanishes for $t > R$ and is equal to unity for $t < \varrho$. Since $u \in U_\Omega(\Omega_R\backslash \operatorname{clos}_\Omega \Omega_\varrho)$, we have

$$c_p(\Omega_R\backslash \operatorname{clos}_\Omega \Omega_\varrho) \le (R-\varrho)^{-p} m_n(\Omega_R\backslash\Omega_\varrho).$$

Consequently,

$$R-\varrho \le K\big[m_n(\Omega_R) - m_n(\Omega_\varrho)\big]^{1/p}\big[m_n(\Omega_\varrho)\big]^{-\alpha}.$$

We define a sequence of numbers $\{\varrho_j\}_{j\ge1}$ by $m_n(\Omega_{\varrho_j}) = 2^j M$. Then $\varrho_{j+1} - \varrho_j \le K2^{j(p^{-1}-\alpha)}$. Summing, we get $\varrho_j \le \varrho_1 + cK2^{j(p^{-1}-\alpha)}$, which together with the definition of the sequence $\{\varrho_j\}$ yields

$$\varrho_j \le \varrho_1 + cK\big(m_n(\Omega_{\varrho_j})\big)^{p^{-1}-\alpha}.$$

Since $\varrho_j \to \infty$ then $\alpha < p^{-1}$. Further we have

$$\varrho_j \le \varrho_1 + cK\big(v_n\varrho_j^n\big)^{p^{-1}-\alpha}$$

and hence $\alpha \le p^{-1} - n^{-1}$. The result follows. $\square$

Let F be a bounded subset of Ω closed in Ω and let $\Lambda_F(\sigma) = \inf s(\partial_i g)$ with the infimum taken over all admissible subsets g of Ω which contain F and with $m_n(g) \ge \sigma$.

An immediate corollary of inequality (6.1.10) is the next proposition.

Proposition 2. *The inequality*

$$p\text{-cap}_\Omega(F) \ge \bigg(\int_{m_n(F)}^\infty \big[\Lambda_F(\sigma)\big]^{-p/(p-1)}\,\mathrm{d}\sigma\bigg)^{1-p} \tag{6.7.5}$$

is valid.

Proposition 3. *If* $\Omega \in \overset{\infty}{\mathscr{J}}_{\alpha+(p-1)/p}$, *then* $\Omega \in \overset{\infty}{\mathscr{I}}_{p,\alpha}$ *and*

$$\overset{\infty}{\mathfrak{A}}_{p,\alpha}(M) \le \bigg(\frac{p-1}{p\,\alpha}\bigg)^{(p-1)/p} \overset{\infty}{\mathfrak{A}}_{1,\alpha+(p-1)/p}(M). \tag{6.7.6}$$

Proof. If $m_n(F) \ge M$ then by (6.7.5) we obtain

$$p\text{-cap}_\Omega(F) \ge \big[\overset{\infty}{\mathfrak{A}}_{1,\alpha+(p-1)/p}(M)\big]^{-p}\bigg(\int_{m_n(F)}^\infty \sigma^{-(\alpha+1-1/p)p/(p-1)}\,\mathrm{d}\sigma\bigg)^{1-p},$$

which is equivalent to (6.7.6).

6.7.2 Embedding $W^1_{p,r}(\Omega) \subset L_q(\Omega)$ $(r > q)$

The proof of the following theorem concerning the embedding of the space $W^1_{p,r}(\Omega) = L^1_p(\Omega) \cap L_r(\Omega)$ into $L_q(\Omega)$ is carried out by an argument similar to that used to prove Theorem 6.3.3. We give it for the reader's convenience since it differs in details.

Theorem. 1. *If $\Omega \in \overset{\infty}{\mathscr{I}}_{p,\alpha}$ and $r > q = \alpha^{-1}$, then for any $u \in W^1_{p,r}(\Omega)$*

$$\|u\|_{L_q(\Omega)} \le C_1 \|\nabla u\|_{L_p(\Omega)} + C_2 \|u\|_{L_r(\Omega)}, \tag{6.7.7}$$

where

$$C_2 = M^{(r-q)/rq}, \qquad C_1 \le p(p-1)^{(1-p)/p}\, \overset{\infty}{\mathfrak{A}}_{p,\alpha}(M).$$

2. *If (6.7.7) holds with $r > q$ then $\Omega \in \overset{\infty}{\mathscr{I}}_{p,\alpha}$ with $\alpha = q^{-1}$. Moreover, $M^{(r-q)/rq} = \varepsilon^{-1} C_2$, $\overset{\infty}{\mathfrak{A}}_{p,\alpha}(M) \le (1-\varepsilon)^{-1} C_1$, where ε is an arbitrary number in $(0,1)$.*

Proof. 1. By Lemma 5.1.2/2 it suffices to prove (6.7.7) for functions in $C^\infty(\Omega)$ with supports in Ω_R for some $R < \infty$. We choose a number T such that

$$m_n(\mathscr{L}_T) \le M \le m_n(\mathscr{N}_T).$$

It can be readily checked that

$$\int_\Omega |u|^q \,\mathrm{d}x = \int_{\mathscr{L}_T} |u|^q \,\mathrm{d}x + \int_0^T m_n(\mathscr{N}_t \backslash \mathscr{L}_T)\,\mathrm{d}\big(t^p\big). \tag{6.7.8}$$

The first summand on the right is estimated by Hölder's inequality

$$\int_{\mathscr{L}_T} |u|^q \,\mathrm{d}x \le M^{1-q/r} \Big(\int_\Omega |u|^r \,\mathrm{d}x \Big)^{q/r}.$$

By (1.3.41), the second integral in (6.7.8) does not exceed

$$\Big\{ \int_0^T \big[m_n(\mathscr{N}_t \backslash \mathscr{L}_T)\big]^{p/q} \,\mathrm{d}\big(t^p\big) \Big\}^{q/p}.$$

Therefore,

$$\|u\|_{L_q(\Omega)} \le M^{1/q-1/r} \|u\|_{L_r(\Omega)} + \Big\{ \int_0^T \big[m_n(\mathscr{N}_t)\big]^{p/q} \,\mathrm{d}\big(t^p\big) \Big\}^{1/p}.$$

Since $m_n(\mathscr{N}_t) \ge M$ for $t \in (0,T)$, it follows that

$$\big[m_n(\mathscr{N}_t)\big]^{1/q} \le \overset{\infty}{\mathfrak{A}}_{p,\alpha}(M) \big[p\text{-cap}_\Omega(\mathscr{N}_t)\big]^{1/p} \le \overset{\infty}{\mathfrak{A}}_{p,\alpha}(M) \big[c_p(\Omega_R \backslash \mathscr{N}_t)\big]^{1/p}. \tag{6.7.9}$$

Applying (6.7.9) and Lemma 6.1.3/2, we obtain

$$\|u\|_{L_q(M)} \le M^{1/q-1/r}\|u\|_{L_r(\Omega)} + \overset{\infty}{\mathfrak{A}}_{p,\alpha}(M)\left[\int_0^\infty c_p(\Omega_R\backslash\mathscr{N}_t)\,\mathrm{d}\big(t^p\big)\right]^{1/p}$$
$$\le M^{1/q-1/r}\|u\|_{L_r(\Omega)} + \overset{\infty}{\mathfrak{A}}_{p,\alpha}(M)\frac{p}{(p-1)^{(p-1)/p}}\|\nabla u\|_{L_p(\Omega)}.$$

2. We put $M = (C_2\varepsilon^{-1})^{rq/(r-q)}$ and consider an arbitrary bounded $F \subset \Omega$ closed in Ω with $m_n(F) \ge M$. Let u be any function in $C^{0,1}(\Omega)$ that vanishes outside some ball and is equal to unity on F. It is clear that

$$\int_\Omega |u|^r\,\mathrm{d}x = \int_0^1 m_n(\mathscr{N}_t)\,\mathrm{d}\big(t^r\big).$$

Since $m_n(\mathscr{N}_t)$ does not increase and $q < r$, by (1.3.41) we have

$$\int_\Omega |u|^r\,\mathrm{d}x \le \left(\int_0^1 \big[m_n(\mathscr{N}_t)\big]^{q/r}\,\mathrm{d}\big(t^q\big)\right)^{r/q}.$$

Taking into account that $m_n(\mathscr{N}_t) \ge m_n(F) \ge M$, we obtain

$$\|u\|_{L_r(\Omega)} \le M^{1/r-1/q}\left(\int_0^1 m_n(\mathscr{N}_t)\,\mathrm{d}\big(t^q\big)\right)^{r/q} = \varepsilon C_2^{-1}\|u\|_{L_q(\Omega)}.$$

Now, (6.7.7) and the preceding inequality imply that

$$\|u\|_{L_q(\Omega)} \le C_1(1-\varepsilon)^{-1}\|\nabla u\|_{L_p(\Omega)}.$$

Having in mind that $u = 1$ on F and minimizing $\|\nabla u\|_{L_p(\Omega)}$, we finally obtain

$$\big[m_n(F)\big]^{1/q} \le C_1(1-\varepsilon)^{-1}\big[p\text{-cap}_\Omega(F)\big]^{1/p}.$$

The theorem is proved. □

The first part of the Theorem and Proposition 6.7.1/3 imply the next corollary.

Corollary. *If* $\Omega \in \overset{\infty}{\mathscr{J}}_\alpha$, $\alpha \le 1$, $p \ge 1$, $r > q$, $p(1-\alpha) < 1$, $q = p/[1 - p(1-\alpha)]$ *then* (6.7.7) *is valid for any* $u \in W_{p,r}^1(\Omega)$.

6.7.3 Example of a Domain in the Class $\overset{\infty}{\mathscr{J}}_{p,\alpha}$

Example. Consider the "paraboloid"

$$\Omega = \big\{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n-1}^2 < a x_n^{2\beta},\ 0 < x_n < \infty\big\}, \tag{6.7.10}$$

where $1 > \beta > 0$ and $a = \text{const} > 0$ (Fig. 25).

First we show that

$$0 < \lim_{M\to\infty} \sup \frac{[m_n(g)]^\alpha}{s(\partial_i g)} < \infty, \tag{6.7.11}$$

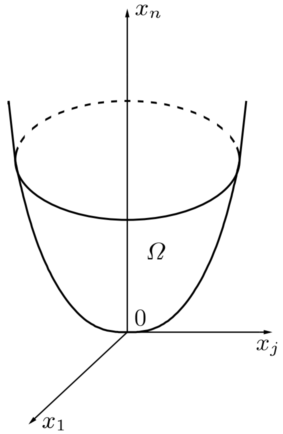

Fig. 25.

where the supremum is taken over all admissible subsets g of the domain Ω with $m_n(g) \geq M$ and $\alpha = \beta(n-1)/(\beta(n-1)+1)$. Taking the symmetrization of g with respect to the ray Ox_n and repeating the proof of Lemma 5.2.1/1 we obtain that the ball B, orthogonal to $\partial\Omega$, has the smallest area $\partial_i g$ among all sets g with the fixed volume M. After a routine calculation we obtain

$$\lim_{M\to\infty} \frac{[m_n(B)]^\alpha}{s(\partial_i B)} = \frac{(1-\alpha)^\alpha}{v_{n-1}^{1-\alpha}}.$$

So (6.7.11) follows.

Thus, $\Omega \in \overset{\infty}{\mathscr{I}}_{\beta(n-1)/(\beta(n-1)+1)}$ and by Corollary 6.7.2 and the second part of Theorem 6.7.2 we have $\Omega \in \overset{\infty}{\mathscr{I}}_{p,\alpha}$ with $p < \beta(n-1)+1$ and $\alpha = p^{-1} - (\beta(n-1)+1)^{-1}$. We show that this value of α is the largest possible. In fact, let $\Omega(\mathscr{X}) = \{x \in \Omega : x_n \leq \mathscr{X}\}$. It is clear that

$$p\text{-cap}_\Omega\big(\Omega(\mathscr{X})\big) \leq \int_{\Omega\setminus\Omega(\mathscr{X})} \big|\nabla\big[(\mathscr{X}/x_n)^{\beta(n-1)}\big]\big|^p \,\mathrm{d}x = c\mathscr{X}^{\beta(n-1)+1-p}$$

and hence

$$\frac{[m_n(\Omega(\mathscr{X}))]^\alpha}{[p\text{-cap}_\Omega(\Omega(\mathscr{X}))]^{1/p}} \geq c\,\mathscr{X}^{[\beta(n-1)+1](\alpha-\gamma)} \xrightarrow{\mathscr{X}\to\infty} \infty$$

for $\alpha > p^{-1} - [\beta(n-1)+1]^{-1} = \gamma$.

Thus, for the domain (6.7.10) and for $\beta(n-1) > p-1 \geq 0$, the inequality (6.7.7) holds with $r > q = [\beta(n-1)+1]p/[\beta(n-1)+1-p]$. This value of q cannot be reduced. We note that it exceeds the limit exponent $np/(n-p)$ in the Sobolev theorem for $\beta < 1$.

6.7.4 Space $\overset{(0)}{L}{}_p^1(\Omega)$ and Its Embedding into $L_q(\Omega)$

Let $\overset{(0)}{L}{}_p^1(\Omega)$ denote the completion of the set of functions in $C^\infty(\Omega)\cap L_p^1(\Omega)$ with bounded supports with respect to the norm $\|\nabla u\|_{L_p(\Omega)}$. (Here and elsewhere in this section $m_n(\Omega)=\infty$.)

According to Lemma 5.1.2/2, the space $\overset{(0)}{L}{}_p^1(\Omega)$ coincides with the completion of $W_{p,r}^1(\Omega)$ (r is an arbitrary positive number) with respect to the norm $\|\nabla u\|_{L_p(\Omega)}$.

Removing the conditions $m_n(g)\ge M$, $m_n(F)\ge M$ in the definitions of the classes $\overset{\infty}{\mathscr{J}}_\alpha$, $\overset{\infty}{\mathscr{I}}_{p,\alpha}$ (i.e., putting $M=0$) we obtain the definitions of the classes $\overset{\infty}{\mathscr{J}}_\alpha(0)$, $\overset{\infty}{\mathscr{I}}_{p,\alpha}(0)$.

Remark 6.3.1 implies $\alpha\ge 1/p-1/n$ provided $n\ge p$ and $\Omega\in\overset{\infty}{\mathscr{I}}_{p,\alpha}$. On the other hand, according to Proposition 6.6, $\alpha\le 1/p-1/n$. Thus, only the class $\overset{\infty}{\mathscr{I}}_{p,\,1/p-1/n}(0)$ is not empty. (For example, the domain (6.7.10) with $\beta=1$ and the space $\mathbb{R}^n$ are contained in this class.)

Taking the latter into account and mimicking the proof of Theorem 2.3.3 with minor modifications, we arrive at the following theorem.

Theorem. *The inequality*

$$\|u\|_{L_q(\Omega)}\le C\|\nabla u\|_{L_p(\Omega)},\quad p\ge 1 \tag{6.7.12}$$

is valid for all $u\in\overset{(0)}{L}{}_p^1(\Omega)$ *if and only if* $n>p$, $q=pn/(n-p)$ *and* $\Omega\in\overset{\infty}{\mathscr{I}}_{p,1/p-1/n}(0)$.

The best constant in (6.7.12) *satisfies*

$$\overset{\infty}{\mathfrak{A}}_{p,1/p-1/n}(0)\le C\le p(p-1)^{(1-p)/p}\,\overset{\infty}{\mathfrak{A}}_{p,1/p-1/n}(0).$$

Remark. We note that the inclusion $\Omega\in\overset{\infty}{\mathscr{I}}_{p,1/p-1/n}(0)$, $n>p$, does not imply the Poincaré-type inequality

$$\inf_{c\in\mathbb{R}^1}\|u-c\|_{L_{pn/(n-p)}(\Omega)}\le C\|\nabla u\|_{L_p(\Omega)},\quad u\in L_p^1(\Omega). \tag{6.7.13}$$

In fact, consider the domain in Fig. 26, which is the union of the two cones $\{x:|x'|<x_n+1\}$ and $\{x:|x'|<1-x_n\}$, $x'=(x_1,\dots,x_{n-1})$. Each of the cones is in the class $\overset{\infty}{\mathscr{I}}_{p,1/p-1/n}(0)$. Hence their union Ω is in the same class (cf. Proposition 6.3.1/1). However, the left-hand side in (6.7.13) is infinite for a smooth function that is odd in x_n, vanishes for $0<x_n<1$ and is equal to unity for $x_n>2$, and which, obviously, belongs to $L_p^1(\Omega)$.

At the same time, by the Theorem, the inclusion $\Omega\in\overset{\infty}{\mathscr{I}}_{p,1/p-1/n}(0)$ is equivalent to the inequality

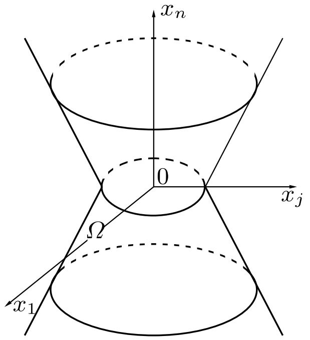

Fig. 26.

$$\|u\|_{L_{pn/(n-p)}(\Omega)} \le C\|\nabla u\|_{L_p(\Omega)}, \quad u \in W^1_{p,pn/(n-p)}(\Omega). \tag{6.7.14}$$

In particular, from this it follows that (6.7.13) implies $\Omega \in \overset{\infty}{\mathscr{I}}_{p,1/p-1/n}(0)$.

The preceding Theorem shows as well that the norm in $L_{pn/(n-p)}(\Omega)$ cannot be replaced by the norm in $L_q(\Omega)$ with $q \ne pn/(n-p)$ in (6.7.13).

6.7.5 Poincaré-Type Inequality for Domains with Infinite Volume

The following assertion gives a description of domains for which (6.7.13) is valid.

Here (and only here) we shall assume that the condition of boundedness of the sets G and F is omitted in the definition of p-conductivity.

Theorem. *Let $m_n(\Omega) = \infty$. The inequality (6.7.13) holds for all $u \in L^1_p(\Omega)$, $p \ge 1$, if and only if Ω is a connected open set in $\overset{\infty}{\mathscr{I}}_{p,1/p-1/n}(0)$ and*

the finiteness of the p-conductivity of the conductor
$K = G\backslash F$ in Ω implies either $m_n(F) < \infty$ or $m_n(\Omega\backslash G) < \infty$. (6.7.15)

For $p = 1$ the condition (6.7.12) is equivalent to the following:

if G is an open subset of Ω such that $\partial_i G$ is a smooth
manifold and $s(\partial_i G) < \infty$ then either $m_n(G) < \infty$ or $m_n(\Omega\backslash G) < \infty$. (6.7.16)

(We recall that $\overset{\infty}{\mathscr{I}}_{p,1-1/n}(0) = \overset{\infty}{\mathscr{J}}_{1-1/n}(0)$.)

Proof. The necessity of the inclusion $\Omega \in \overset{\infty}{\mathscr{I}}_{p,1/p-1/n}(0)$ was noted at the end of Remark 6.7.4. The necessity of the connectedness of Ω is obvious.

We prove that (6.7.13) implies (6.7.15). Let v be an arbitrary function in $U_\Omega(K)$, where K is the conductor $G\backslash F$ with $c_p(K) < \infty$. We put $u = \max\{0, \min\{v, 1\}\}$ in (6.7.13). Then

$$\left[|c|^q m_n(\Omega\backslash G) + |1-c|^q m_n(F)\right]^{1/q} \le C\|\nabla u\|_{L_p(\Omega)}$$

with $q = pn/(n-p)$. Therefore, either $m_n(\Omega\backslash G) < \infty$ or $m_n(F) < \infty$.

Now let $p = 1$. We prove the necessity of (6.7.16). First we note that the condition that G is bounded was not used in the proof of Lemma 5.2.2. Let G be an open subset of Ω such that $\partial_i G$ is smooth manifold and $s(\partial_i G) < \infty$. We insert any function w_m from the sequence constructed in Lemma 5.2.2 in (6.7.13) in place of u. Then, starting with some index m, for any compactum $e \subset G$, we have

$$\left(|c_m|^{n/(n-1)} m_n(\Omega\backslash G) + |1-c_m|^{n/(n-1)} m_n(e)\right)^{(n-1)/n} \le C\|\nabla w_m\|_{L_1(\Omega)},$$

where $c_m = \text{const}$. Since

$$\limsup_{m\to\infty} \|\nabla w_m\|_{L_1(\Omega)} = s(\partial_i G) < \infty,$$

we have either $m_n(\Omega\backslash G) < \infty$, or $c_m = 0$ and

$$\left[m_n(e)\right]^{(n-1)/n} \le C s(\partial_i G).$$

Consequently, (6.7.16) holds.

The sufficiency is proved in several steps.

Lemma 1. 1. *Let $\Omega \in \overset{\infty}{\mathscr{I}}_{p,q}(0)$ with $q = pn/(n-p)$, $n > p$. Then*

$$\left[m_n(F)\right]^{1-p/n} \le \text{const}\, c_p(K), \tag{6.7.17}$$

holds for all conductors $K = G\backslash F$, where G is an open subset of Ω and F is a subset of Ω with finite volume, closed in Ω.

2. *Let $\Omega \in \overset{\infty}{\mathscr{J}}_{1-1/n}(0)$. Then*

$$\left[m_n(G)\right]^{1-1/n} \le \text{const}\, s(\partial_i G) \tag{6.7.18}$$

holds for all open sets $G \subset \Omega$ such that $\partial_i G$ is a smooth manifold and $m_n(G) < \infty$.

Proof. 1. Since $\Omega \in \overset{\infty}{\mathscr{I}}_{p,q}(0)$, it follows that

$$\left[m_n(H)\right]^{1-p/n} \le \text{const}\, c_p(G\backslash H)$$

for any bounded set $H \subset F$ that is closed in Ω. Now (6.7.17) follows from $c_p(G\backslash H) \le c_p(G\backslash F)$ and $m_n(F) = \sup_H m_n(H)$.

2. Let $\Omega \in \overset{\infty}{\mathscr{I}}_{1-1/n}(0)$, $m_n(G) < \infty$, $s(\partial_i G) < \infty$. We note that we did not use the boundedness of G in the proof of Lemma 5.2.2 and insert the sequence constructed in this lemma into (6.7.14). Passing to the limit as $m \to \infty$, we obtain

$$\big[m_n(e)\big]^{1-1/n} \le \text{const}\, s(\partial_i G)$$

for any compactum $e \subset G$. □

Lemma 2. 1. *If $\Omega \in \overset{\infty}{\mathscr{I}}_{p,q}(0)$ with $q = pn/(n-p)$, $n > p$, and (6.7.15) holds, then for any $u \in C^\infty(\Omega) \cap L^1_p(\Omega)$ there exists a unique number c such that*

$$m_n\big(\{x : |u(x) - c| \ge \varepsilon\}\big) < \infty \quad \text{for all } \varepsilon > 0. \tag{6.7.19}$$

2. *The same is true for $p = 1$ provided $\Omega \in \overset{\infty}{\mathscr{I}}_{1-1/n}(0)$ and (6.7.16) holds.*

Proof. 1. We introduce the sets $A_t = \{x : u(x) > t\}$, $B_t = \{x : u(x) \ge t\}$, $C_t = \Omega \backslash A_t$, $D_t = \Omega \backslash B_t$ and put

$$c = \inf\big\{t : m_n(A_t) < \infty\big\}. \tag{6.7.20}$$

Suppose that $c = +\infty$. Then $m_n(A_t) = \infty$ for all $t \in \mathbb{R}^1$ and $m_n(C_t) < \infty$ for all t by

$$(T - t)^p c_p(A_t \backslash B_T) \le \|\nabla u\|^p_{L_p(\Omega)} \tag{6.7.21}$$

and (6.7.12). According to (6.7.17), we have

$$\big[m_n(C_t)\big]^{1-p/n} \le \text{const}\, c_p(D_T \backslash C_t) \tag{6.7.22}$$

for all $T > t$. Since

$$(T - t)^p c_p(D_T \backslash C_t) \le \|\nabla u\|^p_{L_p(\Omega)},$$

the right-hand side in (6.7.22) tends to zero as $T \to +\infty$. Consequently, $m_n(C_t) = 0$ for all t and $c < +\infty$.

Now let $c = -\infty$. Then $m_n(A_t) < \infty$ for all $t \in \mathbb{R}^1$. Applying (6.7.17) to the conductor $A_t \backslash B_T$ with $T > t$, we arrive at

$$\big[m_n(B_T)\big]^{1-p/n} \le \text{const}\, c_p(A_t \backslash B_T).$$

Since by (6.7.21) the right-hand side tends to zero as $t \to -\infty$, we see that $m_n(B_T) = 0$ for all T. Thus $c > -\infty$.

Now we prove (6.7.19). From definition (6.7.20) it follows that

$$m_n\big(\{x : u - c \ge \varepsilon\}\big) < \infty \tag{6.7.23}$$

for $\varepsilon > 0$. On other hand, (6.7.20) gives $m_n(\{x : u - c \ge -\varepsilon/2\}) = \infty$.

Since the p-conductivity of the conductor $A_{c-\varepsilon/2} \backslash B_{c-\varepsilon}$ is finite (cf. (6.7.21)), by (6.7.15) we have

$$m_n(\{x : u < c - \varepsilon\}) < \infty. \tag{6.7.24}$$

Inequalities (6.7.23) and (6.7.24) are equivalent to (6.7.19). The uniqueness of the constant c is an obvious corollary of the condition (6.7.15) and the finiteness of the conductivity of any conductor $A_t \backslash B_T$, $t > T$. The first part of the Lemma is proved.

2. Now let $p = 1$. The identity

$$\int_\Omega |\nabla u| \, dx = \int_{-\infty}^{+\infty} s(\partial A_t) \, dt, \quad u \in C^\infty(\Omega) \cap L_1^1(\Omega)$$

implies

$$s(\partial A_t) < \infty \quad \text{for almost all } t \tag{6.7.25}$$

and

$$\liminf_{t \to -\infty} s(\partial A_t) = \liminf_{t \to +\infty} s(\partial A_t) = 0. \tag{6.7.26}$$

Further, it suffices to duplicate the argument in the proof of the first part of the Theorem using (6.7.25) in place of the finiteness of the conductivity of the conductor $A_t \backslash B_T = D_T \backslash C_t$, and (6.7.26) in place of the convergence to zero of $c_p(A_t \backslash B_T)$ as $t \to +\infty$ or $T \to -\infty$. The lemma is proved. □

We complete the proof of sufficiency in the Theorem. Let $u \in C^\infty(\Omega) \cap L_p^1(\Omega)$. According to Lemma 6.1.1/1 we may in addition assume that $u \in L_\infty(\Omega)$.

We put

$$u_\varepsilon(x) = \begin{cases} u(x) - c - \varepsilon & \text{if } u(x) > c + \varepsilon, \\ 0 & \text{if } |u(x) - c| \le \varepsilon, \\ u(x) - c + \varepsilon & \text{if } u(x) < c - \varepsilon, \end{cases}$$

where c is the constant specified in Lemma 2. From (6.7.19) and the boundedness of u it follows that $u \in L_{pn/(n-p)}(\Omega)$. Hence u_ε can be inserted into (6.7.14). Passing to the limit as $\varepsilon \to +0$, we arrive at (6.7.13). The theorem is proved. □

6.8 Compactness of the Embedding $L_p^1(\Omega) \subset L_q(\Omega)$

In this section we obtain the necessary and sufficient conditions for the sets bounded in $L_p^1(\Omega)$ to be compact in $L_q(\Omega)$. Here Ω is *a domain with finite volume.*

6.8.1 Class $\mathring{\mathscr{I}}_{p,\alpha}$

As before, by $\mathfrak{A}_{p,\alpha}(M)$ we mean the constant in the definition of the class $\mathscr{I}_{p,\alpha}$.

Definition. The domain Ω is contained in *the class* $\mathring{\mathscr{I}}_{p,\alpha}$ provided that $\mathfrak{A}_{p,\alpha}(M) \to 0$ as $M \to 0$.

The equality (6.3.4) implies that $\Omega \in \mathring{\mathscr{I}}_{1,\alpha}$ if and only if

$$\lim_{M\to 0} \sup_{\{g: m_n(g) \le M\}} \frac{[m_n(g)]^\alpha}{s(\partial_i g)} = 0 \tag{6.8.1}$$

(as before, here g designates an admissible subset of Ω).

The value of α in the definition of $\mathring{\mathscr{I}}_{p,\alpha}$ exceeds $1/p - 1/n$ since

$$c_p(B_{2\varrho}\backslash \bar{B}_\varrho) = \text{const}\, \varrho^{n-p}$$

and hence

$$\frac{[m_n(B_\varrho)]^{1/p-1/n}}{[c_p(B_{2\varrho}\backslash \bar{B}_\varrho)]^{1/p}} = \text{const} > 0.$$

6.8.2 Compactness Criteria

Theorem 1. *Let $m_n(\Omega) < \infty$. The embedding operator of $L_p^1(\Omega)$ into $L_{q^*}(\Omega)$, $1 < p \le q^* < \infty$, is compact if and only if $\Omega \in \mathring{\mathscr{I}}_{p,\alpha}$ for $p\alpha \le 1$, where $\alpha^{-1} = q^*$.*

Proof. Sufficiency. Let u be an arbitrary function in $C^\infty(\Omega) \cap L_p^1(\Omega) \cap L_\infty(\Omega)$ with bounded support. (By Corollary 5.1.2, the set of such functions is dense in $L_p^1(\Omega)$.) Let

$$T = \inf\{t : m_n(\mathscr{N}_t) \le M\}.$$

Obviously,

$$\|u\|_{L_{q^*}(\Omega)} \le c\big(\big\|(|u| - T)_+\big\|_{L_{q^*}(\Omega)} + T\big[m_n(\Omega)\big]^{1/q^*}\big).$$

By Corollary 6.3.3 we have

$$\big\|(|u| - T)_+\big\|_{L_{q^*}(\Omega)} \le \delta(M)\|\nabla u\|_{L_p(\Omega)},$$

where $\delta(M) = c\mathfrak{A}_{p,\alpha}(M)$.

Let Ω_M denote a bounded subdomain of Ω with $C^{0,1}$ boundary and with $m_n(\Omega\backslash\Omega_M) < M/2$. Since $m_n(\mathscr{N}_T) \ge M$, it follows that $m_n(\mathscr{N}_T \cap \Omega_M) \ge M/2$. Consequently,

$$\|u\|_{L_r(\Omega_M)} \ge 2^{-1/r} T M^{1/r}$$

and we arrive at

$$\|u\|_{L_{q^*}(\Omega)} \le c\delta(M)\|\nabla u\|_{L_p(\Omega)} + cM^{-1/r}\big[m_n(\Omega)\big]^{1/q^*}\|u\|_{L_r(\Omega_M)}. \tag{6.8.2}$$

By Corollary 5.1.2 the latter is valid for all $u \in L_p^1(\Omega)$.

Since Ω_M is a domain with a smooth boundary and compact closure, the embedding operator of $L_p^1(\Omega_M)$ into $L_r(\Omega_M)$ is compact. Let $\{u_m\}_{m\geq 1}$ with $\|u_m\|_{L_p^1(\Omega)} = 1$ be a Cauchy sequence in $L_r(\Omega_M)$. Then (6.8.2) implies

$$\|u_m - u_l\|_{L_{q^*}(\Omega)} \leq c\delta(M) + cM^{-1/r}\big[m_n(\Omega)\big]^{1/q^*}\|u_m - u\|_{L_r(\Omega_M)}. \quad (6.8.3)$$

Given any $\varepsilon > 0$ we can find an M such that $c\,\delta(M) < \varepsilon/2$. Next we choose a large enough number N_ε so that the second term in (6.8.3) does not exceed $\varepsilon/2$ for $m, l > N_\varepsilon$. Then $\|u_m - u_l\|_{L_{q^*}(\Omega)} < \varepsilon$ for $m, l > N_\varepsilon$ and hence $\{u_m\}$ is a Cauchy sequence in $L_{q^*}(\Omega)$.

Necessity. Suppose the embedding operator of $L_p^1(\Omega)$ into $L_{q^*}(\Omega)$ is compact. Then the elements of the unit sphere in $L_p^1(\Omega)$ have absolutely equicontinuous norms in $L_{q^*}(\Omega)$. Thus, for all $u \in L_p^1(\Omega)$,

$$\|u\|_{L_{q^*}(G)} \leq \varepsilon(M)\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_1(\omega)}\big),$$

where ω is a bounded subdomain of Ω, $\bar{\omega} \subset \Omega$, G is an arbitrary open subset of Ω with $m_n(G) \leq M$ and $\varepsilon(M)$ tends to zero as $M \to 0$.

Suppose that the function $u \in L_p^1(\Omega)$ vanishes outside G. Then

$$\|u\|_{L_{q^*}(\Omega)} \leq \varepsilon(M)\big[1 - \varepsilon(M)M^{1-1/q^*}\big]^{-1}\|\nabla u\|_{L_p(\Omega)}.$$

It remains to use the second part of Corollary 6.3.3. The theorem is proved. □

Theorem 2. *Let $m_n(\Omega) < \infty$. The embedding operator of $L_p^1(\Omega)$ into $L_{q^*}(\Omega)$, $p > 1$, $0 < q^* < p$, is compact if and only if it is bounded, i.e., $\Omega \in \mathscr{H}_{p,\alpha}$, where $\alpha^{-1} = q^*$.*

Proof. Let $\Omega \in \mathscr{H}_{p,\alpha}$. Then $\mathfrak{B}_{p,\alpha}(\frac{1}{2}m_n(\Omega)) < \infty$ and since $\nu_{M,p}(t) \geq \nu_{m_n(\Omega)/2}(t)$ for $t < M$ we have $\mathfrak{B}_{p,\alpha}(M) \to 0$ as $M \to 0$. It remains to notice that the proof of sufficiency in Theorem 1 together with Corollary 6.4.2/1 imply (6.8.3) with $\delta(M) = c\mathfrak{B}_{p,\alpha}(M)$. The proof is complete. □

6.8.3 Sufficient Conditions for Compactness of the Embedding $L_p^1(\Omega) \subset L_{q^*}(\Omega)$

The inequality

$$\mathfrak{A}_{p_1,\alpha_1}(M) \leq c\mathfrak{A}_{p,\alpha}(M)$$

with $p_1 > p \geq 1$, $\alpha_1 - p_1^{-1} = \alpha - p^{-1}$ implies the embedding

$$\mathring{\mathscr{I}}_{p,\alpha} \subset \mathring{\mathscr{I}}_{p_1,\alpha_1}.$$

In particular,

$$\mathring{\mathscr{J}}_{\alpha+1-p^{-1}} \stackrel{\text{def}}{=} \mathring{\mathscr{I}}_{1,\alpha+1-p^{-1}} \subset \mathring{\mathscr{I}}_{p,\alpha}.$$

The preceding leads to the following corollary.

Corollary 1. *If* $\Omega \subset \overset{\circ}{\mathscr{J}}_\alpha$ *and* $p(1-\alpha) < 1$, $1 \le p \le q^* = p/[1 + p(1-\alpha)]$ *then the embedding operator of* $L_p^1(\Omega)$ *into* $L_{q^*}(\Omega)$ *is compact.*

By Proposition 6.4.4 the requirement

$$\int_0^M \left(\frac{\tau}{\lambda_M(\tau)} \right)^{p/(\alpha p - 1)} \mathrm{d}\tau < \infty \qquad (6.8.4)$$

implies $\mathfrak{B}_{p,\alpha}(M) \to 0$ as $M \to 0$. Therefore we have the next statement.

Corollary 2. *If the integral* (6.8.4) *converges, then the embedding operator of* $L_p^1(\Omega)$ *into* $L_{q^*}(\Omega)$ ($q^* = \alpha^{-1}$, $\alpha p > 1$) *is compact.*

Corollaries 1 and 2 immediately imply the following rougher assertion.

Corollary 3. *If* $\Omega \in \mathscr{J}_\alpha$ *and* $p(1-\alpha) \le 1$, $p \ge 1$, $1 \le q < p/[1 + p(\alpha - 1)]$ *then the embedding operator of* $L_p^1(\Omega)$ *into* $L_q(\Omega)$ *is compact.*

Clearly, the space $L_p^1(\Omega)$ can be replaced by $W_{p,r}^1(\Omega)$ with $r \le q^*$ in Theorems 6.8.2/1 and 6.8.2/2 and Corollaries 1–3.

6.8.4 Compactness Theorem for an Arbitrary Domain with Finite Volume

The positiveness of the function λ_M (cf. Lemma 5.2.4) and the estimate (6.3.11) imply $\nu_{M,p}(t) > 0$ for $t > 0$. Hence there exists a nondecreasing positive continuous function φ on $(0, m_n(\Omega)]$ such that $\varphi(t)/\nu_{M,p}(t)$ tends to zero as $t \to +0$.

Theorem. *Let* Ω *be an arbitrary domain with finite volume. Then from any bounded sequence in* $L_p^1(\Omega)$ *we can select a subsequence* $\{u_m\}_{m \ge 1}$ *with*

$$\int_0^\infty \varphi\big[m_n\{x : \big|u_m(x) - u_k(x)\big| \ge t\}\big] \,\mathrm{d}\big(t^p\big) \xrightarrow{m,k \to \infty} 0$$

and therefore any bounded subset of $L_p^1(\Omega)$ *is compact in an* n*-dimensional Lebesgue measure.*

Proof. Let u be a function in $C^\infty(\Omega) \cap L_p^1(\Omega) \cap L_\infty(\Omega)$ with bounded support. Obviously,

$$\int_0^\infty \varphi\big[m_n(\mathscr{N}_t)\big] \,\mathrm{d}\big(t^p\big) \le \int_T^\infty \varphi\big[m_n(\mathscr{N}_t)\big] \,\mathrm{d}\big(t^p\big) + T^p \varphi\big[m_n(\Omega)\big]. \qquad (6.8.5)$$

Here $T = \inf\{t : m_n(\mathscr{N}_t) \le \mu\}$, where μ is a sufficiently small positive number that is independent of u. The right-hand side in (6.8.5) does not exceed

$$\sup_{0 < \tau \le \mu} \frac{\varphi(\tau)}{\nu_{M,p}(\tau)} \int_T^\infty c_p(\mathscr{L}_T \backslash \mathscr{N}_t) \,\mathrm{d}\big(t^p\big) + c\varphi\big[m_n(\Omega)\big] \mu^{-p} \left(\int_{\Omega_\mu} |u| \,\mathrm{d}x \right)^p,$$

where Ω_μ is the domain specified in the proof of Theorem 6.8.2 with M replaced by μ. By Lemma 6.1.3/3 we have

$$\int_0^\infty \varphi\big[m_n(\mathscr{N}_t)\big]\, \mathrm{d}\big(t^p\big) \le \sup_{0\le\tau\le\mu} \frac{\varphi(\tau)}{\nu_{M,p}(\tau)} \|\nabla u\|^p_{L_p(\Omega)} + c\varphi\big[m_n(\Omega)\big]\mu^{-p}\|u\|^p_{L(\Omega_\mu)}. \tag{6.8.6}$$

By Corollary 5.1.2, the last inequality extends to encompass all functions in $L_p^1(\Omega)$. It remains to apply the arguments used at the end of the proof of sufficiency in Theorem 6.8.2/1.

6.8.5 Examples of Domains in the class $\mathring{\mathscr{I}}_{p,\alpha}$

Example 1. The estimates (6.3.15) imply that the domain

$$\Omega = \big\{x : (x_1^2 + \cdots + x_{n-1}^2)^{1/2} < f(x_n),\ 0 < x_n < a\big\},$$

in Example 6.3.6/1 is contained in $\mathring{\mathscr{I}}_{p,\alpha}$ if and only if

$$\lim_{x\to+0} \bigg(\int_0^x \big[f(t)\big]^{n-1}\,\mathrm{d}t\bigg)^{\alpha p/(p-1)} \int_x^a \big[f(t)\big]^{(1-n)/(p-1)}\,\mathrm{d}t = 0.$$

By (5.3.5) and Corollary 6.8.3/1, a sufficient condition for Ω to belong to $\mathring{\mathscr{I}}_{p,\alpha}$ is

$$\lim_{x\to+0} \big[f(x)\big]^{1-n} \bigg(\int_0^x \big[f(t)\big]^{n-1}\,\mathrm{d}t\bigg)^{\alpha+1-1/p} = 0. \tag{6.8.7}$$

Since f does not decrease, we see that (6.8.7) holds for $\alpha p = 1$ and also for $\alpha p < 1$ provided $\lim_{x\to+0} x^\sigma f(x) = 0$, $\sigma = (p\alpha + p - 1)/(n-1)(\alpha p - 1)$.

Example 2. The domain $\Omega = \{x : 0 < x_n < \infty, (x_1^2 + \cdots + x_{n-1}^2)^{1/2} < f(x_n)\}$ in Example 6.3.6/2 is contained in $\mathring{\mathscr{I}}_{p,\alpha}$ if and only if

$$\lim_{x\to+\infty} \bigg(\int_x^{+\infty} \big[f(\tau)\big]^{n-1}\,\mathrm{d}\tau\bigg)^{\alpha p/(p-1)} \int_0^x \big[f(\tau)\big]^{(1-n)/(p-1)}\,\mathrm{d}\tau = 0 \tag{6.8.8}$$

(cf. estimates (6.3.18)). By (5.3.8) and (6.3.11) the last equality holds if

$$\lim_{x\to+\infty} \big[f(x)\big]^{1-n} \bigg(\int_x^\infty \big[f(\tau)\big]^{n-1}\,\mathrm{d}\tau\bigg)^{\alpha+1-1/p} = 0. \tag{6.8.9}$$

In particular, $\Omega \in \mathring{\mathscr{I}}_{p,1/p}$ provided

$$f(\tau) = e^{-\beta(\tau)}, \qquad \beta'(\tau) \to +\infty \quad \text{as } \tau \to +\infty.$$

In the case $f(\tau) = e^{-c\tau}$, the domain under consideration is contained in $\mathscr{I}_{p,1/p}$ and does not belong to $\mathring{\mathscr{I}}_{p,1/p}$.

Similarly, a necessary and sufficient condition for the spiral in Examples 5.3.3/3 and 6.3.6/3 to be in $\mathring{\mathscr{I}}_{p,1/p}$ is that

$$\lim_{\theta\to+\infty}\Big(\int_\theta^\infty \delta(\varphi)\,\mathrm{d}\varphi\Big)^{\alpha p/(p-1)}\int_0^\theta \big[\delta(\varphi)\big]^{1/(1-p)}\,\mathrm{d}\varphi = 0.$$

A simpler sufficient condition is that

$$\Big(\int_\theta^{+\infty} \delta(\varphi)\,\mathrm{d}\varphi\Big)^{\alpha+1-1/p} = o\big(\delta(\theta)\big) \quad \text{as } \theta \to +\infty.$$

In particular, $\Omega \in \mathring{\mathscr{I}}_{p,1/p}$ if $\delta(\varphi) = e^{-\beta(\varphi)}$, $\beta'(\varphi) \to +\infty$ as $\varphi \to +\infty$, and $\Omega \in \mathscr{I}_{p,1/p}\backslash\mathring{\mathscr{I}}_{p,1/p}$ if $\delta(\varphi) = e^{-c\varphi}$.

6.9 Embedding $L_p^l(\Omega) \subset L_q(\Omega)$

We present sufficient conditions for the boundedness and compactness of the embedding operator of $L_p^l(\Omega)$ into $L_{q^*}(\Omega)$, which are simple corollaries of Theorems 6.3.3 and 6.4.2.

Theorem 1. *If $\Omega \in \mathscr{I}_{p,\alpha}$, $1-1/l < p\alpha \le 1$ or $\Omega \in \mathscr{H}_{p,\alpha}$, $p\,\alpha > 1$ then the embedding operator of $L_p^l(\Omega)$ into $L_{q^*}(\Omega)$, $q^* = p/(1-l+pl\alpha)$ is bounded.*

The proof is by induction on the number of derivatives l. In addition, we must use the embeddings

$$\mathscr{I}_{p,\alpha} \subset \mathscr{I}_{p_1,\alpha_1}, \qquad \mathscr{H}_{p,\alpha} \subset \mathscr{H}_{p_1,\alpha_1},$$

with $p_1 > p \ge 1$, $\alpha_1 - p_1^{-1} = \alpha - p^{-1}$ (cf. Corollaries 6.2 and 6.4.2/2).

In particular, Theorem 1 guarantees the continuity of the embedding operator of $L_p^l(\Omega)$ $(lp < n)$ into $L_{q^*}(\Omega)$ with the same $q^* = pn/(n-lp)$ as in the Sobolev theorem for domains of the class $\mathscr{I}_{p,1/p-1/n}$.

Theorem 1 and Proposition 6.3.5/2 imply the following corollary.

Corollary 1. *Let $\Omega \in \mathscr{J}_\alpha$, $1-1/n \le \alpha \le 1$, $lp(1-\alpha) \le 1$. Then $L_p^l(\Omega) \subset L_{q^*}(\Omega)$ where $q^* = p/(1-p\,l(1-\alpha))$ for $p\,l(1-\alpha) < 1$ and q^* is arbitrary for $pl(1-\alpha) = 1$.* (Then the exponent $q^* = pn/(n-pl)$ corresponds to $\alpha = 1-1/n$.)

Example 1. Since the domain

$$\Omega^{(\lambda)} = \big\{x : x_1^2 + \cdots + x_{n-1}^2 < x_n^{2\lambda}, 0 < x_n < 1\big\}, \quad \lambda \ge 1,$$

belongs to the class $\mathscr{J}_\alpha$ with $\alpha = \lambda(n-1)/(\lambda(n-1)+1)$, (cf. Example 5.3.3/1), we have $L_p^l(\Omega^{(\lambda)}) \subset L_{q^*}(\Omega^{(\lambda)})$, where $1+\lambda(n-1) > pl$ and

$$q^* = p\big(\lambda(n-1)+1\big)/\big(1+\lambda(n-1)-pl\big). \tag{6.9.1}$$

The example of the function $u(x) = x_n^\nu$ with $\nu = l+\varepsilon-(1+\lambda(n-1))/p$ (ε is small positive number) shows that the exponent q^* cannot be increased.

Similarly to Theorem 1 we obtain the following theorem stating some conditions for the compactness of the embedding $L_p^l(\Omega) \subset L_{q^*}(\Omega)$ for domains with finite measure m_n.

Theorem 2. *If $\Omega \in \mathring{\mathscr{I}}_{p,\alpha}$, $1-1/l < p\alpha \le 1$, or $\Omega \in \mathscr{H}_{p,\alpha}$, $p\alpha > 1$, then the embedding operator of $L_p^l(\Omega)$ into $L_{q^*}(\Omega)$, $q^* = p/(1-pl(1-\alpha))$, is compact.*

This theorem and Corollary 6.8.3/1 imply the next statement.

Corollary 2. *If $m_n(\Omega) < \infty$ and $\Omega \in \mathring{\mathscr{J}}_\alpha$ where $1-1/n < \alpha \le 1$, $lp(1-\alpha) < 1$, then the embedding operator of $L_p^l(\Omega)$ into $L_q(\Omega)$, $q < p/(1-pl(1-\alpha))$, is compact.*

From this we immediately obtain the following coarser sufficient conditions formulated in terms of $\mathscr{J}_\alpha$.

Corollary 3. *If $m_n(\Omega) < \infty$ and $\Omega \in \mathscr{J}_\alpha$, $1-1/n \le \alpha \le 1$, $lp(1-\alpha) \le 1$, then the embedding operator of $L_p^1(\Omega)$ into $L_q(\Omega)$, $q < p/(1-pl(1-\alpha))$ is compact.*

Example 2. Consider the domain in Example 1. By Corollary 3 the embedding operator of $L_p^l(\Omega^{(\lambda)})$ into $L_q(\Omega^{(\lambda)})$ is compact if

$$q < p\big(\lambda(n-1)+1\big)/\big(1+\lambda(n-1)-pl\big) \tag{6.9.2}$$

(we suppose $1+\lambda(n-1) \ge pl$). We cannot put the equality sign in (6.9.2). In fact, let $\eta \in C_0^\infty(0,3)$, $\eta = 1$ on $(1,2)$. Obviously, the family of functions $\{u_\varepsilon\}_{\varepsilon>0}$, where

$$u_\varepsilon(x) = \varepsilon^{l-(\lambda(n-1)+1)/p}\eta(x/\varepsilon),$$

is bounded in $L_p^l(\Omega^{(\lambda)})$, but not compact in $L_{q^*}(\Omega^{(\lambda)})$ with q^* specified by (6.9.1).

6.10 Applications to the Neumann Problem for Strongly Elliptic Operators

Here we present some applications of the previous results to the study of the solvability and the discreteness of the spectrum of the Neumann problem in domains with irregular boundaries.

6.10.1 Second-Order Operators

Let Ω be a domain with finite volume in $\mathbb{R}^n$ and let a_{ij} $(i, j = 1, \dots, n)$ be real measurable functions in Ω, $a_{ij} = a_{ji}$. Suppose that there exists a constant $c \geq 1$ such that

$$c^{-1}|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq c|\xi|^2$$

for almost all $x \in \Omega$ and for all vectors $\xi = (\xi_1, \dots, \xi_n)$.

We define the operator A_q, $1 \leq q < \infty$ of the Neumann problem for the differential operator

$$u \to -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)$$

by the following conditions: 1. $u \in W^1_{2,q}(\Omega)$, $A_q u \in L_{q'}(\Omega)$, $1/q' + 1/q = 1$; 2. for all $v \in W^l_{2,q}(\Omega)$ the equality

$$\int_\Omega v A_q u \,\mathrm{d}x = \int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,\mathrm{d}x \tag{6.10.1}$$

is valid. The mapping $u \to A_q u$ is closed. It is clear that the range $R(A_q)$ is contained in the set $L_{q'}(\Omega) \ominus 1$ of functions in $L_{q'}(\Omega)$ that are orthogonal to unity in Ω.

Lemma. $R(A_q) = L_{q'}(\Omega) \ominus 1$ *if and only if for all* $v \in W^1_{2,q}(\Omega)$ *the "generalized Poincaré inequality"*

$$\inf_{c\in\mathbb{R}^1} \|v - c\|_{L_q(\Omega)} \leq k\|\nabla v\|_{L_2(\Omega)} \tag{6.10.2}$$

holds.

Proof. Sufficiency. By Lemma 5.1.2/2, the set $W^1_{2,q}(\Omega)$ is dense in $L^1_2(\Omega)$. Thus, if (6.10.2) holds for all $v \in W^1_{2,q}(\Omega)$ then it also holds for all $v \in L^1_2(\Omega)$. Therefore, the functional $v \to \int_\Omega vf\,\mathrm{d}x$ is bounded in $L^1_2(\Omega)$ and can be expressed in the form

$$\int_\Omega a_{ij}(\partial v/\partial x_j)(\partial u/\partial x_i)\,\mathrm{d}x, \quad u \in L^1_2(\Omega), \tag{6.10.3}$$

for an arbitrary function $f \in L_{q'}(\Omega)$. Since by (6.10.2) $L^1_2(\Omega) \subset L_q(\Omega)$, then $u \in W^1_{2,q}(\Omega)$ and hence $A_q u = f$.

Necessity. Let $f \in L_{q'}(\Omega) \ominus 1$, $v \in L^1_2(\Omega) \cap L_q(\Omega)$, $\|\nabla v\|_{L_2(\Omega)} = 1$. Since $R(A_q) = L_{q'}(\Omega) \ominus 1$, the functional

$$f \to v(f) = \int_\Omega fv\,\mathrm{d}x,$$

defined on $L_{q'} \ominus 1$, can be expressed in the form (6.10.3). Therefore, $|v(f)| \leq C\|\nabla u\|_{L_2(\Omega)}$ and the functionals $v(f)$ are bounded for each $f \in L_{q'}(\Omega)$. Thus, they are totally bounded, i.e., (6.10.2) holds for all $v \in W^1_{2,q}(\Omega)$. The lemma is proved. □

Theorems 6.3.3, 6.4.2 and the above lemma imply the following criterion for the solvability of the problem $A_q u = f$ for all $f \in L_{q'}(\Omega) \ominus 1$.

Theorem 1. *$R(A_q) = L_{q'}(\Omega) \ominus 1$ if and only if $\Omega \in \mathscr{I}_{2,1/q}$ for $q \geq 2$ and $\Omega \in \mathscr{H}_{2,1/q}$ for $q \leq 2$.*

Let a be a real function in $L_\infty(\Omega)$ such that $a(x) \geq \text{const} > 0$ for almost all $x \in \Omega$. Then the operator $u \to A_q u + au$ has the same domain as A_q. Consider the Neumann problem $A_q u + au = f$ where $f \in L_{q'}(\Omega)$. If $q' \geq 2$ then its solvability is a trivial consequence of the continuity of the functional $\int_\Omega f v \, \mathrm{d}x$ in the space $W_2^1(\Omega)$ with the inner product

$$\int_\Omega \big(a_{ij}(\partial v/\partial x_j)(\partial u/\partial x_i) + avu\big) \, \mathrm{d}x.$$

If $q' < 2$ then a necessary and sufficient condition for solvability is

$$\|v\|_{L_q(\Omega)} \leq C \|v\|_{W_2^1(\Omega)}$$

for all $v \in W_2^1(\Omega)$.

The preceding theorem and Theorem 6.3.3 directly imply the next result.

Theorem 2. *$R(A_q + aI) = L_{q'}(\Omega)$ with $q' < 2$ if and only if $\Omega \in \mathscr{I}_{2,1/q}$.*

By Rellich's lemma the problem of requirements on Ω for the discreteness of the spectrum of the operator $A \stackrel{\text{def}}{=} A_2$ is equivalent to the study of the compactness of the embedding $W_2^1(\Omega) \subset L_2(\Omega)$. Therefore, from Theorem 6.8.2 we immediately obtain the following criterion.

Theorem 3. *The spectrum of the operator A is discrete if and only if $\Omega \in \mathring{\mathscr{I}}_{2,1/2}$.*

Sufficient conditions for a set to be in $\mathscr{I}_{2,1/q}$, $\mathscr{H}_{2,1/q}$, $\mathring{\mathscr{I}}_{2,1/2}$ as well as examples of domains belonging to these classes were presented in previous sections of this chapter.

6.10.2 Neumann Problem for Operators of Arbitrary Order

In this subsection we limit consideration to strongly elliptic operators with the range in $L_2(\Omega)$.

Let Ω be a bounded subdomain of $\mathbb{R}^n$. Let i, j denote multi-indices of order not higher than l, $l \geq 1$, and let a_{ij} denote bounded complex-valued measurable functions in Ω.

Suppose that for all $u \in L_2^l(\Omega)$

$$\operatorname{Re}\int_\Omega \sum_{|i|=|j|=l} a_{ij} D^i u \overline{D^j u}\, \mathrm{d}x \geq C\|\nabla_l u\|^2_{L_2(\Omega)}, \tag{6.10.4}$$

where $D^i = \{\partial^{|i|}/\partial x_1^{i_1}, \dots, \partial x_n^{i_n}\}$. Let the operator A of the Neumann problem for the differential operator

$$u \to (-1)^l \sum_{|i|=|j|=l} D^i\big(a_{ij} D^j u\big)$$

be defined by the following conditions: 1. $u \in W_2^l(\Omega)$, $Au \in L_2(\Omega)$, 2. for all $v \in W_2^l(\Omega)$

$$\int_\Omega \bar{v} A u\, \mathrm{d}x = \int_\Omega \sum_{|i|=|j|=l} a_{ij} D^j u \overline{D^i v}\, \mathrm{d}x.$$

It is clear that the range $R(A)$ is contained in the orthogonal complement $L_2(\Omega) \ominus \mathscr{P}_{l-1}$ where $\mathscr{P}_{l-1}$ is the space of polynomials of degree not higher than $l-1$.

If for all $v \in L_2^l(\Omega)$

$$\inf_{\Pi \in \mathscr{P}_{l-1}} \|v - \Pi\|_{L_2(\Omega)} \leq k\|\nabla_l v\|_{L_2(\Omega)}, \tag{6.10.5}$$

then $R(A) = L_2(\Omega) \ominus \mathscr{P}_{l-1}$ (cf. Lions and Magenes [500], 9.1, Chap. 2). By a simple argument using induction on the number of derivatives, we show that (6.10.5) follows from the Poincaré inequality

$$\inf_{c \in \mathbb{R}^1} \|u - c\|_{L_2(\Omega)} \leq k\|\nabla v\|_{L_2(\Omega)}.$$

Thus we have the following assertion.

Theorem 1. *If $\Omega \in \mathscr{I}_{2,1/2}$ then $R(A) = L_2(\Omega) \ominus \mathscr{P}_{l-1}$; i.e., the Neumann problem $Au = f$ is solvable for all $f \in L_2(\Omega) \ominus \mathscr{P}_{l-1}$.*

We can pose the Neumann problem for the more general operator

$$u \to (-1)^l \sum_{|i|,|j| \leq l} D^i\big(a_{ij} D^j u\big)$$

in a similar way. We define the operator B of this problem by the conditions: 1. $u \in V_2^l(\Omega)$, $Bu \in L_2(\Omega)$; 2. for all $v \in V_2^l(\Omega)$

$$\int_\Omega \bar{v} B u\, \mathrm{d}x = \int_\Omega \sum_{|i|,|j| \leq l} a_{ij} D^j u \overline{D^i v}\, \mathrm{d}x. \tag{6.10.6}$$

Theorem 2. *If $\Omega \in \mathring{\mathscr{I}}_{2,1/2}$ then $R(B + \lambda I) = L_2(\Omega)$ for sufficiently large values of* $\operatorname{Re}\lambda$. *Moreover, the operator $(B + \lambda I)^{-1} : L_2(\Omega) \to V_2^l(\Omega)$ is compact.*

To prove the theorem we need the following lemma.

Lemma. *If* $\Omega \in \mathring{\mathscr{I}}_{2,1/2}$, *then, for all* $u \in W_2^l(\Omega)$ *and for any* $\varepsilon > 0$,

$$\sum_{k=0}^{l-1} \|\nabla_k u\|_{L_2(\Omega)} \le \varepsilon \|\nabla_l u\|_{L_2(\Omega)} + C(\varepsilon)\|u\|_{L_2(\Omega)}. \tag{6.10.7}$$

Proof. By (6.8.2) we have

$$\|u\|_{L_2(\Omega)} \le c\,\mathfrak{A}_{2,1/2}(M)\|\nabla u\|_{L_2(\Omega)} + c\biggl(\frac{m_n(\Omega)}{M}\biggr)^{1/2}\|u\|_{L_2(\Omega_M)},$$

where Ω_M is a subdomain of Ω with boundary of the class C^1 and such that $2m_n(\Omega\backslash\Omega_M) < M$. Therefore

$$\|\nabla_k u\|_{L_2(\Omega)} \le c\,\mathfrak{A}_{2,1/2}(M)\|\nabla_{k+1} u\|_{L_2(\Omega)} + c\biggl(\frac{m_n(\Omega)}{M}\biggr)^{1/2}\|\nabla_k u\|_{L_2(\Omega_M)}$$

for all $k = 0, 1, \dots, l-1$. Since the boundary of Ω_M is in C^1, it follows that

$$\|\nabla_k u\|_{L_2(\Omega_M)} \le \varepsilon\|\nabla_{k+1} u\|_{L_2(\Omega_M)} + C^{(0)}(\varepsilon)\|u\|_{L_2(\Omega_M)}$$

for all $\varepsilon > 0$. Therefore

$$\|\nabla_k u\|_{L_2(\Omega)} \le c\,\mathfrak{A}_{2,1/2}(M)\|\nabla_{k+1} u\|_{L_2(\Omega)} + C^{(1)}(M)\|u\|_{L_2(\Omega_M)}.$$

Applying this inequality with indices $k, k+1, \dots, l-1$, we obtain

$$\|\nabla_k u\|_{L_2(\Omega)} \le c\bigl[\mathfrak{A}_{2,1/2}(M)\bigr]^{l-k}\|\nabla_l u\|_{L_2(\Omega)} + C^{(2)}(M)\|u\|_{L_2(\Omega_M)}.$$

It remains to note that $\mathfrak{A}_{2,1/2}(M) \to 0$ as $M \to 0$. The lemma is proved. □

We established, incidentally, that (6.10.7) is valid for some $\varepsilon > 0$ provided $\Omega \in \mathscr{I}_{2,1/2}$, i.e., $\mathfrak{A}_{2,1/2}(M) < \infty$.

Proof of Theorem 2. By (6.10.4) we have

$$\operatorname{Re}\int_\Omega \sum_{|i|,|j|\le l} a_{ij} D^i u \overline{D^j u}\,\mathrm{d}x \ge C\|\nabla_l u\|^2_{L_2(\Omega)} - C_1\sum_{k=0}^{l-1}\|\nabla_k u\|^2_{L_2(\Omega)}.$$

Applying (6.10.7) with $\varepsilon = C/2C_1$ we obtain

$$\begin{aligned}\operatorname{Re}\int_\Omega \biggl(\sum_{|i|,|j|\le l} a_{ij} D^i u \overline{D^j u} + \lambda|u|^2\biggr)\mathrm{d}x \\ \ge \frac{1}{2}C_1\|\nabla_l u\|^2_{L_2(\Omega)} + (\operatorname{Re}\lambda - C_2)\|u\|^2_{L_2(\Omega)}.\end{aligned}$$

Thus the "coercive inequality"

$$\sum_{k=0}^{l} \|\nabla_k u\|^2_{L_2(\Omega)} \leq \text{const Re} \int_\Omega \bigg(\sum_{|i|,|j|\leq l} a_{ij} D^i u \overline{D^j u} + \lambda |u|^2 \bigg) \, dx$$

holds for $\operatorname{Re} \lambda > C_2$. This implies (cf., for instance, Lions and Magenes [500], Chap. 2, 9.1) the unique solvability of the equation $Bu + \lambda u = f$ for all $f \in L_2(\Omega)$. The compactness of $(B + \lambda I)^{-1}$ results from Theorem 6.9/2. The proof is complete. □

6.10.3 Neumann Problem for a Special Domain

The monograph by Courant and Hilbert [216] contains the following example of a domain for which the Poincaré inequality is false.

Let Ω be the union of the square

$$Q = \big\{ (x, y) : 0 < x < 2, \; -1 < y < 1 \big\}$$

and the sequence of symmetrically situated squares Q_m, Q_{-m}, $m = 1, 2, \dots$, connected with Q by the necks S_m, S_{-m} (Fig. 27). Let the side lengths of the squares Q_m, Q_{-m} as well as the heights of the necks be equal to $\varepsilon_m = 2^{-m}$. Let the widths of the necks be ε_m^α.

In this subsection we show that *the Poincaré inequality holds only for* $\alpha \leq 3$ *and the Rellich lemma holds only for* $\alpha < 3$. *Hence the Neumann problem considered in Sect. 6.10.1 is solvable in* $L_2^1(\Omega)$ *for any right-hand side in* $L_2^1(\Omega) \ominus 1$ *if and only if* $\alpha \leq 3$ *and the spectrum of this problem is discrete if and only if* $\alpha < 3$.

We introduce the sequence of functions $\{u_m\}_{m\geq 1}$ defined by $u_m = \pm \varepsilon_m^{-1}$ on $Q_{\pm m}$, $u_m(x, y) = \varepsilon_m^{-2}(y \mp 1)$ on $S_{\pm m}$, $u_m = 0$ outside $S_m \cup S_{-m} \cup Q_m \cup Q_{-m}$. This sequence satisfies

$$\iint_\Omega u_m \, dx \, dy = 0, \qquad \iint_\Omega (\nabla u_m)^2 \, dx \, dy = 2\varepsilon_m^{\alpha-3}, \qquad \iint_\Omega u_m^2 \, dx \, dy > 2.$$

Thus the Poincaré inequality is false for Ω if $\alpha > 3$ and the embedding operator of $W_2^1(\Omega)$ into $L_2(\Omega)$ is not compact if $\alpha \geq 3$.

Consider a nonnegative function u equal to zero outside $Q_m \cup S_m$ (m is a fixed natural number) and infinitely differentiable in $\overline{Q_m \cup S_m}$. We introduce the notation

$$\mathscr{E}_t = \big\{ (x, y) : u = t \big\}, \qquad \mathscr{H}_t = \big\{ (x, y) : u < t \big\}, \qquad \mathscr{N}_t = \big\{ (x, y) : u \geq t \big\}.$$

If $t \in (0, \infty)$ satisfies

$$s(\mathscr{E}_t) \geq 2^{-(\alpha+1)m/2}, \tag{6.10.8}$$

then

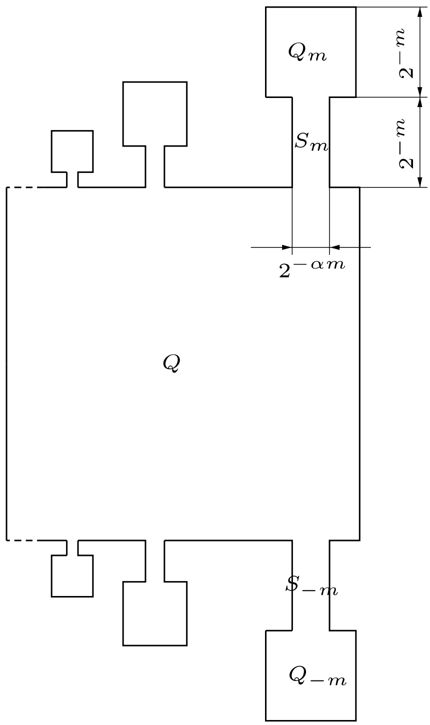

Fig. 27.

$$m_2(\mathscr{N}_t) \le c2^{-(3-\alpha)m/2} s(\mathscr{E}_t). \tag{6.10.9}$$

The set of levels t for which (6.10.8) is valid will be denoted by $\mathfrak{P}$. Duplicating with minor modification the arguments presented in Sect. 4.5, we can show that one of the following cases occurs for $t \in C\mathfrak{P}$:

$$s(\mathscr{E}_t) \ge 2^{-\alpha m}, \tag{6.10.10}$$

$$m_2(\mathscr{N}_t) < c2^{-(1+\alpha)m}; \tag{6.10.11}$$

$$s(\mathscr{E}_t) \ge 2^{-\alpha m}, \tag{6.10.12}$$

$$m_2(\mathscr{H}_t) < c2^{-(1+\alpha)m}; \tag{6.10.13}$$

$$s(\mathscr{E}_t) < 2^{-\alpha m}, \tag{6.10.14}$$

$$m_2(\mathscr{N}_t) \le c2^{-(3-\alpha)m/2} s(\mathscr{E}_t). \tag{6.10.15}$$

This implies that the set of all levels t (up to a set of measure zero) can be represented as the union $\mathfrak{R}' \cup \mathfrak{R}'' \cup \mathfrak{R}'''$ where $\mathfrak{R}'$ is the set of those t for which (6.10.14) and (6.10.15) are valid, $\mathfrak{R}'$ and $\mathfrak{R}'''$ are the set of levels for which (6.10.10) and (6.10.11) and (6.10.12) and (6.10.13) hold, respectively.

The function

$$t \to \psi(t) = \int_0^t \left(\int_{\mathscr{E}_\tau} |\nabla u| \, \mathrm{d}s \right)^{-1} \mathrm{d}\tau$$

admits the estimate

$$\psi(t) \le -\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} m_2(\mathscr{N}_\tau) \frac{\mathrm{d}\tau}{[s(\mathscr{E}_\tau)]^2}$$

(cf. Lemma 6.1.3/2). We express the right-hand side as the sum

$$\int_{\mathfrak{R}'} + \int_{\mathfrak{R}''} + \int_{\mathfrak{R}'''} .$$

The inequality (6.10.15) implies

$$\int_{\mathfrak{R}'} \le -c2^{-(3-\alpha)m} \int_0^t \big[m_2(\mathscr{N}_\tau)\big]^{-2} \frac{\mathrm{d}}{\mathrm{d}\tau} m_2(\mathscr{N}_\tau)\,\mathrm{d}\tau \le c2^{-(3-\alpha)m}/m_2(\mathscr{N}_t).$$

Using (6.10.10) and (6.10.11), we obtain

$$\int_{\mathfrak{R}''} \le 2^{\alpha m} \sup_{\tau\in\mathfrak{R}''} m_2(\mathscr{N}_\tau)\,\mathrm{d}\tau \le 2^{(\alpha-1)m}.$$

Similarly, by (6.10.12) and (6.10.13),

$$\int_{\mathfrak{R}'''} \le 2^{\alpha m} \sup_{\tau\in\mathfrak{R}'''} m_2(\mathscr{N}_\tau) \le 2^{(\alpha-1)m}.$$

Consequently,

$$\psi(t) \le c\bigg(2^{(\alpha-1)m} + \frac{2^{-(3-\alpha)m}}{m_2(\mathscr{N}_t)}\bigg). \tag{6.10.16}$$

Since $m_2(\mathscr{N}_t) \le 2^{-2m}$, (6.10.16) yields

$$m_2(\mathscr{N}_t)\psi(t) \le c2^{-(3-\alpha)m}. \tag{6.10.17}$$

Therefore,

$$\iint_{Q_m\cup S_m} u^2\,\mathrm{d}x\,\mathrm{d}y = 2\int_0^\infty tt'_\psi m_2(\mathscr{N}_{t(\psi)})\,\mathrm{d}\psi \le c2^{-(3-\alpha)m}\int_0^\infty tt'_\psi \frac{\mathrm{d}\psi}{\psi}.$$

The integral on the right-hand side does not exceed

$$\begin{aligned}\bigg(\int_0^\infty t^2\frac{\mathrm{d}\psi}{\psi^2}\bigg)^{1/2}\bigg(\int_0^\infty (t'_\psi)^2\,\mathrm{d}\psi\bigg)^{1/2} &\le 2\int_0^\infty (t'_\psi)^2\,\mathrm{d}\psi\\ &= 2\iint_{Q_m\cup S_m} (\nabla u)^2\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Hence

$$\iint_{Q_m\cup S_m} u^2\,\mathrm{d}x\,\mathrm{d}y \le c2^{-(3-\alpha)m}\iint_{Q_m\cup S_m} (\nabla u)^2\,\mathrm{d}x\,\mathrm{d}y. \tag{6.10.18}$$

It is clear that this inequality is valid for any function $u \in L_2^1(\Omega)$ that is equal to zero outside $Q_m \cup S_m$ and that a similar estimate holds for negative indices m.

Now let v be an arbitrary function in $L_2^1(\Omega)$ and let η be a "truncating" function equal to zero in Q, to unity in Q_m and in Q_{-m}, and linear in S_m and S_{-m} $(m = 1, 2, \dots)$. Then

$$\iint_{\Omega} v^2 \,\mathrm{d}x\,\mathrm{d}y \le \sum_{|k|\ge N} \left(\iint_{Q_k \cup S_k} (v\eta)^2 \, dx\,\mathrm{d}y + \iint_{S_k} v^2 \,\mathrm{d}x\,\mathrm{d}y \right) + \iint_{\Omega_N} v^2 \,\mathrm{d}x\,\mathrm{d}y, \tag{6.10.19}$$

where $\Omega_N = \Omega \ \bigcup_{|k|\ge N}(Q_k \cup S_k)$ and N is an arbitrary positive integer. From (6.10.18) we obtain

$$\iint_{Q_m \cup S_m} (v\eta)^2 \,\mathrm{d}x\,\mathrm{d}y \le c2^{-(3-\alpha)m} \left[\iint_{Q_m \cup S_m} (\nabla v)^2 \,\mathrm{d}x\,\mathrm{d}y + 2^{2m} \iint_{S_m} v^2 \,\mathrm{d}x\,\mathrm{d}y \right]. \tag{6.10.20}$$

We estimate the second integral on the right. We can easily check that

$$\iint_{S_m} |v| \,\mathrm{d}x\,\mathrm{d}y \le 2^{-m} \iint_{S_m} |\nabla v| \,\mathrm{d}x\,\mathrm{d}y + 2^{-m} \int_{\partial S_m \cap \partial Q} |v| \,\mathrm{d}x.$$

Hence

$$\iint_{S_m} v^2 \,\mathrm{d}x\,\mathrm{d}y \le \frac{1}{2} \iint_{S_m} v^2 \,\mathrm{d}x\,\mathrm{d}y + 2^{-2m-1} \iint_{S_m} (\nabla v)^2 \,\mathrm{d}x\,\mathrm{d}y + 2^{-m} \int_{\partial S_m \cup \partial Q} v^2 \,\mathrm{d}x.$$

Since $s(\partial S_m \cap \partial Q) = 2^{-\alpha m}$, by Hölder's inequality,

$$\iint_{S_m} v^2 \,\mathrm{d}x\,\mathrm{d}y \le c2^{-2m} \iint_{S_m} (\nabla v)^2 \,\mathrm{d}x\,\mathrm{d}y + c2^{-2m} \|v\|^2_{L_{2\alpha/(\alpha-1)}(\partial S_m \cap \partial Q)}. \tag{6.10.21}$$

Therefore (6.10.20) implies

$$\iint_{Q_m \cup S_m} (v\eta)^2 \,\mathrm{d}x\,\mathrm{d}y \le c2^{-(3-\alpha)m} \left[\iint_{Q_m \cup S_m} (\nabla v)^2 \,\mathrm{d}x\,\mathrm{d}y + \|v\|^2_{L_{2\alpha/(\alpha-1)}(\partial S_m \cap \partial Q)} \right]. \tag{6.10.22}$$

Using (6.10.21) and (6.10.22), from (6.10.19) we obtain

$$\iint_{\Omega} v^2 \,\mathrm{d}x\,\mathrm{d}y \le c2^{-(3-\alpha)N} \sum_{|k|\ge N} \Big[\iint_{Q_k\cup S_k} (\nabla v)^2 \,\mathrm{d}x\,\mathrm{d}y + \|v\|^2_{L_{2\alpha/(\alpha-1)}(\partial S_k\cap\partial Q)}\Big] + \iint_{\Omega_N} v^2 \,\mathrm{d}x\,\mathrm{d}y.$$

Again, by Hölder's inequality,

$$\begin{aligned}\sum_{|k|\ge N} &\|v\|^2_{L_{2\alpha/(\alpha-1)}(\partial S_k\cap\partial Q)}\\ &\le \sum_{|k|\ge N} 2^{-(\alpha-1)|k|/2}\|v\|^2_{L_{4\alpha/(\alpha-1)}(\partial S_k\cap\partial Q)}\\ &\le \Big(\sum_{|k|\ge N} 2^{-|k|(\alpha-1)\alpha/(\alpha+1)}\Big)^{(\alpha+1)/2\alpha}\\ &\quad\times\Big(\sum_{|k|\ge N}\int_{\partial S_k\cap\partial Q} |v|^{4\alpha/(\alpha-1)}\,\mathrm{d}x\Big)^{(\alpha-1)/2\alpha}\\ &\le c\|v\|^2_{L_{4\alpha/(\alpha-1)}(\partial Q)}.\end{aligned}$$

Thus

$$\iint_{\Omega} v^2 \,\mathrm{d}x\,\mathrm{d}y \le c2^{-(3-\alpha)N} \iint_{\Omega} (\nabla v)^2 \,\mathrm{d}x\,\mathrm{d}y + c\|v\|^2_{L_{4\alpha/(\alpha-1)}(\partial Q)} + c\|v\|^2_{L_2(\Omega_N)}. \tag{6.10.23}$$

Since for any $r \in (1,\infty)$

$$\|v\|_{L_r(\partial Q)} \le c\big(\|\nabla v\|_{L_2(\Omega)} + \|v\|_{L_2(\Omega)}\big),$$

(6.10.23) yields

$$\|v\|_{L_2(\Omega)} \le c\,2^{-(3-\alpha)N/2}\|\nabla v\|_{L_2(\Omega)} + c\|v\|_{L_2(\Omega_N)}. \tag{6.10.24}$$

The boundary of Ω_N is of the class $C^{0,1}$, and so the embedding operator of $W_2^1(\Omega_N)$ into $L_2(\Omega_N)$ is compact. This and (6.10.24) imply the compactness of the embedding operator of $W_2^1(\Omega)$ into $L_2(\Omega)$ for $\alpha < 3$. Finally, for $\alpha \le 3$ from (6.10.24) we get the Poincaré inequality

$$\inf_{\gamma\in\mathbb{R}^1} \|v-\gamma\|_{L_2(\Omega)} \le c\big(\|\nabla v\|_{L_2(\Omega)} + \inf_{\gamma\in\mathbb{R}^1} \|v-\gamma\|_{L_2(\Omega_1)}\big) \le c\,\|\nabla v\|_{L_2(\Omega)}.$$

6.10.4 Counterexample to Inequality (6.10.7)

We shall show that the validity of inequality (6.10.7) for some $\varepsilon > 0$ does not imply its validity for all $\varepsilon > 0$.

Let Ω be the domain considered in Sect. 6.10.3. Suppose the width of the necks S_m and S_{-m} is equal to 2^{-3m}. In Sect. 6.10.3 we showed that in this case $\Omega \in \mathscr{I}_{2,1/2}$. So (6.10.7) holds for some $\varepsilon > 0$.

Consider the sequence of functions $\{u_m\}_{m\geq 1}$ defined by $u_m = 0$ outside $Q_m \cup S_m$, $u_m(x,y) = 4^m(y-1)^2$ on S_m, $u_m(x,y) = 2^{m+1}(y-1) - 1$ in Q_m. We can easily see that

$$\iint_\Omega |\nabla u_m|^2 \, dx \, dy \leq 4\big(1 + 4^{-m}\big), \qquad \iint_\Omega |\nabla_2 u_m|^2 \, dx \, dy = 4,$$
$$\iint_\Omega u_m^2 \, dx \, dy \geq 4^{-m}.$$

By (6.10.7) with $l = 2$ we have

$$2\big(1 + 4^{-m}\big)^{1/2} \leq 2\varepsilon + k(\varepsilon) 2^{-m}, \quad m = 1, 2, \ldots,$$

which fails for some $\varepsilon > 0$.

6.11 Inequalities Containing Integrals over the Boundary

6.11.1 Embedding $W^1_{p,r}(\Omega, \partial\Omega) \subset L_q(\Omega)$

The content of the present subsection is related to that of Sect. 5.6 where the case $p = 1$ is considered. The space $W^1_{p,r}(\Omega, \partial\Omega)$ and the class $K_{\alpha,\beta}$ are defined in Sect. 5.6.1.

Theorem 5.6.3 implies the following sufficient condition for the continuity of the embedding $W^1_{p,r}(\Omega, \partial\Omega) \subset L_q(\Omega)$ for $p > 1$.

Theorem 1. *If $\Omega \in K_{\alpha,\beta}$ with $\alpha \leq 1$, $p(1-\alpha) < 1$, $\beta \geq \alpha$ then for all $u \in W^1_{p,r}(\Omega, \partial\Omega)$*

$$\|u\|_{L_q(\Omega)} \leq c \|u\|_{W^1_{p,r}(\Omega, \partial\Omega)}, \tag{6.11.1}$$

where $q = p/(1 - p(1-\alpha))$, $r = p\alpha/\beta(1 - p(1-\alpha))$.

Proof. By Theorem 5.6.3 for all $v \in C^\infty(\Omega) \cap C(\bar{\Omega})$ with bounded supports we have

$$\left(\int_\Omega |v|^{1/\alpha} \, dx\right)^\alpha \leq C\left(\|\nabla v\|_{L(\Omega)} + \left(\int_{\partial\Omega} |v|^{1/\beta} \, ds\right)^\beta\right).$$

We put $v = |u|^{q\alpha}$ and see that by Hölder's inequality

$$\int_\Omega |u|^{(p-1)/(1-p(1-\alpha))} |\nabla u| \, dx \leq \|\nabla u\|_{L_p(\Omega)} \left(\int_\Omega |u|^{p/(1-p(1-\alpha))} \, dx\right)^{1-1/p}.$$

Hence

$$\|u\|_{L_q(\Omega)} \le C_1\big(\|\nabla u\|^{1/p\alpha-1/\alpha+1}_{L_p(\Omega)}\|u\|^{(p-1)/p\alpha}_{L_q(\Omega)} + \|u\|_{L_r(\partial\Omega)}\big),$$

and (6.11.1) follows. □

Since any set Ω is contained in $K_{1-1/n,1}$, we obtain the following corollary.

Corollary 1. *The inequality*

$$\|u\|_{L_{pn/(n-p)}(\Omega)} \le c\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_{p(n-1)/(n-p)}(\partial\Omega)}\big) \tag{6.11.2}$$

holds for all $W^1_{p,p(n-1)/(n-p)}(\Omega, \partial\Omega)$ *with* $p < n$ *for an arbitrary open set* Ω.

Replacing u by $|u|^r$ and applying Hölder's inequality in (5.6.20) we arrive at the next assertion.

Corollary 2. *The following refined Friedrichs type inequality*

$$\|u\|_{L_q(\Omega)} \le C\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_r(\partial\Omega)}\big) \tag{6.11.3}$$

holds for $(n-p)r \le p(n-1)$, $r \ge 1$, $q = rn/(n-1)$ *for an arbitrary open set* Ω *with finite volume.*

We show that the exponent $q = rn/(n-1)$ on the left in (6.11.3) cannot be improved provided Ω is not subject to additional conditions.

Example. Let the domain Ω be the union of the semiball $B^- = \{x : x_n < 0, |x| < 1\}$, the sequence of balls $\mathscr{B}_m$ $(m = 1, 2, \dots,)$ and thin pipes $\mathscr{C}_m$ connecting $\mathscr{B}_m$ with B^- (Fig. 28). Let ϱ_m be the radius of the ball $\mathscr{B}_m$ and let h_m be the height of $\mathscr{C}_m$. Let u_m denote a piecewise linear function equal to unity in $\mathscr{B}_m$ and to zero outside $\mathscr{B}_m \cup \mathscr{C}_m$. Suppose that there exists a constant Q such that

$$\|u_m\|_{L_q(\Omega)} \le Q\big(\|\nabla u_m\|_{L_p(\Omega)} + \|u_m\|_{L_r(\partial\Omega)}\big)$$

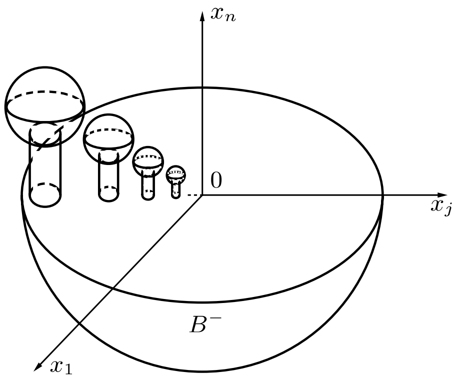

Fig. 28.

for all u_m. This implies the estimate

$$\big[m_n(\mathscr{B}_m)\big]^{1/q} \le Q\big(h_m^{-1}\big[m_n(\mathscr{C}_m)\big]^{1/p} + \big[s(\partial\mathscr{B}_m \cup \partial\mathscr{C}_m)\big]^{1/r}\big).$$

Since the first term on the right can be made arbitrarily small by diminishing the width of $\mathscr{C}_m$, it follows that $\varrho_m^{n/q} = O(\varrho_m^{(n-1)/r})$. Hence $q \le rn/(n-1)$.

To formulate a necessary and sufficient condition for the validity of (6.11.1) we need the following modification of the p-conductivity.

Let K be a conductor in Ω. We put

$$\tilde{c}_p(K) = \inf\Big\{\int_\Omega |\nabla f|^p\, \mathrm{d}x : f \in C^\infty(\Omega)\cap C(\bar{\Omega}),\ f \ge 1 \text{ on } F,\ f \le 0 \text{ on } \Omega\backslash G\Big\}. \tag{6.11.4}$$

Similarly to Lemma 6.1.1/2 we can prove that $\tilde{c}_p(K)$ can be expressed as follows:

$$\tilde{c}_p(K) = \inf\Big\{\int_\Omega |\nabla f|^p\, \mathrm{d}x : f \in C^\infty(\Omega)\cap C(\bar{\Omega}),\ f = 1 \text{ on } F,\ f = 0 \text{ on } \Omega\backslash G\Big\}. \tag{6.11.5}$$

Theorem 2. *A necessary and sufficient condition for the validity of (6.11.1) with $p > 1$, $q \ge p \ge r$ is*

$$\big[m_n(F)\big]^{1/q} \le \mathrm{const}\big(\big[\tilde{c}_p(K)\big]^{1/p} + \big[s(\partial_e G)\big]^{1/r}\big),$$

where K is any conductor $G\backslash F$ in Ω.

This assertion can be proved similarly to Theorem 2.3.9.

The same argument as in the proof of Theorem 6.8.2/1 leads to the following criterion for compactness.

Theorem 3. *A necessary and sufficient condition for compactness of the embedding operator of $W_{p,r}^1(\Omega, \partial\Omega)$ into $L_q(\Omega)$ with $m_n(\Omega) < \infty$ and $q \ge p \ge r$ is*

$$\lim_{M\to 0} \sup\Big\{\frac{[m_n(F)]^{1/q}}{[\tilde{c}_p(K)]^{1/p} + [s(\partial_e G)]^{1/r}} : m_n(G) \le M\Big\} = 0. \tag{6.11.6}$$

This and Corollary 2 imply the following assertion.

Corollary 3. *Let $(n-p)r \le p(n-1)$, $r \ge 1$, $q < rn/(n-1)$. Then the embedding operator of $W_{p,r}^1(\Omega, \partial\Omega)$ into $L_q(\Omega)$ is compact for arbitrary open set Ω with finite volume.*

The example of the present subsection shows that there exist domains with finite volume for which the embedding $W_{p,r}^1(\Omega, \partial\Omega) \subset L_{rn/(n-1)}(\Omega)$ is not compact.

6.11.2 Classes $\mathscr{I}_{p,\alpha}^{(n-1)}$ and $\mathscr{J}_{\alpha}^{(n-1)}$

Definition 1. Let $\tilde{W}_p^1(\Omega)$ be the completion of the set of functions in $C^\infty(\Omega)\cap C(\bar{\Omega})$ with bounded supports with respect to the norm in $W_p^1(\Omega)$.

Definition 2. We say that Ω is contained in the *the class* $\mathscr{I}_{p,\alpha}^{(n-1)}$, $p \geq 1$, $\alpha \geq (n-p)/p(n-1)$, if there exists a constant $R>0$ such that

$$\mathfrak{M}_{p,\alpha}(R) \overset{\text{def}}{=} \sup \frac{[s(\partial_e F)]^\alpha}{[\tilde{c}_p(K)]^{1/p}} < \infty.$$

Here $\partial_e F = \partial F \cap \partial\Omega$ and the supremum is taken over the set of all conductors $K = G\backslash F$ in Ω with $G = \Omega \cap B_R(x)$, $x \in \partial\Omega$.

The restriction $\alpha \geq (n-p)/p(n-1)$ is due to the fact that the class $\mathscr{I}_{p,\alpha}^{(n-1)}$ contains only sets Ω with boundary having the $(n-1)$-dimensional Hausdorff measure zero provided $\alpha < (n-p)/p(n-1)$.

Proposition 1. *If* $\alpha < (n-p)/p(n-1)$, *then either* $\mathfrak{M}_{p,\alpha}(R) = \infty$ *identically, or* $s(\partial\Omega) = 0$.

Proof. Let $\Omega \in \mathscr{I}_{p,\alpha}^{(n-1)}$, $s(\partial\Omega) > 0$ and let ε be a small enough positive number. We construct a covering of $\partial\Omega$ by open balls $B_{r_j}(x_j)$ with $r_j < \varepsilon$ and

$$\sum_j r_j^{n-1} \leq cs(\partial\Omega).$$

Then

$$\sum_j r_j^{n-1} \leq c \sum_j s\big(B_{r_j}(x_j) \cap \partial\Omega\big),$$

and hence, for at least one ball, we have

$$r_j^{n-1} \leq cs\big(B_{r_j}(x_j) \cap \partial\Omega\big). \tag{6.11.7}$$

Consider the conductor $K_j = G_j \backslash F_j$ where $G_j = \Omega \cap B_{2r_j}(x_j)$ and $F_j = \Omega \cap \overline{B_{r_j}(x_j)}$. Since $c_p(K_j) \leq p\text{-cap}(\overline{B_{r_j}(x_j)}, B_{2r_j}(x_j)) = c\, r_j^{n-p}$ (cf. Sect. 2.2.4), by definition of the class $\mathscr{I}_{p,\alpha}^{(n-1)}$ and by estimate (6.11.7) we obtain $r_j^{(n-1)p\alpha} \leq \text{const}\, r_j^{n-p}$. Noting that $r_j < \varepsilon$ and ε is small, we obtain $(n-1)p\,\alpha \geq n-p$. The proposition is proved. □

Definition 3. The set Ω is contained in *the class* $\mathscr{J}_\alpha^{(n-1)}$ if there exists a constant $M \in (0, m_n(\Omega))$ such that

$$\mathfrak{R}_\alpha(M) \overset{\text{def}}{=} \sup\Big\{ \frac{[s(\partial_e g)]^\alpha}{s(\partial_i g)} : g \text{ is an admissible subset of } \Omega \text{ with } m_n(g) \leq M \Big\}$$

is finite.

We introduce the function

$$\mathfrak{P}_{p,\alpha}(M) = \sup \frac{[s(\partial_e F)]^\alpha}{[\tilde{c}_p(K)]^{1/p}},$$

where the supremum is taken over the collection of all conductors $K = G\backslash F$ in Ω with $m_n(G) \le M$.

In the same way as we proved Proposition 6.6 we can prove the following assertion.

Proposition 2. *The inequality*

$$\mathfrak{P}_{p,\gamma}(M) \le c\big[\mathfrak{A}_{p,\beta}(M)\big]^{1-\gamma/\alpha}\big[\mathfrak{R}_\alpha(M)\big]^{\gamma/\alpha}$$

holds for $\gamma = \alpha\beta p/(p-1+p\beta)$, where $\mathfrak{A}_{p,\beta}$ is the function in the definition of the class $\mathscr{I}_{p,\beta}$ (cf. Sect. 6.3.1).

From this proposition it follows that

$$\mathscr{J}_\alpha^{(n-1)} \cap \mathscr{I}_{p,\beta} \subset \mathscr{I}_{p,\gamma}^{(n-1)}. \tag{6.11.8}$$

6.11.3 Examples of Domains in $\mathscr{I}_{p,\alpha}^{(n-1)}$ and $\mathscr{J}_\alpha^{(n-1)}$

Example 1. Let $x' = (x_1, \dots, x_{n-1})$ and let

$$\Omega = \big\{x : x' \in \mathbb{R}^{n-1},\ -\infty < x_n < |x'|^{-\lambda}\big\}, \quad 0 < \lambda < n-2.$$

Let g also denote an arbitrary admissible subset of Ω with $s(\partial_i g) < 1$. We have

$$s(\partial_e g) = \int_{\mathrm{Pr}(\partial_e g)} \big(1 + \lambda^2 |x'|^{-2(\lambda+1)}\big)^{1/2}\,\mathrm{d}x',$$

where Pr is the orthogonal projection mapping onto the plane $x_n = 0$. It is clear that the supremum of the integral

$$\int_{\mathscr{E}} \big(1 + \lambda^2 |x'|^{-2(\lambda+1)}\big)^{1/2}\,\mathrm{d}x',$$

taken over all subsets $\mathscr{E}$ of the plane $x_n = 0$ with a fixed $(n-1)$-dimensional measure, is attained at the $(n-1)$-dimensional ball centered at the origin. Therefore

$$s(\partial_e g) \le c\int_0^\varrho r^{n-2}\big(1 + \lambda^2 r^{-2(\lambda+1)}\big)^{1/2}\,\mathrm{d}r,$$

where

$$\varrho = \big[v_{n-1}^{-1} s\big(\mathrm{Pr}(\partial_e g)\big)\big]^{1/(n-1)}.$$

This implies

$$s(\partial_e g) \le cs\big(\mathrm{Pr}(\partial_e g)\big)^{(n-2-\lambda)/(n-1)}.$$

However, since

$$s\big(\mathrm{Pr}(\partial_e g)\big) \le s(\partial_i g),$$

we have

$$\big[s(\partial_e g)\big]^{(n-1)/(n-2-\lambda)} \le cs(\partial_i g).$$

In Example 6.3.6/2 we showed that Ω is contained in $\mathscr{I}_{p,\beta}$ with $\beta = (n-1+(p-1)\lambda)/(n-1-\lambda)p$. Using (6.11.8) we obtain that $\Omega \in \mathscr{I}^{(n-1)}_{p,\gamma}$ with $\gamma = (n-1+(p-1)\lambda)/p(n-2-\lambda)$.

This value of γ is the best possible which is checked using the sequence of conductors $K_m = G_m \backslash F_m$, where $G_m = \{x \in \Omega : 0 < x_n < 2m^{-1}\}$ and $F_m = \{x \in \Omega : 0 < x_n < m^{-1}\}$ (cf. Example 6.6).

Example 2. A similar argument shows that the set

$$\Omega = \big\{x : |x'| < x_n^\lambda,\ 0 < x_n < \infty\big\}$$

with $\lambda \ge 1$ is contained in $\mathscr{I}^{(n-1)}_{\lambda(n-1)/(\lambda(n-2)+1)}$. Since by Example 6.3.6/1, $\Omega \in \mathscr{I}_{p,\beta}$ with $\beta = (\lambda(n-1)+1-p)/p(\lambda(n-1)+1)$, we see that (6.11.8) implies $\Omega \in \mathscr{I}^{(n-1)}_{p,\gamma}$ with $\gamma = (\lambda(n-1)+1-p)/p(\lambda(n-2)+1)$.

6.11.4 Estimates for the Norm in $L_q(\partial\Omega)$

Theorem 1. *Let $s(\partial\Omega) < \infty$.*

1. *If $\Omega \in \mathscr{I}^{(n-1)}_{p,\alpha}$ with $\alpha p \le 1$, then for all functions $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ with bounded supports the inequality*

$$\|u\|_{L_q(\partial\Omega)} \le C\|u\|_{W_p^1(\Omega)} \tag{6.11.9}$$

holds with $q = \alpha^{-1}$ and with constant C that is independent of u.

2. *If for the same set of functions u inequality (6.11.9) holds with $1/q \ge (n-p)/p(n-1)$, then $\Omega \in \mathscr{I}^{(n-1)}_{p,1/q}$.*

Proof. 1. We construct a covering of $\partial\Omega$ by equal open balls $B_R(x_i)$, $x_i \in \partial\Omega$, such that the multiplicity of the covering is finite and depends only on n. Let $\{\eta_i\}$ be a partition of unity subordinate to this covering with $|\nabla\eta_i| \le c\,R^{-1}$.

Duplicating with obvious modifications the proof of Theorem 2.3.3, we obtain

$$\|u\eta_i\|^q_{L_q(\partial\Omega)} \le c \sup_{F\subset\Omega\cap B_R(x_i)} \frac{s(\partial_e F)}{[\tilde{c}_p(G\backslash F)]^{q/p}} \big\|\nabla(u\eta_i)\big\|^q_{L_p(\Omega)},$$

where $G = \Omega \cap B_R(x_i)$. Summing over i, we arrive at

$$\|u\|_{L_q(\partial\Omega)} \le c\mathfrak{M}_{p,1/q}(R)\big(\|\nabla u\|_{L_p(\Omega)} + R^{-1}\|u\|_{L_p(\Omega)}\big). \tag{6.11.10}$$

2. We show that $\mathfrak{P}_{p,1/q}(M)$ is bounded for some small $M > 0$.

Consider an arbitrary conductor $K = G \backslash F$ in Ω with $m_n(G) \le M$, where M is a constant which will be chosen at the end of the proof.

We insert any function f of the class specified in (6.11.5) into (6.11.9). Then

$$\begin{aligned}\|f\|_{L_q(\partial\Omega)} &\le C\big(\|\nabla f\|_{L_p(\Omega)} + \|f\|_{L_p(\Omega)}\big)\\ &\le C\big(\|\nabla f\|_{L_p(\Omega)} + M^{1/p-(n-1)/nq}\|f\|_{L_{qn/(n-1)}(\Omega)}\big). \quad (6.11.11)\end{aligned}$$

By Corollary 6.11.1/1

$$\|f\|_{L_{qn/(n-1)}(\Omega)} \le C\big(\|\nabla f\|_{L_p(\Omega)} + \|f\|_{L_q(\partial\Omega)}\big). \quad (6.11.12)$$

The inequalities (6.11.11) and (6.11.12) imply

$$\|f\|_{L_q(\partial\Omega)} \le C\big(\|\nabla f\|_{L_p(\Omega)} + M^{1/p-(n-1)/nq}\|f\|_{L_q(\partial\Omega)}\big).$$

If from the very beginning the constant M is chosen to be so small that

$$2CM^{1/p-(n-1)/nq} < 1,$$

then

$$\|f\|_{L_q(\partial\Omega)} \le 2C\|\nabla f\|_{L_p(\Omega)}.$$

Minimizing the right-hand side, we obtain

$$\big[s(\partial_e F)\big]^{1/q} \le 2C\big[\tilde{c}_p(K)\big]^{1/p}.$$

The proof of Theorem 1 implies the following assertion.

Corollary. *Let $s(\partial\Omega) < \infty$ and $p\,\alpha \le 1$. The class $\mathscr{I}_{p,\alpha}^{(n-1)}$ can be defined by the condition $\mathfrak{P}_{p,\alpha}(M)$ is finite for some $M > 0$.*

Theorem 2. *Let $s(\partial\Omega) < \infty$, $m_n(\Omega) < \infty$.*

1. *If $\Omega \in \mathscr{I}_{p,\alpha}^{(n-1)}$, $p\,\alpha < 1$, then for all functions $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ with bounded supports the inequality*

$$\|u\|_{L_q(\partial\Omega)} \le C\|u\|_{W_{p,p}^1(\Omega,\partial\Omega)} \quad (6.11.13)$$

holds with $q = \alpha^{-1}$ and with a constant C independent of u.

2. *If for any u of the same class (6.11.13) holds with $q > p$ then $\Omega \in \mathscr{I}_{p,\alpha}^{(n-1)}$, $\alpha = q^{-1}$.*

The first part of the theorem follows from (6.11.3) and Theorem 1, the proof of the second part is similar to that of the second part of Theorem 1.□

Theorem 3. *Let Ω be a domain with $s(\partial\Omega) < \infty$, $m_n(\Omega) < \infty$ and let $q \ge p$. The inequality*

$$\inf_{c\in\mathbb{R}^1} \|u-c\|_{L_q(\partial\Omega)} \leq C\|\nabla u\|_{L_p(\Omega)} \tag{6.11.14}$$

holds for any function u in $C^\infty(\Omega)\cap C(\bar{\Omega})$ with bounded support and with $q \geq p$ if only if $\Omega \in \mathscr{I}^{(n-1)}_{p,1/q}$.

Proof. Sufficiency. Let $K = G\backslash F$, $m_n(G) \leq M$. By Theorem 6.11.1/2 and Corollary 6.11.1/3 we have

$$m_n(F) \leq \text{const}\big(\tilde{c}_p(K) + s(\partial_e G)\big).$$

This and the corollary of the present subsection imply

$$m_n(F) \leq \text{const}\,\tilde{c}_p(K).$$

In other words, Ω is contained in $\mathscr{I}_{p,1/p}$. By Theorem 6.4.3/2 and Lemma 5.2.3/1, for all $u \in L^1_p(\Omega)$, we have

$$\inf_{c\in\mathbb{R}^1} \|u-c\|_{L_p(\Omega)} \leq C\|\nabla u\|_{L_p(\Omega)}.$$

It remains to refer to Theorem 1.

Necessity. Substituting any function f of the class specified in (6.11.5) into (6.11.14), we obtain

$$\min_{c\in\mathbb{R}^1}\big\{|1-c|^q s(\partial_e F) + |c|^q s(\partial\Omega\backslash\partial_e G)\big\} \leq C^q[\tilde{c}_p(K)]^{q/p}.$$

This leads to the estimate

$$\frac{s(\partial_e F)s(\partial\Omega\backslash\partial_e G)}{\{[s(\partial_e F)]^{1/(q-1)} + [s(\partial\Omega\backslash\partial_e G)]^{1/(q-1)}\}^{q-1}} \leq C^q\big[\tilde{c}_p(K)\big]^{q/p}.$$

Therefore,

$$s(\partial_e F) \leq 2^{q-1}C^q\big[\tilde{c}_p(K)\big]^{q/p}$$

provided $2s(\partial_e G) \leq s(\partial\Omega)$. It remains to take any ball of sufficiently small radius with center at $\partial\Omega$ as G. The theorem is proved. □

6.11.5 Class $\mathring{\mathscr{I}}^{(n-1)}_{p,\alpha}$ and Compactness Theorems

Definition. The set Ω is contained in *the class* $\mathring{\mathscr{I}}^{(n-1)}_{p,\alpha}$ if

$$\lim_{R\to 0} \mathfrak{M}_{p,\alpha}(R) = 0,$$

where $\mathfrak{M}_{p,\alpha}(R)$ is the same as in Proposition 6.11.2/1.

In the proof of Proposition 6.11.2/1 we showed that $\alpha > (n-p)/p(n-1)$ provided $s(\partial\Omega) > 0$ and the class $\mathring{\mathscr{I}}^{(n-1)}_{p,\alpha}$ is not empty.

Example. Consider the domains

$$\Omega_1 = \{x : |x'| < 1,\ 1 < x_n < |x'|^{-\lambda}\}, \quad 0 < \lambda < n-2,$$
$$\Omega_2 = \{x : |x'| < x_n^\lambda,\ 0 < x_n < 1\}, \quad \lambda \geq 1.$$

In Examples 6.11.3/1 and 6.11.3/2 we actually showed that $\Omega_1 \in \mathscr{I}_{p,\gamma_1}^{(n-1)}$ and $\Omega_2 \in \mathscr{I}_{p,\gamma_2}^{(n-1)}$, where $\gamma_1 = (n-1+(p-1)\lambda)/(n-2-\lambda)p$ and $\gamma_2 = (\lambda(n-1)+1-p)/(\lambda(n-2)+1)p$ as well as that $\Omega_i \notin \mathring{\mathscr{I}}_{p,\gamma_1}$. Consequently, $\Omega_i \in \mathring{\mathscr{I}}_{p,\alpha_i}$ $(i=1,2)$ if and only if $\alpha_i > \gamma_i$.

Theorem 1. *Let $s(\partial\Omega) < \infty$ and $m_n(\Omega) < \infty$. The set of functions in $C^\infty(\Omega) \cap C(\bar{\Omega})$ having bounded supports and contained in the unit ball of the space $W_p^1(\Omega)$ is relatively compact in $L_q(\partial\Omega)$, $q \geq p$, if and only if $\Omega \in \mathring{\mathscr{I}}_{p,1/q}^{(n-1)}$.*

Proof. Sufficiency. Let $\Omega \in \mathring{\mathscr{I}}_{p,1/q}^{(n-1)}$, $q \geq p$. If $\|u\|_{W_p^1(\Omega)} \leq 1$, by Theorem 6.11.4/1 we have

$$\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_p(\partial\Omega)} \leq \text{const}.$$

By Corollary 6.11.1/3 the embedding operator of $W_{p,p}^1(\Omega, \partial\Omega)$ into $L_p(\Omega)$ is compact and the unit ball in $W_p^1(\Omega)$ is a compact subset of $L_p(\Omega)$.

Given any positive number ε we can find an R such that $\mathfrak{M}_{p,1/q}(R) < \varepsilon$. Hence, by (6.11.10), for all $u \in C^\infty(\Omega) \cap C(\Omega)$ with bounded supports we have

$$\|u\|_{L_q(\partial\Omega)} \leq \varepsilon\|\nabla u\|_{L_p(\Omega)} + C(\varepsilon)\|u\|_{L_p(\Omega)}.$$

Now the result follows by a standard argument.

Necessity. Let Θ be the set of functions specified in the statement of the theorem. Since the traces on $\partial\Omega$ of functions in Θ form a compact subset of $L_q(\partial\Omega)$, given any $\varepsilon > 0$, we can find an R such that

$$\Big(\int_{B_R(x)\cap\partial\Omega} |u|^q \, ds\Big)^{1/q} \leq \varepsilon$$

for all $u \in \Theta$ and for all balls $B_R(x)$. Let u be an arbitrary function in $C^\infty(\Omega) \cap C(\bar{\Omega})$ with support in $B_R(x)$. We have

$$\|u\|_{L_p(B_R(x)\cap\partial\Omega)} \leq \varepsilon\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega)}\big).$$

Since by Corollary 6.11.1/2

$$\|u\|_{L_p(\Omega)} \leq C\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_p(\partial\Omega)}\big),$$

it follows that

$$\Big(\int_{B_R(x)\cap\partial\Omega}|u|^q\,ds\Big)^{1/q}\le\varepsilon C\Big(\|\nabla u\|_{L_p(\Omega)}+\Big(\int_{B_R(x)\cap\partial\Omega}|u|^p\,ds\Big)^{1/p}\Big).$$

Thus, if ε is small enough, then

$$\Big(\int_{B_R(x)\cap\partial\Omega}|u|^q\,ds\Big)^{1/q}\le 2\varepsilon\,C\|\nabla u\|_{L_p(\Omega)}.$$

Let K be the conductor $(B_R(x)\cap\Omega)\backslash F$. Substituting any function f of the class specified by the formula (6.11.5) into the latter inequality, we obtain

$$\big[s(\partial_e F)\big]^{1/q}\le 2\varepsilon C\big[\tilde{c}_p(K)\big]^{1/p}.\qquad\square$$

While proving Theorem 1, we also obtained the following result.

Theorem 2. *Let $s(\partial\Omega)<\infty$ and $m_n(\Omega)<\infty$. The set of functions in $C^\infty\cap C(\bar{\Omega})$ having bounded supports and contained in the unit ball of the space $W^1_{p,p}(\Omega,\partial\Omega)$ is relatively compact in $L_q(\partial\Omega)$, $q\ge p$, if and only if $\Omega\in\overset{\circ}{\mathscr{I}}{}^{(n-1)}_{p,1/q}$.*

6.11.6 Criteria of Solvability of Boundary Value Problems for Second-Order Elliptic Equations

In Sect. 6.10.1 we established necessary and sufficient conditions for the solvability of the Neumann problem with homogeneous boundary data for uniformly elliptic second-order equations in the energy space as well as criteria for the discreteness of the spectrum of this problem. The theorems of the present section enable us to obtain similar results for the problem

$$\begin{aligned} Lu &\equiv -\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big)+au=f \quad \text{in } \Omega,\\ Mu &\equiv a_{ij}\frac{\partial u}{\partial x_i}\cos(\nu,x_j)+bu=\varphi \quad \text{on } \partial\Omega,\end{aligned} \tag{6.11.15}$$

where ν is an outward normal to $\partial\Omega$. Here a and b are real functions, $a\in L_\infty(\Omega)$, $b\in L_\infty(\partial\Omega)$ and $a_{ij}=a_{ji}$.

In what follows we assume that $s(\partial\Omega)<\infty$, $m_n(\Omega)<\infty$ and that either both a and b are separated from zero and positive or they vanish identically. We assume for the moment that $f\in L_1(\Omega)$ and $\varphi\in L_1(\partial\Omega)$. The exact formulation of the problem is as follows.

We require a function in $W^1_2(\Omega,\partial\Omega)$ such that

$$\int_\Omega\Big(a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}+auv\Big)dx+\int_{\partial\Omega}buv\,ds=\int_\Omega fv\,dx+\int_{\partial\Omega}\varphi v\,ds, \tag{6.11.16}$$

where v is an arbitrary function in $C(\bar{\Omega})\cap W^1_2(\Omega,\partial\Omega)$ with bounded support.

This formulation is correct since by definition of the space $W_2^1(\Omega, \partial\Omega)$ and (6.11.3) the integrals on the left in (6.11.16) converge.

The case $b = 0$, $\varphi = 0$ was studied in Sect. 6.10.1. The same argument as in Sect. 6.10.1 together with Theorems 6.11.1/1–6.11.1/3, 6.11.4/1–6.11.4/3, 6.11.5/1, and 6.11.5/2 leads to the following result.

Theorem 1. 1. *If $a = 0$, $b = 0$, $f = 0$, $q' = q/(q-1) \le 2$, then the problem (6.11.15) is solvable for all $\varphi \in L_{q'}(\partial\Omega)$, orthogonal to unity on $\partial\Omega$, if and only if $\Omega \in \mathscr{I}_{2,1/q}^{(n-1)}$.*

2. *If $\inf a > 0$, $b = 0$ and $f = 0$ then the problem (6.11.15) is solvable for all $\varphi \in L_{q'}(\partial\Omega)$, $q' \le 2$ if and only if $\Omega \in \mathscr{I}_{2,1/q}^{(n-1)}$.*

3. *If $a = 0$, $\inf b > 0$ and $f = 0$ then the problem (6.11.15) is solvable for all $\varphi \in L_2(\partial\Omega)$ for an arbitrary Ω. Under the same assumptions on a, b, and f a necessary and sufficient condition for the solvability of (6.11.15) for all $\varphi \in L_{q'}(\partial\Omega)$, $q' < 2$, is the inclusion $\Omega \in \mathscr{I}_{2,1/q}^{(n-1)}$.*

In each of these cases the solution of (6.11.15) is contained in $W_{2,q}^1(\Omega, \partial\Omega)$.

4. *Let $\varphi = 0$ and $\inf b > 0$. The problem (6.11.15) is solvable for all $f \in L_{q'}(\Omega)$, $q' \ge 2n/(n+1)$, for an arbitrary set Ω. A necessary and sufficient condition for the solvability of this problem for all $f \in L_{q'}(\Omega)$, $q' < 2n/(n+1)$, is the condition of Theorem 6.11.1/2 with $p = r = 2$.*

Theorem 2. 1. *Under the assumptions 1–3 of the previous theorem, a necessary and sufficient condition for the compactness of the inverse operator*

$$L_q(\partial\Omega) \to W_{2,2}^1(\Omega, \partial\Omega), \quad q \le 2,$$

of the problem (6.11.15) is $\Omega \in \overset{\circ}{\mathscr{I}}{}_{2,1/q}^{(n-1)}$.

2. *If the assumption 4 of Theorem 1 is valid, the inverse operator*

$$L_q(\partial\Omega) \to W_{2,2}^1(\Omega, \partial\Omega)$$

of the problem (6.11.6) is compact for any set Ω provided $q' > 2n/(n+1)$.

A necessary and sufficient condition for the compactness of this operator for $q' \le 2n/(n+1)$ is the condition of Theorem 6.11.1/3 with $p = r = 2$.

In the case $q = 2$, Theorem 2 yields necessary and sufficient conditions for the discreteness of the spectrum of the problems

$$\begin{aligned} Lu &= 0 \quad \text{in } \Omega, \qquad Mu = \lambda u \quad \text{on } \partial\Omega, \\ Lu &= \lambda u \quad \text{in } \Omega, \qquad Mu = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

An extension of the results in the present subsection to the mixed boundary problem

$$Lu = f \quad \text{on } \Omega \backslash E, \qquad Mu = \varphi \quad \text{on } \partial\Omega \backslash E, \qquad u = 0 \quad \text{on } E,$$

where E is a subset of $\bar{\Omega}$, is a simple exercise.

6.12 Comments to Chap. 6

Section 6.1. Conductivity (i.e., 2-conductivity) was studied by Pólya and Szegö [666]. This notion was applied to embedding theorems by the author [528]. Here the presentation follows the author's paper [547].

Sections 6.2–6.4. The content of these sections except Sect. 6.4.5 and 6.4.7 is taken from the author's paper [547].

In the paper by Amick [45] the decomposition of the space $[L_2(\Omega)]^N$ into two orthogonal subspaces of solenoidal vector fields and of gradients of functions in $W_2^1(\Omega)$ are studied. This decomposition plays an important role in the mathematical theory of viscous fluids (cf. Ladyzhenskaya [474]). According to the Amick theorem [45], this decomposition is possible for a bounded domain Ω if and only if the spaces $W_2^1(\Omega)$ and $L_2^1(\Omega)$ coincide. By Theorem 6.4.3/2 of the present book, the last property is equivalent to the inclusion $\Omega \in \mathscr{I}_{2,1/2}$.

In connection with Corollary 6.3.4 we note that Buckley and Koskela [147] proved that $L_p^1(\Omega) \subset L_{2p/(2-p)}(\Omega)$, $2 > p \geq 1$, where Ω is a bounded simply connected domain, only if Ω is a John domain. The sufficiency of the last condition is due to Bojarski [122]. In other words, $\Omega \in \mathscr{I}_{2,2p/(2-p)}$ if and only if Ω is a John domain.

Section 6.5. The result of this section for $p = 2$ was obtained by the author [537].

Section 6.6 was first published in the author's book [552].

Section 6.7. Most of this section (6.7.1–6.7.4 except Proposition 6.7.1/1) is borrowed from the author's paper [547]. Proposition 6.7.1/1 as well as the content of Section 6.7.5 follow the author's book [552].

Sections 6.8–6.9 are part of the author's paper [547].

Section 6.10 is partly contained in the author's paper [537]. Sections 6.10.2 and 6.10.4 were published in the author's book [552]. The equivalence of the Poincaré inequality and the solvability of the Neumann problem is well known. The same pertains to the interconnection of conditions for the discreteness of the spectrum and the theorems on compactness (cf. Deny and Lions [234], Nečas [630], Lions and Magenes [500], and others).

Section 6.11 is borrowed from the author's book [552]. Payne and Weinberger [656] showed that the optimal constant C in the Poincaré inequality (6.11.14) with $p = q = 2$ and for convex domains, formulated in terms of diameter is equal to $\pi^{-1} \operatorname{diam} \Omega$. A sharp form of Corollary 6.11.1/1 with optimal constants multiplying the quantities $\|\nabla u\|_{L_p(\Omega)}$ and $\|u\|_{L_{p(n-1)/(n-p)}(\partial\Omega)}$ has recently been established in Maggi and Villani [512]. The conductivity $\tilde{c}_2$ was applied to boundary value problems for the Laplacian in very general domains by Arendt and Warma [51], Biegert and Warma [97, 98] et al. (In these papers the set function $\tilde{c}_2$ is called relative capacity.)

There are a number of papers, where for special classes of domains (without the cone property or unbounded), manifolds, and metric spaces, theorems on the continuity and the compactness of the embedding operator

$W_p^l(\Omega) \to L_q(\Omega)$, as well as necessary conditions for these properties are proved: J.-L. Lions [499]; Stampacchia [719]; Björup [108]; Campanato [165]; Globenko [311]; Andersson [48, 49]; Hurd [388]; R.A. Adams [22, 23]; Edmunds [247]; Fraenkel [285]; Reshetnyak [676]; Gol'dshtein and Reshetnyak [316]; Jerison [401]; Hurri [389, 390]; Martio [525]; Egnell, Pacella, and Tricarico [254]; Evans and Harris [262]; Hurri-Syrjänen [391, 392]; Chua [189]; Evans and Harris [263]; Smith and Stegenga [708]; Stanoyevitch [721]; Stanoyevitch and Stegenga [722, 723]; Buckley and Koskela [147]; Maheux and Saloff-Coste [513]; Koskela and Stanoyevitch [460]; Labutin [470, 471, 473]; Hajłasz and Koskela [343]; Ross [684]; Kilpeläinen and Malý [422]; Besov [90–92]; Edmunds and Hurri-Syrjänen [251]; Hajłasz [341]; Burenkov and Davies [157]; Koskela, Onninen. and Tyson [459]; Koskela and Onninen [458]; R.A. Adams and Fournier [25]; Coulhon and Koskela [215]; Harjulehto and Hästö [352]; Maz'ya and Poborchi [574–576]; Björn and Shanmugalingam [107]; Martin and M. Milman [523]; et al.

Let, for example, Ω be a cusp domain of the form

$$\Omega = \big\{x = (y, z) \in \mathbb{R}^n : z \in (0, 1), |y| < \varphi(z)\big\}, \quad n \geq 2,$$

where φ is an increasing Lipschitz continuous function on $[0, 1]$ such that $\varphi(0) = \lim_{z\to 0} \varphi'(z) = 0$. Poborchi and the author showed [449] that for $1 < p \leq q < \infty$, $l = 1, 2, \dots$, the space $W_p^l(\Omega)$ is continuously embedded in $L_q(\Omega)$ if and only if the quantities A_0, A_1 are finite, where $A_\gamma = \sup_{z\in(0,1)} A_\gamma(z)$,

$$A_\gamma(z) = \left(\int_0^z (z-t)^{q(l-1)(1-\gamma)} \varphi(t)^{n-1}\,\mathrm{d}t\right)^{\frac{1}{q}} \times \left(\int_z^1 \varphi(t)^{\frac{n-1}{1-p}} (t-z)^{\frac{p(l-1)\gamma}{p-1}}\,\mathrm{d}t\right)^{\frac{p-1}{p}}.$$

This embedding is compact if and only if $\lim_{z\to+0} A_0(z) = \lim_{z\to+0} A_1(z) = 0$. The continuity of the embedding operator:

$$W_1^l(\Omega) \to L_q(\Omega), \quad 1 \leq q < \infty,$$

is equivalent to the inequality $\sup\{A(z) : z \in (0, 1)\} < \infty$ with

$$A(z) = \varphi(z)^{1-n}\left(\int_0^z (z-t)^{(l-1)q} \varphi(t)^{n-1}\,\mathrm{d}t\right)^{1/q}$$

and its compactness to $\lim_{z\to+0} A(z) = 0$.

Furthermore, the inequality

$$\int_0^1 \frac{z^{(l-1)p/(p-1)}}{\varphi(z)^{(n-1)/(p-1)}}\,\mathrm{d}z < \infty, \quad p \in (1, \infty)$$

is necessary and sufficient for the continuity of the embedding $W_p^l(\Omega) \to C(\Omega) \cap L_\infty(\Omega)$, and this embedding is automatically compact. The space $W_1^l(\Omega)$ is continuously embedded in $C(\Omega) \cap L_\infty(\Omega)$ if and only if

$$\sup\bigl\{z^{l-1}\varphi(z)^{1-n} : z \in (0,1)\bigr\} < \infty,$$

and this embedding is compact if and only if

$$\lim_{z\to+0} z^{l-1}\varphi(z)^{1-n} = 0.$$

Recently a great deal of attention has been paid to the Sobolev-type embeddings for the so-called λ-John domains.

Let $\lambda \ge 1$. A bounded domain $\Omega \subset \mathbb{R}^n$ is λ-John, if there is a constant $C > 0$ and a distinguished point $x_0 \in \Omega$ such that every $x \in \Omega$ can be joined to x_0 by a rectifiable arc $\gamma \subset \Omega$ along which

$$\operatorname{dist}(y, \partial\Omega) \ge C\bigl|\gamma(x,y)\bigr|^\lambda, \quad y \in \gamma,$$

where $|\gamma(x,y)|$ is the length of the portion of γ joining x to y. Clearly the class of λ-John domains increases with λ and coincides with the class of John domains for $\lambda = 1$.

Hajłasz and Koskela [344] found an exact exponent q for $p = 1$ and “almost exact” q for $p > 1$ providing the continuity of the embedding $L_p^1(\Omega) \subset L_q(\Omega)$ for a λ-John domain. This embedding with exact exponent q was established by Kilpeläinen and Malý [422]. It was shown in the last work that for a λ-John domain the following estimate holds:

$$\inf_{t\in(-\infty,+\infty)} \biggl(\int_\Omega |u-t|^q \varrho(x)^a \,\mathrm{d}x\biggr)^{1/q} \le C\biggl(\int_\Omega |\nabla u|^p \varrho(x)^b dx\biggr)^{1/p}, \quad C = \text{const},$$

where

$$\begin{aligned} &\varrho(x) = \operatorname{dist}(x,\partial\Omega), \qquad b \ge 1-n, \qquad a > -n,\\ &1 \le p \le q < \infty, \qquad q(n-p) \le np,\\ &(n+a)q^{-1} \ge \bigl(\lambda(n+b-1)+1\bigr)p^{-1} - 1, \end{aligned}$$

and u is an arbitrary function on Ω, such that $\varrho^{b/p}\,\nabla u \in L_p(\Omega)$.

This result was generalized by Besov [91] to weighted Sobolev spaces of any order. In particular, Besov [91] proved the continuity of the embedding $L_p^l(\Omega) \subset L_q(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a λ-John domain,

$$1 < p < q < \infty, \qquad l - \bigl(1+\lambda(n-1)\bigr)/p + n/q \ge 0.$$

The exponent q cannot be made greater (see Labutin [473]).

Embeddings of the "anisotropic" space $W_p^{\boldsymbol{l}}(\Omega)$ for special classes of domains are considered in the book by Besov, Il'in, and Nikolsky ([94], Sect. 12, Chap. 3), and in the papers by Besov [88, 89] and Trushin [765, 766].

An interesting approach to the study of boundary value problems and Sobolev spaces for domains with fractal boundaries was proposed by Davies [223]. He showed that properties of functions due to the irregularities of the boundary are the same as those due to the singularities of the coefficients of some Riemannian metrics on the unit ball. For instance, there exists a Lipschitz equivalence between the Koch snowflake domain with the Euclidean metric and the unit disk D with the metric

$$\mathrm{d}s^2 = \bigl(1 - |x|\bigr)^{-2\gamma}\bigl(\mathrm{d}x_1^2 + \mathrm{d}x_2^2\bigr),$$

where

$$\dim(\partial D) = (1-\gamma)^{-1} = \frac{\log 4}{\log 3}$$

(see Theorem 5.1 in [223]).

7

Continuity and Boundedness of Functions in Sobolev Spaces

If a domain Ω has the cone property, then by the Sobolev theorem any function u in $W_p^l(\Omega)$, $pl > n$, coincides almost everywhere with a continuous function in Ω, and

$$\|u\|_{L_\infty(\Omega)} \le C\|u\|_{W_p^l(\Omega)},$$

where the constant C does not depend on u.

A simple example of the function $u(x) = x_1^\mu$, $\mu > 0$, defined on the plane domain $\Omega = \{x : 0 < x_1 < 1,\ 0 < x_2 < x_1^\nu\}$, $\nu > 1$, shows that the cone property is essential for the validity of Sobolev's theorem. We can naturally expect that for sets with "bad" boundaries the embedding $W_p^l(\Omega) \subset L_\infty(\Omega) \cap C(\Omega)$ is valid in some cases under stronger requirements on p and l.

In this chapter we study the classes of domains Ω for which the embedding operator of $W_p^l(\Omega)$ into $L_\infty(\Omega) \cap C(\Omega)$ is bounded or compact. Some theorems we prove give necessary and sufficient conditions ensuring the continuity of these operators and related integral inequalities (see Sects. 7.1–7.3). These criteria are formulated in terms of the p-conductivity.

To be more precise, let $y \in \Omega$ and let $\Omega_\varrho(y)$ be the intersection of Ω with the ball $B_\varrho(y)$. For any conductor of the form $\Omega_\varrho(y)\backslash\{y\}$ we introduce the function

$$\gamma_p(\varrho) = \inf_{y\in\Omega} c_p\big(\Omega_\varrho(y)\backslash\{y\}\big)$$

and show that the embedding operator $W_p^1(\Omega) \to C(\Omega) \cap L_\infty(\Omega)$ is continuous if and only if $\gamma_p(\varrho)$ is not identically zero.

Inequalities for functions with derivatives in Birnbaum–Orlicz spaces in nonsmooth domains are studied in Sect. 7.4. Compactness of the embedding $W_p^1(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$ is fully characterized in Sect. 7.5 by the condition $\lim \gamma_p(\varrho) = \infty$ as $\varrho \to +0$. In that section we show by a counterexample that one should not expect that for domains with bad boundaries the simultaneous fulfillment of both continuity and compactness of this embedding operator always takes place.

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More visible sufficient conditions for continuity and compactness of the above-mentioned embedding operator, given in terms of the isoperimetric function $\lambda(\mu)$, can be found in Sects. 7.1 and 7.5.

The continuity and compactness criteria are generalized to Sobolev spaces of an arbitrary integer order in Sect. 7.6 with the help of the so-called (p,l)-conductivity.

7.1 The Embedding $W_p^1(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$

7.1.1 Criteria for Continuity of Embedding Operators of $W_p^1(\Omega)$ and $L_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$

Let y be an arbitrary point in the domain Ω and let $\varrho > 0$.

Here and in the next two subsections we consider only conductors of the form $\Omega_\varrho(y)\backslash\{y\}$. Further, we introduce the function

$$\gamma_p(\varrho) = \inf_{y\in\Omega} c_p\big(\Omega_\varrho(y)\backslash\{y\}\big), \quad p > n, \tag{7.1.1}$$

on $(0,+\infty)$, where c_p is the p-conductivity introduced in Chap. 6. Obviously, γ_p does not increase and vanishes for $\varrho > \operatorname{diam}(\Omega)$. The condition $p > n$ in the definition of γ_p is justified by the fact that the infimum on the right in (7.1.1) equals zero for $p \le n$ by (2.2.13).

Noting that the function $u(x) = (1 - \varrho^{-1}|x-y|)_+$ is contained in the class $U_\Omega(\Omega_\varrho(y)\backslash\{y\})$ we obtain $\gamma_p(\varrho) \le c\varrho^{n-p}$.

Theorem 1. *The embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is bounded if and only if $\gamma_p \not\equiv 0$.*

Proof. Sufficiency. Let u be any function in $C^\infty(\Omega) \cap W_p^1(\Omega)$ and let y be a point in Ω such that $u(y) \ne 0$. Let R denote a positive number for which $\gamma_p(R) > 0$ and let ϱ denote an arbitrary number in $(0, R]$. We put

$$v(x) = \eta\big((x-y)/\varrho\big)u(x)/u(y),$$

where $\eta \in C_0^\infty(B_1)$, $\eta(0) = 1$. Since $v(y) = 1$ and $v(x) = 0$ outside $\Omega_\varrho(y)$, we have

$$c_p\big(\Omega_\varrho(y)\backslash\{y\}\big) \le \int_\Omega |\nabla v|^p \,\mathrm{d}x,$$

and therefore

$$\big|u(y)\big|^p c_p\big(\Omega_p(y)\backslash\{y\}\big) \le c\bigg(\int_{\Omega_\varrho(y)} |\nabla u|^p \,\mathrm{d}x + \varrho^{-p}\int_{\Omega_\varrho(y)} |u|^p \,\mathrm{d}x\bigg). \tag{7.1.2}$$

Thus the sufficiency of $\gamma_p \not\equiv 0$ follows.

Necessity. For all $u \in W_p^1(\Omega)$, let

$$\|u\|_{L_\infty(\Omega)} \le C\|u\|_{W_p^1(\Omega)}. \tag{7.1.3}$$

Inserting an arbitrary $u \in T_\Omega(\Omega_\varrho(y)\backslash\{y\})$ into (7.1.3) we obtain

$$1 \le C\big(\|\nabla u\|_{L_p(\Omega)} + v_n^{1/p}\varrho^{n/p}\big).$$

If ϱ is small enough then

$$(2C)^{-p} \le \int_\Omega |\nabla u|^p \,\mathrm{d}x.$$

Minimizing the preceding integral over $T_\Omega(\Omega_\varrho(y)\backslash\{y\})$ we obtain

$$c_p\big(\Omega_\varrho(y)\backslash\{y\}\big) \ge (2C)^{-p}.$$

The theorem is proved. □

The next assertion similar to Theorem 1 holds for the space $L_p^1(\Omega)$.

Theorem 2. *Let $m_n(\Omega) < \infty$. The embedding operator of $L_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is bounded if and only if $\gamma_p \not\equiv 0$.*

Proof. We only need to prove the sufficiency of $\gamma_p \not\equiv 0$. By virtue of Lemma 5.1.2/2 we need to derive the inequality

$$\|u\|_{L_\infty(\Omega)} \le C\|u\|_{L_p^1(\Omega)} \tag{7.1.4}$$

for functions in $L_p^1(\Omega) \cap L_\infty(\Omega)$. Let ω denote a bounded set with $\bar{\omega} \subset \Omega$. The estimate (7.1.3) implies

$$\|u\|_{L_\infty(\Omega)} \le C\big(\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_\infty(\Omega)}\big(m_n(\Omega\backslash\omega)\big)^{1/p}\big).$$

Choosing ω to satisfy $2C(m_n(\Omega\backslash\omega))^{1/p} < 1$, we arrive at (7.1.4). The theorem is proved. □

Remark 1. Let $\tilde{L}_p^l(\Omega)$ and $\tilde{W}_p^l(\Omega)$ denote the completions of the spaces $C^\infty(\Omega) \cap C(\bar{\Omega}) \cap L_p^l(\Omega)$ and $C^\infty(\Omega) \cap C(\bar{\Omega}) \cap W_p^l(\Omega)$ with respect to the norms in $L_p^l(\Omega)$ and $W_p^l(\Omega)$.

If we replace $\gamma_p(\varrho)$ in Theorems 1 and 2 by

$$\tilde{\gamma}_p(\varrho) = \inf_{y\in\Omega} \tilde{c}_p\big(\Omega_\varrho(y)\backslash\{y\}\big),$$

where $\tilde{c}_p$ is the p-conductivity defined by (6.11.5), then we obtain analogous assertions for the spaces $\tilde{W}_p^l(\Omega)$ and $\tilde{L}_p^l(\Omega)$.

The following theorem contains two-sided estimates for the constants in inequality (7.1.5).

Let $\sigma_p(\mu)$ denote the infimum of $c_p(K)$ taken over the set of conductors $K = G\backslash F$ in Ω with $m_n(G) \le \mu$.

Since $c_p(K)$ is a nondecreasing function of F, we may assume F to be a point.

Theorem 3. 1. *If $\sigma_p(\mu) \neq 0$ for some $\mu < m_n(\Omega)$, then for all $u \in L_p^1(\Omega) \cap L_q(\Omega)$, $p > n$, $0 < q < \infty$,*

$$\|u\|_{L_\infty(\Omega)} \le k_1 \|\nabla u\|_{L_p(\Omega)} + k_2 \|u\|_{L_q(\Omega)}, \tag{7.1.5}$$

where $k_1 \le [\sigma_p(\mu)]^{-1/p}$, $k_2 \le \mu^{-1/q}$.

2. *If for any $u \in L_p^1(\Omega) \cap L_q(\Omega)$ inequality (7.1.5) holds, then $\sigma_p(\mu) \ge (2k_1)^{-p}$ with $\mu = (2k_2)^{-q}$.*

Proof. 1. It suffices to derive (7.1.5) for all functions in $C^\infty(\Omega) \cap L_p^1(\Omega) \cap L_q(\Omega)$. We choose a positive number t such that

$$m_n\big(\{x : |u(x)| > t\}\big) \le \mu, \qquad m_n\big(\{x : |u(x)| \ge t\}\big) \ge \mu.$$

Let $T > t$ and $\{x : |u(x)| \ge T\} \neq \varnothing$. By the definition of p-conductivity, for the conductor

$$K_{t,T} = \big\{x : |u(x)| > t\big\} \backslash \big\{x : |u(x)| \ge T\big\}$$

we have

$$(T-t)^p c_p(K_{t,T}) \le \int_\Omega \big|\nabla |u|\big|^p \, dx = \int_\Omega |\nabla u|^p \, dx.$$

Consequently,

$$(T-t)^p \sigma_p(\mu) \le \int_\Omega |\nabla u|^p \, dx.$$

Hence

$$T \le \big[\sigma_p(\mu)\big]^{-1/p} \|\nabla u\|_{L_p(\Omega)} + \mu^{-1/q} \|u\|_{L_q(\{x : |u(x)| \ge t\})}$$

and (7.1.5) follows.

2. Suppose (7.1.5) is valid. We put $\mu = (2k_2)^{-q}$ and consider an arbitrary conductor $K = G\backslash F$ with $m_n(G) \le \mu$. Let $\{u_m\}$ be a sequence of functions in $T_\Omega(K)$ such that

$$\|\nabla u_m\|^p_{L_p(\Omega)} \to c_p(K).$$

Clearly,

$$k_2 \|u_m\|_{L_q(\Omega)} \le k_2 \big[m_n(G)\big]^{1/q} \le k_2 \mu^{1/q} = \frac{1}{2}.$$

Moreover, by (7.1.5) we have

$$1 \le 2k_1 \|\nabla u_m\|_{L_p(K)} \to 2k_1 \big[c_p(K)\big]^{1/p}.$$

Consequently, $\sigma_p(\mu) \ge (2k_1)^{-p}$. The theorem is proved. □

Remark 2. Theorems 1 and 3 imply that the conditions $\gamma_p \equiv 0$ and $\sigma_p \equiv 0$ are equivalent.

7.1.2 Sufficient Condition in Terms of the Isoperimetric Function for the Embedding $W_p^1(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$

From Corollary 6.1.3/2 and the definition of the functions σ_p and λ_M it immediately follows that

$$\sigma_p(\mu) \geq \left(\int_0^\mu \frac{\mathrm{d}\tau}{[\lambda_M(\tau)]^{p/(p-1)}} \right)^{1-p}, \tag{7.1.6}$$

where $\mu \leq M$, which together with Theorem 7.1.1/3 yields the following sufficient condition for the embedding $W_p^1(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$.

Theorem. *If for some $M < m_n(\Omega)$*

$$\int_0^M \frac{\mathrm{d}\mu}{[\lambda_M(\mu)]^{p/(p-1)}} < \infty, \tag{7.1.7}$$

then the embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is bounded.

This implies the following obvious corollary.

Corollary. *If $\Omega \in \mathscr{J}_\alpha$ and $p(1-\alpha) > 1$, then the embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is bounded.*

Example. Consider the domain

$$\Omega = \left\{ x : (x_1^2 + \cdots + x_{n-1}^2)^{1/2} < f(x_n),\ 0 < x_n < a \right\}, \tag{7.1.8}$$

in Example 5.3.3/1. From (5.3.5) it follows that the convergence of the integral (7.1.7) is equivalent to the condition

$$\int_0^a \frac{\mathrm{d}\tau}{[f(\tau)]^{(n-1)/(p-1)}} < \infty. \tag{7.1.9}$$

We show that $\sigma_p(\mu) \equiv 0$, i.e., $W_p^1(\Omega)$ is not embedded in $C(\Omega) \cap L_\infty(\Omega)$ if (7.1.9) fails. Let

$$F = \Omega \cap \{0 < x_n \leq \varepsilon\}, \qquad G = \Omega \cap \{x : 0 < x_n < \delta\},$$

where $\delta > \varepsilon$ and K is the conductor $G \backslash F$. We introduce the function $u \in U_\Omega(K)$ which vanishes outside G, and is equal to unity on F and to

$$\int_{x_n}^\delta \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}} \left(\int_\varepsilon^\delta \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}} \right)^{-1}$$

on $G \backslash F$. Obviously,

$$c_p(K) \leq \int_\Omega |\nabla u|^p \, \mathrm{d}x = c \left(\int_\varepsilon^\delta \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}} \right)^{1-p}. \tag{7.1.10}$$

Therefore, $c_p(K) \to 0$ as $\varepsilon \to 0$ and hence $\sigma_p \equiv 0$ provided (7.1.9) diverges.

If $f(\tau) = c\tau^\beta$, $\beta \geq 1$, then Ω is contained in $\mathscr{J}_\alpha$ with $\alpha = \beta(n-1)/(\beta(n-1)+1)$ (cf., for instance, Example 5.3.3/1) and consequently $W_p^1(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$ for $p > 1 + \beta(n-1)$.

The condition (7.1.9) implies that the embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is bounded for all $p > n$ if for small τ the function f is defined as

$$f(\tau) = \tau h\bigl(\log(1/\tau)\bigr),$$

where $h(s) \to 0$ and $s^{-1} \log h(s) \to 0$ as $s \to +\infty$. (Of course, this domain does not have the cone property.)

If for any point P in Ω we can construct a "quasiconic body" situated in Ω and specified in some coordinate system with the origin at P by (7.1.8) where f is subject to (7.1.9), then, obviously, $\gamma_p(\varrho) \not\equiv 0$ and the embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is bounded.

7.1.3 Isoperimetric Function and a Brezis–Gallouët–Wainger-Type Inequality

Theorem. *Let Ω be a domain with $m_n(\Omega) < \infty$ and let*

$$\int_0^{m_n(\Omega)/2} \frac{\mathrm{d}\mu}{[\lambda(\mu)]^{p'}} < \infty,$$

where $p + p' = pp'$ and $\lambda(\mu)$ denotes $\lambda_M(\mu)$ for $M = m_n(\Omega)/2$. Furthermore, let for some $r \in (1, p)$

$$\int_0^{m_n(\Omega)/2} \frac{\mathrm{d}\mu}{[\lambda(\mu)]^{r'}} = \infty.$$

Then for any $\varepsilon \in (0, m_n(\Omega)/2)$ and $u \in L_p^1(\Omega)$

$$\operatorname*{osc}_{\Omega} u \leq c(p,r)\biggl\{\biggl(\int_0^{\varepsilon} \frac{\mathrm{d}\mu}{[\lambda(\mu)]^{p'}}\biggr)^{1/p'} \|\nabla u\|_{L_p(\Omega)} + \biggl(\int_{\varepsilon}^{m_n(\Omega)/2} \frac{d\mu}{[\lambda(\mu)]^{r'}}\biggr)^{1/r'} \|\nabla u\|_{L_r(\Omega)}\biggr\}. \qquad (7.1.11)$$

Proof. Let the numbers T and t be chosen so that

$$m_n\{x : u(x) > T\} \leq m_n(\Omega)/2 \leq m_n\{x : u(x) \geq T\},$$

$$m_n\{x : u(x) > t\} \leq \varepsilon \leq m_n\{x : u(x) \geq t\}.$$

Furthermore, let $S := \operatorname{ess\,sup}_\Omega u$. Then

$$(S-t)^p c_p(K_{t,S}) \le \int |\nabla u|^p \,\mathrm{d}x$$

and

$$(t-T)^r c_r(K_{T,t}) \le \int |\nabla u|^r \,\mathrm{d}x.$$

Now it follows from Corollary 6.1.3/2, that

$$c_p(K_{t,S}) \ge \left(\int_0^\varepsilon \frac{\mathrm{d}\mu}{[\lambda(\mu)]^{p'}}\right)^{1-p}$$

and

$$c_r(K_{T,t}) \ge \left(\int_\varepsilon^{m_n(\Omega)/2} \frac{\mathrm{d}\mu}{[\lambda(\mu)]^{r'}}\right)^{1-r}.$$

This leads to the estimate

$$\begin{aligned}\operatorname*{ess\,sup}_{\Omega} u - T \le &\left(\int_0^\varepsilon \frac{\mathrm{d}\mu}{[\lambda(\mu)]^{p'}}\right)^{1/p'} \|\nabla u\|_{L_p(u\ge T)} \\ &+ \left(\int_\varepsilon^{m_n(\Omega)/2} \frac{\mathrm{d}\mu}{[\lambda(\mu)]^{r'}}\right)^{1/r'} \|\nabla u\|_{L_r(u\ge T)}.\end{aligned}$$

An analogous estimate for $T - \operatorname{ess\,inf}_\Omega u$ is proved in the same way. Adding both estimates we arrive at (7.1.11). □

We specify this theorem for domains in the class $\mathscr{J}_{1/r'}$.

Corollary. *Let* $p > r > 1$, $m_n(\Omega) < \infty$ *and let* $\Omega \in \mathscr{J}_{1/r'}$, *i.e.,*

$$\lambda(\mu) \ge C\mu^{1/r'}.$$

Then

$$\operatorname*{osc}_{\Omega} u \le c_0\left(\varepsilon^{(p-r)/pr}\|\nabla u\|_{L_p(\Omega)} + \left(\log\frac{m_n(\Omega)}{2\varepsilon}\right)^{1/r'}\|\nabla u\|_{L_r(\Omega)}\right), \quad (7.1.12)$$

where $\varepsilon \in (0, m_n(\Omega)/2)$ *and* c_0 *depends only on* C, p, *and* r.

Remark. Minimizing the right-hand side of (7.1.12) in ε we see that

$$\operatorname*{osc}_{\Omega} u \le c_1\big(1 + \big|\log\big(c_2\|\nabla u\|_{L_p(\Omega)}\big)\big|\big)^{1/r'}, \quad (7.1.13)$$

provided

$$\|\nabla u\|_{L_r(\Omega)} = 1.$$

For the cusp (7.1.8) with $f(x_n) = x_n^\beta$, $\beta > 1$, we have by (5.3.7) that

$$\lambda(\mu) \sim \mu^{\beta(n-1)/(\beta(n-1)+1)},$$

and we may take $r = \beta(n-1)+1$.

This example can be used to show that the exponent $1/r'$ of the power of the logarithm in (7.1.12) is sharp by setting

$$u(x) = \frac{\log(x_n + \delta)^{-1}}{(\log \delta^{-1})^{1/[1+\beta(n-1)]}}$$

with a small $\delta > 0$ into (7.1.12).

7.2 Multiplicative Estimate for Modulus of a Function in $W_p^1(\Omega)$

7.2.1 Conditions for Validity of a Multiplicative Inequality

It is well known that the estimate

$$\|u\|_{L_\infty(\Omega)} \le C \|u\|_{W_p^1(\Omega)}^{n/p} \|u\|_{L_p(\Omega)}^{1-n/p} \tag{7.2.1}$$

holds for $p > n$ provided the domain Ω has the cone property. This is a particular case of the general multiplicative Gagliardo–Nirenberg inequalities (cf. Sects. 1.4.7, 1.4.8). The following theorem contains a necessary and sufficient condition for validity of the estimate

$$\|u\|_{L_\infty(\Omega)} \le C \|u\|_{W_p^1(\Omega)}^{1/(r+1)} \|u\|_{L_p(\Omega)}^{r/(r+1)}, \tag{7.2.2}$$

where r is a positive number.

Theorem. *If for some $r > 0$*

$$\liminf_{\mu \to +0} \mu^r \sigma_p(\mu) > 0, \tag{7.2.3}$$

then (7.2.2) holds for all $u \in W_p^1(\Omega)$.

Conversely, if (7.2.2) holds for all $u \in W_p^1(\Omega)$, then Ω satisfies (7.2.3).

Proof. Sufficiency. By (7.2.3) there exists a constant M such that

$$\mu^r \sigma_p(\mu) \ge \varkappa = \text{const} > 0$$

for $\mu \in (0, M]$. Therefore, by Theorem 7.1.1/3,

$$\|u\|_{L_\infty(\Omega)} \le \mu^{r/p} \varkappa^{-1/p} \|\nabla u\|_{L_p(\Omega)} + \mu^{-1/p} \|u\|_{L_p(\Omega)}. \tag{7.2.4}$$

The minimum value of the right-hand side over μ is attained for

$$\mu^* = \bigl(\varkappa^{1/p} r^{-1} \|u\|_{L_p(\Omega)} / \|\nabla u\|_{L_p(\Omega)}\bigr)^{p/(r+1)}$$

and it is equal to

$$c\varkappa^{-1/p(r+1)}\|\nabla u\|_{L_p(\Omega)}^{1/(r+1)}\|u\|_{L_p(\Omega)}^{r/(r+1)}.$$

If $\mu^* \le M$ then (7.2.2) follows. If $\mu^* > M$, then

$$\varkappa^{1/p}r^{-1}\|u\|_{L_p(\Omega)} \ge M^{1+1/r}\|\nabla u\|_{L_p(\Omega)},$$

and (7.2.4) implies

$$\|u\|_{L_\infty(\Omega)} \le c_1\varkappa^{-1/p(r+1)}\|\nabla u\|_{L_p(\Omega)}^{1/(r+1)}\|u\|_{L_p(\Omega)}^{r/(r+1)} + cM^{-1/p}\|u\|_{L_p(\Omega)}.$$

The inequality (7.2.2) is proved.

Necessity. We put $M = (2\,C^{r+1})^{-p/(r+1)}$. We may assume that the constant C in (7.2.2) is so large that $M < m_n(\Omega)$. Consider an arbitrary conductor $K = G\backslash F$ with $m_n(G) \le \mu \le M$. From (7.2.2), for any $u \in T_\Omega(K)$, we have

$$1 \le C\mu^{1/p(r+1)}\big(\|\nabla u\|_{L_p(\Omega)} + M^{1/p}\big)^{1/(r+1)}.$$

Therefore, $2C^{r+1}\mu^{r/p}\|\nabla u\|_{L_p(\Omega)} \ge 1$. Minimizing the left-hand side over $T_\Omega(K)$ we obtain

$$\big(2C^{r+1}\big)^p\mu^r\sigma_p(\mu) \ge 1.$$

The theorem is proved. □

Theorems 1 and 7.1.2 imply the following sufficient condition for the validity of inequality (7.2.2).

Corollary. *If $\Omega \in \mathscr{J}_\alpha$ with $1 > \alpha \ge 1 - 1/n$ and $p(1-\alpha) > 1$, then for any $u \in W_p^1(\Omega)$ inequality (7.2.2) is valid with $r = p(1-\alpha) - 1$.*

Proof. Since $\Omega \in \mathscr{J}_\alpha$, then $\lambda_M(\mu) \ge C_M\mu^\alpha$ for $\mu < M$ where C_M is a positive constant. This and (7.1.6) imply

$$\sigma_p(\mu) \ge C_M^p\big(1 - p\alpha/(p-1)\big)^{p-1}\mu^{1-p(1-\alpha)}.$$

The result follows.

Example. For $f(x_n) = c\,x_n^\beta$, $\beta \ge 1$, the domain (7.1.8) is in the class $\mathscr{J}_{\beta(n-1)/(\beta(n-1)+1)}$ and hence by the Corollary inequality (7.2.2) is valid for $p > 1 + \beta(n-1)$ with $r = (p - 1 - \beta(n-1))/(1 + \beta(n-1))$. This exponent is the best possible since (7.1.6) implies

$$\sigma_p\big(\delta^{\beta(n-1)+1}\big) \le c_1\bigg(\int_0^{c\delta}\xi^{-\beta(n-1)/(p-1)}\,\mathrm{d}\xi\bigg)^{1-p} = c_2\delta^{1+\beta(n-1)-p},$$

where δ is any sufficiently small positive number.

7.2.2 Multiplicative Inequality in the Limit Case $r = (p - n)/n$

Inequality (7.2.2) becomes (7.2.1) for $r = (p - n)/n$. In this particular case a necessary and sufficient condition can be expressed in terms of the function γ_p defined by (7.1.1).

Theorem. *The inequality (7.2.1) holds for all $u \in W_p^1(\Omega)$ if and only if*

$$\liminf_{\varrho \to +0} \varrho^{p-n} \gamma_p(\varrho) > 0. \tag{7.2.5}$$

Proof. Sufficiency. Let r be so small that $\varrho^{p-n}\gamma_p(\varrho) > \delta > 0$ for $\varrho < r$. By virtue of (7.1.2) we have

$$c\delta^{1/p} \|u\|_{L_\infty(\Omega)} \le \varrho^{1-n/p} \|\nabla u\|_{L_p(\Omega)} + \varrho^{-n/p} \|u\|_{L_p(\Omega)} \tag{7.2.6}$$

for $\varrho \le r$ with $c > 0$. The minimum of the right-hand side in (7.2.6) over $\varrho > 0$ is attained at

$$\varrho^* = n(p - n)^{-1} \|u\|_{L_p(\Omega)} / \|\nabla u\|_{L_p(\Omega)}$$

and is equal to

$$c\|\nabla u\|^{n/p}_{L_p(\Omega)} \|u\|^{1-n/p}_{L_p(\Omega)}.$$

If $\varrho^* \le r$ then (7.2.1) follows. If $\varrho^* > r$ then

$$\|u\|_{L_p(\Omega)} \ge (p - n) r\, n^{-1} \|\nabla u\|_{L_p(\Omega)}$$

and (7.2.6) implies

$$c\delta^{1/p} \|u\|_{L_\infty(\Omega)} \le \|\nabla u\|^{n/p}_{L_p(\Omega)} \|u\|^{1-n/p}_{L_p(\Omega)} + r^{-n/p} \|u\|_{L_p(\Omega)}.$$

Thus the sufficiency of the condition (7.2.5) is proved.

Necessity. We insert an arbitrary $u \in T_\Omega(\Omega_\varrho(y)\backslash\{y\})$ into (7.2.1). Since

$$\|u\|^p_{L_p(\Omega)} \le c\varrho^n, \qquad \|u\|^p_{W_p^1(\Omega)} \le c\big(c_p\big(\Omega_\varrho(y)\backslash\{y\}\big) + \varrho^n\big),$$

by (7.2.1) we have

$$C^{-p} \le c\big(c_p\big(\Omega_\varrho(y)\backslash\{y\}\big) + \varrho^n\big)^{n/p} \varrho^{n(p-n)/p}.$$

Consequently,

$$C^{-p^2/n} \le c\big(\varrho^{p-n}\gamma_p(\varrho) + \varrho^p\big).$$

It remains to pass to the lower limit as $\varrho \to +0$. □

In the next proposition we give a sufficient condition for (7.2.6) which generalizes the cone property. Let $y \in \Omega$ and let $S_\varrho(y)$ denote the "sector"

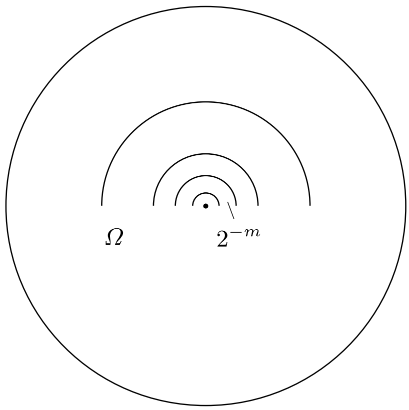

Fig. 29.

$\{x : |x-y| < \varrho, |x-y|^{-1}(x-y) \in \omega(y)\}$, where $\omega(y)$ is a measurable subset of the $(n-1)$-dimensional unit sphere.

Proposition. *Suppose that there exist positive constants R and δ such that any point y in the set Ω can be placed at the vertex of the sector $S_R(y)$ contained in Ω, satisfying the condition $s(\omega(y)) > \delta$. Then* (7.2.5) *holds.*

Proof. Let $0 < \varrho < R$ and let (r, θ) be spherical coordinates centered at y. Obviously,

$$c_p\big(\Omega_\varrho(y)\backslash\{y\}\big) \geq \inf \int_{S_\varrho(y)} |\nabla u|^p \, \mathrm{d}x,$$

where the infimum is taken over all functions $u \in C^{0,1}(\overline{S_\varrho(y)})$ with $u(y) = 1$, $u(\varrho, \theta) = 0$ for $\theta \in \omega(y)$. It remains to note that

$$\begin{aligned}\int_{S_\varrho(y)} |\nabla u|^p \, \mathrm{d}x &\geq \int_{\omega(y)} \mathrm{d}\theta \int_0^\varrho \left|\frac{\partial u}{\partial r}\right|^p r^{n-1} \, \mathrm{d}r \\ &\geq \int_{\omega(y)} \mathrm{d}\theta \left|\int_0^\varrho \frac{\partial u}{\partial r} \, \mathrm{d}r\right|^p \left(\int_0^\varrho r^{(1-n)/(p-1)} \, \mathrm{d}r\right)^{1-p} \\ &> \left(\frac{p-n}{p-1}\right)^{p-1} \delta \varrho^{n-p}.\end{aligned}$$

□

We shall consider a domain that does not satisfy the condition of Proposition and for which (7.2.5) is nevertheless true.

Example. Let Ω be the domain in Fig. 29. Further, let $\delta_m = 2^{-m}$, $Q_m = \{x : \delta_{m+1} < |x| < \delta_m\} \cap \Omega$, $y \in Q_m$. Let u denote a function in $T_\Omega(\Omega_\varrho(y)\backslash\{y\})$ such that

$$\int_\Omega |\nabla u|^p \, \mathrm{d}x \leq c_p\big(\Omega_\varrho(y)\backslash\{y\}\big) + \varepsilon, \quad \varepsilon > 0.$$

We note that

$$\big|u(\xi)-u(\eta)\big|^p \le c|\xi-\eta|^{p-2}\int_{Q_j}|\nabla u|^p\,\mathrm{d}x, \tag{7.2.7}$$

for any points $\xi,\eta\in Q_j$, $j=1,2,\dots$. (This estimate is invariant with respect to a similarity transformation and so it suffices to limit consideration to Q_1. However, inequality (7.2.7) for Q_1 is contained in Theorem 1.4.5, part (f).) We begin with the case $\varrho<\delta_m$ when $Q_m\cap\partial B_\varrho(y)\neq\varnothing$. Let $j=m$, $\xi=y$ and $\eta\in Q_m\cap\partial B_\varrho(y)$ in (7.2.7). Then

$$1\le c\varrho^{p-2}\big(c_p\big(\Omega_\varrho(y)\backslash\{y\}\big)+\varepsilon\big).$$

Next suppose $\varrho\ge\delta_m$. For all $\xi\in Q_j\cap\{x=(x_1,x_2):x_2<0\}$ we have

$$\big|u(\xi)-u(0)\big|^p\le c|\xi|^{p-2}\int_\Omega|\nabla u|^p\,\mathrm{d}x.$$

Using (7.2.7), we find that the last inequality holds for all $\xi\in Q_j$. Consequently,

$$\Big(\operatorname*{osc}_{\Omega_{2\varrho}(0)}u\Big)^p\le c\varrho^{p-2}\int_\Omega|\nabla u|^p\,\mathrm{d}x.$$

Noting that $\Omega_{2\varrho}(0)\supset\Omega_\varrho(y)$, we finally obtain

$$1=\Big(\operatorname*{osc}_{\Omega_\varrho(y)}u\Big)^p\le c\varrho^{p-2}\big(c_p\big(\Omega_\varrho(y)\backslash\{y\}\big)+\varepsilon\big).$$

7.3 Continuity Modulus of Functions in $L_p^1(\Omega)$

The following assertion is an obvious corollary of the definition of p-conductivity.

Theorem. *Let $m_n(\Omega)<\infty$, Λ be a nondecreasing continuous function on $[0,\infty)$ and let u be an arbitrary function in $L_p^1(\Omega)$. In order that for almost all $x,y\in\Omega$ the inequality*

$$\big|u(x)-u(y)\big|\le\Lambda\big(|x-y|\big)\|\nabla u\|_{L_p(\Omega)} \tag{7.3.1}$$

be valid it is necessary and sufficient that

$$c_p(K)\ge\big[\Lambda\big(\mathrm{dist}(\partial_i F,\partial_i G)\big)\big]^{-p} \tag{7.3.2}$$

for any conductor $K=G\backslash F$.

Since the conductivity is a nonincreasing function of G and nondecreasing function of F, the last condition is equivalent to

$$c_p\big[\big(\Omega\backslash\{x\}\big)\backslash\{y\}\big] \geq \big[\Lambda\big(|x-y|\big)\big]^{-p}, \quad x, y \in \Omega. \tag{7.3.3}$$

We say that the class $\dot{u} = \{u + \text{const}\}$ is contained in the space $C_\Lambda(\Omega)$ if

$$\sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{\Lambda(|x-y|)} < \infty.$$

Thus the embedding operator of $\dot{L}_p^1(\Omega)$ into $C_\Lambda(\Omega)$ is continuous if and only if (7.3.3) holds.

Example. Further in this section we consider the domain Ω already studied in Examples 5.3.3/1, 5.5.2, 6.3.6/1, and 7.1.2. Here we show that for this domain the embedding operator of $L_p^1(\Omega)$ into $C_\Lambda(\Omega)$ is continuous if and only if

$$\Lambda(t) \geq k\bigg(\int_0^t \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}}\bigg)^{1-1/p}, \quad k = \text{const} > 0. \tag{7.3.4}$$

Proof. Necessity. We insert the function u equal to unity for $x_n < \varepsilon$, to zero for $x_n > t$ and to

$$\int_{x_n}^t \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}} \bigg(\int_\varepsilon^t \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}}\bigg)^{-1}$$

for $\varepsilon \leq x_n \leq t$ into (7.3.1). (Here $\varepsilon > 0$ and $t \in (\varepsilon, a)$.) Then

$$1 \leq v_{n-1}\big[\Lambda(t)\big]^p \bigg(\int_\varepsilon^t \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}}\bigg)^{p-1},$$

which becomes (7.3.4) as $\varepsilon \to +0$.

The *sufficiency* of (7.3.4) is a simple corollary of the inequality

$$\big|u(x)-u(0)\big| \leq k\bigg(\int_0^{x_n} \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}}\bigg)^{1-1/p} \|\nabla u\|_{L_p(\Omega)}. \tag{7.3.5}$$

(We note that Theorem 1.1.6/1 implies the density of $C^\infty(\bar{\Omega})$ in $L_p^1(\Omega)$ for the domain Ω under consideration.)

To prove (7.3.5), we need the following lemma.

Lemma. *Let*

$$\Omega_b = \big\{x = (x', x_n) : |x'| < f(x_n),\ 0 < x_n < b\big\}$$

and let u be a function in $C^\infty(\bar{\Omega}_b)$ with $u(0) = 0$ and $u(x) \geq 1$ for $x_n = b$. Then

$$\int_{\Omega_b} |\nabla u|^p \,\mathrm{d}x \geq k\bigg(\int_0^b \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}}\bigg)^{1-p}. \tag{7.3.6}$$

Proof. It suffices to establish (7.3.6) under the assumption that $u = 0$ for $x_n < \varepsilon$, $u = 1$ for $x_n > b - \varepsilon$ where ε is a small positive number. Then

$$\int_{\Omega_b} |\nabla u|^p \, \mathrm{d}x \geq c_p(K_\varepsilon),$$

where $K_\varepsilon = G_\varepsilon \backslash F_\varepsilon$, $F_\varepsilon = \operatorname{clos}_\Omega \Omega_\varepsilon$, $G_\varepsilon = \Omega_{b-\varepsilon}$. To estimate $c_p(K_\varepsilon)$ from below we make use of (7.1.6) and (5.3.5). These inequalities are applicable despite the fact that the measure of the set G_ε is large. In fact, extending f to $[b, 2b]$ we obtain the enlarged domain Ω_{2b} such that $2m_n(G_\varepsilon) \leq m_n(\Omega_{2b})$ with no modification of the conductor K_ε. We have

$$c_p(K_\varepsilon) \geq k \biggl(\int_\varepsilon^{b-\varepsilon} \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}} \biggr)^{1-p}, \tag{7.3.7}$$

which together with (7.3.6) completes the proof of the lemma. □

Proof of the inequality (7.3.5). First we note that smooth functions in the closure of the domain

$$g_x = \bigl\{ y \in \Omega : x_n - \bigl[2f'(a)\bigr]^{-1} f(x_n) < y_n < x_n,\ x_n < a \bigr\}$$

satisfy the inequality

$$\bigl| u(z) - u(y) \bigr| \leq C |z - y|^{1-n/p} \|\nabla u\|_{L_p(g_x)}, \tag{7.3.8}$$

where z, y are arbitrary points in g_x and C is a constant that is independent of x. The latter is a corollary of the Sobolev theorem on the embedding of L_p^1 into $C^{1-n/p}$ for domains with smooth boundaries.

Let $u \in C^\infty(\bar{\Omega}_a)$, $u(0) = 0$, $u(x) = 1$ at some $x \in \bar{\Omega}_a$. By (7.3.8),

$$C \bigl[f(x_n) \bigr]^{n-p} \max_{y \in \bar{g}_x} \bigl| 1 - u(y) \bigr|^p \leq \int_{\Omega_a} |\nabla u|^p \, \mathrm{d}x.$$

Therefore,

$$\int_{\Omega_a} |\nabla u|^p \, \mathrm{d}x \geq C x_n^{n-p},$$

provided $2 \min u < 1$ in g_x. This and the obvious estimate

$$\biggl(\int_0^{x_n} \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}} \biggr)^{p-1} \geq k \biggl(\int_0^{x_n} \xi^{(1-n)/(p-1)} \, \mathrm{d}\xi \biggr)^{p-1} = k_1 x_n^{p-n}$$

imply (7.3.5).

Next we assume that $2u(y) \geq 1$ for all $y \in g_x$. Then the function $2u$ satisfies the conditions of the Lemma with

$$b = x_n - \bigl[2f'(a)\bigr]^{-1} f(x_n),$$

and thus

$$\int_{\Omega_a} |\nabla u|^p \,\mathrm{d}x \geq k\bigg(\int_0^b \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}}\bigg)^{1-p}$$
$$\geq k\bigg(\int_0^{x_n} \frac{\mathrm{d}\xi}{[f(\xi)]^{(n-1)/(p-1)}}\bigg)^{1-p}.$$

Inequality (7.3.5) follows. □

7.4 Boundedness of Functions with Derivatives in Birnbaum–Orlicz Spaces

Most of the results of the previous sections in this chapter can be generalized to the space of functions with finite integral

$$\int_\Omega \Phi\big(|\nabla u|\big)\,\mathrm{d}x, \tag{7.4.1}$$

where Φ is a convex function. For this purpose we must introduce the conductivity generated by the integral (7.4.1).

Here we consider only a sufficient condition for the boundedness of functions with finite integral (7.3.1) which is formulated in terms of the function λ. We also state some corollaries of this condition.

Lemma. *If $u \in C^\infty(\Omega)$, then, for almost all t,*

$$\int_{\mathscr{E}_t} \frac{ds}{|\nabla u|} = -\frac{\mathrm{d}}{\mathrm{d}t} m_n(\mathscr{L}_t), \tag{7.4.2}$$

where $\mathscr{E}_t = \{x : u(x) = t\}$ and $\mathscr{L}_t = \{x : u(x) > t\}$.

Proof. The equality (7.4.2) follows from the identity

$$\int_{\tau \geq u > t} \mathrm{d}x = \int_t^\tau \mathrm{d}\xi \int_{\mathscr{E}_\xi} \frac{\mathrm{d}s}{|\nabla u|},$$

which in turn results from Theorem 1.2.4. □

Theorem. *Let Φ be a convex nonnegative function with $\Phi(0) = 0$ and let Ψ be the complementary function of Φ (cf. Sect. 2.3.3). If Ω has finite volume and*

$$\int_0^{m_n(\Omega)/2} \Psi(1/\lambda(\mu))\,\mathrm{d}\mu < \infty, \tag{7.4.3}$$

then any function $u \in C^\infty(\Omega)$ with the finite integral (7.4.1) is bounded.

Proof. Let τ denote a number such that

$$2m_n(\mathscr{N}_\tau) \geq m_n(\Omega), \qquad 2m_n(\mathscr{L}_\tau) \leq m_n(\Omega),$$

where $\mathscr{N}_\tau = \{x : u(x) \geq \tau\}$. We introduce the notation

$$m(t) = m_n(\mathscr{L}_t), \qquad h(t) = s(\mathscr{E}_t).$$

By the inequality $\alpha\beta \leq \Phi(\alpha) + \Psi(\beta)$ with α, $\beta > 0$, for $u(x) \geq \tau$ we have

$$\begin{aligned} u(x) - \tau &= \int_\tau^{u(x)} \frac{h(t)}{m'(t)} \frac{m'(t)}{h(t)} \,\mathrm{d}t \\ &\leq -\int_\tau^{u(x)} \Phi\Big(\frac{h(t)}{-m'(t)}\Big) m'(t)\,\mathrm{d}t - \int_\tau^{u(x)} \Psi\Big(\frac{1}{h(t)}\Big) m'(t)\,\mathrm{d}t. \end{aligned}$$

Using (7.4.2) together with Jensen's inequality, we obtain

$$\begin{aligned} -\Phi\Big(\frac{h(t)}{-m'(t)}\Big) m'(t) &= \Phi\bigg[\frac{1}{\int_{\mathscr{E}_t} \frac{\mathrm{d}s}{|\nabla u|}} \int_{\mathscr{E}_t} |\nabla u| \frac{\mathrm{d}s}{|\nabla u|}\bigg] \int_{\mathscr{E}_t} \frac{\mathrm{d}t}{|\nabla u|} \\ &\leq \int_{\mathscr{E}_t} \Phi\big(|\nabla u|\big) \frac{\mathrm{d}s}{|\nabla u|}. \end{aligned}$$

Consequently,

$$\big(u(x) - \tau\big)_+ \leq \int_{\mathscr{N}_\tau} \Phi\big(|\nabla u|\big)\,\mathrm{d}x + \int_0^{m_n(\Omega)/2} \Psi\big(1/\lambda(\mu)\big)\,\mathrm{d}\mu.$$

A similar estimate is valid for $(\tau - u(x))_+$. Therefore, u is bounded and

$$\operatorname{osc} u \leq \int_\Omega \Phi\big(|\nabla u|\big)\,\mathrm{d}x + 2\int_0^{m_n(\Omega)/2} \Psi\big(1/\lambda(\mu)\big)\,\mathrm{d}\mu. \qquad \square$$

Corollary. *If $\Omega \in \mathscr{J}_\alpha$, $\alpha < 1$, and*

$$\int_1^\infty \Psi(t) t^{-1-1/\alpha}\,\mathrm{d}t < \infty,$$

then any function $u \in C^\infty(\Omega)$ with the finite integral (7.4.1) is bounded.
In particular, $u \in L_\infty(\Omega)$ if

$$\int_\Omega |\nabla u|^{1/(1-\alpha)} \bigg(\prod_{k=1}^m \log_+^k |\nabla u|\bigg)^{\alpha/(1-\alpha)} \big(\log_+^{m+1} |\nabla u|\big)^r \,\mathrm{d}x < \infty, \qquad (7.4.4)$$

where $m \geq 0$, $r > \alpha/(1-\alpha)$ and $\log_+^k$ is the k-times iterated $\log_+$. (For $m = 0$ the expression in the first parentheses in (7.4.4) is absent.)

Proof. The first assertion follows from the theorem just proved and from the definition of $\mathscr{J}_\alpha$.

To prove the second assertion we must use the fact that the convex function

$$\Psi(t) = \alpha t^{1/\alpha} \left(\prod_{k=1}^{m} \log^k t \right)^{-1} \left(\log^{m+1} t\right)^{-r(1-\alpha)/\alpha}$$

is equivalent to the complementary function of

$$(1-\alpha) t^{1/(1-\alpha)} \left(\prod_{k=1}^{m} \log^k t \right)^{\alpha/(1-\alpha)} \left(\log^{m+1} t\right)^{r}$$

for large t (see Krasnosel'skiĭ and Rutickiĭ [168]). □

The Sobolev theorem on the embedding $W_p^1(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$ for $p > n$ can be refined for domains having the cone property on the basis of the previous Corollary. Namely, if $\Omega \in \mathscr{J}_{1-1/n}$, then the continuity and the boundedness of functions in Ω result from the convergence of the integral

$$\int_\Omega |\nabla u|^n \left(\prod_{k=1}^{m} \log_+^k |\nabla u| \right)^{n-1} \left(\log_+^{m+1} |\nabla u|\right)^{n-1+\varepsilon} \mathrm{d}x, \quad \varepsilon > 0.$$

We shall show that one cannot put $r = \alpha/(1-\alpha)$ in (7.4.4).

Example. Consider the domain Ω in Examples 5.3.3/1 and 5.5.2. By (5.3.5), the condition (7.4.3) is equivalent to

$$\int_0^1 \Psi\big(\big[f(\tau)\big]^{1-n}\big)\big[f(\tau)\big]^{n-1} \mathrm{d}\tau < \infty.$$

Let $f(\tau) = c\tau^\beta$, $\beta \geq 1$. As already noted, $\Omega \in \mathscr{J}_\alpha$ with $\alpha = \beta(n-1)/(\beta(n-1)+1)$. The function $u(x) = \log_+^{m+3} x_n^{-1}$, $m \geq 0$, is unbounded in Ω. On the other hand, for small $x_n > 0$ we have

$$\begin{aligned} &|\nabla u|^{\beta(n-1)+1} \left(\prod_{k=1}^{m+1} \log_+^k |\nabla u| \right)^{\beta(n-1)} \\ &\quad \leq c x_n^{-\beta(n-1)-1} \left(\log x_n^{-1}\right)^{-1} \left(\log^2 x_n^{-1}\right)^{-1} \cdots \\ &\qquad \times \left(\log^{m+1} x_n^{-1}\right)^{-1} \left(\log^{m+2} x_n^{-1}\right)^{-\beta(n-1)-1}. \end{aligned}$$

Therefore,

$$\int_\Omega |\nabla u|^{1/(1-\alpha)} \left(\prod_{k=1}^{m+1} \log_+^k |\nabla u| \right)^{\alpha/(1-\alpha)} \mathrm{d}x < \infty.$$

7.5 Compactness of the Embedding $W_p^1(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$

7.5.1 Compactness Criterion

Let γ_p be the function defined by (7.1.1) and let Ω be a domain with finite volume.

Theorem. *The condition*

$$\lim_{\varrho \to +0} \gamma_p(\varrho) = \infty \tag{7.5.1}$$

is necessary and sufficient for the compactness of the embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$.

Proof. Sufficiency. From (7.1.2) for small $\varrho > 0$ it follows that

$$\|u\|^p_{L_\infty(\Omega)} \le c\big[\gamma_p(\varrho)\big]^{-1} \|\nabla u\|^p_{L_p(\Omega)} + C(\varrho)\|u\|^p_{L_p(\Omega)},$$

where $C(\varrho) < \infty$ for each $\varrho > 0$. We fix a small number $\varrho > 0$ and we denote an open set such that $\bar\omega_\varrho \subset \Omega$, $2C(\varrho)\, m_n(\Omega \backslash \omega_\varrho) < 1$ by ω_ϱ. Then

$$\|u\|^p_{L_\infty(\Omega)} \le 2c\big[\gamma_p(\varrho)\big]^{-1} \|\nabla u\|^p_{L_p(\Omega)} + 2C(\varrho)\|u\|^p_{L_p(\omega_\varrho)}.$$

Consider the unit ball in $W_p^1(\Omega)$ and select a sequence $\{u_m\}$ in this ball that converges in $L_p(\omega_\varrho)$. Then

$$\limsup_{k,l \to \infty} \|u_k - u_l\|^p_{L_\infty(\Omega)} \le 2c\big[\gamma_p(\varrho)\big]^{-1}. \tag{7.5.2}$$

Taking into account that $\gamma_p(\varrho) \to \infty$ as $\varrho \to 0$ and passing to the subsequence $\{u_{m_k}\}$, we obtain a sequence convergent in $L_\infty(\Omega) \cap C(\Omega)$.

Necessity. Let the embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ be compact and let

$$\gamma_p(\varrho) = \inf_{y \in \Omega} c_p\big(\Omega_\varrho(y) \backslash \{y\}\big) < A. \tag{7.5.3}$$

We construct a sequence ϱ_k that converges to zero and a sequence of points $y_k \in \Omega$ such that

$$c_p\big(\Omega_{\varrho_k}(y_k) \backslash \{y_k\}\big) < A, \quad k = 1, 2, \dots. \tag{7.5.4}$$

Since

$$\lim_{\varrho \to o} c_p\big(B_\varrho(y) \backslash \{y\}\big) = \infty$$

for $p > n$ (cf. Sect. 2.2.4), the limit points of the sequence $\{y_k\}$ are located on $\partial\Omega$. From (7.5.4) it follows that there exists a sequence of functions $u_k \in$

$T_\Omega(\Omega_{\varrho_k}(y_k)\backslash\{y_k\})$ with

$$\int_\Omega |\nabla u_k|^p \, dx < A.$$

Since $0 \le u_k \le 1$, this sequence is bounded in $W_p^1(\Omega)$ and hence it is compact in $C(\Omega) \cap L_\infty(\Omega)$. Therefore, given any $\varepsilon > 0$ we can find a number N such that $\|u_m - u_k\|_{L_\infty(\Omega)} < \varepsilon$ for all $m, k \ge N$. In particular, $|u_N(y_N) - u_k(y_N)| < \varepsilon$ for all $k > N$. On the other hand, since y_N is not a limit point of $\{y_k\}$ and $u_k(x) = 0$ outside Ω_{ϱ_k}, $u_N(y_N) = 1$, we have $|u_N(y_N) - u_k(y_N)| = 1$ for sufficiently large k. Thus assumption (7.5.3) is false. The theorem is proved. □

Remark. Replacing $\gamma_p(\varrho)$ by $\tilde{\gamma}_p(\varrho)$ defined in Remark 7.1.1/1 in the last theorem, we obtain a necessary and sufficient condition for the compactness of the embedding $\tilde{W}_p^1(\Omega) \subset C(\bar{\Omega})$. We actually proved in the theorem that (7.5.1) is necessary and sufficient for the compactness of the embedding $L_p^1(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$.

7.5.2 Sufficient Condition for the Compactness in Terms of the Isoperimetric Function

Theorem. *If the integral (7.1.7) converges for some M, then the embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is compact.*

Proof. The definition of the function σ_p implies

$$c_p\big(\Omega_\varrho(y)\backslash\{y\}\big) \ge \sigma_p\big(m_n\big(\Omega_\varrho(y)\big)\big)$$

for all $y \in \Omega$. This and (7.1.6) yield

$$c_p\big(\Omega_\varrho(y)\backslash\{y\}\big) \ge \bigg(\int_0^{m_n(\Omega_\varrho(y))} \frac{\mathrm{d}\tau}{[\lambda_M(\tau)]^{p/(p-1)}}\bigg)^{1-p}.$$

Since $m_n(\Omega_\varrho(y)) \le v_n \varrho^n$, from the definition of γ_p we obtain

$$\gamma_p(\varrho) \ge \bigg(\int_0^{v_n\varrho^n} \frac{\mathrm{d}\tau}{[\lambda_M(\tau)]^{p/(p-1)}}\bigg)^{1-p}.$$

Now the required assertion follows from Theorem 7.5.1.

Example. In Example 7.1.2 we noted that for

$$\Omega = \big\{x : |x'| < f(x_n),\ 0 < x_n < a\big\}$$

condition (7.1.7) is equivalent to convergence of the integral (7.1.9). Therefore for such Ω the embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is compact if and only if (7.1.9) holds.

7.5.3 Domain for Which the Embedding Operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is Bounded but not Compact

The embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is simultaneously bounded and compact for the domain in Example 7.5.2. By Theorems 1.4.5 and 1.4.6/2 this is also valid if Ω has the cone property.

The situation can be different with domains having "bad" boundaries. As an example we shall consider a domain for which the function γ_p is not identically zero and is bounded. Theorems 7.1.2 and 7.5.1 imply that for such a domain the embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is bounded without being compact.

Example. First we show that the domain depicted in Fig. 30 satisfies $\gamma_p(\varrho) \not\equiv 0$ for $p > 2$. Consider the conductor $\Omega_\varrho(y)\backslash\{y\}$ where ϱ is small enough and $y \in \Omega$. If $y \in Q$ then

$$c_p\big(Q_\varrho(y)\backslash\{y\}\big) \geq c\varrho^{2-p},$$

where $Q_\varrho(y) = B_\varrho(y) \cap Q$ (cf. Proposition 7.2.2) and hence

$$c_p\big(\Omega_\varrho(y)\backslash\{y\}\big) \geq c\varrho^{2-p}. \tag{7.5.5}$$

Let y be in the rectangle R_m and let $G = B_\varrho(y) \cap (Q \cup R_m)$. By the definition of p-conductivity,

$$c_p\big(\Omega_\varrho(y)\backslash\{y\}\big) \geq c_p\big(G\backslash\{y\}\big). \tag{7.5.6}$$

Take an arbitrary function $u \in T_\Omega(G\backslash\{y\})$. Let

$$\mathscr{N}_t = \big\{x \in \Omega : u(x) \geq t\big\}, \qquad \mathscr{E}_t = \big\{x \in \Omega : u(x) = t\big\}.$$

We only need to consider those levels t for which $\mathscr{E}_t$ is a smooth curve. If $m_2(\mathscr{N}_t) \geq 2\varepsilon_m^{p'}$, where $p' = p/(p-1)$, then

$$m_2(\mathscr{N}_t \cap Q) \geq \varepsilon_m^{p'} \geq m_2(\mathscr{N}_t \cap R_m).$$

Since $Q \in \mathscr{J}_{1/2}$, in the case $m_2(\mathscr{N}_t) \geq 2\varepsilon_m^{p'}$ we have

$$\big[s(\mathscr{E}_t)\big]^2 \geq cm_2(\mathscr{N}_t \cap Q) \geq \frac{1}{2}cm_2(\mathscr{N}_t). \tag{7.5.7}$$

Let $m_2(\mathscr{N}_t) < 2\varepsilon_m^{p'}$. If the set $\mathscr{E}_t$ contains a component connecting points of the polygonal lines abc and def (Fig. 31), then we can easily see that $2s(\mathscr{E}_t) \geq s(\partial\Omega \cap \bar{\mathscr{N}_t})$ and by the isoperimetric inequality we have

$$2\pi^{1/2}\big[m_2(\mathscr{N}_t)\big]^{1/2} \leq s(\partial\Omega \cap \bar{\mathscr{N}_t}) + s(\mathscr{E}_t) \leq 3s(\mathscr{E}_t). \tag{7.5.8}$$

Thus either

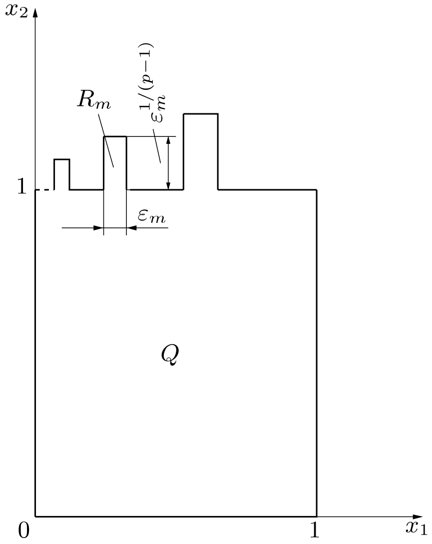

Fig. 30.

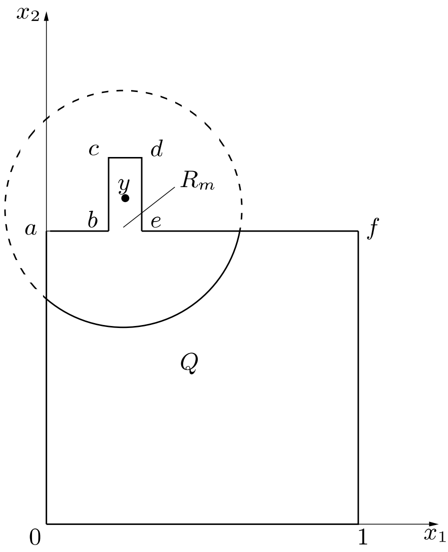

Fig. 31.

$$s(\mathscr{E}_t) \geq \varepsilon_m \quad \text{or} \quad \big[m_2(\mathscr{N}_t)\big]^{1/2} \leq c_0 s(\mathscr{E}_t)$$

provided $m_2(\mathscr{N}_t) < 2\varepsilon_m^{p'}$.

Next we proceed to the estimate of $c_p(G\backslash\{y\})$. By Corollary 6.1.3 we obtain

$$c_p\big(G\backslash\{y\}\big) \geq \inf\left[-\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} m_2(\mathscr{N}_t) \frac{\mathrm{d}t}{[s(\mathscr{E}_t)]^{p'}}\right]^{1-p}. \tag{7.5.9}$$

We express the integral on the right-hand side of (7.5.9) as the sum of integrals over the sets A_1, A_2, A_3, where

$$A_1 = \big\{t : m_2(\mathscr{N}_t) \geq 2\varepsilon_m^{p'}\big\},$$
$$A_2 = \big\{t : s(\mathscr{E}_t) \geq \varepsilon_m\big\}\backslash A_1,$$
$$A_3 = \big\{t : \big[m_2(\mathscr{N}_t)\big]^{1/2} \leq c_0 s(\mathscr{E}_t)\big\}\backslash A_1.$$

From (7.5.7) it follows that

$$\int_{A_1} \leq \left(\frac{c}{2}\right)^{p/2}\left[-\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} m_2(\mathscr{N}_t) \frac{\mathrm{d}t}{[m_2(\mathscr{N}_t)]^{p'/2}}\right] \leq c_1\big[m_2(G)\big]^{(p-2)/2(p-1)}.$$

The integral over A_2 admits the obvious estimate

$$\int_{A_2} \leq \varepsilon_m^{p/(1-p)}\left[-\int_{A_2} \frac{\mathrm{d}}{\mathrm{d}t} m_2(\mathscr{N}_t)\,\mathrm{d}t\right] \leq 2,$$

and the integral over A_3 is estimated by (7.5.8) as follows:

$$\int_{A_3} \le \left(\frac{3}{2\sqrt{\pi}}\right)^{p'} \left[-\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} m_2(\mathcal{N}_t)\frac{\mathrm{d}t}{[m_2(\mathcal{N}_t)]^{p'/2}}\right] \le c_2\big[m_2(G)\big]^{(p-2)/2(p-1)}.$$

The last three inequalities and (7.5.9) lead to $c_p(G\backslash\{y\}) \ge \text{const}$, which together with (7.5.6) and (7.5.5) yield the required result.

We show that the embedding operator of $W_p^1(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is not compact. We define the sequence of functions $\{u_m\}_{m\ge1}$ as follows: $u_m(x_1,x_2) = \varepsilon_m^{1/(1-p)}(x_2-1)$ if $(x_1,x_2) \in R_m$, $u_m(x_1,x_2) = 0$ if $(x_1,x_2) \in Q$. These functions are uniformly bounded in $W_p^1(\Omega)$ since

$$\|\nabla u_m\|_{L_p(\Omega)} = 1, \qquad \|u_m\|_{L_p(\Omega)} = c\varepsilon_m^{1/(p-1)}.$$

However, $\|u_m - u_k\|_{L_\infty(\Omega)} = 2$ for $m \ne k$ and the sequence $\{u_m\}$ is not compact in $C(\Omega) \cap L_\infty(\Omega)$.

7.6 Generalizations to Sobolev Spaces of an Arbitrary Integer Order

7.6.1 The (p,l)-Conductivity

Let G be an open subset of the set Ω and let F be a subset of G that is closed in Ω. We define the (p,l)-conductivity of the conductor $G\backslash F$ by

$$c_{p,l}(G\backslash F) = \inf \|\nabla_l u\|^p_{L_p(\Omega)}, \tag{7.6.1}$$

where the infimum is taken over all functions $u \in C^\infty(\Omega)$ that are equal to zero on $\Omega\backslash G$ and to unity on F.

Proposition 1. *If $pl > n$, $p > 1$, or $l \ge n$, $p = 1$, then*

$$c_{p,l}(B_R\backslash \bar{B}_\varrho) \sim R^{n-pl} \tag{7.6.2}$$

for $R > 2\varrho$.

Proof. If $R = 1$, then (7.6.2) follows from the Sobolev inequality

$$\|u\|_{L_\infty(B_1)} \le c\|\nabla_l u\|_{L_p(B_1)}, \quad u \in C_0^\infty(B_1).$$

The general case can be reduced to $R = 1$ by a similarity transformation.

Proposition 2. *If $n = pl$ and $p > 1$, then*

$$c_{p,l}(B_R\backslash \bar{B}_\varrho) \sim \left(\log \frac{R}{\varrho}\right)^{1-p} \tag{7.6.3}$$

for $R > 2\varrho$.

We shall establish (7.6.3) in the proof of Proposition 13.1.2/2 in the following.

Proposition 3. *If $pl \le n$, $p > 1$ or $l < n$, $p = 1$, then for $R > 2\varrho$,*

$$c_{p,l}(B_R \backslash \bar{B}_\varrho) \sim \varrho^{n-pl}. \tag{7.6.4}$$

Proof. It is clear that

$$c_{p,l}\big(\mathbb{R}^n \backslash \bar{B}_\varrho\big) \le c_{p,l}(B_R \backslash \bar{B}_\varrho) \le c_{p,l}(B_{2\varrho} \backslash B_\varrho).$$

It suffices to show that the rightmost and the leftmost of these functions are equivalent to ϱ^{n-pl}. A similarity transformation reduces the proof to the case $\varrho = 1$ where the required assertion follows from the Sobolev inequality

$$\|u\|_{L_{pn/(n-pl)}} \le c\|\nabla_l u\|_{L_p}, \quad u \in C_0^\infty. \qquad \square$$

In the present section we shall consider only conductors of the form $G \backslash \{y\}$ with $y \in G$. Propositions 2 and 3 along with the definition of the (p,l)-conductivity imply that $c_{p,l}(G \backslash \{y\})$ is identically zero provided $pl \le n$, $p > 1$ or $l < n$, $p = 1$. By Proposition 1,

$$c_{p,l}\big(B_\varrho(y) \backslash \{y\}\big) = c\varrho^{n-pl}, \quad c = \text{const} > 0,$$

for $pl > n$, $p > 1$ or for $l \ge n$, $p = 1$.

7.6.2 Embedding $L_p^l(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$

Theorem. *Let Ω be a domain in $\mathbb{R}^n$. The embedding operator of $L_p^l(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is continuous if and only if*

$$\inf_{y \in \Omega \backslash \bar{\omega}} c_{p,l}\big((\Omega \backslash \bar{\omega}) \backslash \{y\}\big) > 0 \tag{7.6.5}$$

for some open set ω with compact closure $\bar{\omega} \subset \Omega$.

Proof. Sufficiency. Let ω' be a bounded open set with smooth boundary and such that $\bar{\omega} \subset \omega'$, $\overline{\omega'} \subset \Omega$. Let η denote a function in $C^\infty(\Omega)$ which is equal to unity outside ω' and to zero on ω. Further, let u be any function in $C^\infty(\Omega) \cap L_p^l(\Omega)$.

We fix an arbitrary point $y \in \Omega \backslash \omega'$ for which $u(y) \ne 0$ and put $v(x) = \eta(x)u(x)/u(y)$, $x \in \Omega$. Since $v(y) = 1$ and $v(x) = 0$ outside the set $G = \Omega \backslash \bar{\omega}$, we have

$$c_{p,l}\big(G \backslash \{y\}\big) \le \|\nabla_l v\|^p_{L_p(\Omega)}.$$

Therefore,

$$\big|u(y)\big|^p c_{p,l}\big(G\backslash\{y\}\big) \le c\sum_{k=0}^{l}\big\|\nabla_l(u\eta)\big\|^p_{L_p(\Omega)} \le c\|\nabla_l u\|^p_{L_p(\Omega)} + C\|u\|^p_{W_p^{l-1}(\omega')}.$$

Thus

$$\sup_{\Omega\backslash\omega'} |u|^p \le c\Big(\inf_{y\in G} c_{p,l}\big(G\backslash\{y\}\big)\Big)^{-1}\big(\|\nabla_l u\|^p_{L_p(\Omega)} + C\|u\|^p_{W_p^{l-1}(\omega')}\big). \qquad (7.6.6)$$

The estimate for $|u|$ in $\overline{\omega'}$ follows from the Sobolev theorem on the embedding of W_p^l into C for domains with smooth boundaries.

Necessity. Let the inequality

$$\sup_{\Omega} |u| \le C\big(\|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\omega)}\big) \qquad (7.6.7)$$

hold for all $u \in C^\infty(\Omega)\cap L_p^l(\Omega)$, where ω is a domain with compact closure $\bar\omega$, $\bar\omega \subset \Omega$. Consider any conductor $G\backslash\{y\}$ where $G = \Omega\backslash\bar\omega$ and $y \in G$. The insertion of an arbitrary function $u \in C^\infty(\Omega)\cap L_p^1(\Omega)$, equal to unity at y and to zero outside G, into (7.6.7) yields

$$1 \le \sup_{\Omega} |u| \le c\|\nabla u\|_{L_p(\Omega)}.$$

Minimizing the last norm, we obtain $c_{p,l}(G\backslash\{y\}) \ge C^{-p}$. □

7.6.3 Embedding $V_p^l(\Omega) \subset C(\Omega)\cap L_\infty(\Omega)$

Now we present a direct extension of Theorem 7.1.2 to the space $V_p^l(\Omega)$.

Let $y \in \Omega$ and let ϱ be a positive number. Consider the conductor $\Omega_\varrho(y)\backslash\{y\}$. Further, let $pl > n$ or $l = n$, $p = 1$ and

$$c^*_{p,l}\big(\Omega_\varrho(y)\backslash\{y\}\big) = \inf\sum_{k=1}^{l} \|\nabla_k u\|^p_{L_p(\Omega)}, \qquad (7.6.8)$$

where the infimum is taken over all infinitely differentiable functions in the class $V_p^l(\Omega)$ that are equal to zero in $\Omega\backslash B_\varrho(y)$ and to one at y.

Theorem. *The embedding operator of $V_p^l(\Omega)$ into $C(\Omega)\cap L_\infty(\Omega)$ is continuous if and only if*

$$\inf_{y\in\Omega} c^*_{p,l}\big(\Omega_\varrho(y)\backslash\{y\}\big) \not\equiv 0. \qquad (7.6.9)$$

Proof. Sufficiency. Let u be an arbitrary function in $C^\infty(\Omega)\cap V_p^l(\Omega)$ and let $y \in \Omega$ be such that $u(y) \ne 0$. Further, let

$$\inf_{y\in\Omega} c^*_{p,l}\big(\Omega_\varrho(y)\backslash\{y\}\big) > 0$$

for some ϱ and let $\eta \in C_0^\infty(B_1)$. Consider the function

$$v(x) = \eta\big((x-y)/\varrho\big)u(x)/u(y).$$

Since $v(y) = 1$ and $v(x) = 0$ outside $\Omega_\varrho(y)$, we have

$$c^*_{p,l}\big(\Omega_\varrho(y)\backslash\{y\}\big) \le \sum_{k=1}^{l} \varrho^{p(k-l)} \|\nabla_k v\|^p_{L_p(\Omega)}.$$

Consequently,

$$\big|u(y)\big|^p \inf_{y\in\Omega} c^*_{p,l}\big(\Omega_\varrho(y)\backslash\{y\}\big) \le c \sum_{k=0}^{l} \varrho^{p(k-l)} \|\nabla_k u\|^p_{L_p(\Omega)}.$$

Necessity. For all infinitely differentiable functions in $V^l_p(\Omega)$, let the inequality

$$\sup_{\Omega} |u| \le C \sum_{k=0}^{l} \|\nabla_k u\|_{L_p(\Omega)} \tag{7.6.10}$$

hold. Consider any conductor $\Omega_\varrho(y)\backslash\{y\}$ with $y \in \Omega$. We insert an arbitrary function in the definition of $c^*_{p,l}(\Omega_\varrho(y)\backslash\{y\})$ into (7.6.10). Obviously,

$$\|u\|_{L_p(\Omega)} \le \big(v_n \varrho^n\big)^{1/p} \sup_{\Omega_\varrho(y)} |u|.$$

Therefore, if $\varrho < v_n^{-1/n}(2C)^{-p/n}$, then

$$\sup_{\Omega_\varrho(y)} |u| \le 2C \sum_{k=1}^{l} \|\nabla_k u\|_{L_p(\Omega_\varrho(y))}$$

and thus

$$1 \le 2C \sum_{k=1}^{l} \|\nabla_k u\|_{L_p(\Omega_\varrho(y))}.$$

Minimizing the right-hand side over $V_\Omega(\Omega_\varrho(y)\backslash\{y\})$ we obtain

$$c^*_{p,l}\big(\Omega_\varrho(y)\backslash\{y\}\big) \ge c\, C^{-p}.$$

The theorem is proved. □

7.6.4 Compactness of the Embedding $L^l_p(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$

Now we present a necessary and sufficient condition for the compactness of the embedding $L^l_p(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$.

Theorem. *Let $m_n(\Omega) < \infty$. The embedding operator of $L_p^l(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is compact if and only if*

$$\lim_{\nu\to\infty} \inf_{y\in G_\nu} c_{p,l}\big(G_\nu\backslash\{y\}\big) = \infty \tag{7.6.11}$$

for some monotone sequence of bounded open sets $\{\omega_\nu\}_{\nu\geq 1}$ such that $\bar{\omega}_\nu \subset \Omega$ and $\omega_\nu \to \Omega$. Here $G_\nu = \Omega\backslash\bar{\omega}_\nu$.

Proof. Sufficiency. Let ω'_ν be an open set with $\bar{\omega}_\nu \subset \omega'_\nu$, $\overline{\omega'_\nu} \subset \Omega$. By (7.6.6) we have

$$\sup_{\Omega} |u|^p \leq c\Big(\inf_{y\in G_\nu} c_{p,l}\big(G_\nu\backslash\{y\}\big)\Big)^{-1}\big(\|\nabla_l u\|^p_{L_p(\Omega)} + C\|u\|^p_{W_p^{l-1}(\omega'_\nu)}\big) + \sup_{\omega'_\nu}|u|^p.$$

It remains to use the compactness of the embedding of $L_p^l(\Omega)$ into $W_p^{l-1}(\omega'_\nu)$ and into $C(\overline{\omega'_\nu})$ along with (7.6.11) (see the proof of sufficiency in Theorem 7.5.1).

Necessity. Suppose that

$$\lim_{\nu\to\infty} \inf_{y\in G_\nu} c_{p,l}\big(G_\nu\backslash\{y\}\big) < A = \text{const} \tag{7.6.12}$$

for an increasing sequence of open sets $\{\omega_\nu\}_{\nu\geq 1}$. We endow $L_p^l(\Omega)$ with the norm

$$\|u\|_{L_p^l(\Omega)} = \|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\omega_1)}.$$

By (7.6.12) there exist sequences $\{y_\nu\}$ and $\{u_\nu\}$, $u_\nu \in V_\Omega(G_\nu\backslash\{y_\nu\})$, such that $\|\nabla_l u_\nu\|_{L_p(\Omega)} < A^{1/p}$. Since the sequence $\{u_\nu\}_{\nu\geq 1}$ is compact in $C(\Omega)\cap L_\infty(\Omega)$, given $\varepsilon > 0$, there exists a number N such that $\sup_\Omega |u_\mu - u_\nu| < \varepsilon$ for a $\mu, \nu > N$. Using $u_\nu(y_\nu) = 1$, we obtain

$$\big|u_\mu(y_\nu) - 1\big| < \varepsilon, \quad \mu, \nu > N. \tag{7.6.13}$$

Further, since $\omega_\mu \uparrow \Omega$, the point y_ν is contained in ω_μ for a fixed $\nu > N$ and for all large enough μ. Therefore, $u_\mu(y_\nu) = 0$ which contradicts (7.6.13). The theorem is proved. □

We note that we derived (7.6.11) in the proof of necessity for any monotonic sequence of bounded open sets ω_ν with $\bar{\omega}_\nu \subset \Omega$, $\bigcup_\nu \omega_\nu = \Omega$.

7.6.5 Sufficient Conditions for the Continuity and the Compactness of the Embedding $L_p^l(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$

We present a sufficient condition for the boundedness and the compactness of the embedding operator of $L_p^l(\Omega)$ into $C(\Omega)\cap L_\infty(\Omega)$ which generalizes (7.1.7).

Theorem 1. *Let $m_n(\Omega) < \infty$, $p \geq 1$, l, and let l be a positive integer. Also let $\Omega \in \mathscr{J}_\alpha$, where $1 > \alpha > (n-1)/n$ and $pl(1-\alpha) > 1$. Then*

$$\|u\|_{L_\infty(\Omega)} \le C\|u\|_{L^l_p(\Omega)}, \tag{7.6.14}$$

and the embedding operator of $L^l_p(\Omega)$ *into* $C(\Omega) \cap L_\infty(\Omega)$ *is compact.*

Proof. By Theorem 7.1.2/1,

$$\|u\|_{L_\infty(\Omega)} \le C\big(\|\nabla u\|_{L_{pl}(\Omega)} + \|u\|_{L_p(\omega)}\big), \tag{7.6.15}$$

where ω is an open set, $\bar\omega \subset \Omega$ and C is a constant independent of u. Since $p(l-1)(1-\alpha) < 1$ and $pl = p/[1-p(l-1)(1-\alpha)]$, we have by Corollary 6.9/1 that

$$\begin{aligned}\sum_{i=1}^n \Big\|\frac{\partial u}{\partial x_i}\Big\|_{L_{pl}(\Omega)} &\le C\sum_{i=1}^n\Big(\Big\|\nabla_{l-1}\frac{\partial u}{\partial x_i}\Big\|_{L_p(\Omega)} + \Big\|\frac{\partial u}{\partial x_i}\Big\|_{L_p(\omega)}\Big)\\ &\le C_1\big(\|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\omega)}\big).\end{aligned} \tag{7.6.16}$$

Combining (7.6.15) and (7.6.16) we arrive at (7.6.14).

The compactness of the embedding $L^l_p(\Omega) \subset C(\Omega) \cap L_\infty(\Omega)$ follows from (7.6.16) and Theorem 7.5.2 in which p is replaced by $p\,l$ and the condition (7.1.7) is replaced by $\Omega \in \mathscr{J}_\alpha$.

Theorem 2. *If* Ω *is a domain with finite volume contained in* $\mathscr{J}_\alpha$ *with* $1 > \alpha > (n-1)/n$ *and* $p\,l(1-\alpha) > 1$, *then for all* $u \in W^l_p(\Omega)$

$$\|u\|_{L_\infty(\Omega)} \le C\|u\|^{1/(r+1)}_{W^l_p(\Omega)}\|u\|^{r/(r+1)}_{L_p(\Omega)}, \quad r = pl(1-\alpha) - 1.$$

For the proof it suffices to use Corollary 7.2.1 with p replaced by $p\,l$ and then to apply (7.6.16).

Example. (Cusp) Let Ω be the domain in Examples 5.3.3/1 and 7.1.2 with $f(\tau) = c\tau^\beta, \beta \ge 1$. Then the condition $\Omega \in \mathscr{J}_\alpha$ in Theorem 1 is equivalent to $pl > \beta(n-1)+1$ and therefore the embedding operator of $L^l_p(\Omega)$ into $C(\Omega) \cap L_\infty(\Omega)$ is compact provided $p\,l > \beta(n-1)+1$. If the inequality sign is replaced here by equality then the operator fails to be bounded. In fact, the function $u(x) = \log|\log x_n|$ is not in $L_\infty(\Omega)$ and belongs to $L^l_p(\Omega)$ for $p\,l = \beta(n-1)+1$.

In Maz'ya and Poborchi [575] one can find the following more general and exhaustive result concerning the same cuspidal domain with Lipschitz and increasing f on $[0,1]$, such that

$$f(0) = \lim_{x\to 0} f'(x) = 0.$$

If $p \in (1,\infty)$ and $l = 1, 2, \ldots$, then the continuity of the embedding operator $W^l_p(\Omega) \to C(\Omega) \cap L_\infty(\Omega)$ is equivalent to

$$\int_0^1 \frac{t^{(l-1)p/(p-1)}}{f(t)^{(n-1)/(p-1)}}\,\mathrm{d}t < \infty. \tag{7.6.17}$$

This embedding operator is compact. The space $W_1^l(\Omega)$ is embedded into $C(\Omega) \cap L_\infty(\Omega)$ if and only if

$$\sup\left\{\frac{t^{l-1}}{f(t)^{n-1}} : t \in (0,1)\right\} < \infty,$$

and the compactness of the same embedding operator is equivalent to

$$\lim_{t\to 0} \frac{t^{l-1}}{f(t)^{n-1}} = 0.$$

7.6.6 Embedding Operators for the Space $W_p^l(\Omega) \cap \mathring{W}_p^k(\Omega)$, $l > 2k$

In Sect. 1.6 we showed that for $l \le 2k$ the space $W_p^l(\Omega) \cap \mathring{W}_p^k(\Omega)$ satisfies the Sobolev-type theorems for arbitrary bounded domains. Here we consider the case $l > 2k$ where, according to Sect. 1.6.4, some additional requirements on Ω are necessary.

By Corollary 6.9/1 and Theorem 7.6.5/2, the inclusion $\Omega \in \mathscr{J}_\alpha$ with $(n-1)/n \le \alpha < 1$ implies the compactness of the embedding $W_p^l(\Omega) \subset W_q^m(\Omega)$ where $q^{-1} = p^{-1} - (l-m)(1-\alpha)$ if $1 > p(l-m)(1-\alpha)$, q is an arbitrary positive number if $1 = p(l-m)(1-\alpha)$ and $q = \infty$ if $1 < p(l-m)(1-\alpha)$.

In the following we show that in the case $m \ge 2k$ the preceding result is also the best possible for the space $W_p^l(\Omega) \cap \mathring{W}_p^k(\Omega)$. We present an example of a domain $\Omega \in \mathscr{J}_\alpha$ for which the continuity of the embedding $W_p^l(\Omega) \cap \mathring{W}_p^k(\Omega) \subset W_q^m(\Omega)$ implies the inequality $q^{-1} \ge p^{-1} - (l-m)(1-\alpha)$.

Let Ω be the union of the semi-ball $B^- = \{x = (y,z) : z < 0, |x| < 2\}$ and the sequence of disjoint semi-ellipsoids

$$e_i^+ = \{x = (y,z) : z > 0,\ \delta_i^{-2\gamma} z^2 + \delta_i^{-2}|y - O_i|^2 < 1\},$$

where $0 < y < 1$, $\delta_i = 2^{-i-1}$, $|O - O_i| = 3 \cdot 2^{-i}$ (see Fig. 32).

In e_i^+ we define the function

$$w_i(x) = \left(1 - \delta_i^{-2\gamma} z^2 - \delta_i^{-2}|y - O_i|^2\right)^k \eta(z/\delta^\gamma),$$

where η is a smooth function on $(0, +\infty)$, $\eta(z) = 0$ near $z = 0$ and $\eta = 1$ on the half-axis $(\frac{1}{2}, +\infty)$. We assume that each w_i, $i = 1, 2, \ldots$, is extended to $\Omega \backslash e_i^+$.

We can easily check that

$$\|\nabla_s w_i\|_{L_q(\Omega)} \sim \begin{cases} \delta_i^{-s+(n-1+\gamma)/q}, & s < 2k, \\ \delta_i^{2k(\gamma-1)-s\gamma+(n-1+\gamma)/q}, & s \ge 2k. \end{cases}$$

Let the space $W_p^l(\Omega) \cap \mathring{W}_p^k(\Omega)$ be continuously embedded into $W_q^m(\Omega)$, $m \ge 2k$. Then

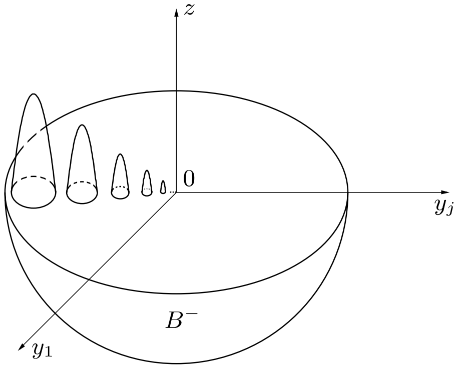

Fig. 32.

$$\|\nabla_m w_i\|_{L_q(\Omega)} \le C\|w_i\|_{W^l_p(\Omega)},$$

or equivalently,

$$\delta_i^{\gamma(m-l)+(1/p-1/q)(n-1+\gamma)} \ge cC^{-1}.$$

Consequently, $1/q \ge 1/p - (l-m)(1-\alpha)$ for $\alpha = (n-1)/(n-1+\gamma)$.

Next we show that $\Omega \in \mathscr{J}_\alpha$. We introduce the sets

$$e^+ = \{\xi = (\eta, \zeta) \in \mathbb{R}^n : \zeta > 0,\ \zeta^2 + \eta^2 < 1\},$$
$$\gamma^+ = e^+ \cap \{\xi : \zeta < \delta_i^{1-\gamma}\}.$$

Since the embedding operator of $W_1^1(e^+)$ into $L_s(e^+)$ with $s = 1/\alpha$ is continuous, for all $v \in W_1^1(e^+)$ we have

$$\|v\|_{L_2(e^+)} \le c\big(\|\nabla v\|_{L_1(e^+)} + \delta_i^{\gamma-1}\|v\|_{L_1(\gamma^*)}\big).$$

Let $u \in W_1^1(\Omega)$. Applying the last inequality to the function $v(\zeta) = u(\delta_i\eta + O_i, \delta_i^\gamma\zeta)$, we obtain

$$\|u\|_{L_1(e_i^+)} \le c\big(\|\nabla u\|_{L_1(e_i^+)} + \delta_i^{-1}\|u\|_{L_1(\gamma_i^+)}\big),$$

where $\gamma_i^+ = \{x \in e_i^+ : z < \delta_i\}$. Consequently

$$\|u\|_{L_s(\Omega)} \le c\left(\sum_{i=1}^\infty \delta_i^{-1}\|u\|_{L_1(\gamma_i^+)} + \|\nabla u\|_{L_1(\Omega)} + \|u\|_{L_1(B^-)}\right). \qquad (7.6.18)$$

Let γ_i^- denote the mirror image of γ_i^+ with respect to the plane $z = 0$. It is clear that

$$\delta_i^{-1}\int_{\gamma_i^+}|u|\,\mathrm{d}x \le \int_{\gamma_i^+\cup\gamma_i^-}|\nabla u|\,\mathrm{d}x + \delta_i^{-1}\int_{\gamma_i^-}|u|\,\mathrm{d}x.$$

This and (7.6.18) imply

$$\|u\|_{L_s(\Omega)} \le c\biggl(\|\nabla u\|_{L_1(\Omega)} + \int_{B^-} |u| \frac{\mathrm{d}x}{|x|}\biggr).$$

Since the second term on the right does not exceed $c\|u\|_{W_1^1(B^-)}$ we have

$$\inf_{c\in\mathbb{R}^1} \|u-c\|_{L_s(\Omega)} \le c\|\nabla u\|_{L_1(\Omega)}$$

and the inclusion $\Omega \in \mathscr{J}_\alpha$ follows from Theorem 5.2.3.

7.7 Comments to Chap. 7

Most of the results of this chapter were announced in Maz'ya [528] and published with proofs in [543].

The p-conductivity criterion of the embedding

$$W_p^1(\Omega) \subset C(\Omega) \cap L_\infty(\Omega) \tag{7.7.1}$$

in Theorem 7.1.1/1 was used by Labutin [473] to obtain the following necessary and sufficient conditions on functions G on $\mathbb{R}$ such that $G \circ u \in W_p^1(\Omega)$ for all $u \in W_p^1(\Omega)$:

(i) $G' \in L_\infty(\mathbb{R})$ if the embedding (7.7.1) fails,
(ii) $G' \in L_\infty(\mathbb{R}, \mathrm{loc})$ if the embedding (7.7.1) holds.

In the case $r = n/(n-1)$, if the boundary is Lipschitz, the inequality (7.1.13) is a particular case of the Brezis–Gallouët–Wainger inequality [141], [146], which has important applications to the theory of nonlinear evolution equations (see for instance, Chap. 13 in M.E. Taylor [747], Brezis and Gallouët [141], et al.).

Buckley and Koskela [148] showed that the inequality (7.3.1) with $\Lambda(t) = t^{1-2/p}$, $p > 2$, is equivalent to (1.5.9).

8

Localization Moduli of Sobolev Embeddings for General Domains

In the present chapter we study noncompact embeddings

$$E_{p,q}(\Omega) : L^1_p(\Omega) \to L_q(\Omega),$$

where Ω is a connected open set in $\mathbb{R}^n$, $n > 1$, and $1 \le p \le q < \infty$. In the opposite case $p > q$, the boundedness of $E_{p,q}$ implies compactness (see Remark 5.5.2 and Theorem 6.8.2/2), which makes this case of no interest for us in this chapter. Here we define new measures of noncompactness of $E_{p,q}$ and characterize their mutual relations as well as their relations with "local" isoperimetric and isoconductivity constants. To describe our results we need to introduce some notation that will be frequently used in this chapter.

We supply $L^1_p(\Omega)$ with the norm

$$\|u\|_{L^1_p(\Omega)} = \|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_p(\omega)},$$

where ∇ stands for the gradient and ω is a nonempty open set with compact closure $\overline{\omega} \subset \Omega$. It is standard that a change of ω leads to an equivalent norm in $L^1_p(\Omega)$. *In this chapter we often omit Ω in notations of spaces and norms if it causes no ambiguity.*

Among other things, we study the essential norm of the embedding operator $E_{p,q} : L^1_p \to L_q$, i.e., the number

$$\mathrm{ess}\|E_{p,q}\| = \inf \|E_{p,q} - T\|_{L^1_p \to L_q},$$

where the infimum is taken over all compact operators $T : L^1_p \to L_q$.

Another characteristic of $E_{p,q}$ to be dealt with later in this chapter is defined by

$$\mathbb{C}(E_{p,q}) = \inf C,$$

where C is a positive constant such that there exist $\rho > 0$ and $K > 0$ subjected to the inequality

$$\|u\|_{L_q(\Omega)} \le C\|u\|_{L^1_p(\Omega)} + K\|u\|_{L_1(\Omega_\rho)} \quad \text{for all } u \in L^1_p \tag{8.0.1}$$

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Grundlehren der mathematischen Wissenschaften 342,
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with

$$\Omega_\rho = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \rho\}.$$

Together with the norm $\|E_{p,q}\|$ of the embedding operator $E_{p,q}$, its essential norm $\operatorname{ess}\|E_{p,q}\|$ and the number $\mathbb{C}(E_{p,q})$, we shall make use of two numbers $\mathcal{M}_1(E_{p,q})$ and $\mathcal{M}_2(E_{p,q})$. The first of them is defined by

$$\mathcal{M}_1(E_{p,q}) = \lim_{s\to 0} \sup\left\{ \frac{\|u\|_{L_q}}{\|u\|_{L_p^1}} : u \in L_p^1,\ u = 0 \text{ on } \Omega_s \right\}.$$

The definition of $\mathcal{M}_2$ is as follows:

$$\mathcal{M}_2(E_{p,q}) = \lim_{\varrho\to 0} \sup_{x\in\partial\Omega} \sup\left\{ \frac{\|u\|_{L_q}}{\|u\|_{L_p^1}} : u \in L_p^1,\ \operatorname{supp} u \subset B(x,\varrho) \right\},$$

where $B(x,\varrho)$ is the open ball with radius ϱ centered at x.

These two characteristics of $E_{p,q}$ differ in the ways of localization of the functions involved and it seems appropriate to call $\mathcal{M}_1$ and $\mathcal{M}_2$ *the localization moduli* of the embedding $E_{p,q}$.

In Sect. 8.1 we show that

$$\operatorname{ess}\|E_{p,q}\| = \mathbb{C}(E_{p,q}) = \mathcal{M}_1(E_{p,q}) \tag{8.0.2}$$

provided $1 \le p \le q < pn/(n-p)$ if $n > p$ and $1 \le q < \infty$ for $p \ge n$. We also prove that the three quantities in (8.0.2) are equal to $\mathcal{M}_2(E_{p,q})$ under the additional assumption $p < q$. The last fact fails if $p = q$ as shown in an example of a domain Ω for which $\mathcal{M}_2(E_{p,q}) = 0$ and

$$\|E_{p,p}\| = \operatorname{ess}\|E_{p,p}\| = \mathcal{M}_1(E_{p,p}) = \mathbb{C}(E_{p,p}) = \infty$$

(see Sect. 8.2).

In Sect. 8.3, we assume that Ω is a bounded C^1 domain and that $q = pn/(n-p)$, and $n > p \ge 1$. In this case we find an explicit value for

$$\mathbb{C}(E_{p,q}) = \mathcal{M}_1(E_{p,q}) = \mathcal{M}_2(E_{p,q}).$$

The results obtained in Sect. 8.1 are readily extended in Sect. 8.4 to the embedding of $L_p^1(\Omega)$ to the space $L_q(\Omega,\mu)$, where μ is a Radon measure.

Next, we turn to domains with a power cusp on the boundary and find explicit formulas for the measures of noncompactness under consideration and apply these results to the Neumann problem for a particular Schrödinger operator (Sects. 8.5 and 8.6).

In the final section we show relations between our measures of noncompactness and local isoconductivity and isoperimetric constants. In particular, we obtain

$$\operatorname{ess}\|E_{1,q}\| = \mathbb{C}(E_{1,q}) = \mathcal{M}_1(E_{1,q}) = \mathcal{M}_2(E_{1,q}) = \lim_{s\to 0} \sup_{g\subset\Omega\setminus\overline{\Omega_s}} \frac{(m_n(g))^{1/q}}{\mathbf{H}_{n-1}(\Omega\cap\partial g)},$$

where $1 < q < n/(n-1)$, g is a relatively closed subset of Ω such that $\Omega \cap \partial g$ is a smooth surface, and $\mathbf{H}_{n-1}$ is the $(n-1)$-dimensional area. This together with the results from Sect. 8.5 yields explicit values of the local isoperimetric constants for power cusps.

8.1 Localization Moduli and Their Properties

Let us discuss relations between the moduli. First of all, obviously,

$$\mathcal{M}_1(E_{p,q}) \geq \mathcal{M}_2(E_{p,q}). \tag{8.1.1}$$

Lemma 1. *Let Ω be a domain in $\mathbb{R}^n$ with $m_n(\Omega) < \infty$. Suppose that $1 \leq p < \infty$ and $1 \leq q < \infty$. Then for any $u \in L^1_p(\Omega)$ the following estimate holds:*

$$\|u\|_{L_q(\Omega)} \leq \big(\mathcal{M}_1(E_{p,q}) + \varepsilon\big)\|u\|_{L^1_p(\Omega)} + C(\varepsilon)\|u\|_{L_{\max\{q,p\}}(\omega_\varepsilon)}, \tag{8.1.2}$$

where ε is an arbitrary positive number, $C(\varepsilon)$ is a positive function of ε, and ω_ε is an open set with smooth boundary and compact closure $\overline{\omega_\varepsilon} \subset \Omega$.

Proof. Let η denote a smooth function on $\mathbb{R}^+$, such that $0 \leq \eta \leq 1$ and $\eta(t) = 0$ for $t \in (0,1)$ and $\eta(t) = 1$ for $t \geq 2$. By $d(x)$ we denote the distance from $x \in \Omega$ to $\partial\Omega$. Let us introduce the cutoff function $H_s(x) = \eta(d(x)/s)$. We write

$$\|u\|_{L_q} \leq \|H_s u\|_{L_q} + \big\|(1-H_s)u\big\|_{L_q},$$

and note that the second term does not exceed

$$\sup\bigg\{\frac{\|v\|_{L_q(\Omega)}}{\|v\|_{L^1_p(\Omega)}} : v \in L^1_p(\Omega),\ \operatorname{supp} v \subset \Omega \setminus \overline{\Omega}_{2s}\bigg\}\big\|(1-H_s)u\big\|_{L^1_p}.$$

Hence, for sufficiently small $s = s(\varepsilon)$

$$\|u\|_{L_q} \leq \big(\mathcal{M}_1(E_{p,q}) + \varepsilon\big)\big(\|u\|_{L^1_p} + \|u\nabla H_s\|_{L_p}\big) + \|H_s u\|_{L_q}.$$

Since the supports of H_s and its derivatives are in Ω_s the result follows. □

Corollary. *Let Ω be a domain in $\mathbb{R}^n$ with $m_n(\Omega) < \infty$. Suppose that $1 \leq p < \infty$ and $1 \leq q < \infty$. Then*

$$\mathcal{M}_1\big(E_{p,q}(\Omega)\big) < \infty,$$

if and only if

$$\|E_{p,q}\|_{L^1_p(\Omega) \to L_q(\Omega)} < \infty.$$

Proof. By Lemma 1, $\mathcal{M}_1(E_{p,q}(\Omega)) < \infty$ implies $\|E_{p,q}\|_{L^1_p(\Omega)\to L_q(\Omega)} < \infty$. The converse is obvious. □

Lemma 2. *Let Ω be a domain in $\mathbb{R}^n$ with $m_n(\Omega) < \infty$.*

(i) *If* $1 \le q < \infty$ *and* $1 \le p < \infty$, *then*

$$\mathcal{M}_1(E_{p,q}) \le \text{ess}\|E_{p,q}\|.$$

(ii) *Let* $1 \le q < pn/(n-p)$ *if* $1 \le p < n$ *and* $1 \le q < \infty$ *if* $n \le p < \infty$. *Then*

$$\text{ess}\|E_{p,q}\| \le \mathcal{M}_1(E_{p,q}).$$

Proof. (i) By F we denote an operator of finite rank acting from $L^1_p(\Omega)$ into $L_q(\Omega)$ and given by

$$Fu = \sum_{1 \le j \le N} c_j(u)\varphi_j,$$

where $\varphi_j \in L_q(\Omega)$ and c_j are continuous functionals on $L^1_p(\Omega)$. Let $\varepsilon > 0$. We choose the operator F to satisfy

$$\varepsilon + \text{ess}\|E_{p,q}\| \ge \|E_{p,q} - F\|. \tag{8.1.3}$$

Without loss of generality, we can assume that the functions φ_j have compact supports in Ω. Hence, there exists a positive $s(\varepsilon)$ such that $Fu = 0$ on $\Omega \backslash \overline{\Omega_{s(\varepsilon)}}$ for all $u \in L^1_p(\Omega)$. Let $s \in (0, s(\varepsilon))$. By (8.1.3), for all $u \in L^1_p(\Omega)$ vanishing on Ω_s,

$$\varepsilon + \text{ess}\|E_{p,q}\| \ge \frac{\|u\|_{L_q(\Omega \backslash \overline{\Omega_{s(\varepsilon)}})}}{\|u\|_{L^1_p(\Omega)}} = \frac{\|u\|_{L_q(\Omega)}}{\|u\|_{L^1_p(\Omega)}}.$$

The required lower estimate for $\text{ess}\|E_{p,q}\|$ follows from the definition of $\mathcal{M}_1(E_{p,q})$.

(ii) Let ε be an arbitrary positive number and let $s > 0$ be so small that

$$\sup\left\{ \frac{\|v\|_{L_q(\Omega)}}{\|v\|_{L^1_p(\Omega)}} : v \in L^1_p(\Omega),\ v = 0 \text{ on } \Omega_s \right\} \le \mathcal{M}_1(E_{p,q}) + \varepsilon.$$

We introduce a domain ω with smooth boundary and compact closure $\overline{\omega}$, $\overline{\omega} \subset \Omega$, such that $m_n(\Omega \setminus \omega) = \delta m_n(\Omega \setminus \overline{\Omega}_s)$ with any chosen $\delta \in (0,1)$. By χ_ω we denote the characteristic function of ω. By the compactness of the embedding $L^1_p(\Omega) \to L_q(\omega)$ we have

$$\text{ess}\|E_{p,q}\| \le \sup_{u \in L^1_p} \frac{\|u - \chi_\omega u\|_{L_q(\Omega)}}{\|u\|_{L^1_p(\Omega)}} = \sup_{u \in L^1_p} \frac{\|u\|_{L_q(\Omega \backslash \omega)}}{\|u\|_{L^1_p(\Omega)}}. \tag{8.1.4}$$

It is obvious that for any positive T

$$|u| \le \min\{T, |u|\} + \big(|u| - T\big)_+,$$

where f_+ means the nonnegative part of f. This implies

$$\|u\|_{L_q(\Omega\setminus\omega)} \le \big\|\min\{T,|u|\}\big\|_{L_q(\Omega\setminus\omega)} + \big\|\big(|u|-T\big)_+\big\|_{L_q(\Omega\setminus\omega)}. \tag{8.1.5}$$

We use the notation $\mathcal{L}_t = \{x : |u(x)| > t\}$ and choose T as

$$T = \inf\{t > 0 : m_n\mathcal{L}_t < m_n(\Omega \setminus \overline{\Omega_s})\}.$$

Then, for $q < \infty$,

$$\begin{aligned}\big\|\min\{T,|u|\}\big\|_{L_q(\Omega\setminus\omega)} &\le T m_n(\Omega\setminus\omega)^{1/q}\\ &= \big(\delta m_n(\Omega\setminus\overline{\Omega_s})\big)^{1/q} T \le \delta^{1/q}\|u\|_{L_q(\Omega)}.\end{aligned}$$

Hence,

$$\sup_{u\in L^1_p(\Omega)} \frac{\|\min\{T,|u|\}\|_{L_q(\Omega\setminus\omega)}}{\|u\|_{L^1_p(\Omega)}} \le \delta^{1/q}\|E_{p,q}\|.$$

Combining this inequality with (8.1.4) and (8.1.5), we arrive at

$$\text{ess}\|E_{p,q}\| \le \sup_{u\in L^1_p(\Omega)} \frac{\|(|u|-T)_+\|_{L_q(\Omega\setminus\omega)}}{\|u\|_{L^1_p(\Omega)}} + \delta^{1/q}\|E_{p,q}\|.$$

Let σ be an arbitrary positive number and let H_σ be the function introduced in the proof of Lemma 1. Then

$$\begin{aligned}\big\|\big(|u|-T\big)_+\big\|_{L_q(\Omega)} &\le \big\|\big(|u|-T\big)_+(1-H_\sigma)\big\|_{L_q(\Omega)}\\ &\quad + \big\|\big(|u|-T\big)_+ H_\sigma\big\|_{L_q(\Omega)}.\end{aligned} \tag{8.1.6}$$

To estimate the first term on the right-hand side, we take σ so small that

$$\sup\left\{\frac{\|v\|_{L_q(\Omega)}}{\|v\|_{L^1_p(\Omega)}} : v\in L^1_p(\Omega);\ v = 0 \text{ on } \Omega_{2\sigma}\right\} \le \mathcal{M}_1(E_{p,q}) + \varepsilon.$$

We normalize u by $\|u\|_{L^1_p(\Omega)} = 1$. Then

$$\begin{aligned}&\big\|\big(|u|-T\big)_+(1-H_\sigma)\big\|_{L_q(\Omega)}\\ &\quad\le \big(\mathcal{M}_1(E_{p,q})+\varepsilon\big)\big(1+\big\|\big(|u|-T\big)_+\nabla H_\sigma\big\|_{L_p(\Omega)}\big).\end{aligned} \tag{8.1.7}$$

Combining (8.1.6) and (8.1.7), we see that

$$\begin{aligned}\big\|(u-T)_+\big\|_{L_q(\Omega)} &\le \big(\mathcal{M}_1(E_{p,q})+\varepsilon\big)\\ &\quad + C(\Omega)\big(\|u\|_{L_q(\mathcal{L}_T\cap\Omega_\sigma)} + \|u\|_{L_p(\mathcal{L}_T\cap\Omega_\sigma)}\big)\end{aligned} \tag{8.1.8}$$

with a constant $C(\Omega)$ depending only on Ω, σ, p, and q but not on s. Using the compactness of the restriction operator $L^1_p(\Omega) \to L_{\max\{p,q\}}(\overline{\Omega}_\sigma)$ and the equality $m_n\mathcal{L}_T \le m_n(\Omega\setminus\overline{\Omega}_s)$, we conclude that the two norms on the right-hand side of (8.1.8) tend to zero as $s \to 0$. Therefore

$$\limsup_{s\to 0}\big\|\big(|u|-T\big)_+\big\|_{L_q(\Omega)} \le \mathcal{M}_1(E_{p,q})+\varepsilon$$

and hence

$$\operatorname{ess}\|E_{p,q}\| \le \mathcal{M}_1(E_{p,q})+\varepsilon+\delta^{1/q}\|E_{p,q}\|.$$

The proof is completed by using the arbitrariness of ε and δ. □

Theorem 1. *Let Ω be a domain in $\mathbb{R}^n$ with $m_n(\Omega)<\infty$. Suppose that $1\le p\le q<pn/(n-p)$ if $1\le p<n$ and $1\le q<\infty$ if $n\le p<\infty$. Then*

$$\operatorname{ess}\|E_{p,q}\| = \mathbb{C}(E_{p,q}) = \mathcal{M}_1(E_{p,q}).$$

Proof. In the case $q<pn/(n-p)$, $n>p$ and q arbitrary when $p\ge n$ the trace map from $L^1_p(\Omega)$ into $L_{\max\{p,q\}}(\omega_\delta)$ is compact. Then from Lions and Magenes [500, Theorem 16.4] it follows that

$$\|u\|_{L_{\max(p,q)}(\omega_\delta)} \le \varepsilon\|u\|_{L^1_p(\Omega)} + C(\varepsilon)\|u\|_{L_1(\omega_\delta)},$$

where $\varepsilon>0$ is an arbitrary small number. This together with (8.1.2) allows us to obtain

$$\mathbb{C}(E_{p,q}) \le \mathcal{M}_1(E_{p,q}).$$

Let $\varepsilon>0$ be an arbitrary positive number. Then

$$\|u\|_{L_q(\Omega)} \le \big(\mathbb{C}(E_{p,q})+\varepsilon\big)\|u\|_{L^1_p(\Omega)} + c(\varepsilon)\|u\|_{L_1(\Omega_{s(\varepsilon)})}$$

and hence

$$\mathcal{M}_1(E_{p,q}) \le \mathbb{C}(E_{p,q})+\varepsilon.$$ □

We shall see that, in dealing with $\mathcal{M}_2(E_{p,q})$, we must distinguish between the cases $p<q$ and $p=q$.

Theorem 2. *Let Ω be a domain in $\mathbb{R}^n$ with $m_n(\Omega)<\infty$ and $1\le p<q<\infty$. Suppose that $q<pn/(n-p)$ for $p<n$ and $1\le q<\infty$ for $p\ge n$. Then*

$$\operatorname{ess}\|E_{p,q}\| = \mathbb{C}(E_{p,q}) = \mathcal{M}_1(E_{p,q}) = \mathcal{M}_2(E_{p,q}).$$

Proof. By (8.1.1) and Theorem 1 it is sufficient to assume that $\mathcal{M}_2(E_{p,q})<\infty$ and to prove the inequality

$$\mathcal{M}_2(E_{p,q}) \ge \mathbb{C}(E_{p,q}).$$

Fix $\rho>0$ and let $\{\mathcal{B}_i\}_{i\ge 1}$ be an open covering of $\overline{\Omega\setminus\Omega_{\rho/2}}$ by balls of radius ρ centered at $\partial\Omega$. Also let the multiplicity of the covering be finite and depend only on n. Obviously the collection $\{\mathcal{B}_i\}_{i\ge 1}$ supplemented by the set $\overline{\Omega}_{\rho/2}$ forms an open covering of Ω which has a finite multiplicity as well. We introduce a family of nonnegative functions $\{\eta_i\}_{i\geq 0}$, such that $\eta_0 \in C_0^\infty(\Omega_{\rho/2})$ and $\eta_i \in C_0^\infty(\mathcal{B}_i)$ for $i \geq 1$, and

$$\sum_{i\geq 0} \eta_i(x)^p = 1 \quad \text{on } \Omega. \tag{8.1.9}$$

The estimates to be obtained in what follows will be first proved for an arbitrary function $u \in L_p^1(\Omega) \cap L_\infty(\Omega)$. Since $L_p^1(\Omega) \cap L_\infty(\Omega)$ by Lemma 1.7.1, these estimates remain valid for $u \in L_p^1(\Omega)$.

Clearly,

$$\|u\|^p_{L_q(\Omega)} = \Big\|\sum_{i\geq 0} |\eta_i u|^p\Big\|_{L_{q/p}(\Omega)} \leq \sum_{i\geq 0} \big\||\eta_i u|^p\big\|_{L_{q/p}(\Omega)}$$
$$= \sum_{i\geq 0} \|\eta_i u\|^p_{L_q(\Omega)}. \tag{8.1.10}$$

Given $\varepsilon > 0$ and sufficiently small ρ, we have

$$\sup_{i\geq 1} \sup\left\{\frac{\|v\|_{L_q(\Omega)}}{\|v\|_{L_p^1(\Omega)}} : v \in L_p^1(\Omega),\ v = 0 \text{ on } \Omega \setminus \mathcal{B}_i\right\} \leq \mathcal{M}_2(E_{p,q}) + \varepsilon.$$

Therefore, the right-hand side in (8.1.10) does not exceed

$$\big(\mathcal{M}_2(E_{p,q}) + \varepsilon\big)^p \sum_{i\geq 1} \|\eta_i u\|^p_{L_p^1(\Omega)} + \|\eta_0 u\|^p_{L_q(\Omega)}.$$

Using the elementary inequality

$$(a+b)^p \leq (1+\varepsilon)a^p + c(\varepsilon)b^p \tag{8.1.11}$$

for positive a and b, we see that

$$\|u\|^p_{L_q(\Omega)} \leq \big(\mathcal{M}_2(E_{p,q}) + \varepsilon\big)^p \Big\{(1+\varepsilon)\sum_{i\geq 1} \|\eta_i \nabla u\|^p_{L_p(\Omega)}$$
$$+\ C(\varepsilon,\rho)\|u\|^p_{L_p(\Omega)}\Big\} + \|\eta_0 u\|^p_{L_q(\Omega)}. \tag{8.1.12}$$

We note further that by (8.1.9) the sum over $i \geq 1$ in (8.1.12) does not exceed $\|\nabla u\|^p_{L_p(\Omega)}$. Since $q^{-1} > p^{-1} - n^{-1}$ and the support of η_0 is separated from $\partial\Omega$, it follows that by the compactness of the restriction mapping $L_p^1(\Omega)$ into $L_q(\Omega_{\rho/2})$ we have the estimate

$$\|\eta_0 u\|_{L_q(\Omega)} \leq \delta\|\eta_0 u\|_{L_p^1(\Omega)} + c(\delta,\rho)\|\eta_0 u\|_{L_1(\Omega)}, \tag{8.1.13}$$

where δ is an arbitrary small number.

Let τ be a positive number independent of ρ. Since $q > p$, we conclude that

$$\|u\|_{L_p(\Omega)} \leq \big(m_n(\Omega \setminus \overline{\Omega}_\tau)\big)^{p^{-1}-q^{-1}} \|u\|_{L_q(\Omega\setminus\overline{\Omega}_\tau)} + \|u\|_{L_p(\Omega_\tau)} \tag{8.1.14}$$

and

$$\|u\|_{L_p(\Omega_\tau)} \leq \varepsilon \|u\|_{L^1_p(\Omega)} + c(\varepsilon,\tau)\|u\|_{L_1(\Omega_\tau)}. \tag{8.1.15}$$

Choosing τ and δ sufficiently small and using (8.1.12)–(8.1.15) we arrive at the inequality

$$\|u\|_{L_q(\Omega)} \leq \big(\mathcal{M}_2(E_{p,q}) + c\varepsilon\big)\|u\|_{L^1_p(\Omega)} + c(\varepsilon)\|u\|_{L_1(\Omega_\tau)}$$

with a constant c independent of ε and u. This completes the proof. □

8.2 Counterexample for the Case $p = q$

In the next example, we show that the condition $p < q$ in Theorem 8.1/2 cannot be removed. To be more precise, for any $p \in (1,\infty)$ we construct a planar domain for which

$$\|E_{p,p}\| = \text{ess}\|E_{p,p}\| = \mathcal{M}_1(E_{p,p}) = \mathbb{C}(E_{p,p}) = \infty$$

and $\mathcal{M}_2(E_{p,p}) = 0$.

Example. (See Fig. 33) Let $\Omega \subset \mathbb{R}^2$ be the union of rectangles

$$\begin{aligned}
A_i &= (0, 1/2] \times (a_{2i-2}, a_{2i}), \quad i > 0,\\
B_0 &= [3/2, 2) \times (0, a_1),\\
B_i &= [3/2, 2) \times (a_{2i-1}, a_{2i+1}), \quad i > 0,\\
C_i &= [1/2, 3/2] \times (a_{i-1}, a_i), \quad i > 0,
\end{aligned}$$

where

$$a_0 = 0 \quad \text{and} \quad a_i = \sum_{n=1}^{i} n^{-p} \quad \text{for } i > 0.$$

Clearly, $m_2(\Omega) < \infty$. For each integer $j > 0$ we define the continuous function $f_j(x)$ as a function that is zero on

$$\Big(\bigcup_{n\leq j} A_n\Big) \cup \Big(\bigcup_{n\leq j} B_n\Big) \cup \Big(\bigcup_{n<2j} C_n\Big),$$

$f_j(x) = i$ and $f_j(x) = i+1$ on A_{j+i} and B_{j+i}, respectively, and linear on C_{2j+i} with $|\nabla f_j| = 1$ for $i \geq 0$. The graph of each function $f_j(x)$ has the shape of a staircase with slope 1 on C_{2j+i}, and landings on A_{j+i} and B_{j+i} for

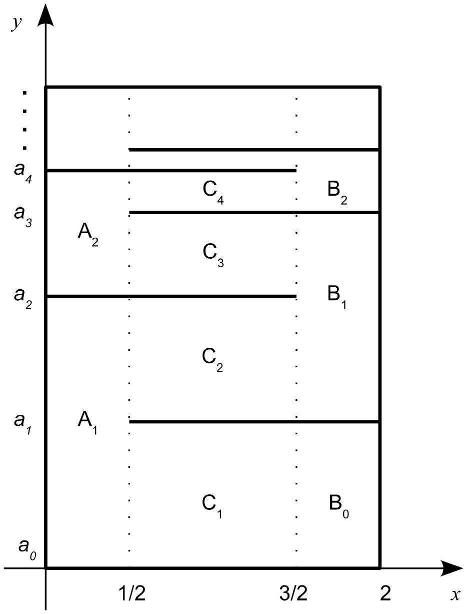

Fig. 33.

$i > 0$. By a simple computation we obtain $\|\nabla f_j\|_{L_p} < \infty$ and $\|f_j\|_{L_p} = \infty$ for each $j > 0$ that implies $\mathcal{M}_1(E_{p,p}) = \infty$. Furthermore, by Theorem 8.1/2 we have $\mathbb{C}(E_{p,p}) = \operatorname{ess}\|E_{p,p}\| = \infty$.

It remains to show that $\mathcal{M}_2(E_{p,p}) = 0$. Let $x \in \partial\Omega$ and $\rho < 1/4$. By $Q(x, \rho)$ we denote the square $(x - \rho, x + \rho)^2$. By the definition of Ω one obtains that $\Omega \cap Q(x, \rho)$ is a union of open disjoint sets $\{I_i\}$, where I_i is either a rectangle or the union of three rectangles.

For $f \in L_p^1(\Omega)$ with supp $f \subset \Omega \cap Q(x, \rho)$, we have

$$\int_{I_i} \big|f(x)\big|^p \,\mathrm{d}x \le (c\rho)^p \int_{I_i} \big|\nabla f(x)\big|^p \,\mathrm{d}x.$$

Summing over $\{I_i\}$, we arrive at

$$\biggl(\int_{\cup I_i} \big|f(x)\big|^p \,\mathrm{d}x\biggr)^{1/p} \le c\rho \biggl(\int_{\cup I_i} \big|\nabla f(x)\big|^p \,\mathrm{d}x\biggr)^{1/p}$$

and then

$$\sup\biggl\{\frac{\|u\|_{L_p}}{\|u\|_{L_p^1}} : u \in L_p^1(\Omega),\ \operatorname{supp} u \subset Q(x, \rho)\biggr\} \le c\rho$$

for every $x \in \partial\Omega$. This implies $\mathcal{M}_2(E_{p,p}) = 0$.

8.3 Critical Sobolev Exponent

Here we show that all our measures of noncompactness can be found explicitly if $\partial\Omega$ has a continuous normal and q is the critical Sobolev exponent $pn/(n-p)$.

Theorem. *Let $n > p \geq 1$ and let Ω be a bounded C^1 domain. Then*

$$\mathbb{C}(E_{p,\frac{pn}{n-p}}) = \mathcal{M}_1(E_{p,\frac{pn}{n-p}}) = \mathcal{M}_2(E_{p,\frac{pn}{n-p}}) = 2^{1/n}c(p,n),$$

where

$$c(p,n) = \pi^{-1/2}n^{-1/p}\left(\frac{p-1}{n-p}\right)^{1-1/p}\left(\frac{\Gamma(n)\Gamma(1+n/2)}{\Gamma(n/p)\Gamma(1+n-n/p)}\right)^{1/n}$$

is the best constant in the Sobolev inequality

$$\|u\|_{L_{\frac{pn}{n-p}}(\mathbb{R}^n)} \leq c\|\nabla u\|_{L_p(\mathbb{R}^n)}. \tag{8.3.1}$$

Proof. Let ζ be a radial function in $C_0^\infty(\mathbb{R}^n)$, $\varepsilon > 0$ and $\zeta_\varepsilon(x) = \zeta(x/\varepsilon)$. By $\mathcal{O}$ we denote an arbitrary point at $\partial\Omega$ and we put $\zeta_{\varepsilon,\mathcal{O}}(x) := \zeta_\varepsilon(x - \mathcal{O})$ into the inequality (8.0.1). We use q^* to denote $pn/(n-p)$. Using the definition of $\mathbb{C}(E_{p,q})$, we obtain

$$\limsup_{\varepsilon\to 0}\frac{\|\zeta_{\varepsilon,\mathcal{O}}\|_{L_{q^*}(\Omega)}}{\|\nabla\zeta_{\varepsilon,\mathcal{O}}\|_{L_p(\Omega)}} \leq \mathbb{C}(E_{p,q^*}).$$

We note the existence of the limit

$$\lim_{\varepsilon\to 0}\frac{\|\zeta_{\varepsilon,\mathcal{O}}\|_{L_{q^*}(\Omega)}}{\|\nabla\zeta_{\varepsilon,\mathcal{O}}\|_{L_p(\Omega)}} = \frac{\|\zeta\|_{L_{q^*}(\mathbb{R}^n_+)}}{\|\nabla\zeta\|_{L_p(\mathbb{R}^n_+)}} = \frac{2^{1/n-1/p}\|\zeta\|_{L_{q^*}(\mathbb{R}^n)}}{2^{-1/p}\|\nabla\zeta\|_{L_p(\mathbb{R}^n)}}.$$

To obtain the lower estimate for $\mathbb{C}(E_{p,q^*})$ one needs only to recall that the best constant in (8.3.1) is attained for a radial function when $p > 1$ and obtained as a limit by a sequence of radial functions for $p = 1$.

Let us turn to the upper estimate for $\mathbb{C}(E_{p,q^*})$. We construct a finite open covering of $\overline{\Omega}$ by balls $\mathcal{B}_i$ of radius ρ with a finite multiplicity depending only on n. Let either $\overline{\mathcal{B}_i} \subset \Omega$ or the center of $\mathcal{B}_i$ be placed on $\partial\Omega$. We introduce a family of nonnegative functions $\{\eta_i\}$ such that $\eta_i \in C_0^\infty(\mathcal{B}_i)$ and

$$\sum_i \eta_i(x)^p = 1 \quad \text{on } \Omega.$$

Obviously,

$$\begin{aligned}\|u\|^p_{L_{q^*}(\Omega)} &= \Big\|\sum_i |\eta_i u|^p\Big\|_{L_{q^*/p}(\Omega)}\\ &\leq \sum_i \big\||\eta_i u|^p\big\|_{L_{q^*/p}(\Omega)} = \sum_i \|\eta_i u\|^p_{L_{q^*}(\Omega)}.\end{aligned} \tag{8.3.2}$$

If $\overline{\mathcal{B}}_i \subset \Omega$, then

$$\|\eta_i u\|_{L_{q^*}(\Omega)} \le c(p,n)\big\|\nabla(\eta_i u)\big\|_{L_p(\Omega)}.$$

Let ε be an arbitrary positive number. Since ρ is sufficiently small and the domain is in the class C^1, one can easily show that

$$\|\eta_i u\|_{L_{q^*}(\Omega)} \le \big(2^{1/n}c(p,n)+\varepsilon\big)\big\|\nabla(\eta_i u)\big\|_{L_p(\Omega)}$$

for the ball $\mathcal{B}_i$ centered at a point at $\partial\Omega$. By using (8.1.11), we see that

$$\begin{aligned}\sum_i \big\|\nabla(\eta_i u)\big\|^p_{L_p(\Omega)} &\le \sum_i \big(\|\eta_i \nabla u\|^p_{L_p(\Omega)} + \|u\nabla\eta_i\|^p_{L_p(\Omega)}\big)\\ &\le (1+\varepsilon)\sum_i \|\eta_i\nabla u\|^p_{L_p(\Omega)}\\ &\quad + C(\varepsilon)\sum_i \|u\nabla\eta_i\|^p_{L_p(\Omega)}. \end{aligned} \tag{8.3.3}$$

We note that the first sum on the right-hand side of (8.3.3) is equal to $\|\nabla u\|^p_{L_p(\Omega)}$. Now, it follows from (8.3.2) that

$$\begin{aligned}\|u\|^p_{L_{q^*}(\Omega)} &\le \big(2^{1/n}c(p,n)+\varepsilon\big)^p\\ &\quad\times\bigg((1+\varepsilon)\|\nabla u\|^p_{L_p(\Omega)} + C(\varepsilon)\sum_i \|u\nabla\eta_i\|^p_{L_p(\Omega)}\bigg). \end{aligned} \tag{8.3.4}$$

Furthermore the second sum in the right-hand side of (8.3.4) can be majorized by $C(\varepsilon,\rho)\|u\|^p_{L_p(\Omega)}$, and then

$$\|u\|^p_{L_{q^*}(\Omega)} \le \big(2^{1/n}c(p,n)+\varepsilon\big)^p\big((1+\varepsilon)\|\nabla u\|^p_{L_p(\Omega)} + C(\varepsilon,\rho)\|u\|^p_{L_p(\Omega)}\big).$$

Since the embedding $E_{p,p}(\Omega)$ is compact, it follows from Lions and Magenes [500, Theorem 16.4] that

$$\begin{aligned}\|u\|^p_{L^p_{q^*}(\Omega)} &\le \big(2^{1/n}c(p,n)+\varepsilon\big)^p\\ &\quad\times\Big((1+\varepsilon)\|\nabla u\|^p_{L_p(\Omega)} + \big(\varepsilon\|u\|_{L^1_p(\Omega)} + C(\varepsilon,\varrho,\tau)\|u\|_{L_1(\Omega_\tau)}\big)^p\Big).\end{aligned}$$

Using (8.1.11) once more, we arrive at (8.0.1). Hence

$$\mathbb{C}(E_{p,q^*}) \le 2^{1/n}c(p,n),$$

which, in combination with the lower estimate for $\mathbb{C}(E_{p,q^*})$, shows that

$$\mathbb{C}(E_{p,q^*}) = 2^{1/n}c(p,n). \tag{8.3.5}$$

Putting an arbitrary $u \in L^1_p(\Omega)$ vanishing outside $\Omega\setminus\overline{\Omega}_s$ into (8.0.1) and taking the limit as $s\to 0$, we conclude that

$$\mathbb{C}(E_{p,q^*}) \geq \mathcal{M}_1(E_{p,q^*}). \tag{8.3.6}$$

Now, let $u \in L^1_p(\Omega)$, $u = 0$ outside the ball $B(x_0, \rho)$ with $x_0 \in \partial\Omega$ and sufficiently small ρ. One can easily construct an extension $\widetilde{u}$ of u onto the whole ball $B(x_0, \rho)$ so that

$$\frac{\|u\|_{L_{q^*}(\Omega)}}{\|u\|_{L^1_p(\Omega)}} \geq 2^{1/n}(1-\varepsilon)\frac{\|\widetilde{u}\|_{L_{q^*}(B(x,\rho))}}{\|\widetilde{u}\|_{L^1_p(B(x,\rho))}}.$$

Choosing u in a such way that its extension $\widetilde{u}$ is an almost optimizing function from the Sobolev inequality (8.3.1), we arrive at

$$\frac{\|u\|_{L_{q^*}(\Omega)}}{\|u\|_{L^1_p(\Omega)}} \geq 2^{1/n}c(p,n)(1-2\varepsilon),$$

and the definition of $\mathcal{M}_2$ yields

$$\mathcal{M}_2(E_{p,q^*}) \geq 2^{1/n}c(p,n). \tag{8.3.7}$$

Combining (8.3.5)–(8.3.7) and the inequality $\mathcal{M}_2 \leq \mathcal{M}_1$, we complete the proof. □

8.4 Generalization

The previous results hold in a more general situation when there is a compact subset of $\partial\Omega$ that is responsible for the loss of compactness of $E_{p,q}$ and the norm in the target space involves a measure.

Let K be a compact subset of $\partial\Omega$ and let $\partial\Omega \setminus K$ be locally Lipschitz (i.e., each point of $\partial\Omega \setminus K$ has a neighborhood $\mathcal{U} \subset \mathbb{R}^n$ such that there exists a quasi-isometric transformation which maps $\mathcal{U} \cap \Omega$ onto a cube). Define

$$\Omega^K_s = \{x \in \Omega : \operatorname{dist}(x, K) > s\}.$$

It is obvious that for each $s > 0$, the embedding

$$L^1_p(\Omega^K_s) \to L_q(\Omega^K_s) \quad \text{is compact}$$

if and only if the embedding

$$L^1_p(\Omega_s) \to L_q(\Omega_s) \quad \text{is compact}. \tag{8.4.1}$$

Let μ be a measure on Ω. We define the embedding operator

$$E_{p,q}(\mu) : L^1_p(\Omega) \to L_q(\Omega, \mu),$$

where

$$L_q(\Omega,\mu) = \left\{ u : \|u\|_{L_{q,\mu}} = \left(\int_\Omega |u|^q \, \mathrm{d}\mu \right)^{1/q} < \infty \right\}.$$

We say that the measure μ is admissible with respect to K if for every $s > 0$ the trace operator $L^1_p(\Omega^K_s) \to L_q(\Omega^K_s, \mu)$ is compact.

Let us note that for $1 < p < q < \infty$ and $p < n$ the admissibility of μ is equivalent to

$$\lim_{\rho\to 0} \sup_{x\in\Omega^K_s} \rho^{q(p-n)/p} \mu\big(B(x,\rho)\big) = 0$$

(see Theorem 1.4.6/1). In the case $1 < p = n < q < \infty$ the admissibility is equivalent to

$$\lim_{\rho\to 0} \sup_{x\in\Omega^K_s} |\log\rho|^{q(p-1)/p} \mu\big(B(x,\rho)\big) = 0$$

by Theorem 11.9.1/4, which will be proved in the sequel.

We introduce the following modified versions of the localization moduli dealt with previously:

$$\mathcal{M}_1\big(E_{p,q}(\mu), K\big) = \lim_{s\to 0} \sup \left\{ \frac{\|u\|_{L_{q,\mu}}}{\|u\|_{L^1_p}} : u \in L^1_p(\Omega),\ v = 0 \text{ on } \Omega \setminus \overline{\Omega^K_s} \right\}$$

and

$$\mathcal{M}_2\big(E_{p,q}(\mu), K\big) = \lim_{\varrho\to 0} \sup_{x\in K} \sup \left\{ \frac{\|u\|_{L_{q,\mu}}}{\|u\|_{L^1_p}} : u \in L^1_p(\Omega),\ \operatorname{supp} u \subset B(x,\varrho) \right\}.$$

In the proofs of Theorem 8.1/1 and Theorem 8.1/2, we replace Ω_s, $L_q(\Omega)$, $\mathcal{M}_1(E_{p,q})$, and $\mathcal{M}_2(E_{p,q})$ by Ω^K_s, $L_q(\Omega,\mu)$, $\mathcal{M}_1(E_{p,q}(\mu), K)$, and $\mathcal{M}_2(E_{p,q}(\mu), K)$, respectively. Then we use (8.4.1) and the definition of the admissible measure μ to obtain the following theorem.

Theorem. *Let K be a compact subset of $\partial\Omega$ such that $\partial\Omega \setminus K$ is locally Lipschitz, and let μ be an admissible measure with respect to K.*

(i) *Let $1 \le p \le q < pn/(n-p)$ for $n > p$ and let $1 \le q < \infty$ for $p \ge n$. Then*

$$\operatorname{ess}\big\|E_{p,q}(\mu)\big\| = \mathbb{C}\big(E_{p,q}(\mu)\big) = \mathcal{M}_1\big(E_{p,q}(\mu), K\big).$$

(ii) *Let $1 \le p < q < pn/(n-p)$ for $n > p$ and let $1 \le q < \infty$ for $p \ge n$. Then*

$$\operatorname{ess}\big\|E_{p,q}(\mu)\big\| = \mathbb{C}\big(E_{p,q}(\mu)\big) = \mathcal{M}_1\big(E_{p,q}(\mu), K\big) = \mathcal{M}_2\big(E_{p,q}(\mu), K\big).$$

8.5 Measures of Noncompactness for Power Cusp-Shaped Domains

In this section we find explicit values of the measures of noncompactness of the embedding $E_{p,q}$ for power cusp-shaped domains.

Let ω be a bounded Lipschitz domain in $\mathbb{R}^{n-1}$. Consider the β-cusp

$$\Omega = \big\{x = (x', x_n) \in \mathbb{R}^n : x' \in x_n^\beta \omega,\ x_n \in (0,1)\big\},$$

where $\beta > 1$.

Theorem. *Let Ω be the β-cusp with $\beta > 1$. Let $p \in [1, \infty)$ and $\gamma \in \mathbb{R}$. We introduce the measure $\mathrm{d}\mu = x_n^{-\gamma}\,\mathrm{d}x$ and set*

$$q = \frac{(\beta(n-1)+1-\gamma)p}{\beta(n-1)+1-p}. \tag{8.5.1}$$

(i) *Let*

$$-\frac{p(\beta-1)(n-1)}{n-p} < \gamma < p \quad \textit{for } 1 < p < n,$$

or

$$\gamma < p < \beta(n-1)+1 \quad \textit{for } n \le p.$$

Then

$$\begin{aligned}
\operatorname{ess}\big\|E_{p,q}(\mu)\big\| &= \mathbb{C}\big(E_{p,q}(\mu)\big) = \mathcal{M}_1\big(E_{p,q}(\mu)\big) = \mathcal{M}_2\big(E_{p,q}(\mu)\big) \\
&= \big(m_{n-1}(\omega)\big)^{\frac{p-q}{pq}} \left(\frac{\beta(n-1)+1-p}{p-1}\right)^{\frac{1}{p}-\frac{1}{q}-1} \left(\frac{p}{q(p-1)}\right)^{\frac{1}{q}} \\
&\quad \times \left(\frac{\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{q}{q-p})\Gamma(p\frac{q-1}{q-p})}\right)^{\frac{q-p}{pq}}.
\end{aligned}$$

(ii) *Let $1 < p < 1+\beta(n-1)$ and $\gamma = p$. Then*

$$\begin{aligned}
\operatorname{ess}\big\|E_{p,p}(\mu)\big\| &= \mathbb{C}\big(E_{p,p}(\mu)\big) = \mathcal{M}_1\big(E_{p,p}(\mu)\big) \\
&= \mathcal{M}_2\big(E_{p,p}(\mu)\big) = p\big(\beta(n-1)+1-p\big)^{-1}.
\end{aligned}$$

(iii) *Let $p = 1$ and $1-\beta < \gamma < 1$. Then*

$$\begin{aligned}
\operatorname{ess}\big\|E_{1,q}(\mu)\big\| &= \mathbb{C}\big(E_{1,q}(\mu)\big) = \mathcal{M}_1\big(E_{1,q}(\mu)\big) \\
&= \mathcal{M}_2\big(E_{1,q}(\mu)\big) = \big(m_{n-1}(\omega)\big)^{\frac{1-q}{q}} \big(\beta(n-1)+1-\gamma\big)^{-1/q},
\end{aligned}$$

where q is given by (8.5.1) with $p = 1$.

Proof. (i) By definition of $\mathcal{M}_2$ we have

$$\mathcal{M}_2\big(E_{p,q}(\mu)\big) \ge \big(m_{n-1}(\omega)\big)^{\frac{p-q}{pq}} \lim_{\gamma\to 0} K_{p,q}(\rho,\beta,\gamma),$$

where

$$K_{p,q}(\rho,\beta,\gamma) := \sup \frac{(\int_0^\rho |v(t)|^q t^{\beta(n-1)-\gamma}\,\mathrm{d}t)^{1/q}}{(\int_0^\rho |v'(t)|^p t^{\beta(n-1)}\,\mathrm{d}t)^{1/p}},$$

the supremum being taken over Lipschitz functions on $[0,\rho]$ vanishing at ρ.

Making the substitution $t=\tau/\lambda$, with $\lambda>0$, we derive

$$K_{p,q}(\rho,\beta,\gamma)=\sup\frac{(\int_0^{\lambda\rho}|v(\tau)|^q\tau^{\beta(n-1)-\gamma}\,\mathrm{d}\tau)^{1/q}}{(\int_0^{\lambda\rho}|v'(\tau)|^p\tau^{\beta(n-1)}\,\mathrm{d}\tau)^{1/p}},$$

where the supremum is taken over all Lipschitz functions on $[0,\lambda\rho]$ vanishing at $\lambda\rho$. This implies that $K_{p,q}(\rho,\beta,\gamma)$ is constant in ρ and then

$$K_{p,q}(\rho,\beta,\gamma)=\sup\frac{(\int_0^{\infty}|v(t)|^q t^{\beta(n-1)-\gamma}\,\mathrm{d}t)^{1/q}}{(\int_0^{\infty}|v'(t)|^p t^{\beta(n-1)}\,\mathrm{d}t)^{1/p}},$$

where the supremum is extended over all Lipschitz functions with compact support on $[0,\infty)$. By substitution of $t=\tau^c$, where $c=(1-p)(\beta(n-1)+1-p)^{-1}$, we obtain

$$K_{p,q}(\rho,\beta,\gamma)$$
$$=\Big(\frac{\beta(n-1)+1-p}{p-1}\Big)^{\frac{q-p}{pq}-1}\sup\frac{(\int_0^{\infty}|v(\tau)|^q\tau^{(\beta(n-1)-\gamma+1)c-1}\,\mathrm{d}\tau)^{\frac{1}{q}}}{(\int_0^{\infty}|v'(\tau)|^p\,\mathrm{d}\tau)^{\frac{1}{p}}},$$

where the supremum is taken over all Lipschitz functions with compact support on $[0,\infty)$. Since

$$\big(\beta(n-1)-\gamma+1\big)c-1=-1-\frac{q(p-1)}{p},$$

it follows by Bliss's inequality (4.6.4) that

$$K_{p,q}(\rho,\beta,\gamma)$$
$$=\Big(\frac{\beta(n-1)+1-p}{p-1}\Big)^{\frac{q-p}{pq}-1}\Big(\frac{p}{q(p-1)}\Big)^{\frac{1}{q}}\Big(\frac{\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{q}{q-p})\Gamma(p\frac{q-1}{q-p})}\Big)^{\frac{q-p}{pq}},$$

which gives the required lower estimate for the essential norm of $E_{p,q}(\mu)$.

We now derive an upper bound for the essential norm of $E_{p,q}(\mu)$. Let us introduce the mean value of u over the cross section $\{(x',x_n):x'\in x_n^\beta\omega\}$ by

$$\overline{u}(x_n):=\frac{1}{m_{n-1}\omega}\int_{\mathbb{R}^{n-1}}\chi_\omega(z)u\big(x_n^\beta z,x_n\big)\,\mathrm{d}z. \tag{8.5.2}$$

Let $u\in L_p^1(\Omega)$. By the triangle inequality

$$\|u\|_{L_q(\Omega,\mu)}\le\|\overline{u}\|_{L_q(\Omega,\mu)}+\|u-\overline{u}\|_{L_q(\Omega,\mu)}. \tag{8.5.3}$$

We estimate the first term on the right-hand side

$$\begin{aligned}\|\overline{u}\|_{L_q(\Omega,\mu)} &= \left(\int_0^1 \int_{x_n^\beta \omega} \left|\overline{u}(x_n)\right|^q \mathrm{d}x' x_n^{-\gamma}\, \mathrm{d}x_n\right)^{1/q} \\ &= (m_{n-1}\omega)^{1/q} \left(\int_0^1 \left|\overline{u}(x_n)\right|^q x_n^{\beta(n-1)-\gamma}\, \mathrm{d}x_n\right)^{1/q} \\ &\le K_{p,q}(1,\beta,\gamma)(m_{n-1}\omega)^{1/q} \left(\int_0^1 \left|(\overline{u})'(x_n)\right|^p x_n^{\beta(n-1)}\, \mathrm{d}x_n\right)^{1/p}.\end{aligned} \tag{8.5.4}$$

Since

$$\begin{aligned}(\overline{u})'(x_n) &= \frac{\partial}{\partial x_n}\left(\frac{1}{m_{n-1}\omega}\int_{\mathbb{R}^{n-1}} \chi_\omega(z) u\big(z x_n^\beta, x_n\big)\, \mathrm{d}z\right) \\ &= \frac{1}{m_{n-1}\omega}\int_{\mathbb{R}^{n-1}} \chi_\omega(z)\big(\partial_n u + \beta x_n^{\beta-1}(z, \nabla_{x'})u\big)\big(z x_n^\beta, x_n\big)\, \mathrm{d}z,\end{aligned}$$

it follows that

$$\left|(\overline{u})'(x_n)\right| \le \left|(\overline{\partial_n u})(x_n)\right| + (\operatorname{diam}\omega)\beta x_n^{\beta-1}\left|(\overline{\nabla_{x'} u})(x_n)\right|.$$

Using the last inequality, we obtain

$$\begin{aligned}&\left(\int_0^\rho \left|(\overline{u})'(x)\right|^p x^{\beta(n-1)}\, \mathrm{d}x\right)^{1/p} \\ &\quad\le \left(\int_0^\rho \left|(\overline{\partial_n u})(x)\right|^p x^{\beta(n-1)}\, \mathrm{d}x\right)^{1/p} \\ &\qquad + (\operatorname{diam}\omega)\beta\rho^{\beta-1}\left(\int_0^\rho \left|(\overline{\nabla_{x'} u})(x)\right|^p x^{\beta(n-1)}\, \mathrm{d}x\right)^{1/p}.\end{aligned} \tag{8.5.5}$$

Since

$$\begin{aligned}&\left(\int_0^\rho \left|\overline{v}(x_n)\right|^p x_n^{\beta(n-1)}\, \mathrm{d}x_n\right)^{1/p} \\ &\quad\le (m_{n-1}\omega)^{-1/p}\left(\int_0^\rho \int_{x_n^\beta\omega} |v|^p\, \mathrm{d}x'\, \mathrm{d}x_n\right)^{1/p},\end{aligned} \tag{8.5.6}$$

we have for sufficiently small ρ

$$\left(\int_0^\rho \left|(\overline{u})'(x_n)\right|^p x_n^{\beta(n-1)}\, \mathrm{d}x_n\right)^{1/p} \le (m_{n-1}\omega)^{-1/p}\|\nabla u\|_{L_p(\Omega_\rho^0)}, \tag{8.5.7}$$

where $\Omega_\rho^0 = \{x \in \Omega : x_n < \rho\}$. Then, for every $\varepsilon > 0$, there exists a sufficiently small $\rho > 0$ such that

$$\|\overline{u}\|_{L_q(\Omega_\rho^0,\mu)} \le \big(K_{p,q}(1,\beta,\gamma)(m_{n-1}\omega)^{\frac{p-q}{pq}} + \varepsilon\big)\|\nabla u\|_{L_p(\Omega_\rho^0)}. \tag{8.5.8}$$

Let us estimate the second term on the right-hand side of (8.5.3). Consider a sequence $\{z_k\}_{k\ge 0}$ given by

$$z_0 \le 1, \qquad z_{k+1} + z_{k+1}^\beta = z_k, \quad k \ge 0.$$

One can easily verify that $z_k \searrow 0$, $z_{k+1}z_k^{-1} \to 1$. Moreover $z_{k+1}^\beta z_k^{-\beta} \to 1$.

Choosing z_0 to be sufficiently small, we obtain $z_{k+1}/2 < z_k < 2z_{k+1}, k \ge 1$. Let us set

$$C_k = \big\{x = (x', x_n) \in \mathbb{R}^n : x_n \in (z_{k+1}, z_k),\ x' \in x_n^\beta\omega\big\}, \quad k \ge 1.$$

It follows from the construction of C_k that

$$\|u - \overline{u}\|_{L_q(C_k,\mu)} \le 2^{\gamma/q} z_k^{-\gamma/q}\|u - \overline{u}\|_{L_q(C_k)}. \tag{8.5.9}$$

We obtain by Sobolev's theorem that

$$\|u - \overline{u}\|_{L_q(C_k)} \le c z_k^{\beta n(\frac{1}{q}-\frac{1}{p})}\big(\|u - \overline{u}\|_{L_p(C_k)} + z_k^\beta\big\|\nabla(u - \overline{u})\big\|_{L_p(C_k)}\big), \tag{8.5.10}$$

where c depends on ω, n, p, q and is independent of k.

By the Poincaré inequality we have

$$\int_{x_n^\beta\omega}\big|u(x', x_n) - \overline{u}(x', x_n)\big|^p\,\mathrm{d}x' \le c x_n^{\beta p}\int_{x_n^\beta\omega}\big|\nabla_{x'}u(x', x_n)\big|^p\,\mathrm{d}x$$

for almost all $x_n \in (0, 1)$. Hence it follows from (8.5.10) and the previous inequality that

$$\|u - \overline{u}\|_{L_q(C_k)} \le c\, z_k^{\beta(1-\frac{n}{p}+\frac{n}{q})}\big(\|\nabla u\|_{L_p(C_k)} + \|\nabla\overline{u}\|_{L_p(C_k)}\big). \tag{8.5.11}$$

We deduce from (8.5.7) that

$$\|\nabla\overline{u}\|_{L_p(C_k)} = \|\partial_n\overline{u}\|_{L_p(C_k)} \le \|\nabla u\|_{L_p(C_k)}. \tag{8.5.12}$$

Combining (8.5.9), (8.5.11), and (8.5.12) implies

$$\|u - \overline{u}\|_{L_q(C_k,\mu)} \le c z_k^{-\frac{\gamma}{q}+\beta(1-\frac{n}{p}+\frac{n}{q})}\|\nabla u\|_{L_p(C_k)}. \tag{8.5.13}$$

Using (8.5.13) and the inequality

$$\Big(\sum_k a_k^q\Big)^{1/q} \le \Big(\sum_k a_k^p\Big)^{1/p}, \quad a_k \ge 0,\ q \ge p,$$

we conclude

$$\left(\sum_{k=l}^{\infty} \|u - \overline{u}\|^q_{L_q(C_k,\mu)} \right)^{1/q} \le c \left(\sum_{k=l}^{\infty} z_k^{[-\frac{\gamma}{q} + \beta(1 - \frac{n}{p} + \frac{n}{q})]p} \|\nabla u\|^p_{L_p(C_k)} \right)^{1/p}.$$

Since $-\frac{\gamma}{q} + \beta(1 - \frac{n}{p} + \frac{n}{q}) > 0$, it follows that for every $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\|u - \overline{u}\|_{L_q(\Omega^0_\rho,\mu)} \le \varepsilon \|\nabla u\|_{L_p(\Omega^0_\rho)}. \tag{8.5.14}$$

Combining (8.5.3), (8.5.8), and (8.5.14) gives the upper estimate for the essential norm of $E_{p,q}(\mu)$.

(ii) Let us recall the Hardy inequality

$$\int_0^\infty \left(\int_x^\infty f(t)\,\mathrm{d}t \right)^p x^{\beta(n-1)-p}\,\mathrm{d}x \le \left(\frac{p}{\beta(n-1)+1-p} \right)^p \int_0^\infty f^p(x) x^{\beta(n-1)}\,\mathrm{d}x, \tag{8.5.15}$$

where $[p/(\beta(n-1)+1-p)]^p$ is the best constant. Then replacing Bliss's inequality in (i) by Hardy's inequality (8.5.15) with appropriate changes in the proof of (i), we obtain (ii).

(iii) As in (ii) we replace Bliss's inequality by Hardy's inequality

$$\left(\int_0^\infty \left(\int_x^\infty f(t)\,\mathrm{d}t \right)^q x^{\beta(n-1)-\gamma}\,\mathrm{d}x \right)^{1/q} \le c \int_0^\infty f(x) x^{\beta(n-1)}\,\mathrm{d}x,$$

with the best constant

$$c = \big(\beta(n-1)+1-\gamma\big)^{-1/q},$$

and repeat the proof of (i). □

8.6 Finiteness of the Negative Spectrum of a Schrödinger Operator on β-Cusp Domains

Let $\Omega \subset \mathbb{R}^n$ be the β-cusp domain with $\beta > 1$. In this section we study the Neumann problem for the Schrödinger operator

$$\begin{aligned} -\Delta_N u - \frac{\alpha}{x_n^2} u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega \setminus \{0\}, \end{aligned} \tag{8.6.1}$$

where $\alpha = \text{const} > 0$ and ν is a normal. The corresponding quadratic form is given by

$$A_\alpha(u,u) = \int_\Omega |\nabla u|^2\,\mathrm{d}x - \alpha \int_\Omega \frac{|u|^2}{x_n^2}\,\mathrm{d}x. \tag{8.6.2}$$

We need the following Hardy-type inequality for functions in Ω.

Lemma. *If $u \in L_2^1(\Omega)$ and $u = 0$ for $x_n = 1$, then*

$$\int_\Omega |u|^2 x_n^{-2}\,\mathrm{d}x \le c \int_\Omega |\nabla u|^2\,\mathrm{d}x. \tag{8.6.3}$$

Proof. Let $X = (s, t)$, where $s \in \mathbb{R}^{n-1}$, $t \in (0, \infty)$ and

$$s = x_n^{-\beta} x', \qquad t = (\beta - 1)^{-1}\big(x_n^{1-\beta} - 1\big).$$

The change of variable $x \to X$ maps Ω onto the half-cylinder $\{X : s \in \omega, t > 0\}$. Put $v(X) = u(x)$. Since

$$|\nabla_x u| \sim |\nabla_X v| x_n(t)^{-\beta}$$

and the Jacobian $|Dx/DX|$ is equal to $c\, x_n(t)^{\beta n}$, the desired inequality (8.6.3) takes the form

$$\int_\Omega \int_0^\infty \big|v(s,t)\big|^2 x_n(t)^{-2+n\beta}\,\mathrm{d}t\,\mathrm{d}s$$
$$\le c \int_\Omega \int_0^\infty |\nabla_X v|^2 x_n(t)^{(n-2)\beta}\,\mathrm{d}t\,\mathrm{d}s. \tag{8.6.4}$$

Setting here $x_n(t) = (1 + (\beta - 1)t)^{-1/(\beta-1)}$, we see that (8.6.4) is a consequence of the one-dimensional Hardy-type inequality

$$\int_0^\infty |w|^2 (1+t)^{-\frac{n\beta - 2}{\beta - 1}}\,\mathrm{d}t \le c \int_0^\infty (w')^2 (1+t)^{-\frac{(n-2)\beta}{\beta-1}}\,\mathrm{d}t$$

with $w(0) = 0$, which follows from Theorem 1.3.2/1. □

As in Sect. 8.5, we denote $\mathrm{d}\mu = x_n^{-2}\,\mathrm{d}x$. Since, by the lemma just proved, the space $L_2^1(\Omega)$ is continuously embedded into $L_2(\Omega, \mu)$ when $\beta > 1$, A_α is well defined on $L_2^1(\Omega)$.

We start by showing the semiboundedness of A_α which guarantees the existence of the Friedrichs extension of (8.6.1).

Theorem 1. *A_α is semibounded if and only if $\alpha \le [(\beta(n-1) - 1)/2]^2$.*

Proof. Let $\overline{u}$ be the mean value of u over the cross section (see (8.5.2)), then by the triangle inequality

$$\|u\|_{L_2(\Omega,\mu)} \le \|\overline{u}\|_{L_2(\Omega,\mu)} + \|u - \overline{u}\|_{L_2(\Omega,\mu)}. \tag{8.6.5}$$

Combining (8.5.4), (8.5.5), and (8.5.6) (with $p = q = 2$ and $\gamma = -2$), we obtain

$$\|\overline{u}\|_{L_2(\Omega^0_\rho,\mu)} \le \operatorname{ess}\big\|E_{2,2}(\mu)\big\|\big(\|\partial_n u\|_{L_2(\Omega^0_\rho)} \\ + (\operatorname{diam}\omega)\beta\rho^{\beta-1}\|\nabla_{x'}u\|_{L_2(\Omega^0_\rho)}\big), \tag{8.6.6}$$

where $\Omega^0_\rho = \{x = (x', x_n) \in \mathbb{R}^n : x \in \Omega \text{ and } x_n < \rho\}$.

Let us estimate the second term on the right-hand side of (8.6.5)

$$\|u - \overline{u}\|^2_{L_2(\Omega^0_\rho,\mu)} = \int_0^\rho \int_{x_n^\beta\omega} \big|u(x', x_n) - \overline{u}(x_n)\big|^2 \,\mathrm{d}x' \frac{\mathrm{d}x_n}{x_n^2}.$$

Using the Poincaré inequality on the cross line, we see that

$$\|u - \overline{u}\|^2_{L_2(\Omega^0_\rho,\mu)} \le \big(m_{n-1}(\omega)\big)^2 \int_0^\rho \int_{x_n^\beta\omega} \big|\nabla_{x'}u(x', x_n)\big|^2 x_n^{2\beta-2} \,\mathrm{d}x' \,\mathrm{d}x_n \\ \le \rho^{2\beta-2}\big(m_{n-1}(\omega)\big)^2\|\nabla_{x'}u\|^2_{L_2(\Omega^0_\rho)}.$$

Therefore, by (8.6.5) and (8.6.6), we obtain

$$\|u\|_{L_2(\Omega^0_\rho,\mu)} \le \operatorname{ess}\|E_{2,2}(\mu)\|\|\nabla u\|_{L_2(\Omega^0_\rho)}$$

for sufficiently small $\rho > 0$.

Then

$$\|u\|^2_{L_2(\Omega,\mu)} \le \|u\|^2_{L_2(\Omega^0_\rho,\mu)} + \|u\|^2_{L_2(\Omega\setminus\Omega^0_\rho,\mu)} \\ \le \big(\operatorname{ess}\big\|E_{2,2}(\mu)\big\|\big)^2\|\nabla u\|^2_{L_2(\Omega^0_\rho)} + c(\rho)\|u\|^2_{L_2(\Omega\setminus\Omega^0_\rho)} \\ \le \big(2/\big(\beta(n-1)-1\big)\big)^2\|\nabla u\|^2_{L_2(\Omega^0_\rho)} + c(\rho)\|u\|^2_{L_2(\Omega)}, \tag{8.6.7}$$

which gives the semiboundedness for $\alpha \le [(\beta(n-1)-1)/2]^2$.

Let $\alpha > [(\beta(n-1)-1)/2]^2$. We set $d = \alpha - [(\beta(n-1)-1)/2]^2 > 0$. It follows from Theorem 8.5 that

$$\lim_{\varrho\to 0} \sup\left\{\frac{\int \frac{|u|^2}{x_n^2}\,\mathrm{d}x}{\int|\nabla u|^2\,\mathrm{d}x} : u \in L^1_p(\Omega),\ \operatorname{supp} u \subset \Omega^0_\rho\right\} = \big[2/\big(\beta(n-1)-1\big)\big]^2.$$

Then there exists a sequence of functions $\{u_i\}_{i=1}^\infty$ such that $\operatorname{supp} u_i \subset \Omega^0_{1/i}$ and

$$\left(\alpha - \frac{d}{2}\right)\int_\Omega \frac{|u_i|^2}{x_n^2}\,\mathrm{d}x > \int_\Omega |\nabla u_i|^2\,\mathrm{d}x.$$

Hence,

$$\int_\Omega |\nabla u_i|^2\,\mathrm{d}x - \alpha\int_\Omega \frac{|u_i|^2}{x_n^2}\,\mathrm{d}x < -\frac{d}{2}\int_\Omega \frac{|u_i|^2}{x_n^2}\,\mathrm{d}x \\ \le -\frac{i^2 d}{2}\int_\Omega |u_i|^2\,\mathrm{d}x.$$

Therefore A_α is not semibounded when $\alpha > [(\beta(n-1)-1)/2]^2$. □

The next theorem gives a condition for finiteness of the negative spectrum of A_α.

Theorem 2. *If* $\alpha < [(\beta(n-1)-1)/2]^2$, *then the negative spectrum of* A_α *is finite.*

Proof. Let $\alpha < [(\beta(n-1)-1)/2]^2$ and M be a linear infinite-dimensional manifold in $L_2^1(\Omega)$. Take

$$\varepsilon < \frac{1-\alpha[2/(\beta(n-1)-1)]^2}{(2+2\alpha)}. \tag{8.6.8}$$

It follows from (8.6.7) that

$$\|u\|^2_{L_2(\Omega,\mu)} \le \big(2/\big(\beta(n-1)-1\big)\big)^2 \|\nabla u\|^2_{L_2(\Omega^0_\rho)} + c(\rho)\|u\|^2_{L_2(\Omega\setminus\Omega^0_\rho)}$$

for sufficiently small $\rho > 0$, where $\Omega^0_\rho = \{x = (x', x_n) \in \mathbb{R}^n : x \in \Omega$ and $x_n < \rho\}$.

Since the restriction $L_2^1(\Omega) \to L_2(\Omega\setminus\Omega^0_\rho)$ is compact, there exists a finite rank operator $F : L_2^1(\Omega) \to L_2(\Omega\setminus\Omega^0_\rho)$, for which

$$\|u - Fu\|_{L_2(\Omega\setminus\Omega^0_\rho)} \le \left(\frac{\varepsilon}{c(\rho)}\right)^{1/2} \|u\|_{L_2^1(\Omega)}.$$

Note that

$$\begin{aligned}\|u\|^2_{L_2(\Omega,\mu)} \le{}& \big(2/\big(\beta(n-1)-1\big)\big)^2 \|\nabla u\|^2_{L_2(\Omega^0_\rho)}\\ &+ c(\rho)\left(\left(\frac{\varepsilon}{c(\rho)}\right)^{1/2}\|u\|_{L_2^1(\Omega)} + \|Fu\|_{L_2(\Omega\setminus\Omega^0_\rho)}\right)^2.\end{aligned}$$

Let $M^\perp \subset M$ be defined by

$$M^\perp := \{u : Fu = 0 \text{ and } u \in M\}.$$

Then $M^\perp$ is a linear infinite-dimensional manifold in $L_2^1(\Omega)$ and for every $u \in M^\perp$ we have

$$\begin{aligned}\|u\|^2_{L_2(\Omega,\mu)} &\le \big(2/\big(\beta(n-1)-1\big)\big)^2 \|\nabla u\|^2_{L_2(\Omega^0_\rho)} + \varepsilon\|u\|^2_{L_2^1(\Omega)}\\ &\le \big[\big(2/\big(\beta(n-1)-1\big)\big)^2 + 2\varepsilon\big]\|\nabla u\|^2_{L_2(\Omega)} + 2\varepsilon\|u\|^2_{L_2(\Omega)}.\end{aligned} \tag{8.6.9}$$

Combining (8.6.2), (8.6.8), and (8.6.9), we obtain for each $u \in M^\perp \subset M$

$$\begin{aligned}A_\alpha(u,u) &= \int_\Omega |\nabla u|^2\,\mathrm{d}x - \alpha\int_\Omega \frac{|u|^2}{x_n^2}\,\mathrm{d}x\\ &\ge \big[\big(2/\big(\beta(n-1)-1\big)\big)^2 + 2\varepsilon\big]^{-1}\int_\Omega \frac{|u|^2}{x_n^2}\,\mathrm{d}x - \alpha\int_\Omega \frac{|u|^2}{x_n^2}\,\mathrm{d}x\\ &\quad - 2\varepsilon\big[\big(2/\big(\beta(n-1)-1\big)\big)^2 + 2\varepsilon\big]^{-1}\int_\Omega |u|^2\,\mathrm{d}x > 0.\end{aligned}$$

Therefore, there does not exist a linear manifold of infinite dimension on which $A_\alpha(u,u) < 0$. This together with Lemma 2.5.4/2 completes the proof. □

8.7 Relations of Measures of Noncompactness with Local Isoconductivity and Isoperimetric Constants

Let E and F denote arbitrary relatively closed disjoint subsets of Ω. We use the p-conductivity of the conductor $K = (\Omega\backslash F)\backslash E$, that is,

$$c_p(K) = \inf\big\{\|\nabla u\|^p_{L_p} : u \ge 1 \text{ on } E \text{ and } u = 0 \text{ on } F, u \text{ is locally Lipschitz in } \Omega\big\}$$

(this definition is equivalent to the one given in Sect. 6.1.1) and we define the local isoconductivity constants

$$S(p,q,\mu,\Omega) = \lim_{s\to 0} \sup\bigg\{\frac{(\mu(E))^{1/q}}{(c_p((\Omega\backslash\overline{\Omega_s})\backslash E))^{1/p}} : E \subset \Omega \setminus \overline{\Omega_s}\bigg\}$$

and

$$\widetilde{S}(p,q,\mu,\Omega) = \lim_{\rho\to 0} \sup_{x\in\partial\Omega} \sup\bigg\{\frac{(\mu(E))^{1/q}}{(c_p((\Omega\cap B(x,\varrho))\backslash E))^{1/p}} : E \subset \Omega\cap B(x,\rho)\bigg\}.$$

Theorem 1. *Let $1 \le p \le q < pn/(n-p)$ if $n > p$ and $1 \le q < \infty$ if $p \ge n$. Then*

$$S(p,q,\mu,\Omega) \le \operatorname{ess}\big\|E_{p,q}(\mu)\big\| \le K(p,q) S(p,q,\mu,\Omega). \tag{8.7.1}$$

If additionally $p < q$, then

$$\widetilde{S}(p,q,\mu,\Omega) \le \operatorname{ess}\big\|E_{p,q}(\mu)\big\| \le K(p,q) \widetilde{S}(p,q,\mu,\Omega), \tag{8.7.2}$$

where

$$K(p,q) = \begin{cases} \Big(\dfrac{\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{q}{q-p})\Gamma(p\frac{q-1}{q-p})}\Big)^{(q-p)/pq}, & \text{when } 1 < p < q;\\ (p-1)^{(1-p)/p}, & \text{when } 1 < p = q;\\ 1, & \text{when } 1 = p \le q. \end{cases}$$

Proof. The left-hand side inequality in (8.7.1) follows immediately from the definition of $\mathcal{M}_1$.

The right-hand side inequality in (8.7.1) is a consequence of the conductor inequality

$$\bigg(\int_0^\infty \big(c_p\big((\Omega\backslash F)\backslash \mathcal{N}_t\big)\big)^{q/p} \,\mathrm{d}(t^q)\bigg)^{1/q} \le K(p,q)\bigg(\int_\Omega |\nabla u|^p \,\mathrm{d}x\bigg)^{1/p},$$

where $u = 0$ on F which is a relatively compact subset of Ω (see Lemma 6.1.3/3). The inequality (8.7.2) follows by the same arguments when $\mathcal{M}_1$ is replaced by $\mathcal{M}_2$. □

In the case $p = 1$ the p-capacity can be replaced by the $(n-1)$-dimensional area $\mathbf{H}_{n-1}$. By g we denote relatively closed subsets of Ω such that $\Omega \cap \partial g$ are smooth surfaces. We introduce the local isoperimetric constants

$$T(q,\mu,\Omega) = \lim_{s\to 0} \sup\left\{ \frac{(\mu(g))^{1/q}}{\mathbf{H}_{n-1}(\Omega\cap\partial g)} : g \subset \Omega \setminus \overline{\Omega_s} \right\}$$

and

$$\widetilde{T}(q,\mu,\Omega) = \lim_{s\to 0} \sup_{x\in\partial\Omega} \sup\left\{ \frac{(\mu(g))^{1/q}}{\mathbf{H}_{n-1}(\Omega\cap\partial g)} : g \subset \Omega \cap B(x,\rho) \right\}.$$

Theorem 2. *If* $1 < q < n/(n-1)$*, then*

$$T(q,\mu,\Omega) = \widetilde{T}(q,\mu,\Omega) = \operatorname{ess}\big\|E_{1,q}(\mu)\big\|,$$

and if $q = 1$*, then*

$$T(q,\mu,\Omega) = \operatorname{ess}\big\|E_{1,q}(\mu)\big\|.$$

Proof. It follows from Lemma 5.2.3/1 that $\mathcal{M}_1(E_{1,q}) = T(q,\Omega)$, which together with Theorem 8.1/1 completes the proof. □

In view of Theorem 8.1/1 and Theorem 8.1/2, the role of the essential norm of $E_{p,q}(\mu)$ can be played by $\mathcal{M}_1(E_{p,q}(\mu))$ and $\mathbb{C}(E_{p,q}(\mu))$ and additionally by $\mathcal{M}_2(E_{p,q}(\mu))$ when $p < q$.

The following corollary, which is an immediate consequence of Theorem 8.5 (iii) and the previous theorem, gives explicit values of the local isoperimetric constants for power cusps.

Corollary. *Let* $\Omega \subset \mathbb{R}^n$ *be the* β*-cusp domain with* $\beta > 1$ *and* $\gamma \in (1-\beta, 1)$*. We introduce the measure* $d\mu = x_n^{-\gamma}\,\mathrm{d}x$ *and set*

$$q = \frac{\beta(n-1)+1-\gamma}{\beta(n-1)}.$$

Then

$$T(q,\mu,\Omega) = \big(m_{n-1}(\omega)\big)^{\frac{1-q}{q}} \big(\beta(n-1)+1-\gamma\big)^{-1/q}.$$

8.8 Comments to Chap. 8

The material of this chapter is borrowed from the paper by Lang and Maz'ya [478]. Lemma 8.6 is a particular case of Lemma 5.1.3/3 in Maz'ya and Poborchi [576].

Various characteristics of noncompact embeddings such as essential norms, limits of the approximation numbers, certain measures of noncompactness, and the constants in the Poincaré-type inequalities, were investigated by Amick [45], Edmunds and Evans [248], Evans and Harris [261] (see Edmunds and Evans [249] and [250] for a detailed account of this development), and Yerzakova [806, 807].

9

Space of Functions of Bounded Variation

In the 1960s the family of spaces of differentiable functions was joined by the space $BV(\Omega)$ of functions whose derivatives are measures in the open set $\Omega \subset \mathbb{R}^n$. This space turned out to be useful in geometric measure theory, the calculus of variations, and the theory of quasilinear partial differential equations. In the present chapter we study the connection between the properties of functions in $BV(\Omega)$ and geometric characteristics of the boundary of Ω.

Sections 9.1 and 9.2 contain well-known basic facts of the theory of $BV(\Omega)$. In Sect. 9.3 we find requirements on Ω in terms of relative isoperimetric inequalities that are necessary and sufficient for any function in $BV(\Omega)$ to admit an extension to $\mathbb{R}^n$ in the class $BV(\Omega)$ and for the boundedness of the extension operator. Some explicit formulas for the norm of the optimal extension operator are found in Sect. 9.4.

We define the notion of a trace on the boundary for a function in $BV(\Omega)$ and give conditions for the integrability of the trace (Sects. 9.5 and 9.6). Some results on the relation between the relative isoperimetric inequalities and the integral inequalities (of the embedding theorem type) for $BV(\Omega)$ similar to those in Chap. 5 are established in Sect. 9.5. We also consider conditions for the validity of the Gauss–Green formula for functions in $BV(\Omega)$ (Sect. 9.6).

9.1 Properties of the Set Perimeter and Functions in $BV(\Omega)$

9.1.1 Definitions of the Space $BV(\Omega)$ and of the Relative Perimeter

The space of functions u that are locally integrable in Ω, whose gradients $\nabla_\Omega u$ (in the sense of distribution theory) are charges in Ω, is called the space $BV(\Omega)$. We denote the variation of the charge over the whole domain Ω by $\|u\|_{BV(\Omega)}$. The perimeter of a set $\mathscr{E}$ relative to Ω is defined by

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$$P_\Omega(\mathscr{E}) = \|\chi_{\mathscr{E}\cap\Omega}\|_{BV(\Omega)}, \tag{9.1.1}$$

where $\chi_{\mathscr{A}}$ is the characteristic function of the set $\mathscr{A}$. (We put $P_\Omega(\mathscr{E}) = \infty$ provided $\chi_{\mathscr{E}\cap\Omega} \notin BV(\Omega)$.)

We introduce the perimeter of $\mathscr{E}$ relative to the closed set $C\Omega = \mathbb{R}^n\backslash\Omega$. Namely, $P_{C\Omega}(\mathscr{E}) = \inf_{G\supset C\Omega} P_G(\mathscr{E})$, where G is an open set.

If $P_{\mathbb{R}^n}(\mathscr{E}) < \infty$, then obviously

$$\begin{aligned} &P_\Omega(\mathscr{E}) = \operatorname{var}\nabla_{\mathbb{R}^n}\chi_{\mathscr{E}}(\Omega), \qquad P_{C\Omega}(\mathscr{E}) = \operatorname{var}\nabla_{\mathbb{R}^n}\chi_{\mathscr{E}}(C\Omega),\\ &P_\Omega(\mathscr{E}) + P_{C\Omega}(\mathscr{E}) = P_{\mathbb{R}^n}(\mathscr{E}). \end{aligned} \tag{9.1.2}$$

We also note that

$$P_\Omega(\mathscr{E}_1) = P_\Omega(\mathscr{E}_2) \tag{9.1.3}$$

if $\mathscr{E}_1 \cap \Omega = \mathscr{E}_2 \cap \Omega$.

Henceforth, in the cases where it causes no ambiguity, we write ∇u instead of $\nabla_\Omega u$, $\nabla_{\mathbb{R}^n} u$ and $P(\mathscr{E})$ instead of $P_{\mathbb{R}^n}(\mathscr{E})$.

9.1.2 Approximation of Functions in $BV(\Omega)$

The definition of the mollification operator $\mathscr{M}_h$ (cf. Sect. 1.1.3) immediately implies the following lemma.

Lemma 1. *If $u \in BV(\Omega)$ and G is an open subset of Ω with $[G]_h \subset \Omega$, where $[G]_h$ is the h-neighborhood of G, then*

$$\|\nabla\mathscr{M}_h u\|_{L_1(\Omega)} \le \operatorname{var}\nabla u\big([G]_h\big). \tag{9.1.4}$$

Lemma 2. *If $u_i \in BV(\Omega)$ and $u_i \to u$ in $L(\Omega, \mathrm{loc})$, then*

$$\|u\|_{BV(\Omega)} \le \liminf_{i\to\infty} \|u_i\|_{BV(\Omega)}. \tag{9.1.5}$$

Proof. It suffices to consider the case

$$\liminf_{i\to\infty} \|u_i\|_{BV(\Omega)} < \infty.$$

For any vector-function φ with components in $\mathscr{D}(\Omega)$ we have

$$\int_\Omega \varphi\nabla u_i\,\mathrm{d}x = -\int_\Omega u_i \operatorname{div}\varphi\,\mathrm{d}x \to -\int_\Omega u\operatorname{div}\varphi\,\mathrm{d}x. \tag{9.1.6}$$

Therefore,

$$\left|\int_\Omega u\operatorname{div}\varphi\,\mathrm{d}x\right| \le \sup|\varphi| \liminf_{i\to\infty}\|u_i\|_{BV(\Omega)},$$

i.e., $\nabla_\Omega u$ is a charge and inequality (9.1.5) holds.

By definition, the sequence $\{\mu_k\}_{k\geq 1}$ of finite charges converges to a charge μ locally weakly in Ω if, for any function $\varphi \in C(\Omega)$ with compact support,

$$\lim_{k\to\infty} \int_\Omega \varphi\mu_k\,(\mathrm{d}x) = \int_\Omega \varphi\mu\,(\mathrm{d}x).$$

Lemma 3. *If $u_i \in BV(\Omega)$, $u_i \to u$ in $L_1(\Omega, \mathrm{loc})$, and $\sup_i \|u\|_{BV(\Omega)} < \infty$, then*

$$\nabla_\Omega u_i \xrightarrow{\text{loc. weak.}} \nabla_\Omega u.$$

Proof. By Lemma 2 we have $u \in BV(\Omega)$. It remains to refer to (9.1.6) and to use the density of $\mathscr{D}(\Omega)$ in the space of continuous functions with compact supports in Ω. □

Theorem. *Let $u \in BV(\Omega)$. Then there exists a sequence of functions $\{u_m\}_{m\geq 1}$ in $C^\infty(\Omega)$ such that $u_m \to u$ in $L_1(\Omega, \mathrm{loc})$ and*

$$\lim_{m\to\infty} \int_\Omega |\nabla u_m|\,\mathrm{d}x = \|u\|_{BV(\Omega)}.$$

If, in addition, $u \in L_1(\Omega)$ then $u_m \to u$ in $L_1(\Omega)$.

Proof. Let $\{\Omega_i\}_{i\geq 0}$ be a sequence of open sets with compact closures $\bar{\Omega}_i \subset \Omega_{i+1}$ and such that $\bigcup_i \Omega_i = \Omega$, $\Omega_0 = \varnothing$. We choose Ω_i so that

$$\operatorname{var} \nabla u\Big(\bigcup_i \partial\Omega_i\Big) = 0.$$

This can be done, for instance, in the following way. Let $\{\Omega_i'\}$ be an arbitrary sequence of open sets with compact closures such that $\bigcup_i \Omega_i' = \Omega$, $\bar{\Omega}_i' \subset \Omega$, $\partial\Omega_i' \cap \partial\Omega_j' = \varnothing$ for $i \neq j$, $\partial\Omega_i' \in C^\infty$. Let Ω_i^t denote the t-neighborhood of Ω_i'. For small t the surface $\partial\Omega_i^t$ is smooth. The set of t for which $\operatorname{var} \nabla u(\partial\Omega_i^t) \neq 0$ is finite or countable as the collection of jumps of a monotonic function. Therefore, for any i we can find an arbitrarily small number t_i such that $\operatorname{var} \nabla u(\partial\Omega_i^{t_i}) = 0$. It remains to put $\Omega_i = \Omega_i^{t_i}$.

We fix a positive integer m and choose bounded open sets D_i, G_i such that $G_i \supset \bar{D}_i \supset (\bar{\Omega}_{i+1}\backslash\Omega_i)$ and

$$\sum_i \operatorname{var} \nabla u\big(G_i\backslash(\bar{\Omega}_{i+1}\backslash\Omega_i)\big) < m^{-1}.$$

Let $\{\alpha_i\}$ be a partition of unity subordinate to the covering $\{D_i\}$ of Ω. We can find small numbers $h_i > 0$ so that $\bar{G}_i \subset [D_i]_{h_i}$ and

$$\|u - \mathscr{M}_{h_i} u\|_{L_1(D_i)} \max|\nabla\alpha_i| \leq m^{-1}2^{-i}. \tag{9.1.7}$$

We put $u_m = \sum \alpha_i \mathscr{M}_{h_i} u$. Then $u_m \to u$ in $L_1(\Omega, \mathrm{loc})$. Also,

$$\|\nabla u_m\|_{L_1(\Omega)} \le \sum_i \|\alpha_i \nabla \mathscr{M}_{h_i} u\|_{L_1(\Omega)} + \Big\|\sum_i \mathscr{M}_{h_i} u \nabla \alpha_i\Big\|_{L_1(\Omega)}. \tag{9.1.8}$$

Since $\operatorname{supp} \alpha_i \subset D_i$, by Lemma 1 the first term does not exceed

$$\sum_i \|\nabla \mathscr{M}_{h_i} u\|_{L_1(D_i)} \le \sum_i \operatorname{var} \nabla u(G_i) \le \sum_i \operatorname{var} \nabla u(\Omega) + m^{-1}.$$

By the equality $\sum_i \nabla \alpha_i = 0$ and (9.1.7), for the second summand on the right in (9.1.8) we have

$$\begin{aligned} \Big\|\sum_i \mathscr{M}_{h_i} u \nabla \alpha_i\Big\|_{L_1(\Omega)} &= \Big\|\sum_i (\mathscr{M}_{h_i} u - u) \nabla \alpha_i\Big\|_{L_1(\Omega)} \\ &\le \sum_i \|\mathscr{M}_{h_i} u - u\|_{L_1(D_i)} \max_{\bar{D}_i} |\nabla \alpha_i| \\ &\le \sum_{i=1}^{\infty} m^{-1} 2^{-i} = m^{-1}. \end{aligned}$$

Consequently,

$$\|\nabla u_m\|_{L_1(\Omega)} \le \operatorname{var} \nabla u(\Omega) + 2m^{-1}$$

and

$$\limsup_{m\to\infty} \|\nabla u_m\|_{L_1(\Omega)} \le \operatorname{var} \nabla u(\Omega).$$

The result follows from the last inequality together with Lemma 2. □

Corollary. *If $u \in BV(\Omega)$, then the functions u^+, u^-, and $|u|$ are also contained in $BV(\Omega)$ and*

$$\|u^+\|_{BV(\Omega)} + \|u^-\|_{BV(\Omega)} = \|u\|_{BV(\Omega)} \ge \big\||u|\big\|_{BV(\Omega)}. \tag{9.1.9}$$

In fact, let $\{u_m\}$ be the sequence of functions in $C^\infty(\Omega)$ introduced in the Theorem. Then $u_m^+ \to u^+$, $u_m^- \to u^-$ in $L_1(\Omega, \mathrm{loc})$ and

$$\|u_m^+\|_{BV(\Omega)} + \|u_m^-\|_{BV(\Omega)} = \big\||u_m|\big\|_{BV(\Omega)} = \|u_m\|_{BV(\Omega)}. \tag{9.1.10}$$

This and Lemma 9.1.2/1 imply

$$\|u^+\|_{BV(\Omega)} + \|u^-\|_{BV(\Omega)} \le \|u\|_{BV(\Omega)}. \tag{9.1.11}$$

Consequently, u^+, $u^- \in BV(\Omega)$. The reverse inequality of (9.1.11) is obvious. Passing to the limit in (9.1.10), we arrive at the inequality in (9.1.9).

9.1.3 Approximation of Sets with Finite Perimeter

By definition, the sequence $\{\mathscr{E}_i\}$ of sets $\mathscr{E}_i \subset \Omega$ converges to $\mathscr{E} \subset \Omega$ if $\chi_{\mathscr{E}_i} \to \chi_{\mathscr{E}}$ in $L_1(\Omega, \mathrm{loc})$.

Lemmas 9.1.2/2 and 9.1.2/3 imply the next assertions.

Lemma 1. *If $\mathscr{E}_k \to \mathscr{E}$, then $P_\Omega(\mathscr{E}) \le \liminf_{k\to\infty} P_\Omega(\mathscr{E}_k)$ and if $\sup_k P_\Omega(\mathscr{E}_k) < \infty$, then*

$$\nabla_\Omega \chi_{\mathscr{E}_k} \xrightarrow{\text{loc. weak.}} \nabla_\Omega \chi_{\mathscr{E}}.$$

Lemma 2. *Let $u \in L_1(\Omega)$ and let $\chi_{\mathscr{E}}$ be the characteristic function of the set $\mathscr{E} \subset \Omega$. Further let $\|\chi_{\mathscr{E}} - u\|_{L_1(\Omega)} \le \varepsilon$. Then for any $t \in [\gamma, 1-\gamma]$, $\gamma > 0$, the inequality*

$$\|\chi_{\mathscr{E}} - \chi_{\mathscr{N}_t}\|_{L_1(\Omega)} \le \varepsilon\gamma^{-1}$$

holds where $\mathscr{N}_t = \{x : u(x) \ge t\}$.

Proof. Obviously,

$$\varepsilon \ge \|\chi_{\mathscr{E}} - u\|_{L_1(\mathscr{E}\backslash\mathscr{N}_t)} + \|u - \chi_{\mathscr{E}}\|_{L_1(\mathscr{N}_t\backslash\mathscr{E})} \ge \int_{\mathscr{E}\backslash\mathscr{N}_t} \big(1 - u(x)\big)\,\mathrm{d}x + \int_{\mathscr{N}_t\backslash\mathscr{E}} u(x)\,\mathrm{d}x.$$

Since $u(x) < t$ for $x \in \mathscr{E}\backslash\mathscr{N}_t$ and $u(x) \ge t$ for $x \in \mathscr{N}_t\backslash\mathscr{E}$, we obtain

$$\varepsilon \ge (1-t)m_n(\mathscr{E}\backslash\mathscr{N}_t) + t m_n(\mathscr{N}_t\backslash\mathscr{E}) \ge \gamma\|\chi_{\mathscr{E}} - \chi_{\mathscr{N}_t}\|_{L_1(\Omega)}. \qquad \square$$

Theorem. *For any measurable set $\mathscr{E} \subset \Omega$ having finite measure m_n there exists a sequence of sets $\mathscr{E}_i \subset \Omega$ for which $\partial\mathscr{E}_i\backslash\partial\Omega$ is a C^∞-smooth submanifold of $\mathbb{R}^n$ (however, not compact, in general). Moreover, $\chi_{\mathscr{E}_i} \to \chi_{\mathscr{E}}$ in $L_1(\Omega)$ and $P_\Omega(\mathscr{E}_i) \to P_\Omega(\mathscr{E})$.*

Proof. If $P(\mathscr{E}) = \infty$, then the result follows. Let $P(\mathscr{E}) < \infty$. Let u_m denote the sequence constructed in Theorem 9.1.2 for $u = \chi_{\mathscr{E}}$. Since $0 \le \chi_{\mathscr{E}} \le 1$, the definition of u_m implies $0 \le u_m \le 1$. Therefore, by Theorem 1.2.4 we have

$$\|\nabla u_m\|_{L_1(\Omega)} = \int_0^1 s\big(\mathscr{E}_t^{(m)}\big)\,\mathrm{d}t, \tag{9.1.12}$$

where $\mathscr{E}_t^{(m)} = \{x : u_m(x) = t\}$. The sets $\mathscr{E}_t^{(m)}$ are C^∞-manifolds for almost all $t \in (0,1)$ (see Corollary 1.2.2). In what follows we consider only such levels t.

Let $\varepsilon > 0$. We choose $m = m(\varepsilon)$ to be so large that

$$\|\chi_{\mathscr{E}} - u_m\|_{L_1(\Omega)} < \varepsilon.$$

Then by Lemma 1,

$$\|\chi_{\mathscr{E}} - \chi_{\mathscr{N}_t^{(m)}}\|_{L_1(\Omega)} \le \varepsilon^{1/2}, \tag{9.1.13}$$

where $\mathscr{N}_t^{(m)} = \{x : u_m(t) \geq t\}$ and $t \in [\varepsilon^{1/2}, 1-\varepsilon^{1/2}]$. Furthermore, for any m there exists a $t = t_m \in (\varepsilon^{1/2}, 1-\varepsilon^{1/2})$ such that

$$(1 - 2\varepsilon^{1/2}) s\big(\mathscr{E}_{t_m}^{(m)}\big) \leq \int_0^1 s\big(\mathscr{E}_t^{(m)}\big)\, \mathrm{d}t. \tag{9.1.14}$$

Inequalities (9.1.13) and (9.1.14), together with the equality

$$P_\Omega(\mathscr{E}) = \lim_{m\to\infty} \int_0^1 s\big(\mathscr{E}_t^{(m)}\big)\, \mathrm{d}t,$$

which follows from (9.1.12) and Theorem 9.1.2, imply $\chi_{\mathscr{N}_{t_m}^{(m)}} \to \chi_{\mathscr{E}}$ in $L_1(\Omega)$ and

$$\limsup_{\varepsilon\to 0} s\big(\mathscr{E}_{t_m}^{(m)}\big) \leq P_\Omega(\mathscr{E}).$$

□

Remark. If $\mathscr{E}$ is a set with compact closure $\bar{\mathscr{E}} \subset \Omega$, then the smooth manifolds constructed in the preceding theorem are compact.

9.1.4 Compactness of the Family of Sets with Uniformly Bounded Relative Perimeters

Theorem. *The collection of sets $\mathscr{E}_\alpha \subset \Omega$ with uniformly bounded relative perimeters $P_\Omega(\mathscr{E}_\alpha)$ is compact.*

Proof. By Theorem 9.1.2, for any $\mathscr{E}_\alpha$ there exists a sequence u_{α_m} that converges to $\chi_{\mathscr{E}_\alpha}$ in $L_1(\Omega, \mathrm{loc})$ and is such that

$$\lim_{m\to\infty} \|\nabla u_{\alpha_m}\|_{L_1(\Omega)} = P_\Omega(\mathscr{E}_\alpha).$$

Lemma 1.4.6 implies that the family $\{u_{\alpha_m}\}$ is compact in $L_1(\omega)$ where ω is an arbitrary open set with compact closure $\bar{\omega} \subset \Omega$ and with a smooth boundary. Therefore, the family $\{\chi_{\mathscr{E}_\alpha}\}$ is compact in $L_1(\omega)$. □

9.1.5 Isoperimetric Inequality

Theorem. *If $\mathscr{E}$ is a measurable subset of $\mathbb{R}^n$ and $m_n(\Omega) < \infty$, then*

$$m_n(\mathscr{E})^{(n-1)/n} \leq n^{-1} v_n^{-1/n} P(\mathscr{E}). \tag{9.1.15}$$

Proof. It suffices to consider the case $P(\mathscr{E}) < \infty$. By Theorem 9.1.3 there exists a sequence of open sets $\mathscr{E}_i$ with C^∞-smooth boundaries $\partial\mathscr{E}_i$ such that $m_n(\mathscr{E}_i) \to m_n(\mathscr{E})$ and $s(\partial\mathscr{E}_i) \to P(\mathscr{E})$, where s is the $(n-1)$-dimensional area. Inequality (9.1.15) is valid for the sets $\mathscr{E}_i$ (cf. Lyusternik [507] et al.). Passing to the limit we arrive at (9.1.15).

9.1.6 Integral Formula for the Norm in $BV(\Omega)$

Lemma. *If u_1 and u_2 are nonnegative functions in $L_1(\Omega)$ then*

$$\int_\Omega |u_1 - u_2|\,\mathrm{d}x = \int_0^\infty m_n\big((\mathscr{L}_t^1\backslash\mathscr{L}_t^2) \cup (\mathscr{L}_t^2\backslash\mathscr{L}_t^1)\big)\,\mathrm{d}t,$$

where $\mathscr{L}_t^i = \{x : x \in \Omega, u_i(x) > t\}$.

Proof. It is clear that

$$\int_\Omega |u_1 - u_2|\,\mathrm{d}x = \int_A (u_1 - u_2)\,\mathrm{d}x + \int_{\Omega\backslash A} (u_2 - u_1)\,\mathrm{d}x = \mathscr{I}_1 + \mathscr{I}_2,$$

where $A = \{x \in \Omega : u_1 > u_2\}$. By Lemma 1.2.3,

$$\mathscr{I}_1 = \int_0^\infty m_n\big(\mathscr{L}_t^1 \cap A\big)\,\mathrm{d}t - \int_0^\infty m_n\big(\mathscr{L}_t^2 \cap A\big)\,\mathrm{d}t.$$

We note that $u_1(x) > u_2(x)$ if $x \in \mathscr{L}_t^1\backslash\mathscr{L}_t^2$. Therefore, $(\mathscr{L}_t^1\backslash\mathscr{L}_t^2) \cap A = \mathscr{L}_t^1\backslash\mathscr{L}_t^2$ and, similarly, $(\mathscr{L}_t^2\backslash\mathscr{L}_t^1) \cap (\Omega\backslash A) = \mathscr{L}_t^2\backslash\mathscr{L}_t^1$. Hence

$$\mathscr{I}_1 = \int_0^\infty m_n\big(\mathscr{L}_t^1\backslash\mathscr{L}_t^2\big)\,\mathrm{d}t.$$

In the same way we obtain

$$\mathscr{I}_2 = \int_0^\infty m_n\big(\mathscr{L}_t^2\backslash\mathscr{L}_t^1\big)\,\mathrm{d}t.$$

This completes the proof. □

Theorem. *For any function u that is locally integrable in Ω we have*

$$\|u\|_{BV(\Omega)} = \int_{-\infty}^{+\infty} P_\Omega(\mathscr{L}_t)\,\mathrm{d}t, \tag{9.1.16}$$

where $\mathscr{L}_t = \{x : u(x) > t\}$.

Proof. By Corollary 9.1.2 we may assume $u \geq 0$. According to Lemma 1.2.3, for any smooth vector-function φ with compact support in Ω,

$$\int_\Omega u \operatorname{div}\varphi\,\mathrm{d}x = \int_0^\infty \mathrm{d}t\bigg(\int_\Omega \chi_{\mathscr{L}_t} \operatorname{div}\varphi\,\mathrm{d}x\bigg).$$

Therefore,

$$\left|\int_\Omega u \operatorname{div}\varphi\,\mathrm{d}x\right| \leq \max|\varphi| \,{}_*\!\!\int_0^\infty P_\Omega(\mathscr{L}_t)\,\mathrm{d}t,$$

where $\int_*$ is the lower Lebesgue integral. Hence

$$\|u\|_{BV(\Omega)} \leq {}_*\!\int_0^\infty P_\Omega(\mathscr{L}_t)\,\mathrm{d}t. \tag{9.1.17}$$

If $\|u\|_{BV(\Omega)} = \infty$ then (9.1.16) follows. Let $u \in BV(\Omega)$. Consider the sequence $\{u_m\}$ constructed in the proof of Theorem 9.1.2. Note that $u_m \geq 0$. Let $\{\omega_i\}$ be a sequence of open sets ω_i with compact closures $\bar{\omega}_i \subset \Omega$ and such that $\bigcup_i \omega_i = \Omega$. Since $u_m \to u$ in $L_1(\Omega, \mathrm{loc})$, by the Lemma we obtain

$$\int_{\omega_i} |u_m - u|\,\mathrm{d}x = \int_0^\infty m_n\big(\big((\mathscr{L}_t^m \backslash \mathscr{L}_t) \cup (\mathscr{L}_t \backslash \mathscr{L}_t^m)\big) \cap \omega_i\big)\,\mathrm{d}t \to 0,$$

where $\mathscr{L}_t^m = \{x \in \Omega : u_m(x) > t\}$. Therefore, for almost all t and for all i,

$$m_n\big(\big((\mathscr{L}_t^m \backslash \mathscr{L}_t) \cup (\mathscr{L}_t \backslash \mathscr{L}_t^m)\big) \cap \omega_i\big) \xrightarrow{m\to\infty} 0.$$

The latter means that $\mathscr{L}_t^m \to \mathscr{L}_t$ for almost all t. Hence by Lemma 9.1.3/1 we have

$$\int_0^{*\infty} P_\Omega(\mathscr{L}_t) \leq \int_0^{*\infty} \liminf_{m\to\infty} P_\Omega(\mathscr{L}_t^m)\,\mathrm{d}t \leq \liminf_{m\to\infty} \int_0^{*\infty} P_\Omega(\mathscr{L}_t^m)\,\mathrm{d}t,$$

where $\int^*$ is the upper Lebesgue integral. According to (1.2.6), the last integral is equal to $\|\nabla u_m\|_{L_1(\Omega)}$ and thus

$$\int_0^{*\infty} P_\Omega(\mathscr{L}_t)\,\mathrm{d}t \leq \liminf_{m\to\infty} \|\nabla u_m\|_{L_1(\Omega)} = \|u\|_{BV(\Omega)},$$

which together with (9.1.17) completes the proof. □

9.1.7 Embedding $BV(\Omega) \subset L_q(\Omega)$

The contents of this subsection are closely connected with Chap. 5. First we note that by Theorem 9.1.2 the inequality

$$\inf_{c\in\mathbb{R}^1} \|u - c\|_{L_q(\Omega)} \leq C\|\nabla u\|_{L_1(\Omega)}, \quad u \in L_1(\Omega),$$

implies

$$\inf_{c\in\mathbb{R}^1} \|u - c\|_{L_q(\Omega)} \leq C\|u\|_{BV(\Omega)}, \quad u \in BV(\Omega).$$

Therefore, for the domain Ω with finite volume, by Theorem 5.2.3 the last inequality (for $q \geq 1$) holds for $u \in BV(\Omega)$ if and only if $\Omega \in \mathscr{J}_\alpha$, $\alpha = q^{-1}$.

In the same way we can establish theorems similar to Theorem 5.5.2/1 and 6.3.3 for the space $BV(\Omega)$. By Theorem 9.1.3 the definitions of the classes $\mathscr{J}_\alpha$ and $\mathring{\mathscr{J}}_\alpha$ can be formulated in terms of the ratio

$$\left[m_n(\mathscr{E})\right]^\alpha / P_\Omega(\mathscr{E}),$$

where $\mathscr{E}$ is a measurable subset of Ω. The area minimizing function λ_M introduced in Sect. 5.2.4 can be defined as the infimum of the numbers $P_\Omega(\mathscr{E})$ taken over the collection of measurable sets $\mathscr{E} \subset \Omega$ with $\mu \le m_n(\mathscr{E}) \le M$.

Further we note that by Lemma 5.2.1/1 for the unit ball Ω and for any $\mathscr{E} \subset \Omega$, the inequality

$$\min\left\{m_n(\mathscr{E}), m_n(\Omega \backslash \mathscr{E})\right\} \le \frac{1}{2} v_n v_{n-1}^{n/(1-n)} \left[P_\Omega(\mathscr{E})\right]^{n/(n-1)}, \tag{9.1.18}$$

holds with the best possible constant.

9.2 Gauss–Green Formula for Lipschitz Functions

9.2.1 Normal in the Sense of Federer and Reduced Boundary

For fixed x, $\nu \in \mathbb{R}^n$, $\nu \ne 0$ we put

$$A^+ = \left\{y : (y-x)\nu > 0\right\}, \qquad A^- = \left\{y : (y-x)\nu < 0\right\},$$

$$A^0 = \left\{y : (y-x)\nu = 0\right\}.$$

Definition 1. The unit vector ν is called *the* (*outward*) *normal in the sense of Federer to the set* $\mathscr{E}$ *at the point* x if

$$\begin{aligned} &\lim_{\varrho \to 0} \varrho^{-n} m_n\left(\mathscr{E} \cap B_\varrho(x) \cap A^+\right) = 0, \\ &\lim_{\varrho \to 0} \varrho^{-n} m_n\left(C\mathscr{E} \cap B_\varrho(x) \cap A^-\right) = 0. \end{aligned} \tag{9.2.1}$$

Definition 2. The set of those points $x \in \partial\mathscr{E}$ for which normals to $\mathscr{E}$ exist is called the reduced boundary $\partial^*\mathscr{E}$ of $\mathscr{E}$.

9.2.2 Gauss–Green Formula

The remainder of Sect. 9.2 contains the proof of the following assertion.

Theorem 1. *If* $P(\mathscr{E}) < \infty$*, then the set* $\partial^*\mathscr{E}$ *is measurable with respect to* s *and* $\operatorname{var} \nabla\chi_{\mathscr{E}}$*. Moreover,* $\operatorname{var} \nabla\chi_{\mathscr{E}}(\partial\mathscr{E} \backslash \partial^*\mathscr{E}) = 0$ *and, for any* $\mathfrak{B} \subset \partial^*\mathscr{E}$*,*

$$\nabla\chi_{\mathscr{E}}(\mathfrak{B}) = -\int_{\mathfrak{B}} \nu(x) s(\mathrm{d}x), \qquad \operatorname{var} \nabla\chi_{\mathscr{E}}(\mathfrak{B}) = s(\mathfrak{B}). \tag{9.2.2}$$

These formulas immediately imply the next assertion.

Theorem 2. (The Gauss–Green formula). *If $P(\mathscr{E}) < \infty$ and u is a Lipschitz function in $\mathbb{R}^n$ with compact support, then*

$$\int_{\mathscr{E}} \nabla u(x)\,\mathrm{d}x = \int_{\partial^* \mathscr{E}} u(x)\nu(x)s(\mathrm{d}x). \tag{9.2.3}$$

It suffices to prove (9.2.3) for smooth functions. By the definition of $\nabla\chi_{\mathscr{E}}$ we have

$$\int_{\mathbb{R}^n} \chi_{\mathscr{E}} \nabla u\,\mathrm{d}x = -\int_{\mathbb{R}^n} u\nabla\chi_{\mathscr{E}}(\mathrm{d}x),$$

which together with (9.2.2) implies the result. □

Remark. In particular, from Theorem 1 it follows that $P(\mathscr{E}) \le s(\partial\mathscr{E})$. The case $P(\mathscr{E}) < s(\partial\mathscr{E})$ is not excluded. Moreover, the perimeter can be finite whereas $s(\partial\mathscr{E}) = \infty$. As an example, it suffices to consider the plane disk B_1 from which a sequence of segments of infinite total length has been removed. In this case $\partial^*\mathscr{E} = \partial B_1$.

9.2.3 Several Auxiliary Assertions

Lemma 1. *Let the vector charge μ concentrated on $\mathscr{E} \subset \mathbb{R}^n$ satisfy the condition $|\mu(\mathscr{E})| = \operatorname{var}\mu(\mathscr{E})$. Then $\mu = a\varphi$ where a is a constant vector and φ is a scalar nonnegative measure.*

Proof. Since the charge μ is absolutely continuous with respect to $\nu = \operatorname{var}\mu$ then the derivative $\mathrm{d}\mu/\mathrm{d}\nu = f = (f_1, \dots, f_n)$ exists ν almost everywhere. The equality $|\mu(\mathscr{E})| = \operatorname{var}\mu(\mathscr{E})$ implies that $|f| < 1$ is impossible on a set of positive measure ν. In fact, then we have $|\mu(E)| < \operatorname{var}\mu(E)$ for some E with $\nu(E) > 0$. Since $|\mu(\mathscr{E}\backslash E)| \le \operatorname{var}\mu(\mathscr{E}\backslash E)$ then

$$\left|\mu(\mathscr{E})\right| \le \left|\mu(E)\right| + \left|\mu(\mathscr{E}\backslash E)\right| < \operatorname{var}\mu(E) + \operatorname{var}\mu(\mathscr{E}\backslash E) = \operatorname{var}\mu(\mathscr{E}),$$

and we arrive at a contradiction. Since $|f| \le 1$ ν almost everywhere, it follows that $|f| = 1$ ν almost everywhere and since $|\mathrm{d}\mu_i/\mathrm{d}\nu| \le 1$, we have

$$\mu_i(\mathscr{E}) = \int \frac{\mathrm{d}\mu_i}{\mathrm{d}\nu}\,\mathrm{d}\nu$$

by the absolute continuity of μ_i with respect to ν. Therefore,

$$\left|\mu(\mathscr{E})\right| = \left[\sum_i \mu_i(\mathscr{E})^2\right]^{1/2} = \left[\sum_i \left(\int_{\mathscr{E}} \frac{\mathrm{d}\mu_i}{\mathrm{d}\nu}\,\mathrm{d}\nu\right)^2\right]^{1/2} = \left[\sum_i \left(\int f_i\,\mathrm{d}\nu\right)^2\right]^{1/2}.$$

The condition $\nu(\mathscr{E}) = |\mu(\mathscr{E})|$ means that the Minkowski inequality

$$\left[\sum_i\left(\int f_i\,\mathrm{d}\nu\right)^2\right]^{1/2} \le \int\left(\sum_i f_i^2\right)^{1/2}\mathrm{d}\nu$$

holds with the equality sign. Then the functions $x \to f_i(x)$ do not change sign and are proportional with coefficients independent of x for ν almost all x. □

Lemma 2. *If $P(\mathscr{E}) < \infty$ and the equality $\nabla\chi_{\mathscr{E}} = a\varphi$ is valid in the ball B_ϱ, where a is a constant vector and φ is a scalar measure, then, up to a set of the measure m_n zero, we have*

$$B_\varrho \cap \mathscr{E} = \big\{x \in B_\varrho : (x-y)a > 0\big\},$$

where y is a point in B_ϱ.

Proof. We may assume that $a = (1, 0, \dots, 0)$. Mollifying $\chi_{\mathscr{E}}$ with radius ε, we obtain

$$\frac{\partial}{\partial x_1}\mathscr{M}_\varepsilon\chi_{\mathscr{E}} \ge 0, \qquad \frac{\partial}{\partial x_i}\mathscr{M}_\varepsilon\chi_{\mathscr{E}} = 0 \quad (i = 2, \dots, m),$$

in the ball $B_{\varrho-\varepsilon}$. Hence the function $\mathscr{M}_\varepsilon\chi_{\mathscr{E}}$ does not depend on $x_2, \dots, x_n$ and does not decrease in x_1 in $B_{\varrho-\varepsilon}$. Therefore, the same is true for the limit function $\chi_{\mathscr{E}}$. □

Lemma 3. *If $P(\mathscr{E}) < \infty$, then, for almost all $\varrho > 0$,*

$$\nabla_{B_\varrho}\chi_{\mathscr{E}}(B_\varrho) = -\frac{1}{\varrho}\int_{\mathscr{E}\cap\partial B_\varrho} x\,\mathrm{d}s(x).$$

Proof. For all $\varrho > 0$ except, at most, a countable set we have $\operatorname{var}\nabla\chi_{\mathscr{E}}(\partial B_\varrho) = 0$. Suppose ϱ is not contained in that exceptional set. Let $\eta_\varepsilon(t)$ denote a piecewise linear continuous function on $(0, \infty)$, equal to 1 for $t \le \varrho$ and vanishing for $t > \varrho + \varepsilon$, $\varepsilon > 0$.

Since

$$\int_{\mathbb{R}^n} \chi_{\mathscr{E}}(x)\big[\eta_\varepsilon\big(|x|\big)\big]\,\mathrm{d}x = -\int_{\mathbb{R}^n} \eta_\varepsilon\big(|x|\big)\nabla\chi_{\mathscr{E}}(\mathrm{d}x),$$

we have

$$\frac{1}{\varepsilon}\int_{B_{\varrho+\varepsilon}\setminus B_\varrho} \chi_{\mathscr{E}}(x)\frac{x}{|x|}\,\mathrm{d}x = -\nabla\chi_{\mathscr{E}}(B_\varrho) - \int_{B_{\varrho+\varepsilon}\setminus B_\varrho} \eta_\varepsilon\big(|x|\big)\nabla\chi_{\mathscr{E}}(\mathrm{d}x). \quad (9.2.4)$$

By virtue of $\operatorname{var}\nabla\chi_{\mathscr{E}}(\partial B_\varrho) = 0$, the last integral converges to zero as $\varepsilon \to +0$. The left-hand side of (9.2.4) has the limit equal to

$$\varrho^{-1}\int_{\partial B_\varrho} xs(\mathrm{d}x)$$

for almost all ϱ. The result follows. □

Lemma 4. *If* $P(\mathscr{E}) < \infty$, *then* $P(\mathscr{E} \cap B_r(x)) < \infty$ *for any* $x \in \mathbb{R}^n$ *and for almost all* $r > 0$. *Moreover,*

$$P\big(\mathscr{E} \cap B_r(x)\big) = P_{B_r(x)}(\mathscr{E}) + s\big(\mathscr{E} \cap \partial B_r(x)\big).$$

Proof. We assume that x is located at the origin. By Theorem 9.1.3, there exists a sequence of polyhedra Π_i such that $\Pi_i \to \mathscr{E}$ and $P(\Pi_i) \to P(\mathscr{E})$. Using the Fubini theorem, we obtain $s(\Pi_i \cap \partial B_r) \to s(\mathscr{E} \cap \partial B_r)$ for almost all $r > 0$. Then

$$\limsup_{i\to\infty} P(\Pi_i \cap B_r) \le \lim_{i\to\infty} P(\Pi_i) + \lim_{i\to\infty} s(\Pi_i \cap \partial B_r) = P(\mathscr{E}) + s(\mathscr{E} \cap \partial B_r)$$

and thus $P(\mathscr{E} \cap B_r) < \infty$. By Lemma 9.1.3/1, there exists a sequence of polyhedra $\{\Pi_i\}$ such that

$$\nabla\chi_{\Pi_i \cap B_r} \to \nabla\chi_{\mathscr{E}\cap B_r}, \qquad \nabla\chi_{\Pi_i} \to \nabla\chi_{\mathscr{E}}. \tag{9.2.5}$$

Let the number r satisfy the equality

$$\limsup_{i\to\infty} \operatorname{var} \nabla\chi_{\Pi_i}(\partial B_r) = 0$$

(which can fail only for a countable set of values r). Then (9.2.5) implies

$$\nabla_{B_r}\chi_{\Pi_i} \to \nabla_{B_r}\chi_{\mathscr{E}}. \tag{9.2.6}$$

By the convergence $s(\Pi_i \cap \partial B_r) \to s(\mathscr{E} \cap \partial B_r)$ we find that the set functions μ_i defined by

$$\mu_i(\mathfrak{B}) = \int_{\partial B_r} \chi_{\mathfrak{B}\cap\Pi_i} \nu \, \mathrm{d}s,$$

where ν is the outward normal to B_r, weakly converge to μ, where

$$\mu(\mathfrak{B}) = \int_{\partial B_r} \chi_{\mathfrak{B}\cap\mathscr{E}} \nu \, \mathrm{d}s.$$

Obviously, $\nabla\chi_{\Pi_i \cap B_r} = \nabla_{B_r}\chi_{\Pi_i} + \mu_i$. Passing here to the limit and taking into account (9.2.5), (9.2.6), and the weak convergence of μ_i to μ, we arrive at $\nabla\chi_{\mathscr{E}\cap B_r} = \nabla_{B_r}\chi_{\mathscr{E}} + \mu$. Since the set function $\nabla_{B_r}\chi_{\mathscr{E}}$ is supported on B_r and $\operatorname{supp}\mu \subset \partial B_r$, the result follows from the last identity. □

9.2.4 Study of the Set N

Let N denote the set of points $x \in \partial\mathscr{E}$ that satisfy the following conditions:

(a) $\operatorname{var} \nabla\chi_{\mathscr{E}}(B_\varrho(x)) > 0$ for all $\varrho > 0$,

(b) the limit

$$\xi = \lim_{\varrho \to 0} \frac{\nabla \chi_{\mathscr{E}}(B_\varrho(x))}{\operatorname{var} \nabla \chi_{\mathscr{E}}(B_\varrho(x))}$$

exists and $|\xi| = 1$.

We note that the vector $\xi(x)$ exists and that $|\xi(x)| = 1$ almost everywhere with respect to the measure $\operatorname{var} \nabla \chi_{\mathscr{E}}$. Moreover,

$$\nabla \chi_{\mathscr{E}}(\mathfrak{B}) = \int_{\mathfrak{B} \cap N} \xi(x) \operatorname{var} \chi_{\mathscr{E}}(\mathrm{d}x). \tag{9.2.7}$$

Lemma. *If $P(\mathscr{E}) < \infty$ and $x \in N$, then*

$$\liminf_{\varrho \to 0} \varrho^{-n} m_n\bigl(\mathscr{E} \cap B_\varrho(x)\bigr) > 0, \tag{9.2.8}$$

$$\liminf_{\varrho \to 0} \varrho^{-n} m_n\bigl(C\mathscr{E} \cap B_\varrho(x)\bigr) > 0, \tag{9.2.9}$$

$$\limsup_{\varrho \to 0} \varrho^{1-n} P_{B_\varrho(x)}(\mathscr{E}) < \infty. \tag{9.2.10}$$

Proof. By definition of ξ,

$$P_{B_\varrho(x)}(\mathscr{E}) = \operatorname{var} \nabla \chi_{\mathscr{E}}\bigl(B_\varrho(x)\bigr) \le 2\bigl|\nabla \chi_{\mathscr{E}}\bigl(B_\varrho(x)\bigr)\bigr|$$

for sufficiently small ϱ. By Lemma 9.2.3/3 the right-hand side of this inequality does not exceed $2s(\mathscr{E} \cap \partial B_\varrho(x))$. Using Lemma 9.2.3/1, we have

$$P\bigl(\mathscr{E} \cap B_\varrho(x)\bigr) = P_{B_\varrho(x)}(\mathscr{E}) + s\bigl(\mathscr{E} \cap \partial B_\varrho(x)\bigr).$$

Hence

$$P\bigl(\mathscr{E} \cap B_\varrho(x)\bigr) \le 3s\bigl(\mathscr{E} \cap \partial B_\varrho(x)\bigr). \tag{9.2.11}$$

This estimate, together with the isoperimetric inequality (9.1.15), shows that

$$m_n\bigl(\mathscr{E} \cap B_\varrho(x)\bigr)^{(n-1)/n} \le c \frac{\mathrm{d}}{\mathrm{d}\varrho} m_n\bigl(\mathscr{E} \cap B_\varrho(x)\bigr) \tag{9.2.12}$$

for sufficiently small ϱ. The property (a) of the set N and Lemma 9.2.3/4 imply $P(\mathscr{E} \cap B_\varrho(x)) > 0$; therefore,

$$m_n\bigl(\mathscr{E} \cap B_\varrho(x)\bigr) > 0.$$

Since the function $\varrho \to m_n(\mathscr{E} \cap B_\varrho(x))$ is absolutely continuous, it follows from (9.2.12) that $c_1 \varrho^n \le m_n(\mathscr{E} \cap B_\varrho(x))$ for almost all ϱ.

It is clear that the last inequality is actually true for all ϱ. Thus (9.2.8) follows. Replacing $\mathscr{E}$ by $C\mathscr{E}$ in the previous argument we arrive at (9.2.9).

From (9.2.11) we have

$$P\bigl(\mathscr{E} \cap B_\varrho(x)\bigr) \le c\varrho^{n-1}$$

for almost all ϱ, which together with Lemma 9.2.3/1 yields

$$P_{B_\varrho(x)}(\mathscr{E}) \le c\varrho^{n-1}$$

for all ϱ. □

Theorem. *If $P(\mathscr{E}) < \infty$ and $x \in N$, then the normal ν at x exists and $\nu = \xi$. Moreover, for any $\varepsilon > 0$*

$$\lim_{\varrho\to 0} \varrho^{1-n} \operatorname{var} \nabla\chi_{\mathscr{E}}\big(B_\varrho(x) \cap \big[A^0\big]_{\varrho\varepsilon}\big) = v_{n-1}, \tag{9.2.13}$$

where $A^0 = \{y : (y-x)\nu = 0\}$ and $[\ ]_\varepsilon$ is the ε-neighborhood.

Proof. It suffices to check that any sequence $\varrho > 0$ contains a subsequence such that equalities (9.2.1) and (9.2.13) are valid. Let $\delta\mathscr{E}$ denote the set obtained from $\mathscr{E}$ by the similarity transformation with center x and coefficient δ. We may assume that x is located at the origin. Clearly, $P_{B_\varrho}(\mathscr{E}) = \varrho^{n-1}P_{B_1}(\varrho^{-1}\mathscr{E})$. By the Lemma the relative perimeters $P_{B_1}(\varrho^{-1}\mathscr{E})$ are uniformly bounded. Consequently, by Theorem 9.1.4, there exists a sequence $\varrho_i > 0$ such that the sequence of sets $B_1 \cap \varrho_i^{-1}\mathscr{E}$ converges to some set D. Moreover, Lemma 9.1.3/1 yields

$$\nabla_{B_1}\chi_{\varrho_i^{-1}\mathscr{E}} \to \nabla_{B_1}\chi_D.$$

Thus, for all $r \in (0,1)$ except, at most, a countable set we have

$$\nabla_{B_1}\chi_{\varrho_i^{-1}\mathscr{E}}(B_r) = \nabla\chi_{\varrho_i^{-1}\mathscr{E}}(B_r) \xrightarrow{i\to\infty} \nabla\chi_D(B_r). \tag{9.2.14}$$

By definition of the set N we obtain

$$\lim_{i\to\infty} \frac{|\nabla\chi_{\varrho_i^{-1}\mathscr{E}}(B_r)|}{\operatorname{var}\nabla\chi_{\varrho_i^{-1}\mathscr{E}}(B_r)} = \lim_{i\to\infty} \frac{|\nabla\chi_{\mathscr{E}}(B_{\varrho_i r})|}{\operatorname{var}\nabla\chi_{\mathscr{E}}(B_{\varrho_i r})} = 1.$$

Comparing the latter equalities with (9.2.14) and taking into account the semicontinuity of the variation under the weak convergence we obtain

$$\big|\nabla\chi_D(B_r)\big| = \lim_{i\to\infty}\big|\nabla\chi_{\varrho_i^{-1}\mathscr{E}}(B_r)\big| = \lim_{i\to\infty} \operatorname{var}\nabla\chi_{\varrho_i^{-1}\mathscr{E}}(B_r) \ge \operatorname{var}\nabla\chi_D(B_r). \tag{9.2.15}$$

Hence, by virtue of Lemmas 9.2.3/1 and 9.2.3/2, we conclude that the set $D \cap B_r$ coincides with $\{y \in B_r : y\nu < b\}$ up to the set of measure m_n zero.

We show that $b = 0$. In fact, if $b < 0$ then

$$0 = m_n(D \cap B_{|b|}) = \lim_{i\to\infty} m_n\big(\varrho_i^{-1}\mathscr{E} \cap B_{|b|}\big) = \lim_{i\to\infty} \varrho_i^{-1} m_n(\mathscr{E} \cap B_{|b|}),$$

which contradicts (9.2.8). Similarly, $b > 0$ contradicts (9.2.9). From the convergence

$$B_r \cap \varrho_i^{-1}\mathscr{E} \to B_r \cap D = B_r \cap A^-,$$

it follows that equalities of the form (9.2.1) are valid for the sequence $\{\varrho_i r\}$, where $\{\varrho_i\}$ is a subsequence of any given sequence $\varrho \to 0$ and r is arbitrarily close to unity. Hence (9.2.1) is true.

It remains to prove (9.2.13). We choose a subsequence ϱ_i such that

$$\operatorname{var} \nabla \chi_{\varrho_i^{-1}\mathscr{E}} \to \mu,$$

where μ satisfies the inequality

$$\mu(\mathfrak{B}) \geq \operatorname{var} \nabla \chi_D(\mathfrak{B})$$

for any $\mathfrak{B} \subset B_1$. Furthermore, (9.2.15) implies the existence of numbers $r < 1$, arbitrarily close to unity, such that

$$\mu(B_r) = \big|\nabla \chi_D(B_r)\big| \leq \operatorname{var} \nabla \chi_D(B_r).$$

Therefore, $\mu = \operatorname{var} \nabla \chi_D$. Now, for almost all $\varepsilon > 0$ and $r \in (0,1)$, we have

$$\begin{aligned} \lim_{i\to\infty} (\varrho_i r)^{1-n} \operatorname{var} \nabla \chi_{\mathscr{E}}\big(B_{\varrho_i r} \cap \big[A^0\big]_{\varrho_i \varepsilon}\big) &= \lim_{i\to\infty} \operatorname{var} \nabla \chi_{\varrho_i^{-1}\mathscr{E}}\big(B_r \cap \big[A^0\big]_\varepsilon\big) \\ &= \operatorname{var} \nabla \chi_D\big(B_r \cap \big[A^0\big]_\varepsilon\big) = r^{n-1} v_{n-1}, \end{aligned}$$

and (9.2.13) follows. □

9.2.5 Relations Between var$\boldsymbol{\nabla \chi_{\mathscr{E}}}$ and $\boldsymbol{s}$ on $\boldsymbol{\partial\mathscr{E}}$

Theorem 9.2.4 implies $\partial^*\mathscr{E} \supset N$. Moreover, since $\operatorname{var} \nabla \chi_{\mathscr{E}}(\mathbb{R}^n \backslash N) = 0$, it follows that $\operatorname{var} \nabla \chi_{\mathscr{E}}(\mathbb{R}^n \backslash \partial^*\mathscr{E}) = 0$ and thus the sets N and $\partial^*\mathscr{E}$ are measurable relative to $\operatorname{var} \nabla \chi_{\mathscr{E}}$.

Next we need the following well-known general assertion.

Lemma 1. *Let μ be a measure in $\mathbb{R}^n$ and let, for all points x in the μ-measurable set $\mathfrak{B}$, the following inequality hold:*

$$\limsup_{\varrho\to 0} \varrho^{1-n} \mu\big(B_\varrho(x)\big) \geq \beta > 0,$$

where β does not depend on x. Then $\beta s(\mathfrak{B}) \leq c(n)\mu(\mathfrak{B})$.

Proof. For any $\varepsilon > 0$ there exists an open set G such that

$$\mu(G \backslash \mathfrak{B}) + \mu(\mathfrak{B} \backslash G) < \varepsilon.$$

By the definition of the Hausdorff measure s, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$s(G) \leq c_1(n) \sum \varrho_i^{n-1} + \varepsilon, \tag{9.2.16}$$

for any covering of G by balls B_{ϱ_i} with $\varrho_i < \delta$. For any $x \in \mathfrak{B} \cap G$ consider the family of balls $B_\varrho(x) \subset G$, $3\varrho < \delta$, such that

$$\varrho^{1-n}\mu\big(B_\varrho(x)\big) \geq \beta/2. \tag{9.2.17}$$

By Theorem 1.2.1/1, there exists a sequence of mutually disjoint balls $B_{\varrho_i}(x_i)$ satisfying the condition $\bigcup_i B_{3\varrho_i}(x_i) \supset \mathfrak{B} \cap G$. Then

$$\sum \mu\big(B_{\varrho_i}(x_i)\big) \leq \mu(G) \leq \mu(\mathfrak{B}) + \varepsilon.$$

By (9.2.16) and (9.2.17) we have

$$\begin{aligned} s(G) &\leq c_1(n)\sum(3\varrho_i)^{n-1} + \varepsilon \leq c_2(n)\sum \varrho_i^{n-1} + \varepsilon \\ &\leq c_2(n)\beta^{-1}\sum \mu\big(B_{\varrho_i}(x_i)\big) + \varepsilon \leq c_3(n)\beta^{-1}\big(\mu(\mathfrak{B}) + \varepsilon\big) + \varepsilon. \end{aligned}$$

□

Definition 9.2.1/1 and the Fubini theorem imply the following lemma.

Lemma 2. *If $P(\mathscr{E}) < \infty$ and $x \in \partial^*\mathscr{E}$, then*

$$\liminf_{\varrho\to 0} \varrho^{1-n} s\big(\mathscr{E} \cap \partial B_\varrho(x) \cap A^+\big) = 0,$$

where the lower limit is taken over any subset of measure 1 in the interval $0 < \varrho < 1$.

Lemma 3. *If $P(\mathscr{E}) < \infty$ and $x \in \partial^*\mathscr{E}$, then*

$$\limsup_{\varrho\to 0} \varrho^{1-n} \operatorname{var} \nabla\chi_{\mathscr{E}}\big(B_\varrho(x)\big) \geq v_{n-1}.$$

Proof. Let Q be a subset of the interval $0 < \varrho < 1$ on which the identity in Lemma 9.2.3/1 is valid. By Lemmas 9.2.3/3 and 2 we have

$$\begin{aligned} &\limsup_{\varrho\to\infty} \varrho^{1-n} \operatorname{var} \nabla\chi_{\mathscr{E}}\big(B_\varrho(x)\big) \\ &\quad\geq \limsup_{\varrho\to 0} \varrho^{1-n}\big|\nabla\chi_{\mathscr{E}}\big(B_\varrho(x)\big)\big| \geq \limsup_{Q\ni\varrho\to 0} \varrho^{-n}\left|\int_{\mathscr{E}\cap\partial B_\varrho(x)} x\,\mathrm{d}s\right| \\ &\quad= \lim_{\varrho\to 0} \varrho^{-n}\left|\int_{A^-\cap\partial B_\varrho(x)} x\,\mathrm{d}s\right| = v_{n-1}. \end{aligned}$$

The result follows. □

Taking into account the equality $\operatorname{var}\nabla\chi_{\mathscr{E}}(\partial^*\mathscr{E}\backslash N) = 0$, from Lemmas 1 and 3 we obtain the next assertion.

Lemma 4. *If $P(\mathscr{E}) < \infty$, then $s(\partial^*\mathscr{E}\backslash N) = 0$.*

Now to prove Theorem 9.2.2/1 it suffices to verify that the vector measures $\nu\,\mathrm{d}s$ and $\operatorname{var}\nabla\chi_{\mathscr{E}}(\mathrm{d}x)$ coincide on N.

Lemma 5. *Let $P(\mathscr{E}) < \infty$ and let the set $\mathfrak{B} \cap N$ be measurable relative to $\operatorname{var}\nabla\chi_{\mathscr{E}}$. Then $s(\mathfrak{B}) \geq \operatorname{var}\nabla\chi_{\mathscr{E}}(\mathfrak{B})$.*

Proof. The function

$$x \to f_\varrho(x) = v_{n-1}^{-1} \varrho^{1-n} \operatorname{var} \nabla \chi_{\mathscr{E}}\big(B_\varrho(x)\big)$$

is lower semicontinuous. This is a consequence of the relation

$$B_\varrho(x) \backslash B_\varrho(x_1) \to \varnothing \quad \text{as } x_1 \to x.$$

Therefore, $f_\varrho(x)$ is measurable. Let $\varrho_i \to 0$. By Theorem 9.2.4, the sequence f_{ϱ_i} converges to $f(x) \equiv 1$ on N. By the Egorov theorem, for any $\varepsilon > 0$ there exists a set $N_\varepsilon \subset N$ such that $\operatorname{var} \nabla \chi_{\mathscr{E}}(N_\varepsilon) < \varepsilon$ and the sequence f_{ϱ_i} converges uniformly on $N \backslash N_\varepsilon$. Therefore there exists a $\delta > 0$ such that

$$\operatorname{var} \nabla \chi_{\mathscr{E}}\big(B_r(x)\big) \le (1+\varepsilon) v_{n-1} r^{n-1}, \tag{9.2.18}$$

for all $x \in N \backslash N_\varepsilon$ and $r \in (0, \delta)$.

By definition of the measure s, there exists a finite covering of $N \backslash N_\varepsilon$ by balls $B_{r_i}(x_i)$ with $r_i < \delta$, such that

$$(1+\varepsilon) s\big(\mathfrak{B} \cap (N \backslash N_\varepsilon)\big) \ge v_{n-1} \sum r_i^{n-1}.$$

This and (9.2.18) imply

$$\begin{aligned}
\operatorname{var} \nabla \chi_{\mathscr{E}}(\mathfrak{B}) &\le \varepsilon + \operatorname{var} \nabla \chi_{\mathscr{E}}\big(\mathfrak{B} \cap (N \backslash N_\varepsilon)\big) \\
&\le \varepsilon + \sum_i \operatorname{var} \nabla \chi_{\mathscr{E}}\big(B_{r_i}(x)\big) \le \varepsilon + (1+\varepsilon) v_{n-1} \sum_i r_i^{n-1} \\
&\le \varepsilon + (1+\varepsilon)^2 s\big(\mathfrak{B} \cap (N \backslash N_\varepsilon)\big) \le \varepsilon + (1+\varepsilon)^2 s(\mathfrak{B}).
\end{aligned}$$

The result follows since ε is arbitrary. □

The next assertion is a modification of the classical Vitali–Carathéodory covering theorem.

Lemma 6. *Let μ be a finite measure in $\mathbb{R}^n$ concentrated on a set $\mathscr{E}$. Further, let $\mathfrak{M}$ be a family of closed balls having the following property: For each point $x \in \mathscr{E}$ there exists a $\delta(x) > 0$ such that $B_r(x) \in \mathfrak{M}$ for all $r < \delta(x)$ and*

$$\alpha r^k < \mu\big(B_r(x)\big) \le \beta r^k \tag{9.2.19}$$

for some $k > 0$, where $B_r(x)$ is any ball in $\mathfrak{M}$ and α, β are positive constants independent of r and x. Then there exists, at most, a countable family of mutually disjoint balls $\mathscr{B}^{(i)} \in \mathfrak{M}$ such that $\mu(\mathscr{E} \backslash \bigcup_i \mathscr{B}^{(i)}) = 0$.

Proof. We fix a number $a > 1$ and construct a sequence of balls $\mathscr{B}^{(i)} \in \mathfrak{M}$ in the following way. Suppose that $\mathscr{B}^{(1)}, \dots, \mathscr{B}^{(j-1)}$ have already been specified. Then we choose $\mathscr{B}^{(j)}$ to satisfy

$$\mathscr{B}^{(j)} \cap \mathscr{B}^{(i)} = \varnothing \quad \text{for } i < j,$$
$$a\mu\big(\mathscr{B}^{(j)}\big) \ge \sup\big\{\mu\big(B_r(x)\big) : B_r(x) \cap \mathscr{B}^{(i)} = \varnothing, 1 \le i < j\big\}.$$

If the process breaks at some point j then $\mathscr{E} \subset \bigcup_{i=1}^{j} \mathscr{B}^{(i)}$ and the lemma is proved. $\square$

Suppose the sequence $\{\mathscr{B}^{(i)}\}$ is infinite. Let $\mathscr{C}^{(i)}$ denote the closed ball concentric to $\mathscr{B}^{(i)}$ with radius $R_i = Qr_i$ where r_i is the radius of $\mathscr{B}^{(i)}$ and the constant $Q \in (1, \infty)$ will be specified later. Note that from the very beginning we could have constructed the sequence $\{\mathscr{B}^{(i)}\}$ so that $\mathscr{C}^{(i)}$ is contained in $\mathfrak{M}$ simultaneously with $\mathscr{B}^{(i)}$.

We show that

$$\mathscr{E} \subset \mathscr{E} \cap \left[\left(\bigcup_{i=1}^{j-1} \mathscr{B}^{(i)} \right) \cup \left(\bigcup_{i=j}^{\infty} \mathscr{C}^{(i)} \right) \right] \tag{9.2.20}$$

for some Q and for all j. In fact, let $x \in \mathscr{E} \backslash \bigcup_{i=1}^{j-1} \mathscr{B}^{(i)}$. Then there exists a ball $\mathscr{B} \in \mathfrak{M}$ centered at x such that $\mathscr{B} \cap \mathscr{B}^{(i)} = \varnothing$ for $i < j$. Note that we have $\mathscr{B} \cap \mathscr{B}^{(p)} \neq \varnothing$ for some $p \geq j$. Indeed, if $\mathscr{B} \cap \mathscr{B}^{(p)} = \varnothing$ for all p then the constructed sequence $\{\mathscr{B}^{(i)}\}$ satisfies $\mu(\mathscr{B}) \leq a\mu(\mathscr{B}^{(p)})$. Since the balls $\mathscr{B}^{(p)}$ are mutually disjoint, the last inequality contradicts finiteness of the measure μ.

Let the number p be such that $\mathscr{B} \cap \mathscr{B}^{(i)} = \varnothing$ for $i < p$ and $\mathscr{B} \cap \mathscr{B}^{(p)} \neq \varnothing$. Inequalities $\mu(\mathscr{B}) \leq a\mu(\mathscr{B}^{(p)})$ and (9.2.19) imply the estimates

$$\alpha r^k \leq \mu(\mathscr{B}) \leq \alpha\mu\big(\mathscr{B}^{(p)}\big) \leq \alpha\beta r_p^k,$$

where r is the radius of $\mathscr{B}$. Since the balls $\mathscr{B}$ and $\mathscr{B}^{(p)}$ are disjoint, the distance between their centers satisfies

$$\mathrm{d} \leq r + r_p \leq r_p\big(1 + (a\beta/\alpha)^{1/k}\big).$$

Let the constant Q be equal to $1 + (a\beta/\alpha)^{1/k}$. Then $\mathrm{d} \leq R_p$ and hence $x \in \mathscr{C}^{(p)}$. The inclusion (9.2.20) follows.

It remains to note that

$$\mu\left(\mathscr{E} \backslash \bigcup_{i=1}^{j-1} \mathscr{B}^{(i)} \right) \leq \sum_{i=j}^{\infty} \mu\big(\mathscr{C}^{(i)}\big) \leq \beta \sum_{i=j}^{\infty} R_i^k \leq \beta Q^k \sum_{i=j}^{\infty} r_i^k \leq \frac{\beta Q^k}{\alpha} \sum_{i=j}^{\infty} \mu\big(\mathscr{B}^{(i)}\big).$$

Since the series $\sum_{i=1}^{\infty} \mu(\mathscr{B}^{(i)})$ converges, we have

$$\mu\left(\mathscr{E} \backslash \bigcup_{i=1}^{j-1} \mathscr{B}^{(i)} \right) \to 0 \quad \text{as } j \to \infty.$$

Lemma 7. *If $P(\mathscr{E}) < \infty$ and $\mathfrak{B}$ is a subset of N, measurable with respect to* $\operatorname{var} \nabla\chi_{\mathscr{E}}$, *then $\mathfrak{B}$ is s-measurable and*

$$s(\mathfrak{B}) \leq \operatorname{var} \nabla\chi_{\mathscr{E}}(\mathfrak{B}). \tag{9.2.21}$$

Proof. By Theorem 9.2.4, for any $\varepsilon \in (0,1)$ the measure $\mu = \operatorname{var} \nabla\chi_{\mathscr{E}}$ satisfies the conditions of Lemma 6 with $\alpha = v_{n-1}(1-\varepsilon)$, $\beta = v_{n-1}(1+\varepsilon)$, $k = n-1$.

By the definition of the Hausdorff measure, given any ε, there exists a $\delta > 0$ such that

$$s(\mathfrak{B}) \le v_{n-1} \sum r_i^{n-1} + \varepsilon$$

for any finite covering of $\mathfrak{B}$ by balls $B_{r_i}(x_i)$ with $r_i < \delta$.

Let $\{\mathscr{B}^{(i)}\}$ be the sequence of closed balls in Lemma 6. We assume their radii to be less than δ. We choose a finite subsequence $\{\mathscr{B}^{(i)}\}_{i=1}^q$ such that

$$\mu\left(\mathfrak{B} \backslash \bigcup_{i=1}^q \mathscr{B}^{(i)}\right) < \varepsilon.$$

As was shown in the proof of Lemma 1, there exists a finite collection of disjoint open balls $\mathscr{C}^{(j)}$ with radii $\varrho_j < \delta$ such that $\mu(\bigcup_j \mathscr{C}^{(j)}) < \varepsilon$ and the concentric balls $3\mathscr{C}^{(j)}$ with radii $3\varrho_j$ form a covering of $\mathfrak{B} \backslash \bigcup_{i \le q} \mathscr{B}^{(i)}$. Thus

$$\bigcup_j 3\mathscr{C}^{(j)} \cup \left(\bigcup_{i \le q} \mathscr{B}^{(i)}\right) \supset \mathfrak{B}.$$

Now we have

$$s(\mathfrak{B}) \le v_{n-1}\left(3 \sum_j \varrho_j^{n-1} + \sum_{i \le q} r_i^{n-1}\right) + \varepsilon,$$

where r_i is the radius of $\mathscr{B}^{(i)}$. Hence

$$s(\mathfrak{B}) \le (1+\varepsilon)\left[c \sum_j \mu\big(\mathscr{B}^{(j)}\big) + \sum_{i \le q} \mu\big(\mathscr{C}^{(i)}\big)\right] + \varepsilon \le (1+\varepsilon)\big(c\varepsilon + \mu(\mathfrak{B})\big) + \varepsilon,$$

and (9.2.21) follows because ε is arbitrary.

Since inequality (9.2.21) is valid for all μ-measurable sets, this implies that $\mathfrak{B}$ is s-measurable. □

Combining Lemmas 4, 5, and 7, we complete the proof of Theorem 9.2.2/1.

9.3 Extension of Functions in $BV(\Omega)$ onto $\mathbb{R}^n$

With any set $\mathscr{E} \subset \Omega$ we associate the value

$$\tau_\Omega(\mathscr{E}) = \inf_{\mathfrak{B} \cap \Omega = \mathscr{E}} P_{C\Omega}(\mathfrak{B}).$$

Theorem. (a) *If for any function $u \in BV(\Omega)$ there exists an extension $\hat{u} \in BV(\mathbb{R}^n)$ such that*

$$\|\hat{u}\|_{BV\left(\mathbb{R}^n\right)} \le C\|u\|_{BV(\Omega)}, \tag{9.3.1}$$

where C is a constant independent of u, then

$$\tau_\Omega(\mathscr{E}) \le (C-1)P_\Omega(\mathscr{E}) \tag{9.3.2}$$

for any set $\mathscr{E} \subset \Omega$.

(b) *Conversely, if for any $\mathscr{E} \subset \Omega$ the inequality* (9.3.2) *holds with a constant C independent of $\mathscr{E}$, then for any $u \in BV(\Omega)$ there exists an extension $\hat{u} \in BV(\mathbb{R}^n)$ for which* (9.3.1) *is true.*

9.3.1 Proof of Necessity of (9.3.2)

The inequality (9.3.2) is trivial provided $P_\Omega(\mathscr{E}) = \infty$. Let $P_\Omega(\mathscr{E}) < \infty$. By hypothesis there exists an extension $\hat{\chi}_{\mathscr{E}}$ of the characteristic function $\chi_{\mathscr{E}}$ such that

$$\|\hat{\chi}_{\mathscr{E}}\|_{BV\left(\mathbb{R}^n\right)} \le CP_\Omega(\mathscr{E}).$$

This and formula (9.1.16) imply

$$CP_\Omega(\mathscr{E}) \ge \int_{-\infty}^{\infty} P_{\mathbb{R}^n}\big(\{x : \hat{\chi}_{\mathscr{E}} > t\}\big)\,\mathrm{d}t \ge \int_0^1 P_{\mathbb{R}^n}\big(\{x : \hat{\chi}_{\mathscr{E}} > t\}\big)\,\mathrm{d}t.$$

Since $\{x : \hat{\chi}_{\mathscr{E}}(x) > t\} \cap \Omega = \mathscr{E}$ for $t \in (0,1)$, by (9.1.2) and (9.1.3) we obtain

$$\begin{aligned} CP_\Omega(\mathscr{E}) &\ge \inf_{\mathfrak{B}\cap\Omega=\mathscr{E}} P_{\mathbb{R}^n}(\mathfrak{B}) \\ &\ge \inf_{\mathfrak{B}\cap\Omega=\mathscr{E}} P_\Omega(\mathfrak{B}) + \inf_{\mathfrak{B}\cap\Omega=\mathscr{E}} P_{C\Omega}(\mathfrak{B}) \ge P_\Omega(\mathscr{E}) + \tau_\Omega(\mathscr{E}). \end{aligned}$$

Hence (9.3.2) follows.

9.3.2 Three Lemmas on $P_{C\Omega}(\mathscr{E})$

To prove the sufficiency of (9.3.2) we need the following three auxiliary assertions.

Lemma 1. *If $\mathfrak{B} \subset \Omega$, $\tau_\Omega(\mathfrak{B}) < \infty$, and $P_\Omega(\mathfrak{B}) < \infty$, then there exists a set $\mathscr{E} \subset \mathbb{R}^n$ such that $\mathscr{E} \cap \Omega = \mathfrak{B}$ and*

$$P_{C\Omega}(\mathscr{E}) = \tau_\Omega(\mathfrak{B}). \tag{9.3.3}$$

Proof. Let $\{\mathscr{E}_i\}$ be a sequence of subsets of $\mathbb{R}^n$ such that $\mathscr{E}_i \cap \Omega = \mathfrak{B}$ and

$$\lim_{i\to\infty} P_{C\Omega}(\mathscr{E}_i) = \tau_\Omega(\mathfrak{B}). \tag{9.3.4}$$

By (9.3.4), $\sup_i P_{C\Omega}(\mathscr{E}_i) < \infty$ and since $P_\Omega(\mathscr{E}_i) = P_\Omega(\mathfrak{B}) < \infty$, we have $\sup_i P_{\mathbb{R}^n}(\mathscr{E}_i) < \infty$. Hence from Theorem 9.1.4 it follows that there exists a subsequence (for which we retain the notation $\{\mathscr{E}_i\}$) that converges to some set $\mathscr{E}$. By Lemma 9.1.3/1,

$$P(\mathscr{E}) \le \liminf_{i\to\infty} P(\mathscr{E}_i).$$

Taking into account that $\mathscr{E} \cap \Omega = \mathfrak{B}$ as well as equalities (9.1.2) and (9.1.3), we obtain

$$P_{C\Omega}(\mathscr{E}) \le \lim_{i\to\infty} P_{C\Omega}(\mathscr{E}_i) = \tau_\Omega(\mathfrak{B}). \tag{9.3.5}$$

Comparing (9.3.5) with the definition of $\tau_\Omega(\mathfrak{B})$, we arrive at (9.3.3). □

Lemma 2. *Let $\mathscr{E}_1$, $\mathscr{E}_2$ be measurable subsets of $\mathbb{R}^n$. Then*

$$P_{C\Omega}(\mathscr{E}_1 \cap \mathscr{E}_2) + P_{C\Omega}(\mathscr{E}_1 \cup \mathscr{E}_2) \le P_{C\Omega}(\mathscr{E}_1) + P_{C\Omega}(\mathscr{E}_2). \tag{9.3.6}$$

Proof. Let G be an open set, $G \supset C\Omega$. Then by (9.1.16)

$$\begin{aligned} P_G(\mathscr{E}_1) + P_G(\mathscr{E}_2) &\ge \big\|(\chi_{\mathscr{E}_1} + \chi_{\mathscr{E}_2})\big\|_{BV(G)} \\ &= \int_{-\infty}^{\infty} P_G\big(\{x : \chi_{\mathscr{E}_1} + \chi_{\mathscr{E}_2} > t\}\big)\,\mathrm{d}t \\ &= \int_0^1 P_G\big(\{x : \chi_{\mathscr{E}_1} + \chi_{\mathscr{E}_2} > t\}\big)\,\mathrm{d}t \\ &\quad + \int_1^2 P_G\big(\{x : \chi_{\mathscr{E}_1} + \chi_{\mathscr{E}_2} > t\}\big)\,\mathrm{d}t \\ &= P_G(\mathscr{E}_1 \cup \mathscr{E}_2) + P_G(\mathscr{E}_1 \cap \mathscr{E}_2). \end{aligned} \tag{9.3.7}$$

Consider the sequence of open sets G_i such that $G_{i+1} \subset G_i$ and $\bigcap_i G_i = C\Omega$. Since

$$P_{C\Omega}(\mathscr{E}_k) = \lim_{i\to\infty} P_{G_i}(\mathscr{E}_k), \quad k = 1, 2,$$

applying (9.3.7), we obtain (9.3.6). □

Lemma 3. *Let $P_{C\Omega}(\mathscr{E}_k) < \infty$, $k = 1, 2$. We put $\mathfrak{B}_k = \mathscr{E}_k \cap \Omega$. Then*

$$P_{C\Omega}(\mathscr{E}_1 \cap \mathscr{E}_2) = P_{C\Omega}(\mathscr{E}_1), \qquad P_{C\Omega}(\mathscr{E}_1 \cup \mathscr{E}_2) = P_{C\Omega}(\mathscr{E}_2),$$

provided $\mathfrak{B}_1 \subset \mathfrak{B}_2$ and

$$P_{C\Omega}(\mathscr{E}_k) = \tau_\Omega(\mathfrak{B}_k), \quad k = 1, 2. \tag{9.3.8}$$

Proof. Since $\mathscr{E}_1 \cap \mathscr{E}_2 \cap \Omega = \mathfrak{B}_1$ and $(\mathscr{E}_1 \cup \mathscr{E}_2) \cap \Omega = \mathfrak{B}_2$, by the definition of τ_Ω we have

$$\tau_\Omega(\mathfrak{B}_1) \le P_{C\Omega}(\mathscr{E}_1 \cap \mathscr{E}_2), \qquad \tau_\Omega(\mathfrak{B}_2) \le P_{C\Omega}(\mathscr{E}_1 \cup \mathscr{E}_2). \tag{9.3.9}$$

Using (9.3.8) we can rewrite (9.3.6) as

$$P_{C\Omega}(\mathscr{E}_1 \cap \mathscr{E}_2) + P_{C\Omega}(\mathscr{E}_1 \cup \mathscr{E}_2) \le \tau_\Omega(\mathfrak{B}_1) + \tau_\Omega(\mathfrak{B}_2),$$

which together with (9.3.9) proves the lemma. □

9.3.3 Proof of Sufficiency of (9.3.2)

1°. *Plan of Proof.* Starting from $\mathscr{N}_t = \{x : u(x) \ge t\}$ we construct the family of sets $\mathfrak{B}_t$ satisfying the conditions $\mathfrak{B}_t \cap \Omega = \mathscr{N}_t$, $P_{C\Omega}(\mathfrak{B}_t) = \tau_\Omega(\mathscr{N}_t)$, $\mathfrak{B}_t \subset \mathfrak{B}_\tau$ for $t > \tau$.

We first construct $\mathfrak{B}_t$ for a countable set $\{t_i\}$, which is everywhere dense on $(-\infty, \infty)$ (item 2°) and then for all other t (item 3°). Finally, in item 4° we introduce the function $\hat{u}(x) = \sup\{t : x \in \mathfrak{B}_t\}$ and prove that $\hat{u}(x)$ satisfies the conditions of Theorem 9.3.

2°. Since $u \in BV(\Omega)$ then for almost all t we have $P_\Omega(\mathscr{N}_t) < \infty$ by formula (9.1.16). Therefore, we can choose a countable set $\{t_i\}$, $t_i \ne t_j$ for $i \ne j$, which is everywhere dense on $(-\infty, \infty)$ and satisfies $P_\Omega(\mathscr{N}_{t_i}) < \infty$. From (9.3.2) it follows that $\tau_\Omega(\mathscr{N}_{t_i}) < \infty$.

We construct a sequence of sets $\mathfrak{B}_{t_i}$, $i = 1, 2, \dots$, such that

$$\begin{aligned} &\text{(a)} \quad \mathfrak{B}_{t_i} \cap \Omega = \mathscr{N}_{t_i}, \\ &\text{(b)} \quad P_{C\Omega}(\mathfrak{B}_{t_i}) = \tau_\Omega(\mathscr{N}_{t_i}), \\ &\text{(c)} \quad \mathfrak{B}_{t_i} \subset \mathfrak{B}_{t_j}, \quad t_i > t_j. \end{aligned}$$

By Lemma 9.3.2/1 there exists a set $\mathfrak{B}_{t_1}$ satisfying the conditions (a)–(b). Suppose the sets $\mathfrak{B}_{t_1}, \dots, \mathfrak{B}_{t_n}$ have already been constructed so that the conditions (a)–(c) are fulfilled for $i, j = 1, \dots, n-1$. By Lemma 9.3.2/1 there exists a set $\mathfrak{B}^{(n)}$ satisfying (a) and (b). Let t_* be the largest of those numbers t_i, $i = 1, \dots, n-1$, for which $t_i < t_n$ and let t^* be the smallest of those numbers t_i, $i = 1, \dots, n-1$, for which $t_n < t_i$. We put

$$\mathfrak{B}_{t_n} = \left(\mathfrak{B}^{(n)} \cap \mathfrak{B}_{t_*}\right) \cup \mathfrak{B}_{t^*}.$$

It is clear that $\mathfrak{B}_{t_*} \supset \mathfrak{B}_{t_n} \supset \mathfrak{B}_{t^*}$. Hence $\mathfrak{B}_{t_n} \subset \mathfrak{B}_{t_i}$ for $t_n > t_i$ and $\mathfrak{B}_{t_n} \supset \mathfrak{B}_{t_i}$ for $t_n < t_i$, $i = 1, \dots, n-1$. Since

$$\mathfrak{B}^{(n)} \cap \Omega = \mathscr{N}_{t_n}, \qquad \mathfrak{B}_{t_*} \cap \Omega = \mathscr{N}_{t_*} \supset \mathscr{N}_{t_n}, \qquad \mathfrak{B}_{t^*} \cap \Omega = \mathscr{N}_{t^*} \subset \mathscr{N}_{t_n},$$

we have $\mathfrak{B}_{t_n} \cap \Omega = \mathscr{N}_{t_n}$. Applying Lemma 9.3.2/3 to the sets $\mathfrak{B}^{(n)}$, $\mathfrak{B}_{t_*}$, and then to the sets $\mathfrak{B}^{(n)} \cap \mathfrak{B}_{t_*}$, $\mathfrak{B}_{t^*}$, we obtain

$$P_{C\Omega}(\mathfrak{B}_{t_n}) = \tau_\Omega(\mathscr{N}_{t_n}).$$

Thus the collection of sets $\mathfrak{B}_{t_1}, \dots, \mathfrak{B}_{t_n}$ satisfies the conditions (a)–(c) for $i, j = 1, \dots, n$.

3°. Let $t \notin \{t_i\}$. From the set $\{t_i\}$ we select two monotone sequences $\{\alpha_i\}$, $\{\beta_i\}$ such that $\alpha_i < t < \beta_i$ and $\lim_{i\to\infty}\alpha_i = \lim_{i\to\infty}\beta_i = t$.

By Lemma 9.3.2/1, there exists a set $\mathfrak{B}_t^{(0)}$ such that $\mathfrak{B}_t^{(0)} \cap \Omega = \mathscr{N}_t$ and $P_{C\Omega}(\mathfrak{B}_t^{(0)}) = \tau_\Omega(\mathscr{N}_t)$. Consider the sequence of sets $\mathfrak{B}_t^{(k)} = \mathfrak{B}_t^{(0)} \cap \mathfrak{B}_{\alpha_k}$, $k = 1, 2, \dots$. It is clear that $\mathfrak{B}_t^{(k)} \cap \Omega = \mathscr{N}_t$, $\mathfrak{B}_t^{(k+1)} \subset \mathfrak{B}_t^{(k)}$. Using Lemma 9.3.2/3, have $P_{C\Omega}(\mathfrak{B}_t^{(k)}) = \tau_\Omega(\mathscr{N}_t)$. We introduce the notation

$$\tilde{\mathfrak{B}}_t = \bigcap_{k=1}^{\infty} \mathfrak{B}_t^{(k)}.$$

Since $\mathfrak{B}_t^{(k)} \to \tilde{\mathfrak{B}}_t$ as $k \to \infty$, we see that

$$P_{C\Omega}(\tilde{\mathfrak{B}}_t) \le \liminf_{k\to\infty} P_{C\Omega}\big(\mathfrak{B}_t^{(k)}\big) = \tau_\Omega(\mathscr{N}_t).$$

On the other hand, $\tilde{\mathfrak{B}}_t \cap \Omega = \mathscr{N}_t$. Therefore $\tau_\Omega(\mathscr{N}_t) \le P_{C\Omega}(\tilde{\mathfrak{B}}_t)$. Thus $P_{C\Omega}(\tilde{\mathfrak{B}}_t) = \tau_\Omega(\mathscr{N}_t)$.

Next consider the sequence of sets $\mathfrak{C}_t^{(k)} = \tilde{\mathfrak{B}}_t \cup \mathfrak{B}_{\beta_k}$, $k = 1, 2, \dots$. In the same way as when we considered the sets $\mathfrak{B}_t^{(k)}$ we conclude that the set

$$\mathfrak{B}_t = \bigcup_{k=1}^{\infty} \mathfrak{C}_t^{(k)},$$

is measurable and satisfies the conditions

1. $\mathfrak{B}_t \cap \Omega = \mathscr{N}_t$, 2. $P_{C\Omega}(\mathfrak{B}_t) = \tau_\Omega(\mathscr{N}_t)$, 3. $\mathfrak{B}_{\beta_i} \subset \mathfrak{B}_t \subset \mathfrak{B}_{\alpha_i}$,

$i = 1, 2, \dots$. Now let t, τ be arbitrary numbers, $t < \tau$. Then condition 3 implies that $\mathfrak{B}_t \supset \mathfrak{B}_\tau$.

4°. Consider the function $\hat{u}$ defined by $\hat{u}(x) = \sup\{t : x \in \mathfrak{B}_t\}$. We put

$$\mathfrak{A}_t = \big\{x : \hat{u}(x) \ge t\big\}, \qquad \mathfrak{C}_t = \big\{x : \hat{u}(x) > t\big\}.$$

Obviously, $\mathfrak{A}_t \supset \mathfrak{B}_t \supset \mathfrak{C}_t$. The sets $\mathfrak{A}_t \backslash \mathfrak{C}_t$ are mutually disjoint for different t and hence $m_n(\mathfrak{A}_t \backslash \mathfrak{C}_t) = 0$ for almost all t. Thus the sets $\mathfrak{A}_t$, $\mathfrak{C}_t$ are measurable for almost all t. Moreover,

$$P_{\mathbb{R}^n}(\mathfrak{A}_t) = P_{\mathbb{R}^n}(\mathfrak{B}_t) = P_{\mathbb{R}^n}(\mathfrak{C}_t).$$

We prove that $\hat{u}$ is locally integrable. It is well known that the inequality

$$\big(m_n(\mathscr{E})\big)^{(n-1)/n} \le C(R, \varepsilon) P_{B_R}(\mathscr{E}) \tag{9.3.10}$$

is valid for the subset $\mathscr{E}$ of the ball B_R such that $m_n(\mathscr{E}) < m_n(B_R) - \varepsilon$. (In particular, this follows from Lemma 5.2.1/1.) Let the closed ball B_δ be contained in Ω and let $B_R \supset B_\delta$. Then (9.3.10) implies

$$\begin{aligned} m_n(\mathscr{E}) &\le C(R,\delta)\big[P_{B_R}(\mathscr{E}) + m_n(\mathscr{E}\cap B_\delta)\big] \\ &\le C(R,\delta)\big[P_{\mathbb{R}^n}(\mathscr{E}) + m_n(\mathscr{E}\cap B_\delta)\big] \end{aligned}$$

for any set $\mathscr{E}\subset B_R$. Putting $\mathscr{E} = \mathfrak{B}_t\cap B_R$ for $t\ge 0$ and $\mathscr{E} = B_R\backslash\mathfrak{B}_t$ for $t<0$ in the latter inequality and using $P_{C\Omega}(\mathfrak{B}_t) = \tau_\Omega(\mathscr{N}_t)$ and estimate (9.3.2), we obtain

$$\begin{aligned} m_n(\mathfrak{B}_t\cap B_R) &\le C(R,\delta)\big[CP_\Omega(\mathscr{N}_t) + m_n(\mathscr{N}_t\cap B_\delta)\big], \quad t\ge 0, \\ m_n(B_R\backslash\mathfrak{B}_t) &\le C(R,\delta)\big[CP_\Omega(\mathscr{N}_t) + m_n\big((\Omega\backslash\mathscr{N}_t)\cap B_\delta\big)\big], \quad t<0. \end{aligned}$$

Taking into account that $m_n(\mathfrak{B}_t) = m_n(\mathfrak{A}_t)$ for almost all t, from the latter two inequalities we obtain

$$\begin{aligned} &\int_0^\infty m_n(\mathfrak{A}_t\cap B_R)\,\mathrm{d}t + \int_{-\infty}^0 m_n(B_R\backslash\mathfrak{A}_t)\,\mathrm{d}t \\ &\quad\le C(R,\delta)\bigg[C\int_{-\infty}^\infty P_\Omega(\mathscr{N}_t)\,\mathrm{d}t + \int_0^\infty m_n(\mathscr{N}_t\cap B_\delta)\,\mathrm{d}t \\ &\qquad + \int_0^\infty m_n(B_\delta\backslash\mathscr{N}_t)\,\mathrm{d}t\bigg], \end{aligned}$$

which is equivalent to

$$\int_{B_R}|\hat{u}|\,\mathrm{d}x \le C(R,\delta)\bigg[C\|u\|_{BV(\Omega)} + \int_{B_\delta}|u|\,\mathrm{d}x\bigg],$$

whence the local integrability of $\hat{u}$ follows. Applying (9.1.16), (9.3.2), and recalling that $P_{\mathbb{R}^n}(\mathfrak{C}_t) = P_{\mathbb{R}^n}(\mathfrak{B}_t)$ for almost all t, we obtain

$$\begin{aligned} \|\hat{u}\|_{BV(\mathbb{R}^n)} &= \int_{-\infty}^\infty P_{\mathbb{R}^n}(\mathfrak{C}_t)\,\mathrm{d}t = \int_{-\infty}^\infty \big[P_\Omega(\mathfrak{B}_t) + P_{C\Omega}(\mathfrak{B}_t)\big]\,\mathrm{d}t \\ &= \int_{-\infty}^\infty \big[P_\Omega(\mathscr{N}_t) + \tau_\Omega(\mathscr{N}_t)\big]\,\mathrm{d}t \le C\int_{-\infty}^\infty P_\Omega(\mathscr{N}_t)\,\mathrm{d}t = C\|u\|_{BV(\Omega)}, \end{aligned}$$

i.e., $\hat{u}\in BV(\mathbb{R}^n)$ and (9.3.1) is valid.

9.3.4 Equivalent Statement of Theorem 9.3

Theorem 9.3 can be rephrased in terms of the extension operator $A_\Omega : u\to\hat{u}$, which associates with each $u\in BV(\Omega)$ its extension $\hat{u}\in BV(\mathbb{R}^n)$.

We put

$$\|A_\Omega\| = \sup\bigg\{\frac{\|\hat{u}\|_{BV(\mathbb{R}^n)}}{\|u\|_{BV(\Omega)}} : u\in BV(\Omega)\bigg\},$$

and denote by $|\Omega|$ the infimum of those numbers k for which $\tau_\Omega(\mathscr{E})\le kP_\Omega(\mathscr{E})$ for all $\mathscr{E}\subset\Omega$. Now we have the following theorem.

Theorem. *The operator A_Ω exists and is bounded if and only if $|\Omega|<\infty$. Moreover, $\|A_\Omega\|\ge 1+|\Omega|$ for any extension operator A_Ω and there exists an operator A_Ω with $\|A_\Omega\| = 1+|\Omega|$.*

9.3.5 One More Extension Theorem

The condition (9.3.2) in Theorem 9.3 is of a global nature. For example, nonconnected sets Ω do not satisfy it.

This impediment may be removed if we make the requirements on the extension operator less restrictive. Specifically, the following theorem holds.

Theorem. *Let Ω be a bounded open set. For any function $u \in BV(\Omega)$ to have an extension $\hat{u} \in BV(\mathbb{R}^n)$ with*

$$\|\hat{u}\|_{BV(\mathbb{R}^n)} \le K\big(\|u\|_{BV(\Omega)} + \|u\|_{L_1(\Omega)}\big), \tag{9.3.11}$$

where K is independent of u, it is necessary and sufficient that there exists a $\delta > 0$ such that $\tau_\Omega(\mathscr{E}) \le CP_\Omega(\mathscr{E})$ for any $\mathscr{E} \subset \Omega$ with a diameter less than δ, the constant C being independent of $\mathscr{E}$.

Proof. Necessity. Let $\mathscr{E} \subset \Omega$ and let $\chi_{\mathscr{E}}$ be the characteristic function of $\mathscr{E}$ while $\hat{\chi}_{\mathscr{E}}$ is an extension of $\chi_{\mathscr{E}}$ satisfying (9.3.11). We have

$$K\big(P_\Omega(\mathscr{E}) + m_n(\mathscr{E})\big) \ge \|\hat{\chi}_{\mathscr{E}}\|_{BV(\mathbb{R}^n)} \ge \int_0^1 P\big(\{x : \hat{\chi}_{\mathscr{E}} > t\}\big)\,\mathrm{d}t.$$

Since $\{x : \hat{\chi}_{\mathscr{E}} > t\} \cap \Omega = \mathscr{E}$ for $t \in (0, 1)$, it follows that

$$K\big(P_\Omega(\mathscr{E}) + m_n(\mathscr{E})\big) \ge \inf_{\mathfrak{B}\cap\Omega=\mathscr{E}} P(\mathfrak{B}).$$

By the inclusion $\mathscr{E} \subset \mathfrak{B}$, the latter estimate and the isoperimetric inequality (9.1.15) imply

$$K\big(P_\Omega(\mathscr{E}) + m_n(\mathscr{E})\big) \ge nv_n^{1/n}\big(m_n(\mathscr{E})\big)^{(n-1)/n}. \tag{9.3.12}$$

We put $\delta = n/(2K)$. Then from (9.3.12) under the condition $\operatorname{diam}\mathscr{E} < \delta$ it follows that $m_n(\mathscr{E}) \le P_\Omega(\mathscr{E})$. Therefore,

$$2KP_\Omega(\mathscr{E}) \ge \inf_{\mathfrak{B}\cap\Omega=\mathscr{E}} P(\mathfrak{B}) \ge \tau_\Omega(\mathscr{E}).$$

Sufficiency. Consider the partition of unity $\alpha_i(x)$, $i = 1, \dots, \nu$, such that

$$\bigcup_{i=1}^{\nu} \operatorname{supp}\alpha_i \supset \bar{\Omega}, \qquad \operatorname{diam}\operatorname{supp}\alpha_i < \delta,$$

and $|\mathrm{grad}\alpha_i| \le \mathrm{d} = \text{const}$. Let $u \in BV(\Omega)$. We put $\varphi_i = u\alpha_i$ and $\mathscr{N}_t = \{x : |\varphi_i| \ge t\}$. Since for all $t \ne 0$ we have $\operatorname{diam}\mathscr{N}_t < \delta$ then $\tau_\Omega(\mathscr{N}_t) \le CP_\Omega(\mathscr{N}_t)$. Therefore, following the same argument as in the proof of sufficiency in Theorem 9.3 we obtain the function $\hat{\varphi}_i \in BV(\mathbb{R}^n)$ such that $\hat{\varphi}_i = \varphi_i$ in Ω and

$$\|\hat{\varphi}_i\|_{BV(\mathbb{R}^n)} \le (C+1)\|\varphi_i\|_{BV(\Omega)} \le (C+1)\big(\|u\|_{BV(\Omega)} + \mathrm{d}\|u\|_{L_1(\Omega)}\big).$$

We put $\hat{u} = \sum \hat{\varphi}_i$. It is clear that $\hat{u} = u$ in Ω and

$$\|\hat{u}\|_{BV(\mathbb{R}^n)} \le \nu(C+1)\big(\|u\|_{BV(\Omega)} + \mathrm{d}\|u\|_{L_1(\Omega)}\big). \qquad \square$$

9.4 Exact Constants for Certain Convex Domains

By Theorem 9.3 the norm of the extension operator $BV(\Omega) \to BV(\mathbb{R}^n)$ is expressed by the exact constant in the isoperimetric inequality (9.3.2). This constant can be found in some particular cases. For plane convex domains it has a simple geometrical interpretation (cf. Corollary 9.4.4/2). The constant is also easily calculated if Ω is an n-dimensional ball.

9.4.1 Lemmas on Approximations by Polyhedra

Lemma 1. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ and let $\mathscr{E} \subset \Omega$, $P_{\mathbb{R}^n}(\mathscr{E}) < \infty$. Then there exists a sequence of polyhedra Π_k such that $\Pi_k \to \mathscr{E}$ and*

$$\lim_{k\to\infty} P_\Omega(\Pi_k) = P_\Omega(\mathscr{E}), \qquad \lim_{k\to\infty} P_{C\Omega}(\Pi_k) = P_{C\Omega}(\mathscr{E}). \tag{9.4.1}$$

Proof. Let Ω_ε be the domain obtained from Ω by the similarity transformation with coefficient $1+\varepsilon$ and with the center at a fixed point of Ω. We denote the image of $\mathscr{E}$ under the same transformation by $\mathscr{E}_\varepsilon$. It is clear that

$$P_{\Omega_\varepsilon}(\mathscr{E}_\varepsilon) = (1+\varepsilon)^{n-1} P_\Omega(\mathscr{E}), \qquad P_{C\Omega_\varepsilon}(\mathscr{E}_\varepsilon) = (1+\varepsilon)^{n-1} P_{C\Omega}(\mathscr{E}).$$

Hence we can easily obtain that

$$\lim_{\varepsilon\to 0} P_\Omega(\mathscr{E}_\varepsilon) = P_\Omega(\mathscr{E}), \qquad \lim_{\varepsilon\to 0} P_{C\Omega}(\mathscr{E}_\varepsilon) = P_{C\Omega}(\mathscr{E}). \tag{9.4.2}$$

In fact, $(1+\varepsilon)^{n-1} P_\Omega(\mathscr{E}) \geq P_\Omega(\mathscr{E}_\varepsilon)$ and consequently

$$P_\Omega(\mathscr{E}) \geq \liminf_{\varepsilon\to 0} P_\Omega(\mathscr{E}_\varepsilon) \geq P_\Omega(\mathscr{E}).$$

The latter inequality is a corollary of Lemma 9.1.3/1. Since

$$P_\Omega(\mathscr{E}) + P_{C\Omega}(\mathscr{E}) = P(\mathscr{E}), \qquad P_\Omega(\mathscr{E}_\varepsilon) + P_{C\Omega}(\mathscr{E}_\varepsilon) = P(\mathscr{E}_\varepsilon),$$

the first inequality (9.4.2) implies the second. For almost all ε we have

$$\operatorname{var} \nabla \chi_{\mathscr{E}_\varepsilon}(\partial\Omega) = 0. \tag{9.4.3}$$

Let ε be subject to (9.4.3). By Theorem 9.1.3 there exists a sequence of polyhedra $\Pi_{k,\varepsilon}$ such that $\Pi_{k,\varepsilon} \to \mathscr{E}_\varepsilon$, $P(\Pi_{k,\varepsilon}) \to P(\mathscr{E}_\varepsilon)$ as $k \to \infty$. This and Lemma 9.1.2/3 yield

$$\operatorname{var} \nabla \chi_{\Pi_{k,\varepsilon}} \xrightarrow{\text{weakly}} \operatorname{var} \nabla \chi_{\mathscr{E}_\varepsilon}.$$

By (9.4.3) we have

$$\lim_{k\to\infty} \sup \operatorname{var} \nabla\chi_{\Pi_{k,\varepsilon}}(\partial\Omega) \le \operatorname{var} \nabla\chi_{\mathscr{E}_\varepsilon}(\partial\Omega) = 0,$$

and therefore,

$$\lim_{k\to\infty} P_\Omega(\Pi_{k,\varepsilon}) = P_\Omega(\mathscr{E}_\varepsilon), \qquad \lim_{k\to\infty} P_{C\Omega}(\Pi_{k,\varepsilon}) = P_{C\Omega}(\mathscr{E}_\varepsilon). \tag{9.4.4}$$

We choose a sequence of numbers ε_i that satisfy (9.4.3), such that $\varepsilon_i \to 0$ as $i \to \infty$. Then (9.4.2) and (9.4.4) imply

$$\lim_{i\to\infty} \lim_{k\to\infty} P_\Omega(\Pi_{k,\varepsilon_i}) = P_\Omega(\mathscr{E}),$$

$$\lim_{i\to\infty} \lim_{k\to\infty} P_{C\Omega}(\Pi_{k,\varepsilon_i}) = P_{C\Omega}(\mathscr{E}).$$

This concludes the proof. □

In the following we shall use the following elementary assertion.

Lemma 2. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ and let Π be a finite polyhedron. Then $s(\partial\Pi \cap C\Omega) \ge s(\partial\Omega \cap \Pi)$.*

Lemma 3. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ and let $\mathscr{E} \subset \Omega$, $P_\Omega(\mathscr{E}) < \infty$. Then there exists a sequence of polyhedra Π_k such that $\Pi_k \to \mathscr{E}$ and*

$$\lim_{k\to\infty} P_\Omega(\Pi_k \cap \Omega) = P_\Omega(\mathscr{E}), \qquad \lim_{k\to\infty} P_{C\Omega}(\Pi_k \cap \Omega) = P_{C\Omega}(\mathscr{E}).$$

Proof. By Lemma 1, there exists a sequence of polyhedra Π_k, $\Pi_k \to \mathscr{E}$, satisfying (9.4.1). It is clear that $P_\Omega(\Pi_k \cap \Omega) = P_\Omega(\Pi_k)$. According to Lemma 9.3.2/3 we have

$$P_{C\Omega}(\Pi_k \cap \Omega) \le P_{C\Omega}(\Pi_k).$$

Therefore,

$$\lim_{k\to\infty} P_{C\Omega}(\Pi_k \cap \Omega) \le \lim_{k\to\infty} P_{C\Omega}(\Pi_k) = P_{C\Omega}(\mathscr{E}), \tag{9.4.5}$$

$$\lim_{k\to\infty} P_\Omega(\Pi_k \cap \Omega) = \lim_{k\to\infty} P_\Omega(\Pi_k) = P_\Omega(\mathscr{E}). \tag{9.4.6}$$

Since $\Pi_k \cap \Omega \to \mathscr{E}$, we obtain

$$P_{\mathbb{R}^n}(\mathscr{E}) \le \lim_{k\to\infty} P_{\mathbb{R}^n}(\Pi_k \cap \Omega). \tag{9.4.7}$$

From (9.4.6) and (9.4.7) we conclude that

$$P_{C\Omega}(\mathscr{E}) \le \lim_{k\to\infty} P_{C\Omega}(\Pi_k \cap \Omega),$$

which together with (9.4.5) completes the proof. □

9.4.2 Property of $P_{C\Omega}$

Lemma. *Let $P(\Omega) < \infty$ and suppose a normal to Ω exists s-almost everywhere on $\partial\Omega$. Then, for any set $\mathscr{E} \subset \Omega$,*

$$P_{C\Omega}(\mathscr{E}) + P_{C\Omega}(\Omega\backslash\mathscr{E}) = s(\partial\Omega).$$

Proof. By the equality $\chi_\Omega = \chi_\mathscr{E} + \chi_{\Omega\backslash\mathscr{E}}$ we have

$$\operatorname{var}\nabla\chi_\Omega(C\Omega) \le \operatorname{var}\nabla\chi_\mathscr{E}(C\Omega) + \operatorname{var}\nabla\chi_{\Omega\backslash\mathscr{E}}(C\Omega) = P_{C\Omega}(\mathscr{E}) + P_{C\Omega}(\Omega\backslash\mathscr{E}).$$

Since a normal to Ω exists s-almost everywhere on $\partial\Omega$, by Theorem 9.2.2/1 we obtain

$$\operatorname{var}\nabla\chi_\Omega(C\Omega) = P_{\mathbb{R}^n}(\Omega) = s(\partial\Omega).$$

Consequently,

$$s(\partial\Omega) \le P_{C\Omega}(\mathscr{E}) + P_{C\Omega}(\Omega\backslash\mathscr{E}).$$

We prove the reverse inequality. Let $\mathfrak{A}^*$, $\mathfrak{B}^*$ denote the reduced boundaries of the sets $\mathscr{E}$ and $\Omega\backslash\mathscr{E}$, respectively. The sets $\mathfrak{A}^*$ and $\mathfrak{B}^*$ are s-measurable (cf. Theorem 9.2.2/1). We note that the sets $\mathfrak{A}^* \cap \partial^*\Omega$ and $\mathfrak{B}^* \cap \partial^*\Omega$ are disjoint. In fact, suppose there exists a point $x \in \partial^*\Omega$ common to $\mathfrak{A}^*$ and $\mathfrak{B}^*$. Then the volume densities of $\mathscr{E}$ and $\Omega\backslash\mathscr{E}$ at the point x are equal to $1/2$. This is impossible because $x \in \partial^*\Omega$. Consequently,

$$s\bigl(\mathfrak{A}^* \cap \partial^*\Omega\bigr) + s\bigl(\mathfrak{B}^* \cap \partial^*\Omega\bigr) \le s(\partial\Omega).$$

It remains to use the equalities

$$s\bigl(\mathfrak{A}^* \cap \partial^*\Omega\bigr) = s\bigl(\mathfrak{A}^* \cap \partial\Omega\bigr) = P_{C\Omega}(\mathscr{E}),$$
$$s\bigl(\mathfrak{B}^* \cap \partial^*\Omega\bigr) = s\bigl(\mathfrak{B}^* \cap \partial\Omega\bigr) = P_{C\Omega}(\Omega\backslash\mathscr{E})$$

(cf. Theorem 9.2.2/1). The lemma is proved. □

9.4.3 Expression for the Set Function $\tau_\Omega(\mathscr{E})$ for a Convex Domain

Theorem. *If Ω is a bounded convex domain in $\mathbb{R}^n$, then the equality*

$$\tau_\Omega(\mathscr{E}) = \min\bigl[P_{C\Omega}(\mathscr{E}), P_{C\Omega}(\Omega\backslash\mathscr{E})\bigr] \qquad (9.4.8)$$

holds for any set $\mathscr{E} \subset \Omega$ with $P(\mathscr{E}) < \infty$.

Proof. For the sake of definiteness, let

$$P_{C\Omega}(\mathscr{E}) \le P_{C\Omega}(\Omega\backslash\mathscr{E}).$$

Let the set $\mathfrak{B}$ be such that $\mathfrak{B} \cap \Omega = \mathscr{E}$, $P_{C\Omega}(\mathfrak{B}) = \tau_\Omega(\mathscr{E})$. Assume for the moment that $m_n(\mathfrak{B}) < \infty$.

By Lemma 9.4.1/1, we can find a sequence of polyhedra Π_k, $\Pi_k \to \mathfrak{B}$ such that

$$\lim_{k\to\infty} P_\Omega(\Pi_k) = P_\Omega(\mathfrak{B}), \qquad \lim_{k\to\infty} P_{C\Omega}(\Pi_k) = P_{C\Omega}(\mathfrak{B}). \tag{9.4.9}$$

Since $m_n(\mathfrak{B}) < \infty$, the polyhedra Π_k are finite. By Lemma 9.4.1/2 we have $P_{C\Omega}(\Pi_k \cap \Omega) \le P_{C\Omega}(\Pi_k)$. This and (9.4.9) yield

$$\limsup_{k\to\infty} P_{C\Omega}(\Pi_k \cap \Omega) \le P_{C\Omega}(\mathfrak{B}). \tag{9.4.10}$$

Using $\Pi_k \cap \Omega \to \mathfrak{B} \cap \Omega$, we obtain

$$P(\mathfrak{B} \cap \Omega) \le \liminf_{k\to\infty} P(\Pi_k \cap \Omega),$$

which together with (9.4.9) implies

$$P_{C\Omega}(\mathscr{E}) = P_{C\Omega}(\mathfrak{B} \cap \Omega) \le \liminf_{k\to\infty} P_{C\Omega}(\Pi_k \cap \Omega).$$

Hence from (9.4.10) it follows that $P_{C\Omega}(\mathscr{E}) \le P_{C\Omega}(\mathfrak{B}) = \tau_\Omega(\mathscr{E})$. Thus $P_{C\Omega}(\mathscr{E}) = \tau_\Omega(\mathscr{E})$.

Now let $m_n(\mathfrak{B}) = \infty$. Since $P_{C\Omega}(\mathfrak{B}) < \infty$, we have $m_n(C\mathfrak{B}) < \infty$ (cf. (9.1.18)). Further we note that

$$P_{C\Omega}(C\mathfrak{B}) = P_{C\Omega}(\mathfrak{B}) = \tau_\Omega(\mathscr{E}) = \tau_\Omega(\Omega \backslash \mathscr{E}).$$

Hence, according to what we proved earlier, $\tau_\Omega(\Omega\backslash\mathscr{E}) = P_{C\Omega}(\Omega\backslash\mathscr{E})$ and therefore, by (9.4.8),

$$\tau_\Omega(\mathscr{E}) = P_{C\Omega}(\Omega\backslash\mathscr{E}) \ge P_{C\Omega}(\mathscr{E}).$$

Since, obviously, $\tau_\Omega(\mathscr{E}) \le P_{C\Omega}(\mathscr{E})$, we conclude that $\tau_\Omega(\mathscr{E}) = P_{C\Omega}(\mathscr{E})$. □

9.4.4 The Function $|\Omega|$ for a Convex Domain

Corollary 1. *Let Ω be a bounded convex domain. Then*

$$|\Omega| = \inf\{k : P_{C\Omega}(\mathscr{E}) \le k P_\Omega(\mathscr{E})\},$$

where $\mathscr{E}$ is any subset of Ω with $P_{C\Omega}(\mathscr{E}) \le \frac{1}{2}s(\partial\Omega)$.

The result follows immediately from Theorem 9.4.3 and Lemma 9.4.2.

Corollary 2. *Let Ω be a bounded convex domain in $\mathbb{R}^2$. Then*

$$|\Omega| = \frac{1}{2h}s(\partial\Omega),$$

where h is the minimum length of those lines whose end points separate $\partial\Omega$ into arcs of equal length.

Proof. We take an arbitrary $\varepsilon > 0$. Let $\mathscr{E}$ be a measurable subset of Ω such that

$$P_\Omega(\mathscr{E}) > 0, \quad P_{C\Omega}(\mathscr{E}) < \frac{1}{2}s(\partial\Omega), \quad \text{and} \quad |\Omega| - \frac{P_{C\Omega}(\mathscr{E})}{P_\Omega(\mathscr{E})} < \varepsilon$$

(cf. Corollary 1). By Lemma 9.4.1/3, we can find a polyhedron Π such that

$$|\Omega| - \frac{P_{C\Omega}(\Pi \cap \Omega)}{P_\Omega(\Pi)} < 2\varepsilon.$$

Let A and B be the set of points of the intersection of $\partial\Omega$ with some component of the boundary of Π. Points A and B can be chosen so that the segment of the component of $\partial\Pi$ being considered, bounded by points A and B, lies entirely in Ω. The segment AB divides Ω into two sets Q and Q'. Let $P_{C\Omega}(Q) \le P_{C\Omega}(Q')$. It is clear that

$$P_{C\Omega}(Q)/AB \ge P_{C\Omega}(\Pi \cap \Omega)/P_\Omega(\Pi)$$

and therefore

$$|\Omega| - \frac{P_{C\Omega}(Q)}{AB} < 2\varepsilon. \tag{9.4.11}$$

If $P_{C\Omega}(Q) = P_{C\Omega}(Q')$ then (9.4.11) implies the required assertion by virtue of the inequality $AB \ge h$ and the fact that ε is arbitrary.

Let $P_{C\Omega}(Q) < P_{C\Omega}(Q')$. We shift the segment AB parallel to itself to a new position $A'B'(A' \in \partial\Omega, B' \in \partial\Omega)$ so that $P_{C\Omega}(Q_1) = P_{C\Omega}(Q_1')$ where Q_1 and Q_1' are the domains into which the segment $A'B'$ divides Ω.

Elementary calculations show that

$$P_{C\Omega}(Q_1)/A'B' \ge P_{C\Omega}(Q)/AB,$$

which together with (9.4.11) proves the corollary. □

Lemma. *Let Ω be the unit ball in $\mathbb{R}^n$. Then*

$$\inf\{P_{C\Omega}(\mathscr{E}) : \mathscr{E} \subset \Omega, P_\Omega(\mathscr{E}) = p = \text{const} \le \omega_n\}$$

equals the area of the spherical part of the boundary of the spherical segment whose base has area p.

Proof. Let $\mathscr{E} \subset \Omega$, $P_\Omega(\mathscr{E}) = p$. By Lemma 9.4.1/3 there exists a sequence of polyhedra Π_k, $\Pi_k \to \mathscr{E}$, such that

$$P_\Omega(\Pi_k) \to p, \qquad P_{C\Omega}(\Pi_k \cap \Omega) \to P_{C\Omega}(\mathscr{E}).$$

We perform the spherical symmetrization of $\Pi_k \cap \Omega$ relative to some ray l with origin at the center of Ω. We obtain the set Π_k', symmetric relative to l, with a piecewise smooth boundary and such that

$$P_\Omega(\Pi'_k) \le P_\Omega(\Pi_k), \qquad P_{C\Omega}(\Pi'_k) = P_{C\Omega}(\Pi_k \cap \Omega).$$

We denote the spherical segment whose spherical part of the boundary is $\partial\Pi'_k \cap \partial\Omega$ by Q_k. It is clear that

$$P_\Omega(Q_k) \le P_\Omega(\Pi'_k), \qquad P_{C\Omega}(Q_k) = P_{C\Omega}(\Pi'_k).$$

Hence the result follows. □

The next assertions can be obtained from the Lemma by simple calculations.

Corollary 3. (a) *If Ω is the unit ball in $\mathbb{R}^n$ then*

$$|\Omega| = \omega_n/2\pi v_{n-1}.$$

(b) *If Ω is the unit ball in $\mathbb{R}^3$ then, for any $\mathscr{E} \subset \Omega$,*

$$4\pi P_\Omega(\mathscr{E}) \ge P_{C\Omega}(\mathscr{E})\big(4\pi - P_{C\Omega}(\mathscr{E})\big).$$

(c) *If Ω is the unit disk then, for any $\mathscr{E} \subset \Omega$,*

$$P_\Omega(\mathscr{E}) \ge 2\sin\left(\frac{1}{2}P_{C\Omega}(\mathscr{E})\right).$$

9.5 Rough Trace of Functions in $BV(\Omega)$ and Certain Integral Inequalities

9.5.1 Definition of the Rough Trace and Its Properties

On the reduced boundary of Ω we define *the rough trace u^* of a function* $u \in BV(\Omega)$. We put

$$u^*(x) = \sup\{t : P(\mathscr{N}_t) < \infty, x \in \partial^*\mathscr{N}_t\},$$

where $x \in \partial^*\Omega$ and $\mathscr{N}_t = \{x \in \Omega : u(x) \ge t\}$. (The supremum of the empty set is assumed to be $-\infty$.)

It is clear that if u has a limit value at a point $x \in \partial^*\Omega$ then

$$u^*(x) = \lim_{y\to x} u(y).$$

Lemma 1. *Let $s(\partial\Omega) < \infty$. Then $P_\Omega(\mathscr{E}) < \infty$ implies $P(\mathscr{E}) < \infty$ for any $\mathscr{E} \subset \Omega$.*

Proof. Since $s(\partial\Omega)$ is finite, we can construct a sequence of polyhedra Π_k, $\Pi_k \subset \Omega$, such that $s(\partial\Pi_k) \le K < \infty$. Since $P_\Omega(\mathscr{E}) < \infty$, we have

$P_\Omega(\mathscr{E} \cap \Pi_k) \le K_1 < \infty$. Moreover, $\mathscr{E} \cap \Pi_k \to \mathscr{E}$. Hence the result follows by Lemma 9.1.3/1. □

Corollary 1. *If* $s(\partial\Omega) < \infty$ *and* $u \in BV(\Omega)$*, then* $P(\mathscr{N}_t) < \infty$ *for almost all* t*.*

Lemma 2. *Let* $s(\partial\Omega) < \infty$ *and* $u \in BV(\Omega)$*. Then the rough trace* u^* *is* s*-measurable on* $\partial^*\Omega$ *and*

$$s\big(\{x \in \partial^*\Omega\colon u^*(x) \ge t\}\big) = s\big(\partial^*\Omega \cap \partial^*\mathscr{N}_t\big) \tag{9.5.1}$$

for almost all t*.*

In fact, instead of (9.5.1) we prove that, for all $t \in \mathbb{R}$ except for a countable subset,

$$s\big(\{x \in \partial^*\Omega\colon u^*(x) \ge t\}\Delta\partial^*\mathscr{N}_t\big) = 0,$$

where $\mathscr{A}\Delta\mathscr{B} = (\mathscr{A} \setminus \mathscr{B})\bigcap(\mathscr{B} \setminus \mathscr{A})$.

Proof. Denote $B_t = \{x \in \partial^*\Omega\colon u^*(x) \ge t\}$, $Y_t = \partial^*\mathscr{N}_t$, and $X_t = B_t \setminus Y_t$. One can see that $Y_t \subset B_t$. Thus, it remains to prove that $s(X_t) = 0$.

The sets Y_t are measurable and the sets X_t are disjoint. It is clear that the inclusions $Y_{t_0} \supset Y_{t_1}$ and $Y_{t_0} \cup X_{t_0} \supset Y_{t_1} \cup X_{t_1}$ hold for $t_0 < t_1$, which implies $Y_{t_0} \supset X_{t_1}$. Thus

$$\Big(\bigcap_{t<t_1} Y_t\Big) \setminus Y_{t_1} \supset X_{t_1}.$$

On the other hand, the sets $(\cap_{t<t_1} Y_t)\setminus Y_{t_1}$ are measurable and disjoint. Therefore $s((\bigcap_{t<t_1} Y_t) \setminus Y_{t_1}) = 0$ for almost all $t_1 \in \mathbb{R}$. This implies that the sets X_t (being subsets of measure zero sets) are measurable and of measure zero for almost all $t \in \mathbb{R}$. Thereby the sets B_t are measurable. □

For a set $\mathscr{E} \subset \Omega$, denote by $\Theta_x(\mathscr{E})$ its relative density at the point x; i.e,

$$\Theta_x(\mathscr{E}) = \lim_{\varrho\to 0} \frac{m_n(\mathscr{E} \bigcap B(x,\varrho))}{m_n(\Omega \bigcap B(x,\varrho))}.$$

In the most interesting case $x \in \partial^*\Omega$, we have

$$\Theta_x(\mathscr{E}) = 2v_n^{-1} \lim_{\varrho\to 0} \varrho^{-n} m_n\big(\mathscr{E} \cap B(x,\varrho)\big).$$

Upper and lower relative densities $\bar{\Theta}_x(\mathscr{E})$ and $\underline{\Theta}_x(\mathscr{E})$ are defined similarly.

Lemma 3. *Let* $P(\Omega) < \infty$*,* $\mathscr{E} \subset \Omega$*,* $P_\Omega(\mathscr{E}) < \infty$*. Then the density of* $\mathscr{E}$ *is equal either* 0 *or* 1 *for* s*-almost all* $x \in \partial^*\Omega$*.*

Proof. It is clear that $\Theta_x(\mathscr{E}) = 1$ for $x \in \partial^*\Omega \bigcap \partial^*\mathscr{E}$ and $\Theta_x(\mathscr{E}) = 0$ for $x \in \partial^*\Omega\setminus\partial^*\mathscr{E}$. We put

$$C_k = \Big\{x : x \in \partial^*\Omega, \frac{1}{k} < \bar{\Theta}_x(\mathscr{E}) < 1\Big\}, \quad k = 2, 3, \ldots.$$

It remains to prove that $s(C_k) = 0$, $k = 2, 3, \dots$. Using the inclusion $C_k \subset \partial^* \Omega$, for $x \in C_k$ we deduce

$$\bar{\Theta}_x(\mathscr{E}) = \frac{2}{v_n} \limsup_{\varrho \to 0} m_n\Big(\mathscr{E} \bigcap B_\varrho(x)\Big) \geq k^{-1}. \tag{9.5.2}$$

By Lemma 1 we have $P(\mathscr{E}) < \infty$. Therefore, (9.1.18) implies

$$m_n\Big(\mathscr{E} \bigcap B_\varrho(x)\Big) \leq C\big[\operatorname{var} \nabla_{\mathbb{R}^n} \chi_{\mathscr{E}}\big(B_\varrho(x)\big)\big]^{n/(n-1)}.$$

Comparing this with (9.5.2), we obtain

$$\limsup_{\varrho \to 0} \varrho^{1-n} \operatorname{var} \nabla \chi_{\mathscr{E}}\big(B_\varrho(x)\big) \geq \left(\frac{v_n}{2kC}\right)^{(n-1)/n}. \tag{9.5.3}$$

Since $C_k \bigcap \partial^* \mathscr{E} = \varnothing$, we have $\operatorname{var} \nabla \chi_{\mathscr{E}}(C_k) = 0$. Thus (9.5.3) along with Lemma 9.2.5/1 yields $s(C_k) = 0$ and the result follows. □

Lemma 4. *For any $u \in BV(\Omega)$ and almost all $x \in \partial^* \Omega$*

$$-u^*(x) = (-u)^*(x). \tag{9.5.4}$$

Proof. This result is equivalent to the fact that for almost all $x \in \partial^* \Omega$

$$\sup\big\{t \colon x \in \partial^* \mathscr{N}_t\big\} = \inf\big\{t \colon x \in \partial^*(\Omega \setminus \mathscr{N}_t)\big\}.$$

This equality means that

$$\sup\big\{t \colon \Theta_{\mathscr{N}_t}(x) = 1\big\} = \inf\big\{t \colon \Theta_{(\Omega \setminus \mathscr{N}_t)}(x) = 1\big\}.$$

In its turn, this is equivalent to the equality

$$\sup\big\{t \colon \Theta_{\mathscr{N}_t}(x) = 1\big\} = \inf\big\{t \colon \Theta_{\mathscr{N}_t}(x) = 0\big\}.$$

Denote by L and R the left and the right terms of the last equality. It is easy to see that $\Theta_{\mathscr{N}_t}(x)$ is a nonincreasing function on t. So $L \leq R$. Consider the set of points x such that $L(x) < R(x)$. It suffices to prove that the s-measure of this set is zero. Let $\{t_i\}_{i=1}^{\infty}$ be a countable set dense in $\mathbb{R}$ such that $P(E_{t_i}) < \infty$. If $L(x) < R(x)$, then there exists t_i such that $L(x) < t_i < R(x)$. Now our statement follows from the equality $s\{x \in \partial^* \Omega \colon 0 < \Theta_{\mathscr{N}_t}(x) < 1\} = 0$. □

Corollary 2. *For any $u \in BV(\Omega)$ and for almost all $x \in \partial^* \Omega$*

$$\big(u^*\big)^+ = \big(u^+\big)^*, \qquad \big(u^*\big)^- = \big(u^-\big)^*, \tag{9.5.5}$$

and as a result $u^* = (u^+)^* - (u^-)^*$.

Proof. The first equality is obvious. The second one follows from Lemma 4. Indeed, $(u^-)^* = ((-u)^+)^* = ((-u)^*)^+ = (-u^*)^+ = (u^*)^-$.

9.5.2 Integrability of the Rough Trace

Theorem. *Suppose that $P(\Omega) < \infty$ and that a normal to Ω exists s-almost everywhere on $\partial\Omega$. In order for any $u \in BV(\Omega)$*

$$\inf_c \int_{\partial\Omega} \big|u^* - c\big| s(\mathrm{d}x) \le k\|u\|_{BV(\Omega)}, \tag{9.5.6}$$

where k is independent of u, it is necessary and sufficient that the inequality

$$\min\big\{P_{C\Omega}(\mathscr{E}), P_{C\Omega}(\Omega\backslash\mathscr{E})\big\} \le kP_\Omega(\mathscr{E}) \tag{9.5.7}$$

holds for any $\mathscr{E} \subset \Omega$.

Proof. Necessity. Let $\mathscr{E} \subset \Omega$, $P_\Omega(\mathscr{E}) < \infty$. By Lemma 9.5.1/1 we have $P(\mathscr{E}) < \infty$. Let $\chi_{\mathscr{E}}$ be the characteristic function of the set $\mathscr{E}$. Then

$$\begin{aligned}
&\inf_c \int_{\partial\Omega} \big|\chi^*_{\mathscr{E}}(x) - c\big| s(\mathrm{d}x) \\
&\quad = \min_c \big\{|1-c| s\big(\partial^*\mathscr{E} \cap \partial^*\Omega\big) + |c| s\big(\partial^*\Omega\backslash\partial^*\mathscr{E}\big)\big\} \\
&\quad = \min\big\{s\big(\partial^*\mathscr{E} \cap \partial^*\Omega\big), s\big(\partial^*\Omega\backslash\partial^*\mathscr{E}\big)\big\} = \min\big\{P_{C\Omega}(\mathscr{E}), P_{C\Omega}(\Omega\backslash\mathscr{E})\big\}.
\end{aligned}$$

(The preceding equality is valid since, by hypothesis, $s(\partial\Omega\backslash\partial^*\Omega) = 0$.)

On the other hand, $\|\chi_{\mathscr{E}}\|_{BV(\Omega)} = P_\Omega(\mathscr{E})$. Applying (9.5.6), we arrive at (9.5.7).

Sufficiency. Let $u \in BV(\Omega)$. It is clear that $s(\partial\Omega \cap \partial^*\mathscr{N}_t)$ is a nonincreasing function of t. In fact, let $x \in \partial^*\Omega \cap \partial^*\mathscr{N}_t$ and let $\tau < t$. We have

$$\begin{aligned}
1 &= \lim_{\varrho\to 0} 2v_n \varrho^{-n} m_n(\Omega \cap B_\varrho) \ge \lim_{\varrho\to 0} 2v_n \varrho^{-n} m_n(\mathscr{N}_\tau \cap B_\varrho) \\
&\ge \lim_{\varrho\to 0} 2v_n \varrho^{-n} m_n(\mathscr{N}_t \cap B_\varrho) = 1,
\end{aligned}$$

i.e., $x \in \partial^*\Omega \cap \partial^*\mathscr{N}_\tau$.

Similarly, $s(\partial\Omega\backslash\partial^*\mathscr{N}_t)$ is a nondecreasing function of t. From (9.1.16) we obtain

$$k\|u\|_{BV(\Omega)} = k\int_{-\infty}^{\infty} P_\Omega(\mathscr{N}_t)\,\mathrm{d}t \ge \int_{-\infty}^{\infty} \min\big\{s\big(\partial\Omega \cap \partial^*\mathscr{N}_t\big), s\big(\partial\Omega\backslash\partial^*\mathscr{N}_t\big)\big\}\,\mathrm{d}t.$$

Putting

$$t_0 = \sup\big\{t : P(\mathscr{N}_t) < \infty, s\big(\partial\Omega \cap \partial^*\mathscr{N}_t\big) \ge s\big(\partial\Omega\backslash\partial^*\mathscr{N}_t\big)\big\},$$

we obtain

$$
\begin{aligned}
k\|u\|_{BV(\Omega)} &\geq \int_{t_0}^{\infty} s\big(\partial\Omega \cap \partial^*\mathscr{N}_t\big)\,\mathrm{d}t + \int_{-\infty}^{t_0} s\big(\partial\Omega\backslash\partial^*\mathscr{N}_t\big)\,\mathrm{d}t \\
&= \int_{t_0}^{\infty} s\big(\{x : u^*(x) \geq t\}\big)\,\mathrm{d}t + \int_{-\infty}^{t_0} s\big(\{x : u^*(x) \leq t\}\big)\,\mathrm{d}t \\
&= \int_{\partial\Omega} \big[u^*(x) - t_0\big]^+ s(\mathrm{d}x) + \int_{\partial\Omega} \big[u^*(x) - t_0\big]^- s(\mathrm{d}x) \\
&= \int_{\partial\Omega} \big|u^*(x) - t_0\big| s(\mathrm{d}x).
\end{aligned}
$$

Consequently,

$$
k\|u\|_{BV(\Omega)} \geq \inf_c \int_{\partial\Omega} \big|u^* - c\big| s(\mathrm{d}x),
$$

which completes the proof. □

From Corollary 9.4.4/1 we obtain that *the best constant in* (9.5.8) *is equal to* $|\Omega|$ *provided* Ω *is a convex domain.* In particular, for a plane convex domain this constant coincides with the ratio of $\frac{1}{2}s(\partial\Omega)$ to the length of the smallest chord dividing $\partial\Omega$ into arcs of equal length (Corollary 9.4.4/2). According to Corollary 9.4.4/3, *the best constant in* (9.5.6) *equals* $\omega_n/2v_{n-1}$ *for the unit ball.*

9.5.3 Exact Constants in Certain Integral Estimates for the Rough Trace

Definition 1. Let $\mathscr{A} \subset \bar{\Omega}$. Let $\tau_{\mathscr{A}}^{(\alpha)}$ denote the infimum of those k for which $[P_{C\Omega}(\mathscr{E})]^\alpha \leq kP_\Omega(\mathscr{E})$ holds for all $\mathscr{E} \subset \Omega$ that satisfy

$$
m_n(\mathscr{E} \cap \mathscr{A}) + s\big(\mathscr{A} \cap \partial^*\mathscr{E}\big) = 0. \tag{9.5.8}
$$

Theorem. *Let* $P(\Omega) < \infty$ *and suppose a normal to* Ω *exists* s*-almost everywhere on* Ω*. Then, for any function* $u \in BV(\Omega)$ *such that* $u(\mathscr{A} \cap \Omega) = 0$ *and* $u^*(\mathscr{A} \cap \partial^*\Omega) = 0$*, the inequality*

$$
\int_{\partial\Omega} \big|u^*\big| s(\mathrm{d}x) \leq \zeta_{\mathscr{A}}^{(1)} \|u\|_{BV(\Omega)} \tag{9.5.9}
$$

holds. Moreover, the constant $\zeta_{\mathscr{A}}^{(1)}$ *is exact.*

Proof. We have

$$
\int_{\partial\Omega} \big|u^*\big| s(\mathrm{d}x) = \int_0^{\infty} s\big(\{x : u^* \geq t\}\big)\,\mathrm{d}t + \int_0^{\infty} s\big(\{x : -u^* \geq t\}\big)\,\mathrm{d}t.
$$

By (9.1.16), the first integral on the right is equal to

$$\int_0^\infty s\big(\partial^* \mathscr{N}_t \cap \partial^* \Omega\big)\,\mathrm{d}t = \int_0^\infty P_{C\Omega}(\mathscr{N}_t)\,\mathrm{d}t.$$

Note that $m_n(\mathscr{A} \cap \mathscr{N}_t) + s(\mathscr{A} \cap \partial^* \mathscr{N}_t) = 0$ for almost all t. Consequently, by the definition of $\zeta^{(1)}_{\mathscr{A}}$,

$$\int_0^\infty s\big(\{x : u^* \geq t\}\big)\,\mathrm{d}t \leq \int_0^\infty P_{C\Omega}(\mathscr{N}_t)\,\mathrm{d}t \leq \zeta^{(1)}_{\mathscr{A}} \int_0^\infty P_\Omega(\mathscr{N}_t)\,\mathrm{d}t.$$

Similarly we have

$$\int_0^\infty s\big(\{x : -u^* \geq t\}\big)\,\mathrm{d}t \leq \int_{-\infty}^0 P_{C\Omega}(\Omega\backslash\mathscr{N}_t)\,\mathrm{d}t \leq \zeta^{(1)}_{\mathscr{A}} \int_{-\infty}^0 P_\Omega(\mathscr{N}_t)\,\mathrm{d}t.$$

Finally,

$$\int_{\partial\Omega} |u^*| s(\mathrm{d}x) \leq \zeta^{(1)}_{\mathscr{A}} \|u\|_{BV(\Omega)}.$$

To see the sharpness of the constant $\zeta^{(1)}_{\mathscr{A}}$ it suffices to put $u = \chi_{\mathscr{E}}$ into (9.5.9), where $\mathscr{E}$ is a set satisfying (9.5.8). □

Definition 2. We introduce the function

$$\zeta_\alpha(S) = \sup\bigg\{\frac{[P_{C\Omega}(\mathscr{E})]^\alpha}{P_\Omega(\mathscr{E})} : \mathscr{E} \subset \Omega, P_\Omega(\mathscr{E}) > 0, P_{C\Omega}(\mathscr{E}) \leq S\bigg\}.$$

The preceding theorem implies the following obvious corollary.

Corollary. *Suppose that $P(\Omega) < \infty$ and that a normal to Ω exists s-almost everywhere on $\partial\Omega$. Then for any $u \in BV(\Omega)$ with $s(\{x : u^*(x) \neq 0\}) \leq S$ the inequality*

$$\int_{\partial\Omega} |u^*| s(\mathrm{d}x) \leq \zeta_1(S) \|u\|_{BV(\Omega)}$$

holds. Moreover, the constant $\zeta_1(S)$ is exact.

From Lemma 9.4.4 it follows that for a ball the function $\zeta_1(S)$ coincides with the ratio of S to the area of the base of the spherical segment whose spherical part of the boundary has the area S.

In particular, $\zeta_1(S) = S/(2\sin(S/2))$ for $n = 2$ and $\zeta_1(S) = 4\pi/(4\pi - S)$ for $n = 3$.

Lemma. *Let Ω be a domain with $P(\Omega) < \infty$ and suppose that a normal to Ω exists s-almost everywhere on $\partial\Omega$. Then*

$$\eta(S) \stackrel{\text{def}}{=} \inf\big\{P_\Omega(\mathscr{E}) : \mathscr{E} \subset \Omega, P_{C\Omega}(\mathscr{E}) \geq S, P_{C\Omega}(\Omega\backslash\mathscr{E}) \geq S\big\} > 0.$$

Proof. Let $\{\mathscr{E}_i\}$ be the minimizing sequence for $\eta(S)$. If

$$\liminf_{i\to\infty} \min\{m_n\mathscr{E}_i, m_n(\Omega\backslash\mathscr{E}_i)\} > 0,$$

then the result follows from Theorem 9.1.3 and Lemma 5.2.4. Let this lower limit be equal to zero and, for the sake of definiteness, let $m_n\mathscr{E}_i \to 0$. Then $m_n(\Omega\backslash\mathscr{E}_i) \to m_n(\Omega)$. By Lemma 9.1.3/1 we obtain

$$\liminf_{i\to\infty} P(\Omega\backslash\mathscr{E}_i) \le P(\Omega) = s(\partial\Omega)$$

and by Lemma 9.4.2,

$$s(\partial\Omega) = P_{C\Omega}(\mathscr{E}_i) + P_{C\Omega}(\Omega\backslash\mathscr{E}_i).$$

Moreover, we always have

$$P(\Omega\backslash\mathscr{E}_i) = P_\Omega(\Omega\backslash\mathscr{E}_i) + P_{C\Omega}(\Omega\backslash\mathscr{E}_i).$$

Thus, for any $\varepsilon > 0$ and for large enough i, we obtain

$$P_\Omega(\Omega\backslash\mathscr{E}_i) \ge P_{C\Omega}(\mathscr{E}_i) - \varepsilon \ge S - \varepsilon,$$

i.e., $\inf P_\Omega(\Omega\backslash\mathscr{E}) \ge S$. The lemma is proved. □

We can easily see that the function η introduced in the preceding lemma is connected with ζ_α by the inequality

$$\zeta_\alpha(S) = S^\alpha/\eta(S).$$

The same lemma immediately implies that $\zeta_\alpha(S)$ is finite for all $S \in (0, s(\partial\Omega))$ provided Ω is a domain with $P(\Omega) = s(\partial\Omega) < \infty$ and $\zeta_\alpha(S) < \infty$ for some $S < s(\partial\Omega)$.

Hence from Theorem 9.5.2 we conclude that (9.5.6) holds if and only if $\zeta_i(S) < \infty$ for some $S \in (0, s(\partial\Omega))$.

9.5.4 More on Integrability of the Rough Trace

Theorem. *Let $P(\Omega) < \infty$ and suppose that a normal to Ω exists s-almost everywhere on $\partial\Omega$. For any function $u \in BV(\Omega)$ to satisfy the inequality*

$$\|u^*\|_{L(\partial\Omega)} \le k\big(\|u\|_{BV(\Omega)} + \|u\|_{L_1(\Omega)}\big), \tag{9.5.10}$$

where the constant k is independent of u, it is necessary and sufficient that there exist a $\delta > 0$ such that for each measurable set $\mathscr{E} \subset \Omega$ with diameter less than δ the inequality

$$P_{C\Omega}(\mathscr{E}) \le k_1 P_\Omega(\mathscr{E}) \tag{9.5.11}$$

holds where k_1 is a constant independent of $\mathscr{E}$.

Proof. The necessity of (9.5.11) easily follows by the insertion of $u = \chi_{\mathscr{E}}$ into (9.5.10) and then by application of the isoperimetric inequality. The sufficiency results from Theorem 9.5.3 if we use a partition of unity (cf. the proof of Theorem 9.2.2/2). □

Remark. If each of the sets Ω_1, Ω_2 satisfies the hypothesis of the preceding theorem then their union has the same property.

The proof follows from formula (9.3.6).

9.5.5 Extension of a Function in $BV(\Omega)$ to $C\Omega$ by a Constant

In the present subsection we assume that $P(\Omega) < \infty$ and $s(\partial\Omega\backslash\partial^*\Omega) = 0$.

We introduce the notation $u_c(x) = u(x)$ for $x \in \Omega$, $u_c(x) = c$ for $x \in C\Omega$ where $c \in \mathbb{R}^1$.

Lemma. *The equality*

$$\|u_c\|_{BV(\mathbb{R}^n)} = \|u\|_{BV(\Omega)} + \big\|u^* - c\big\|_{L_1(\partial\Omega)} \tag{9.5.12}$$

holds.

Proof. We have

$$\begin{aligned}\|u_c\|_{BV(\mathbb{R}^n)} &= \int_0^\infty P\big(\{x : |u_c - c| > t\}\big)\,\mathrm{d}t \\ &= \int_0^\infty P_\Omega\big(\{x : |u - c| > t\}\big)\,\mathrm{d}t \\ &\quad + \int_0^\infty P_{C\Omega}\big(\{x : |u - c| > t\}\big)\,\mathrm{d}t.\end{aligned} \tag{9.5.13}$$

It is clear that

$$\int_0^\infty P_\Omega\big(\{x : |u - c| > t\}\big)\,\mathrm{d}t = \|u\|_{BV(\Omega)}. \tag{9.5.14}$$

Further, since $s(\partial\Omega\backslash\partial^*\Omega) = 0$, we see that

$$\begin{aligned}&\int_0^\infty P_{C\Omega}\big(\{x : |u - c| > t\}\big)\,\mathrm{d}t \\ &= \int_0^\infty s\big(\{x : (u-c)^* > t\}\big)\,\mathrm{d}t + \int_{-\infty}^0 s\big(\{x : (u-c)^* < t\}\big)\,\mathrm{d}t \\ &= \int_{\partial\Omega} \big|(u-c)^*\big| s(\mathrm{d}x) = \int_{\partial\Omega} |u^* - c| s(\mathrm{d}x),\end{aligned}$$

which together with (9.5.13) and (9.5.14) implies (9.5.12). □

Let $\overset{\circ}{BV}(\Omega)$ denote the subset $BV(\Omega)$ which contains functions with

$$\|u_0\|_{BV(\mathbb{R}^n)} = \|u\|_{BV(\Omega)}.$$

Then from (9.5.12) it follows that $u \in \mathring{BV}(\Omega)$ if and only if $u^* = 0$ for the class of domains under consideration. Thus the elements of the quotient-space $BV(\Omega)/\mathring{BV}(\Omega)$ are classes of functions that have the rough traces u^*.

Formula (9.5.12) and Theorem 9.5.2 imply the next assertion.

Corollary 1. *If, for any $\mathscr{E} \subset \Omega$,*

$$\min\bigl\{P_{C\Omega}(\mathscr{E}), P_{C\Omega}(\Omega\backslash\mathscr{E})\bigr\} \le kP_\Omega(\mathscr{E}), \tag{9.5.15}$$

where k is a constant that is independent of $\mathscr{E}$, then there exists a c such that

$$\|u_c\|_{BV(\mathbb{R}^n)} \le (k+1)\|u\|_{BV(\Omega)}. \tag{9.5.16}$$

Conversely, if for each $u \in BV(\Omega)$ there exists a c such that (9.5.16) holds with k independent of u, then (9.5.15) holds for any $\mathscr{E} \subset \Omega$.

Corollary 2. *For the inequality*

$$\|u_0\|_{BV(\mathbb{R}^n)} \le k\bigl(\|u\|_{BV(\Omega)} + \|u\|_{L_1(\Omega)}\bigr), \tag{9.5.17}$$

where k is independent of u, to hold for any $u \in BV(\Omega)$, it is necessary and sufficient that there exists a $\delta > 0$ such that for any measurable set $\mathscr{E} \subset \Omega$ with $\operatorname{diam}\mathscr{E} < \delta$ *the inequality $P_{C\Omega}(\mathscr{E}) \le k_1 P_\Omega(\mathscr{E})$ holds where k_1 is independent of $\mathscr{E}$.*

Proof. The necessity follows immediately from (9.5.12) and the isoperimetric inequality. The sufficiency results from (9.5.12) and Theorem 9.5.4. □

Inequality (9.3.6) implies the following corollary.

Corollary 3. *If each of the sets Ω_1, Ω_2 satisfies the hypothesis of Corollary 2, then their union has the same property.*

In particular, this implies that any function in $BV(\Omega)$ can be extended by zero to the whole space so that (9.5.17) is valid for domains Ω that are finite unions of domains in $C^{0,1}$.

9.5.6 Multiplicative Estimates for the Rough Trace

Let $P(\Omega) < \infty$ and suppose a normal to Ω exists s-almost everywhere on $\partial\Omega$. Let $\mathscr{A}$ denote a subset of $\bar{\Omega}$; by $\zeta^{(\alpha)}_{\mathscr{A}}$, $\zeta_\alpha(S)$ we mean the functions introduced in Definitions 9.5.3/1 and 9.5.3/2.

The following assertion supplements Theorem 9.5.3.

Theorem. 1. *If $\zeta^{(1/q^*)}_{\mathscr{A}} < \infty$, where $q^* \le 1$, then, for any $u \in BV(\Omega)$ such that*

$$u(x) = 0 \quad \text{for } x \in \mathscr{A} \cap \Omega, \qquad u^*(x) = 0 \quad \text{for } x \in \mathscr{A} \cap \partial^* \Omega, \tag{9.5.18}$$

the inequality

$$\|u^*\|_{L_q(\partial\Omega)} \le C \|u\|_{BV(\Omega)}^{1-\varkappa} \|u^*\|_{L_t(\partial\Omega)}^{\varkappa} \tag{9.5.19}$$

holds where $0 < t < q < q^*$,

$$\varkappa = \frac{t(q^* - q)}{q(q^* - t)}, \tag{9.5.20}$$

and $C^{q(q^*-t)/q^*(q-t)} \le c\zeta_{\mathscr{A}}^{(1/q^*)}$.

2. *If for all* $u \in BV(\Omega)$ *satisfying* (9.5.18) *the inequality* (9.5.19) *holds with* $\varkappa$ *specified by* (9.5.20) *and with* $q^* > q$, $q^* > t$, *then*

$$\zeta_{\mathscr{A}}^{(1/q^*)} \le C^{q(q^*-t)/q^*(q-t)}.$$

Proof. 1. Following the same line of reasoning as in the proof of Theorem 9.5.3, we obtain

$$\int_0^\infty \big[s(\Gamma_\tau)\big]^{1/q^*} \, \mathrm{d}\tau \le \zeta_{\mathscr{A}}^{(1/q^*)} \|u\|_{BV(\Omega)}, \tag{9.5.21}$$

where $\Gamma_\tau = \{x \in \partial\Omega : |u^*(x)| \ge \tau\}$.

In Lemma 1.3.5/2 we put $\xi = t^q$, $f(\xi) = s(\Gamma_\tau)$, $b = 1/q^*$, $a \in (1, \infty)$, $\lambda = a(q-t)/q$, and $\mu = (q^* - q)/q^* q$. Then

$$\int_0^\infty s(\Gamma_\tau)\tau^{q-1} \, \mathrm{d}\tau \le c\Big(\int_0^\infty \big[s(\Gamma_\tau)\big]^a \tau^{at-1} \, \mathrm{d}\tau\Big)^{(q^*-q)/q(q^*-t)} \times \Big(\int_0^\infty \big[s(\Gamma_\tau)\big]^{1/q^*} \, \mathrm{d}\tau\Big)^{q^*(q-t)/(q^*-t)}. \tag{9.5.22}$$

Since $a > 1$ and the function $s(\Gamma_\tau)$ does not increase, Lemma 1.3.5/1 can be applied to the first factor. Then we have

$$\int_0^\infty \big[s(\Gamma_\tau)\big]^a \tau^{at-1} \, \mathrm{d}\tau \le c\Big(\int_0^\infty s(\Gamma_\tau)\tau^{t-1} \, \mathrm{d}\tau\Big)^a = c\Big(\int_{\partial\Omega} |u^*(x)|^t s(\mathrm{d}x)\Big)^a. \tag{9.5.23}$$

Combining (9.5.21)–(9.5.23) we arrive at item 1 of the theorem.

2. The lower bound for the constant C results by insertion of $\chi_{\mathscr{E}}$, where $\mathscr{E}$ satisfies (9.5.8), into (9.5.19). □

Let us consider two domains for which we can obtain exact conditions for the boundedness of the function $\zeta_{\mathscr{A}}^{(\alpha)}$.

Example 1. Let $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$ and let

$$\Omega = \{x : 1 < x_n < |x'|^{-\beta}, |x'| < 1\}, \quad 0 < \beta < n-2.$$

Further let $\mathscr{A} = \{x : x_n = 1, |x'| < 1\}$. We show that $\zeta^{(\alpha)}_{\mathscr{A}} < \infty$ for $\alpha = (n-1)/(n-2-\beta)$. (Using the sequence of sets $\mathscr{E}_m = \{x \in \Omega : x_n > m\}$, $m = 1, 2, \dots$, we can prove that this α is the best possible.)

Although Ω is not convex we can apply to it the proof of Lemma 9.3.2/3. Therefore, it suffices to verify the estimate

$$\big[P_\Omega(\mathscr{E})\big]^{(n-1)/(n-2-\beta)} \le c P_\Omega(\mathscr{E}) \tag{9.5.24}$$

for any set $\mathscr{E}$ that is the intersection of a polyhedron with Ω, $\mathscr{E} \cap \mathscr{A} = \varnothing$. Since (9.5.24) has already been obtained in Example 6.11.3/1, the result follows.

Example 2. Using the same argument and referring to Example 6.11.3/2, we can show that for

$$\Omega = \{x : |x'| < x_n^\beta, 0 < x_n < 1\}, \quad \beta \ge 1,$$

and for $\mathscr{A} = \{x : |x'| < 1, x_n = 1\}$ the value $\zeta^{(\alpha)}_{\mathscr{A}}$ is finite for $\alpha = \beta(n-1)/(\beta(n-2)+1)$ and that α is the best as possible.

The theorem of the present subsection implies that boundedness of the value $\zeta_{1/q^*}(S)$ is necessary and sufficient for the validity of (9.5.19) for any function $u \in BV(\Omega)$ with $s(\{x : u^*(x) \ne 0\}) \le S$ (cf. Corollary 9.5.3). Hence we easily conclude that the boundedness of $\zeta_{1/q^*}(S)$ for some $S < P(\Omega)$ is necessary and sufficient for the validity of the inequality

$$\|u^*\|_{L_q(\partial\Omega)} \le \big(C_1\|u\|_{BV(\Omega)} + C_2\|u^*\|_{L_r(\partial\Omega)}\big)^{1-\varkappa} \|u^*\|^{\varkappa}_{L_t(\partial\Omega)}, \tag{9.5.25}$$

where u is any function in $BV(\Omega)$, $r < q^*$ and q^*, q, t, $\varkappa$ are the same as in the theorem of the present subsection (cf. Theorem 6.3.3).

9.5.7 Estimate for the Norm in $L_{n/(n-1)}(\Omega)$ of a Function in $BV(\Omega)$ with Integrable Rough Trace

To conclude this section we prove an assertion similar to Corollary 5.6.3.

Theorem. *Suppose that $P(\Omega) < \infty$ and that a normal to Ω exists s-almost everywhere on $\partial\Omega$. Then for any $u \in BV(\Omega)$ the inequality*

$$\|u\|_{L_{n/(n-1)}(\Omega)} \le n v_n^{-1/n} \big(\|u\|_{BV(\Omega)} + \|u^*\|_{L_1(\partial\Omega)}\big) \tag{9.5.26}$$

is valid. Moreover, the constant $n v_n^{-1/n}$ is exact.

Proof. By (5.6.14) we have

$$\|u\|_{L_{n/(n-1)}(\Omega)} \le \int_0^\infty \big[m_n(\mathscr{N}_t)\big]^{(n-1)/n}\,\mathrm{d}t + \int_{-\infty}^0 \big[m_n(\Omega\backslash\mathscr{N}_t)\big]^{(n-1)/n}\,\mathrm{d}t, \tag{9.5.27}$$

where $\mathscr{N}_t = \{x : u(x) \ge t\}$. By the isoperimetric inequality (9.1.15) we obtain

$$\big[m_n(\mathscr{N}_t)\big]^{(n-1)/n} \le n v_n^{-1/n} P(\mathscr{N}_t) = n v_n^{-1/n}\big[P_\Omega(\mathscr{N}_t) + s\big(\partial^*\mathscr{N}_t \cap \partial^*\Omega\big)\big].$$

Since $s(\partial^*\mathscr{N}_t \cap \partial^*\Omega) = s(\{x : u^* \ge t\})$ for almost all t (Lemma 9.5.1/2), we see that

$$\int_0^\infty \big[m_n(\mathscr{N}_t)\big]^{(n-1)/n}\,\mathrm{d}t \le n v_n^{-1/n}\bigg[\int_0^\infty P_\Omega(\mathscr{N}_t)\,\mathrm{d}t + \int_0^\infty s\big(\{x : u^* \ge t\}\big)\,\mathrm{d}t\bigg].$$

Taking into account that $P_{C\Omega}(\mathscr{N}_t) + P_{C\Omega}(\Omega\backslash\mathscr{N}_t) = P(\Omega)$ by Lemma 9.4.2, we similarly obtain that

$$\begin{aligned}&\int_{-\infty}^0 \big[m_n(\Omega\backslash\mathscr{N}_t)\big]^{(n-1)/n}\,\mathrm{d}t\\ &\qquad \le n v_n^{-1/n}\bigg[\int_{-\infty}^0 P_\Omega(\mathscr{N}_t)\,\mathrm{d}t + \int_{-\infty}^0 s\big(\{x : u^* \le t\}\big)\,\mathrm{d}t\bigg].\end{aligned}$$

Consequently,

$$\begin{aligned}&\bigg[\int_\Omega |u|^{n/(n-1)}\,\mathrm{d}x\bigg]^{(n-1)/n}\\ &\qquad \le n v_n^{-1/n}\bigg\{\int_{-\infty}^\infty P_\Omega(\mathscr{N}_t)\,\mathrm{d}t + \int_0^\infty s\big(\{x : |u^*| \ge t\}\big)\,\mathrm{d}t\bigg\}\\ &\qquad = n v_n^{-1/n}\bigg(\|u\|_{BV(\Omega)} + \int_{\partial\Omega} |u^*|\,s(\mathrm{d}x)\bigg).\end{aligned}$$

The sharpness of the constant in (9.5.26) is a corollary of the fact that (9.5.26) becomes an equality for $u = \chi_{B_\varrho}$ where B_ϱ is a ball in Ω. □

Similarly to the preceding theorem, we can generalize Theorem 5.6.3 to functions in $BV(\Omega)$.

9.6 Traces of Functions in $BV(\Omega)$ on the Boundary and Gauss–Green Formula

9.6.1 Definition of the Trace

Let Ω be an open set in $\mathbb{R}^n$ and let the function u be integrable in a neighborhood of a point $x \in \partial\Omega$. *The upper and the lower traces of the functions u at the point x* are the numbers

$$\bar{u}(x) = \limsup_{\varrho \to 0} \frac{1}{m_n(B_\varrho(x) \cap \Omega)} \int_{B_\varrho(x)\cap\Omega} u(y)\,\mathrm{d}y,$$
$$\underline{u}(x) = \liminf_{\varrho \to 0} \frac{1}{m_n(B_\varrho(x) \cap \Omega)} \int_{B_\varrho(x)\cap\Omega} u(y)\,\mathrm{d}y,$$

respectively.

If $\bar{u}(x) = \underline{u}(x)$, this common value is called *the trace* $\tilde{u}(x)$ *of the function* u *at the point* $x \in \partial\Omega$.

9.6.2 Coincidence of the Trace and the Rough Trace

Lemma. *Let* $u \in BV(\Omega)$, $u \geq 0$ *and let*

$$\int_{\partial^*\Omega} u^*(x) s(\mathrm{d}x) < \infty.$$

Then for any $x \in \partial^*\Omega$ *the inequality*

$$\underline{u}(x) \geq u^*(x) \tag{9.6.1}$$

holds.

Proof. By Theorem 9.5.7, the function u is integrable in Ω and hence the function $\underline{u}(x)$ is defined.

Inequality (9.6.1) is trivial provided $u^*(x) = 0$. Suppose $0 < u^*(x) < \infty$. We take an arbitrary $\varepsilon > 0$ and choose t to satisfy $0 < u^*(x) - t < \varepsilon$ and $P_\Omega(\mathscr{N}_t) < \infty$. Then $x \in \partial^*\mathscr{N}_t$ where $\mathscr{N}_t = \{y : u(y) \geq t\}$.

It is clear that the normal to $\mathscr{N}_t$ at the point x coincides with the normal to Ω. Consequently, we can find $r_0(x) > 0$ such that

$$1 - \varepsilon < \frac{m_n(\mathscr{N}_t \cap B_r(x))}{m_n(\Omega \cap B_r(x))} \leq 1 \tag{9.6.2}$$

for $0 < r < r_0(x)$. Since

$$\int_{B_r(x)\cap\Omega} u(y)\,\mathrm{d}y = \int_0^\infty m_n\big(\mathscr{N}_\tau \cap B_r(x)\big)\,\mathrm{d}\tau,$$

(9.6.2) implies

$$\begin{aligned}\frac{1}{m_n(B_r \cap \Omega)} \int_{B_r\cap\Omega} u(y)\,\mathrm{d}y &\geq \frac{1}{m_n(B_r \cap \Omega)} \int_0^t m_n(\mathscr{N}_\tau \cap B_r)\,\mathrm{d}\tau \\ &\geq t\frac{m_n(\mathscr{N}_t \cap B_r)}{m_n(\Omega \cap B_r)} \geq (1-\varepsilon)t,\end{aligned}$$

which proves (9.6.1) for $u^*(x) < \infty$.

In the case $u^*(x) = \infty$ the arguments are similar. □

Theorem. *Let $P(\Omega) < \infty$ and suppose that a normal to Ω exists s-almost everywhere on $\partial\Omega$. If $u \in BV(\Omega)$ and*

$$\int_{\partial\Omega} |u^*| s(\mathrm{d}x) < \infty,$$

then the trace $\tilde{u}$ of the function u exists s-almost everywhere on $\partial\Omega$ and coincides with the rough trace u^*.

Proof. By Theorem 9.5.7, the function u is integrable in Ω and consequently, the upper and lower traces $\bar{u}$ and $\underline{u}$ are defined.

First, consider the case of a nonnegative function u. Then by Lemma, $\underline{u}(x) \geq u^*(x)$ for all $x \in \partial^*\Omega$. Next we prove that the inequality $\bar{u}(x) \leq u^*(x)$ holds for s-almost all $x \in \partial\Omega$, if $u \geq 0$. Suppose that

$$s\big(\{x \in \partial^*\Omega : \bar{u}(x) > u^*(x)\}\big) > 0.$$

Then there exists a $c > 0$ such that $s(Q) > 0$ where $Q = \{x : x \in \partial^*\Omega, \bar{u}(x) > u^*(x) + c\}$. Recalling the definition of $\bar{u}(x)$, for $x \in Q$ we have

$$c + u^*(x) \leq \limsup_{\varrho \to 0} \frac{1}{m_n(B_\varrho(x) \cap \Omega)} \int_0^\infty m_n\big(\mathscr{N}_t \cap B_\varrho(x)\big)\,\mathrm{d}t.$$

Since $x \in \partial^*\Omega$, we see that

$$\lim_{\varrho \to 0} \frac{2}{v_n \varrho^n} m_n\big(\Omega \cap B_\varrho(x)\big) = 1. \tag{9.6.3}$$

Therefore,

$$\begin{aligned} c + u^*(x) &\leq \frac{2}{v_n} \lim_{\varrho \to 0} \varrho^{-n} \int_0^\infty m_n\big(\mathscr{N}_t \cap B_\varrho(x)\big)\,\mathrm{d}t \\ &\leq \left(\frac{2}{v_n}\right)^{(n-1)/n} \lim_{\varrho \to 0} \varrho^{1-n} \int_0^\infty \big[m_n\big(\mathscr{N}_t \cap B_\varrho(x)\big)\big]^{(n-1)/n}\,\mathrm{d}t. \end{aligned} \tag{9.6.4}$$

The equality (9.6.3) implies

$$m_n\big(\mathscr{N}_t \cap B_\varrho(x)\big) \leq \alpha_\varrho \min\big\{m_n\big(\mathscr{N}_t \cap B_\varrho(x)\big), m_n\big(B_\varrho(x)\backslash\mathscr{N}_t\big)\big\},$$

where α_ϱ does not depend on t and $\alpha_\varrho \to 1$ as $\varrho \to 0$. Applying the relative isoperimetric inequality (9.1.18), we obtain

$$\big[m_n\big(\mathscr{N}_t \cap B_\varrho(x)\big)\big]^{\frac{n-1}{n}} \leq \alpha_\varrho^{\frac{n-1}{n}} \left(\frac{v_n}{2}\right)^{\frac{n-1}{n}} v_{n-1}^{-1} \operatorname{var} \nabla\chi_{\mathscr{N}_t}\big(B_\varrho(x)\big). \tag{9.6.5}$$

Noting that

$$\operatorname{var} \nabla\chi_{\mathscr{N}_t}(B_\varrho) = \operatorname{var} \nabla\chi_{\mathscr{N}_t}(B_\varrho \cap \Omega) + s\big(\partial^* \Omega \cap \partial^* \mathscr{N}_t\big),$$

and integrating (9.6.5) with respect to t, we obtain

$$\begin{aligned}
&\int_0^\infty \big[m_n\big(\mathscr{N}_t \cap B_\varrho(x)\big)\big]^{(n-1)/n}\,\mathrm{d}t \\
&\quad \le \alpha_n^{\frac{n-1}{n}} \left(\frac{v_n}{2}\right)^{\frac{n-1}{n}} v_{n-1}^{-1} \Big\{ \int_0^\infty \operatorname{var} \nabla\chi_{\mathscr{N}_t}\big(B_\varrho(x) \cap \Omega\big)\,\mathrm{d}t \\
&\qquad + \int_0^\infty s\big(\partial^*\Omega \cap \partial^* \mathscr{N}_t\big)\,\mathrm{d}t \Big\} \\
&\quad = \alpha_\varrho^{\frac{n-1}{n}} \left(\frac{v_n}{2}\right)^{\frac{n-1}{n}} v_{n-1}^{-1} \Big\{ \operatorname{var} \nabla u\big(B_\varrho(x)\big) + \int_{\partial^*\Omega \cap B_\varrho(x)} u^*(y) s(\mathrm{d}y) \Big\}.
\end{aligned} \tag{9.6.6}$$

Comparing (9.6.4) and (9.6.6) and taking into account that $\alpha_\varrho \to 1$ as $\varrho \to 0$ we obtain

$$\begin{aligned}
c + u^*(x) \le v_{n-1}^{-1} \Big\{ &\limsup_{\varrho\to 0} \varrho^{1-n} \operatorname{var} \nabla u\big(B_\varrho(x)\big) \\
&+ \limsup_{\varrho\to 0} \varrho^{1-n} \int_{\partial^*\Omega \cap B_\varrho(x)} u^*(y) s(\mathrm{d}y) \Big\}.
\end{aligned} \tag{9.6.7}$$

By (9.2.13),

$$\lim_{\varrho\to 0} \varrho^{1-n} \operatorname{var} \nabla\chi_\Omega\big(B_\varrho(x)\big) = v_{n-1}$$

for s-almost all $x \in \partial^*\Omega$. On the other hand, $\operatorname{var} \nabla\chi_\Omega(B_\varrho) = s(\partial^*\Omega \cap B_\varrho)$. Therefore, for s-almost all $x \in Q$ inequality (9.6.7) can be rewritten in the form

$$\begin{aligned}
c + u^*(x) \le v_{n-1}^{-1} &\limsup_{\varrho\to 0} \varrho^{1-n} \operatorname{var} \nabla u\big(B_\varrho(x)\big) \\
&+ \limsup_{\varrho\to 0} \frac{1}{s(\partial^*\Omega \cap B_\varrho(x))} \int_{\partial^*\Omega \cap B_\varrho(x)} u^*(y) s(\mathrm{d}y).
\end{aligned} \tag{9.6.8}$$

The integral

$$I(\mathscr{E}) = \int_{\mathscr{E}} u^*(y) s(\mathrm{d}y)$$

is absolutely continuous relative to the measure $s(\mathscr{E})$. Thus the derivative

$$\frac{\mathrm{d}I}{\mathrm{d}s}(x) = \lim_{\varrho\to 0} \frac{1}{s(\partial^*\Omega \cap B_\varrho(x))} \int_{\partial^*\Omega \cap B_\varrho(x)} u^*(y) s(\mathrm{d}y) = u^*(x)$$

exists for s-almost all $x \in \partial^*\Omega$ (see, for instance, Hahn and Rosenthal [336], p. 290). Therefore, for s-almost all $x \in Q$ the inequality (9.6.8) can be rewritten as

$$cv_{n-1} \le \limsup_{\varrho\to 0} \varrho^{1-n} \operatorname{var} \nabla u\big(B_\varrho(x)\big). \tag{9.6.9}$$

Since $\operatorname{var} \nabla u(\mathbb{R}^n) < \infty$ and $\operatorname{var} \nabla u(Q) = 0$, Lemma 9.2.5/1 and (9.6.9) imply $s(Q) = 0$. The assertion is proved.

Now let u be an arbitrary function in $BV(\Omega)$. Then the functions

$$u^+ = \frac{1}{2}\big(u + |u|\big), \qquad u^- = \frac{1}{2}\big(|u| - u\big),$$

are also in $BV(\Omega)$. By what we proved previously, the equalities

$$\tilde{u}^+(x) = \big(u^+(x)\big)^*, \qquad \tilde{u}^-(x) = \big(u^-(x)\big)^*, \tag{9.6.10}$$

hold for s-almost everywhere on $\partial^*\Omega$. Consequently, the trace $\tilde{u}$ of the function u exists s-almost everywhere on $\partial^*\Omega$. Moreover,

$$\tilde{u}(x) = \tilde{u}^+(x) - \tilde{u}^-(x). \tag{9.6.11}$$

Further, by (9.5.5) we have

$$u^*(x) = \big(u^+(x)\big)^* - \big(u^-(x)\big)^*. \tag{9.6.12}$$

Comparing equalities (9.6.10)–(9.6.12), we conclude that $\tilde{u}(x) = u^*(x)$. □

9.6.3 Trace of the Characteristic Function

The hypothesis of Theorem 9.6.2 may be weakened for the characteristic function. Namely, the following lemma holds.

Lemma. *Let $P(\Omega) < \infty$, $\mathscr{E} \subset \Omega$, and $P_\Omega(\mathscr{E}) < \infty$. Then the trace of $\tilde{\chi}_{\mathscr{E}}$ of the function $\chi_{\mathscr{E}}$ exists and coincides with $(\chi_{\mathscr{E}})^*$ for s-almost all $x \in \partial^*\Omega$.*

It is just a reformulation of Lemma 9.5.1/3.

9.6.4 Integrability of the Trace of a Function in $BV(\Omega)$

Theorem. *Let $P(\Omega) < \infty$ and suppose that a normal to Ω exists s-almost everywhere on $\partial\Omega$. Then:*

1. *If for any measurable set $\mathscr{E}$ the inequality*

$$\min\big\{P_{C\Omega}(\mathscr{E}), P_{C\Omega}(\Omega\backslash\mathscr{E})\big\} \le kP_\Omega(\mathscr{E}), \tag{9.6.13}$$

where k is independent of $\mathscr{E}$, is valid, then the trace $\tilde{u}$ exists for any $u \in BV(\Omega)$. Moreover,

$$\inf_c \int_{\partial\Omega} |\tilde{u} - c|s(\mathrm{d}x) \le k\|u\|_{BV(\Omega)}. \tag{9.6.14}$$

2. *If the inequality* (9.6.14), *with a constant* k *independent of* u, *holds for any* $u \in BV(\Omega)$ *having a trace* $\tilde{u}$ *on* $\partial\Omega$ *then, for any measurable set* $\mathscr{E} \subset \Omega$, *the estimate* (9.6.13) *is true.*

Proof. 1. By Theorem 9.5.2, the rough trace of u is integrable on $\partial\Omega$. Consequently, by Theorem 9.6.2, s-almost everywhere on $\partial\Omega$ there exists the trace $\tilde{u}$ which coincides with u^*. Therefore (9.5.6) implies (9.6.14).

2. Let $\mathscr{E}$ be the measurable subset of Ω with $P_\Omega(\mathscr{E}) < \infty$. By Lemma 9.6.3 the trace $\tilde{\chi}_{\mathscr{E}}$ of the function $\chi_{\mathscr{E}}$ exists s-almost everywhere and equals $\chi^*_{\mathscr{E}}$. Thus, by inserting $u = \chi_{\mathscr{E}}$ into (9.6.14), we obtain

$$\inf \int_{\partial\Omega} \left|\chi^*_{\mathscr{E}} - c\right| s(\mathrm{d}x) \le k P_\Omega(\mathscr{E}),$$

which is equivalent to (9.6.13) (compare with the proof of necessity in Theorem 9.5.2). The theorem is proved. □

9.6.5 Gauss–Green Formula for Functions in $BV(\Omega)$

Lemma. *For any function* $u \in BV(\Omega)$ *and any measurable set* $\mathfrak{B} \subset \Omega$ *the equality*

$$\nabla u(\mathfrak{B}) = \int_{-\infty}^{\infty} \nabla \chi_{\mathscr{N}_t}(\mathfrak{B})\, \mathrm{d}t \tag{9.6.15}$$

holds where $\mathscr{N}_t = \{x : u(x) \ge t\}$.

Proof. It suffices to prove (9.6.15) for $u \ge 0$. Let φ be an infinitely differentiable function with compact support in Ω. Then

$$-\int_\Omega \varphi \nabla u(\mathrm{d}x) = \int_\Omega u \nabla\varphi(\mathrm{d}x) = \int_\Omega \int_0^\infty \chi_{\mathscr{N}_t}(x)\, \mathrm{d}t \nabla\varphi(\mathrm{d}x).$$

By the Fubini theorem, the double integral equals

$$\int_0^\infty \mathrm{d}t \int_\Omega \chi_{\mathscr{N}_t}(x) \nabla\varphi(\mathrm{d}x).$$

Moreover, we note that

$$\int_\Omega \varphi \nabla u(\mathrm{d}x) = \int_0^\infty \int_\Omega \varphi \nabla \chi_{\mathscr{N}_t}(\mathrm{d}x) = \int_\Omega \varphi\, \mathrm{d}x \int_0^\infty \nabla\chi_{\mathscr{N}_t}\, \mathrm{d}t \tag{9.6.16}$$

for almost all t. Here we may reverse the order of integration since the equality (9.1.16) implies the finiteness of the integral

$$\int_0^\infty \mathrm{d}t \int_\Omega |\varphi| \operatorname{var} \nabla \chi_{\mathscr{N}_t}(\mathrm{d}x).$$

Thus (9.6.15) immediately results from (9.6.16). □

Theorem. (The Gauss–Green Formula). *Let $P(\Omega) < \infty$ and suppose that a normal to Ω exists s-almost everywhere on $\partial\Omega$. Then for any function $u \in BV(\Omega)$ whose rough trace is integrable on the boundary of Ω, the equality*

$$\nabla v(\Omega) = \int_{\partial\Omega} u^*(x)\nu(x)s(\mathrm{d}x)$$

holds where $\nu(x)$ is the normal to Ω at the point x.

Proof. Since $\nabla\chi_{\mathscr{E}}(\mathbb{R}^n) = 0$ for any set $\mathscr{E}$ with $P(\mathscr{E}) < \infty$, by the Lemma we have

$$\nabla u(\Omega) = \int_{-\infty}^{\infty} \nabla\chi_{\mathscr{N}_t}(\Omega)\,\mathrm{d}t = \int_{-\infty}^{\infty} \nabla\chi_{\mathscr{N}_t}\big(\partial\Omega \cap \partial^*\mathscr{N}_t\big)\,\mathrm{d}t.$$

Using $s(\partial\Omega\backslash\partial^*\Omega) = 0$ and the coincidence of the normal to $\mathscr{N}_t$ with that to Ω on $\partial^*\Omega \cap \partial^*\mathscr{N}_t$, we obtain

$$\nabla\chi_{\mathscr{N}_t}\big(\partial\Omega \cap \partial^*\mathscr{N}_t\big) = \int_{\partial^*\Omega\cap\partial^*\mathscr{N}_t} \nu(x)s(\mathrm{d}x) = \nabla\chi_\Omega\big(\partial^*\mathscr{N}_t\big).$$

Thus

$$\begin{aligned}
\nabla u(\Omega) &= \int_0^{\infty} \nabla\chi_\Omega\big(\partial^*\mathscr{N}_t\big)\,\mathrm{d}t + \int_{-\infty}^{0} \nabla\chi_\Omega\big(\partial^*\mathscr{N}_t\big)\,\mathrm{d}t \\
&= \int_0^{\infty} \nabla\chi_\Omega\big(\partial^*\mathscr{N}_t\big)\,\mathrm{d}t - \int_{-\infty}^{0} \nabla\chi_\Omega\big(\partial^*\Omega\backslash\partial^*\mathscr{N}_t\big)\,\mathrm{d}t \\
&= \int_0^{\infty} \nabla\chi_\Omega\big(\{x : u^* \ge t\}\big)\,\mathrm{d}t - \int_{-\infty}^{0} \nabla\chi_\Omega\big(\{x : u^* \le t\}\big)\,\mathrm{d}t \\
&= \int_{\partial\Omega} u^*\nabla\chi_\Omega(\mathrm{d}x) = \int_{\partial\Omega} u^*(x)\nu(x)s(\mathrm{d}x).
\end{aligned}$$

The theorem is proved. □

The example of the disk with a slit

$$\big\{z = \varrho e^{i\theta} : 0 < \varrho < 1, 0 < \theta < 2\pi\big\}$$

and of the function $u(z) = \theta$ shows that the condition $s(\partial\Omega\backslash\partial^*\Omega) = 0$ cannot be omitted under our definition of the trace.

The last theorem and Theorem 9.6.2 immediately imply the following corollary.

Corollary. *Let $P(\Omega) < \infty$, $s(\partial\Omega\backslash\partial^*\Omega) = 0$. If for any $\mathscr{E} \subset \Omega$*

$$\min\big\{P_{C\Omega}(\mathscr{E}), P_{C\Omega}(\Omega\backslash\mathscr{E})\big\} \le kP_\Omega(\mathscr{E}),$$

where k is independent of $\mathscr{E}$, then the trace $\tilde{u}(x)$ exists for any $u \in BV(\Omega)$ and the Gauss–Green formula

$$\nabla u(\Omega) = \int_{\partial\Omega} \tilde{u}(x)\nu(x)s(\mathrm{d}x)$$

holds.

9.7 Comments to Chap. 9

This chapter was written together with Yu.D. Burago.

The first two sections contain well-known facts from the theory of sets with finite perimeter and from the theory of functions in BV. The foundation of this theory was laid by Caccioppoli [160, 161] and De Giorgi [229, 230]. Its further development is due to Krickeberg [464], Fleming [281], Fleming and Rishel [282], and others. The results in Sects. 9.1.3–9.1.5 are due (up to the presentation) to De Giorgi [229, 230]. Theorem 9.1.2 is a modification of a result by Krickeberg [464]. Formula (9.1.16) in 9.1.6 was obtained by Fleming and Rishel [282].

The results of Sect. 9.2 were established by De Giorgi [230] and supplemented by Federer [268].

At the present time the theory of sets with a finite perimeter can be considered as a part of the theory of integral currents (cf. Federer [271], part 4.5).

Sections 9.3–9.6 contain an expanded presentation of the paper by Burago and Maz'ya [150].

Bokowski and Sperner [124] obtained the estimates for the functions $\eta(S)$ and λ_M for convex domains by the radii of inscribed and circumscribed balls.

For various facts concerning isoperimetric inequalities see the book by Burago and Zalgaller [151] and the review paper by Osserman [648].

Souček [718] studied the properties of functions whose derivatives of order l are charges.

In connection with the contents of the present chapter see also the paper by Volpert [781].

The following observation concerns the integrability of the trace of a function in $BV(\Omega)$.

The requirement (9.6.13) has a global character: It is not satisfied, for example, by any nonconnected set Ω. This is essential because it is a question of inequality (9.6.14). If we, however, put (9.6.14) in the following (equivalent, provided Ω is connected) form

$$\int_{\partial\Omega} |\tilde{u}| s(\mathrm{d}x) \le \text{const}\ \|u\|_{BV(\Omega)},$$

then the role of inequality (9.6.13) is taken by the local requirement

$$\sup_{x\in\Omega} \lim_{\rho\to+0} \sup\left\{\frac{P_{C\Omega}(E)}{P_\Omega(E)} : E \subset \Omega \cap B(x,\varrho)\right\} < \infty. \tag{9.7.1}$$

The following result on continuation to the boundary of the domain was established by Anzellotti and Giaquinta [50]:

If condition (9.7.1) is fulfilled, then for any function $\varphi \in L_1(\partial^*\Omega)$ there exists a function $u \in W_1^1(\Omega)$ whose trace on $\partial^*\Omega$ equals φ and we have the inequality

$$\|u\|_{W_1^1(\Omega)} \le C\|\varphi\|_{L_1(\partial^*\Omega)},$$

where C depends on Ω but not on φ.

Thus, for domains Ω subject to condition (9.7.1) the spaces of traces of $\partial^*\Omega$ of functions in $BV(\Omega)$ and $W_1^1(\Omega)$ coincide with $L_1(\partial^*\Omega)$.

Recently the basic results of Sects. 9.5 and 9.6 were extended by Burago and Kosovsky [149]: They assumed that $\partial\Omega$ is a $(n-1)$-rectifiable set instead of our conditions that $P(\Omega) < \infty$ and normals in the sense of Federer exist a.e. on $\partial\Omega$. It means that now the results of Sects. 9.5 and 9.6 are extended, in particular, to the class of domains with cuts, important for applications.

Baldi and Montefalcone [66] generalized the extension criterion (9.3.2) to a certain class of metric spaces.

A few words about the extension criterion (9.3.2). Roughly speaking, $\tau_\Omega(E)$ may be viewed as the area of soap film placed on the exterior of Ω and suspended on that part of the boundary of E which belongs to $\partial\Omega$. Therefore (9.3.2) has a simple geometric meaning. Let l be an arbitrary closed contour on the boundary of a three-dimensional body Ω and let s_e and s_i be the areas of the films, suspended on l and positioned, respectively, outside and inside Ω. Condition (9.3.2) means that the ratio s_e/s_i is bounded irrespective of l. If $n = 2$ then it is a question of the ratio between the distance between two arbitrary points of $\partial\Omega$ measured outside and inside Ω. This heuristic observation was rigorously justified by Koskela, Miranda, and Shanmugalingam [457] for a bounded simply connected domain $\Omega \subset \mathbb{R}^2$. They showed that Ω *is a BV-extension domain if and only if there exists a constant $C > 0$ such that for all x, $y \in \mathbb{R}^2\backslash\Omega$ there is a rectifiable curve $\gamma \subset \mathbb{R}^2\backslash\Omega$ connecting x and y with length*

$$l(\gamma) \leq C|x - y|.$$

In other words, Ω is a BV-extension domain if and only if the complement of Ω is quasiconvex. A corollary of this result is the necessity of quasiconvexity for Ω to be a W_1^1-extension domain, and it is conjectured in [457] that the same necessary condition holds for all W_p^1-extension Jordan domains in $\mathbb{R}^2$ with an arbitrary $p \in [1, 2]$.

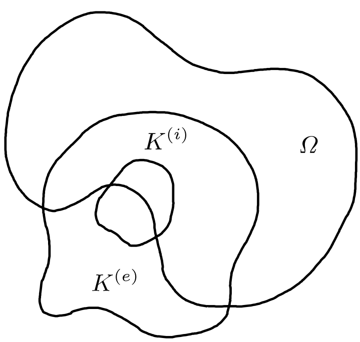

Fig. 34.

A necessary and sufficient condition for the extension of functions in $W_p^1(\Omega)$, $p \neq 2$, to $\mathbb{R}^n$ without changing the class is not known. It is easy to guess the following not very visual condition, which is a direct analog of the Burago–Maz'ya criterion (9.3.2) for the extension of functions in $BV(\Omega)$. For each conductor $K^{(i)}$ in Ω there exists a conductor $K^{(e)}$ in $\mathbb{R}^n \backslash \bar{\Omega}$ abutting to it (cf. Fig. 34) such that

$$c_p\big(K^{(e)}\big) \leq c_p\big(K^{(i)}\big).$$

It cannot be excluded that this requirement is indeed sufficient, but this has not been proved.

10

Certain Function Spaces, Capacities, and Potentials

Section 10.1 is of an auxiliary nature. Here we collect (mostly without proofs) the results of function theory that are applied later or related to the facts used in the sequel. First we discuss the theorems on spaces of functions having "derivatives of arbitrary positive order" (Sect. 10.1). The theory of these spaces is essentially presented in monographs (cf. Stein [724]; Peetre [657]; Nikolsky [639]; Besov, Il'in, and Nikolsky [94]; Triebel [755, 756]; and Runst and Sickel [685]) though in some cases the reader interested in the proofs will have to refer to the original papers.

Section 10.2 is concerned with the Bourgain, Brezis, and Mironescu theorem on the asymptotic behavior of the norm of the Sobolev-type embedding operator $\mathcal{W}_p^s \to L_{pn/(n-sp)}$ as $s \uparrow 1$ and $s \uparrow n/p$. Their result is extended to all values of $s \in (0,1)$ and is supplied with an elementary proof. The relation

$$\lim_{s\downarrow 0} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y = 2p^{-1}\omega_n \|u\|^p_{L_p(\mathbb{R}^n)}$$

is proved.

In Sect. 10.3 we prove the multiplicative Gagliardo–Nirenberg-type inequality

$$\|u\|_{\mathcal{W}^{\theta s}_{p/\theta}} \le c(n)\left(\frac{p}{p-1}\right)^{\theta}\left(\frac{1-s}{1-\theta}\right)^{\theta/p}\|u\|^{\theta}_{\mathcal{W}^s_p}\|u\|^{1-\theta}_{L_\infty},$$

where $0<\theta<1$, $0<s<1$, $1<p<\infty$, and $\|u\|_{\mathcal{W}^s_p}$ is the seminorm in the fractional Sobolev space $\mathcal{W}^s_p(\mathbb{R}^n)$ and show that the dependence of the constant factor in the right-hand side on each of the parameters s, θ, and p is precise in a sense.

In Sect. 10.4, dealing with properties of capacities and nonlinear potentials, we restrict ourselves to the formulation of the results.

V. Maz'ya, *Sobolev Spaces*,
Grundlehren der mathematischen Wissenschaften 342,
DOI 10.1007/978-3-642-15564-2_10,

10.1 Spaces of Functions Differentiable of Arbitrary Positive Order

10.1.1 Spaces w_p^l, W_p^l, b_p^l, B_p^l for $l > 0$

For $p \geq 1$ and integer $l > 0$, let w_p^l denote the completion of the space $\mathscr{D}$ with respect to the norm $\|\nabla_l u\|_{L_p}$. For $p \geq 1$ and noninteger l, we define w_p^l as the completion of $\mathscr{D}$ with respect to the norm

$$\left(\int \|\Delta_y u\|^p_{w_p^{[l]}} |y|^{-n-p\{l\}}\, \mathrm{d}y\right)^{1/p}. \tag{10.1.1}$$

Here and elsewhere $\Delta_y u(x) = u(x+y) - u(x)$, $[l]$ and $\{l\}$ are the integer and fractional parts of l, respectively.

Replacing the norm (10.1.1) by the norm

$$\|u\|_{b_p^l} = \left(\int \|\Delta_y^2 u\|^p_{L_p} |y|^{-n-pl}\, \mathrm{d}y\right)^{1/p}, \quad 0 < l \leq 1, \tag{10.1.2}$$

in the previous definition, we obtain the space b_p^l (here $\Delta_y^2 u(x) = u(x+y) - 2u(x) + u(x-y)$). For $l > 1$ we put $\|u\|_{b_p^l} = \|\nabla u\|_{b_p^{l-1}}$.

Further, let W_p^l and B_p^l be the complements of $\mathscr{D}$ with respect to the norms $\|u\|_{w_p^l} + \|u\|_{L_p}$ and $\|u\|_{b_p^l} + \|u\|_{L_p}$.

For fractional l the norms in w_p^l and b_p^l, as well as the norms in W_p^l and B_p^l, are equivalent. In fact, the identity

$$\begin{aligned} 2\big(u(x+h) - u(x)\big) = &-\big[u(x+2h) - 2u(x+h) + u(x)\big] \\ &+ \big[u(x+2h) - u(x)\big] \end{aligned}$$

implies the estimates

$$\big(2 - 2^l\big)\mathscr{H}_l u \leq \mathscr{G}_l u \leq \big(2 - 2^l\big)\mathscr{H}_l u, \quad 0 < l < 1,$$

where

$$(\mathscr{H}_l u)(x) = \left(\int \big|(\Delta_y u)(x)\big|^p |y|^{-n-pl}\, \mathrm{d}y\right)^{1/p},$$

$$(\mathscr{G}_l u)(x) = \left(\int \big|(\Delta_y^2 u)(x)\big|^p |y|^{-n-pl}\, \mathrm{d}y\right)^{1/p}.$$

The utility of the defined spaces is mostly owing to the following trace theorem.

Theorem 1. *For $p \in [1, \infty)$, $l > 0$, $m = 1, 2, \ldots,$ we have*

$$\|u\|_{b_p^l(\mathbb{R}^n)} \sim \inf_{\{U\}} \big\||y|^{m-\{l\}-p^{-1}} \nabla_{m+[l]} U\big\|_{L_p(\mathbb{R}^{n+1})}, \tag{10.1.3}$$

where $U \in \mathscr{D}(\mathbb{R}^{n+1})$ *is an arbitrary extension of* $u \in \mathscr{D}(\mathbb{R}^n)$ *to the space* $\mathbb{R}^{n+1} = \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}^1\}$.

Proof. We restrict ourselves to the derivation of (10.1.3) for $0 < l \le 1$, $m = 2$.

Let $U \in \mathscr{D}(\mathbb{R}^{n+1})$, $u = U|_{\mathbb{R}^n}$. We put $\bar{\Delta}_t^{(2)} u(x,t) = u(x,2t) - 2u(x,t) + u(x,0)$. It can be easily seen that

$$\bar{\Delta}_t^{(2)} u(x,t) = \int_0^t (t-\tau) \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \big[u(x, t+\tau) + u(x, t-\tau)\big] \,\mathrm{d}\tau. \tag{10.1.4}$$

We can also readily check that

$$\begin{aligned} \bar{\Delta}_t^{(2)} u(x,0) = &-2\bar{\Delta}_{|h|}^{(2)} u\big(x,|h|\big) + 2\Delta_h^{(2)} u\big(x,|h|\big) - \Delta_h^{(2)} u\big(x, 2|h|\big) \\ &+ \bar{\Delta}_{|h|}^{(2)} u\big(x+h,|h|\big) + \bar{\Delta}_{|h|}^{(2)} u\big(x-h,|h|\big). \end{aligned} \tag{10.1.5}$$

We consider only the first and the second summands since the others can be estimated in a similar way. By (10.1.4) we obtain

$$\big\|\bar{\Delta}_{|h|}^{(2)} u\big(\cdot,|h|\big)\big\|_{L_p(\mathbb{R}^n)} \le 2\int_0^{2|h|} y \left\|\frac{\mathrm{d}^2}{\mathrm{d}y^2} U(x,y)\right\|_{L_p(\mathbb{R}^n)} \mathrm{d}y.$$

Therefore (for $l \in (0,1]$, $p \ge 1$),

$$\begin{aligned} &\int_{\mathbb{R}^n} \big\|\bar{\Delta}_{|h|}^{(2)} u\big(\cdot,|h|\big)\big\|_{L_p(\mathbb{R}^n)}^p \frac{\mathrm{d}h}{|h|^{n+pl}} \\ &\quad \le c\int_0^\infty \left(\int_0^{2\varrho} y \left\|\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}(\cdot,y)\right\|_{L_p(\mathbb{R}^n)} \mathrm{d}y\right)^p \frac{\mathrm{d}\varrho}{\varrho^{1+pl}} \\ &\quad \le c\int_0^\infty y^{(2-l)p-1} \left\|\frac{\mathrm{d}^2 U}{\mathrm{d}y^2}(\cdot,y)\right\|_{L_p(\mathbb{R}^n)}^p \mathrm{d}y. \end{aligned} \tag{10.1.6}$$

Next we proceed to the second item in (10.1.5). We have

$$\Delta_h^{(2)} u\big(x,|h|\big) = \int_0^1 (1-\lambda) \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} \big[u\big(x+\lambda h,|h|\big) + u\big(x-\lambda h,|h|\big)\big] \,\mathrm{d}\lambda.$$

By the Minkowski inequality

$$\big\|\Delta_h^{(2)} u\big(\cdot,|h|\big)\big\|_{L_p(\mathbb{R}^n)} \le c|h|^2 \big\|\nabla_{2,x} u\big(\cdot,|h|\big)\big\|_{L_p(\mathbb{R}^n)}.$$

Hence

$$\int_{\mathbb{R}^n} \big\|\Delta_h^{(2)} u\big(\cdot,|h|\big)\big\|_{L_p(\mathbb{R}^n)}^p \frac{\mathrm{d}h}{|h|^{n+pl}} \le c\int_0^\infty y^{(2-l)p-1} \big\|\nabla_{2,x} U(\cdot,y)\big\|_{L_p(\mathbb{R}^n)}^p \,\mathrm{d}y,$$

which together with (10.1.6) yields the upper bound for the norm $\|u\|_{b_p^l}$. We proceed to the lower estimate.

Let $u \in \mathscr{D}(\mathbb{R}^n)$ and let $\varPi$ be the extension operator to the space $\mathbb{R}^{n+1} = \{X = (x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}^1\}$ defined by

$$(\varPi u)(X) = y^{-n} \int_{\mathbb{R}^n} \pi\Big(\frac{\xi - x}{y}\Big) u(\xi)\,\mathrm{d}\xi, \tag{10.1.7}$$

where

$$\pi \in C_0^\infty(B_1), \qquad \int_{\mathbb{R}^n} \pi(x)\,\mathrm{d}x = 1, \quad \pi(x) = \pi(-x).$$

Using the evenness of the function π, for $|\alpha| = 2$ we obtain

$$\begin{aligned}(D_x^\alpha \varPi u)(X) &= y^{-n-2} \int_{\mathbb{R}^n} (D^\alpha \pi)\Big(\frac{\xi - x}{y}\Big) u(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{2} y^{-n-2} \int_{\mathbb{R}^n} (D^\alpha \pi)(h/y) \varDelta_h^{(2)} u(x)\,\mathrm{d}h,\end{aligned}$$

where, as before, $\varDelta_h^{(2)} u(x) = u(x+h) - 2u(x) + u(x-h)$. Therefore for $|\alpha| = 2$ we have

$$\big|(D_x^\alpha \varPi u)(X)\big| \le c y^{-n-2} \int_{B_y} \big|\varDelta_h^{(2)} u(x)\big|\,\mathrm{d}h.$$

Since

$$2(\varPi u)(X) = y^{-n} \int \pi(h/y) \varDelta_y^{(2)} u(x)\,\mathrm{d}h + 2u(x),$$

it follows that

$$\Big|\Big(\frac{\mathrm{d}^2}{\mathrm{d}y^2} \varPi u\Big)(X)\Big| \le c y^{-n-2} \int_{B_y} \big|\varDelta_h^{(2)} u(x)\big|\,\mathrm{d}h.$$

We can easily check that the same estimate is also true for $|\partial^2 \varPi / \partial x_i \partial y|$. So the second derivatives of u are bounded and

$$\begin{aligned}&\int_0^\infty y^{-1+p(2-l)} \big\|(\nabla_2 \varPi u)(\cdot, y)\big\|_{L_p(\mathbb{R}^n)}^p\,\mathrm{d}y \\ &\quad \le \int_0^\infty y^{-1+p(l+n)} \Big\|\int_{B_y} \big|\varDelta_y^{(2)} u(\cdot)\big|\,\mathrm{d}h\Big\|_{L_p(\mathbb{R}^n)}^p\,\mathrm{d}y.\end{aligned} \tag{10.1.8}$$

By the Minkowski inequality the right-hand side does not exceed

$$\begin{aligned}&c \int_0^\infty y^{-1-pl-n} \Big(\int_{B_y} \big\|\varDelta_h^{(2)} u(\cdot)\big\|_{L_p(\mathbb{R}^n)}\,\mathrm{d}h\Big)^p\,\mathrm{d}y \\ &\quad \le c \int_{\mathbb{R}^n} \big\|\varDelta_h^{(2)} u(\cdot)\big\|_{L_p(\mathbb{R}^n)}^p \int_{|h|}^\infty y^{-1-pl-n}\,\mathrm{d}y\,\mathrm{d}h \\ &\quad = c \int_{\mathbb{R}^n} \big\|\varDelta_h^{(2)} u(\cdot)\big\|_{L_p(\mathbb{R}^n)}^p |h|^{-n-pl}\,\mathrm{d}h,\end{aligned}$$

and the required lower estimate for the norm $\|u\|_{b_p^l}$ follows.

Next we show that the function Πu can be approximated by the sequence of extensions $U_k \in \mathscr{D}(\mathbb{R}^{n+1})$ of the function u in the metric

$$\big\| |y|^{2-l-p^{-1}} \nabla_2 U \big\|_{L_p(\mathbb{R}^{n+1})}.$$

Let $\eta_k(X) = \eta(X/k)$ where $\eta \in \mathscr{D}(\mathbb{R}^{n+1})$, $\eta = 1$ for $|X| < k$ and $k = 1, 2 \dots$. Since $(D^\alpha \Pi u)(X) = O((|X|+1)^{-n-|\alpha|})$ for $0 < |\alpha| \le 2$ then $\nabla_2(\Pi u - \eta_k \Pi u) = O((|X|+1)^{-n-2})$. Furthermore,

$$\operatorname{supp}(\Pi u - \eta_k \Pi u) \subset \big\{X \in \mathbb{R}^{n+1} : |X| > k\big\}.$$

Consequently,

$$\big\| |y|^{2-l-p^{-1}} \nabla_2 (\Pi u - \eta_k \Pi u) \big\|_{L_p(\mathbb{R}^{n+1})} = O\big(k^{-l-n+n/p}\big) = o(1)$$

as $k \to \infty$. It remains to approximate each of the functions $\eta_k \Pi u$ by a sequence of mollifications. □

Similarly we can show the following analogous assertion for the space $B_p^l(\mathbb{R}^n)$.

Theorem 2. *For $p \in [1, \infty)$, $l > 0$, $m = 1, 2, \dots$, we have*

$$\|u\|_{B_p^l(\mathbb{R}^n)} \sim \inf_{\{U\}} \Big(\big\| |y|^{m-\{l\}-p^{-1}} \nabla_{m+[l]} U \big\|_{L_p(\mathbb{R}^{n+1})} + \|U\|_{L_p(\mathbb{R}^{n+1})} \Big).$$

Theorems 1 and 2 have a long history. For $p = 2$, $\{l\} = m - \frac{1}{2}$ they were established by Aronszajn [52], Babich and Slobodeckiĭ [60], Slobodeckiĭ [704], and Freud and Králik [291]. The particular case $n = p = 2$, $m = l = 1$ is actually contained in the papers by Douglas [242], 1931, and Beurling [95], 1940. The generalization for $p \ne 2$ is due to Gagliardo [298] for $l = 1 - 1/p$. Theorems 1 and 2 were proved, in a form similar to that above, by Uspenskiĭ [770] (cf. also Lizorkin [504]).

A theory of spaces with "weighted" norms is discussed in the books by Triebel [756], Kufner [465], and in the survey by Besov, Il'in, Kudryavtsev, Lizorkin, and Nikolsky [93].

In conclusion we state the following trace theorem for functions in $w_1^1(\mathbb{R}^{n+1})$, $W_1^1(\mathbb{R}^{n+1})$ restricted to $\mathbb{R}^n$.

Theorem 3. *We have*

$$\|u\|_{L_1(\mathbb{R}^n)} \sim \inf_{\{U\}} \|\nabla U\|_{L_1(\mathbb{R}^{n+1})} \sim \inf_{\{U\}} \|U\|_{W_1^1(\mathbb{R}^{n+1})},$$

where $u \in \mathscr{D}(\mathbb{R}^n)$ and $\{U\}$ is the collection of all extensions of u to $\mathbb{R}^{n+1}$, $U \in \mathscr{D}(\mathbb{R}^{n+1})$.

This assertion is proved in the paper by Gagliardo [298] (cf. also the book by Besov, Il'in, and Nikolsky [94]). For a more recent development of trace theorems see the Comments to the present chapter.

10.1.2 Riesz and Bessel Potential Spaces

With each function $u \in \mathscr{D} = \mathscr{D}(\mathbb{R}^n)$, we associate its Fourier transform

$$\hat{u}(\xi) = Fu(\xi) = (2\pi)^{-n/2} \int e^{ix\xi} u(x)\, dx.$$

The same notation will be retained for the Fourier transform of the distribution u contained in the space $\mathscr{D}'$ dual of $\mathscr{D}$. We denote the convolution of distributions by a star, $*$.

The scales of "fractional" spaces different from the spaces introduced in Sect. 10.1.1 are defined by means of the operators

$$(-\Delta)^{l/2} = F^{-1}|\xi|^l F, \qquad (-\Delta + 1)^{l/2} = F^{-1}\big(1 + |\xi|^2\big)^{l/2} F,$$

where Δ is the Laplace operator.

Namely, let h_p^l and H_p^l ($1 < p < \infty$, $l > 0$) denote the completion of the space $\mathscr{D}$ with respect to the norms

$$\|u\|_{h_p^l} = \big\|(-\Delta)^{l/2} u\big\|_{L_p}, \qquad \|u\|_{H_p^l} = \big\|(-\Delta + 1)^{l/2} u\big\|_{L_p}.$$

The following assertion is the Mikhlin theorem on Fourier integral multipliers [601].

Theorem 1. *Let the function Φ defined on $\mathbb{R}^n \backslash \{0\}$ have the derivatives $\partial^k \Phi(\lambda)/\partial\lambda_{j_1}, \dots, \partial\lambda_{j_k}$, where $0 \le k \le n$ and $1 \le j_1 < j_2 < \dots < j_k \le n$. Further let*

$$|\lambda|^k \left| \frac{\partial^k \Phi(\lambda)}{\partial\lambda_{j_1}, \dots, \partial\lambda_{j_k}} \right| \le M = \text{const}\,.$$

Then for all $u \in L_p$

$$\big\|F^{-1}\Phi F u\big\|_{L_p} \le cM \|u\|_{L_p}, \quad 1 < p < \infty,$$

where c is a constant that depends only on n and p.

Corollary 1. *Let $l = 1, 2, \dots,$ then there exist positive numbers c and C that depend only on n, p, l, such that*

$$c\big\|(-\Delta)^{l/2} u\big\|_{L_p} \le \|\nabla_l u\|_{L_p} \le C\big\|(-\Delta)^{l/2} u\big\|_{L_p} \tag{10.1.9}$$

for all $u \in \mathscr{D}$.

Proof. Let α be a multi-index with $|\alpha| = l$. Then

$$F^{-1}\xi^\alpha F u = F^{-1}\xi^\alpha |\xi|^{-l} |\xi|^l F u.$$

The function $\xi^\alpha |\xi|^{-l}$ satisfies the hypothesis of Theorem 1, which leads to the rightmost estimate in (10.1.9). On the other hand,

$$|\xi|^l = |\xi|^{2l}|\xi|^{-l} = \Big(\sum_{|\alpha|=l} c_\alpha \xi^\alpha \xi^\alpha\Big)|\xi|^{-l},$$

where $c_\alpha = l!/\alpha!$, so

$$F^{-1}|\xi|^l F u = \sum_{|\alpha|=l} c_\alpha F^{-1}\frac{\xi^\alpha}{|\xi|^l}\xi^\alpha F u.$$

Again, applying Theorem 1, we obtain the leftmost estimate in (10.1.9).

The following corollary has a similar proof.

Corollary 2. *Let $l = 1, 2, \dots$. There exist positive numbers c and C that depend only on n, p, l such that*

$$c\|u\|_{W_p^l} \le \big\|(-\Delta + 1)^{l/2} u\big\|_{L_p} \le C\|u\|_{W_p^l}$$

for all $u \in \mathscr{D}$.

Thus, $w_p^l = h_p^l$ and $W_p^l = H_p^l$ provided $p > 1$ and l is an integer.

For the proof of the following theorem see the paper by Havin and Maz'ya [567].

Theorem 2. *Let $pl < n$, $p > 1$. Then $u \in h_p^l$ if and only if*

$$u = (-\Delta)^{-l/2} f \equiv c|x|^{l-n} * f,$$

where $f \in L_p$.

A similar well-known assertion for the space H_p^l is contained in the following theorem.

Theorem 3. *The function u belongs in H_p^l, $p > 1$, if and only if*

$$u = (-\Delta + 1)^{-l/2} f \equiv G_l * f,$$

where $f \in L_p$,

$$G_l(x) = c|x|^{(l-n)/2} K_{(n-l)/2}\big(|x|\big) > 0,$$

K_ν is the modified Bessel function of the third kind.

For $|x| \le 1$ the estimates

$$G_l(x) \le \begin{cases} c|x|^{l-n}, & 0 < l < n, \\ c\log(2/|x|), & l = n, \\ c, & l > n, \end{cases}$$

hold. If $|x| \ge 1$, then

$$G_l(x) \le c|x|^{(l-n-1)/2} e^{-|x|}.$$

The integral operators

$$f \xrightarrow{I_l} |x|^{l-n} * f, \qquad f \xrightarrow{J_l} G_l * f,$$

are called the Riesz potential and the Bessel potential, respectively. Thus Theorems 2 and 3 state that each element of the space $h_p^l(pl < n)\,(H_p^l)$ is the Riesz (Bessel) potential with density in L_p.

Next we formulate the theorem due to Strichartz [728] on equivalent norms in the spaces h_p^l and H_p^l.

Theorem 4. *Let* $\{l\} > 0$ *and let*

$$(\mathscr{D}_{\{l\}}v)(x) = \left(\int_0^\infty \left[\int_{|\theta|<1} \big|v(x+\theta y) - v(x)\big|\,\mathrm{d}\theta\right]^2 y^{-1-2\{l\}}\,\mathrm{d}y\right)^{1/2}. \quad (10.1.10)$$

Then

$$\|u\|_{h_p^l} \sim \|\mathscr{D}_{\{l\}}\nabla_{[l]}u\|_{L_p}, \quad (10.1.11)$$

$$\|u\|_{H_p^l} \sim \|\mathscr{D}_{\{l\}}\nabla_{[l]}u\|_{L_p} + \|u\|_{L_p}. \quad (10.1.12)$$

Two-sided estimates for the L_p-norm of a function u on $\mathbb{R}^1$ by its harmonic extension to $\mathbb{R}^1 \times (0,\infty)$ are due to Littlewood, Paley, Zygmund, and Marcinkiewicz. Let the Littlewood–Paley function $g(u)$ be defined by

$$\big[g(u)\big](x) = \left(\int_0^\infty \big|\nabla U(x,y)\big|^2 y\,\mathrm{d}y\right)^{1/2},$$

where $U(x,y)$ is the Poisson integral of u. The basic result here is the equivalence of the norms $\|u\|_{L_p(\mathbb{R}^1)}$ and $\|g(u)\|_{L_p(\mathbb{R}^1)}$, $1 < p < \infty$. In the book by Stein [724] this equivalence is proved for $\mathbb{R}^n$.

The next theorem by Shaposhnikova [697], similar to Theorems 10.1.1/1 and 10.1.1/2, contains a characterization of $h_p^l(\mathbb{R}^n)$ in terms of extensions to $\mathbb{R}^{n+k}$.

Theorem 5. *The norm of* $u \in \mathscr{D}(\mathbb{R}^n)$ *in* $h_p^l(\mathbb{R}^n)$, $0 < l < 1$ *is equivalent to*

$$\inf_{\{U\}} \left\{\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^k} |y|^{2-2l-k} |\nabla U|^2\,\mathrm{d}y\right)^{p/2} \mathrm{d}x\right\}^{1/p},$$

where the infimum is taken over all extensions $U \in \mathscr{D}(\mathbb{R}^{n+k})$ *of* u *to* $\mathbb{R}^{n+k} = \{(x,y) : x \in \mathbb{R}^n, y \in \mathbb{R}^k\}$.

Similarly,

$$\|u\|_{H_p^l} \sim \inf_{\{U\}} \left\{\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^k} |y|^{2-2l-k} \big(|\nabla U|^2 + |U|^2\big)\,\mathrm{d}y\right)^{p/2} \mathrm{d}x\right\}^{1/p}.$$

10.1.3 Other Properties of the Introduced Function Spaces

The following two theorems, whose proofs can be found in the books by Stein [724], Peetre [657], Nikolsky [639], and Triebel [756], are classical facts of the theory of the spaces b^l_p, B^l_p, h^l_p, and H^l_p.

Theorem 1. *If* $2 \le p < \infty$ *then* $h^l_p \subset b^l_p$, $H^l_p \subset B^l_p$. *If* $1 < p < 2$ *then* $b^l_p \subset h^l_p$, $B^l_p \subset H^l_p$.

Theorem 2. (i) *If* $p \in (1, \infty)$, $l > 0$, *then*

$$\|u\|_{b^l_p(\mathbb{R}^n)} \sim \inf_{\{U\}} \|U\|_{h^{l+1/p}_p(\mathbb{R}^{n+1})}.$$

Here and in (ii) *the notation* $\{U\}$ *has the same meaning as in Theorem* 10.1.1/1.

(ii) *If* $p \in [1, \infty)$, $l > 0$ *then*

$$\|u\|_{b^l_p(\mathbb{R}^n)} \sim \inf_{\{U\}} \|U\|_{b^{l+1/p}_p(\mathbb{R}^{n+1})}.$$

In items (i) *and* (ii) *one can replace* b *and* h *by* B *and* H, *respectively.*

A far-reaching generalization of relations (i) and (ii) is given in the article by Jonsson and Wallin [407] where functions in B^l_p on the so-called d-sets F are extended to $\mathbb{R}^n$. The latter sets are defined by the relation

$$H_d\big(F \cap B(x, \varrho)\big) \sim \varrho^d,$$

which is valid for all $x \in F$ and $\varrho < \delta$, where H_d is d-dimensional Hausdorff measure (cf. Sect. 1.2.4).

Henceforth we denote by $\mu\mathscr{B}$ the ball with radius μr, concentric with the ball $\mathscr{B}$ of radius r. Similarly, with the cube $\mathscr{Q}$ with edge length d we associate the concentric cube $\mu\mathscr{Q}$ with sides parallel to those of $\mathscr{Q}$ and with edge length μd.

Next we state the theorem that is easily derived from (10.1.12) (cf. Strichartz [728]) for the spaces H^l_p, $\{l\} > 0$. For W^l_p and B^l_p it is a simple corollary of the definitions of these spaces.

Theorem 3. *Let* $\{\mathscr{B}^{(j)}\}_{j\ge 0}$ *be a covering of* $\mathbb{R}^n$ *by unit balls that has a finite multiplicity depending only on* n. *Further, let* $O^{(j)}$ *be the center of* $\mathscr{B}^{(j)}$, $O^{(0)} = O$, *and let* $\eta_j(x) = \eta(x - O^{(j)})$ *where* $\eta \in C^\infty_0(2\mathscr{B}^{(0)})$, $\eta = 1$ *on* $\mathscr{B}^{(0)}$. *Then*

$$\|u\|_{S^l_p} \sim \Big(\sum_{j\ge 0} \|u\eta_j\|^p_{s^l_p}\Big)^{1/p}, \tag{10.1.13}$$

where $S^l_p = H^l_p$, W^l_p *or* B^l_p *and* $s^l_p = h^l_p$, w^l_p *or* b^l_p, *respectively.*

We can easily check that for any $v \in C^\infty_0(B_1)$ the inequality

$$\|v\|_{L_p} \le c\|v\|_{s_p^l}, \tag{10.1.14}$$

holds. Therefore the norm $\|u\eta_j\|_{s_p^l}$ in (10.1.13) can be replaced by the equivalent norm $\|u\eta_j\|_{S_p^l}$.

When dealing with the embedding of a Banach space $\mathscr{X}$ into another Banach space $\mathscr{Y}$ (notation: $\mathscr{X} \subset \mathscr{Y}$) we always mean a continuous embedding.

Theorem 4. 1. *If $l = 1, 2, \ldots,$ then $b_1^l \subset w_1^l$.*

2. *If $p > 1$, $l > \lambda \ge 0$, $n > (l - \lambda)p$ and $l - n/p = \lambda - n/\pi$ then $h_p^l \subset h_\pi^\lambda$.*

3. *If $p \ge 1$, $l > \lambda > 0$, $n > (l - \lambda)p$, and $l - n/p = \lambda - n/\pi$ then $b_p^l \subset b_\pi^\lambda$.*

4. *If $p \ge 1$, $l > 0$, $n > lp$, and $1/\pi = 1/p - l/n$ then $b_p^l \subset L_\pi$.*

5. *If $l = 1, 2, \ldots,$ $n \ge l - \lambda$ and $l - n = \lambda - n/\pi$ then $w_1^l \subset b_\pi^\lambda$.*

Replacing the letters b, h, w by B, H, and W in items 1–5 we also obtain true assertions.

6. *If $p > 1$, $l > \lambda \ge 0$, $n = (l - \lambda)p$, then $H_p^l \subset H_\pi^\lambda$ for any $\pi \in (1, \infty)$. If $p > 1$, $lp > n$, then $H_p^l \subset L_\infty \cap C$.*

7. *If $p > 1$, $l > \lambda > 0$, $n = (l - \lambda)p$, then $B_p^l \subset B_\pi^\lambda$ for any $\pi \in (1, \infty)$.*

8. *If $p > 1$, $l > 0$, $n = lp$, then $B_p^l \subset L_\pi$ for any $\pi \in (1, \infty)$.*

9. *If $p > 1$, $lp > n$, then $B_p^l \subset L_\infty \cap C$.*

Various proofs of assertions 1–4 and 6–9 can be found in the monographs mentioned at the beginning of the present subsection. The proof of assertion 5 is due to Solonnikov [717]. The embedding in 2 is an immediate corollary of the continuity of the operator $(-\Delta)^{(\lambda - l)/2} : L_p \to L_\pi$ established by Sobolev [712]. Item 1 follows from the inequalities

$$\|\nabla_l u\|_{L_1(\mathbb{R}^n)} \le c_1 \|\nabla_{l+1} U\|_{L_1(\mathbb{R}^{n+1})} \le c_2 \|y \nabla_{l+2} U\|_{L_1(\mathbb{R}^{n+1})}$$

and from Theorem 10.1.1/1. (Here $U \in \mathscr{D}(\mathbb{R}^{n+1})$ is an arbitrary extension of the function $u \in \mathscr{D}(\mathbb{R}^n)$.) The same theorem together with the inequality

$$\big\||y|^{1-\{\lambda\}-1/\pi} \nabla_{[\lambda]+1} U\big\|_{L_\pi(\mathbb{R}^{n+1})} \le c \big\||y|^{-\{l\}} \nabla_{[l]+1} U\big\|_{L_1(\mathbb{R}^{n+1})},$$

which results from Corollary 2.1.7/4, leads to (3) for $p = 1$. The same result for $p > 1$ follows from the embedding

$$h_p^{l+1/p}(\mathbb{R}^{n+1}) \subset h_\pi^{\lambda+1/\pi}(\mathbb{R}^{n+1})$$

(cf. item 2 above and part (i) of Theorem 2).

The corresponding assertions for the spaces H_p^l and B_p^l can be obtained in a similar way. The embedding in item 6 easily follows from the definition of the Bessel potential. The properties in items 7 and 9 result from item 6 applied to the space $H_p^{l+1/p}(\mathbb{R}^{n+1})$, and item 8 is a corollary of item 7.

10.2 Bourgain, Brezis, and Mironescu Theorem Concerning Limiting Embeddings of Fractional Sobolev Spaces

10.2.1 Introduction

Let $s \in (0,1)$ and let $p \geq 1$. We introduce the space $\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)$ as the completion of $C_0^\infty(\mathbb{R}^n)$ in the norm

$$\left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p}.$$

We also need the space $\mathcal{W}_\perp^{s,p}(Q)$ of functions defined on the cube $Q = \{x \in \mathbb{R}^n : |x_i| < 1/2, 1 \leq i \leq n\}$, which are orthogonal to 1 and have the finite norm

$$\left(\int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p}.$$

The main result of the recent article by Bourgain, Brezis, and Mironescu [139] is the inequality

$$\|u\|_{L_q(Q)}^p \leq c(n) \frac{1-s}{(n-sp)^{p-1}} \|u\|_{\mathcal{W}_\perp^{s,p}(Q)}^p, \tag{10.2.1}$$

where $u \in \mathcal{W}_\perp^{s,p}(Q)$, $1/2 \leq s < 1$, $sp < n$, $q = pn/(n-sp)$, and $c(n)$ depends only on n.

Figuring out a similar estimate for functions in $\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)$, valid for the whole interval $0 < s < 1$, one can anticipate the appearance of the factor $s(1-s)$ in the right-hand side since the norm in $\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)$ blows up both as $s \uparrow 1$ and $s \downarrow 0$. The following theorem shows that this is really the case.

Theorem. *Let $n \geq 1$, $p \geq 1$, $0 < s < 1$, and $sp < n$. Then, for an arbitrary function $u \in \mathring{\mathcal{W}}_p^s(\mathbb{R}^n)$, there holds*

$$\|u\|_{L_q(\mathbb{R}^n)}^p \leq c(n,p) \frac{s(1-s)}{(n-sp)^{p-1}} \|u\|_{\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)}^p, \tag{10.2.2}$$

where $q = pn/(n-sp)$ and $c(n,p)$ is a function of n and p.

From this theorem, one can derive inequality (10.2.1) for all $s \in (0,1)$ with a constant c depending both on n and p (Corollary 10.2.2/1). In the case $s \geq 1/2$ considered in [139], one has $1 < p < 2n$ and therefore the dependence of the constant c on p can be eliminated. Thus, we arrive at the Bourgain–Brezis–Mironescu result and extend it to the values $s < 1/2$.

The proof given in [139] relies upon some advanced harmonic analysis and is quite complicated. Our proof of (10.2.2) is straightforward and rather simple. It is based upon an estimate of the best constant in a Hardy-type inequality for the norm in $\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)$, which is obtained in Theorem 10.2.2 and is of independent interest.

In Theorem 10.2.4 we derive a formula for $\lim_{s\downarrow 0} s\|u\|^p_{\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)}$ that complements an analogous formula for $\lim_{s\uparrow 1}(1-s)\|u\|^p_{\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)}$ found in [138].

10.2.2 Hardy-Type Inequalities

Theorem. *Let $n \ge 1$, $p \ge 1$, $0 < s < 1$, and $sp < n$. Then, for an arbitrary function $u \in \mathring{\mathcal{W}}_p^s(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \bigl|u(x)\bigr|^p \frac{\mathrm{d}x}{|x|^{sp}} \le c(n,p) \frac{s(1-s)}{(n-sp)^p} \|u\|^p_{\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)}. \tag{10.2.3}$$

Proof. Let

$$\psi(h) = \omega_n^{-1} n(n+1)\bigl(1-|h|\bigr)_+,$$

where $h \in \mathbb{R}^n$ and the subscript plus stands for the nonnegative part of a real-valued function. We introduce the standard extension of u onto $\mathbb{R}^{n+1}_+ = \{(x,z) : x \in \mathbb{R}^n, z > 0\}$

$$U(x,z) := \int_{\mathbb{R}^n} \psi(h) u(x+zh)\,\mathrm{d}h.$$

A routine majoration implies

$$\bigl|\nabla U(x,z)\bigr| \le \frac{n(n+1)(n+2)}{z\omega_n} \int_{|h|<1} \bigl|u(x+zh) - u(x)\bigr|\,\mathrm{d}h.$$

Hence, and by Hölder's inequality, one has

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} z^{-1+p(1-s)} \bigl|\nabla U(x,z)\bigr|^p \,\mathrm{d}x\,\mathrm{d}z \\
&\quad \le \frac{n}{\omega_n}(n+1)^p (n+2)^p \int_0^\infty z^{-1-ps} \\
&\qquad \times \int_{|h|<1} \int_{\mathbb{R}^n} \bigl|u(x+zh) - u(x)\bigr|^p \,\mathrm{d}x\,\mathrm{d}h\,\mathrm{d}z.
\end{aligned} \tag{10.2.4}$$

Setting $\eta = zh$ and changing the order of integration, one can rewrite (10.2.4) as

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} z^{-1+p(1-s)} \bigl|\nabla U(x,z)\bigr|^p \,\mathrm{d}x\,\mathrm{d}z \\
&\quad \le \frac{n(n+1)^p (n+2)^p}{\omega_n (sp+n)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{10.2.5}$$

By Hardy's inequality

$$\int_0^{|x|} z^{-1-sp}\Big|\int_0^z \varphi(\tau)\,\mathrm{d}\tau\Big|^p\,\mathrm{d}z \le s^{-p}\int_0^{|x|} z^{-1+p(1-s)}\big|\varphi(z)\big|^p\,\mathrm{d}z,$$

one has

$$\begin{aligned}\frac{|u(x)|^p}{|x|^{sp}} &= p(1-s)\int_0^{|x|} z^{-1+p(1-s)}\,\mathrm{d}z\frac{|u(x)|^p}{|x|^p}\\ &\le p(1-s)\int_0^{|x|} z^{-1-sp}\,\mathrm{d}z\bigg(\int_0^z\bigg(\Big|\frac{\partial U}{\partial\tau}(x,\tau)\Big| + \frac{|U(x,\tau)|}{|x|}\bigg)\,\mathrm{d}\tau\bigg)^p\\ &\le \frac{p(1-s)}{s^p}\int_0^{|x|} z^{-1+p(1-s)}\bigg(\Big|\frac{\partial U}{\partial z}(x,z)\Big| + \frac{|U(x,z)|}{|x|}\bigg)^p\,\mathrm{d}z.\end{aligned}$$

Now, the integration over $\mathbb{R}^n$ and Minkowski's inequality imply

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x\\ &\quad\le \frac{p(1-s)}{s^p}\bigg(\bigg(\int_{\mathbb{R}^n}\int_0^\infty z^{-1+p(1-s)}\Big|\frac{\partial U}{\partial z}(x,z)\Big|^p\,\mathrm{d}z\,\mathrm{d}x\bigg)^{1/p} + A\bigg)^p, \quad (10.2.6)\end{aligned}$$

where

$$A := \bigg(\int_{\mathbb{R}^n}\int_0^{|x|} z^{-1+p(1-s)}|x|^{-p}\big|U(x,z)\big|^p\,\mathrm{d}z\,\mathrm{d}x\bigg)^{1/p}.$$

Clearly,

$$A^p \le 2^{p/2}\int_{\mathbb{R}^n}\mathrm{d}x\int_0^\infty z^{-1+p(1-s)}\frac{|U(x,z)|^p}{(x^2+z^2)^{p/2}}\,\mathrm{d}z\,\mathrm{d}x,$$

which does not exceed

$$2^{p/2}\int_{S^n_+}(\cos\theta)^{-1+p(1-s)}\int_0^\infty |U|^p\rho^{n-1-sp}\,\mathrm{d}\rho\,\mathrm{d}\sigma, \qquad (10.2.7)$$

where $\rho = (x^2+z^2)^{1/2}$, $\cos\theta = z/\rho$, $\mathrm{d}\sigma$ is an element of the surface area on the unit sphere S^n and S^n_+ is the upper half of S^n. Using Hardy's inequality

$$\int_0^\infty |U|^p\rho^{n-1-sp}\,\mathrm{d}\rho \le \Big(\frac{p}{n-sp}\Big)^p\int_0^\infty\Big|\frac{\partial U}{\partial\rho}\Big|^p\rho^{n-1+p(1-s)}\,\mathrm{d}\rho,$$

one arrives at the estimate

$$A^p \le \Big(\frac{2^{1/2}p}{n-sp}\Big)^p\int_0^\infty\int_{\mathbb{R}^n} z^{-1+p(1-s)}\big|\nabla U(x,z)\big|^p\,\mathrm{d}x\,\mathrm{d}z.$$

Combining this with (10.2.6), one obtains

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x \le \frac{p(1-s)}{s^p}\Big(1+\frac{2^{1/2}p}{n-sp}\Big)^p \times \int_0^\infty \int_{\mathbb{R}^n} z^{-1+p(1-s)}\big|\nabla U(x,z)\big|^p\,\mathrm{d}x\,\mathrm{d}z,$$

which, along with (10.2.5), gives

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x \le \frac{(1-s)}{(n-sp)^p}\frac{p(n+2p)^{3p}}{\omega_n s^p}\|u\|^p_{\mathring{\mathcal{W}}^s_p(\mathbb{R}^n)}. \tag{10.2.8}$$

To justify (10.2.3) we need to improve (10.2.8) for small values of s. Clearly,

$$\frac{\omega_n}{2^{sp}sp}\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x = \int_{\mathbb{R}^n}\int_{|x-y|>2|x|}\frac{\mathrm{d}y}{|x-y|^{n+sp}}\big|u(x)\big|^p\,\mathrm{d}x.$$

Since $|x-y|>2|x|$ implies $2|y|/3<|x-y|<2|y|$, we obtain

$$\Big(\frac{\omega_n}{2^{sp}sp}\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x\Big)^{1/p} \le \Big(\int_{\mathbb{R}^n}\int_{|x-y|>|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p} + \Big(\omega_n\frac{3^{sp}-1}{2^{sp}sp}\int_{\mathbb{R}^n}\frac{|u(y)|^p}{|y|^{sp}}\,\mathrm{d}y\Big)^{1/p}.$$

Hence,

$$\Big(\frac{\omega_n}{2^{sp}sp}\Big)^{1/p}\big(1-\big(3^{sp}-1\big)^{1/p}\big)\Big(\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x\Big)^{1/p} \le 2^{-1/p}\|u\|_{\mathring{\mathcal{W}}^s_p(\mathbb{R}^n)}.$$

Let δ be an arbitrary number in $(0,1)$. If $s\le(4p)^{-1}\delta^p$, we conclude

$$\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x \le \frac{2^{sp-1}sp}{\omega_n(1-\delta)^p}\|u\|^p_{\mathring{\mathcal{W}}^s_p(\mathbb{R}^n)}. \tag{10.2.9}$$

Setting $\delta=2^{-1}$ and comparing this inequality with (10.2.8), we arrive at (10.2.3) with

$$c(n,p)=\omega_n^{-1}(n+2p)^{3p}p^{p+2}2^{(n+1)(n+2)}.$$

The proof is complete. □

From Theorem 10.2.2, we shall deduce an inequality, analogous to (10.2.3), for functions defined on the cube Q. Unlike (10.2.3), this inequality contains no factor s on the right-hand side, which is not surprising because, for smooth u, the norm $\|u\|_{\mathcal{W}^{s,p}_\perp(Q)}$ tends to a finite limit as $s\downarrow 0$.

Corollary 1. *Let $n\ge1$, $p\ge1$, $0<s<1$, and $sp<n$. Then any function $u\in\mathcal{W}^{s,p}_\perp(Q)$ satisfies*

$$\int_Q\big|u(x)\big|^p\frac{\mathrm{d}x}{|x|^{sp}} \le c(n,p)\frac{1-s}{(n-sp)^p}\|u\|^p_{\mathcal{W}^{s,p}_\perp(Q)}. \tag{10.2.10}$$

Proof. Let us preserve the notation u for the extension by a reflection of $u \in \mathcal{W}^{s,p}_{\perp}(Q)$ to the cube $3Q$, where aQ stands for the cube obtained from Q by dilation with the coefficient a. We choose a cutoff function η, equal to 1 on Q and vanishing outside $2Q$, say, $\eta(x) = \prod_{i=1}^n \min\{1, 2(1-x_i)_+\}$. By the previous Theorem, it is enough to prove that

$$\|\eta u\|^p_{\mathring{\mathcal{W}}^s_p(\mathbb{R}^n)} \le s^{-1} c(n,p) \|u\|^p_{\mathcal{W}^{s,p}_{\perp}(Q)}. \tag{10.2.11}$$

Clearly, the norm in the left-hand side is majorized by

$$\begin{aligned}&\Big(\int_{3Q}\int_{3Q} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\eta(y)^p\,\mathrm{d}y\Big)^{1/p}\\ &+\Big(\int_{3Q}\int_{3Q} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x|u(y)|^p\,\mathrm{d}y\Big)^{1/p}\\ &+\Big(2\int_{3Q}\int_{\mathbb{R}^n\setminus 3Q} \frac{\mathrm{d}y}{|x-y|^{n+sp}}\big|(\eta u)(x)\big|^p\,\mathrm{d}x\Big)^{1/p}.\end{aligned}$$

The first term does not exceed $6^{n/p}\|u\|_{\mathcal{W}^{s,p}_{\perp}(Q)}$; the second term is not greater than

$$\begin{aligned}&2n^{1/2}\Big(\int_{3Q}\int_{3Q} \frac{\mathrm{d}x}{|x-y|^{n-p(1-s)}}\big|u(y)\big|^p\,\mathrm{d}y\Big)^{1/p}\\ &\quad\le n3^{2+n/p}\Big(\frac{\omega_n}{p(1-s)}\Big)^{1/p}\|u\|_{L_p(Q)},\end{aligned}$$

and the third one is dominated by

$$\Big(2\int_{2Q}\int_{|x-y|>1/2} \frac{\mathrm{d}y}{|x-y|^{n+sp}}\big|u(x)\big|^p\,\mathrm{d}x\Big)^{1/p} \le \Big(\frac{2^{n+1+p}}{sp}\Big)^{1/p}\|u\|_{L_p(Q)}.$$

Adding these estimates, one obtains

$$\begin{aligned}\|\eta u\|_{\mathcal{W}^{s,p}_0(\mathbb{R}^n)} &\le 6^{n/p}\|u\|_{\mathring{\mathcal{W}}^s_p(Q)}\\ &\quad + n3^{2+n/p}p^{-1/p}\big(s^{-1/p}+(1-s)^{-1/p}\big)\|u\|_{L_p(Q)}.\end{aligned} \tag{10.2.12}$$

We preserve the notation u for the mirror extension of u onto $\mathbb{R}^n$. Recalling that $u\perp 1$ on Q, we have

$$\begin{aligned}\int_Q\big|u(x)\big|^p\,\mathrm{d}x &\le \int_Q\int_Q\big|u(x)-u(y)\big|^p\,\mathrm{d}x\,\mathrm{d}y\\ &\le \int_{2Q}\mathrm{d}h\int_Q\big|u(x+h)-u(x)\big|^p\,\mathrm{d}x.\end{aligned} \tag{10.2.13}$$

Let U be the extension of u to $\mathbb{R}^{n+1}_+$. For any $z > 0$ and $h \in 2Q$

$$\begin{aligned}
&\big\|u(\cdot + h) - u(\cdot)\big\|_{L_p(Q)} \\
&\quad \le \bigg\| \int_0^z \frac{\partial U}{\partial \tau}(\cdot + h, \tau)\,\mathrm{d}\tau \bigg\|_{L_p(Q)} + \bigg\| \int_0^z \frac{\partial U}{\partial \tau}(\cdot, \tau)\,\mathrm{d}\tau \bigg\|_{L_p(Q)} \\
&\qquad + \big\|U(\cdot + h, z) - U(\cdot, z)\big\|_{L^p(Q)} \\
&\quad \le 2\int_0^z \bigg\| \frac{\partial U}{\partial \tau}(\cdot, \tau) \bigg\|_{L_p(3Q)} \mathrm{d}\tau + \big\|U(\cdot + h, z) - U(\cdot, z)\big\|_{L_p(Q)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&p^{-1/p}(1-s)^{-1/p}|h|^{1-s}\big\|u(\cdot + h) - u(\cdot)\big\|_{L_p(Q)} \\
&\quad = \bigg(\int_0^{|h|} \big\|u(\cdot + h) - u(\cdot)\big\|^p_{L_p(Q)} z^{-1+p(1-s)}\,\mathrm{d}z \bigg)^{1/p} \\
&\quad \le \alpha + \beta, \qquad (10.2.14)
\end{aligned}$$

where

$$\alpha^p = \big(2|h|\big)^p \int_0^{|h|} \bigg(\int_0^z \bigg\| \frac{\partial U}{\partial \tau}(\cdot, \tau) \bigg\|_{L_p(3Q)} \mathrm{d}\tau \bigg)^p z^{-1-ps}\,\mathrm{d}z$$

and

$$\beta^p = \int_0^{|h|} \big\|U(\cdot + h, z) - U(\cdot, z)\big\|^p_{L_p(Q)} \, z^{-1+p(1-s)}\,\mathrm{d}z.$$

Using Hardy's inequality already referred to at the beginning of the proof of the previous Theorem, we arrive at

$$\alpha^p \le \bigg(\frac{2|h|}{s} \bigg)^p \int_0^{|h|} \bigg\| \frac{\partial U}{\partial z}(\cdot, z) \bigg\|^p_{L_p(3Q)} z^{-1+p(1-s)}\,\mathrm{d}z.$$

The trivial inequality

$$\int_Q \big|U(x+h, z) - U(x, z)\big|^p \,\mathrm{d}x \le |h|^p \int_{3nQ} \big|\nabla U(x, z)\big|^p \,\mathrm{d}x$$

implies

$$\beta^p \le |h|^p \int_0^{|h|} \big\|\nabla U(\cdot, z)\big\|^p_{L_p(3nQ)} z^{-1+p(1-s)}\,\mathrm{d}z.$$

We put the just-obtained estimates for α and β into (10.2.14) and deduce

$$\begin{aligned}
&\big\|u(\cdot + h) - u(\cdot)\big\|^p_{L_p(Q)} \\
&\quad \le p(1-s)\bigg(\frac{2}{s} + 1 \bigg)^p |h|^{ps} \int_0^{|h|} \big\|\nabla U(\cdot, z)\big\|^p_{L_p(2nQ)} z^{-1+p(1-s)}\,\mathrm{d}z.
\end{aligned}$$

Noting that $|h| \le \sqrt{n}$ for $h \in 2Q$, we find

$$|h|^{-ps} \big\| u(\cdot + h) - u(\cdot) \big\|^p_{L_p(Q)} \\ \le p(1-s)\Big(\frac{2}{s}+1\Big)^p \int_0^{\sqrt{n}} \int_{3nQ} \big|\nabla U(x,z)\big|^p \,\mathrm{d}x z^{-1+p(1-s)}\,\mathrm{d}z. \tag{10.2.15}$$

Now, let U be the same extension of u onto $\mathbb{R}^{n+1}_+$ as in the beginning of the proof of Theorem 10.2.2. Repeating with obvious changes the standard argument in the proof of the previous Theorem, which leads to (10.2.5), we conclude that the integral over $(0,\sqrt{n})\times 3nQ$ in the right-hand side of (10.2.15) is majorized by

$$c_0 \int_0^{\sqrt{n}} \int_{|\chi|<1} \int_{3nQ} \big|u(x+z\chi) - u(x)\big|^p \,\mathrm{d}x\,\mathrm{d}\chi\, z^{-1-ps}\,\mathrm{d}z \\ \le \frac{c_0}{n+ps} \int_{|\eta|<\sqrt{n}} |\eta|^{-n-ps} \int_{3nQ} \big|u(x+\eta) - u(x)\big|^p \,\mathrm{d}x\,\mathrm{d}\eta,$$

where

$$c_0 = \frac{n}{\omega_n}(n+1)^p(n+2)^p.$$

Therefore,

$$|h|^{-ps} \big\| u(\cdot + h) - u(\cdot) \big\|^p_{L_p(Q)} \\ \le c_0 3^{n+2p} n^{2n} \frac{1-s}{s^p} \int_Q \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y. \tag{10.2.16}$$

Let $s > 1/2$. It follows from (10.2.13) that

$$\int_Q \big|u(x)\big|^p \,\mathrm{d}x \le n^{(n+ps)/2} \int_Q \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} \,\mathrm{d}x\,\mathrm{d}y.$$

This inequality together with (10.2.16) shows that for all $s \in (0,1)$

$$\|u\|_{L_p(Q)} \le (4n)^{4n}(1-s)^{1/p} \|u\|_{\mathcal{W}^{s,p}_\perp(Q)}.$$

Combining this inequality with (10.2.12), we arrive at (10.2.11) and hence complete the proof. □

Corollary 2. *Let* $0 < s < 1$ *and* $p \ge 1$. *Then there holds*

$$\sup |h|^{-s} \big\| u(\cdot + h) - u(\cdot) \big\|_{L_q(Q)} \le c(n,p)(1-s)^{1/p} \|u\|_{\mathcal{W}^{s,p}_\perp(Q)}.$$

Proof. The result follows from the well-known embedding $B^s_p(Q) \subset B^s_{p,\infty}(Q)$ if $s \le 1/2$ and from (10.2.16) if $s > 1/2$. □

10.2.3 Sobolev Embeddings

Proof of Theorem 10.2.1. It is well known that the fractional Sobolev norm of order $s \in (0,1)$ is nonincreasing with respect to the symmetric rearrangement of functions decaying to zero at infinity (see Wik [796], Almgren and Lieb [41], Theorem 9.2, and Cianchi [196]). Let $v(|x|)$ denote the rearrangement of $|u(x)|$. Then

$$\|u\|_{L_q(\mathbb{R}^n)} = \left(\frac{\omega_n}{n} \int_0^\infty v(r)^q \, \mathrm{d}\big(r^n\big) \right)^{1/q}, \tag{10.2.17}$$

where ω_n is the area of the unit sphere ∂B_1. Recalling that an arbitrary nonnegative nonincreasing function f on the semi-axis $(0, \infty)$ satisfies

$$\int_0^\infty f(t)^\lambda \, \mathrm{d}\big(t^\lambda\big) \le \lambda \int_0^\infty \left(\int_0^t f(\tau) \, \mathrm{d}\tau \right)^{\lambda-1} f(t) \, \mathrm{d}t = \left(\int_0^\infty f(t) \, \mathrm{d}t \right)^\lambda, \quad \lambda \ge 1$$

(see Hardy, Littlewood and Pólya [350]), one finds that the right-hand side in (10.2.17) does not exceed

$$\left(\frac{\omega_n}{n} \right)^{1/q} \left(\int_0^\infty v(r)^p \, \mathrm{d}\big(r^{n-sp}\big) \right)^{1/p} = \frac{(n-sp)^{1/p}}{n^{1/q} \omega_n^{s/n}} \left(\int_{\mathbb{R}^n} v\big(|x|\big)^p \frac{\mathrm{d}x}{|x|^{sp}} \right)^{1/p}.$$

We now see that (10.2.2) results from inequality (10.2.3). □

Corollary. *Let $n \ge 1$, $p \ge 1$, $0 < s < 1$, and $sp < n$. Then any function $u \in \mathcal{W}_\perp^{s,p}(Q)$ satisfies*

$$\|u\|^p_{L_q(Q)} \le c(n,p) \frac{1-s}{(n-sp)^{p-1}} \|u\|^p_{\mathcal{W}_\perp^{s,p}(Q)}.$$

Proof. Let η be the same cutoff function as in Corollary 10.2.2/1. The result follows by combining inequality (10.2.11) with Theorem 10.2.1, where u is replaced by ηu.

10.2.4 Asymptotics of the Norm in $\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)$ as $s \downarrow 0$

Theorem. *For any function $u \in \bigcup_{0<s<1} \mathring{\mathcal{W}}_p^s(\mathbb{R}^n)$, there exists the limit*

$$\lim_{s \downarrow 0} s \|u\|^p_{\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)} = 2p^{-1} \omega_n \|u\|^p_{L_p(\mathbb{R}^n)}.$$

Proof. Since δ can be chosen arbitrarily small, inequality (10.2.9) implies

$$\liminf_{s \downarrow 0} s \|u\|^p_{\mathring{\mathcal{W}}_p^s(\mathbb{R}^n)} \ge 2p^{-1} \omega_n \|u\|^p_{L_p(\mathbb{R}^n)}. \tag{10.2.18}$$

Let us majorize the upper limit. By (10.2.18), it suffices to assume that $u \in L_p(\mathbb{R}^n)$. Clearly,

$$s\|u\|^p_{\mathring{\mathcal{W}}^s_p(\mathbb{R}^n)} \le 2\Big\{\Big(s\int_{\mathbb{R}^n}\int_{|y|\ge 2|x|}\frac{\mathrm{d}y}{|x-y|^{n+sp}}\big|u(x)\mathbb{R}^n\big|^p\,\mathrm{d}x\Big)^{1/p}.$$
$$+\Big(s\int_{\mathbb{R}^n}\big|u(y)\big|^p\int_{|y|\ge 2|x|}\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{n+sp}}\Big)^{1/p}\Big\}^p$$
$$+2s\int_{\mathbb{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y.$$

The first term in braces does not exceed

$$\Big(s\int_{\mathbb{R}^n}\int_{|y|\ge|x|}\frac{\mathrm{d}y}{|x-y|^{n+sp}}\big|u(x)\big|^p\,\mathrm{d}x\Big)^{1/p}=\frac{\omega_n^{1/p}}{p^{1/p}}\Big(\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x\Big)^{1/p},$$

hence its $\limsup_{s\downarrow 0}$ is dominated by $\omega_n^{1/p}p^{-1/p}\|u\|_{L_p(\mathbb{R}^n)}$. The second term in braces is not greater than

$$s^{1/p}\Big(2^{n+sp}\int_{\mathbb{R}^n}\frac{|u(y)|^p}{|y|^{n+sp}}\,\mathrm{d}y\int_{|x|<|y|/2}\mathrm{d}x\Big)^{1/p}$$
$$=2^s\Big(\frac{s}{p}\omega_n\Big)^{1/p}\Big(\int_{\mathbb{R}^n}\frac{|u(y)|^p}{|y|^{sp}}\,\mathrm{d}y\Big)^{1/p},$$

so it tends to zero as $s\downarrow 0$.

We claim that

$$\limsup_{s\downarrow 0} s\int_{\mathbb{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y=0. \tag{10.2.19}$$

By assumption of the Theorem, $u\in\mathring{\mathcal{W}}^\tau_p(\mathbb{R}^n)$ for a certain $\tau\in(0,1)$. Let N be an arbitrary number greater than 1 and let $s<\tau$. We have

$$2s\int_{\mathbb{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\le 2sN^{p(\tau-s)}\int_{\mathbb{R}^n}\int_{\substack{|x|<|y|<2|x|\\ |x-y|\le N}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\tau p}}\,\mathrm{d}x\,\mathrm{d}y$$
$$+2s\int_{\mathbb{R}^n}\int_{\substack{|x|<|y|<2|x|\\ |x-y|>N}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y.$$

The first term on the right-hand side tends to zero as $s\downarrow 0$ and the second one does not exceed

$$2^{p+1}s\int_{|x|>N/3}\int_{|x-y|>N}\frac{\mathrm{d}y}{|x-y|^{n+sp}}\big|u(x)\big|^p\,\mathrm{d}x\le c(n,p)\int_{|x|>N/3}\big|u(x)\big|^p\,\mathrm{d}x,$$

which is arbitrarily small if N is sufficiently large. The proof is complete. □

Remark. Since the proof of the Theorem just proved holds for vector-valued functions, one can write

$$\lim_{s\downarrow 0} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\nabla u(x)-\nabla u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = 2p^{-1}\omega_n \int_{\mathbb{R}^n} \big|\nabla u(x)\big|^p\,dx, \quad (10.2.20)$$

for any function u such that

$$\nabla u \in \bigcup_{0<s<1} \mathring{\mathcal{W}}_p^s(\mathbb{R}^n).$$

Formula (10.2.20) complements the following relation that was established in Bourgain, Brezis, and Mironescu [138]:

$$\lim_{s\uparrow 1}(1-s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy$$
$$= \int_{\partial B_1} |\cos\theta|^p\,d\sigma \int_{\mathbb{R}^n} \big|\nabla u(x)\big|^p\,dx, \quad (10.2.21)$$

where θ is the angle deviation from the vertical.

10.3 On the Brezis and Mironescu Conjecture Concerning a Gagliardo–Nirenberg Inequality for Fractional Sobolev Norms

10.3.1 Introduction

Let $s \in (0,1)$ and let $1 < p \le \infty$. We introduce the space $\mathcal{W}_p^s(\mathbb{R}^n)$ of functions in $\mathbb{R}^n$ with the finite seminorm

$$\|u\|_{\mathcal{W}_p^s} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{1/p}.$$

The relation (10.2.21) motivated Brezis and Mironescu to conjecture the following Gagliardo–Nirenberg-type inequality:

$$\|u\|_{\mathcal{W}_{2p}^{s/2}} \le c(n,p)(1-s)^{1/2p} \|u\|_{\mathcal{W}_p^s}^{1/2} \|u\|_{L_\infty}^{1/2} \quad (10.3.1)$$

(see [144], Remark 5). In [144] one can also read, “It would be of interest to establish

$$\|u\|_{\mathcal{W}_{p/\theta}^{\theta s}} \le c\|u\|_{\mathcal{W}_p^s}^{\theta} \|u\|_{L_\infty}^{1-\theta}, \quad 0<\theta<1, \quad (10.3.2)$$

with control of the constant c, in particular when $s \uparrow 1$”.

In the next subsection we prove that (10.3.2) holds with

$$c = c(n,p,\theta)(1-s)^{\theta/p},$$

which, obviously, contains inequality (10.3.1) predicted by Brezis and Mironescu. Our proof is straightforward and rather elementary. In concluding Remarks 1 and 2 we show that the dependence of c on each of the parameters s, θ, and p is sharp in a certain sense.

10.3.2 Main Theorem

Theorem. *For all* $u \in \mathcal{W}_p^s \cap L_\infty$ *there holds the inequality*

$$\|u\|_{\mathcal{W}^{\theta s}_{p/\theta}} \le c(n)\Big(\frac{p}{p-1}\Big)^{\theta}\Big(\frac{1-s}{1-\theta}\Big)^{\theta/p}\|u\|^{\theta}_{\mathcal{W}^s_p}\|u\|^{1-\theta}_{L_\infty}, \tag{10.3.3}$$

where $0 < s < 1$, $1 < p < \infty$, *and* $0 < \theta < 1$.

Proof. Clearly,

$$\|u\|_{\mathcal{W}^{\theta s}_{p/\theta}} \le \max\big\{2^{\theta/p}, 2^{1-\theta}\big\}\|u\|^{1-\theta}_{L_\infty}\|u\|^{\theta}_{\mathcal{W}^s_p}. \tag{10.3.4}$$

Hence it suffices to prove (10.3.3) only for $s \ge 1/2$.

Let $B_r(x) = \{\xi \in \mathbb{R}^n : |\xi - x| < r\}$ and $B_r(0) = B_r$. We introduce the mean value $\overline{u}_{x,y}$ of u over the ball $\mathcal{B}_{x,y} := B_{|x-y|/2}((x+y)/2)$. Since

$$\big|u(x) - u(y)\big|^{p/\theta} \le 2^{-1+p/\theta}\big(|u(x) - \overline{u}_{x,y}|^{p/\theta} + \big|\overline{u}_{x,y} - u(y)\big|^{p/\theta}\big),$$

it follows that

$$\|u\|_{\mathcal{W}^{\theta s}_{p/\theta}} \le 2\Big(\int_{\mathbb{R}^n} D(x)^{p/\theta}\,\mathrm{d}x\Big)^{\theta/p}, \tag{10.3.5}$$

where

$$D(x) = \Big(\int_{\mathbb{R}^n} \frac{|u(x) - \overline{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}}\,\mathrm{d}y\Big)^{\theta/p}.$$

We note that

$$\int_{|x-y|>\delta} \frac{|u(x) - \overline{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}}\,\mathrm{d}y \le \frac{2^{p/\theta}\omega_n}{ps}\|u\|^{p/\theta}_{L_\infty}\delta^{-ps}. \tag{10.3.6}$$

Let U be an arbitrary extension of u onto $\mathbb{R}^{n+1}_+ = \{(x,z) : x \in \mathbb{R}^n, z > 0\}$ such that $\nabla U \in L_1(\overline{\mathbb{R}^{n+1}_+}, \mathrm{loc})$. By $\overline{U}_{x,y}(z)$ we denote the mean value of $U(\cdot, z)$ in $\mathcal{B}_{x,y}$. Using the identity

$$|x-y|^{(1-s)p} = p(1-s)\int_0^{|x-y|} z^{-1+p(1-s)}\,\mathrm{d}z,$$

we find

$$\begin{aligned}
&\int_{|x-y|<\delta} \frac{|u(x) - \overline{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}}\,\mathrm{d}y \\
&= p^{1/\theta}(1-s)^{1/\theta}\int_{|x-y|<\delta}\Big(\int_0^{|x-y|} z^{-1+p(1-s)}\big|u(x) - \overline{u}_{x,y}\big|^p\,\mathrm{d}z\Big)^{1/\theta} \\
&\quad\times \frac{\mathrm{d}y}{|x-y|^{n+ps+(1-s)p/\theta}} \\
&\le 3^{-1+p/\theta}p^{1/\theta}(1-s)^{1/\theta}(\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3),
\end{aligned} \tag{10.3.7}$$

where

$$\mathcal{J}_1 := \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} \big| u(x) - U(x,z) \big|^p \, \mathrm{d}z \right)^{1/\theta} \times \frac{\mathrm{d}y}{|x-y|^{n+ps+(1-s)p/\theta}},$$

$$\mathcal{J}_2 := \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} |\overline{U}_{x,y}(z) - \overline{u}_{x,y}|^p \, \mathrm{d}z \right)^{1/\theta} \times \frac{\mathrm{d}y}{|x-y|^{n+ps+(1-s)p/\theta}},$$

and

$$\mathcal{J}_3 := \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} \big| U(x,z) - \overline{U}_{x,y}(z) \big|^p \, \mathrm{d}z \right)^{1/\theta} \times \frac{\mathrm{d}y}{|x-y|^{n+ps+(1-s)p/\theta}}.$$

Clearly,

$$\begin{aligned}
\mathcal{J}_1 &\le \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} \left(\int_0^z \left| \frac{\partial U(x,t)}{\partial t} \right| \mathrm{d}t \right)^p \mathrm{d}z \right)^{1/\theta} \times \frac{\mathrm{d}y}{|x-y|^{n+ps+(1-s)p/\theta}} \\
&\le \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1-ps} \left(\int_0^z \left| \frac{\partial U(x,t)}{\partial t} \right| \mathrm{d}t \right)^p \mathrm{d}z \right)^{1/\theta} \times \frac{\mathrm{d}y}{|x-y|^{n-ps(1-\theta)/\theta}}.
\end{aligned}$$

By Hardy's inequality

$$\int_0^a z^{-1-sp} \left| \int_0^z \varphi(t) \, \mathrm{d}t \right|^p \mathrm{d}z \le s^{-p} \int_0^a z^{-1+p(1-s)} \big| \varphi(z) \big|^p \, \mathrm{d}z,$$

one has

$$\begin{aligned}
\mathcal{J}_1 &\le s^{-p/\theta} \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} \left| \frac{\partial U(x,z)}{\partial z} \right|^p \mathrm{d}z \right)^{1/\theta} \frac{\mathrm{d}y}{|x-y|^{n-ps(1-\theta)/\theta}} \\
&\le \frac{\theta \omega_n}{s^{p/\theta} ps(1-\theta)} \left(\int_0^\infty z^{-1+p(1-s)} \left| \frac{\partial U(x,z)}{\partial z} \right|^p \mathrm{d}z \right)^{1/\theta} \delta^{ps(1-\theta)/\theta}. \qquad (10.3.8)
\end{aligned}$$

Duplicating the same argument, we conclude that $\mathcal{J}_2$ does not exceed

$$s^{-p/\theta} \int_{|x-y|<\delta} \frac{\mathrm{d}y}{|x-y|^{n-ps(1-\theta)/\theta}} \left(\int_0^{|x-y|} z^{-1+p(1-s)} \left| \frac{\partial \overline{U}_{x,y}(z)}{\partial z} \right|^p \mathrm{d}z \right)^{1/\theta}. \tag{10.3.9}$$

Let $\mathcal{M}$ denote the n-dimensional Hardy–Littlewood maximal operator

$$(\mathcal{M}f)(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r(x)} \bigl|f(\xi)\bigr| \,\mathrm{d}\xi.$$

Using the obvious inequality

$$\left| \frac{\partial \overline{U}_{x,y}(z)}{\partial z} \right| \le \left(\mathcal{M} \frac{\partial U}{\partial z} \right)(x,z),$$

we find from (10.3.9)

$$\mathcal{J}_2 \le \frac{\theta \omega_n}{s^{p/\theta} ps(1-\theta)} \left(\int_0^\infty z^{-1+p(1-s)} \left(\mathcal{M} \frac{\partial U}{\partial z} \right)^p \mathrm{d}z \right)^{1/\theta} \delta^{ps(1-\theta)/\theta}. \tag{10.3.10}$$

To estimate $\mathcal{J}_3$ we use the Sobolev-type integral representation in the form given in Akilov and Kantorovich [38], Chap. 10, Sect. 3

$$U(x,z) - \overline{U}_{x,y}(z) = \sum_{k=1}^n \int_{\mathcal{B}_{x,y}} \frac{b_k(\xi,x)}{|x-\xi|^{n-1}} \frac{\partial U(\xi,z)}{\partial \xi_k} \,\mathrm{d}\xi, \tag{10.3.11}$$

where $b_k(\xi,x)$ are continuous functions for $x \ne \xi$ admitting the estimate

$$\bigl|b_k(\xi,x)\bigr| \le \frac{|x-y|^n}{n|\mathcal{B}_{x,y}|}.$$

Clearly, (10.3.11) implies the estimate

$$\bigl|U(x,z) - \overline{U}_{x,y}(z)\bigr| \le \frac{2^n n^{1/2}}{\omega_n} \int_{B_r(x)} \frac{|\nabla_\xi U(\xi,z)|}{|x-\xi|^{n-1}} \,\mathrm{d}\xi,$$

where $r = |x-y|$. Integrating by parts we find

$$\begin{aligned} &\int_{B_r(x)} \frac{|\nabla_\xi U(\xi,z)|}{|x-\xi|^{n-1}} \,\mathrm{d}\xi \\ &\quad = r^{1-n} \int_{B_r(x)} \bigl|\nabla_\xi U(\xi,z)\bigr| \,\mathrm{d}\xi + (n-1) \int_0^r \frac{\mathrm{d}s}{s^n} \int_{B_s(x)} \bigl|\nabla_\xi U(\xi,z)\bigr| \,\mathrm{d}\xi \\ &\quad \le n|x-y| \bigl(\mathcal{M}|\nabla U|\bigr)(x,z). \end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{J}_3 &\le \left(\frac{2^n n^{3/2}}{\omega_n}\right)^{p/\theta} \int_{|x-y|<\delta} \left(\int_0^{|x-y|} z^{-1+p(1-s)} \big(\mathcal{M}|\nabla U|\big)^p \,\mathrm{d}z\right)^{1/\theta} \\ &\qquad \times \frac{\mathrm{d}y}{|x-y|^{n-ps(1-\theta)/\theta}} \\ &\le \frac{(2^n n^{3/2})^{p/\theta}\theta}{\omega_n^{(p-\theta)/\theta} ps(1-\theta)} \left(\int_0^\infty z^{-1+p(1-s)} \big(\mathcal{M}|\nabla U|\big)^p \,\mathrm{d}z\right)^{1/\theta} \\ &\qquad \times \delta^{ps(1-\theta)/\theta}. \end{aligned} \tag{10.3.12}$$

Here and in the sequel, for the sake of brevity, by $\mathcal{M}|\nabla U|$ we mean $(\mathcal{M}|\nabla U|)(x, z)$. Putting estimates (10.3.8), (10.3.10), and (10.3.12) into (10.3.7), we arrive at

$$\begin{aligned}&\int_{|x-y|<\delta} \frac{|u(x)-\overline{u}_{x,y}|^{p/\theta}}{|x-y|^{n+ps}} \,\mathrm{d}y \\ &\quad \le c(n)\frac{(1-s)^{1/\theta}}{1-\theta} \left(\int_0^\infty z^{-1+p(1-s)} \big(\mathcal{M}|\nabla U|\big)^p \,\mathrm{d}z\right)^{1/\theta} \delta^{ps(1-\theta)/\theta}.\end{aligned}$$

This estimate together with (10.3.6) implies that $D(x)$ is majorized by

$$c(n)\left(\|u\|_{L_\infty}\delta^{-\theta s} + \left(\frac{1-s}{1-\theta}\right)^{1/p} \left(\int_0^\infty z^{-1+p(1-s)} \big(\mathcal{M}|\nabla U|\big)^p \,\mathrm{d}z\right)^{1/p} \delta^{s(1-\theta)}\right).$$

Minimizing the right-hand side, we conclude that

$$D(x) \le c(n)\left(\frac{1-s}{1-\theta}\right)^{\theta/p} \|u\|_{L_\infty}^{1-\theta} \left(\int_0^\infty z^{-1+p(1-s)} \big(\mathcal{M}|\nabla U|\big)^p \,\mathrm{d}z\right)^{\theta/p}.$$

Hence and by (10.3.5)

$$\begin{aligned}\|u\|_{\mathcal{W}^{\theta s}_{p/\theta}} &\le c(n)\left(\frac{1-s}{1-\theta}\right)^{\theta/p} \|u\|_{L_\infty}^{1-\theta} \\ &\qquad \times \left(\int_{\mathbb{R}^n}\int_0^\infty z^{-1+p(1-s)} \big(\mathcal{M}|\nabla U|\big)^p \,\mathrm{d}z\,\mathrm{d}x\right)^{\theta/p}.\end{aligned}$$

Since

$$\|\mathcal{M}u\|_{L_p} \le \frac{n8^n p}{\omega_n(p-1)}\|u\|_{L_p}$$

(see Iwaniec [399], Sect. 2.5), it follows that the norm $\|u\|_{\mathcal{W}^{\theta s}_{p/\theta}}$ does not exceed

$$c(n)\left(\frac{p}{p-1}\right)^{\theta} \left(\frac{1-s}{1-\theta}\int_0^\infty \int_{\mathbb{R}^n} z^{-1+p(1-s)} \big|\nabla U(x,z)\big|^p \,\mathrm{d}x\,\mathrm{d}z\right)^{\theta/p} \|u\|_{L_\infty}^{1-\theta}. \tag{10.3.13}$$

Now we define U by the formula

$$U(x,z) := \int_{\mathbb{R}^n} \psi(h) u(x+zh)\,\mathrm{d}h, \tag{10.3.14}$$

where

$$\psi(h) = \omega_n n(n+1)\big(1-|h|\big)_+,$$

with the subscript plus standing for the nonnegative part of a real-valued function. It follows directly from (10.3.14) that

$$\big|\nabla U(x,z)\big| \le \frac{n(n+1)(n+2)}{z\omega_n} \int_{|h|<1} \big|u(x+zh)-u(x)\big|\,\mathrm{d}h.$$

Hence and by Hölder's inequality

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} z^{-1+p(1-s)} \big|\nabla U(x,z)\big|^p\,\mathrm{d}x\,\mathrm{d}z \\
&\quad \le \frac{n}{\omega_n}(n+1)^p(n+2)^p \int_0^\infty z^{-1-ps} \int_{|h|<1}\int_{\mathbb{R}^n} \\
&\qquad \times \big|u(x+zh)-u(x)\big|^p\,\mathrm{d}x\,\mathrm{d}h\,\mathrm{d}z.
\end{aligned} \tag{10.3.15}$$

We have

$$\begin{aligned}
&\int_0^\infty z^{-1-ps} \int_{|h|<1} \big|u(x+zh)-u(x)\big|^p\,\mathrm{d}h\,\mathrm{d}z \\
&\quad = \int_0^\infty z^{-1-ps-n} \int_0^z \rho^{n-1}\,\mathrm{d}\rho \int_{\partial B_1} \big|u(x+\rho\theta)-u(x)\big|^p\,\mathrm{d}\theta \\
&\quad = (ps+n)^{-1} \int_0^\infty \rho^{-ps-1}\,\mathrm{d}\rho \int_{\partial B_1} \big|u(x+\rho\theta)-u(x)\big|^p\,\mathrm{d}\theta.
\end{aligned}$$

Thus,

$$\int_0^\infty \int_{\mathbb{R}^n} z^{-1+p(1-s)} \big|\nabla U(x,z)\big|^p\,\mathrm{d}x\,\mathrm{d}z \le \frac{n(n+1)^p(n+2)^p}{\omega_n(ps+n)} \|u\|^p_{\mathcal{W}^s_p}. \tag{10.3.16}$$

Combining (10.3.16) with (10.3.13) we complete the proof.

Remark 1. Let

$$\|u\|_{\mathcal{W}^1_p} = \left(\int_{\mathbb{R}^n} \big|\nabla u(x)\big|^p\,\mathrm{d}x\right)^{1/p}.$$

As a particular case of a more general inequality, Brezis and Mironescu [144] obtained the inequality

$$\|u\|_{\mathcal{W}^\theta_{p/\theta}} \le c\|\nabla u\|^\theta_{L_p} \|u\|^{1-\theta}_{L_\infty}, \quad 0<\theta<1.$$

They commented on this in the following way: “We do not know any elementary (i.e., without the Littlewood-Paley machinery) proof of (10.3.2) when $s = 1$.” Obviously, the earlier proof of (10.3.3), complemented by the reference to formula (10.2.21), provides an elementary proof of the inequality

$$(1-\theta)^{\theta/p}\|u\|_{\mathcal{W}^{\theta}_{p/\theta}} \le c(n,p)\|\nabla u\|^{\theta}_{L_p}\|u\|^{1-\theta}_{L_\infty}.$$

The factor $(1-\theta)^{\theta/p}$ controls the blow-up of the norm in $\mathcal{W}^{\theta}_{p/\theta}$ as $\theta \uparrow 1$.

Remark 2. Note that passing to the limit as $p \to \infty$ in both sides of (10.3.3) one obtains inequality (10.3.2) with $p = \infty$ and with a certain finite constant c. Let us consider the case $p \to 1$ when the constant factor in (10.3.3) tends to infinity. It follows from (10.3.3) that the best value of $c(n,p,\theta)$ in the inequality

$$\|u\|_{\mathcal{W}^{\theta s}_{p/\theta}} \le c(n,p,\theta)(1-s)^{\theta/p}\|u\|^{\theta}_{\mathcal{W}^{s}_{p}}\|u\|^{1-\theta}_{L_\infty}, \tag{10.3.17}$$

admits the upper estimate

$$\limsup_{p\downarrow 1}(p-1)^{\theta}c(n,p,\theta) \le c(n)(1-\theta)^{-\theta}. \tag{10.3.18}$$

Now we obtain the analogous lower estimate

$$\liminf_{p\downarrow 1}(p-1)^{\theta}c(n,p,\theta) \ge 1. \tag{10.3.19}$$

In fact, the characteristic function χ of the ball B_1 belongs to $\mathcal{W}^{s}_{p}$ and $\mathcal{W}^{\theta s}_{p/\theta}$ if and only if $sp < 1$, and there holds

$$\|\chi\|_{\mathcal{W}^{\theta s}_{p/\theta}} = \|\chi\|^{\theta}_{\mathcal{W}^{s}_{p}}.$$

Putting $u = \chi$ into (10.3.17), where $s = p^{-1} - \varepsilon$ with an arbitrarily small $\varepsilon > 0$, we obtain

$$1 \le c(n,p,\theta)\big((p-1)/p\big)^{\theta/p},$$

which implies (10.3.19). Thus, the growth $O((p-1)^{-\theta})$ of the constant in (10.3.3) as $p \downarrow 1$ is the best possible.

10.4 Some Facts from Nonlinear Potential Theory

10.4.1 Capacity $\mathrm{cap}(e, S^l_p)$ and Its Properties

With each function space $S^l_p = H^l_p, W^l_p, B^l_p, h^l_p, w^l_p$, and b^l_p introduced in Sect. 10.1, we associate a set function called *the capacity*. Namely, for any compactum $e \subset \mathbb{R}^n$ we put

$$\mathrm{cap}\big(e, S^l_p\big) = \inf\big\{\|u\|^p_{S^l_p} : u \in C^\infty_0,\ u \ge 1 \text{ on } e\big\}.$$

If E is an arbitrary subset of $\mathbb{R}^n$, then we call the numbers

$$\underline{\mathrm{cap}}(E, S_p^l) = \sup\{\mathrm{cap}(e, S_p^l) : e \subset E, e \text{ is a compactum}\},$$
$$\overline{\mathrm{cap}}(E, S_p^l) = \inf\{\underline{\mathrm{cap}}(G, S_p^l) : G \supset E, G \text{ is an open set}\},$$

the *inner* and *outer capacities* of E, respectively.

Theorem 10.1.3/3 implies the relation

$$\mathrm{cap}(e, S_p^l) \sim \sum_{i \geq 0} \mathrm{cap}(e \cap \mathscr{B}^{(i)}, S_p^l), \tag{10.4.1}$$

where $\{\mathscr{B}^{(i)}\}$ is the sequence of balls introduced in Theorem 10.1.3/3. Similar quasi-additivity relations for capacities were studied by D.R. Adams [6] and Aikawa [34, 35].

We state certain well-known properties of the capacity $\mathrm{cap}(\cdot, S_p^l)$, where $S_p^l = H_p^l$ or h_p^l, $p > 1$ (cf. Reshetnyak [674], Meyers [596], Maz'ya and Havin [567]).

1. *If the set $e \subset \mathbb{R}^n$ is compact then for each $\varepsilon > 0$ there exists an open set $G \subset \mathbb{R}^n$ such that $G \supset e$ and*

$$\mathrm{cap}(e', S_p^l) < \mathrm{cap}(e, S_p^l) + \varepsilon,$$

where e' is an arbitrary compact subset of G.

2. *If the set $e \subset \mathbb{R}^n$ is compact, then*

$$\overline{\mathrm{cap}}(e, S_p^l) = \mathrm{cap}(e, S_p^l).$$

3. *If $E_1 \subset E_2 \subset \mathbb{R}^n$ then*

$$\underline{\mathrm{cap}}(E_1, S_p^l) \leq \underline{\mathrm{cap}}(E_2, S_p^l), \qquad \overline{\mathrm{cap}}(E_1, S_p^l) \leq \overline{\mathrm{cap}}(E_2, S_p^l).$$

4. *If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets in $\mathbb{R}^n$ and $E = \bigcup_k E_k$, then*

$$\overline{\mathrm{cap}}(E, S_p^l) \leq \sum_{k=1}^{\infty} \overline{\mathrm{cap}}(E_k, S_p^l).$$

It is well known that *any analytic* (*in particular, any Borel*) *set $E \subset \mathbb{R}^n$ is measurable with respect to the capacity* $\mathrm{cap}(\cdot, S_p^l)$ (i.e., $\overline{\mathrm{cap}}(E, S_p^l) = \underline{\mathrm{cap}}(E, S_p^l)$) (cf. Meyers [596], Maz'ya and Havin [567]).

We introduce one more capacity

$$c_{k,p}(E) = \inf\Big\{\|f\|_{L_p}^p : f \in L_p,\ f \geq 0 \text{ and}$$
$$\int k(|x-y|) f(y)\, \mathrm{d}y \geq 1 \text{ for all } x \in E\Big\},$$

where k is a positive decreasing continuous function on the half-axis $(0, +\infty)$ (cf. Meyers [596]).

We list some connections between the two capacities.

(i) *If* $S^l_p = H^l_p$ *or* h^l_p *and* k *is the Bessel or Riesz kernel, then*

$$c_{k,p}(E) = c\,\mathrm{cap}\big(E, S^l_p\big).$$

(ii) *If* $\mathrm{diam}\, E \le 1$ *and* $pl < n$, *then*

$$\mathrm{cap}\big(E, H^l_p\big) \sim \mathrm{cap}\big(E, h^l_p\big) \tag{10.4.2}$$

(cf. Adams and Meyers [16]). Relations similar to (10.4.2) are also valid for other pairs of spaces, e.g., B^l_p, b^l_p and W^l_p, w^l_p.

(iii) *If* $1 < p < \infty$, *then*

$$\mathrm{cap}\big(E, H^l_p\big) \sim \mathrm{cap}\big(E, B^l_p\big)$$

(see Proposition 4.4.4 in D.R. Adams and Hedberg [15]).

(iv) *If* $E \subset \mathbb{R}^n$, $l > 0$, $1 < p < \infty$ *then*

$$\mathrm{cap}\big(E, B^l_p(\mathbb{R}^n)\big) \sim \mathrm{cap}\big(E, H^{l+1/p}_p(\mathbb{R}^{n+1})\big) \sim \mathrm{cap}\big(E, B^{l+1/p}_p(\mathbb{R}^{n+1})\big)$$

(cf. Sjödin [703]).

(v) *Let* m, $l > 0$, $1 < p$, *and* $q < \infty$. *For* $E \subset \mathbb{R}^n$ *the inequality*

$$\big[\mathrm{cap}\big(E, h^m_q\big)\big]^{n-lp} \le c\big[\mathrm{cap}\big(E, h^l_p\big)\big]^{n-mq}, \quad mq < lp < n,$$

is valid.

(vi) *If, in addition,* $E \subset B_1$, *then*

$$\big[\mathrm{cap}\big(E, H^m_q\big)\big]^{n-lp} \le c\big[\mathrm{cap}\big(E, H^l_p\big)\big]^{n-mq}, \quad mq < lp < n,$$
$$\left[\log \frac{c_0}{\mathrm{cap}(E, H^m_q)}\right]^{1-p} \le c\,\mathrm{cap}\big(E, H^l_p\big), \quad mq < lp = n,$$
$$\big[\mathrm{cap}\big(E, H^m_q\big)\big]^{p-1} \le c\big[\mathrm{cap}\big(E, H^l_p\big)\big]^{q-1}, \quad mq = lp = n,\ p \le q.$$

Putting $E = B_r$, we conclude that all power exponents here are exact. Items (v) and (vi) are due to Adams and Hedberg [14].

10.4.2 Nonlinear Potentials

Nonlinear potentials were introduced in the article by Maz'ya and Havin [566] and their theory has turned out to be a useful tool in various areas. It was developed in the papers by Maz'ya and Havin [567], D.R. Adams and Meyers [17], Hedberg and Wolff [372], D.R. Adams [9], et al. The reader interested in a detailed exposition of the nonlinear potential theory can find it in the comprehensive monograph by D.R. Adams and Hedberg [15]. Here we collect some facts from this theory.

Let $p \in (1, \infty)$, $n > pl$. Each nonnegative measure μ given on the Borel σ-algebra of the space $\mathbb{R}^n$ generates the function $U_{p,l}\mu$ defined on $\mathbb{R}^n$ by

$$(U_{p,l}\mu)(x) = \int_{\mathbb{R}^n} |x-y|^{l-n} \left(\int_{\mathbb{R}^n} |z-y|^{l-n} \, \mathrm{d}\mu(z) \right)^{1/(p-1)} \mathrm{d}y, \qquad (10.4.3)$$

or equivalently,

$$U_{p,l}\mu = I_l (I_l \mu)^{p'-1}, \quad p + p' = pp'.$$

For $p = 2$, by changing the order of integration in (10.4.3) and taking into account the composition formula

$$\int |y-z|^{l-n} |y-x|^{l-n} \, \mathrm{d}y = \mathrm{const}\, |z-x|^{2l-n},$$

(cf. Landkof [477]), we obtain

$$(U_{2,l}\mu)(x) = c \int \frac{\mathrm{d}\mu(z)}{|z-x|^{n-2l}}.$$

The function $U_{2,l}\mu$ is the Riesz potential of order $2l$ (for $l = 1$ it is the Newton potential). Similarly, $U_{p,l}\mu$ is called the nonlinear Riesz potential ((p,l)-potential).

The nonlinear Bessel potential is defined as

$$V_{p,l}\mu = J_l (J_l \mu)^{p'-1}.$$

The potentials $U_{p,l}\mu$ and $V_{p,l}\mu$ satisfy the following rough maximum principle for nonlinear potentials.

Proposition 1. *Let $P\mu$ be one of the potentials $U_{p,l}\mu$ or $V_{p,l}\mu$. Then there exists a constant $\mathfrak{M}$ that depends only on n, p, and l, such that*

$$(P\mu)(x) \le \mathfrak{M} \sup\big\{ (P\mu)(x) : x \in \operatorname{supp} \mu \big\}.$$

This assertion was proved in the papers by Maz'ya and Havin [567] and D.R. Adams and Meyers [16]. It is well known that we can take $\mathfrak{M} = 1$ for $p = 2$, $l \le 1$ (cf. Landkof [477]). In general, this is impossible even for $p = 2$ (cf. Landkof [477]).

The next assertion contains basic properties of the so-called (p,l)-capacitary measure (cf. Meyers [596], Maz'ya and Havin [567]).

Proposition 2. *Let E be a subset of $\mathbb{R}^n$. If $\overline{\mathrm{cap}}(E, h_p^l) < \infty$, then there exists a unique measure μ_E with the following properties:*

1. $\|I_l \mu_E\|^{p/(p-1)}_{L_{p/(p-1)}} = \overline{\mathrm{cap}}(E, h_p^l)$;

2. $(U_{p,l}\mu_E)(x) \geq 1$ *for* (p,l)*-quasi all* $x \in E$ (*the notion "*(p,l)*-quasi everywhere" means everywhere except for a set of zero outer capacity* $\overline{\text{cap}}(\cdot, h_p^l)$);
3. $\operatorname{supp} \mu_E \subset \bar{E}$;
4. $\mu_E(\bar{E}) = \overline{\text{cap}}(E, h_p^l)$;
5. $(U_{p,l}\mu_E)(x) \leq 1$ *for all* $x \in \operatorname{supp} \mu_E$.

The measure μ_E is called *the capacitary measure of* E and $U_{p,l}\mu_E$ is called *the capacitary potential of* E. The last proposition remains valid after the replacement of h_p^l by H_p^l, of $U_{p,l}\mu$ by $V_{p,l}\mu$, and of I_l by J_l. Besides, we note that the capacity $\text{cap}(e, S_p^l)$ (for $S_p^l = h_p^l$ or H_p^l) can be defined as

$$\text{cap}\big(e, S_p^l\big) = \sup_{x \in e}\big\{\mu(e) : \operatorname{supp} \mu \subset e, (P\mu)(x) \leq 1\big\}, \tag{10.4.4}$$

where $P = U_{p,l}$ or $V_{p,l}$ (cf. Maz'ya and Havin [567]). Next we present some pointwise estimates for (p,l)-potentials that are obvious in the linear case and nontrivial in the nonlinear case.

Proposition 3. (Maz'ya and Havin [567] and D.R. Adams [4]).
(i) *If* $2 - l/n < p < n/l$, *then*

$$(V_{p,l}\mu)(x) \leq c \int_0^\infty \left(\frac{\mu(B(x,\varrho))}{\varrho^{n-lp}}\right)^{1/(p-1)} e^{-c_0\varrho}\,\frac{\mathrm{d}\varrho}{\varrho}. \tag{10.4.5}$$

(ii) *If* $p > 1$ *and* $\varphi(\varrho) = \sup_x \mu(B(x,\varrho))$, *then*

$$(V_{p,l}\mu)(x) \leq c \int_0^\infty \left(\frac{\varphi(\varrho)}{\varrho^{n-lp}}\right)^{1/(p-1)} e^{-c_0\varrho}\,\frac{\mathrm{d}\varrho}{\varrho}. \tag{10.4.6}$$

The same estimates, without the factor $e^{-c_0\varrho}$, *are valid for the potential* $U_{p,l}\mu$.

It is almost obvious that the following estimate, opposite to (10.4.5),

$$(V_{p,l}\mu)(x) \geq c \int_0^\infty \left(\frac{\mu(B(x,\varrho))}{\varrho^{n-lp}}\right)^{1/(p-1)} e^{-c_0\varrho}\,\frac{\mathrm{d}\varrho}{\varrho}, \tag{10.4.7}$$

holds for all $p \in (1,\infty)$, $l > 0$, whereas (10.4.5) is not true for $p \leq 2 - l/n$.
Wolff showed (cf. Hedberg and Wolff [372]) that for $pl < n$ the inequality

$$\|I_l\mu\|^{p/(p-1)}_{L_{p/(p-1)}} \leq c \int W_{p,l}\mu\,\mathrm{d}\mu, \tag{10.4.8}$$

where

$$(W_{p,l}\mu)(x) = \int_0^\infty \left(\frac{\mu(B(x,\varrho))}{\varrho^{n-lp}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

is valid. The analogous inequality for Bessel potentials is

$$\|J_l\mu\|^{p/(p-1)}_{L_{p/(p-1)}} \leq c \int S_{p,l}\mu \, \mathrm{d}\mu, \tag{10.4.9}$$

where $pl \leq n$ and

$$(S_{p,l}\mu)(x) = \int_0^\infty \left(\frac{\mu(B(x,\varrho))}{\varrho^{n-lp}}\right)^{1/(p-1)} e^{-c_0\varrho} \frac{\mathrm{d}\varrho}{\varrho}. \tag{10.4.10}$$

The estimates converse to (10.4.8) and (10.4.9) follow immediately from (10.4.7).

Remark. The failure of the inequality (10.4.5) for $p \leq 2-l/n$ caused serious difficulties in attempts at a satisfactory generalization of the basic facts of the classical potential theory to the nonlinear case.

Using inequalities (10.4.8) and (10.4.9) Hedberg (cf. the previously mentioned paper by Hedberg and Wolff) managed to surmount this difficulty by virtue of an analog of the nonlinear potential theory in which the roles of $U_{p,l}\mu$ and $V_{p,l}\mu$ are played by certain nonlinear potentials that are equivalent to $W_{p,l}\mu$ and $S_{p,l}\mu$ (see also D.R. Adams and Hedberg [15], Sects. 4.4 and 4.5).

Upper pointwise estimates similar to (10.4.5) are obtained for the case $1 < p \leq 2-l/n$ under the additional assumption that the potential is bounded. Namely, the following proposition is true.

Proposition 4. (D.R. Adams and Meyers [16]).
(i) *If $1 < p < 2 - l/n$ and $(U_{p,l}\mu)(x) \leq K$ for all $x \in \mathbb{R}^n$, then*

$$(U_{p,l}\mu)(x) \leq cK^\gamma \int_0^\infty \left(\frac{\mu(B(x,\varrho))}{\varrho^{n-lp}}\right)^{(n-1)/(n-lp)} \frac{\mathrm{d}\varrho}{\varrho}, \tag{10.4.11}$$

where $\gamma = ((2-p)n-l)/(n-lp)$.
(ii) *If $p = 2 - l/n$ and $(U_{p,l}\mu)(x) \leq K$ for all $x \in \mathbb{R}^n$, then*

$$(U_{p,l}\mu)(x) \leq c \int_0^\infty \left(\frac{\mu(B(x,\varrho))}{\varrho^{n-lp}} \log\left(cK^{p-1}\frac{\varrho^{n-lp}}{\mu(B(x,\varrho))}\right)\right)^{p'-1} \frac{\mathrm{d}\varrho}{\varrho}. \tag{10.4.12}$$

(*The condition $(U_{p,l}\mu)(x) \leq K$ for all $x \in \mathbb{R}^n$ implies the estimate*

$$\mu\big(B(x,\varrho)\big) \leq e^{-1}aK^{p-1}\varrho^{n-lp}.)$$

10.4.3 Metric Properties of Capacity

The following relations are useful (cf. Meyers [596]).

If $pl < n$ and $0 < \varrho < 1$, then

$$\mathrm{cap}(B_\varrho, H^l_p) \sim \varrho^{n-pl}. \tag{10.4.13}$$

If $pl < n$ and $0 < \varrho < \infty$, then

$$\operatorname{cap}\bigl(B_\varrho, h_p^l\bigr) = c\varrho^{n-pl}. \qquad (10.4.14)$$

For $pl = n$, $0 < \varrho \le 1$ we have

$$\operatorname{cap}\bigl(B_\varrho, H_p^l\bigr) \sim (\log 2/\varrho)^{1-p}. \qquad (10.4.15)$$

If $p\,l > n$, then $\operatorname{cap}(\{x\}, H_p^l) > 0$. Thus, only the empty set has zero capacity if $pl > n$.

The following equivalence relations for the capacity of a parallelepiped were obtained by D.R. Adams [7].

Proposition 1. *Let* $0 < a_1 \le a_2 \le \cdots \le a_n$, $a = (a_1, a_2, \dots, a_n)$ *and let* $Q(a) = \{x \in \mathbb{R}^n : |x_j| \le a_j, j = 1, \dots, n\}$.

(i) *If* $k - 1 < lp < k$, $k = 1, \dots, n$, *then*

$$\operatorname{cap}\bigl(Q(a), h_p^l\bigr) \sim a_k^{k-lp} \prod_{j=k+1}^{n} a_j.$$

(*Here the product equals unity provided* $k = n$.)

(ii) *If* $lp = k$, $k = 1, 2, \dots, n-1$, *then*

$$\operatorname{cap}\bigl(Q(a), h_p^l\bigr) \sim \min\left\{\left(\log \frac{a_{k+1}}{a_k}\right)^{1-p}, 1\right\} \prod_{j=k+1}^{n} a_j.$$

Similar two-sided estimates hold for $\operatorname{cap}(Q(a), H_p^l)$.

If T is a quasi-isometric mapping of $\mathbb{R}^n$ onto itself, then $\operatorname{cap}(TE, S_p^l) \sim \operatorname{cap}(E, S_p^l)$, where $S_p^l = H_p^l$ or h_p^l. This is a simple corollary of (10.4.4).

Meyers [597] showed that $\operatorname{cap}(PE, S_p^l) \le \operatorname{cap}(E, S_p^l)$ provided P is a projection $\mathbb{R}^n \to \mathbb{R}^k$, $k < n$, and $S_p^l = H_p^l$ or h_p^l.

For any set $E \subset \mathbb{R}^n$ and for a nondecreasing positive function φ on $[0, \infty)$ we define *the Hausdorff* φ*-measure*

$$H(E, \varphi) = \lim_{\varepsilon \to +0} \inf_{\{\mathscr{B}^{(i)}\}} \sum_i \varphi(r_i),$$

where $\{\mathscr{B}^{(i)}\}$ is any covering of the set E by open balls $\mathscr{B}^{(i)}$ with radii $r_i < \varepsilon$. If $\varphi(t) = t^q$, then d is called the dimension of the Hausdorff measure. The d-dimensional Hausdorff measure $H_d(E)$ is equal to $v_d H(E, t^d)$ (cf. Sect. 1.2.4). For $d = n$ the measure H_n coincides with the n-dimensional Lebesgue measure m_n.

The following propositions contain noncoinciding, but in a certain sense, exact necessary and sufficient conditions for positiveness of the capacity formulated in terms of the Hausdorff measures.

Proposition 2. *Let $1 < p \le n/l$ and let φ be a nonnegative nondecreasing function on $[0, \infty)$ with $\varphi(0) = 0$ and*

$$\int_0^\infty \left(\frac{\varphi(t)}{t^{n-pl}} \right)^{1/(p-1)} \frac{dt}{t} < \infty. \tag{10.4.16}$$

Then for any Borel set E in $\mathbb{R}^n$ with positive Hausdorff φ-measure we have

$$\operatorname{cap}\bigl(E, H_p^l\bigr) > 0.$$

(The last fact is a corollary of (10.4.6); cf. Maz'ya and Havin [567].)

Proposition 3. *Let E be a Borel set in $\mathbb{R}^n$.*

1. *If $n > pl$ and $H_{n-pl}(E) < \infty$, then* $\operatorname{cap}(E, S_p^l) = 0$, *where $S_p^l = h_p^l$ or H_p^l.*
2. *If $n = pl$ and $H(E, \varphi) < \infty$, where $\varphi(r) = |\log r|^{1-p}$, then* $\operatorname{cap}(E, H_p^l) = 0$ (cf. Meyers [597], Maz'ya and Havin [567]).

Next we present one more sufficient condition for the vanishing of $\operatorname{cap}(E, H_p^l)$ (Maz'ya and Havin [567]).

Proposition 4. *Let $\mathscr{N}$ be a measurable nonnegative function on $[0, \infty)$. Suppose, for any positive r, the set E can be covered by at most $\mathscr{N}(r)$ closed balls whose radii do not exceed r.*

If

$$\int_0 \bigl[\mathscr{N}(r)\bigr]^{1/(1-p)} r^{(n-pl)/(1-p)-1}\, dr = \infty,$$

then $\operatorname{cap}(E, H_p^l) = 0$.

Using Propositions 2 and 4, we can give a complete description of the n-dimensional Cantor sets E with positive $\operatorname{cap}(E, H_p^l)$.

Let $\mathscr{L} = \{l_j\}_{j=1}^\infty$ be a decreasing sequence of positive numbers such that $2l_{j+1} < l_j$ $(j = 1, 2, \dots,)$ and let $\varDelta_1$ be a closed interval with length l_1. Let e_1 denote a set contained in $\varDelta_1$, which equals the union of two closed intervals $\varDelta_2$ and $\varDelta_3$ with length l_2 and which contains both ends of the interval $\varDelta_1$. We put $E_1 = \underbrace{e_1 \times e_1 \times \dots \times e_1}_{n\text{-times}}$. Next we repeat the procedure with the intervals $\varDelta_2$ and $\varDelta_3$ (here the role of l_2 passes to l_3) and thus obtain four closed intervals with length l_3. Let their union be denoted by e_2; $E_2 = \underbrace{e_2 \times e_2 \times \dots \times e_2}_{n\text{-times}}$ and so on.

We put

$$E(\mathscr{L}) = \bigcap_{j=1}^\infty E_j.$$

Proposition 5. (Maz'ya and Havin [567]). *The following properties are equivalent:*

(i) $\operatorname{cap}(E(\mathscr{L}), H_p^l) > 0$;

(ii) $\sum_{j\geq 1} 2^{jn/(1-p)} l_j^{(n-pl)/(1-p)} < \infty \quad$ *for* $n > pl$;

$$\sum_{j\geq 1} 2^{jn/(1-p)} \log \frac{l_j}{l_{j+1}} < \infty \quad \textit{for } n = pl.$$

Sharp results on metric properties of capacities generated by the Besov spaces $B^l_{p,q}$, $0 < p < \infty$, $0 < q \leq \infty$, $0 < lp < n$, were obtained by Netrusov in [633].

10.4.4 Refined Functions

The function φ in H_p^l is called *refined or* (p,l)*-refined* if there exists a sequence of functions $\{\varphi_m\}_{m\geq 1}$ in $\mathscr{D}$ that converges to φ in H_p^l and such that for each $\varepsilon > 0$ there exists an open set ω with $\operatorname{cap}(\omega, H_p^l) < \varepsilon$ and $\varphi_m \to \varphi$ uniformly on $\mathbb{R}^n\backslash\omega$.

Another (equivalent) definition is as follows: The function $\varphi \in H_p^l$ is called refined if for each $\varepsilon > 0$ there exists an open set ω such that $\operatorname{cap}(\omega, H_p^l) < \varepsilon$ and the restriction of φ to $\mathbb{R}^n\backslash\omega$ is continuous.

We list the basic properties of refined functions.

(i) *If* $\varphi \in H_p^l$ *then there exists a refined function* $\tilde{\varphi}$ *that coincides with* φ *almost everywhere* (*with respect to* n*-dimensional Lebesgue measure*) *in* $\mathbb{R}^n$.

(ii) *If* φ_1 *and* φ_2 *are refined functions that coincide almost everywhere* (*with respect to* n*-dimensional Lebesgue measure*), *then* φ_1 *and* φ_2 *coincide quasi-everywhere.*

(iii) *Each sequence of refined functions in* H_p^l *that converges to a refined function* φ *in* H_p^l *contains a subsequence that converges to* φ *quasi-everywhere.*

For the proofs of these assertions see the paper by Havin and the author [567], where references to the earlier literature are given.

For $pl > n$ these properties become trivial since $H_p^l \subset C$.

The following result due to Bagby and Ziemer [62] shows that a function in H_p^l coincides with a function in C^m $(m \leq l)$ outside some set that is small with respect to the corresponding capacity.

Proposition. *Let* $u \in H_p^l$, $1 < p < \infty$ *and let* m *be an integer* $0 \leq m \leq l$. *Then for each* $\varepsilon > 0$ *there exists a function* $u_\varepsilon \in C^m$ *and an open set* ω *such that* $\operatorname{cap}(\omega, H_p^{l-m}) < \varepsilon$ *and* $u(x) = u_\varepsilon(x)$ *for all* $x \in \mathbb{R}^n\backslash\omega$.

Swanson proved that every function in the Bessel potential space H_p^l may be approximated in capacity and norm by smooth functions in $C^{m,\lambda}$, $0 < m + \lambda < l$ [734].

In conclusion we add to the previously mentioned literature on nonlinear potentials the lectures by D.R. Adams [8], which also contain a survey of some other problems that we do not touch upon here, and also D.R. Adams' very interesting reminiscences [12].

10.5 Comments to Chap. 10

Section 10.1. Besides the literature cited in Sect. 10.1, in which embedding, trace, and extension theorems are established for various fractional-order and more general spaces, we also cite (with no intention of claiming completeness) the papers by Solonnikov [716]; Besov [87]; Golovkin and Solonnikov [323]; Aronszajn, Mulla, and Szeptycki [53]; Taibleson [737, 738]; Golovkin [321, 322]; Volevič and Panejah [782]; Il'in [396, 397]; Burenkov [152, 153]; Kudryavtsev [467]; the first chapter of the book by Hörmander [384]; and the book by Gel'man and the author [305].

Here we mention some works dedicated to trace and extension theorems for Besov and Sobolev spaces. The following result concerning the case $0 < p < \infty$ can be found in Jawerth [400] and Sect. 2.7 in Triebel's book [757].

Theorem. *The trace operator $B^s_{p,q}(\mathbb{R}^n) \to B^{s-1/p}_{p,q}(\mathbb{R}^{n-1})$ is continuous if $s > 1/p$, $p \geq 1$ and for $s > 1 - n + n/p$, $p < 1$. Moreover, this operator has a right inverse that is bounded from $B^{s-1/p}_{p,q}(\mathbb{R}^{n-1})$ to $B^s_{p,q}(\mathbb{R}^n)$ for every $s \in \mathbb{R}$.*

The borderline case $s = 1 - n + n/p$, $0 < p < 1$, was studied by Johnsen [403].

A. Jonsson [405] (see also A. Jonsson and H. Wallin [408]) obtained generalizations of Theorems 10.1.1/1 and 10.1.1/2 for traces on the so-called d-sets in $\mathbb{R}^n$, i.e., the sets that support a measure μ subject to $\mu(B(x,r)) \sim r^d$.

Explicit formulas for linear continuous extension operators $B^s_{p,q}(\Omega) \to B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\Omega) \to F^s_{p,q}(\mathbb{R}^n)$, where B and F stand for Besov and Triebel–Lizorkin spaces, $0 < p < \infty$, $0 < q \leq \infty$, and Ω is a Lipschitz domain, are given by Rychkov [686].

A description of the traces of the functions in $W^1_p(\Omega)$ to the boundary of a cusp domain was given by Poborchi and the author ([576], Chap. 7). Let Ω be a domain in $\mathbb{R}^n$ and $p \in [1, \infty)$. By $TW^1_p(\Omega)$ we mean the space of the traces $u|_{\partial\Omega}$ of the functions $u \in W^1_p(\Omega)$. The norm in this space is defined by

$$\|f\|_{TW^1_p(\Omega)} = \inf\bigl\{\|u\|_{W^1_p(\Omega)} : u \in W^1_p(\Omega),\ u|_{\partial\Omega} = f\bigr\}.$$

According to Gagliardo's theorem, $TW^1_p(\Omega) = W^{1-1/p}_p(\partial\Omega)$ for $p \in (1, \infty)$ and $TW^1_1(\Omega) = L_1(\partial\Omega)$ if Ω is a bounded Lipschitz domain. When Ω has cusps on the boundary, Gagliardo's theorem generally fails. Consider a typical domain with an outward cusp

$$\Omega = \bigl\{x = (y,z) \in \mathbb{R}^n : z \in (0,1),\ |y| < \varphi(z)\bigr\}, \quad n > 2,$$

where φ is an increasing Lipschitz continuous function on $[0,1]$ such that $\varphi(0) = \lim_{z\to 0} \varphi'(z) = 0$.

It turns out that the boundary values of functions in $W_p^1(\Omega)$ and in $W_p^1(\mathbb{R}^n \setminus \overline{\Omega})$ can be characterized in terms of the finiteness of the norm

$$\langle f\rangle_{p,\partial\Omega} = \left(\int_{\partial\Omega} \left|f(x)\right|^p q(x)\,\mathrm{d}s_x \right. \\ \left. + \iint_{\partial\Omega\times\partial\Omega} \left|f(x) - f(\xi)\right|^p Q(x,\xi)\,\mathrm{d}s_x\,\mathrm{d}s_\xi\right)^{1/p},$$

where q and Q are nonnegative weight functions and ds_x, ds_ξ the area elements on $\partial\Omega$.

Let $p \in (1,\infty)$ and let f be a function on $\partial\Omega$ vanishing outside a small neighborhood of the origin. Then f belongs to $TW_p^1(\Omega)$ if and only if $\langle f\rangle_{p,\partial\Omega} < \infty$, where $x = (y,z)$, $\xi = (\eta,\zeta)$, $0 \le q(x) \le \mathrm{const}\,\varphi'(z)$,

$$Q(x,\xi) = \begin{cases} |x-\xi|^{2-n-p} & \text{if } |z-\zeta| < \varphi(z) + \varphi(\zeta),\ z,\zeta \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

In addition, the norm $\langle f\rangle_{p,\partial\Omega}$ is equivalent to $\|f\|_{TW_p^1(\Omega)}$.

A necessary and sufficient condition for f to belong to $TW_p^1(\mathbb{R}^n \setminus \overline{\Omega})$ is that $\langle f\rangle_{p,\partial\Omega} < \infty$ with

$$q(x) = \begin{cases} \varphi(z)^{1-p} & \text{for } 1 < p < n-1, \\ (\varphi(z)|\log(\varphi(z)/z)|)^{1-p} & \text{for } p = n-1, \\ \varphi(z)^{2-n} & \text{for } p > n-1, \end{cases}$$

and $Q(x,\xi) \ne 0$ only if $z,\zeta \in (0,1)$. For these pairs $x,\xi \in \partial\Omega$, Q is defined as follows. If $p < n-1$, then

$$Q(x,\xi) = |x-\xi|^{2-n-p}.$$

In the cases $p \ge n-1$ and $|x-\xi| < \varphi(z) + \varphi(\zeta)$ the weight Q is determined by the same formula. Finally, if $|x-\xi| \ge \varphi(z) + \varphi(\zeta)$, then

$$Q(x,\xi) = \frac{(\varphi(z)+\varphi(\zeta))^{2(1-p)}}{|x-\xi|}\left(\log\left(1 + \frac{|x-\xi|}{\varphi(z)+\varphi(\zeta)}\right)\right)^{-p}, \quad p = n-1,$$

and

$$Q(x,\xi) = |x-\xi|^{n-p-2}\big(\varphi(z)\varphi(\zeta)\big)^{2-n}, \quad p > n-1.$$

The norm $\langle f\rangle_{p,\partial\Omega}$ with these weights q, Q is equivalent to $\|f\|_{TW_p^1(\mathbb{R}^n\setminus\overline{\Omega})}$. If $p = n-1$, some additional restrictions are imposed on φ (not excluding power cusps). A function f defined on $\partial\Omega$ and vanishing outside a small neighborhood of the origin is in $TW_1^1(\Omega)$ if and only if $\langle f\rangle_{1,\partial\Omega} < \infty$ with

$$0 \le q(x) \le \text{const}\, \varphi'(z),$$

$$Q(x,\xi) = \begin{cases} (\varphi(z)+\varphi(\zeta))^{1-n} & \text{for } z,\zeta \in (0,1),\ |z-\zeta| < \varphi(z)+\varphi(\zeta), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the norms $\|f\|_{TW_1^1(\mathbb{R}^n\setminus\overline{\Omega})}$ and $\langle f\rangle_{1,\partial\Omega}$ are equivalent.

The characterization of the space $TW_1^1(\mathbb{R}^n\setminus\overline{\Omega})$ is the same as for Lipschitz domains

$$TW_1^1\big(\mathbb{R}^n \setminus \overline{\Omega}\big) = L_1(\partial\Omega).$$

In all the cases mentioned previously there exists a bounded extension operator

$$TW_p^1(\Omega) \to W_p^1(\Omega) \quad \text{and} \quad TW_p^1\big(\mathbb{R}^n \setminus \overline{\Omega}\big) \to W_p^1\big(\mathbb{R}^n \setminus \overline{\Omega}\big).$$

This operator is linear for $p > 1$ and nonlinear for $p = 1$. One can easily obtain from the previous results that the space $TW_p^1(\mathbb{R}^n \setminus \overline{\Omega})$ is continuously embedded into $TW_p^1(\Omega)$ for $p \in [1,\infty)$, $n > 2$. Hence the space of the traces on $\partial\Omega$ of the functions in $W_p^1(\mathbb{R}^n)$ coincides with $TW_p^1(\mathbb{R}^n \setminus \overline{\Omega})$.

Shvartsman [699] described the restrictions of the space $W_p^1(\mathbb{R}^n)$, $p > n$, to an *arbitrary closed subset* $S \subset \mathbb{R}^n$ via certain doubling measures supported on S. Observe that Dyn'kin [245, 246] conjectured that *every compact set* $S \subset \mathbb{R}^n$ *carries a nontrivial doubling measure* μ. He constructed a doubling measure on every compact set $S \subset \mathbb{R}$ that satisfies a certain "porosity" condition. Dyn'kin's conjecture was subsequently proved by Volberg and Konyagin [780]. Moreover, Volberg and Konyagin showed that every compact set in $\mathbb{R}^n$ carries a nontrivial measure μ satisfying the following condition: There exists a constant $C = C_\mu > 0$ such that, for all $x \in S$, $0 < r \le 1$ and $1 \le k \le 1/r$, we have $\mu(B(x,kr)) \le C_\mu k^n \mu(B(x,r))$. Using their argument, Luukkainen and Saksman [511] extended this result to all closed subsets of $\mathbb{R}^n$. Jonsson [406] showed that it may also be assumed that $\mu(B(x,1)) \sim 1$ for all $x \in S$.

Jonsson and Wallin [405, 408] obtained generalizations of Theorems 1 and 2 for the family of d-sets in $\mathbb{R}^n$, $0 < n-d < p$, $1 < p < \infty$.

Let

$$\rho_S(x,y) := \inf\Big\{\operatorname{diam} B : B \text{ is a ball},\ B \ni x,y,\ \frac{1}{15}B \subset \mathbb{R}^n\setminus S\Big\}, \quad x,y \in S.$$

Shvartsman [699] proved that for every function $f \in C(S)$ its trace norm in $W_p^1(\mathbb{R}^n)|_S$, $p > n$, i.e., the quantity

$$\|f\|_{W_p^1(\mathbb{R}^n)|_S} := \inf\big\{\|F\|_{W_p^1(\mathbb{R}^n)} : F \in W_p^1(\mathbb{R}^n) \text{ and continuous},\ F|_S = f\big\}$$

can be calculated as follows:

$$\|f\|_{W_p^1(\mathbb{R}^n)|_S} \sim \|f\|_{L_p(\mu)} + \sup_{0<t\le 1}\left(\iint_{|x-y|<t} \frac{|f(x)-f(y)|^p}{t^{p-n}\mu(B(x,t))^2}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right)^{\frac{1}{p}}$$
$$+\left(\iint_{\rho_S(x,y)<1} \frac{|f(x)-f(y)|^p}{\rho_S(x,y)^{p-n}\mu(B(x,\rho_S(x,y)))^2}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\right)^{\frac{1}{p}}.$$

Sections 10.2 and 10.3. For the historical background we refer to the introductions to these sections. The material is borrowed from the papers by Maz'ya and Shaposhnikova [584, 585]. Results related to (10.2.3) and (10.3.1) were subsequently obtained by Kolyada and Lerner [445], M. Milman [606], and Karadzhov, M. Milman, and Xiao [414].

Section 10.4. The references are given in the text. Here we mention additionally the paper by Maz'ya and Havin [569], where several applications of the (p, l)-capacity and nonlinear potentials to the theory of exceptional sets are considered (the uniqueness and approximation of analytic functions, convergence of Fourier series, and removal of singularities of analytic and polyharmonic functions). Properties of capacities in Birnbaum–Orlicz spaces were studied by Aissaoui and Benkirane [37].

11

Capacitary and Trace Inequalities for Functions in $\mathbb{R}^n$ with Derivatives of an Arbitrary Order

11.1 Description of Results

According to Corollary 2.3.4, for $q \ge p > 1$, the inequality

$$\|u\|_{L_q(\mathbb{R}^n,\mu)} \le A\|\nabla u\|_{L_p(\mathbb{R}^n)}, \quad u \in C_0^\infty, \tag{11.1.1}$$

follows from the isocapacitary inequality

$$\big(\mu(E)\big)^{p/q} \le p^{-p}(p-1)^{p-1}A^p \operatorname{cap}\big(E, w_p^1\big).$$

Here and henceforth E is an arbitrary Borel set in $\mathbb{R}^n$ and w_p^1 is the completion of C_0^∞ with respect to the norm $\|\nabla u\|_{L_p}$.

On the other hand, if (11.1.1) is valid for any $u \in C_0^\infty$, then

$$\big(\mu(E)\big)^{p/q} \le A^p \operatorname{cap}\big(E, w_p^1\big)$$

for all $E \subset \mathbb{R}^n$.

The present chapter contains similar results in which the role of w_p^1 is played by the spaces H_p^l, h_p^l, W_p^l, w_p^l, B_p^l, and b_p^l.

Namely, let S_p^l be any one of these spaces. Then the best constant in

$$\|u\|_{L_q(\mu)} \le A\|u\|_{S_p^l}, \quad u \in C_0^\infty, \tag{11.1.2}$$

where $q \ge p$, is equivalent to the best constant in the "isoperimetric" inequality

$$\big(\mu(E)\big)^{p/q} \le B \operatorname{cap}\big(E, S_p^l\big). \tag{11.1.3}$$

The estimate $A \ge B$ immediately follows from the definition of capacity. The reverse estimate is a deeper fact, its proof being based on the capacitary inequality

$$\int_0^\infty \operatorname{cap}\big(\mathcal{N}_t, S_p^l\big)t^{p-1}\,\mathrm{d}t \le C\|u\|_{S_p^l}^p, \tag{11.1.4}$$

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where $u \in S_p^l$, $\mathcal{N}_t = \{x : |u(x)| \geq t\}$ and C is a constant independent of u. We dealt with inequalities of a similar nature in the previous chapter, but only in the case $l \leq 2$. In Sect. 11.2 we present two proofs of inequality (11.1.4) with $l \in (0, \infty)$ and different fields of application and in Sect. 1.3 we discuss criteria for the validity of embedding theorems formulated in terms of isocapacitary inequalities. Section 1.4 is dedicated to a counterexample showing that the capacitary inequality for the norm in $L_2^2(\Omega)$ may fail for some domain.

We might ask if it is possible to replace arbitrary sets E in (11.1.3) by balls. From the D.R. Adams Theorem 1.4.1 it follows that this is so for the Riesz potential space $S_p^l = h_p^l$, $pl < n$. The condition given by D.R. Adams is

$$\mu\big(B(x,\varrho)\big) \leq C\varrho^s, \tag{11.1.5}$$

where $s = q(n/p - l)$ and $B(x,\varrho)$ is any ball with center x and radius ϱ.

Thus, inequality (11.1.5) with $q > p$ implies the isoperimetric inequality (11.1.3) for any set E.

In Sect. 11.7 we give a direct proof of more general assertions of this kind. Namely, for any ball $B(x,r)$, let

$$\mu\big(B(x,r)\big) \leq \varPhi\big(\operatorname{cap}\big(B_r, h_p^l\big)\big), \tag{11.1.6}$$

where $B_r = B(0,r)$, $\varPhi$ is an increasing function subject to some additional requirements and μ is a measure in $\mathbb{R}^n$. Then for all Borel sets $E \subset \mathbb{R}^n$

$$\mu(E) \leq c\varPhi\big(c \operatorname{cap}\big(E, h_p^l\big)\big). \tag{11.1.7}$$

By this theorem along with the equivalence of (11.1.2) and (11.1.3), we show in Sect. 11.8 that inequalities similar to (11.1.6) are necessary and sufficient for the validity of estimates for traces of Riesz and Bessel potentials in Birnbaum–Orlicz spaces $L_M(\mu)$ and, in particular, in $L_q(\mu)$. Besides, this gives a new proof of the aforementioned D.R. Adams theorem where no interpolation is used. Another corollary, of interest in its own right, claims that the inequality

$$\|u\|_{L_q(\mu)} \leq c\|u\|_{H_p^l},$$

where $q > p > 1$, $lp = n$ is fulfilled if and only if

$$\mu\big(B(x,r)\big) \leq c|\log r|^{-q/p'}$$

for all balls $B(x,r)$ with radii $r \in (0, \frac{1}{2})$.

Next we state some other results relating the conditions for (11.1.2).

(a) If $S_p^l = H_p^l$, $pl < n$, $q > p$, then (11.1.2) is valid simultaneously with (11.1.5), where $0 < \varrho < 1$ (see Sect. 11.8).

(b) In the case $pl > n$ a necessary and sufficient condition for (11.1.2) with $S_p^l = H_p^l$ is

$$\sup\big\{\mu\big(B(x,1)\big) : x \in \mathbb{R}^n\big\} < \infty$$

(see Sect. 11.8).

(c) For $q = p$, condition (11.1.5) is not sufficient for (11.1.2) (cf. Remark 11.8/2). So in this case, which is probably the most important for applications, we have to deal with a less explicit isocapacitary condition (11.1.3). However, there are ball and pointwise criteria for (11.1.2) with $q = p$ due to Kerman and Sawyer and Maz'ya and Verbitsky. They are treated in Sect. 1.5.

(*d*) In Sect. 11.6 we give the following necessary and sufficient condition for the validity of (11.1.2) provided that $p > 1$ and $p > q > 0$:

$$\int_0^\infty \left(\frac{t}{\varkappa(t)}\right)^{q/(p-q)} \mathrm{d}t < \infty,$$

where $\varkappa(t) = \inf\{\mathrm{cap}(F, S^l_p) : \mu(F) \geq t\}$ (cf. Theorem 11.6.1/1). A noncapacitary criterion for (11.1.2), where $p > q$, obtained by Cascante, Ortega, and Verbitsky, is formulated in the same Sect. 1.6.

(e) In the case $pl > n$, $p > q$ a necessary and sufficient condition for (11.1.2) can be written in the essentially simpler form

$$\sum_i \big(\mu\big(\mathcal{Q}^{(i)}\big)\big)^{p/(p-q)} < \infty,$$

where $\{\mathcal{Q}^{(i)}\}$ is the sequence of closed cubes with edge length 1, which forms the coordinate grid in $\mathbb{R}^n$ (cf. Sect. 11.6.2).

(f) We note also that in the case $q = 1$, $p > 1$, the inequality (11.1.2) with $S^l_p = h^l_p$ or $S^l_p = H^l_p$ is equivalent to the inclusion $I_l\mu \in L_{p'}$ or $J_l\mu \in L_{p'}$, respectively (here I_l and J_l are the Riesz and the Bessel potentials) (cf. Sect. 11.6.2).

(g) In Sect. 11.10 we consider the case $p = 1$. For $S^l_1 = b^l_1$ in addition to Theorem 1.4.3 it is shown that (11.1.2) holds simultaneously with (11.1.5), where $p = 1$, $q \geq 1$. If $S^l_1 = B^l_1$, we have to add the condition $\varrho \in (0, 1)$ in (11.1.5). We recall that according to Theorem 1.4.3 the same pertains to the cases $s^l_1 = w^l_1$, $S^l_1 = W^l_1$.

(h) Using the interpretation of b^l_p and B^l_p, $p > 1$ as the trace spaces of the corresponding potential spaces, we obtain theorems on b^l_p and B^l_p from the theorems concerning h^l_p and H^l_p (cf. Remark 11.8/3).

Section 11.9 contains some applications of the results obtained in Sects. 11.2–11.7. In particular, we present necessary and sufficient conditions for the compactness of the embedding operator of H^l_p into $L_q(\mu)$. In Sect. 11.9 we also state some corollaries to previous theorems. They concern the negative spectrum of the operator $(-\Delta)^l - p(x)$, $p(x) \geq 0$, $x \in \mathbb{R}^n$.

Integral inequalities with certain seminorms of a function, involving arbitrary measures, on the left-hand side and the L_1 norm of the gradient on the right-hand side are characterized in Sect. 11.10. The concluding Sect. 11.11 is devoted to the study of the inequality

$$\left(\int_\Omega \int_\Omega \big|u(x) - u(y)\big|^q \mu(\mathrm{d}x, \mathrm{d}y)\right)^{1/q} \leq C\|\nabla u\|_{L_p(\Omega)},$$

with $q \geq p > 1$.

11.2 Capacitary Inequality of an Arbitrary Order

11.2.1 A Proof Based on the Smooth Truncation of a Potential

We use the Hardy–Littlewood maximal operator $\mathcal{M}$ defined by

$$(\mathcal{M}g)(x) = \sup_{r>0} \frac{1}{m_n B_r} \int_{B(x,r)} \big|g(\xi)\big| \,\mathrm{d}\xi. \tag{11.2.1}$$

Lemma 1. (Hedberg [365]) *Let f be a nonnegative function and let I_α be the Riesz potential of order α. For all α and β such that $0 < \alpha < \beta < n$ and almost all $x \in \mathbb{R}^n$ there holds the inequality*

$$(I_\alpha f)(x) \le c\big((I_\beta f)(x)\big)^{\alpha/\beta}\big((\mathcal{M}f)(x)\big)^{1-\alpha/\beta}, \quad 0 < \alpha < \beta < n. \tag{11.2.2}$$

Proof. Let t be an arbitrary positive number to be chosen later. We make use of the equality

$$\int_{B_t(x)} \frac{f(y)\,\mathrm{d}y}{|x-y|^{n-\alpha}} = (n-\alpha)\int_0^t \int_{B_s(x)} f(y)\,\mathrm{d}y \frac{\mathrm{d}s}{s^{n-\alpha+1}} + t^{\alpha-n}\int_{B_t(x)} f(y)\,\mathrm{d}y, \tag{11.2.3}$$

which is readily checked by changing the order of integration on the right-hand side. Hence

$$\int_{B_t(x)} \frac{f(y)\,\mathrm{d}y}{|x-y|^{n-\alpha}} \le ct^\alpha(\mathcal{M}f)(x). \tag{11.2.4}$$

Clearly we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_t(x)} \frac{f(y)\,\mathrm{d}y}{|x-y|^{n-\alpha}} &\le t^{\alpha-\beta}\int_{\mathbb{R}^n\setminus B_t(x)} \frac{f(y)\,\mathrm{d}y}{|x-y|^{n-\beta}} \\ &\le t^{\alpha-\beta}(I_\beta f)(x).\end{aligned} \tag{11.2.5}$$

Adding this inequality and (11.2.3), we obtain

$$(I_\alpha f)(x) \le ct^\alpha(\mathcal{M}f)(x) + t^{\alpha-\beta}(I_\beta f)(x).$$

Minimization of the right-hand side in t completes the proof. □

Lemma 2. *Let l be an integer, $0 < l < n$, $I_l f = |x|^{l-n} * f$, where $f \ge 0$ and let F be a function in $C^l(0,\infty)$ such that*

$$t^{k-1}\big|F^{(k)}(t)\big| \le Q, \quad k = 0, \dots, l.$$

Then

$$\big|\nabla_l F(I_l f)\big| \le cQ\big(\mathcal{M}f + |\nabla_l I_l f|\big)$$

almost everywhere in $\mathbb{R}^n$.

Proof. We have

$$\begin{aligned}\big|\nabla_l F(u)\big| &\le c\sum_{k=1}^{l}\big|F^{(k)}(u)\big|\sum_{j_1+\dots+j_k=l}|\nabla_{j_1}u|\cdots|\nabla_{j_k}u|\\ &\le cQ\sum_{k=1}^{l}\sum_{j_1+\dots+j_k=l}\frac{|\nabla_{j_1}u|}{u^{1-j_1/l}}\cdots\frac{|\nabla_{j_k}u|}{u^{1-j_k/l}}.\end{aligned}$$

Since $|\nabla u|\le I_{l-s}f$, it follows that

$$\big|\nabla_l F(u)\big|\le cQ\Bigg(|\nabla_l I_l f|+\sum_{k=1}^{l}\sum_{j_1+\dots+j_k=l}{}' \frac{I_{l-j_1}f\cdots I_{l-j_k}f}{(I_l f)^{1-j_1/l}\cdots(I_l f)^{1-j_k/l}}\Bigg),$$

where the sum $\sum'$ is taken over all collections of numbers $j_1,\dots,j_k$ less than l. The result follows from (11.2.2). □

Let w be a nonnegative function in $\mathbb{R}^n$ satisfying the Muckenhoupt condition

$$\sup_{\mathcal{Q}}\bigg(\frac{1}{m_n\mathcal{Q}}\int_{\mathcal{Q}}w^p\,\mathrm{d}x\bigg)\bigg(\frac{1}{m_n\mathcal{Q}}\int_{\mathcal{Q}}w^{-p'}\,\mathrm{d}x\bigg)^{p-1}<\infty, \tag{11.2.6}$$

where the supremum is taken over all cubes $\mathcal{Q}$. This condition ensures the continuity of the operators $\mathcal{M}$ and $\nabla_l I_l$ in the space of functions φ with the finite norm $\|w\varphi\|_{L_p}$ (cf. Muckenhoupt [621], Coifman and Fefferman [210]).

Theorem. *Let $p>1$, $l=1,2,\dots,lp<n$. Inequality (11.1.4), where S_p^l is the completion of C_0^∞ with respect to the norm $\|w\nabla_l u\|_{L_p}$, holds.*

Proof. Let $u\in C_0^\infty(\mathbb{R}^n)$, $u=I_l f$, $v=I_l|f|$. We can easily see that $v\in C^l(\mathbb{R}^n)$ and $v(x)=O(|x|^{l-n})$ as $|x|\to\infty$. Since $v(x)\ge|u(x)|$, putting $t_j=2^{-j}$ $(j=0,\pm1,\dots,)$, we have

$$\int_0^\infty \operatorname{cap}\big(\mathcal{N}_t,S_p^l\big)\,\mathrm{d}\big(t^p\big)\le c\sum_{j=-\infty}^{\infty}2^{-pj}\gamma_j, \tag{11.2.7}$$

where $\gamma_j=\operatorname{cap}(\{x:v(x)\ge t_j\},S_p^l)$.

We shall use an operator of "smooth level truncation." Let $\alpha\in[0,1]$ be a nondecreasing function, vanishing near 0, and equal to one in a neighborhood of 1, and let us introduce the function $f\in C^\infty((0,\infty))$ given on $[t_{j+1},t_j]$ by

$$F(u)=t_{j+1}+\alpha\big((u-t_{j+1})/(t_j-t_{j+1})\big)(t_j-t_{j+1}).$$

Then we obtain by definition of the capacity that

$$2^{-p_j}\gamma_j\le c\big\|F(v)\big\|^p_{S_p^l},$$

which implies

$$\sum_{j=-\infty}^{+\infty} 2^{-p_j}\gamma_j \le c\big\|F(v)\big\|^p_{S^l_p}.$$

By Lemma 2, the preceding norm is majorized by

$$c\big(\big\|w\mathcal{M}|f|\big\|_{L_p} + \big\|w\nabla_l I_l|f|\big\|_{L_p}\big). \tag{11.2.8}$$

Since the weight function w satisfies (11.2.6), we can use the boundedness of the operators $\mathcal{M}$ and $\nabla_l I_l$ in the weighted L_p to show that the sum (11.2.8) does not exceed

$$c\|wf\|_{L_p} = c_1\big\|w(-\Delta)^l I_l u\big\|_{L_p} \le c\|w\nabla_l u\|_{L_p}.$$

The theorem is proved. □

Corollary. *Inequality* (11.1.4) *holds, where* $S^l_p = b^l_p$, $p > 1$, $l > 0$.

Proof. Let U be an arbitrary extension of $u \in C_0^\infty(\mathbb{R}^n)$ to the space $\mathbb{R}^{n+1} = \{\mathcal{X} = (x_n, x_{n+1}) : x \in \mathbb{R}^n, x_{n+1} \in \mathbb{R}^1\}$. According to Theorem 10.1.1/1,

$$\|u\|_{b^l_p(\mathbb{R}^n)} \sim \inf_{\{U\}} \|U\|_{\mathring{L}_p^{[l]+1}(\mathbb{R}^{n+1}, x_{n+1}^{1-[l]-1/p})},$$

where $\mathring{L}^k_p(\mathbb{R}^{n+1}, w)$ is the completion of $C_0^\infty(\mathbb{R}^{n+1})$ with respect to the norm $\|w\nabla_k u\|_{L_p(\mathbb{R}^{n+1})}$. Consequently,

$$\mathrm{cap}\big(e, b^l_p(\mathbb{R}^n)\big) \sim \mathrm{cap}\big(e, L_p^{[l]+1}\big(\mathbb{R}^{n+1}, x_{n+1}^{1-\{l\}-1/p}\big)\big).$$

We can easily check that the function $\mathcal{X} \to w(\mathcal{X}) = x_{n+1}^{1-[l]-1/p}$ satisfies (11.2.6). Therefore, the last theorem yields

$$\int_0^\infty \mathrm{cap}\big(\mathcal{N}_t, b^l_p(\mathbb{R}^n)\big)\,\mathrm{d}\big(t^p\big) \le c\|U\|^p_{L_p^{[l]+1}(\mathbb{R}^{n+1}, x_{n+1}^{1-[l]-1/p})}.$$

We complete the proof by minimizing the right-hand side over all extensions of u to $\mathbb{R}^{n+1}$. □

11.2.2 A Proof Based on the Maximum Principle for Nonlinear Potentials

Let $K\mu$ be the linear Bessel or Riesz potential of order l with density μ and let $K(K\mu)^{p'-1}$ be the nonlinear potential generated by K. Further, let $\mathfrak{M}$ denote the constant in the rough maximum principle for the potential $K(K\mu)^{p'-1}$ (cf. Proposition 10.4.2/1).

Theorem. *Inequality*

$$\int_0^\infty \mathrm{cap}\big(\mathcal{N}_t, S^l_p\big)t^{p-1}\,\mathrm{d}t \le C\|u\|^p_{S^l_p} \tag{11.2.9}$$

holds, where S_p^l *is either* H_p^l *or* h_p^l $(pl < n)$*. The best constant* C *in* (11.2.9) *satisfies*

$$\begin{aligned} &C \le (p')^{p-1}\mathfrak{M} \quad \textit{if } p \ge 2,\\ &C \le (p')^p p^{-1}\mathfrak{M}^{p-1} \quad \textit{if } p < 2. \end{aligned} \tag{11.2.10}$$

(Note that for $p = 2$ both estimates give $C \le 2\mathfrak{M}$.)

Proof. For the sake of brevity, let $c(t) = \operatorname{cap}(\mathcal{N}_t, S_p^l)$. It suffices to assume $u = Kf$, $f \ge 0$, $f \in L_p$. Let μ_t denote the capacitary measure of $\mathcal{N}_t$ (cf. Proposition 10.4.2/2).

We choose a finite number N of arbitrary values $t_1, \dots, t_N$ satisfying $0 < t_1 <, \dots, < t_N$ and introduce the step function σ_N on $(0, \infty)$ by

$$\sigma_N = \begin{cases} 0 & \text{for } t \in (0, t_1),\\ t_j & \text{for } t_j \le t < t_{j+1}, j = 1, \dots, N-1,\\ t_N & \text{for } t \ge t_N. \end{cases}$$

Now the sum

$$\sum_{1 \le j \le N-1} f(t_j)(t_{j+1} - t_j),$$

where f is a function defined on $(0, \infty)$, can be written as the Stieltjes integral

$$\int_0^\infty f(t)\, \mathrm{d}\sigma_N(t).$$

To obtain (11.2.9), it suffices to prove the inequality

$$\int_0^\infty c(t) t^{p-1}\, \mathrm{d}\sigma_N(t) \le C\|u\|^p_{S_p^l}, \tag{11.2.11}$$

with a constant C subject to (11.2.10). The left-hand side in the last inequality does not exceed

$$\int_0^\infty \int Kf\, \mathrm{d}\mu_t\, t^{p-2}\, \mathrm{d}\sigma_N(t) = \int f\, \mathrm{d}x \int_0^\infty K\mu_t t^{p-2}\, \mathrm{d}\sigma_N(t),$$

which is majorized by

$$\|f\|_{L_p} \left\| \int_0^\infty t^{p-2} K\mu_t\, \mathrm{d}\sigma_N(t) \right\|_{L_{p'}}.$$

Thus, to get the result it suffices to obtain the estimate

$$\int \left(\int_0^\infty t^{p-2} K\mu_t\, \mathrm{d}\sigma_N(t) \right)^{p'} \mathrm{d}x \le C^{p'-1} \int_0^\infty c(t) t^{p-1}\, \mathrm{d}\sigma_N(t). \tag{11.2.12}$$

First, we note that by the maximum principle

$$\int (K\mu_\tau)^{p'-1} K\mu_t \, \mathrm{d}x \le \mathfrak{M} c(t) \tag{11.2.13}$$

for all $\tau \in (0, \infty)$. Next, we consider separately the cases $p \ge 2$ and $p < 2$. Let $p \ge 2$. The left-hand side in (11.2.12) can be written as

$$p' \iint_0^\infty K\mu_\tau \Big(\int_\tau^\infty K\mu_t t^{p-2} \, \mathrm{d}\sigma_N(t) \Big)^{p'-1} \tau^{p-2} \, \mathrm{d}\sigma_N(\tau) \, \mathrm{d}x.$$

By virtue of the Hölder inequality this expression is majorized by

$$p' \Big(\iint_0^\infty \tau^{p-1} (K\mu_\tau)^{p'} \, \mathrm{d}\sigma_N(\tau) \, \mathrm{d}x \Big)^{2-p'}$$
$$\times \Big(\iint_0^\infty (K\mu_\tau)^{p'-1} \int_\tau^\infty K\mu_t t^{p-2} \, \mathrm{d}\sigma_N(t) \, \mathrm{d}\sigma_N(\tau) \, \mathrm{d}x \Big)^{p'-1},$$

which by (11.2.13) does not exceed

$$p' \mathfrak{M}^{p'-1} \Big(\int_0^\infty \|K\mu_t\|_{L_{p'}}^{p'} t^{p-1} \, \mathrm{d}\sigma_N(t) \Big)^{2-p'} \Big(\int_0^\infty c(t) t^{p-1} \, \mathrm{d}\sigma_N(t) \Big)^{p'-1}.$$

Thus (11.2.11) follows for $p \ge 2$.

Let $p < 2$. The left-hand side in (11.2.12) is equal to

$$p' \iint_0^\infty K\mu_t t^{p-2} \, \mathrm{d}\sigma_N(t) \Big(\int_0^t K\mu_\tau \tau^{p-2} \, \mathrm{d}\sigma_N(\tau) \Big)^{p'-1} \mathrm{d}x.$$

Hence, by Minkowski's inequality, it is majorized by

$$p' \int_0^\infty \Big(\int_0^t \Big(\int (K\mu_\tau)^{p'-1} K\mu_t \, \mathrm{d}x \Big)^{p-1} \tau^{p-2} \, \mathrm{d}\sigma_N(\tau) \Big)^{p'-1} t^{p-2} \, \mathrm{d}\sigma_N(t).$$

Estimating this value by (11.2.13), we obtain that it is majorized by

$$p' \mathfrak{M} \int_0^\infty c(t) \Big(\int_0^t \tau^{p-2} \, \mathrm{d}\tau \Big)^{p'-1} t^{p-2} \, \mathrm{d}\sigma_N(t),$$

and (11.2.11) follows for $p < 2$. □

11.3 Conditions for the Validity of Embedding Theorems in Terms of Isocapacitary Inequalities

We state the generalization of Theorem 2.3.3 to the case of Bessel and Riesz potential spaces in $\mathbb{R}^n$. We omit the proof since it duplicates that of Theorem 2.3.3.

Theorem. *The best constant in the inequality*

$$\big\||u|^p\big\|_{L_M(\mu)} \le A\|u\|^p_{S^l_p}, \tag{11.3.1}$$

where $S^l_p = h^l_p$ for $pl < n$ or $S^l_p = H^l_p$ for $pl \le n$, $p \in (1,\infty)$, is equivalent to

$$B = \sup\bigg\{\frac{\mu(E)N^{-1}(1/\mu(E))}{\mathrm{cap}(E, S^l_p)} : E \subset \mathbb{R}^n, \mathrm{cap}\big(E, S^l_p\big) > 0\bigg\}.$$

Namely, $B \le A \le pBC$, where C is the constant in (11.1.4) (cf. Theorem 11.2.2)*.*

This assertion immediately implies the following corollary.

Corollary. *The best constant $C_{p,q}$ in*

$$\|u\|_{L_q(\mu)} \le C_{p,q}\|u\|_{S^l_p}, \tag{11.3.2}$$

where $q \ge p > 1$ and S^l_p is one of the spaces in the preceding theorem, satisfies

$$B_{p,q} \le C_{p,q} \le B_{p,q}(pC)^{1/p}.$$

Here

$$B_{p,q} = \sup\bigg\{\frac{\mu(E)^{p/q}}{\mathrm{cap}(E, S^l_p)} : E \subset \mathbb{R}^n, \mathrm{cap}\big(E, S^l_p\big) > 0\bigg\}$$

and C is the constant in (11.1.4)*.*

A theorem due to D.R. Adams [5] states that inequality (11.3.2) with $q = p > 1$, $lp < n$, and $S^l_p = h^l_p$ holds if and only if, for all compact sets $e \subset \mathbb{R}^n$,

$$\|I_l\mu_e\|^{p'}_{L_{p'}} \le \mathrm{const}\ \mu(e), \tag{11.3.3}$$

where μ_e is the restriction of the measure μ to e.

This result follows from the preceding corollary and the next proposition.

Proposition 1. *Let $p \in (1,\infty)$, $lp < n$. Then we have the relation $Q \sim R$ where*

$$Q = \sup_e \frac{\mu(e)}{\mathrm{cap}(e, h^l_p)}, \qquad R = \sup_e \frac{\|I_l\mu_e\|^p_{L_{p'}}}{[\mu(e)]^{p-1}},$$

and the suprema are taken over all compacta e in $\mathbb{R}^n$.

Proof. For any $u \in C^\infty_0$, $u \ge 1$ on e, we obtain

$$\mu(e) \le \int u(x)\,\mathrm{d}\mu_e(x) \le \big\|(-\Delta)^{-l/2}\mu_e\big\|_{L_{p'}}\big\|(-\Delta)^{l/2}u\big\|_{L_p},$$

which can be rewritten as

$$\mu(e) \le c\|I_l \mu_e\|_{L_{p'}} \|u\|_{h_p^l}.$$

Taking the minimum of the right-hand side over all functions u, we obtain

$$\mu(e) \le cR^{1/p}\mu(e)^{1/p'}\left[\operatorname{cap}\big(e, h_p^l\big)\right]^{1/p}.$$

On the other hand, by the Corollary,

$$\int |u|^p \, \mathrm{d}\mu \le cQ\|u\|^p_{h_p^l}.$$

Therefore,

$$\left|\int u \, \mathrm{d}\mu_e\right|^p \le cQ\mu(e)^{p-1}\big\|(-\Delta)^{l/2}u\big\|^p_{L_{p'}},$$

which yields

$$\|I_l \mu_e\|_{L_{p'}} \le cQ^{1/p}\mu(e)^{1/p'}.$$

Thus $h \le cQ$. The proof is complete. □

In the same way, we can obtain the relation

$$\sup_e \frac{\mu(e)}{\operatorname{cap}(e, H_p^l)} \sim \sup_e \frac{\|J_l \mu_e\|^p_{L_{p'}}}{[\mu(e)]^{p-1}},$$

where e is either an arbitrary compactum in $\mathbb{R}^n$ or a compactum with a diameter not exceeding unity.

An important refinement of Proposition 1 due to Kerman and Sawyer [420] will be considered in Sect. 11.5.

To conclude the present section, we note that with the same arguments as in the proof of Proposition 1 together with Theorem 11.2.1 we arrive at the following proposition.

Proposition 2. *Let $p > 1$, $l = 1, 2, \ldots$, $lp < n$ and let w be a nonnegative function that satisfies the Muckenhoupt condition* (11.2.6)*. Then the best constant C_p in*

$$\|u\|_{L_p(\mu)} \le C_p\|w\nabla_l u\|_{L_p}, \quad u \in C_0^\infty,$$

satisfies the relation

$$C_p \sim \sup_e \frac{\|\frac{1}{w} I_l \mu_e\|^p_{L_{p'}}}{[\mu(e)]^{p-1}},$$

where e is an arbitrary compactum in $\mathbb{R}^n$.

11.4 Counterexample to the Capacitary Inequality for the Norm in $L_2^2(\Omega)$

Let Ω be a domain in $\mathbb{R}^n$ and let ω denote a subdomain of Ω with compact closure in Ω. In this section we use the norm

$$\|u\|_{L^1_p(\Omega)} = \|u\|_{L_p(\omega)} + \|\nabla_l u\|_{L_p(\Omega)}.$$

Let F be a relatively closed subset of Ω. By the capacity of F generated by the space $L^l_p(\Omega)$ we mean the set function

$$\operatorname{cap}\bigl(F; L^l_p(\Omega)\bigr) = \inf\bigl\{\|u\|^p_{L^l_p(\Omega)} : u \in C^\infty(\Omega), u|_F \geq 1\bigr\}.$$

Let μ be a Borel measure on Ω, $p \in (1,\infty)$, and suppose that the trace inequality

$$\|u\|_{L_p(\Omega,\mu)} \leq C\|u\|_{L^l_p(\Omega)}, \quad C = \text{const},$$

holds for all $u \in C^\infty(\Omega)$. Then the isocapacitary inequality

$$\mu(F) \leq D \operatorname{cap}\bigl(F; L^l_p(\Omega)\bigr), \quad D = \text{const}, \tag{11.4.1}$$

easily follows from the definition of $\operatorname{cap}(F; L^l_p(\Omega))$ for any relatively closed $F \subset \Omega$ with $D = C^p$. One can readily show that (11.4.1) is also sufficient for the trace inequality to be valid for all $u \in C^\infty(\Omega)$ if the capacitary inequality

$$\int_0^\infty \operatorname{cap}\bigl(\bigl\{x \in \Omega : \bigl|u(x)\bigr| \geq t\bigr\}; L^l_p(\Omega)\bigr)\, \mathrm{d}\bigl(t^p\bigr) \leq c\|u\|^p_{L^l_p(\Omega)}, \tag{11.4.2}$$

has been established for all $u \in C^\infty(\Omega)$. Indeed, in view of Lemma 1.2.3 we have

$$\int_\Omega \bigl|u(x)\bigr|^p d\mu = \int_0^\infty \mu(\mathscr{N}_t)\mathrm{d}\bigl(t^p\bigr),$$

where

$$\mathscr{N}_t = \bigl\{x \in \Omega : |u(x)| \geq t\bigr\}.$$

By applying the above isocapacitary inequality with respect to $F = \mathscr{N}_t$ and using the (11.4.2), one obtains the trace inequality with $C^p = cD$.

We now remark that the isocapacitary inequality really holds true for $l = 1, 2$, $p \in (1,\infty)$ and "nice" domains Ω, say, $\Omega \in C^{0,1}$. In this case it is a consequence of Stein's extension theorem and the validity of the capacitary inequality for the norm in $W^l_p(\mathbb{R}^n)$, (see Sect. 10.2).

First, we note that if $p \in [1,\infty)$, the capacitary inequality (11.4.2) holds for the norm in $L^1_p(\Omega)$ without restrictions on Ω.

Theorem. *Let Ω be a domain in $\mathbb{R}^n$. If $u \in C^\infty(\Omega) \cap L^1_p(\Omega), 1 \leq p < \infty$, then*

$$\int_0^\infty \operatorname{cap}\bigl(\mathscr{N}_t; L^1_p(\Omega)\bigr)\, \mathrm{d}\bigl(t^p\bigr) \leq c(p)\|u\|^p_{L^1_p(\Omega)}.$$

Proof. Since $\operatorname{cap}(\mathscr{N}_t; L^1_p(\Omega))$ is a nonincreasing function of t, the integral on the left-hand side of the required inequality does not exceed

$$S = \bigl(2^p - 1\bigr) \sum_{j=-\infty}^{\infty} 2^{pj} \operatorname{cap}\bigl(\mathscr{N}_{2^j}; L^1_p(\Omega)\bigr).$$

Given any $\varepsilon \in (0,1)$, let λ_ε be a function in $C^\infty(\mathbb{R}^1)$ such that $0 \le \lambda_\varepsilon \le 1, 0 \le \lambda'_\varepsilon \le 1+\varepsilon, \lambda_\varepsilon = 0$ in a neighborhood of $(-\infty, 0]$ and $\lambda_\varepsilon = 1$ in a neighborhood of $[1,\infty)$. Putting

$$u_j(x) = \lambda_\varepsilon\big(2^{1-j}|u(x)| - 1\big),$$

we observe that $u_j \in C^\infty(\Omega), u_j(x) = 1$ for $x \in \mathscr{N}_{2^j}$, $\operatorname{supp} u_j \subset \mathscr{N}_{2^{j-1}}$. Hence

$$S \le 2^{p-1}\big(2^p - 1\big)(S_1 + S_2),$$

where

$$S_1 = \sum_{j=-\infty}^{\infty} 2^{pj} \int_{\mathscr{N}_{2^{j-1}} \setminus \mathscr{N}_{2^j}} |\nabla u_j|^p \,\mathrm{d}x,$$

$$S_2 = \sum_{j=-\infty}^{\infty} 2^{pj} \int_{\omega \cap \mathscr{N}_{2^{j-1}}} |u_j|^p \,\mathrm{d}x.$$

Clearly $|\nabla u_j| \le (1+\varepsilon)2^{1-j}|\nabla u|$, and

$$S_1 \le c \sum_{j=-\infty}^{\infty} \int_{\mathscr{N}_{2^{j-1}} \setminus \mathscr{N}_{2^j}} |\nabla u|^p \,\mathrm{d}x = c\|\nabla u\|^p_{L_p(\Omega)},$$

with $c = (1+\varepsilon)^p 2^p$.

To bound the sum S_2, we note that $|u_j| \le 1$ and that the function $(0,\infty) \ni t \mapsto m_n(\omega \cap \mathscr{N}_t)$ is nonincreasing. Therefore, the general term of the sum is not greater than

$$2^p\big(1 - 2^{-p}\big)^{-1} \int_{2^{j-2}}^{2^{j-1}} m_n(\omega \cap \mathscr{N}_t)\,\mathrm{d}\big(t^p\big).$$

Thus

$$2^{-p}\big(1 - 2^{-p}\big)S_2 \le \int_0^\infty m_n(\omega \cap \mathscr{N}_t)\,\mathrm{d}\big(t^p\big) = \int_\omega \big|u(x)\big|^p \,\mathrm{d}x.$$

Here Lemma 1.2.3 with $\mu = m_n|_\omega$ has been used at the last step. By letting ε tend to zero, we arrive at the required capacitary inequality with $c = 2^{3p-1}$. □

In this section we show that the capacitary inequality for the norm in $L^l_p(\Omega)$ fails for $l > 1$ unless some restrictions are imposed on Ω. We describe a bounded domain $\Omega \subset \mathbb{R}^2$ and a Borel measure μ on Ω such that there is no constant $C > 0$ for which the trace inequality

$$\|u\|_{L_2(\Omega,\mu)} \le C\|u\|_{L^2_2(\Omega)}$$

holds for all $u \in C^\infty(\Omega)$ in spite of the fact that the inequality

$$\mu(F) \le \operatorname{const} \operatorname{cap}\big(F; L^2_2(\Omega)\big) \tag{11.4.3}$$

is true for all sets $F \subset \Omega$ closed in Ω. According to what was said previously, the capacitary inequality in $L^2_2(\Omega)$ fails for the same domain.

Before we proceed to the construction of Ω, we prove two auxiliary assertions. In this section c designates various absolute positive constants.

Lemma 1. *Let*

$$T_\varepsilon = \big\{(x,y) \in \mathbb{R}^2 : |x| < y < \varepsilon\big\}$$

and let $u \in W_2^2(T_\delta)$, *where* $0 < 2\varepsilon < \delta \le 1$. *Then*

$$\|\nabla u\|_{L_2(T_\varepsilon)} \le c\varepsilon |\log \varepsilon|^{1/2} \big(\delta^{-2}\|u\|_{L_2(T_\delta)} + \|\nabla_2 u\|_{L_2(T_\delta)}\big).$$

Proof. It will suffice to consider the case $\delta = 1$. Here the required estimate is a consequence of the inequality

$$\|v\|_{L_2(T_\varepsilon)} \le c\varepsilon |\log \varepsilon|^{1/2} \|\nabla v\|_{L_2(T_1)},$$

where $v \in W_2^1(T_1), v(x,y) = 0$ for $r = (x^2+y^2)^{1/2} \ge 1$. Passing to the polar coordinates $(x,y) = (r,\theta)$, we observe that

$$v(r,\theta)^2 = \left(\int_r^1 v_\varrho(\varrho,\theta)\,\mathrm{d}\varrho\right)^2 \le |\log r| \int_0^1 v_\varrho(\varrho,\theta)^2 \varrho\,\mathrm{d}\varrho,$$

if $r \in (0,1), \theta \in (\pi/4, 3\pi/4)$. Thus,

$$\|v\|^2_{L_2(T_\varepsilon)} \le \int_{\pi/4}^{3\pi/4} \mathrm{d}\theta \int_0^{\varepsilon\sqrt{2}} r|\log r|\,\mathrm{d}r \int_0^1 v_\varrho(\varrho,\theta)^2 \varrho\,\mathrm{d}\varrho,$$

and the result follows. □

Lemma 2. *Let* $0 < 2\varepsilon < \delta \le 1$ *and suppose that* $u \in W_2^2(T_\delta)$ *satisfies*

$$\|\nabla_2 u\|_{L_2(T_\delta)} \le 1, \qquad \|u\|_{L_\infty(T_\varepsilon)} \le 1, \qquad \|\nabla u\|_{L_2(T_\varepsilon)} \le \varepsilon|\log\varepsilon|^{1/2}.$$

Then

$$\big|u(x,y)\big| \le c\big(1 + y|\log\varepsilon|^{1/2}\big)$$

for all $(x,y) \in T_\delta$.

Proof. There exists a linear function $\ell(x,y) = ax + by + d$ such that the generalized Poincaré inequality

$$\delta^{-2}\|u - \ell\|_{L_2(T_\delta)} + \delta^{-1}\big\|\nabla(u-\ell)\big\|_{L_2(T_\delta)} \le c\|\nabla_2 u\|_{L_2(T_\delta)}$$

is valid. Therefore, by the Sobolev embedding and Lemma 1

$$\|u - \ell\|_{L_\infty(T_\delta)} \le c\delta, \qquad \big\|\nabla(u-\ell)\big\|_{L_2(T_\varepsilon)} \le c\varepsilon|\log\varepsilon|^{1/2}. \tag{11.4.4}$$

By the assumptions of the lemma, we obtain

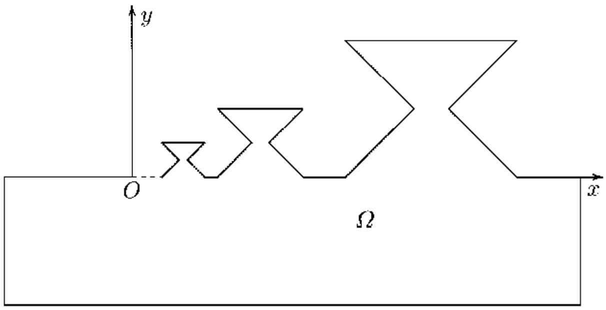

Fig. 35.

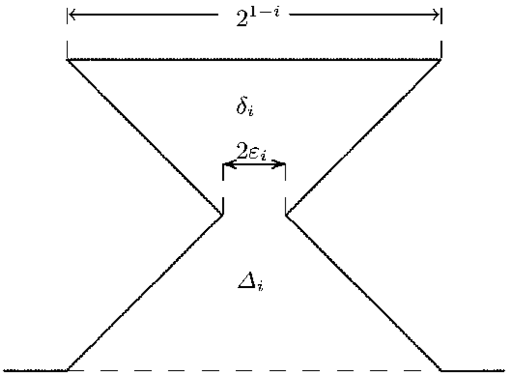

Fig. 36.

$$\|\ell\|_{L_\infty(T_\varepsilon)} \le c, \qquad \|\nabla \ell\|_{L_2(T_\varepsilon)} \le c\varepsilon|\log\varepsilon|^{1/2}.$$

In other words,

$$|a| + |b| \le c|\log\varepsilon|^{1/2}, \qquad |d| \le c.$$

By using (11.4.4), we arrive at the desired estimate thus concluding the proof. □

We now turn to the counterexample. Let $\{\sigma_i\}_{i\ge1}$ be a sequence of open nonoverlapping intervals of length 2^{1-i} lying in interval (0,4) of the real axis. For definiteness, σ_i are supposed to concentrate near the origin, i.e., for any small $\varepsilon > 0$, interval $(\varepsilon, 4)$ contains only a finite number of σ_i. By $\{\Delta_i\}_{i\ge1}$ we denote the sequence of open right isosceles triangles situated over the axis Ox with hypotenuses σ_i.

Let δ_i be the triangle symmetric to Δ_i with respect to the line $y = 2^{-i} - \varepsilon_i$, $\varepsilon_i \in (0, 2^{-i})$ being specified in the following. Furthermore, let $R = [-1,4] \times [-1,0]$. We define Ω to be the interior of the union of R and all the triangles Δ_i and δ_i (see Figs. 35 and 36).

To introduce the measure μ, we use the points

$$A_{ij} = \big(\gamma_i, 2^{-i} - \varepsilon_i + 2^j h_i\big), \quad j = 1, 2, \dots, 2^i;\ i = 1, 2, \dots,$$

where γ_i is the abscissa of the middle point of σ_i,

$$h_i = |\log_2 \varepsilon_i|^{-1/2}, \qquad \log_2 |\log_2 \varepsilon_i| = 2^{i+2}.$$

Let F be an arbitrary Borel subset of Ω and χ_F its characteristic function. The required measure μ is defined by

$$\mu(F) = \sum_{i\geq 1} \sum_{j=1}^{2^i} 2^{-2j-i/2} \chi_F(A_{ij}).$$

Let us verify (11.4.3). To be more specific, we introduce the norm in $L_2^2(\Omega)$ by

$$\|u\|_{L_2^2(\Omega)} = \|\nabla_2 u\|_{L_2(\Omega)} + \|u\|_{L_2((-1,4)\times(-1,0))}.$$

We deduce (11.4.3) by using the estimate

$$\operatorname{cap}\big(\{A_{ij}\}; L_2^2(\Omega)\big) \geq c 2^{-2j}, \quad j = 1, 2, \dots, 2^i, \tag{11.4.5}$$

which will be checked later. Let F be a relatively closed subset of Ω and let $m(F)$ be the minimum value of j such that there exists a point A_{ij} in the set F. Put $m(F) = \infty$ if there are no points A_{ij} in F. Then (11.4.5) and the definition of μ imply

$$\operatorname{cap}\big(F; L_2^2(\Omega)\big) \geq c 2^{-2m}, \qquad \mu(F) \leq c 2^{-2m}.$$

Hence μ satisfies (11.4.3).

Inequality (11.4.5) is a consequence of the estimate

$$\big|u(A_{ij})\big| \leq c 2^j,$$

where u is an arbitrary function in $L_2^2(\Omega)$ normalized by $\|u\|_{L_2^2(\Omega)} = 1$. Let G be the interior of the union $R \bigcup_{i=1}^\infty \Delta_i$. Clearly, $G \subset \Omega$ and G possesses the cone property. By Lemma 1 and the Sobolev embedding $L_2^2(G) \subset L_\infty(G)$, we have

$$\|\nabla u\|_{L_2(t_i)} \leq c \varepsilon_i |\log \varepsilon_i|^{1/2}, \qquad \|u\|_{L_\infty(t_i)} \leq c,$$

where $t_i = \delta_i \cap \Delta_i$. Next, Lemma 2 applied to the triangles δ_i and $\delta_i \cap \{(x, y) : y < 2^{-i} - \varepsilon_i\}$ yields

$$\big|u(x, y)\big| \leq c\big(1 + \big|y - 2^{-i} + 2\varepsilon_i\big| |\log \varepsilon_i|^{1/2}\big),$$

for all $(x, y) \in \delta_i$. Since the ordinate of the point A_{ij} is $2^{-i} - \varepsilon_i + 2^j h_i$, inequality $|u(A_{ij})| \leq c 2^j$ follows.

We now define a function f such that

$$f \in L_2^2(\Omega) \cap C(\Omega), \quad f \notin L_2(\Omega, \mu).$$

Put $f(x,y) = 0$ for $y \le 0$ and

$$f(x,y) = 2^{-i/4} h_i \eta\bigl(2^i y\bigr)\bigl(y + \varepsilon_i - 2^{-i}\bigr)\log_2\bigl(2^{-i} - y\bigr),$$

on the set

$$\tau_i = \bigl\{(x,y) \in \Delta_i : y \le 2^{-i} - \varepsilon_i\bigr\},$$

where $\eta \in C^\infty(0,\infty), \eta(t) = 0$ for $t \le 1/4, \eta(t) = 1$ for $t \ge 1/2$. The linear extension of this function f to the set $\{(x,y) \in \delta_i, y > 2^{-i} - \varepsilon_i\}$ will be also denoted by f, i.e.,

$$f(x,y) = -2^{-i/4} h_i^{-1}\bigl(y - 2^{-i} + \varepsilon_i\bigr)$$

for $(x,y) \in \delta_i, y > 2^{-i} - \varepsilon_i$. Clearly supp $\nabla_2 f$ is placed in the set $\cup_{i\ge 1}\overline{\tau}_i$. Furthermore, the estimate

$$|\nabla_2 f| \le c 2^{-i/4} h_i \max\bigl\{2^i i, \bigl(2^{-i} - y\bigr)^{-1}\bigr\}$$

holds for $(x,y) \in \Delta_i$. Therefore

$$\begin{aligned}\|\nabla_2 f\|^2_{L_2(\Omega)} &= \sum_{i\ge 1} \|\nabla_2 f\|^2_{L_2(\Delta_i)} \\ &\le c\sum_{i\ge 1} 2^{-i/2} h_i^2\bigl(i^2 + |\log_2 \varepsilon_i|\bigr) \le c\sum_{i\ge 1} 2^{-i/2} < \infty\end{aligned}$$

and $f \in L_2^2(\Omega)$. At the same time

$$\|f\|^2_{L_2(\Omega,\mu)} = \sum_{i\ge 1}\sum_{j=1}^{2^i} 2^{-i/2-2j} f(A_{ij})^2 = \sum_{i\ge 1} 1 = \infty.$$

The result follows. □

11.5 Ball and Pointwise Criteria

It was shown by Kerman and Sawyer [420] (see also Sawyer [691]) that it is enough to assume in (11.3.3) with $S_p^l = h_p^l$ that $E = B$, where $B = B(x,r)$ is a ball (or cube) in $\mathbb{R}^n$.

Theorem 1. *Let $p > 1$ and $0 < l < n$. The trace inequality (11.3.3) holds if and only if*

$$\int_B \bigl[I_l(\chi_B \, \mathrm{d}\mu)\bigr]^{p'} \, \mathrm{d}x \le C\mu(B) \tag{11.5.1}$$

for every ball B.

The following criterion for (11.3.3) with $S_p^l = h_p^l$ formulated in purely pointwise form was obtained by Maz'ya and Verbitsky [591].

Theorem 2. *Let* $1 < p < \infty$ *and let* $0 < l < n$*. Let* μ *be a nonnegative Borel measure on* $\mathbb{R}^n$*. Then the following statements are equivalent.*

(i) *The trace inequality* (11.3.3) *or, equivalently*

$$\|I_l f\|_{L_p(\mathrm{d}\mu)} \le C\|f\|_{L_p} \tag{11.5.2}$$

holds for all $f \in L_p(\mathbb{R}^n)$*.*

(ii) $I_l\mu < \infty$ *a.e. and*

$$I_l\big[(I_l\mu)^{p'}\big](x) \le C I_l\mu(x) \quad \text{a.e.} \tag{11.5.3}$$

(iii) *The trace inequality* (11.5.2) *holds with* μ *replaced by the absolutely continuous measure* $\mathrm{d}\nu = (I_l\mu)^{p'}\,\mathrm{d}x$*, or, equivalently,*

$$\nu(E) = \int_E (I_l\mu)^{p'}\,\mathrm{d}x \le C\,\mathrm{cap}\big(E; h_p^l\big), \tag{11.5.4}$$

where C *is a constant which is independent of a compact set* E*.*

Both Theorems 1 and 2 will be proved in the present section.

We shall need the following "integration by parts" lemma which is an analog of the elementary inequality

$$\left(\sum_{k=1}^{\infty} a_k\right)^p \le p\sum_{k=1}^{\infty} a_k\left(\sum_{j=k}^{\infty} a_j\right)^{p-1},$$

where $1 \le p < \infty$ and $\{a_k\}$ is a sequence of nonnegative numbers. A similar statement for more general integral operators is proved in Verbitsky and Wheeden [776]. (See also Kalton and Verbitsky [412] where a discrete analog with sharp constants is used.)

Lemma. *Let* $0 < l < n$ *and* $1 \le p < \infty$*. Let* μ *be a positive Borel measure on* $\mathbb{R}^n$ *and let* $f \in L^1_{\mathrm{loc}}(\mathrm{d}\mu)$*,* $f \ge 0$*. Suppose that* $I_l(f\,\mathrm{d}\mu)(x) < \infty$*. Then*

$$\big[I_l(f\,\mathrm{d}\mu)(x)\big]^p \le C I_l\big[f\big(I_l(f\,\mathrm{d}\mu)\big)^{p-1}\,\mathrm{d}\mu\big](x), \tag{11.5.5}$$

where C *is a constant that depends only on* μ*,* p*, and* n*.*

Proof. For convenience we ignore the normalization constant c_n in the definition of I_l and denote by C constants that depend only on l, n, and p. For a fixed $x \in \mathbb{R}^n$ such that $A = I_l(f\,\mathrm{d}\mu)(x) < \infty$, we set

$$B = I_l\big[f\big(I_\mu(f\,\mathrm{d}\mu)\big)^{p-1}\,\mathrm{d}\mu\big](x).$$

Then (11.5.5) may be rewritten as $A^p \le CB$.

We first consider the more difficult case $1 < p \le 2$. Clearly

$$A^p = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-l}}\left[\int_{\mathbb{R}^n} \frac{f(z)}{|x-z|^{n-l}}\,\mathrm{d}\mu(z)\right]^{p-1}\mathrm{d}\mu(y) \le I + II,$$

where

$$I = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-l}} \left[\int_{\{z:|y-z|\le 2|x-z|\}} \frac{f(z)}{|x-z|^{n-l}} \, \mathrm{d}\mu(z) \right]^{p-1} \mathrm{d}\mu(y)$$

and

$$II = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-l}} \left[\int_{\{z:|y-z|>2|x-z|\}} \frac{f(z)}{|x-z|^{n-l}} \, \mathrm{d}\mu(z) \right]^{p-1} \mathrm{d}\mu(y).$$

To estimate I, notice that in the inside integral we have

$$|x-z|^{l-n} \le C|y-z|^{l-n}.$$

Hence

$$I \le C \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-l}} \left[\int_{\mathbb{R}^n} \frac{f(z)}{|y-z|^{n-l}} \, \mathrm{d}\mu(z) \right]^{p-1} \mathrm{d}\mu(y) = CB.$$

We estimate II using Hölder's inequality with exponents $1/(p-1)$ and $1/(2-p)$, which gives $II \le A^{2-p} III^{p-1}$, where by Fubini's theorem,

$$\begin{aligned} III &= \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-l}} \left[\int_{\{z:|y-z|>2|x-z|\}} \frac{f(z)}{|x-z|^{n-l}} \, \mathrm{d}\mu(z) \right] \mathrm{d}\mu(y) \\ &= \int_{\mathbb{R}^n} \frac{f(z)}{|x-z|^{n-l}} \left[\int_{\{y:|y-z|>2|x-z|\}} \frac{f(y)}{|x-y|^{n-l}} \, \mathrm{d}\mu(y) \right] \mathrm{d}\mu(z). \end{aligned}$$

We now estimate the function in square brackets on the right-hand side, which we denote by

$$G(z) = \int_{\{y:|y-z|>2|x-z|\}} \frac{f(y)}{|x-y|^{n-l}} \, \mathrm{d}\mu(y).$$

Obviously, $G(z) \le A$. On the other hand, $G(z) \le C I_l(f \, \mathrm{d}\mu)(z)$. Indeed, by the triangle inequality,

$$\{y : |y-z| > 2|x-z|\} \subset \{y : |y-z| \le 2|x-y|\}$$

and hence

$$G(z) \le \int_{\{y:|y-z|\le 2|x-y|\}} \frac{f(y)}{|x-y|^{n-l}} \, \mathrm{d}\mu(y) \le C I_l(f \, \mathrm{d}\mu)(z).$$

Combining the preceding inequalities, we get

$$G(z) \le C A^{2-p} \big[I_l(f \, \mathrm{d}\mu)(z) \big]^{p-1}.$$

From this it follows that $III \le C A^{2-p} B$, and thus

$$II \le A^{2-p} III^{p-1} \le C A^{p(2-p)} B^{p-1}.$$

Combining the estimates for I and II, we obtain

$$A^p \le CB + CA^{p(2-p)}B^{p-1},$$

which yields $A^p \le CB$ for $1 < p \le 2$.

We now show that the inequality $A^p \le CB$ for $p > 2$ follows from the case $p = 2$ considered previously. By Hölder's inequality with exponents $p-1$ and $(p-1)/(p-2)$ we have

$$\begin{aligned} A^2 &= \big[I_l(f\,\mathrm{d}\mu)(x)\big]^2 \le CI_l\big[f\big(I_l(f\,\mathrm{d}\mu)\big)\,\mathrm{d}\mu\big] \\ &\le C\big[I_l\big[f\big(I_l(f\,\mathrm{d}\mu)\big)\big]^{p-1}\,\mathrm{d}\mu\big]^{\frac{1}{p-1}}\big[I_l(f\,\mathrm{d}\mu)\big]^{\frac{p-1}{p-2}} = CB^{\frac{1}{p-1}}A^{\frac{p-2}{p-1}}, \end{aligned}$$

which clearly implies $A^p \le CB$. The proof of the Lemma is complete. □

Remark. It is easy to see that the above Lemma is true without the restriction $I_l(f\,\mathrm{d}\mu)(x) < \infty$, i.e., the right-hand side of (11.5.5) is infinite if the same is true for the left-hand side (see Verbitsky and Wheeden [776]).

We now show that the pointwise condition

$$I_l\big[(I_l\mu)^{p'}\big](x) \le cI_l\mu(x) < \infty \quad \text{a.e.} \tag{11.5.6}$$

implies both of the trace inequalities

$$\|I_l f\|_{L_p(\mathrm{d}\mu)} \le C\|f\|_{L_p}, \quad \forall f \in L_p\big(\mathbb{R}^n\big) \tag{11.5.7}$$

and

$$\|I_l f\|_{L_p(\mathrm{d}\nu)} \le C\|f\|_{L_p}, \quad \forall f \in L_p\big(\mathbb{R}^n\big), \tag{11.5.8}$$

where ν is defined by

$$\mathrm{d}\nu = (I_l\mu)^{p'}\,\mathrm{d}x. \tag{11.5.9}$$

Proposition 1. *Let $1 < p < \infty$. Let μ be a positive Borel measure on $\mathbb{R}^n$ and let ν be defined by* (11.5.9)*. Suppose that the pointwise condition* (11.5.6) *holds. Then both* (11.5.7) *and* (11.5.8) *hold.*

Proof. Without loss of generality we may assume that $f \ge 0$, and that f is uniformly bounded and compactly supported. By the above Lemma with $\mathrm{d}\mu = \mathrm{d}x$ and Fubini's theorem,

$$\|I_l f\|^p_{L_p(\mathrm{d}\mu)} \le C\int_{\mathbb{R}^n} I_l\big[f(I_l f)^{p-1}\big]\,\mathrm{d}\mu = C\int_{\mathbb{R}^n} f(I_l f)^{p-1}(I_l\mu)\,\mathrm{d}x.$$

Here C is the constant in (2.1) that depends only on p, l, and n. From this, by Hölder's inequality, we get

$$\|I_l f\|^p_{L_p(\mathrm{d}\mu)} \le C\|f\|_{L_p}\|I_l f\|^{p-1}_{L_p(\mathrm{d}\nu)}. \tag{11.5.10}$$

The preceding inequality shows that (11.5.8)⇒(11.5.7) for any μ.

Repeating the above argument with ν in place of μ, we obtain

$$\|I_l f\|^p_{L_p(\mathrm{d}\nu)} \le C\|f\|_{L_p}\|I_l f\|^{p-1}_{L_p(\mathrm{d}\nu_1)}, \tag{11.5.11}$$

where by (11.5.6)

$$\mathrm{d}\nu_1 = (I_l\nu)^{p'}\,\mathrm{d}x = \big[I_l(I_l\mu)^{p'}\big]^{p'}\,\mathrm{d}x \le c^{p'}\,\mathrm{d}\nu.$$

Here c is the constant in (11.5.5). Hence by (11.5.11) and the preceding estimate

$$\|I_l f\|^p_{L_p(\mathrm{d}\nu)} \le C\|f\|_{L_p}\|I_l f\|^{p-1}_{L_p(\mathrm{d}\nu)}. \tag{11.5.12}$$

Assuming that $\|I_l f\|_{L_p(\mathrm{d}\nu)} < \infty$, we get

$$\|I_l f\|_{L_p(\mathrm{d}\nu)} \le C\|f\|_{L_p},$$

which proves (11.5.8). Now (11.5.10) combined with the preceding estimate yields (11.5.7).

It remains to check that $\|I_l f\|_{L_p(\mathrm{d}\nu)} < \infty$, which follows easily from (11.5.6) and the assumption that f is bounded and compactly supported. Indeed, by (11.5.6) $I_l\mu < \infty$ a.e., which implies that $I_l\mu$ is locally integrable. Since $I_\alpha f$ is obviously bounded, it follows that

$$\begin{aligned}\|I_l f\|^p_{L_p(\mathrm{d}\nu)} &= \int_{\mathbb{R}^n} (I_l f)^p (I_l\mu)^{p'}\,\mathrm{d}x \le C\int_{\mathbb{R}^n} I_l f (I_l\mu)^{p'}\,\mathrm{d}x\\ &= C\int_{\mathbb{R}^n} f I_l\big[(I_l\mu)^{p'}\big]\,\mathrm{d}x \le C\int_{\mathbb{R}^n} f I_l\mu\,\mathrm{d}x < \infty.\end{aligned}$$

□

Proposition 2. *Let $1 < p < \infty$. Let μ be a positive Borel measure on $\mathbb{R}^n$. Then the pointwise condition (11.5.6) is equivalent to the Kerman–Sawyer condition*

$$\int_B \big[I_l(\chi_B\,\mathrm{d}\mu)\big]^{p'}\,\mathrm{d}x \le C\mu(B) \tag{11.5.13}$$

for every ball $B = B(x,r)$.

Proof. By Proposition 1 it follows that the pointwise condition (11.5.6) implies the trace inequality (11.5.7). By duality, (11.5.7) is equivalent to the inequality

$$\big\|I_l(g\,\mathrm{d}\mu)\big\|_{L_{p'}} \le C\|g\|_{L_{p'}(\mathrm{d}\mu)}, \quad \forall g \in L_{p'}(\mathrm{d}\mu). \tag{11.5.14}$$

Letting $g = \chi_B$, $B = B(x,r)$, we see that (11.5.13) holds.

To prove the converse, we shall show that (11.5.13) implies

$$\nu\big(B(x,r)\big) = \int_{B(x,r)} (I_l\mu)^{p'}\,\mathrm{d}x \le Cr^n \int_r^\infty \frac{\mu(B(x,t))}{t^n}\,\frac{\mathrm{d}t}{t} \tag{11.5.15}$$

for every ball $B(x,r)$, which readily gives (11.5.6).

Note that, clearly,

$$I_l(\chi_{B(x,r)}\,\mathrm{d}\mu) \geq C\frac{\mu(B(x,r))}{r^{n-l}}\chi_{B(x,r)}, \tag{11.5.16}$$

and hence it follows from (11.5.13) that

$$\mu\big(B(x,r)\big) \leq Cr^{n-lp}. \tag{11.5.17}$$

In particular, the right-hand side of (11.5.15) is finite and hence $I_l\mu < \infty$ a.e.

The proof of (11.5.15) is based on the decomposition $\mathrm{d}\mu = \mathrm{d}\mu_1 + \mathrm{d}\mu_2$, where

$$\mathrm{d}\mu_1(x) = \chi_{B(x,2r)}\,\mathrm{d}\mu(x) \quad \text{and} \quad \mathrm{d}\mu_2(x) = (1-\chi_{B(x,2r)})\,\mathrm{d}\mu(x).$$

For $y \in B(x,r)$,

$$I_l\mu_2(y) \leq CI_l\mu_2(x),$$

where C depends only on l and n. Hence by (11.5.13) and the preceding estimate,

$$\begin{aligned}\nu\big(B(x,r)\big) &= \int_{B(x,r)} (I_l\mu)^{p'}\,\mathrm{d}y \leq C\int_{B(x,r)} (I_l\mu_1)^{p'}\,\mathrm{d}y + C\int_{B(x,r)} (I_l\mu_2)^{p'}\,\mathrm{d}y \\ &\leq C\mu\big(B(x,2r)\big) + Cr^n\big[I_l\mu_2(x)\big]^{p'}.\end{aligned}$$

Clearly,

$$\begin{aligned}\big[I_l\mu_2(x)\big]^{p'} &= C\bigg[\int_{2r}^{\infty}\frac{\mu(B(x,t))}{t^{n-l}}\frac{\mathrm{d}t}{t}\bigg]^{p'} \\ &= Cp'\int_{2r}^{\infty}\frac{\mu(B(x,t))}{t^{n-l}}\bigg(\int_t^{\infty}\frac{\mu(B(x,s))}{s^{n-l}}\frac{\mathrm{d}s}{s}\bigg)^{p'-1}\frac{\mathrm{d}t}{t}.\end{aligned}$$

Now by (11.5.16) it follows that $\mu(B(x,s)) \leq Cs^{n-lp}$, and hence

$$\bigg(\int_t^{\infty}\frac{\mu(B(x,s))}{s^{n-l}}\frac{\mathrm{d}s}{s}\bigg)^{p'-1} \leq Ct^{-l}.$$

From this we obtain

$$\big[I_l\mu_2(x)\big]^{p'} \leq C\int_{2r}^{\infty}\frac{\mu(B(x,t))}{t^n}\frac{\mathrm{d}t}{t}.$$

Combining these estimates, we get

$$\begin{aligned}\nu\big(B(x,r)\big) &\leq C\mu\big(B(x,2r)\big) + Cr^n\int_{2r}^{\infty}\frac{\mu(B(x,t))}{t^n}\frac{\mathrm{d}t}{t} \\ &\leq Cr^n\int_r^{\infty}\frac{\mu(B(x,t))}{t^n}\frac{\mathrm{d}t}{t}.\end{aligned}$$

This proves (11.5.15).

To obtain (11.5.6), we notice that by (11.5.15) and Fubini's theorem,

$$\begin{aligned} I_l\nu(x) &= C\int_0^\infty \frac{\nu(B(x,r))}{r^{n-l}}\frac{\mathrm{d}r}{r} \\ &\le C\int_0^\infty r^l \int_r^\infty \frac{\mu(B(x,t))}{t^n}\frac{\mathrm{d}t}{t}\frac{\mathrm{d}r}{r} = C\int_0^\infty \frac{\mu(B(x,t))}{t^n}\left(\int_0^t r^l\frac{\mathrm{d}r}{r}\right)\frac{\mathrm{d}t}{t} \\ &= C\int_0^\infty \frac{\mu(B(x,t))}{t^{n-l}}\frac{\mathrm{d}t}{t} = CI_l\mu(x). \end{aligned}$$

Hence

$$I_l\nu(x) = I_l\big[(I_l\mu)^{p'}\big](x) \le CI_l\mu(x) < \infty \quad \text{a.e.} \qquad \square$$

We now are in a position to complete the proof of Theorems 11.5/1 and 11.5/2.

By Proposition 1 it follows that (11.5.6)⇒(11.5.7) and (11.5.8). As was observed in the proof of Proposition 11.5/1, it follows from Lemma 11.5 that (11.5.8)⇒(11.5.7). Since (11.5.7)⇒(11.5.13) and (11.5.13)⇔(11.5.6) by Proposition 1, we have shown that the following conditions are equivalent:

$$(11.5.7) \Leftrightarrow (11.5.8) \Leftrightarrow (11.5.6) \Leftrightarrow (11.5.13).$$

The proofs of Theorems 1 and 2 are complete. $\square$

11.6 Conditions for Embedding into $L_q(\mu)$ for $p > q > 0$

In the present section we discuss the necessary and sufficient conditions for the validity of (11.3.2) for $p > q > 0$, $p > 1$.

11.6.1 Criterion in Terms of the Capacity Minimizing Function

Theorem 1. *Let $p \in (1,\infty), 0 < q < p, l > 0$. The inequality*

$$\|u\|_{L_q(\mathbb{R}^n,\mu)} \le C_{p,q}\|u\|_{S_p^l}, \tag{11.6.1}$$

where S_p^l is the same as in Theorem 11.3, holds for all $u \in C_0^\infty(\mathbb{R}^n)$ if and only if

$$D_{p,q} = \int_0^{\mu(\mathbb{R}^n)} \left(\frac{t^{p/q}}{\varkappa(t)}\right)^{\frac{q}{p-q}}\frac{\mathrm{d}t}{t} < \infty,$$

where

$$\varkappa(t) = \inf\big\{\mathrm{cap}\big(F, S_p^l\big) : F \text{ is a compactum in } \mathbb{R}^n, \mu(F) \ge t\big\},$$
$$t \in \big(0, \mu(\mathbb{R}^n)\big).$$

Proof. Sufficiency. We prove the inequality

$$\|u\|_{L_q(\mu)} \le (4pC)^{1/p}\biggl(\frac{p}{p-q}\int_0^{\mu(\mathbb{R}^n)}\Bigl(\frac{t}{\varkappa(t)}\Bigr)^{q/(p-q)}\,\mathrm{d}t\biggr)^{(p-q)/pq}\|u\|_{S_p^l}. \tag{11.6.2}$$

Let $\mathfrak{S}$ be any sequence of open sets $\{g_j\}_{j=-\infty}^{+\infty}$ such that $\bar g_{j+1} \subset g_j$. We put $\mu_j = \mu(g_j)$, $\gamma_j = \mathrm{cap}(g_j, S_p^l)$ and

$$\mathcal{D}_{p,q} = \sup_{\{\mathfrak{S}\}}\biggl(\sum_{j=-\infty}^{+\infty}\Bigl(\frac{(\mu_j-\mu_{j+1})^{1/q}}{\gamma^{1/p}}\Bigr)^{pq/(p-q)}\biggr)^{(p-q)/pq}.$$

We show that the best constant in (11.6.1) satisfies

$$C_{p,q} \le (4pC)^{1/p}\mathcal{D}_{p,q}, \tag{11.6.3}$$

where C is the constant in (11.1.4).

Let $f \in C_0^\infty$, $f \ge 0$, $u = Kf$, $g_j = \{x : (Kf)(x) > \alpha^j\}$, where $\alpha > 1$. Obviously,

$$\|u\|^q_{L_q(\mu)} \le \sum_j \alpha^{q(j+1)}\bigl[\mu(g_j)-\mu(g_{j+1})\bigr] = \sum_j \alpha^{q(j+1)}\frac{\mu_j-\mu_{j+1}}{\gamma_j^{q/p}}\gamma_j^{q/p}.$$

By Hölder's inequality the last sum does not exceed

$$\mathcal{D}^q_{p,q}\Bigl(\sum_j \alpha^{p(j+1)}\gamma_j\Bigr)^{q/p}.$$

Next we note that

$$\sum_j \alpha^{p(j+1)}\gamma_j \le \frac{p\alpha^{2p}}{\alpha^p-1}\sum_j\int_{\alpha^{j-1}}^{\alpha^j}\mathrm{cap}\bigl(\mathcal{N}_t, S_p^l\bigr)t^{p-1}\,\mathrm{d}t.$$

Putting $\alpha^p = 2$ in this inequality, we obtain

$$\sum_j \alpha^{p(j+1)}\gamma_j \le 4p\int_0^\infty \mathrm{cap}\bigl(\mathcal{N}_t, S_p^l\bigr)t^{p-1}\,\mathrm{d}t.$$

Consequently,

$$\|u\|_{L_q(\mu)} \le (4pC)^{1/p}\mathcal{D}_{p,q}\|u\|_{S_p^l}, \tag{11.6.4}$$

and (11.6.3) follows.

The inequality (11.6.2) results immediately from (11.6.4) and the estimates

$$\begin{aligned}\sum_j\Bigl(\frac{(\mu_j-\mu_{j+1})^{p/q}}{\gamma_j}\Bigr)^{q/(p-q)} &\le \sum_j \frac{\mu_j^{p/(p-q)}-\mu_{j+1}^{p/(p-q)}}{\gamma_j^{q/(p-q)}}\\ &\le \int_0^{\mu(\mathbb{R}^n)}\frac{\mathrm{d}(t^{p/(p-q)})}{(\varkappa(t))^{q/(p-q)}}.\end{aligned}$$

The proof of sufficiency is complete.

Necessity. First we remark that (11.6.1) implies $\varkappa(t) > 0$ for $t > 0$. Let j be any integer satisfying $2^j < \mu(\mathbb{R}^n)$. Then there exists a compact set $F_j \subset \mathbb{R}^n$ such that

$$\mu(F_j) \geq 2^j, \qquad \operatorname{cap}\big(F_j, S_p^l\big) \leq 2\varkappa\big(2^j\big).$$

Let φ_j be a function for which

$$\chi_{F_j} \leq \mathcal{K}_l * \varphi_j, \qquad \varphi_j \geq 0, \qquad \|\varphi_j\|^p_{L_p(\mathbb{R}^n)} \leq 4\varkappa\big(2^j\big),$$

where $\mathcal{K}_l$ is either the Riesz or the Bessel kernel.

Suppose that s is the integer for which $2^s < \mu(\mathbb{R}^n) \leq 2^{s+1}$ if $\mu(\mathbb{R}^n)$ is finite and s is an arbitrary integer otherwise. Let $r < s$ be another integer and let

$$\varphi_{r,s} = \max_{r \leq j \leq s} \beta_j \varphi_j, \quad \beta_j = \big(2^j/\varkappa\big(2^j\big)\big)^{1/(p-q)}.$$

Furthermore, we put

$$f_{r,s} = \mathcal{K}_l * \varphi_{r,s}.$$

Clearly,

$$\|f_{r,s}\|^p_{S_p^l} = \|\varphi_{r,s}\|^p_{L_p} \leq \sum_{j=r}^{s} \beta_j^p \|\varphi_j\|^p_{L_p} \leq 4 \sum_{j=r}^{s} \beta_j^p \varkappa\big(2^j\big).$$

To obtain a lower bound for the norm of $f_{r,s}$ in $L_q(\mathbb{R}^n, \mu)$, we introduce the nonincreasing rearrangement $(f_{r,s})^*$ of $f_{r,s}$

$$(f_{r,s})^*(t) = \inf\big\{\tau > 0 : \mu\big(\{|f_{r,s}| > \tau\}\big) \leq t\big\}, \quad t \in \big(0, \mu\big(\mathbb{R}^n\big)\big).$$

Since $f_{r,s}|_{F_j} \geq \beta_j$ and $\mu(F_j) \geq 2^j$, the inequality

$$\mu\big(\{|f_{r,s}| > \tau\}\big) < 2^j$$

implies $\tau \geq \beta_j$. Hence

$$f^*_{r,s}(t) \geq \beta_j \quad \text{for } t \in \big(0, 2^j\big),\ r \leq j \leq s,$$

which gives

$$\|f_{r,s}\|^q_{L_q(\mathbb{R}^n,\mu)} = \int_0^{\mu(\mathbb{R}^n)} \big(f^*_{r,s}(t)\big)^q \,dt \geq \sum_{j=r}^{s} \int_{2^{j-1}}^{2^j} \big(f^*_{r,s}(t)\big)^q \,dt \geq \sum_{j=r}^{s} \beta_j^q 2^{j-1}.$$

Next, we note that if (11.6.1) is valid for all $u \in C_0^\infty(\mathbb{R}^n)$, then (11.6.1) is valid for all $u \in S_p^l$. In particular, for $u = f_{r,s}$ one obtains

$$C_{p,q} \geq c \frac{(\sum_{j=r}^s \beta_j^q 2^j)^{1/q}}{(\sum_{j=r}^s \beta_j^p \varkappa(2^j))^{1/p}} = c \left(\sum_{j=r}^{s} \frac{2^{pj/(p-q)}}{(\varkappa(2^j))^{q/(p-q)}} \right)^{1/q-1/p}.$$

Letting $r \to -\infty$ and using the monotonicity of $\varkappa$, we have

$$C_{p,q} \geq c\biggl(\sum_{j=-\infty}^{s}\Bigl(\frac{2^j}{\varkappa(2^j)}\Bigr)^{\frac{q}{p-q}} 2^j\biggr)^{1/q-1/p} \geq c\biggl(\int_0^{2^s}\Bigl(\frac{t}{\varkappa(t)}\Bigr)^{\frac{q}{p-q}}\,\mathrm{d}t\biggr)^{1/q-1/p}$$

with c independent of s. Letting $s \to \infty$, we arrive at the estimate $C_{p,q} \geq cD_{p,q}^{1/q-1/p}$ for $\mu(\mathbb{R}^n) = \infty$. In the case of finite $\mu(\mathbb{R}^n)$ we have

$$\begin{aligned}&\int_{2^s}^{\mu(\mathbb{R}^n)} \bigl(t/\varkappa(t)\bigr)^{q/(p-q)}\,\mathrm{d}t \\ &\qquad\leq 2\int_{2^{s-1}}^{2^s}\bigl(2t/\varkappa(2t)\bigr)^{q/(p-q)}\,\mathrm{d}t \leq c\int_{2^{s-1}}^{2^s}\bigl(t/\varkappa(t)\bigr)^{q/(p-q)}\,\mathrm{d}t,\end{aligned}$$

and again the inequality $C_{p,q} \geq cD_{p,q}^{1/q-1/p}$ is true. The proof is complete. □

The proof of sufficiency implies the following assertion.

Corollary. *Let the assumptions of the above theorem be fulfilled. If A is a Borel subset of $\mathbb{R}^n$, then*

$$\Bigl(\int_A |u|^q\,\mathrm{d}\mu\Bigr)^{1/q} \leq c\biggl(\int_0^{\mu(A)}\Bigl(\frac{t^{p/q}}{\varkappa(t)}\Bigr)^{\frac{q}{p-q}}\frac{\mathrm{d}t}{t}\biggr)^{(p-q)/pq}\|u\|_{S_p^l}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ with c depending only on p, q.

Proof. Let μ_A be the restriction of μ to the set A. Then it is readily seen that $\varkappa_\mu(t) \leq \varkappa_{\mu_A}(t)$. The result follows by applying the sufficiency statement of the theorem with respect to the measure μ_A on $\mathbb{R}^n$. □

The following criterion of the inequality

$$\|u\|_{L_q(\mu)} \leq c\|u\|_{h_p^l}, \tag{11.6.5}$$

with $q < p$, whose statement involves no capacity, was established by Cascante, Ortega, and Verbitsky [175] for $q > 1$ and by Verbitsky [775] for $q > 0$ (see [176]).

This criterion is formulated in terms of the nonlinear potential

$$(\mathcal{W}_{l,p}\mu)(x) = \int_0^\infty \Bigl[\frac{\mu(B(x,r))}{r^{n-lp}}\Bigr]^{p'-1}\frac{\mathrm{d}r}{r},$$

which appeared in Maz'ya and Havin [567] and was used by Wolff in [372] in his proof of inequality (10.4.8).

Theorem 2. *Let $0 < q < p < \infty$ and $0 < l < n$. Then the inequality* (11.6.5) *holds if and only if*

$$\int \bigl(\mathcal{W}_{l,p}\mu(x)\bigr)^{\frac{q(p-1)}{p-q}}\,\mathrm{d}\mu(x) < \infty.$$

11.6.2 Two Simple Cases

For $pl > n$ the necessary and sufficient condition for the validity of (11.3.2), where $S_p^l = H_p^l$, W_p^l or B_p^l, can be written in the following simple way.

Theorem 1. *If $pl > n$, $p > q$ and $C_{p,q}$ is the best constant in (11.3.2), then*

$$C_{p,q} \sim \Big(\sum_{i\geq 0} \mu\big(\mathcal{Q}^{(i)}\big)^{p/(p-q)}\Big)^{(p-q)/pq}, \tag{11.6.6}$$

where $\{\mathcal{Q}^{(i)}\}$ is the sequence of closed cubes with edge length 1 forming the coordinate grid in $\mathbb{R}^n$.

Proof. Let $O^{(i)}$ be the center of $\mathcal{Q}^{(i)}$, $O^{(0)} = O$, and let $2\mathcal{Q}^{(i)}$ be the concentric cube of $\mathcal{Q}^{(i)}$ with edges parallel to those of $\mathcal{Q}^{(0)}$ and having edge length 2. We put $\eta_i(x) = \eta(x - O^{(i)})$, where $\eta \in C_0^\infty(2\mathcal{Q}^{(0)})$, $\eta = 1$ on $\mathcal{Q}^{(0)}$. We have

$$\begin{aligned} \|u\|^q_{L_q(\mu)} &\leq \sum_{i\geq 0} \|u\|^q_{C(\overline{\mathcal{Q}^{(i)}})} \mu\big(\mathcal{Q}^{(i)}\big) \\ &\leq \Big(\sum_{i\geq 0} \mu\big(\mathcal{Q}^{(i)}\big)^{p/(p-q)}\Big)^{1-q/p} \Big(\sum_{i\geq 0} \|u\|^p_{C(\overline{\mathcal{Q}^{(i)}})}\Big)^{q/p}. \end{aligned} \tag{11.6.7}$$

Next we note that for $pl > n$

$$\|u\|_{C(\overline{\mathcal{Q}^{(i)}})} \leq \|u\eta_i\|_{S_p^l}, \tag{11.6.8}$$

where $S_p^l = h_p^l$, w_p^l or b_p^l (cf. (10.1.14) and items 6 and 9 of Theorem 10.1.3/4). Now (10.1.14), (11.6.7), and (11.6.8) imply the upper bound for $C_{p,q}$.

To obtain the lower bound for $C_{p,q}$ it suffices to insert the function

$$u_N(x) = \sum_{i=0}^N \mu\big(\mathcal{Q}^{(i)}\big)\eta_i(x), \quad N = 1, 2, \dots,$$

into (11.3.2). This concludes the proof. □

The constant $C_{p,q}$ can be easily calculated for $q = 1$.

Theorem 2. *Let S_p^l be either H_p^l or h_p^l. Then*

$$C_{p,1} = \|K\mu\|_{L_{p'}},$$

where $K\mu$ is either the Riesz or the Bessel potential.

Proof. Let $|u| \leq Kf$, $f \geq 0$ and $\|f\|_{L_p} = \|u\|_{S_p^l}$. We have

$$\int |u| \,\mathrm{d}\mu \leq \int fK\mu \,\mathrm{d}x \leq \|f\|_{L_p} \|K\mu\|_{L_{p'}},$$

which gives $C_{p,1} \leq \|K\mu\|_{L_{p'}}$. The reverse inequality follows by the substitution of $u = K(K\mu)^{1/(p-1)}$ into (11.3.2) with $q = 1$. □

11.7 Cartan-Type Theorem and Estimates for Capacities

In this section we establish the equivalence of inequalities of the type (11.1.6) and (11.1.7). This follows from a theorem giving an estimate for the set where the functions $\mathcal{W}_{p,l}\mu$ and $S_{p,l}\mu$, introduced in Sect. 10.4.2, majorize a given value. Such estimates were first obtained for harmonic functions by Cartan [172] (cf. also Nevanlinna [635]). For linear Riesz potentials they are given in Landkof [477].

The same scheme is used here for the nonlinear case.

Lemma. *Let $1 < p \le n/l$ and let μ be finite measure in $\mathbb{R}^n$. Let φ denote an increasing function on $[0, +\infty)$ with $\varphi(0) = 0$, $\varphi(r) = \varphi(r_0) = \mu(\mathbb{R}^n)$ for $r > r_0$. Further let D be the set $\{x \in \mathbb{R}^n : (P\mu)(x) > Y[\varphi]\}$, where $P\mu = \mathcal{W}_{p,l}\mu$ for $pl > n$, $P\mu = S_{p,l}\mu$ for $pl = n$, and*

$$Y[\varphi] = \begin{cases} \int_0^\infty (\frac{\varphi(r)}{r^{n-lp}})^{p-1} \frac{\mathrm{d}r}{r} & \text{for } 1 < p < n/l, \\ \int_0^\infty (\varphi(r))^{p'-1} \mathrm{e}^{-br} r^{-1}\,\mathrm{d}r & \text{for } p = n/l. \end{cases}$$

Then D can be the covered by a sequence of balls of radii $r_k \le r_0$ such that

$$\sum_k \varphi(r_k) \le c\mu(\mathbb{R}^n). \tag{11.7.1}$$

Proof. First, consider the case $1 < p < n/l$. Let $x \in D$. Suppose $\mu(B(x,r)) \le \varphi(r)$ for all $r > 0$. Then

$$(\mathcal{W}_{p,l}\mu)(x) = \int_0^\infty \left(\frac{\mu(B(x,r))}{r^{n-lp}}\right)^{p'-1} \frac{\mathrm{d}r}{r} \le \int_0^\infty \left(\frac{\varphi(r)}{r^{n-lp}}\right)^{p'-1} \frac{\mathrm{d}r}{r}.$$

The latter means that $x \notin D$. This contradiction shows that given any $x \in D$ there exists an $r = r(x) \in (0, r_0)$ such that $\varphi(r) < \mu(B(x,r)) \le \mu(\mathbb{R}^n)$. Applying Theorem 1.2.1/1 we select a covering $\{B(x_k, r_k)\}$, $k = 1, 2, \ldots$, of D with finite multiplicity $c = c(n)$ in the union of balls $\{B(x_k, r(x))\}$, $x \in D$. It is clear that

$$\sum_k \varphi(r_k) < \sum_k \mu\big(B(x_k, r_k)\big) \le c\mu\big(\mathbb{R}^n\big),$$

and the result follows for $1 < p < n/l$. For $p = n/l$ the proof is the same. □

In the next theorem we denote by Φ a nonnegative increasing function on $[0, +\infty)$ such that $t\Phi(t^{-1})$ decreases and tends to zero as $t \to \infty$. Further, for all $u > 0$, let

$$\int_u^{+\infty} \Psi(t) t^{-1}\,\mathrm{d}t \le c\Psi(u), \tag{11.7.2}$$

where

$$\Psi(v) = \begin{cases} (v\Phi(v^{-1}))^{p'-1} & \text{for } 1 < p < n/l, \\ v(\Phi(v^{1-p}))^{p'-1} & \text{for } p = n/l. \end{cases}$$

Theorem. *Let $p \in (1, n/l]$ and let μ be a finite measure in $\mathbb{R}^n$. Further let m be a positive number such that*

$$m^{p-1} > \mu(\mathbb{R}^n) \quad \textit{for } p = n/l.$$

Then the set $G = \{x \in \mathbb{R}^n : (P\mu)(x) > m\}$ can be covered by a sequence of balls $\{B(x_k, r_k)\}$ with

$$\sum_k \Phi\big(\mathrm{cap}\big(B_{r_k}, S_p^l\big)\big) < c\Phi\big(cm^{1-p}\mu\big(\mathbb{R}^n\big)\big). \tag{11.7.3}$$

Here $S_p^l = h_p^l$ for $lp < n$ and $S_p^l = H_p^l$ for $lp = n$.

Proof. Let $\varkappa = \mathrm{cap}(B_1, h_p^l)$ for $n > lp$. For $n = lp$ we define $\varkappa$ as

$$\varkappa = \min\big\{t : \mathrm{cap}\big(B_r, H_p^l\big) \le t|\log r|^{1-p},\ r < \mathrm{e}^{-1}\big\}.$$

Further let $Q = \mu(\mathbb{R}^n)$. In the Lemma we put $\varphi(r) = Q$ for $r > r_0$ and

$$\varphi(r) = \begin{cases} Q\Phi(\varkappa r^{n-lp})/\Phi(\varkappa r_0^{n-lp}) & \text{if } pl < n, r \le r_0, \\ Q\Phi(\varkappa|\log r|^{1-p})/\Phi(\varkappa|\log r_0|^{1-p}) & \text{if } pl = n, r \le r_0. \end{cases}$$

Here and henceforth r_0 is a number that will be specified later to satisfy the inequality $m > Y[\varphi]$ (the number $Y[\varphi]$ was defined in the Lemma).

1. Let $1 < p < n/l$. We have

$$Y[\varphi] = \int_0^{r_0} \left(\frac{\varphi(r)}{r^{n-lp}}\right)^{p'-1} \frac{\mathrm{d}r}{r} + Q^{p'-1}\frac{p-1}{n-lp} r_0^{(n-lp)/(1-p)}.$$

We show that the integral on the right does not exceed

$$cQ^{p'-1} r_0^{(n-lp)/(1-p)}.$$

This is equivalent to the inequality

$$\big(\Phi\big(\varkappa r_0^{n-lp}\big)\big)^{1-p'} \int_0^{r_0} \left(\frac{\Phi\big(\varkappa r^{n-lp}\big)}{r^{n-lp}}\right)^{p'-1} \frac{\mathrm{d}r}{r} \le c r_0^{(n-lp)/(1-p)}.$$

Putting $\varkappa r^{pl-n} = t$, $\varkappa r_0^{pl-n} = t_0$, we rewrite the latter as

$$\int_{t_0}^{\infty} \big(t\Phi\big(t^{-1}\big)\big)^{p'-1} t^{-1}\,\mathrm{d}t \le c\big(t_0\Phi\big(t_0^{-1}\big)\big)^{p'-1},$$

which is fulfilled by virtue of (11.7.2). Thus

$$Y[\varphi] < cQ^{p'-1} r_0^{(n-lp)/(1-p)},$$

and the inequality $Y[\varphi] < m$ is satisfied provided we put

$$r_0^{n-lp} = \left(cm^{-1}\right)^{p-1} Q.$$

We introduce the set $D = \{x \in \mathbb{R}^n : (P\mu)(x) > Y[\varphi]\}$, which is open by the lower semicontinuity of $P\mu$. Since $m > cY[\varphi]$ then $G \subset D$. Let $\{B(x_k, r_k)\}$ be the sequence of balls constructed in the Lemma for the set D by the function φ specified here. Inequality (11.7.1) can be rewritten as

$$\sum_k \Phi\left(\varkappa r_k^{n-lp}\right) \leq c\Phi\left(cm^{1-p}Q\right).$$

Thus we obtain the covering of G by balls $\{B(x_k, r_k)\}$ satisfying (11.7.3).

2. Let $p = n/l$ and let $r_0 < 1/e$. We have

$$Y[\varphi] = \int_0^{r_0} \left(\varphi(r)\right)^{p'-1} \mathrm{e}^{-cr} r^{-1}\, \mathrm{d}r + Q^{p'-1} \int_{r_0}^{\infty} \mathrm{e}^{-cr} r^{-1}\, \mathrm{d}r. \qquad (11.7.4)$$

The second integral is majorized by

$$\int_{r_0}^{\infty} \mathrm{e}^{-br} r^{-1}\, \mathrm{d}r < \int_{r_0}^{1} r^{-1}\, \mathrm{d}r + \int_1^{\infty} \mathrm{e}^{-cr}\, \mathrm{d}r \leq \left(1 + c^{-1}e^{-c}\right) |\log r_0|.$$

We show that the first integral on the right in (11.7.4) does not exceed $cQ^{p'-1} \times |\log r_0|$. In other words, we prove that

$$\left(\Phi\left(\varkappa |\log r_0|^{1-p}\right)\right)^{1-p'} \int_0^{r_0} \left(\Phi\left(\varkappa |\log r|^{1-p}\right)\right)^{p'-1} \frac{\mathrm{d}r}{r} < c|\log r_0|.$$

Putting $\varkappa|\log r| = t$, $\varkappa|\log r_0| = t_0$, we rewrite the preceding inequality as

$$\int_{t_0}^{\infty} \left(\Phi\left(t^{1-p}\right)\right)^{p'-1} \mathrm{d}t \leq ct_0 \left(\Phi\left(t_0^{1-p}\right)\right)^{p'-1},$$

which holds by (11.7.2). Therefore there exists a constant $c \in (1, \infty)$ such that

$$Y[\varphi] < cQ^{p'-1} |\log r_0|.$$

Thus the inequality $Y[\varphi] < m$ is satisfied provided we set

$$|\log r_0|^{1-p} = \left(cm^{-1}\right)^{p-1} Q.$$

The completion of the proof follows the same line of reasoning as for $p \in (1, n/l)$. □

Remark 1. The proof of the theorem shows that in the case $pl = n$ we can take the radii of the balls, covering G, to be less than $1/e$.

Corollary 1. *Let* $1 < p \le n/l$ *and let* Φ *be the function defined just before the last theorem. Further let* K *be a compactum in* $\mathbb{R}^n$ *with* $\operatorname{cap}(K, S_p^l) > 0$ *where* $S_p^l = h_p^l$ *for* $pl < n$ *and* $S_p^l = H_p^l$ *for* $pl = n$*. Then there exists a covering of* K *by balls* $B(x_k, r_k)$ *such that*

$$\sum_k \Phi\big(\operatorname{cap}\big(B_{r_k}, S_p^l\big)\big) < c\Phi\big(c \operatorname{cap}\big(K, S_p^l\big)\big), \tag{11.7.5}$$

where c *is a constant that depends on* n*,* p*,* l*, and on the function* Φ*. In the case* $pl = n$ *we may assume that* $r_k \le \mathrm{e}^{-1}$*.*

Proof. We limit consideration to the case $pl < n$. For $pl = n$ the argument is the same. We need nonlinear potentials $\mathcal{V}_{p,l}\mu$ defined by (4.4.4) in the book by D.R. Adams and Hedberg [15]. (They used the notation $\mathcal{V}^{\mu}_{l,p}$.) These potentials are comparable with $W_{p,l}\mu$ defined in Sect. 10.4 (see (4.5.3) in [15]).

We put

$$C(K) = \inf\bigg\{ \int \mathcal{V}_{p,l}\mu \, \mathrm{d}\mu : \mathcal{V}_{p,l}\mu \ge 1 \ (p,l)\text{-quasi everywhere on } K \bigg\}.$$

By (10.4.8) the capacities $C(K)$ and $\operatorname{cap}(K, h_p^l)$ are equivalent. In the paper by Hedberg and Wolff [372] it was shown that the extremal measure μ_K for the previous variational problem exists and that $C(K) = \mu_K(K)$. We introduce the set $G_\varepsilon = \{x \in \mathbb{R}^n : \mathcal{V}_{p,l}\mu_K(x) \ge 1 - \varepsilon\}$, where $\varepsilon > 0$. Since $\mathcal{V}_{p,l}\mu_K(x) \ge 1$ for (p,l)-quasi-every $x \in K$, then $E \subset G_\varepsilon \cup E_0$ where $\operatorname{cap}(E_0, h_p^l) = 0$.

By the Theorem there exists a covering of G_ε by balls $B(x_j, r_j)$ for which (11.7.3) is valid with $m = 1 - \varepsilon$ and $\mu(\mathbb{R}^n) = \operatorname{cap}(K, h_p^l)$. Since $\Psi(t)/t$ is integrable on $[1, +\infty)$, the function $\varphi(r) = \Phi(\operatorname{cap}(B_r, h_p^l))$ satisfies (10.4.16). This and Proposition 10.4.3/2 imply that the set E_0 has zero Hausdorff φ-measure. Therefore, E_0 can be covered by balls $B(y_i, \varrho_i)$ so that

$$\sum_i \Phi\big(\operatorname{cap}\big(B_{\varrho_i}, h_p^l\big)\big) < \varepsilon.$$

The balls $B(x_j, r_j)$ and $B(y_i, \varrho_i)$ form the required covering. □

Corollary 2. *Let* $p \in (1, n/l]$ *and let* $S_p^l = h_p^l$ *for* $lp < n$*,* $S_p^l = H_p^l$ *for* $lp = n$*. Further, let* Φ *be the function defined just before the Theorem. If a measure* μ *is such that*

$$\mu\big(B(x, \varrho)\big) \le \Phi\big(c \operatorname{cap}\big(B_\varrho, S_p^l\big)\big), \tag{11.7.6}$$

then, for any Borel set E *with the finite capacity* $\operatorname{cap}(E, S_p^l)$*, the inequality*

$$\mu(E) \le c\Phi\big(c \operatorname{cap}\big(E, S_p^l\big)\big), \tag{11.7.7}$$

where c *is a constant that depends on* n*,* p*,* l*, and* Φ *is valid.*

Proof. It suffices to derive (11.7.7) for a compactum E. According to Corollary 1 there exists a covering of E by balls $B(x_k, r_k)$ satisfying (11.7.5). Using the additivity of μ as well as estimate (11.7.6), we obtain

$$\mu(E) \le \mu\Big(\bigcup_k B(x_k, r_k)\Big) \le \sum_k \mu\big(B(x_k, r_k)\big)$$
$$\le \sum_k \Phi\big(c \operatorname{cap}\big(B_{r_k}, S_p^l\big)\big) < c\Phi\big(c \operatorname{cap}\big(E, S_p^l\big)\big).$$

The result follows. □

Remark 2. According to (10.4.2) we have the equivalence $\operatorname{cap}(E, H_p^l) \sim \operatorname{cap}(E, h_p^l)$ if $\operatorname{diam} E \le 1$. Therefore under the additional requirement $\operatorname{diam} E \le 1$ we may also put $S_p^l = H_p^l$ in Corollary 2 for $pl < n$.

To prove this assertion we need to verify that the measure $\mathbb{R}^n \supset A \to \mu_1(A) = \mu(A \cap E)$ satisfies (11.7.6).

Let $\operatorname{diam} E \le 1$ and, for all $r \in (0, 1)$, let

$$\mu\big(B(x, r)\big) \le \Phi\big(\operatorname{cap}\big(B_r, H_p^l\big)\big). \tag{11.7.8}$$

For $r < 1$ we have

$$\mu_1\big(B(x, r)\big) = \mu\big(B(x, r) \cap E\big) \le \mu\big(B(x, r)\big)$$
$$\le \Phi\big(\operatorname{cap}\big(B_r, H_p^l\big)\big) \le \Phi\big(c \operatorname{cap}\big(B_r, h_p^l\big)\big).$$

In the case $r \ge 1$

$$\mu_1\big(B(x, r)\big) \le \mu\big(B(y, 1)\big)$$

for any $y \in E$. Hence, using (11.7.8) and the monotonicity of the capacity, we obtain

$$\mu_1\big(B(x, r)\big) \le \Phi\big(\operatorname{cap}\big(B_1, H_p^l\big)\big) \le \Phi\big(c \operatorname{cap}\big(B_r, h_p^l\big)\big).$$

Thus the measure μ_1 satisfies (11.7.6).

11.8 Embedding Theorems for the Space S_p^l (Conditions in Terms of Balls, $p > 1$)

Theorem. *Let M be a convex function and let N be the complementary function of M. Further let Φ be the inverse function of $t \to tN^{-1}(1/t)$ subject to condition (11.7.2). Then:*

(α) *The best constant A in (11.3.1) with $S_p^l = h_p^l$, $lp < n$, is equivalent to*

$$C_1 = \sup\big\{\varrho^{lp-n} \mu\big(B(x, \varrho)\big) N^{-1}\big(1/\mu\big(B(x, \varrho)\big)\big) : x \in \mathbb{R}^n, \varrho > 0\big\}.$$

(β) *The best constant A in (11.3.1) with $S_p^l = H_p^l$ is equivalent to*

$$C_2 = \sup\{\varrho^{lp-n}\mu(B(x,\varrho))N^{-1}(1/\mu(B(x,\varrho))) : x \in \mathbb{R}^n,\ 0<\varrho<1\}$$

if $pl < n$ *and to*

$$C_3 = \sup\left\{|\log\varrho|^{p-1}\mu(B(x,\varrho))N^{-1}(1/\mu(B(x,\varrho))) : x \in \mathbb{R}^n,\ 0<\varrho<\frac{1}{2}\right\}$$

if $pl = n$.

The proof immediately follows from Theorem 11.3 and the equivalence $B \sim C_j, j = 1,2,3$, obtained in Corollary 11.7/2 and Remark 11.7/2.

Remark 1. We can easily see that in the case $pl > n$ the constant A in (11.3.1) with $S^l_p = H^l_p$ is equivalent to

$$C_4 = \sup\{\mu(B(x,1))N^{-1}(1/\mu(B(x,1))) : x \in \mathbb{R}^n\}.$$

Indeed, let $\{\eta^{(j)}\}$ be a partition of unity subordinate to a covering of $\mathbb{R}^n$ by unit balls $\{\mathcal{B}^{(j)}\}$ with finite multiplicity. From the definition of the norm in $L_M(\mu)$ and the Sobolev theorem on embedding H^l_p into L_∞ we obtain

$$\begin{aligned}\||u|^p\|_{L_M(\mu)} &\le c\sum_j \||u\eta^{(j)}|^p\|_{L_M(\mu)} \\ &\le c_1\sum_j \|\chi(\cdot,\mathcal{B}^{(j)})\|_{L_M(\mu)}\|u\eta^{(j)}\|^p_{H^l_p} \le c_1 C_4\sum_j\|u\eta^{(j)}\|^p_{H^l_p}.\end{aligned}$$

The last sum does not exceed $c\|u\|^p_{H^l_p}$ (cf. Theorem 10.1.3/3). Hence $A \le c_2 C_4$. The opposite estimate follows from (11.3.1) by the substitution of the function $\eta \in C^\infty_0(B(x,2))$, $\eta = 1$ on $B(x,1)$.

Now the D.R. Adams Theorem 1.4.1 follows from (α) of the previous theorem where $M(t) = t^{q/p}$, $q > p$.

Remark 2. We show that the condition (11.1.5) with $s = n - pl$ is not sufficient for (11.1.2) to hold in the case $q = p$. Let $q = p$, $n > pl$. We choose a Borel set E with finite positive $(n-pl)$-dimensional Hausdorff measure. We can take E to be closed and bounded (since any Borel set of positive Hausdorff measure contains a bounded subset having the same property). By the Frostman theorem (see Carleson [168], Theorem 1, Ch. 2) there exists a measure $\mu \ne 0$ with support in E such that

$$\mu(B(x,\varrho)) \le c\varrho^{n-pl}, \tag{11.8.1}$$

where c is a constant that is independent of x and ϱ. By Proposition 10.4.3/3, $\operatorname{cap}(E, H^l_p) = 0$. On the other hand, from (11.1.2) it follows that $\mu(E) \le A\operatorname{cap}(E, H^l_p)$ and hence $\mu(E) = 0$. This contradiction shows that (11.1.2) fails although (11.8.1) holds.

Setting $M(t) = t^{q/p}$ in the previous theorem, we obtain the following result for the case $lp = n$.

Corollary 1. *If $lp = n$, $q > p > 1$ then the exact constant A in*

$$\|u\|_{L_q(\mu)} \le A\|u\|_{H_p^l} \tag{11.8.2}$$

is equivalent to

$$C_5 = \sup\Big\{|\log\varrho|^{p-1}\big[\mu\big(B(x,\varrho)\big)\big]^{p/q} : x \in \mathbb{R}^n, 0 < \varrho < \frac{1}{2}\Big\}.$$

From the theorem of the present section we easily obtain the following assertion relating the case $lp = n$, $M(t) = \exp(t^{p'-1}) - 1$ and measure μ of "power type".

Corollary 2. *The inequality*

$$\big\||u|^p\big\|_{L_{\exp(t^{p'-1}-1)}} \le A\|u\|^p_{S_p^l}$$

holds if and only if for $0 < \varrho \le 1$

$$\mu\big(B(x,\varrho)\big) \le c\varrho^\beta,$$

with a certain $\beta > 0$.

Proof. Since $N'(t) = (\log t)^{p-1}(1 + o(1))$ as $t \to \infty$ we have $\Phi^{-1}(t) = tN^{-1}(1/t) = (\log t)^{1-p}(1 + o(1))$. Hence, $\log\Phi(t) = -t^{p'-1}(1 + o(1))$. Obviously, Φ satisfies the condition (11.7.2). Now it remains to use $\text{cap}(B_\varrho, H_p^l) \sim |\log\varrho|^{1-p}$ with $\varrho \in (0, \frac{1}{2})$ and to apply the Theorem. The proof is complete. □

Remark 3. Since $B_p^l(\mathbb{R}^n)$ is the space of traces on $\mathbb{R}^n$ of functions in $H_p^{l+1/p}(\mathbb{R}^{n+1})$, the Theorem and Corollary 1 still hold if the space $H_p^l(\mathbb{R}^n)$ is replaced by $B_p^l(\mathbb{R}^n)$.

Remark 4. We can obtain assertions similar to the Theorem and Corollaries 1 and 2 by replacing u by $\nabla_k u$ in the left-hand sides of inequalities (11.3.1) and (11.8.2). For example, the generalization of Corollary 1 runs as follows.

If $(l - k)p = n$, $q > p > 1$ then the best constant in

$$\|\nabla_k u\|_{L_q(\mu)} \le A\|u\|_{H_p^l} \tag{11.8.3}$$

is equivalent to C_5.

The estimate $A \le cC_5$ needs no additional arguments. To prove the reverse inequality we place the origin at an arbitrary point of the space and put

$$u(x) = x_1^k \zeta\Big(\frac{\log|x|}{\log\varrho}\Big),$$

where $\varrho \in (0, \frac{1}{2})$ and $\zeta \in C^\infty(\mathbb{R}^1)$, $\zeta(t) = 1$ for $t > 1$, $\zeta(t) = 0$ for $t < \frac{1}{2}$ into (11.8.3). It is clear that supp $u \subset B_{\varrho^{1/2}}$ Using standard but rather cumbersome estimates we can show that, for $|x| < \frac{1}{2}$,

$$(D_l u)(x) \leq c|x|^{-n/p}\big|\log|x|\big|^{-1},$$

where $D_l u$ is the Strichartz function (10.1.10) for $\{l\} > 0$ and $D_l u = |\nabla_l u|$ for $\{l\} = 0$. Hence from Theorem 10.1.2/4 we have

$$\|u\|^p_{H^l_p} \leq c|\log \varrho|^{1-p}.$$

On the other hand,

$$\|\nabla_k u\|^q_{L_q(\mu)} \geq k!\mu(B_\varrho).$$

Consequently, $A \geq cC_5$.

11.9 Applications

11.9.1 Compactness Criteria

The theorems proved in the present chapter imply necessary and sufficient conditions for compactness of embedding operators of the spaces H^l_p, h^l_p, W^l_p, w^l_p, B^l_p, and b^l_p into $L_q(\mu)$. The proof of these results follows by standard arguments (compare with Theorems 2.4.2/1 and 2.4.2/2), so we restrict ourselves to the next four statements. The first two theorems are based on Corollary 11.3.

Theorem 1. *Let $p > 1$, $pl < n$ and let s^l_p be any one of the spaces h^l_p, w^l_p, b^l_p. Any set of functions in C^∞_0, bounded in s^l_p, is relatively compact in $L_q(\mu)$ if and only if*

$$\lim_{\delta\to 0} \sup\left\{\frac{\mu(e)}{\operatorname{cap}(e, s^l_p)} : e \subset \mathbb{R}^n,\ \operatorname{diam} e \leq \delta\right\} = 0, \tag{11.9.1}$$

$$\lim_{\varrho\to\infty} \sup\left\{\frac{\mu(e)}{\operatorname{cap}(e, s^l_p)} : e \subset \mathbb{R}^n\backslash B_\varrho\right\} = 0, \tag{11.9.2}$$

where $B_\varrho = \{x : |x| < \varrho\}$.

Theorem 2. *Let $p > 1$, $pl \leq n$, and let S^l_p be any of the spaces H^l_p, W^l_p, and B^l_p. A set of functions in C^∞_0, bounded in S^l_p, is relatively compact in $L_q(\mu)$ if and only if condition (11.9.1) holds and*

$$\lim_{\varrho\to\infty} \sup\left\{\frac{\mu(e)}{\operatorname{cap}(e, S^l_p)} : e \subset \mathbb{R}^n\backslash B_\varrho,\ \operatorname{diam} e \leq 1\right\} = 0. \tag{11.9.3}$$

Theorems 3 and 4 below follow from Theorem 11.8 and Corollary 11.8/1, respectively.

Theorem 3. *Let* $p \geq 1$, $l > 0$, $pl < n$. *Further let* $1 \leq q < \infty$ *if* $p = 1$ *and* $p < q < \infty$ *if* $p > 1$. *Then the set* $\{u \in C_0^\infty : \|u\|_{W_p^l} \leq 1\}$ *is relatively compact in* $L_q(\mu)$ *if and only if*

$$\text{(i)} \quad \lim_{\delta \to +0} \sup_{x;\varrho \in (0,\delta)} \varrho^{l-n/p} \big[\mu\big(B(x,\varrho)\big)\big]^{1/q} = 0,$$

$$\text{(ii)} \quad \lim_{|x| \to \infty} \sup_{\varrho \in (0,1)} \varrho^{l-n/p} \big[\mu\big(B(x,\varrho)\big)\big]^{1/q} = 0.$$

Theorem 4. *Let* $p > 1$, $l > 0$, $pl = n$ *and* $q > p$. *Then the set* $\{u \in C_0^\infty : \|u\|_{W_p^l} \leq 1\}$ *is relatively compact in* $L_q(\mu)$ *if and only if*

$$\text{(i)} \quad \lim_{\delta \to 0} \sup_{x;\varrho \in (0,\delta)} |\log \varrho|^{1-1/p} \big[\mu\big(B(x,\varrho)\big)\big]^{1/q} = 0,$$

$$\text{(ii)} \quad \lim_{|x| \to \infty} \sup_{2\varrho < 1} |\log \varrho|^{1-1/p} \big[\mu\big(B(x,\varrho)\big)\big]^{1/q} = 0.$$

11.9.2 Equivalence of Continuity and Compactness of the Embedding $H_p^l \subset L_q(\mu)$ for $p > q$

Here we prove the following assertion.

Theorem. *Let* $1 < p < \infty$, $0 < q < p$ *and* $l > 0$. *Suppose that* μ *is a finite Borel measure on* $\mathbb{R}^n$. *Then the compactness of the trace operator* $H_p^l \to L_q(\mu)$ *is equivalent to its continuity and to the inequality* $D_{p,q} < \infty$, *where* $D_{p,q}$ *is defined in Theorem* 11.6.1/1.

This theorem will be obtained as a consequence of the compactness properties of the convolution maps which are studied in the present subsection. We consider integral convolution operators of the form $f \mapsto K * f$, where

$$(K * f)(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, \mathrm{d}y$$

and K is the kernel of convolution. In the following we abbreviate $\|\cdot\|_{L_p(\mathbb{R}^n)} = \|\cdot\|_p$, $L_p(\mathbb{R}^n) = L_p$. It will be shown that under some conditions on K and μ the continuity of the convolution map $L_p \to L_q(\mu)$ implies its compactness.

Lemma 1. *Let* $p \in (1,\infty)$ *and* $K \in L_{p'}, p' = p/(p-1)$. *Suppose that* μ *is a finite Borel measure on* $\mathbb{R}^n$ *having bounded support. Then the operator* $L_p \ni f \mapsto K * f \in L_q(\mu)$ *is compact for any* $q \in (0,\infty)$.

Proof. An application of Hölder's inequality yields

$$\|u\|_\infty \le \|K\|_{p'}\|f\|_p, \quad u = K * f.$$

Furthermore

$$\big|u(x+h) - u(x)\big| \le \big\|K(\cdot + h) - K(\cdot)\big\|_{p'}\|f\|_p$$

for any $x, h \in \mathbb{R}^n$. Hence it follows that the elements of the set $\{K * f : \|f\|_p \le 1\}$ are uniformly bounded and equicontinuous on $\mathbb{R}^n$. Since the support of μ is bounded, the required result is a consequence of the Arzela–Ascoli compactness theorem. □

We continue the analysis of compactness properties of convolution operators with the following lemma.

Lemma 2. *Let K be a nonnegative function in L_1 whose support has a finite volume. Suppose μ is a finite Borel measure on $\mathbb{R}^n$ with bounded support. If $p \in (1, \infty), 0 < q < p$, then the continuity of convolution operator $L_p \ni f \mapsto K * f \in L_q(\mu)$ implies its compactness.*

Proof. Given $\varepsilon > 0$, put $K_\varepsilon(x) = K(x)$ if $K(x) \le \varepsilon^{-1}$ and $K_\varepsilon(x) = 0$ otherwise. Let $K^{(\varepsilon)} = K - K_\varepsilon$. According to Lemma 1, the operator $f \mapsto K_\varepsilon * f$ is compact as an operator $L_p \to L_q(\mu)$.

Suppose the map $f \mapsto K * f \in L_q(\mu)$ is not compact. Then the norm of the operator

$$f \mapsto \mathcal{K}_\varepsilon f = K^{(\varepsilon)} * f \in L_q(\mu)$$

is bounded below uniformly with respect to $\varepsilon \in (0, \infty)$. Thus, there is a constant $c_0 > 0$ such that for any $\varepsilon > 0$ there exists a function $f^{(\varepsilon)} \in L_p$ satisfying

$$\big\|f^{(\varepsilon)}\big\|_p \le 1, \qquad \big\|K^{(\varepsilon)} * f^{(\varepsilon)}\big\|_{L_q(\mu)} \ge c_0.$$

Since the operator $\mathcal{K}_\varepsilon : L_p \to L_q(\mu)$ is continuous, we can assume without loss of generality that $f^{(\varepsilon)}$ is a nonnegative continuous function on $\mathbb{R}^n$ with compact support.

Put $\varepsilon_1 = 1, f_1 = f^{(\varepsilon_1)}$. Since $\lim_{\delta \to +0} \|K^{(\delta)}\|_1 = 0$ and

$$\sup_{x \in \mathbb{R}^n} \big|\big(K^{(\delta)} * f_1\big)(x)\big| \le \big\|K^{(\delta)}\big\|_1 \|f_1\|_\infty,$$

it follows that $\|K^{(\delta)} * f_1\|_{L_q(\mu)} \to 0$ as $\delta \to 0$. Hence there is a number $\varepsilon_2 \in (0, \varepsilon_1)$ for which

$$\big\|\big(K^{(\varepsilon_1)} - K^{(\varepsilon_2)}\big) * f_1\big\|_{L_q(\mu)} \ge c_0/2.$$

Letting $f_2 = f^{(\varepsilon_2)}$ and arguing as before, one can select $\varepsilon_3 \in (0, \varepsilon_2)$ subject to

$$\big\|\big(K^{(\varepsilon_2)} - K^{(\varepsilon_3)}\big) * f_2\big\|_{L_q(\mu)} \ge c_0/2.$$

Continuing this process, we construct a decreasing sequence $\{\varepsilon_i\}_{i\geq 1}$ of positive numbers and a sequence $\{f_i\}_{i\geq 1} \subset C_0(\mathbb{R}^n)$ such that $\|f_i\|_p \leq 1$, $f_i \geq 0$ and

$$\|h_i\|_{L_q(\mu)} \geq c_0/2, \quad i = 1, 2, \dots,$$

where $h_i = (K^{(\varepsilon_i)} - K^{(\varepsilon_{i+1})}) * f_i \geq 0$. For $m = 1, 2, \dots$ define

$$g_m(x) = m^{-1/p} \max_{1\leq i\leq m} f_i(x).$$

Then

$$\|g_m\|_p \leq m^{-1/p} \left\| \left(\sum_{i=1}^{m} f_i^p \right)^{1/p} \right\|_p \leq 1$$

and also

$$\|K * g_m\|_{L_q(\mu)} \geq m^{-1/p} \left\| \sum_{i=1}^{m} h_i \right\|_{L_q(\mu)}.$$

Since

$$\left(\sum_{i=1}^{m} h_i \right)^q \geq \min\{1, m^{q-1}\} \sum_{i=1}^{m} h_i^q, \quad q > 0,$$

we have

$$\|K * g_m\|_{L_q(\mu)} \geq m^{-1/p} \min\{1, m^{1-1/q}\} \left(\sum_{i=1}^{m} \|h_i\|^q_{L_q(\mu)} \right)^{1/q}$$

and hence

$$\|K * g_m\|_{L_q(\mu)} \geq \frac{c_0}{2} \min\{m^{1/q-1/p}, m^{1-1/p}\}.$$

However, the expression on the right is unbounded as $m \to \infty$, and this contradicts the continuity of the operator $f \mapsto K * f \in L_q(\mu)$. Lemma 2 is proved. □

Proof of Theorem. Let B_k be the ball of radius k centered at the origin, $k = 1, 2, \dots$. Let μ_k denote the restriction of μ to B_k. Clearly, for any $l > 0$ the Bessel kernel can be expressed as the sum of two nonnegative kernels, one of which has compact support and belongs to L_1 while the other is in L_r for all $r > 0$. According to the previous lemmas any subset of $C_0^\infty(\mathbb{R}^n)$, bounded in H_p^l, is relatively compact in $L_q(\mu_k)$. Furthermore, by Corollary 11.6.1

$$\left(\int_{\mathbb{R}^n\setminus B_k} |u|^q \,\mathrm{d}\mu \right)^{1/q} \leq c(p,q) \left(\int_0^{\mu(\mathbb{R}^n\setminus B_k)} \left(\frac{t}{\varkappa(t)} \right)^{q/(p-q)} \mathrm{d}t \right)^{1/q-1/p} \|u\|_{H_p^l}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. Thus for the same u we have

$$\|u\|_{L_q(\mu)} \leq \delta_k \|u\|_{H_p^l} + \mathrm{const} \|u\|_{L_q(\mu_k)}, \quad k = 1, 2, \dots,$$

where δ_k is a sequence of positive numbers converging to zero. By using the last estimate and a diagonal method, one can select from any sequence in $C_0^\infty(\mathbb{R}^n)$, bounded in H_p^l, a subsequence convergent in $L_q(\mu)$. □

11.9.3 Applications to the Theory of Elliptic Operators

Corollary 11.3 has immediate applications to the spectral theory of elliptic operators. Let

$$S_h = h(-\Delta)^l - p(x), \quad x \in \mathbb{R}^n,$$

where $p \geq 0$ and h is a positive number. The results in Sect. 2.5 concerning the Schrödinger operator $-h\Delta - p(x)$ have natural analogs for the operators S_h. Duplicating with minor modifications the proofs of Theorems 2.5.3–2.5.6 and other assertions in Sect. 2.5, we can derive conditions for the semiboundedness of the operator S_h as well as conditions for the discreteness, finiteness, or infiniteness of the negative part of its spectrum. Here we present the statements of two typical theorems of this kind.

Theorem 1. *Let $n > 2l$ and let $\mathfrak{M}$ denote the constant in the rough maximum principle for the Riesz potential of order $2l$.*

1. *If*

$$\lim_{\delta\to 0} \sup\left\{\frac{\int_e p(x)\,\mathrm{d}x}{\operatorname{cap}(e, h_2^l)} : \operatorname{diam} e \leq \delta\right\} < (4\mathfrak{M})^{-1},$$

then the operator S_1 is semibounded.

2. *If the operator S_1 is semibounded, then*

$$\lim_{\delta\to 0} \sup\left\{\frac{\int_e p(x)\,\mathrm{d}x}{\operatorname{cap}(e, h_2^l)} : \operatorname{diam} e \leq \delta\right\} \leq 1.$$

Theorem 2. *Let $n > 2l$. The conditions*

$$\lim_{\delta\to 0} \sup\left\{\frac{\int_e p(x)\,\mathrm{d}x}{\operatorname{cap}(e, h_2^l)} : \operatorname{diam} e \leq \delta\right\} = 0,$$

$$\lim_{\varrho\to \infty} \sup\left\{\frac{\int_e p(x)\,\mathrm{d}x}{\operatorname{cap}(e, h_2^l)} : e \subset \mathbb{R}^n \backslash B_\varrho, \operatorname{diam} e \leq 1\right\} = 0,$$

are necessary and sufficient for the semiboundedness of the operator S_h and for the discreteness of its negative spectrum for all $h > 0$.

11.9.4 Criteria for the Rellich–Kato Inequality

By the basic Rellich–Kato theorem [673], [415], the self-adjointness of $-\Delta + V$ in $L_2(\mathbb{R}^n)$ is guaranteed by the inequality

$$\|Vu\|_{L_2(\mathbb{R}^n)} \leq a\|\Delta u\|_{L_2(\mathbb{R}^n)} + b\|u\|_{L_2(\mathbb{R}^n)}, \tag{11.9.4}$$

where $a < 1$ and u is an arbitrary function in $C_0^\infty(\mathbb{R}^n)$.

Let $n \leq 3$. The Sobolev embedding $W_2^2(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n)$ and an appropriate choice of the test function in (11.9.4) show that (11.9.4) holds if and only if there is a sufficiently small constant $c(n)$ such that

$$\sup_{x\in\mathbb{R}^n}\int_{B_1(x)}\bigl|V(y)\bigr|^2\,\mathrm{d}y\le c(n). \tag{11.9.5}$$

Here and elsewhere $B_r(x)=\{y\in\mathbb{R}^n:|y-x|<r\}$. This class of potentials is known as Stummel's class S_n, which is defined also for higher dimensions by

$$\lim_{r\downarrow 0}\biggl(\sup_x\int_{B_r(x)}|x-y|^{4-n}\bigl|V(y)\bigr|^2\,\mathrm{d}y\biggr)=0\quad\text{for } n\ge 5,$$

$$\lim_{r\downarrow 0}\biggl(\sup_x\int_{B_r(x)}\log\bigl(|x-y|^{-1}\bigr)\bigl|V(y)\bigr|^2\,\mathrm{d}y\biggr)=0\quad\text{for } n=4$$

(see Stummel [731]). Although the condition $V\in S_n$ is sufficient for (11.9.4) for every n, it does not seem quite natural for $n\ge 4$. As a matter of fact, it excludes the simple potential $V(x)=c|x|^{-2}$ obviously satisfying (11.9.4) if the factor c is small enough.

If $n\ge 5$, a characterization of (11.9.4) (modulo best constants) results directly from a necessary and sufficient condition for the inequality

$$\|Vu\|_{L_2(\mathbb{R}^n)}\le C\|\Delta u\|_{L_2(\mathbb{R}^n)},\quad u\in C_0^\infty\bigl(\mathbb{R}^n\bigr),$$

contained in Theorem 11.9.3/1: *The Rellich-Kato condition* (11.9.4) *holds in the case $n\ge 4$ if and only if there is a sufficiently small $c(n)$ subject to*

$$\sup_{\{e:\operatorname{diam} e\le 1\}}\frac{\int_F|V(y)|^2\,\mathrm{d}y}{\operatorname{cap}(e,h_2^2)}\le c(n) \tag{11.9.6}$$

(*the values of $c(n)$ in the sufficiency and necessity parts are different*). *For $2m=n$ one should add $\|u\|^2_{L_2(\mathbb{R}^n)}$ to the last integral.*

The condition

$$\sup_{\{e:\operatorname{diam} e\le \delta\}}\frac{\int_F|V(y)|^2\,\mathrm{d}y}{\operatorname{cap}(e,h_2^2)}\to 0\quad\text{as }\delta\to 0 \tag{11.9.7}$$

is necessary and sufficient for (11.9.4) *to hold with an arbitrary a and $b=b(a)$.*

An obvious necessary condition for (11.9.4) is

$$\begin{cases}\sup_{r\le 1,x\in\mathbb{R}^n} r^{4-n}\int_{B_r(x)}|V(y)|^2\,\mathrm{d}y\le c(n) & \text{for } n\ge 5,\\ \sup_{r\le 1,x\in\mathbb{R}^n}(\log\frac{2}{r})^{-1}\int_{B_r(x)}|V(y)|^2\,\mathrm{d}y\le c(n) & \text{for } n=4,\end{cases} \tag{11.9.8}$$

where $c(n)$ is sufficiently small. Standard lower estimates of cap_2 by m_n combined with the criterion (11.9.6) give the sufficient condition

$$\begin{cases}\sup_{\{e:\operatorname{diam} e\le 1\}}(m_n e)^{(4-n)/n}\int_e|V(y)|^2\,\mathrm{d}y\le c(n) & \text{for } n\ge 5,\\ \sup_{\{e:\operatorname{diam} e\le 1\}}(\log\frac{v_n}{m_n e})^{-1}\int_e|V(y)|^2\,\mathrm{d}y\le c(n) & \text{for } n=4.\end{cases} \tag{11.9.9}$$

Though sharp and looking similar, (11.9.8) and (11.9.9) are not equivalent.

Finally, we observe that *the inequality*

$$\|Vu\|_{L_2(\mathbb{R}^n)} \le C\|\Delta u\|^{\tau}_{L_2(\mathbb{R}^n)}\|u\|^{1-\tau}_{L_2(\mathbb{R}^n)}$$

holds for a certain $\tau \in (0,1)$ *and every* $u \in C_0^\infty(\mathbb{R}^n)$ *if and only if for all* $r \in (0,1)$

$$\sup_x \int_{B_r(x)} \left|V(y)\right|^2 \mathrm{d}y \le Cr^{n-4\tau}$$

(see Theorem 1.4.7).

11.10 Embedding Theorems for $p = 1$

11.10.1 Integrability with Respect to a Measure

The aim of the present subsection is to prove the following theorem which complements Theorem 1.4.3.

Theorem 1. *Let* k *be a nonnegative integer,* $0 < l - k \le n$, $1 \le q < \infty$. *Then the best constant* A *in*

$$\|\nabla_k u\|_{L_q(\mathbb{R}^n,\mu)} \le A\|u\|_{b_1^l} \tag{11.10.1}$$

is equivalent to

$$K = \sup_{x,\varrho>0} \varrho^{l-k-n}\mu\big(B(x,\varrho)\big)^{1/q}.$$

Proof. (α) We show that $A \ge cK$. We put $u(\xi) = (x_1 - \xi_1)^k\varphi(\varrho^{-1}(x-\xi))$, where $\varphi \in C_0^\infty(B_2)$, $\varphi = 1$ on B_1, into (11.10.1). Since

$$\|\nabla_k u\|^q_{L_q(\mathbb{R}^n,\mu)} \ge k!\mu\big(B(x,\varrho)\big), \tag{11.10.2}$$

$$\|u\|_{b_1^l} = c\varrho^{n-l+k}, \tag{11.10.3}$$

then $A \ge cK$.

(β) We prove that $A \le cK$. Let $q > 1$. By Theorem 11.8 and Remark 11.8/2 we have

$$\|\nabla_k u\|_{L_q(\mathbb{R}^n,\mu)} \le c \sup_{x,\varrho} \frac{\mu(B(x,\varrho))^{1/q}}{\varrho^{k-(l-n+n/t)+n/t}}\|u\|_{b_t^{l-n-n/t}},$$

where t is a number sufficiently close to unity, $t > 1$. It remains to apply items (iii) and (iv) of Theorem 10.1.3/4.

Next we show that $A \le cK$ for $q = 1$. It suffices to consider the case $k = 0$. Let $l \in (0,1)$. According to Corollary 2.1.6,

$$\|u\|_{L_1(\mathbb{R}^n,\mu)} \le cK \int_{\mathbb{R}^{n+1}} |y|^{-l}\big|\nabla U(z)\big|\,\mathrm{d}z,$$

where $U \in C_0^\infty(\mathbb{R}^{n+1})$ is an arbitrary extension of a function u to $\mathbb{R}^{n+1}$. Taking into account the relation

$$\|u\|_{b_1^l} \sim \inf_U \int_{\mathbb{R}^{n+1}} |y|^{-l} \big|\nabla U(z)\big| \,\mathrm{d}z$$

contained in Theorem 10.1.1/1, we arrive at $A \leq cK$.

If $l = 1$, then by Theorem 1.4.3

$$\|u\|_{L_1(\mathbb{R}^n,\mu)} \leq cK \|\nabla_2 U(z)\|_{L(\mathbb{R}^{n+1})}.$$

Minimizing the right-hand side over all U we conclude that $A \leq cK$ for the space b_1^1.

Suppose the estimate $A \leq cK$ is established for $l \in (N-2, N-1)$, where N is an integer $N \geq 2$. We prove it for all $l \in (N-1, N]$. We have

$$\int |u| \,\mathrm{d}\mu(x) = c \int \left| \int \frac{(\xi - x)\nabla_\xi u(\xi)}{|\xi - x|^n} \,\mathrm{d}\xi \right| \mathrm{d}\mu(x) \leq c \int |\nabla u| I_1 \mu \,\mathrm{d}x,$$

where $I_1\mu = |x|^{1-n} * \mu$. By the induction hypothesis, the latter integral does not exceed

$$c \sup_{x,r} \left(r^{l-1-n} \int_{B(x,r)} I_1 \mu(\xi) \,\mathrm{d}\xi \right) \|\nabla u\|_{b_1^{l-1}}.$$

By Lemma 1.4.3 with $q = 1$ the last supremum is majorized by cK. The theorem is proved. $\square$

Remark 1. We substitute the function u defined by $u(x) = \eta(x/\varrho)$ where $\eta \in C_0^\infty(\mathbb{R}^n)$, $\varrho > 0$, into (11.10.1). Let $\varrho \to \infty$. Then (11.10.1) is not fulfilled for $l - k > n$ provided $\mu \neq 0$.

For $l - k = n$, $q < \infty$ inequality (11.10.1) holds if and only if $\mu(\mathbb{R}^n) < \infty$.

Theorem 2. *Let $0 < k < l$, $l - k \leq n$, $1 \leq q < \infty$. The best constant C_0 in*

$$\|\nabla_k u\|_{L_q(\mathbb{R}^n,\mu)} \leq C_0 \|u\|_{B_1^l} \tag{11.10.4}$$

is equivalent to

$$K_0 = \sup_{x;\varrho\in(0,1)} \varrho^{l-k-n} \mu\big(B(x,\varrho)\big)^{1/q}.$$

Proof. The estimate $C_0 \geq cK_0$ follows in the same way as $C \geq cK$ in Theorem 1. To prove the reverse inequality we use the partition of unity $\{\varphi_j\}_{j\geq 1}$ subordinate to the covering of $\mathbb{R}^n$ by open balls with centers at the nodes of a sufficiently fine coordinate grid and apply Theorem 1 to the norm $\|\nabla_k(\varphi_j u)\|_{L_q(\mathbb{R}^n,\mu_j)}$ where μ_j is the restriction of μ to the support of φ_j. Then

$$\int |\nabla_k u|^q \,\mathrm{d}\mu \leq c \sum_j \int \big|\nabla_k(\varphi_j u)\big|^q \,\mathrm{d}\mu_j$$
$$\leq cK_0^q \sum_j \|\varphi_j u\|_{b_1^l}^{q_l} \leq cK_0^q \Big(\sum_j \|\varphi_j u\|_{b_1^l} \Big)^q.$$

(Here we made use of the inequality $\sum a_i^q \le (\sum a_i)^q$, where $a_i \ge 0$, $q \ge 1$.) Now a reference to Theorem 10.1.3/3 completes the proof. □

Remark 2. For $l - k \ge n$ the best constant in (11.10.4) is equivalent to one of the following values:

$$\sup_{x\in\mathbb{R}^n} \big[\mu\big(B(x,1)\big)\big]^{1/q} \quad \text{if } q \ge 1,$$

$$\Big(\sum_{i\ge 0} \mu\big(\mathcal{Q}^{(i)}\big)^{(1-q)^{-1}}\Big)^{(1-q)q^{-1}} \qquad \text{if } 0 < q < 1,$$

where $\{\mathcal{Q}^{(i)}\}$ is the same sequence as in Theorem 11.6.2/1. The proof is contained in Remark 11.8/1 and Theorem 11.6.2/1.

11.10.2 Criterion for an Upper Estimate of a Difference Seminorm (the Case $p = 1$)

Let us consider the seminorm

$$\langle u\rangle_{q,\mu} = \Big(\int_\Omega\int_\Omega \big|u(x) - u(y)\big|^q \mu(\mathrm{d}x, \mathrm{d}y)\Big)^{1/q}, \tag{11.10.5}$$

where Ω is an open subset of a Riemannian manifold and μ is a nonnegative measure on $\Omega \times \Omega$, locally finite outside the diagonal $\{(x, y) : x = y\}$. By definition, the product $0 \cdot \infty$ equals zero.

In this section, first, we characterize both μ and Ω subject to the inequality

$$\langle u\rangle_{q,\mu} \le C\|\nabla u\|_{L_1(\Omega)}, \tag{11.10.6}$$

where $q \ge 1$ and u is an arbitrary function in $C^\infty(\Omega)$. We show that (11.10.6) is equivalent to a somewhat unusual relative isoperimetric inequality.

Theorem. *Inequality (11.10.6) holds for all $u \in C^\infty(\Omega)$ with $q \ge 1$ if and only if for any open subset g of Ω, such that $\Omega \cap \partial g$ is smooth, the relative isoperimateric inequality*

$$\big(\mu(g, \Omega\backslash\bar{g}) + \mu(\Omega\backslash\bar{g}, g)\big)^{1/q} \le Cs(\Omega \cap \partial g) \tag{11.10.7}$$

holds with the same value of C as in (11.10.6).

Proof. Sufficiency. Denote by u_+ and u_- the positive and negative parts of u, so that $u = u_+ - u_-$. We notice that

$$\langle u\rangle_{q,\mu} \le \langle u_+\rangle_{q,\mu} + \langle u_-\rangle_{q,\mu} \tag{11.10.8}$$

and

$$\int_\Omega |\nabla u|\,\mathrm{d}x = \int_\Omega |\nabla u_+|\,\mathrm{d}x + \int_\Omega |\nabla u_-|\,\mathrm{d}x. \tag{11.10.9}$$

First, we obtain (11.10.6) separately for $u = u_+$ and $u = u_-$. Let $a > b$ and let $\chi_t(a, b) = 1$ if $a > t > b$ and $\chi_t(a, b) = 0$ otherwise.

Clearly,

$$\langle u\rangle_{q,\mu} = \left(\int_\Omega\int_\Omega\left|\int_{u(x)}^{u(y)} \mathrm{d}t\right|^q \mu(\mathrm{d}x, \mathrm{d}y)\right)^{1/q}$$
$$= \left(\int_\Omega\int_\Omega\left|\int_0^\infty \big(\chi_t\big(u(x), u(y)\big) + \chi_t\big(u(y), u(x)\big)\big)\,\mathrm{d}t\right|^q \mu(\mathrm{d}x, \mathrm{d}y)\right)^{1/q}.$$

By Minkowski's inequality,

$$\langle u\rangle_{q,\mu} \le \int_0^\infty \left(\int_\Omega\int_\Omega \big(\chi_t\big(u(x), u(y)\big) + \chi_t\big(u(y), u(x)\big)\big)^q \mu(\mathrm{d}x, \mathrm{d}y)\right)^{1/q} \mathrm{d}t$$
$$= \int_0^\infty \left(\int_\Omega\int_\Omega \big(\chi_t\big(u(x), u(y)\big) + \chi_t\big(u(y), u(x)\big)\big)\mu(\mathrm{d}x, \mathrm{d}y)\right)^{1/q} \mathrm{d}t$$
$$= \int_0^\infty \big(\mu(\mathcal{L}_t, \Omega\backslash\mathcal{N}_t) + \mu(\Omega\backslash\mathcal{N}_t, \mathcal{L}_t)\big)^{1/q}\,\mathrm{d}t,$$

where $\mathcal{L}_t = \{x \in \Omega : u(x) > t\}$ and $\mathcal{N}_t = \{x \in \Omega : u(x) \ge t\}$.

By (11.10.7) and the co-area formula (1.2.6), the last integral does not exceed

$$C\int_0^\infty s\big(\{x \in \Omega : u(x) = t\}\big)\,\mathrm{d}t = C\int_\Omega \big|\nabla u(x)\big|\,\mathrm{d}x.$$

Therefore,

$$\langle u_\pm\rangle_{q,\mu} \le C\int_\Omega \big|\nabla u_\pm(x)\big|\,\mathrm{d}x,$$

and the reference to (11.10.8) and (11.10.9) completes the proof of sufficiency.

Necessity. Let $\{w_m\}$ be the sequence of locally Lipschitz functions in Ω constructed in Lemma 5.2.2 with the following properties:

1. $w_m(x) = 0$ in $\Omega\backslash g$,
2. $w_m(x) \in [0, 1]$ in Ω,
3. for any compactum $K \subset g$ there exists an integer $N(e)$ such that $w_m(x) = 1$ for $x \in K$ and $m \ge N(e)$,
4. the limit relation holds

$$\limsup_{m\to\infty} \int_\Omega \big|\nabla w_m(x)\big|\,\mathrm{d}x = s(\Omega \cap \partial g).$$

By Theorem 1.1.5/1, the inequality (11.10.6) holds for all locally Lipschitz functions. Therefore,

$$\langle w_m\rangle_{q,\mu} \le C\|\nabla w_m\|_{L_1(\Omega)}, \tag{11.10.10}$$

and by property 4,

$$\limsup_{m\to\infty} \langle w_m\rangle_{q,\mu} \le Cs(\Omega \cap \partial g). \tag{11.10.11}$$

Further,

$$\begin{aligned}\langle w_m\rangle^q_{q,\mu} = &\int_{x\in g}\int_{y\in\Omega\backslash g} w_m(x)^q\mu(\mathrm{d}x,\mathrm{d}y)\\ &+\int_{x\in\Omega\backslash g}\int_{y\in g} w_m(y)^q\mu(\mathrm{d}x,\mathrm{d}y)\\ &+\int_g\int_g \big|w_m(x)-w_m(y)\big|^q\mu(\mathrm{d}x,\mathrm{d}y),\end{aligned}$$

which implies

$$\langle w_m\rangle^q_{q,\mu} \ge \int_g w_m(x)^q\mu(\mathrm{d}x,\Omega\backslash\bar g)+\int_g w_m(y)^q\mu(\Omega\backslash\bar g,\mathrm{d}y).$$

This, along with property 3, leads to

$$\liminf_{m\to\infty}\langle w_m\rangle^q_{q,\mu}\ge \mu(g,\Omega\backslash\bar g)+\mu(\Omega\backslash g,\bar g).$$

Combining this relation with (11.10.10) and (11.10.11), we arrive at (11.10.7). □

Corollary 1. (*One-Dimensional Case*). *Let*

$$\Omega=(\alpha,\beta),\quad \textit{where } -\infty\le\alpha<\beta\le\infty.$$

The inequality

$$\left(\int_\Omega\int_\Omega\big|u(x)-u(y)\big|^q\mu(\mathrm{d}x,\mathrm{d}y)\right)^{1/q}\le C\int_\Omega\big|u'(x)\big|\,\mathrm{d}x \tag{11.10.12}$$

with $q\ge 1$ holds for all $u\in C^\infty(\Omega)$ if and only if

$$\big(\mu(I,\Omega\backslash\bar I)+\mu(\Omega\backslash\bar I,I)\big)^{1/q}\le 2C \tag{11.10.13}$$

for all open intervals I such that $\bar I\subset\Omega$, and

$$\big(\mu(I,\Omega\backslash\bar I)+\mu(\Omega\backslash\bar I,I)\big)^{1/q}\le C \tag{11.10.14}$$

for all open intervals $I\subset\Omega$ such that $\bar I$ contains one of the end points of Ω.

Proof. Necessity follows directly from (11.10.7) by setting $g=I$. Let us check the *sufficiency* of (11.10.13). Represent an arbitrary open set g of Ω as the union on nonoverlapping open intervals I_k. Then by (11.10.13) and (11.10.14)

$$\begin{aligned}&\big(\mu(g,\Omega\backslash\bar g)+\mu(\Omega\backslash\bar g,g)\big)^{1/q}\\ &\quad=\left(\sum_k\big(\mu(I_k,\Omega\backslash\bar g)+\mu(\Omega\backslash\bar g,I_k)\big)\right)^{1/q}\\ &\quad\le\sum_k\big(\mu(I_k,\Omega\backslash\bar g)+\mu(\Omega\backslash\bar g,I_k)\big)^{1/q}\le C\sum_k s(\Omega\cap\partial I_k),\end{aligned}$$

which is the same as (11.10.7). The result follows from the Theorem. □

Remark 1. Suppose that the class of admissible functions in the Theorem is diminished by the requirement $u = 0$ *in a neighborhood of a closed subset* F of $\bar{\Omega}$. Then the same proof leads to the same criterion (11.10.7) with the only difference that the admissible sets g should be at a positive distance from F. For the example $F = \partial\Omega$, i.e., for the inequality (11.10.6) with any $u \in C_0^\infty(\Omega)$, the necessary and sufficient condition (11.10.7) becomes the isoperimetric inequality

$$\big(\mu(g, \Omega\backslash\bar{g}) + \mu(\Omega\backslash\bar{g}, g)\big)^{1/q} \le Cs(\partial g) \tag{11.10.15}$$

for all open sets g with a smooth boundary and compact closure $\bar{g} \subset \Omega$. If, in particular, in Corollary 1, the criterion of the validity of (11.10.12) for all $u \in C_0^\infty(\Omega)$ is the inequality (11.10.13) for every interval I, $\bar{I} \subset \Omega$. In the case $u = 0$ near one of the end points $\Omega = (\alpha, \beta)$, one should require both (11.10.13) and (11.10.14), but the intervals I should be at a positive distance from that end point.

Needless to say, the condition (11.10.7) is simplified as follows for a symmetric measure μ, i.e., under the assumption $\mu(\mathcal{E}, \mathcal{F}) = \mu(\mathcal{F}, \mathcal{E})$:

$$\mu(g, \Omega\backslash\bar{g})^{1/q} \le 2^{-1/q} Cs(\Omega \cap \partial g),$$

for the same open sets g as in the Theorem.

Remark 2. The integration domain $\Omega \times \Omega$ in (11.10.5) excludes inequalities for integrals taken over $\partial\Omega$. This can be easily avoided assuming additionally that μ is defined on compact subsets of $\bar{\Omega} \times \bar{\Omega}$ and that $u \in C(\bar{\Omega}) \cap C^\infty(\Omega)$. Then, with the same proof, one obtains the corresponding criterion, similar to (11.10.7):

$$\big(\mu(\bar{g}, \bar{\Omega}\backslash\bar{g}) + \mu(\bar{\Omega}\backslash\bar{g}, \bar{g})\big)^{1/q} \le Cs(\Omega \cap \partial g).$$

As an application, consider the inequality

$$\int_{\partial\Omega}\int_{\partial\Omega} \big|u(x) - u(y)\big| s(\mathrm{d}x) s(\mathrm{d}y) \le C \int_\Omega |\nabla u| \,\mathrm{d}x, \tag{11.10.16}$$

which holds if and only if

$$s(\partial\Omega \cap \partial g) s(\partial\Omega\backslash\partial g) \le 2^{-1} Cs(\Omega \cap \partial g), \tag{11.10.17}$$

for the same sets g as in the Theorem.

By Corollary 9.4.4/3,

(i) If Ω is the unit ball in $\mathbb{R}^3$, then

$$4\pi s(\Omega \cap \partial g) \ge s(\partial\Omega \cap \partial g) s(\partial\Omega\backslash\partial g),$$

and

(ii) If Ω is the unit disk on the plane, then

$$s(\Omega \cap \partial g) \geq 2 \sin\Big(\frac{1}{2} s(\partial\Omega \cap \partial g)\Big).$$

Moreover, the last two inequalities are sharp. Hence, the inequality (11.10.16) holds with the best constant $C = 8\pi$ if $\Omega = B$. In case (ii),

$$s(\Omega \cap \partial g) \geq 2^{-1} \min_{0\leq\varphi\leq\pi} \frac{\sin\varphi}{\varphi(\pi-\varphi)} s(\partial\Omega \cap \partial g) s(\partial\Omega \backslash \partial g).$$

Since the last minimum equals π^{-1}, it follows that the best value of C in the inequality (11.10.16) for the unit disk is 4π.

Example. We deal with functions in $\mathbb{R}^n$ and prove the inequality

$$\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^q}{|x-y|^{n+\alpha q}}\, dx\, dy\Big)^{1/q} \leq C \int_{\mathbb{R}^n} |\nabla u|\, dx, \tag{11.10.18}$$

where $u \in C_0^\infty(\mathbb{R}^n)$, $n > 1$, $0 < \alpha < 1$, and $q = n/(n-1+\alpha)$.

Let us introduce the set function

$$g \to \mathrm{i}(g) := \int_g \int_{\mathbb{R}^n\backslash g} \frac{dx\, dy}{|x-y|^{n+\alpha q}}.$$

By the Theorem we only need to prove the isoperimetric inequality

$$\big(\mathrm{i}(g)\big)^{\frac{n-1}{n-\alpha q}} \leq c(\alpha, n) s(\partial g), \tag{11.10.19}$$

for $q = n/(n-1+\alpha)$. Let Δ be the Laplace operator in $\mathbb{R}^n$. If $u = r^\lambda$, we may write

$$\Delta u = \frac{1}{r^{n-1}}\big(r^{n-1}u_r\big)_r = \lambda(\lambda+n-2)r^{\lambda-2}.$$

Setting $\lambda = 2-n-\alpha q$, we arrive at

$$\Delta_y |x-y|^{2-n-\alpha q} = (n-2+\alpha q)|x-y|^{-n-\alpha q}.$$

Using (1.4.13) and Example 2.1.5/2, we obtain

$$\begin{aligned}
\mathrm{i}(g) &= \frac{1}{\alpha q(n-2+\alpha q)} \int_g \int_{\mathbb{R}^n\backslash g} \Delta_y |x-y|^{2-n-\alpha q}\, dy\, dx \\
&= \frac{1}{\alpha q(n-2+\alpha q)} \int_g \int_{\partial g} \frac{\partial}{\partial \nu_y} |x-y|^{2-n-\alpha q}\, dy\, dx \\
&\leq \frac{1}{\alpha q} \int_{\partial g} \int_g |x-y|^{n-1+\alpha q}\, dx\, ds_y \\
&\leq \frac{n v_n^{1-\frac{1-\alpha q}{n}}}{\alpha q(1-\alpha q)} (m_n g)^{\frac{1-\alpha q}{n}} s(\partial g) \leq \frac{(n v_n)^{1-\frac{1-\alpha q}{n-1}}}{\alpha q(1-\alpha q)} s(\partial g)^{1+\frac{1-\alpha q}{n-1}}.
\end{aligned}$$

Since

$$1 - \alpha q = \frac{(n-1)(1-\alpha)}{n-1+\alpha},$$

inequality (11.10.19) follows.

Remark 3. Inequality (11.10.18) can be interpreted as the embedding

$$\mathring{L}_1^1(\mathbb{R}^n) \subset \mathring{W}_q^\alpha(\mathbb{R}^n),$$

where $\mathring{L}_1^1(\mathbb{R}^n)$ is the completion of the space $C_0^\infty(\mathbb{R}^n)$ in the norm $\|\nabla u\|_{L_1(\mathbb{R}^n)}$ and $\mathring{W}_q^\alpha(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ in the fractional Sobolev norm

$$\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^q}{|x-y|^{n+\alpha q}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/q}.$$

We can simplify the criteria (11.10.7) for $\Omega = \mathbb{R}^n$, replacing arbitrary sets g by arbitrary balls $B(x,\rho)$ similarly to Theorem 1.4.2/2, where the norm

$$\|u\|_{L_q(\mu)} = \Big(\int_{\mathbb{R}^n} |u|^q\,\mathrm{d}\mu\Big)^{1/q}$$

is treated in place of $\langle u\rangle_{q,\mu}$. Unfortunately, the best constant in the sufficiency part will be lost.

Corollary 2. (i) *If $q \ge 1$ and*

$$\sup_{x\in\mathbb{R}^n,\rho>0} \rho^{(1-n)q}\big(\mu\big(B(x,\rho),\mathbb{R}^n\backslash B(x,\rho)\big) + \mu\big(\mathbb{R}^n\backslash B(x,\rho), B(x,\rho)\big)\big) < \infty, \tag{11.10.20}$$

then the inequality

$$\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|u(x)-u(y)\big|^q \mu(\mathrm{d}x,\mathrm{d}y)\Big)^{1/q} \le C\|\nabla u\|_{L_1(\mathbb{R}^n)} \tag{11.10.21}$$

holds for all $u \in C^\infty(\mathbb{R}^n)$ and

$$C^q \le c^q \sup_{x\in\mathbb{R}^n,\rho>0} \rho^{(1-n)q}\big(\mu\big(B(x,\rho),\mathbb{R}^n\backslash B(x,\rho)\big) + \mu\big(\mathbb{R}^n\backslash B(x,\rho), B(x,\rho)\big)\big), \tag{11.10.22}$$

where c depends only on n.

(ii) *If* (11.10.21) *holds for all $u \in C^\infty(\mathbb{R}^n)$, then*

$$C^q \ge \omega_n^{-q} \sup_{x\in\mathbb{R}^n,\rho>0} \rho^{(1-n)q}\big(\mu\big(B(x,\rho),\mathbb{R}^n\backslash B(x,\rho)\big) + \mu\big(\mathbb{R}^n\backslash B(x,\rho), B(x,\rho)\big)\big).$$

Proof. Let g be an arbitrary open set in $\mathbb{R}^n$ with smooth boundary and let $\{B(x_j, \rho_j)\}$ be a covering of g subject to

$$\sum_j \rho_j^{n-1} \le cs(\partial g), \tag{11.10.23}$$

where c depends only on n (see Theorem 1.2.2/2). Then

$$\begin{aligned}\mu\big(g, \mathbb{R}^n\backslash g\big) &\le \sum_j \mu\big(B(x_j, \rho_j), \mathbb{R}^n\backslash g\big) \le \Big(\sum_j \mu\big(B(x_j, \rho_j), \mathbb{R}^n\backslash g\big)^{1/q}\Big)^q \\ &\le \Big(\sum_j \mu\big(B(x_j, \rho_j), \mathbb{R}^n\backslash B(x_j, \rho_j)\big)^{1/q}\Big)^q \le (cB)^q \Big(\sum_j \rho_j^{n-1}\Big)^q,\end{aligned}$$

where B is the value of the supremum in (11.10.20). This and (11.10.23) imply

$$\mu\big(g, B(x_j, \rho_j)\big) \le \big(cBs(\partial g)\big)^q.$$

Similarly

$$\mu\big(\mathbb{R}^n\backslash g, g\big) \le \big(cBs(\partial g)\big)^q,$$

and the result follows from the Theorem.

The assertion (ii) stems from (11.10.7) by setting $g = B(x, \rho)$. □

11.10.3 Embedding into a Riesz Potential Space

It is possible to obtain a similar criterion for the embedding

$$\mathring{L}_1^1 \subset \mathring{R}_q^\alpha(\mu), \tag{11.10.24}$$

where $0 < \alpha < 1$, $q \ge 1$ and $R_q^\alpha(\mu)$ is the completion of the space C_0^∞ in the norm

$$\langle u\rangle_{R_q^\alpha(\mu)} = \left\| \int_{\mathbb{R}^n} \frac{\nabla u(y)}{|x-y|^{n+\alpha-1}}\, \mathrm{d}y \right\|_{L_q(\mu)}, \quad q \ge 1,$$

where μ is a measure in $\mathbb{R}^n$.

In the case $q > 1$ and $\mu = \mathrm{mes}_n$ this norm is equivalent to the norm

$$\big\|(-\varDelta)^{\alpha/2} u\big\|_{L_q}$$

in the space of Riesz potentials of order α with densities in L_q. We shall see that embedding (11.10.24) is equivalent to the isoperimetric inequality of a new type.

Theorem. *Let $q \ge 1$ and $0 < \alpha < 1$. The inequality*

$$\langle u\rangle_{R_q^\alpha(\mu)} \le C \int_{\mathbb{R}^n} \varPhi |\nabla u|\, \mathrm{d}x, \tag{11.10.25}$$

where Φ is a continuous nonnegative function, holds for all $u \in C_0^\infty$ if and only if, for any bounded open $g \in \mathbb{R}^n$ with smooth boundary ∂g, the isoperimetric inequality holds:

$$\left\| \int_{\partial g} \frac{\nu_y \, \mathrm{d}s_y}{|x-y|^{n+\alpha-1}} \right\|_{L_q(\mu)} \le C \int_{\partial g} \Phi(x) \, \mathrm{d}s_x. \tag{11.10.26}$$

Proof. Necessity of (11.10.26) follows by the substitution of a mollification of a characteristic function of g into (11.10.26).

Let us prove the sufficiency. By the co-area formula,

$$\langle u \rangle_{R_q^\alpha(\mu)} = \left\| \int_{-\infty}^{\infty} \mathrm{d}t \int_{E_t} \frac{\nu_y \, \mathrm{d}s_y}{|x-y|^{n+\alpha-1}} \right\|_{L_q(\mu)}.$$

Hence it follows from Minkowski's inequality and (11.10.26) that

$$\begin{aligned} \langle u \rangle_{R_q^\alpha(\mu)} &\le \int_{-\infty}^{\infty} \left\| \int_{E_t} \frac{\nu_y \, \mathrm{d}s_y}{|x-y|^{n+\alpha-1}} \right\|_{L_q(\mu)} \mathrm{d}t \\ &\le C \int_{-\infty}^{\infty} \int_{E_t} \Phi(x) \, \mathrm{d}s_x \, \mathrm{d}t = C \int_{\mathbb{R}^n} \Phi |\nabla u| \, \mathrm{d}x. \end{aligned}$$

The result follows. □

Remark. The set of the inequalities of type (11.10.27) is not void. Let us show, for example, that for

$$q = \frac{n}{n-1-\alpha}, \quad n \ge 2, \; 0 < \alpha < 1,$$

there holds the isoperimetric inequality

$$\left\| \int_{\partial g} \frac{\nu_y \, \mathrm{d}s_y}{|x-y|^{n+\alpha-1}} \right\|_{L_q} \le c(n,\alpha) s(\partial g). \tag{11.10.27}$$

In fact, since $q \le 2$, we have the well-known inequality

$$\|u\|_{h_q^\alpha} \le c \|u\|_{b_q^\alpha};$$

the norm in b_q^α on the right-hand side does not exceed $c\|\nabla u\|_{L_1}$ and it remains to refer to Theorem 11.10.3.

11.11 Criteria for an Upper Estimate of a Difference Seminorm (the Case $p > 1$)

11.11.1 Case $q > p$

Now we deal with the inequality

$$\langle u\rangle_{q,\mu} \le C\|\nabla u\|_{L_p(\Omega)}, \tag{11.11.1}$$

where $q > p > 1$, and show that it is equivalent to a certain isocapacitary inequality.

The capacity to appear in the present context is defined as follows. Let F_1 and F_2 be nonoverlapping subsets of Ω, closed in Ω. The p-capacity of the pair (F_1, F_2) with respect to Ω is given by

$$\operatorname{cap}_p(F_1, F_2; \Omega) = \inf_{\{u\}} \int_\Omega |\nabla u(x)|^p \, dx,$$

where $\{u\}$ is the set of all $u \in C^\infty(\Omega)$, such that $u \ge 1$ on F_1 and $u \le 0$ on F_2. Obviously, this capacity does not change if F_1 and F_2 change places. In fact, $\operatorname{cap}_p(F_1, F_2; \Omega)$ is nothing but the p-conductivity of the conductor $K = (\Omega\backslash F_1)\backslash F_2$ (see Sect. 6.1.1). We use the new notation only in this section to emphasize the symmetric dependence of the following geometric criteria on two sets F_1 and F_2.

Furthermore, if F is a closed set in $\mathbb{R}^n$ and $F \subset G$, where G is an open set, such that $\bar{G} \subset \Omega$, then $\operatorname{cap}_p(F, \Omega\backslash G; \Omega)$ coincides with the p-capacity $\operatorname{cap}_p(F; G)$ defined in Sect. 2.2.1.

Theorem. *Inequality (11.11.1) with $p \in (1, q)$ holds for all $u \in C^\infty(\Omega)$ if and only if for any pair (F_1, F_2) of nonoverlapping sets, closed in Ω,*

$$\mu(F_1, F_2)^{p/q} \le \mathcal{B} \operatorname{cap}_p(F_1, F_2; \Omega), \tag{11.11.2}$$

where $c_1 C \le \mathcal{B} \le c_2 C$ with positive c_1 and c_2 depending only on p and q. In the sufficiency part we may assume that F_1 and F_2 are sets with smooth $\Omega \cap \partial F_i$.

In the proof of this theorem, we use the inequality

$$\Big(\int_{\mathbb{R}_+} |f(\psi)|^q \psi^{-1-q/p'} \, d\psi\Big)^{1/q} \le c\|f'\|_{L_p(\mathbb{R}_+)} \tag{11.11.3}$$

(see (4.6.7)) and the inequality

$$\Big(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{|f(\psi)-f(\phi)|^q}{|\psi-\phi|^{2+q/p'}} \, d\phi \, d\psi\Big)^{1/q} \le c\|f'\|_{L_p(\mathbb{R}_+)}, \tag{11.11.4}$$

where $q > p > 1$, $p' = p/(p-1)$ and f is an arbitrary absolutely continuous function on $\bar{\mathbb{R}}_+$.

A short argument leading to (11.11.4) is as follows. Clearly, (11.11.4) results from the same inequality with $\mathbb{R}$ in place of $\mathbb{R}_+$, which follows, in its turn, from the estimate

$$\|f\|_{B_q^{1-(q-p)/pq}(\mathbb{R})} \le c\|f\|_{W_p^1(\mathbb{R})} \tag{11.11.5}$$

by dilation with a coefficient λ and the limit passage as $\lambda \to 0_+$. (The standard notations B and W for Besov and Sobolev spaces with nonhomogeneous norms are used in (11.11.5).) To obtain (11.11.5), we recall the well-known Sobolev-type inequality

$$\|h\|_{L_{p'}(\mathbb{R})} \le c\|h\|_{B_{q'}^{(q-p)/pq}(\mathbb{R})},$$

(see Theorem 4′, Sect. 5.1 of Stein [724]) and put $h = (-\Delta + 1)^{-1/2} f$, which shows that

$$\|f\|_{W_{p'}^{-1}(\mathbb{R})} \le c\|f\|_{B_{q'}^{-1+(q-p)/pq}(\mathbb{R})}. \tag{11.11.6}$$

By duality, (11.11.6) is equivalent to (11.11.5).

With (11.11.4) at hand, we return to the Theorem.

Proof. Sufficiency. Arguing as at the beginning of the proof of Theorem 11.10.2, we see that it suffices to prove (11.11.1) for a nonnegative u. By the definition of the Lebesgue integral

$$\int_\Omega u \,\mathrm{d}\nu = \int_{\mathbb{R}_+} \nu(\mathcal{N}_\tau)\,\mathrm{d}\tau = \int_{\mathbb{R}_+} \nu(\mathcal{L}_\tau)\,\mathrm{d}\tau,$$

where ν is a measure, and therefore

$$\int_\Omega P(u)\,\mathrm{d}\nu = \int_{\mathbb{R}_+} \nu(\mathcal{N}_\tau)\,\mathrm{d}P(\tau), \tag{11.11.7}$$

where P is a nondecreasing function on $\mathbb{R}_+$. Putting here $u = 1/v$ and $Q(\tau) = P(\tau^{-1})$, we deduce

$$\int_\Omega Q(u)\,\mathrm{d}u = -\int_{\mathbb{R}_+} \nu(\Omega\backslash\mathcal{L}_\tau)\,\mathrm{d}Q(\tau), \tag{11.11.8}$$

where Q is nonincreasing. We obtain

$$\begin{aligned}
&\int_\Omega\int_\Omega \bigl|u(x) - u(y)\bigr|^q \mu(\mathrm{d}x,\mathrm{d}y)\\
&\quad= \int_\Omega\int_\Omega \bigl(u(x) - u(y)\bigr)_+^q \mu(\mathrm{d}x,\mathrm{d}y)\\
&\qquad+ \int_\Omega\int_\Omega \bigl(u(y) - u(x)\bigr)_+^q \mu(\mathrm{d}x,\mathrm{d}y)\\
&\quad= \int_\Omega\int_\Omega \bigl(u(x) - u(y)\bigr)_+^q \bigl(\mu(\mathrm{d}x,\mathrm{d}y) + \mu(\mathrm{d}y,\mathrm{d}x)\bigr).
\end{aligned}$$

By (11.11.7) and (11.11.8), the last double integral is equal to

$$\begin{aligned}
&q\int_{\mathbb{R}_+}\int_\Omega \bigl(t - u(y)\bigr)_+^{q-1} \bigl(\mu(\mathcal{N}_\tau,\mathrm{d}y) + \mu(\mathrm{d}y,\mathcal{N}_\tau)\bigr)\,\mathrm{d}\tau\\
&\quad= q(q-1)\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} (\tau - \sigma)_+^{q-2} \bigl(\mu(\mathcal{N}_\tau,\Omega\backslash\mathcal{L}_\sigma) + \mu(\Omega\backslash\mathcal{L}_\sigma,\mathcal{N}_\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}\sigma.
\end{aligned}$$

Now, (11.11.2) implies

$$\langle u\rangle_{q,\mu}^q \le 2q(q-1)\mathcal{B}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}(\tau-\sigma)_+^{q-2}\,\mathrm{cap}(\mathcal{N}_\tau, \Omega\backslash\mathcal{L}_\sigma;\Omega)\,\mathrm{d}\tau\,\mathrm{d}\sigma,$$

and using the function $\psi \to t(\psi)$, the inverse of (4.7.18), we arrive at the inequality

$$\begin{aligned}\langle u\rangle_{q,\mu}^q \le\ & 2q(q-1)\mathcal{B}^{q/p}\\ &\times\int_{\mathbb{R}_+}\int_0^\psi \big(t(\psi)-t(\phi)\big)^{q-2}\big(\mathrm{cap}(\mathcal{N}_{t(\psi)}, \Omega\backslash\mathcal{L}_{t(\phi)};\Omega)\big)^{q/p}\\ &\times t'(\phi)t'(\psi)\,\mathrm{d}\phi\,\mathrm{d}\psi.\end{aligned}$$

By Lemma 2.2.2/1, for $\psi > \phi$

$$\mathrm{cap}(\mathcal{N}_{t(\psi)}, \Omega\backslash\mathcal{L}_{t(\phi)};\Omega) \le (\psi-\phi)^{1-p},$$

and therefore,

$$\begin{aligned}\langle u\rangle_{q,\mu}^q \le\ & 2q(q-1)\mathcal{B}^{q/p}\\ &\times\int_{\mathbb{R}_+}\int_0^\psi (\psi-\phi)^{-q/p'}\big(t(\psi)-t(\phi)\big)^{q-2}t'(\phi)t'(\psi)\,\mathrm{d}\phi\,\mathrm{d}\psi.\end{aligned}\tag{11.11.9}$$

Integrating by parts twice on the right-hand side of (11.11.9), we obtain

$$\begin{aligned}\langle u\rangle_{q,\mu}^q &\le 2\mathcal{B}^{q/p}\frac{q}{p'}\bigg(\Big(\frac{q}{p'}+1\Big)\int_{\mathbb{R}_+}\int_0^\psi \frac{(t(\psi)-t(\phi))^q}{(\psi-\phi)^{2+q/p'}}\,\mathrm{d}\phi\,\mathrm{d}\psi\\ &\qquad + \int_{\mathbb{R}_+}\psi^{-q/p'}t(\psi)^{q-1}t'(\psi)\,\mathrm{d}\psi\bigg)\\ &= \mathcal{B}^{q/p}\frac{q}{p'}\bigg(\Big(\frac{q}{p'}+1\Big)\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{|t(\psi)-t(\phi)|^q}{|\psi-\phi|^{2+q/p'}}\,\mathrm{d}\phi\,\mathrm{d}\psi\\ &\qquad + \frac{1}{p'}\int_{\mathbb{R}_+}t(\psi)^q\psi^{-1-q/p'}\,\mathrm{d}\psi\bigg).\end{aligned}$$

Hence, we deduce from (11.11.3) and (11.11.4) that

$$\langle u\rangle_{q,\mu} \le c\mathcal{B}^{1/p}\|t'\|_{L_p(\mathbb{R}_+)},\tag{11.11.10}$$

where c depends only on p and q. It remains to refer to (4.6.6).

Necessity. Let F_1 and F_2 be subsets of Ω, closed in Ω. We take an arbitrary function $u \in C^\infty(\Omega)$, such that $u \ge 1$ on F_1 and $u \le 0$ on F_2, and put it into (11.11.1). Then

$$\mu(F_1, F_2;\Omega)^{p/q} \le \bigg(\int_{F_1}\int_{F_2}\big|u(x)-u(y)\big|^q\mu(\mathrm{d}x,\mathrm{d}y)\bigg)^{1/q} \le C\int_\Omega |\nabla u|^p\,\mathrm{d}x.$$

It remains to minimize the right-hand side to obtain

$$\mu(F_1, F_2; \Omega)^{p/q} \le C \operatorname{cap}_p(F_1, F_2; \Omega).$$

The result follows. □

A direct consequence of the Theorem and the isocapacitary inequality for $\operatorname{cap}_p(F; G)$ (see (2.2.10) and (2.2.11)) is the following sufficient condition for (11.11.1) formulated in terms of the n-dimensional Lebesgue measure:

$$\mu(F, \Omega\backslash G) \le c\left(\log \frac{m_n(G)}{m_n(F)}\right)^{q(1-n)/n} \quad \text{if } p = n, \tag{11.11.11}$$

and

$$\mu(F, \Omega\backslash G) \le c\left|m_n(G)^{(p-n)/n(p-1)} - m_n(F)^{(p-n)/n(p-1)}\right|^{1-p}, \tag{11.11.12}$$

if $p \ne n$. Choosing two concentric balls situated in Ω as the sets F_1 and $\Omega\backslash F_2$ in (11.11.2) and using the explicit formulas for the p-capacity of spherical condensers (see Sect. 2.2.4) we see that the inequalities (11.11.11) and (11.11.12), with concentric balls F and G placed in Ω, are necessary for (11.11.1).

In the one-dimensional case the Theorem can be written in a much simpler form.

Corollary. *Let*

$$\Omega = (\alpha, \beta), \quad -\infty \le \alpha < \beta \le \infty.$$

The inequality

$$\left(\int_\Omega \int_\Omega \left|u(x) - u(y)\right|^q \mu(\mathrm{d}x, \mathrm{d}y)\right)^{1/q} \le C\left(\int_\Omega \left|u'(x)\right|^p \mathrm{d}x\right)^{1/p}, \tag{11.11.13}$$

holds for every $u \in C^\infty(\Omega)$ if and only if, for all pairs of intervals I and J of the three types

$$I = [x-d, x+d] \quad \textit{and} \quad J = (x-d-r, x+d+r),$$

$$I = (\alpha, x] \quad \textit{and} \quad J = (\alpha, x+r), \tag{11.11.14}$$

$$I = [x, \beta] \quad \textit{and} \quad J = (x-r, \beta), \tag{11.11.15}$$

where d and r are positive and $J \subset \Omega$, we have

$$r^{(p-1)/p}\big(\mu(I, \Omega\backslash J)\big)^{1/q} \le B, \tag{11.11.16}$$

where B does not depend on I and J.

Proof. The necessity of (11.11.16) follows directly from that in the Theorem and the inequality

$$\operatorname{cap}_p(I, \Omega\backslash J; \Omega) \le 2r^{1-p}$$

(see Lemma 2.2.2/2).

Let us prove the sufficiency. By G_1 we mean an open subset of Ω such that $F_1 \subset G_1$ and $\bar{G}_1 \subset \Omega\backslash F_2$. Connected components of $\Omega\backslash F_2$ will be denoted by J_k. Let J_k contain the closed convex hull $\tilde{I}_k$ of those connected components of G_1 which are situated in J_k.

Then

$$\mu(F_1, F_2)^{p/q} \le \mu(G_1, F_2)^{p/q} \le \Big(\sum_k \mu(\tilde{I}_k, \Omega\backslash J_k)\Big)^{p/q} \le \sum_k \mu(\tilde{I}_k, \Omega\backslash J_k)^{p/q},$$

and since by (11.11.16)

$$\mu(\tilde{I}_k, \Omega\backslash J_k)^{p/q} \le B^p \big(\operatorname{dist}\{\tilde{I}_k, \mathbb{R}\backslash J_k\}\big)^{1-p},$$

we obtain

$$\mu(F_1, F_2)^{p/q} \le B^p \sum_k \big(\operatorname{dist}\{\tilde{I}_k, \mathbb{R}\backslash J_k\}\big)^{1-p}. \tag{11.11.17}$$

Consider an arbitrary function $u \in C^\infty(\Omega)$, such that $u = 1$ on G_1 and $u = 0$ on F_2. Clearly, $u = 0$ on ∂J_k. We have

$$\int_\Omega |u'|^p \, \mathrm{d}x \ge \sum_k \int_{J_k} |u'|^p \, \mathrm{d}x \ge \sum_k \int_{J_k} |\tilde{u}_k'|^p \, \mathrm{d}x, \tag{11.11.18}$$

where $\tilde{u}_k = u$ on $J_k\backslash\tilde{I}_k$, $\tilde{u}_k = 1$ on $\tilde{I}_k$, and $\tilde{u}_k = 0$ on ∂J_k. Hence

$$\int_\Omega |u'|^p \, \mathrm{d}x \ge \sum_k \big(\operatorname{dist}\{\tilde{I}_k, \mathbb{R}\backslash J_k\}\big)^{1-p}.$$

Comparing this estimate with (11.11.17), we arrive at

$$\int_\Omega |u'|^p \, \mathrm{d}x \ge \mu(F_1, F_2)^{p/q},$$

and minimizing the integral in the left-hand side over all functions u, we obtain (11.11.2). □

Remark. It is straightforward but somewhat cumbersome to obtain a more general criterion by replacing the seminorm on the right-hand side of (11.11.13) with

$$\Big(\int_\Omega \big|u'(x)\big|^p \sigma(\mathrm{d}x)\Big)^{1/p}, \tag{11.11.19}$$

where σ is a measure in Ω. In fact, one can replace σ by its absolutely continuous part $(\mathrm{d}\sigma^*/\mathrm{d}x)\,\mathrm{d}x$ and further, roughly speaking, the criterion will follow from the Corollary by the change of variable $x \to \xi$, where

$$\mathrm{d}\xi = \big(\mathrm{d}\sigma^*/\mathrm{d}x\big)^{1/(1-p)}\,\mathrm{d}x.$$

Restricting myself to this hint, the author leaves the details to the interested reader.

11.11.2 Capacitary Sufficient Condition in the Case $q = p$

In the marginal case $q = p$ the condition (11.11.2) in Theorem 11.11.1, being necessary, is not generally sufficient. In fact, let $n = 1$, $\Omega = \mathbb{R}$, and

$$\mu(\mathrm{d}x, \mathrm{d}y) = \frac{\mathrm{d}x\, \mathrm{d}y}{|x-y|^{p+1}}.$$

Then as shown in the proof of Corollary 11.10.2/2, (11.11.2) is equivalent to (11.11.16), and (11.11.16) holds since

$$\begin{aligned}\mu(I, \mathbb{R}\backslash J) &= \int_{|t-x|<d} \mathrm{d}t \int_{|\tau-x|>d+r} \frac{\mathrm{d}\tau}{|t-\tau|^{p+1}} \\ &= \int_{|t|<d} \mathrm{d}t \int_{|\tau|>d+r} \frac{\mathrm{d}\tau}{|t-\tau|^{p+1}} \leq c r^{1-p},\end{aligned}$$

and the same estimate holds for I and J defined by (11.11.14) and (11.11.15).

On the other hand, (11.10.15) fails because

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x)-u(y)|^p}{|x-y|^{p+1}}\, \mathrm{d}x\, \mathrm{d}y = \infty$$

for every nonconstant function u.

In the next theorem we give a sufficient condition for (11.11.1) with $q = p > 1$ formulated in terms of an isocapacitary inequality.

Theorem. *Given $p \in (1, \infty)$ and a positive, vanishing at infinity, nonincreasing absolutely continuous function ν on $\mathbb{R}_+$, such that*

$$S := \sup_{\tau>0} \biggl(\int_0^\tau \bigl|\nu'(\sigma)\bigr|^{1/(1-p)} \frac{\mathrm{d}\sigma}{\sigma} \biggr)^{p-1} \int_\tau^\infty \bigl|\nu'(\sigma)\bigr| \frac{\mathrm{d}\sigma}{\sigma} < \infty.$$

Suppose that

$$\mu(F_1, F_2) \leq \nu\Bigl(\bigl(\mathrm{cap}_p(F_1, F_2; \Omega) \bigr)^{1-p} \Bigr) \tag{11.11.20}$$

for all nonoverlapping sets F_1 and F_2 closed in Ω. Assume also that

$$\mathcal{K} := \int_0^\infty \bigl|\nu'(\sigma)\bigr| \sigma^{p-1}\, \mathrm{d}\sigma < \infty. \tag{11.11.21}$$

Then

$$\langle u \rangle_{p,\mu} \leq 2^{1/p} p \biggl(\frac{S}{(p-1)^{p-1}} \biggr)^{1/pp'} \mathcal{K}^{1/p} \|\nabla u\|_{L_p(\Omega)} \tag{11.11.22}$$

for all $u \in C^\infty(\Omega)$.

Proof. We assume that $\nabla u \in L_p(\Omega)$ and the integral in (11.11.22) involving derivatives of ν is convergent. Arguing as in the proof of Theorem 11.11.1 and using (11.11.20) instead of (11.11.2), we obtain

$$\langle u\rangle^p_{p,\mu} \le 2p(p-1)\int_0^\infty \int_\phi^\infty \nu(\psi-\phi)\big(t(\psi)-t(\phi)\big)^{p-2} t'(\psi)\,\mathrm{d}\psi t'(\phi)\,\mathrm{d}\phi. \tag{11.11.23}$$

Owing to (11.11.21), we can integrate by parts in the inner integral in (11.11.23) and obtain

$$\langle u\rangle^p_{p,\mu} \le 2p\int_0^\infty \int_\phi^\infty \big|\nu'(\psi-\phi)\big|\big(t(\psi)-t(\phi)\big)^{p-1}\,\mathrm{d}\psi\, t'(\phi)\,\mathrm{d}\phi$$
$$= 2p\int_0^\infty \int_0^\psi \big|\nu'(\psi-\phi)\big|\big(t(\psi)-t(\phi)\big)^{p-1} t'(\phi)\,\mathrm{d}\phi\,\mathrm{d}\psi.$$

By Hölders inequality

$$\langle u\rangle^p_{p,\mu} \le 2p\int_0^\infty \mathcal{A}(\phi)^{1/p'}\mathcal{B}^{1/p}\,\mathrm{d}\phi, \tag{11.11.24}$$

where

$$\mathcal{A} = \int_0^\psi \frac{|\nu'(\psi-\phi)|}{\psi-\phi}\big(t(\psi)-t(\phi)\big)^p\,\mathrm{d}\phi$$

and

$$\mathcal{B} = \int_0^\psi \big|\nu'(\psi-\phi)\big|(\psi-\phi)^{p-1}\big|t'(\psi)\big|^p\,\mathrm{d}\phi.$$

Using Theorem 1.3.2/1 concerning a two-weight Hardy inequality, we obtain

$$\mathcal{A} \le \frac{p^p}{(p-1)^{p-1}} S\mathcal{B},$$

which together with (11.11.24) gives

$$\langle u\rangle^p_{p,\mu} \le 2p^p(p-1)^{(1-p)/p'} S^{1/p'}\int_0^\infty\int_0^\phi \big|\nu'(\psi-\phi)\big|(\psi-\phi)^{p-1}\big|t'(\psi)\big|^p\,\mathrm{d}\phi\,\mathrm{d}\psi.$$

Changing the order of integration, we arrive at

$$\langle u\rangle_{p,\mu} \le 2^{1/p}p\big((p-1)^{1-p}S\big)^{1/pp'}\mathcal{K}^{1/p}\|t'\|_{L_p(\mathbb{R}_+)}.$$

It remains to apply (4.6.6).

Remark 1. If the requirement

$$u = 0 \quad \text{on a neighborhood of a closed subset } E \text{ of } \bar{\Omega}$$

is added in the formulations of Theorems 11.11.1 and the theorem just proved, the same proofs give conditions for the validity of (11.11.1), similar to (11.11.2) and (11.11.16). The only new feature is the restriction

$$\Omega \cap \partial(\Omega\backslash F_2) \quad \text{is at a positive distance from } E.$$

In the important particular case $E = \partial\Omega$, which corresponds to zero Dirichlet data on $\partial\Omega$, the conditions (11.11.2) and (11.11.20) become

$$\mu(F, \Omega\backslash G)^{p/q} \le B\,\mathrm{cap}_p(F; G) \tag{11.11.25}$$

and

$$\mu(F, \Omega\backslash G) \le \nu\big(\big(\mathrm{cap}_p(F; G)\big)^{1-p}\big), \tag{11.11.26}$$

respectively, where F is closed and G is open, $G \supset F$, and the closure of G is compact and situated in Ω. The capacity $\mathrm{cap}_p(F; G)$ is defined with $\Omega = G$ in Sect. 2.3.1.

Using lower estimates for the p-capacity in terms of area minimizing functions, one obtains sufficient conditions from (11.11.2), (11.11.16), (11.11.25), and (11.11.26) formulated in terms of the area minimizing function $\mathscr{C}$, introduced in Definition 2.1.4, with $\Phi(\cdot, v) = v$. For example, by (11.11.25) and (11.11.26), inequalities (11.11.2) and (11.11.20) hold for all $u \in C_0^\infty(\Omega)$ if, respectively,

$$\mu(F, \Omega\backslash G)^{p/q} \le B\bigg(\int_{m_n(F)}^{m_n(\Omega\backslash G)} \frac{\mathrm{d}v}{\mathscr{C}(v)^{p/(p-1)}}\bigg)$$

and

$$\mu(F, \Omega\backslash G) \le \nu\bigg(\int_{m_n(F)}^{m_n(\Omega\backslash G)} \frac{\mathrm{d}v}{\mathscr{C}(v)^{p/(p-1)}}\bigg),$$

where F and G are the same as in (11.11.25) and (11.11.26).

By obvious modifications of the proof of sufficiency in Corollary 11.10.2, one deduces the following assertion from the Theorem in this subsection.

Corollary. (One-Dimensional Case) *With the notation used in Corollary 11.11.1, suppose that*

$$\mu(I, \Omega\backslash J) \le \nu(r).$$

Then there exists a positive constant c depending only on p and such that

$$\langle u\rangle_{p,\mu} \le cS^{1/pp'}K^{1/p}\|u'\|_{L_p(\Omega)}$$

for all $u \in C^\infty(\Omega)$.

Remark 2. Let us show that the condition $K < \infty$, which appeared in the Theorem of this subsection, is sharp. Suppose that there exists a positive constant C independent of u and such that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \big|u(t) - u(\tau)\big|^p \nu''(t-\nu)\,\mathrm{d}t\,\mathrm{d}\tau \le C\int_{\mathbb{R}} \big|u'(t)\big|^p\,\mathrm{d}t, \tag{11.11.27}$$

where ν is a convex function in $C^2(\mathbb{R})$. We take an arbitrary $N > 0$ and put $u(t) = \min\{|t|, N\}$ into (11.11.27). Then

$$\int_0^{N/2}\int_\tau^N (t-\tau)^p\nu''(t-\tau)\,\mathrm{d}t\,\mathrm{d}\tau \le 2CN,$$

and setting here $t=\tau+s$, we obtain

$$\frac{1}{2}pN\int_0^{N/2} s^{p-1}\big|\nu'(s)\big|\,\mathrm{d}s \le p\int_0^{N/2}\int_0^{N-\tau} s^{p-1}\big|\nu'(s)\big|\,\mathrm{d}s\,\mathrm{d}\tau \le 2CN.$$

Hence $K\le 4p^{-1}C$.

Remark 3. It seems appropriate, in conclusion, to say a few words about the lower estimate for the difference seminorm $\langle u\rangle_{p,\mu}$, similar to the classical Sobolev inequality

$$\left(\int_\Omega |u|^q\nu(\mathrm{d}x)\right)^{1/q} \le C\langle u\rangle_{p,\mu}, \tag{11.11.28}$$

where Ω is a subdomain of a Riemannian manifold, μ and ν are measures in $\Omega\times\Omega$ and Ω, respectively, and u is an arbitrary function in $C_0^\infty(\Omega)$. Suppose that $q\ge p\ge 1$. Then a condition, necessary and sufficient for (11.11.28), is the isocapacitary inequality

$$\sup_{\{F\}}\frac{\nu(F)^{p/q}}{\mathrm{cap}_{p,\mu}(F;\Omega)}<\infty, \tag{11.11.29}$$

where F is an arbitrary compact set in Ω and the capacity is defined by

$$\mathrm{cap}_{p,\mu}(F;\Omega)=\inf\big\{\langle u\rangle_{p,\mu}^p : u\in C_0^\infty(\Omega), u\ge 1 \text{ on } F\big\}.$$

The necessity of (11.11.28) is obvious and the sufficiency results directly from the inequality

$$\int_0^\infty \mathrm{cap}_{p,\mu}(\mathcal{N}_t;\Omega)\,\mathrm{d}\big(t^p\big) \le c(p)\langle u\rangle_{p,\mu}^p \tag{11.11.30}$$

(see (4.1.5)).

Although providing a universal characterization of (11.11.28), the condition (11.11.29) does not seem satisfactory when dealing with concrete measures and domains. This is related even to the one-dimensional case (cf. Problem 2 in Kufner, Maligranda, and Persson [468]). As an example of a more visible criterion, consider the measure μ on $\mathbb{R}^n\times\mathbb{R}^n$ given by

$$\mu(\mathrm{d}x,\mathrm{d}y)=|x-y|^{-n-p\alpha}\,\mathrm{d}x\,\mathrm{d}y, \tag{11.11.31}$$

with $0<\alpha<1$ and $\alpha p<n$. This measure generates a seminorm in the homogeneous Besov space $b_p^\alpha(\mathbb{R}^n)$. With this particular choice of μ, we have by Theorem 11.10.1/1 and Remark 11.8/3 that (11.11.28) holds with $q>p>1$ and $q\ge p=1$ if and only if

$$\sup_{x\in\mathbb{R}^n,\rho>0}\frac{\nu(B(x,\rho))^{p/q}}{\rho^{n-p\alpha}}<\infty. \tag{11.11.32}$$

The inequality (11.11.32) is the same as (11.11.29) with $F=B(x,\rho)$ and μ is defined in (11.11.31).

11.12 Comments to Chap. 11

Section 11.2. Inequalities similar to (11.1.4) were proved in the author's paper [543], where (11.1.4) (and even a stronger estimate in which the role of the capacity of the set Q_t is played by the capacity of the condenser $Q_t \backslash Q_{2t}$) were derived for $l = 1$ and $l = 2$. In the more difficult case $l = 2$, the proof was based on the procedure of "smooth truncation" of the potential near equipotential surfaces (see Proposition 3.7). By combining this procedure with the Hedberg inequality (11.2.2). D.R. Adams [5] established (11.1.4) for the Sobolev space W_p^l for any integer l. The proof of D.R. Adams is presented in Sect. 11.2.1. The same tools together with Theorem 10.1.1/1 on traces of functions in the weighted Sobolev space were used by the author to derive (11.1.4) for functions in W_p^l for all $p > 1$, $l > 0$. This implies the validity of (11.1.4) for the Bessel potential space H_p^l for all fractional $l > 0$ but only for $p \geq 2$. The latter restriction was removed by Dahlberg [219] whose proof is also based on "smooth truncation" and on subtle estimates for potentials with nonnegative density. Hansson [347, 348] found a new proof of (11.1.4) for spaces of potentials that uses no truncation. Hansson's approach is suitable for a wide class of potentials with general kernels. In Sect. 11.2.2 we presented the author's proof (cf. [554]) of inequality (11.1.4) based on Hansson's idea [347], but apparently simpler. In [20] D.R. Adams and Xiao proved the inequalities

$$\int_\theta^\infty \operatorname{cap}\bigl(\mathscr{N}_t, b^l_{p,q}\bigr)\,\mathrm{d}t^p \leq C\|u\|^p_{b^l_{p,q}}, \quad 1 < q \leq p < \infty,$$

$$\int_0^\infty \bigl(\operatorname{cap}\bigl(\mathscr{N}_t, b^l_{p,q}\bigr)\bigr)^{q/p}\,\mathrm{d}t^q \leq C\|u\|^q_{b^l_{p,q}}, \quad 1 < p \leq q < \infty,$$

where cap is the capacity associated with the space $b^l_{p,q}(\mathbb{R}^n)$. The case $q = p$ is due to Maz'ya [548] and the case $1 \leq p \leq q < \infty$ was considered by Wu [797].

Section 11.3. The equivalence of (11.1.1) and an isocapacitary inequality was discovered by the author in 1962 for the particular case $p = q = 2$, $l = 1$ (cf. Maz'ya [531, 534]). Results of this kind were later obtained in the papers by Maz'ya [543, 548], D.R. Adams [5], Maz'ya and Preobrazhenski [577], and others.

Section 11.4. The counterexample to the isocapacitary inequality in the norm $L_2^2(\Omega)$ is taken from the paper by Maz'ya and Netrusov [572].

Section 11.5. We follow the survey-article [775] by Verbitsky.

Section 11.6. The capacitary characterization of the embedding in the case $q > p > 0$ in Sect. 11.6.1 is due to the author [556] and the author and Netrusov [572].

Sections 11.7–11.8. The presentation mostly follows the paper by the author and Preobrazhenski [577]. In comparison with this paper the requirements on the function Φ are made less restrictive for $1 < p < 2 - l/n$ by virtue of the results due to Hedberg and Wolff [372], which appeared later.

Remark 11.8/2 proved by D.R. Adams [5] and Corollary 11.8/2 was established earlier by D.R. Adams [2] by a different method.

Inequalities related to Corollary 11.8/2, with sharp constants are available. In fact, Cianchi [201] obtained the following far-reaching result concerning the Yudovich inequality (see the Comments to Sect. 1.4).

Theorem. *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$. Let μ be a positive Borel measure on Ω such that*

$$\sup\bigl\{\varrho^{-\beta}\mu\bigl(B(x,\varrho)\cap\Omega\bigr) : x \in \mathbb{R}^n, 0 < \varrho < 1\bigr\} < \infty \tag{11.12.1}$$

for some $\beta \in (0, n]$. Then there exists a constant $C = C(\Omega, \mu)$ such that

$$\int_\Omega \exp\left(\frac{nv_n^{\frac{1}{n}}\beta^{\frac{n-1}{n}}|u(x)|}{\|\nabla u\|_{L_n(\Omega)}}\right)^{\frac{n}{n-1}} \mathrm{d}\mu(x) \leq C \tag{11.12.2}$$

for every $u \in \mathscr{D}(\Omega)$. The constant $nv_n^{\frac{1}{n}}\beta^{\frac{n-1}{n}}$ in (11.12.2) is sharp, i.e., (11.12.2) fails if $nv_n^{\frac{1}{n}}\beta^{\frac{n-1}{n}}$ is replaced by any larger constant, whenever there exist $x_0 \in \Omega$ and ρ_1, $C_1 > 0$ such that

$$\varrho^{-\beta}\mu\bigl(B(x,\varrho)\cap\Omega\bigr) \geq C_1, \quad 0 < \varrho \leq \varrho_1. \tag{11.12.3}$$

The case when μ is the Lebesgue measure in this theorem was established by Moser [618] and extended to the space of Bessel potentials and to higher derivatives of integer order by D.R. Adams [11]. The existence of an optimizer was proved in the planar case by Carleson and Chang [169]. The general case of an arbitrary μ can be found in Cianchi [201], where inequalities for functions not necessarily vanishing on the boundary are taken into account (see also Cianchi [198]). In particular, trace inequalities on $\partial\Omega$ of Moser's type are obtained in [201]. Note that a general optimal capacitary inequality implying Moser's result can be found in Sect. 4.6 (see also Maz'ya [561]). An extension of the Yudovich inequality to the case when the derivatives belong to the Lorentz spaces $L^{n,q}$, where $1 < q \leq n$ is due to Hudson and Leckband [386].

Capacitary inequalities were used by Schechter in the study of estimates of the form

$$M_0^{-1}\left(\int M_0(u)\,\mathrm{d}\mu(x)\right) \leq CM^{-1}\left(\int M\bigl((1-\Delta)^m u\bigr)\,\mathrm{d}\nu(x)\right)$$

for $u \in C^\infty(\mathbb{R}^n)$, where M_0, M are convex functions and μ, ν are measures [692].

Section 11.9. Here various applications of the results obtained in Chap. 11 are given. Section 11.9.2 is borrowed from Maza'ya and Poborchi [576], Sect. 8.6.

Theorems of the present chapter were applied to the problem of the description of classes of multipliers in various spaces of differentiable functions (see the book by Maz'ya and Shaposhnikova [588]). Here we restrict ourselves to mentioning just a few results.

By *multipliers* acting from a function space S_1 into another one S_2 we mean functions such that multiplication by them sends S_1 into S_2. Thus, with the two spaces S_1 and S_2 we associate a third one, *the space of multipliers* $M(S_1 \to S_2)$. If $S_1 = S_2 = S$ we will employ the notation $M(S)$.

The norm of the operator of multiplication by the multiplier γ is taken as the norm of γ in $M(S_1 \to S_2)$.

We list some equivalent normings of multipliers in pairs of Sobolev spaces.

Let $m > l$, $mp < n$, and further, either $q > p > 1$ or $q \geq p = 1$. Then

$$\|\gamma\|_{M(W_p^m(\mathbb{R}^n) \to W_q^l(\mathbb{R}^n))} \sim \sup_{x \in \mathbb{R}^n; 0<r<1} \big(r^{m-n/p} \| \operatorname{grad}_l \gamma\|_{L_q(B_r(x))} + \|\gamma\|_{L_1(B_1(x))}\big). \quad (11.12.4)$$

For the formulation of an analogous result for $q = p > 1$, we require the capacity $c_{l,p}$, as introduced in Sect. 10.4.1.

If $m > l$ and $p > 1$, then

$$\|\gamma\|_{M(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n))} \sim \sup_{\{e \subset \mathbb{R}^n, \operatorname{diam} e \leq 1\}} \frac{\| \operatorname{grad}_l \gamma\|_{L_p(e)}}{[c_{m,p}(e)]^{1/p}} + \sup_{x \in \mathbb{R}^n} \|\gamma\|_{L_1(B_1(x))}. \quad (11.12.5)$$

For $m = l$ the second term to the right in relations (11.12.4) and (11.12.5) has to be replaced by $\|\gamma\|_{L_\infty(\mathbb{R}^n)}$.

If $mp = n$ then the first term to the right in (11.12.4) takes the form

$$\sup_{x \in \mathbb{R}^n; 0<r<1} |\log 2/r|^{(p-1)/p} \| \operatorname{grad}_l \gamma\|_{L_q(B_r(x))},$$

and if $mp > n$ the space $M(W_p^m(\mathbb{R}^n) \to W_q^l(\mathbb{R}^n))$ coincides with *the space* $W_q^l(\mathbb{R}^n, \text{unif})$ of functions with the finite norm

$$\sup_{x \in \mathbb{R}^n} \|\gamma\|_{W_q^l(B_1(x))}.$$

Multipliers have a number of useful properties. For instance, the traces of multipliers are multipliers of traces. More exactly, each function in $MW_p^{l-1/p}(\partial\Omega)$ can be continued to Ω as a function in $MW_p^l(\Omega)$ (we assume that the boundary of Ω is smooth). The converse statement, that traces of functions in $MW_p^l(\Omega)$ belong to $MW_p^{l-1/p}(\partial\Omega)$, is trivial.

Sections 11.10 and 11.11. The results of these sections are due to the author [551, 560, 564].

12

Pointwise Interpolation Inequalities for Derivatives and Potentials

In the present chapter we consider various pointwise interpolation inequalities for derivatives of integer and fractional order and apply them to some weighted integral inequalities. In Sect. 12.1 we prove, among others, the estimate

$$\mathcal{M}I_z f(x) \le c\big(\mathcal{M}I_\zeta f(x)\big)^{\Re z/\Re\zeta}\big(\mathcal{M}f(x)\big)^{1-\Re z/\Re\zeta} \tag{12.0.1}$$

for all $x \in \mathbb{R}^n$ and $0 < \Re z < \Re\zeta < n$. Here I_z is the Riesz potential and $\mathcal{M}$ is the Hardy–Littlewood maximal operator. As an application we easily derive weighted multiplicative inequalities of the Gagliardo–Nirenberg type for complex powers of $-\Delta$ and $1-\Delta$.

Section 12.2 is mostly dedicated to the proof of the following sharp inequality and its corollaries

$$\big|\nabla u(x)\big| \le n(n+1)D_\omega(\nabla u; x)\Phi\bigg(\frac{\mathcal{M}^\diamond u(x)}{nD_\omega(\nabla u; x)}\bigg), \tag{12.0.2}$$

where

$$D_\omega(\mathbf{v}; x) = \sup_{r>0}\frac{1}{\omega(r)}\bigg|\mathbf{v}(x) - \fint_{\partial B_r(x)}\mathbf{v}(y)\,\mathrm{d}s_y\bigg|, \tag{12.0.3}$$

$\mathcal{M}^\diamond$ is the maximal operator defined by

$$\mathcal{M}^\diamond f(x) = \sup_{r>0}\bigg|\fint_{B_r(x)}\frac{y-x}{|y-x|}f(y)\,\mathrm{d}y\bigg|, \tag{12.0.4}$$

and Φ is a certain strictly increasing function generated by the continuity modulus ω. The barred integral stands for the mean value. A simple corollary of (12.0.2) is the inequality

$$\big|u'(x)\big|^2 \le \frac{8}{3}\big(\mathcal{M}^\diamond u\big)(x)\big(\mathcal{M}^\diamond u''\big)(x) \tag{12.0.5}$$

with the best constant.

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Grundlehren der mathematischen Wissenschaften 342,
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One of the results in Sect. 12.3 is the estimate

$$(D_{p,\alpha}u)(x) \le c\big(\big(\mathcal{M}\big|u-u(x)\big|^p\big)(x)\big)^{(1-\alpha)/p}\big(\big(\mathcal{M}|\nabla u|^q\big)(x)\big)^{\alpha/q}, \qquad (12.0.6)$$

where $0<\alpha<1$, $p\in[1,\infty)$, $q\ge \max\{1, pn/(n+p)\}$, and

$$(D_{p,\alpha}u)(x) = \bigg(\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^p}{|y-x|^{n+\alpha p}}\,\mathrm{d}y\bigg)^{1/p}.$$

Section 12.3 also contains simple proofs of weighted norm multiplicative inequalities of the Gagliardo–Nirenberg type for integer and fractional derivatives based on pointwise interpolation inequalities.

We conclude this chapter showing in Sect. 12.4 that (12.0.6) is useful in the question of continuity of the composition operator $u\to f(u)$ in a fractional Sobolev space.

12.1 Pointwise Interpolation Inequalities for Riesz and Bessel Potentials

12.1.1 Estimate for the Maximal Operator of a Convolution

Lemma. *Let k be a nonnegative, nonincreasing function in $L_1(0,\infty)$ and let $g\in L_1(\mathbb{R}^n,\mathrm{loc})$. Assume that $\mathcal{M}g(x)<\infty$ for some $x\in\mathbb{R}^n$. Then the integral*

$$(Kg)(x) = \int_{\mathbb{R}^n} k\big(|y-x|\big)g(y)\,\mathrm{d}y,$$

is absolutely convergent and

$$\mathcal{M}Kg(x) \le 2^{n+1}\int_{\mathbb{R}^n} k\big(|\xi|\big)\,\mathrm{d}\xi\ \mathcal{M}g(x)$$

Proof. Let $x=0$ and $r\in(0,\infty)$. We have

$$\begin{aligned}\fint_{B_r}\big|(Kg)(y)\big|\,\mathrm{d}y \le 2^n\fint_{B_{2r}}\big|g(z)\big|\int_{B_r(z)}k\big(|y|\big)\,\mathrm{d}y\,\mathrm{d}z&\\ +\fint_{B_r}\int_{\mathbb{R}^n\setminus B_{2r}}k\big(|z-y|\big)\big|g(z)\big|\,\mathrm{d}z\,\mathrm{d}y.&\qquad (12.1.1)\end{aligned}$$

The first term on the right in (12.1.1) is majorized by

$$2^n\int_{\mathbb{R}^n}k\big(|y|\big)\,\mathrm{d}y\fint_{B_{2r}}\big|g(z)\big|\,\mathrm{d}z. \qquad (12.1.2)$$

Since in the second term $|z-y|\ge|z|-r\ge|z|/2$, we have $k(|z-y|)\le k(|z|/2)$ and it follows that this term does not exceed

$$\int_{\mathbb{R}^n} k\big(|z|/2\big)\big|g(z)\big|\,\mathrm{d}z$$
$$= \int_0^\infty \int_{B_{2t}} \big|g(\xi)\big|\,\mathrm{d}\xi\,\big|\mathrm{d}k(t)\big| \le 2^n v_n \int_0^\infty t^n\big|\mathrm{d}k(t)\big|\,\mathcal{M}g(0)$$
$$= 2^n \int_{\mathbb{R}^n} k\big(|\xi|\big)\,\mathrm{d}\xi\,\mathcal{M}g(0).$$

This together with (12.1.2) completes the proof. □

12.1.2 Pointwise Interpolation Inequality for Riesz Potentials

Let z be a complex number with $\Re z < n$. The Riesz potential I_z is defined by

$$I_z f(x) = c\int_{\mathbb{R}^n} \frac{f(y)}{|y-x|^{n-z}}\,\mathrm{d}y,$$

where the constant c is chosen in such a way that $I_z = (-\Delta)^{-z/2}$, i.e.,

$$I_z f(x) = F^{-1}_{\xi\to x}|\xi|^{-z}F_{x\to\xi}f,$$

with F denoting the Fourier transform in $\mathbb{R}^n$.

Theorem. *Let $f \in L_{1,loc}(\mathbb{R}^n)$ and let $0 < \Re z < \Re\zeta < n$. Then*

$$\mathcal{M}I_z f(x) \le c\big(\mathcal{M}I_\zeta f(x)\big)^{\Re z/\Re\zeta}\big(\mathcal{M}f(x)\big)^{1-\Re z/\Re\zeta}. \tag{12.1.3}$$

Proof. We introduce a function χ in the Schwartz space S such that $F\chi = 1$ in a neighborhood of the origin, and the functions

$$P(x) = c_1\ F^{-1}_{\xi\to x}\big(|\xi|^{\zeta-z}F\chi(\xi)\big), \tag{12.1.4}$$

$$Q(x) = c_2\ F^{-1}_{\xi\to x}\big(|\xi|^{-z}\big(1-F\chi(\xi)\big)\big). \tag{12.1.5}$$

It is then evident that

$$I_z f(0) = P * I_\zeta f(0) + Q * f(0). \tag{12.1.6}$$

Let m be a positive integer such that

$$0 < m - \Re\zeta + \Re z \le 1.$$

Since

$$P(x) = c\sum_{|\alpha|=m}\frac{m!}{\alpha!}\partial_x^\alpha\int_{\mathbb{R}^n}\chi(y)\partial_x^\alpha|x-y|^{2m-n-\zeta+z}\,\mathrm{d}y,$$

where ∂_x is the gradient, we have

$$\left|P(x)\right| \le c\big(|x|+1\big)^{-n-\Re\zeta+\Re z}.$$

Therefore, $P \in L_1(\mathbb{R}^n)$ and by Lemma 12.1.1,

$$\mathcal{M}(P * I_\zeta f)(0) \le c\,\mathcal{M} I_\zeta f(0). \tag{12.1.7}$$

We observe that the function $|\xi|^{-z}(1 - F\chi(\xi))$ is smooth, so for $|y| \ge 1$ and for sufficiently large N

$$\left|Q(y)\right| \le c(N)|y|^{-N}.$$

For $|y| < 1$

$$\left|Q(y)\right| \le c|y|^{-n+\Re z} + \left|I_z\chi(y)\right|.$$

Since the second term on the right is bounded, the last two estimates imply $Q \in L_1(\mathbb{R}^n)$. By Lemma 12.1.1,

$$\mathcal{M}(Q * f)(0) \le c\,\mathcal{M} f(0).$$

Combining this with (12.1.7) and (12.1.6), we find

$$\mathcal{M} I_z f(0) \le c\big(\mathcal{M} I_\zeta f(0) + \mathcal{M} f(0)\big).$$

The dilation $y \to y/t$ with an arbitrary $t > 0$ implies

$$\mathcal{M} I_z f(0) \le c\big(t^{\Re(z-\zeta)}\mathcal{M} I_\zeta f(0) + t^{\Re z}\mathcal{M} f(0)\big),$$

and it remains to minimize the right-hand side in t. □

12.1.3 Estimates for $|J_{-w}\chi_\rho|$

Let z be a complex number. The Bessel potential J_z is defined by $J_z = (-\Delta + 1)^{-z/2}$, i.e.,

$$J_z f(x) = F^{-1}_{\xi\to x}\big(1 + |\xi|^2\big)^{-z/2} F_{x\to\xi} f.$$

Another formula for J_z is

$$J_z f(x) = c\int_{\mathbb{R}^n} G_z(x-y) f(y)\,\mathrm{d}y, \tag{12.1.8}$$

where

$$G_z(x) = c|x|^{(z-n)/2} K_{(n-z)/2}\big(|x|\big).$$

K_ν is the modified Bessel function of the third kind. We formulate some estimates of the kernel G_z used in the following (see Aronszajn [53]).

For $|x| \le 1$ one has

$$\left|\nabla_k G_z(x)\right| \le \begin{cases} c\log(2/|x|) & \text{for } z = n+k, k \text{ even},\\ c(|x|^{\Re z-n-k}+1) & \text{for other values of } z. \end{cases} \tag{12.1.9}$$

Further, for $|x| > 1$

$$\left|\nabla_k G_z(x)\right| \le c|x|^{(\Re z-n-1)/2} e^{-|x|}. \tag{12.1.10}$$

As in Sect. 12.1.2, by χ we denote a function in the Schwartz space S such that $F\chi = 1$ in B_1. We put

$$\chi_\rho(x) = (m_n B_\rho)^{-1}\chi(x/\rho).$$

Lemma. *The following inequalities are valid:*

$$\left|(-\Delta+1)^{w/2}\chi_\rho(x)\right| \le \begin{cases} c_L|x|^{-L} & \text{for } |x|>1 \text{ and arbitrary } L>0,\\ c((|x|+\rho)^{-n-\Re w}+1) & \text{for } |x|\le 1 \text{ and } w\ne -n,\\ c|\log(2|x|+\rho)| & \text{for } |x|\le 1 \text{ and } w=-n. \end{cases}$$

(*In the trivial case of even positive w sharper estimates hold.*)

Proof. For $|x| > 1$ and arbitrarily large L we have

$$\begin{aligned} &|x|^{2L}\left|(-\Delta+1)^{w/2}\chi_\rho(x)\right| \\ &\quad = \left|F^{-1}_{\xi\to x}(-\Delta_\xi)^L\big(\xi^2+1\big)^{w/2}F_{x\to\xi}\chi_\rho\right| \\ &\quad \le c\int_{\mathbb{R}^n}\sum_{k=0}^{2L}\left|\nabla_{k,\xi}\big(\xi^2+1\big)^{w/2}\right|\left|\nabla_{2L-k,\xi}F_{x\to\xi}\chi_\rho\right|\,\mathrm{d}\xi. \end{aligned} \tag{12.1.11}$$

By using $F_{x\to\xi}\chi_\rho = (F\chi)(\rho\xi)$ we find

$$|\nabla_{2L-k}F_{x\to\xi}\chi_\rho| \le \frac{c_L\rho^{2L-k}}{(\rho|\xi|+1)^N},$$

where N is an arbitrarily large positive number. Therefore, the integral on the right in (12.1.11) is majorized by

$$\begin{aligned} &c\sum_{k=0}^{2L}\rho^{2L-k}\int_{\mathbb{R}^n\setminus B_{1/\rho}}\big(|\xi|+1\big)^{\Re w-k}\big(\rho|\xi|+1\big)^{-N}\,\mathrm{d}\xi \\ &\quad + c\int_{B_{1/\rho}}\big(|\xi|+1\big)^{\Re(\zeta-z)-2L}\big(\rho|\xi|+1\big)^{-N}\,\mathrm{d}\xi \le c\rho^{2L-\Re(\zeta-z)-n}+c. \end{aligned}$$

Hence

$$\left|(-\Delta+1)^{w/2}\chi_\rho(x)\right| \le c|x|^{-2L} \quad \text{for } |x|>1. \tag{12.1.12}$$

For $|x| < \rho$

$$\begin{aligned} &\left|(-\Delta+1)^{w/2}\chi_\rho(x)\right| \\ &\quad \le c\bigg(\int_{\mathbb{R}^n\setminus B_{1/\rho}}|\xi|^{\Re w}\big(\rho|\xi|\big)^{-L}\,\mathrm{d}\xi + \rho^{-\Re w}\int_{B_{1/\rho}}\mathrm{d}\xi\bigg) \le c\rho^{-n-\Re w}. \end{aligned} \tag{12.1.13}$$

It remains to estimate the left-hand side of (12.1.13) for $\rho < |x| < 1$. We start with $\Re w \le 0$. The case $w = 0$ is trivial. If $w \ne 0$ we write

$$(-\Delta + 1)^w \chi_\rho(x) = \int_{\mathbb{R}^n} G_{-w}(x-y)\chi_\rho(y)\,\mathrm{d}y. \tag{12.1.14}$$

We divide the integration domain into $B_2(x)$ and $\mathbb{R}^n\backslash B_2(x)$. By (12.1.10) the integral over $\mathbb{R}^n\backslash B_2(x)$ is dominated by

$$c\rho^{-n}\int_{\mathbb{R}^n\backslash B_2(x)} |x-y|^{-N}\Big(\frac{\rho}{|y|}\Big)^L \mathrm{d}y \le c\rho^{L-n}\int_{\mathbb{R}^n\backslash B_1} \frac{\mathrm{d}y}{|y|^{N+L}}.$$

The estimates of the integral over $B_2(x)$ are also straightforward. Let, for example, $\Re w > -n$. By (12.1.9) the integral in question is majorized by

$$\begin{aligned}
&c\rho^{-n}\int_{B_3} |x-y|^{-n-\Re w}\Big(\frac{\rho}{|y|+\rho}\Big)^L \mathrm{d}y\\
&\quad\le c\rho^{L-n}\int_{B_3\backslash B_{2|x|}} \frac{\mathrm{d}y}{|y|^{n+\Re w+L}} + c\frac{\rho^{L-n}}{|x|^L}\int_{B_{2|x|}\backslash B_{|x|/2}} |x-y|^{-n-\Re w}\,\mathrm{d}y\\
&\qquad + c\rho^{L-n}|x|^{-n-\Re w}\int_{B_{|x|/2}} \frac{\mathrm{d}y}{(|y|+\rho)^L}\\
&\quad\le c|x|^{-n-\Re w}.
\end{aligned}$$

Setting $s = [\Re w] > 0$ and $t = \{\Re w\}$ we proceed by induction in s. Let the required estimates be proved for $[\Re w] < s$. We use the identity

$$(-\Delta+1)^{\frac{s+t}{2}} = (-\Delta)^{s+1}(-\Delta+1)^{\frac{t-s}{2}-1} + \sum_{k=0}^{s}(-1)^{s-k}\binom{s+1}{k}(-\Delta+1)^{k-1+\frac{t-s}{2}},$$

which can be verified directly. By induction hypothesis the functions

$$(-\Delta+1)^{k-1+\frac{t-s}{2}+i\Im w}$$

have the majorant $c|x|^{-n-s-t}$ for $\rho < |x| < 1$. Hence we are left with estimating the function

$$\begin{aligned}
&(-\Delta)^{s+1}(-\Delta+1)^{\frac{t-s}{2}-1+i\Im w}\chi_\rho(x)\\
&\quad = c\sum_{|\alpha|=s+1}\frac{(s+1)!}{\alpha!}\int_{\mathbb{R}^n}\partial_y^\alpha\chi_\rho(y)\partial_x^\alpha G_{2+2s-w}(x-y)\,\mathrm{d}y.
\end{aligned} \tag{12.1.15}$$

We introduce a cutoff function $\kappa \in C_0^\infty(B_4\backslash B_{1/4})$, $\kappa = 1$ on $B_2\backslash B_{1/2}$. Suppose $n + t > 1$. The remaining case $n = 1$, $t = 0$ will be dealt with separately. Integrating by parts, we have

$$\left|\int_{\mathbb{R}^n}\left(1-\kappa\left(\frac{y}{|x|}\right)\right)\partial_y^\alpha\chi_\rho(y)\partial_x^\alpha G_{2+2s-w}(x-y)\,\mathrm{d}y\right|$$
$$\le c\sum_{k=0}^{s+1}\int_{\mathbb{R}^n}\left|\chi_\rho(y)\right|\left|\nabla_{k,y}\left(1-\kappa\left(\frac{y}{|x|}\right)\right)\right|\left|\nabla_{2s+2-k}G_{2+2s-w}(x-y)\right|\mathrm{d}y. \tag{12.1.16}$$

By (12.1.9) and (12.1.10) for $k\le s+1$

$$\left|\nabla_{2s+2-k}G_{2+2s-w}(x-y)\right|\le c|x-y|^{-n-s-t+k}. \tag{12.1.17}$$

(Note that $-n-s-t+k$ is negative since $n+t>1$.) Hence the sum on the right-hand side of (12.1.16) is majorized by

$$c\Bigg(\int_{B_{|x|/2}\cup(\mathbb{R}^n\setminus B_{2|x|})}\left|\chi_\rho(y)\right||x-y|^{-n-s-t}\,\mathrm{d}y$$
$$+\sum_{k=1}^{s+1}|x|^{-k}\left(\int_{B_{4|x|}\setminus B_{2|x|}}+\int_{B_{|x|/2}\setminus B_{|x|/4}}\right)\left|\chi_\rho(y)\right||x-y|^{-n-s-t+k}\,\mathrm{d}y\Bigg). \tag{12.1.18}$$

Clearly,

$$\int_{B_{|x|/2}}\left|\chi_\rho(y)\right||x-y|^{-n-s-t}\,\mathrm{d}y\le c|x|^{-n-s-t}$$

and

$$\int_{\mathbb{R}^n\setminus B_{2|x|}}\left|\chi_\rho(y)\right||x-y|^{-n-s-t}\,\mathrm{d}y\le c_L\left(\frac{\rho}{|x|}\right)^L|x|^{-n-s-t}.$$

Similarly, the integrals over $B_{4|x|}\setminus B_{2|x|}$ and $B_{|x|/2}\setminus B_{|x|/4}$ in (12.1.18) are estimated by

$$c_L\left(\frac{\rho}{|x|}\right)^L|x|^{-n-s-t+k}.$$

Thus

$$\left|\int_{\mathbb{R}^n}\left(1-\kappa\left(\frac{y}{|x|}\right)\right)\partial_y^\alpha\chi_\rho(y)\partial_x^\alpha G_{2+2s-w}(x-y)\,\mathrm{d}y\right|\le c|x|^{-n-s-t}. \tag{12.1.19}$$

To estimate

$$\left|\int_{\mathbb{R}^n}\kappa\left(\frac{y}{|x|}\right)\partial_y^\alpha\chi_\rho(y)\partial_x^\alpha G_{2+2s-w}(x-y)\,\mathrm{d}y\right| \tag{12.1.20}$$

we use (12.1.17) for $k=s+1$ and obtain that for any sufficiently large $K>0$ (12.1.20) does not exceed

$$c_K \rho^{-n-s-1}\Big(\frac{\rho}{|x|}\Big)^K \int_{B_{4|x|}\setminus B_{|x|/4}} |x-y|^{1-t-n}\,\mathrm{d}y$$
$$\leq c_K\Big(\frac{\rho}{|x|}\Big)^{K-n-s-1} |x|^{-t-n-s}.$$

Combining this with (12.1.19), we find

$$\big|(-\Delta)^{s+1}(-\Delta+1)^{\frac{t-s}{2}-1+i\Im w}\chi_\rho(x)\big| \leq c_L|x|^{-n-s-t}\Big(\frac{\rho}{|x|}\Big)^L$$

for $\rho<|x|<1$. Thus, for $n+t>1$ the result follows.

Let $n=1, t=0$. As before, we have to estimate the function (12.1.15), that is, the integral

$$\int_{\mathbb{R}^1} \partial_y^{s+1}\chi_\rho(y)\partial_x^{s+1}G_{2+s}(x-y)\,\mathrm{d}y$$
$$= \int_{\mathbb{R}^1}\Big(1-\kappa\Big(\frac{y}{|x|}\Big)\Big)\partial_y^s\chi_\rho(y)\partial_x^{s+2}G_{2+s}(x-y)\,\mathrm{d}y$$
$$+\int_{\mathbb{R}^1}\kappa\Big(\frac{y}{|x|}\Big)\partial_y^s\chi_\rho(y)\partial_x^{s+2}G_{2+s}(x-y)\,\mathrm{d}y. \tag{12.1.21}$$

The first integral on the right does not exceed

$$c\sum_{k=0}^{s}\int_{\mathbb{R}^1}\big|\chi_\rho(y)\big|\Big|\partial_y^k\Big(1-\kappa\Big(\frac{y}{|x|}\Big)\Big)\Big|\big|\partial_x^{2s+2-k}G_{2+s}(x-y)\big|\,\mathrm{d}y.$$

In view of (12.1.17) the argument previously used to estimate the right-hand side of (12.1.16) gives the majorant $c|x|^{-1-s}$ for the last sum.

We pass to the second integral on the right in (12.1.21). Restricting ourselves to $x>0$ without loss of generality, we can rewrite it as

$$\int_{x/4}^{4x}\Big(\kappa\Big(\frac{y}{x}\Big)\partial_y^s\chi_\rho(y)-\kappa(1)\partial_x^s\chi_\rho(x)\Big)\partial_x^{s+2}G_{s+2}(x-y)\,\mathrm{d}y$$
$$+\kappa(1)\partial_x^s\chi_\rho(x)\int_{-3x}^{3x/4}\partial_t^{s+2}G_{s+2}(t)\,\mathrm{d}t. \tag{12.1.22}$$

It is enough to assume that s is odd. We have

$$\partial_t^{s+2}G_{s+2}(t) = \mathrm{const}\,t^{-1}+O(1)\quad \text{as } t\to 0,$$

because $\xi^{s+2}(\xi^2+1)^{-(s+2)/2}$ is asymptotically equal to $\operatorname{sgn}\xi$ as $|\xi|\to\infty$. Therefore, the second term in (12.1.22) does not exceed $c_L x^{-1-s}(\rho/x)^{L-1-s}$ and the first one is dominated by

$$c\int_{x/4}^{4x}\frac{|\kappa(\frac{y}{x})\partial_y^s\chi_\rho(y)-\kappa(1)\partial_x^s\chi_\rho(x)|}{|x-y|}\,\mathrm{d}y$$
$$\le cx\sup_{(x/4,4x)}\left|\partial_y\left(\kappa\left(\frac{y}{x}\right)\partial_y^s\chi_\rho(y)\right)\right|\le c_L x^{-1-s}\left(\frac{\rho}{x}\right)^L.$$

Thus the second integral on the right-hand side of (12.1.21) has the majorant $c_L|x|^{-1-s}(\rho/x)^L$. Hence the function (12.1.21) is dominated by $c|x|^{-1-s}$, which completes the proof. □

12.1.4 Estimates for $|J_{-w}(\delta-\chi_\rho)|$

Lemma. (i) *For an arbitrarily large L there exists a constant c_L such that for $|x|>\rho$*

$$\left|(-\Delta+1)^{w/2}(\delta-\chi_\rho)(x)\right|\le c_L\left(\frac{\rho}{|x|}\right)^L\rho^{-n-\Re w}.$$

Here and elsewhere δ stands for the Dirac function.

(ii) *There exists a constant c such that for $|x|<\rho$*

$$\left|(-\Delta+1)^{w/2}(\delta-\chi_\rho)(x)\right|\le\begin{cases}c|x|^{-\Re w-n} & \text{for } \Re w\ge -n,\ w\ne -n,\\ c\log(2\rho|x|^{-1}) & \text{for } w=-n,\\ c\rho^{-\Re w-n} & \text{for } \Re w<-n.\end{cases}$$

Proof. (i) We need to estimate the absolute value of

$$\varphi_\rho(x)=F^{-1}_{\xi\to x}\big((\xi^2+1)^{w/2}\big(1-(F\chi)(\rho\xi)\big)\big).$$

Let $|x|>\rho$ and let N be a sufficiently large, positive integer. We have

$$|x|^{2N}|\varphi_\rho(x)|\le\int_{\mathbb{R}^n}\left|\Delta_\xi^N\big((\xi^2+1)^{w/2}\big(1-(F\chi)(\rho\xi)\big)\big)\right|\mathrm{d}\xi$$
$$\le c\int_{\mathbb{R}^n}\sum_{k=0}^{2N}\left|\nabla_{k,\xi}\big(1-(F\chi)(\rho\xi)\big)\right|\frac{\mathrm{d}\xi}{(\xi^2+1)^{(2N-k-\Re w)/2}}.$$

Since $F\chi=1$ in a neighborhood of the origin,

$$|x|^{2N}|\varphi_\rho(x)|\le c\int_{\mathbb{R}^n}\sum_{k=0}^{2N}\frac{\rho^k\,\mathrm{d}\xi}{(\xi^2+\rho)^{2N-k-\Re w}}=c\rho^{2N-n-\Re w},$$

which gives the result.

(ii) Now let $|x|<\rho$. In the case $\Re w\ge -n$, $w\ne -n$ the assertion follows from (12.1.9) and Lemma 12.1.3. Setting $\varXi=\rho\xi$ and $X=x/\rho$ we obtain

$$\varphi_\rho(x) = \rho^{-w-n} F^{-1}_{\Xi\to X}\big((\Xi^2+\rho^2)^{w/2}(1-(F\chi)(\Xi))\big).$$

In the case $\Re w < -n$

$$|\varphi_\rho(x)| \le c\rho^{-\Re w-n}\int_{\mathbb{R}^n}(\Xi^2+1)^{\Re w/2}\,\mathrm{d}\Xi = c\rho^{-\Re w-n}.$$

In the remaining case $\Re w = -n$ we notice that for $\Xi \subset \operatorname{supp}(1-F\chi)$

$$(\Xi^2+\rho^2)^{w/2} = (\Xi^2+1)^{w/2} + O\big(|\Xi|+1\big)^{-n-1},$$

uniformly with respect to $\rho \in (0,1)$. Consequently,

$$\varphi_\rho(x) = \rho^{-w-n} F^{-1}_{\Xi\to X}\big((\Xi^2+1)^{w/2}(1-(F\chi)(\Xi)) + O(1)\big)$$

and therefore

$$|\varphi_\rho(x)| \le |G_{-w}(X)| + |J_{-w}\chi_\rho(X)|.$$

The second term on the right is bounded uniformly with respect to $\rho \in (0,1)$ and the first term is $O(\log(2|X|^{-1}))$ if $w=-n$, and is bounded if $\Re w = -n$, $\Im w \ne 0$. Hence $\varphi_\rho(x)$ admits the required estimates for $|x|<\rho$. $\square$

12.1.5 Pointwise Interpolation Inequality for Bessel Potentials

Theorem. *Let* $f \in L_1(\mathbb{R}^n, \mathrm{loc})$ *and let* $0 < \Re z < \Re\zeta$. *Then*

$$\mathcal{M}J_z f(x) \le c\big(\mathcal{M}J_\zeta f(x)\big)^{\Re z/\Re\zeta}\big(\mathcal{M}f(x)\big)^{1-\Re z/\Re\zeta} \qquad (12.1.23)$$

for all $x \in \mathbb{R}^n$.

Proof. We introduce the functions

$$P_\rho = (-\Delta+1)^{(\zeta-z)/2}\chi_\rho, \qquad (12.1.24)$$

$$Q_\rho = J_z(\delta-\chi_\rho), \qquad (12.1.25)$$

where $\rho \in (0,1)$. Clearly,

$$J_z f = P_\rho * J_\zeta f + Q_\rho * f. \qquad (12.1.26)$$

We claim that

$$\fint_{B_r} |(P_\rho * J_\zeta f)(y)|\,\mathrm{d}y \le c\rho^{-\Re(\zeta-z)}\mathcal{M}J_\zeta f(0) \qquad (12.1.27)$$

for all $r>0$ and $\rho \in (0,1)$. In fact,

$$\int_{\mathbb{R}^n}|P_\rho(|t|)|\,\mathrm{d}t \le c\bigg(\int_{\mathbb{R}^n}\frac{\mathrm{d}\xi}{(|\xi|+1)^L} + \int_{\mathbb{R}^n}\frac{\mathrm{d}\xi}{(|\xi|+\rho)^{n+\Re(\zeta-z)}}\bigg) \le c\rho^{-\Re(\zeta-z)}.$$

Hence (12.1.27) follows from Lemma 12.1.1.

By Lemma 12.1.1,

$$\big|Q_\rho(x)\big| \le \begin{cases} c\rho^{\Re z-n}(\rho/|x|)^{n+1} & \text{for } |x|>\rho, \\ c|x|^{\Re z-n} & \text{for } |x|<\rho. \end{cases}$$

Therefore

$$\int_{\mathbb{R}^n} \big|Q_\rho(x)\big|\,\mathrm{d}x \le c\rho^{\Re z}$$

and by Lemma 12.1.1

$$\fint_{B_r} \big|(Q_\rho * f)(y)\big|\,\mathrm{d}y \le c\rho^{\Re z}\mathcal{M}f(0). \tag{12.1.28}$$

Combining this with (12.1.27), we arrive at

$$\mathcal{M}J_z f(0) \le c\big(\rho^{-\Re(\zeta-z)}\mathcal{M}J_\zeta f(0) + \rho^{\Re z}\mathcal{M}f(0)\big) \tag{12.1.29}$$

for all $\rho \in (0,1)$. If

$$\mathcal{M}J_\zeta f(0) < \frac{\Re z}{\Re(\zeta - z)}\mathcal{M}f(0),$$

then the minimum of the right-hand side of (12.1.29) on $(0,1]$ is attained at

$$\rho = \left(\frac{\Re(\zeta-z)\,\mathcal{M}J_\zeta f(0)}{\Re z\,\mathcal{M}f(0)}\right)^{1/\Re\zeta} \in (0,1)$$

and is equal to

$$c\big(\mathcal{M}J_\zeta f(0)\big)^{\Re z/\Re\zeta}\big(\mathcal{M}f(0)\big)^{1-\Re z/\Re\zeta},$$

which gives (12.1.23). If

$$\mathcal{M}J_\zeta f(0) \ge \frac{\Re z}{\Re(\zeta - z)}\mathcal{M}f(0),$$

we have by (12.1.29)

$$\mathcal{M}J_z f(0) \le c\big(\mathcal{M}J_\zeta f(0) + \mathcal{M}f(0)\big) \le c_1\mathcal{M}J_\zeta f(0).$$

Since by Lemma 12.1.1

$$\mathcal{M}J_\zeta f(0) \le c\mathcal{M}f(0),$$

we arrive at (12.1.23). The proof is complete. □

Remark 1. Inequality (12.1.3) can be deduced from (12.1.23) by dilation $x \to x/\epsilon$ and by passage to the limit as $\epsilon \to 0$. However, we included an independent proof of (12.1.3) because it is simpler than that of (12.1.23).

Remark 2. If $\Im z = 0$ and $f \geq 0$, one has

$$\mathcal{M} I_z f(x) \leq c I_z f(x) \quad \text{and} \quad \mathcal{M} J_z f(x) \leq c J_z f(x)$$

(see, in particular, Lemma 12.1.1). This means that for real z, ζ, and nonnegative f inequalities (12.1.3) and (12.1.23) are equivalent to Hedberg's inequalities (cf. [365] and [15], Sect. 3.1), which contains no operator $\mathcal{M}$ in front of the potentials.

Obvious changes in the proofs of Theorems 12.1.2 and 12.1.5 give the following generalization of (12.1.3) and (12.1.23).

Proposition. *Let $f \in L^1_{loc}(\mathbb{R}^n)$.*
(i) *If $k < \Re\zeta < n$ then for all $x \in \mathbb{R}^n$*

$$(\mathcal{M}\nabla_k I_\zeta f)(x) \leq c\big((\mathcal{M} I_\zeta f)(x)\big)^{1-k/\Re\zeta}\big((\mathcal{M} f)(x)\big)^{k/\Re\zeta}. \tag{12.1.30}$$

(ii) *If $k < \Re\zeta$ then for all $x \in \mathbb{R}^n$*

$$(\mathcal{M}\nabla_k J_\zeta f)(x) \leq c\big((\mathcal{M} J_\zeta f)(x)\big)^{1-k/\Re\zeta}\big((\mathcal{M} f)(x)\big)^{k/\Re\zeta}. \tag{12.1.31}$$

12.1.6 Pointwise Estimates Involving $\mathcal{M}\nabla_k u$ and $\Delta^l u$

Here we obtain two multiplicative inequalities involving integer powers of the Laplacian.

Corollary. *Let k and l be positive integers. Then for all $x \in \mathbb{R}^n$*

$$(\mathcal{M}\nabla_k u)(x) \leq \big((\mathcal{M} u)(x)\big)^{1-\frac{k}{2l}}\big((\mathcal{M}\Delta^l u)(x)\big)^{\frac{k}{2l}}, \quad k < 2l, \tag{12.1.32}$$

$$(\mathcal{M}\nabla_k u)(x) \leq \big((\mathcal{M} u)(x)\big)^{1-\frac{k}{2l+1}}\big((\mathcal{M}\nabla\Delta^l u)(x)\big)^{\frac{k}{2l+1}}, \quad k < 2l+1. \tag{12.1.33}$$

Proof. Setting $\zeta = l$ and $u = J_l f$ in (12.1.31), we arrive at

$$(\mathcal{M}\nabla_k u)(x) \leq \big((\mathcal{M} u)(x)\big)^{1-\frac{k}{l}}\big((\mathcal{M}(-\Delta+1)^{l/2} u)(x)\big)^{\frac{k}{l}}.$$

Now (12.1.32) follows by the dilation $x \to x\varepsilon$ and passage to the limit as $\varepsilon \to 0$.

By (12.1.32) with k replaced by $k-1$ we have

$$\Big(\mathcal{M}\nabla_{k-1}\frac{\partial u}{\partial x_i}\Big)(x) \leq \bigg(\Big(\mathcal{M}\frac{\partial u}{\partial x_i}\Big)(x)\bigg)^{1-\frac{k-1}{2l}}\big((\mathcal{M}\nabla\Delta^l u)(x)\big)^{\frac{k-1}{2l}}, \tag{12.1.34}$$

where $i = 1, 2, \ldots, n$. Using (12.1.32) once more with $k = l = 1$ we obtain

$$\left(\mathcal{M}\frac{\partial u}{\partial x_i}\right)(x) \le c\big((\mathcal{M}u)(x)\big)^{1/2}\big((\mathcal{M}\Delta u)(x)\big)^{1/2}. \tag{12.1.35}$$

To estimate $\mathcal{M}\Delta u$ we write $\nabla u = \mathbf{v}$ and note that by (12.1.32)

$$(\mathcal{M}\,\mathrm{div}\,\mathbf{v})(x) \le c\big((\mathcal{M}\mathbf{v})(x)\big)^{1-\frac{1}{2l}}\big((\mathcal{M}\Delta^l\mathbf{v})(x)\big)^{\frac{1}{2l}},$$

which implies

$$(\mathcal{M}\Delta u)(x) \le c\big((\mathcal{M}\nabla u)(x)\big)^{1-\frac{1}{2l}}\big(\big(\mathcal{M}\nabla\Delta^l u\big)(x)\big)^{\frac{1}{2l}}.$$

Combining this inequality with (12.1.35) we find

$$(\mathcal{M}\nabla u)(x) \le c\big((\mathcal{M}u)(x)\big)^{\frac{2l}{2l+1}}\big(\big(\mathcal{M}\nabla\Delta^l u\big)(x)\big)^{\frac{1}{2l+1}},$$

which together with (12.3.13) completes the proof of (12.1.33). □

12.1.7 Application: Weighted Norm Interpolation Inequalities for Potentials

Let w be a nonnegative measurable function on $\mathbb{R}^n$. By $L_p(w\,\mathrm{d}x)$ we mean the space of measurable functions f with the finite norm

$$\|f\|_{L_p(w\,\mathrm{d}x)} = \left(\int_{\mathbb{R}^n}\big|f(x)\big|^p w(x)\,\mathrm{d}x\right)^{1/p}. \tag{12.1.36}$$

Proposition. *Let* $1 < q < \infty$, $1 < p < \infty$, *and let*

$$\frac{1}{s} = \left(1 - \frac{\Re z}{\Re\zeta}\right)\frac{1}{p} + \frac{\Re z}{\Re\zeta}\frac{1}{q}.$$

Further, let the weight function w *belong to the Muckenhoupt class* $A_{\min\{q,p\}}$. *Then*

$$\|I_z f\|_{L_s(w\,\mathrm{d}x)} \le c\|I_\zeta f\|^{\Re z/\Re\zeta}_{L_q(w\,\mathrm{d}x)}\|f\|^{1-\Re z/\Re\zeta}_{L_p(w\,\mathrm{d}x)}, \quad 0 < \Re z < \Re\zeta < n, \tag{12.1.37}$$

$$\|J_z f\|_{L_s(w\,\mathrm{d}x)} \le c\|J_\zeta f\|^{\Re z/\Re\zeta}_{L_q(w\,\mathrm{d}x)}\|f\|^{1-\Re z/\Re\zeta}_{L_p(w\,\mathrm{d}x)}, \quad 0 < \Re z < \Re\zeta. \tag{12.1.38}$$

Proof. By Theorem 12.1.2

$$\|I_z f\|_{L_s(w\,\mathrm{d}x)} \le c\left(\int_{\mathbb{R}^n}\big(\mathcal{M}I_\zeta f(x)\big)^{s\Re z/\Re\zeta}\big(\mathcal{M}f(x)\big)^{s(1-\Re z/\Re\zeta)}w(x)\,\mathrm{d}x\right)^{1/s},$$

which by Hölder's inequality is majorized by

$$c\|\mathcal{M}I_\zeta f\|^{\Re z/\Re\zeta}_{L_q(w\,\mathrm{d}x)}\|\mathcal{M}f\|^{1-\Re z/\Re\zeta}_{L_p(w\,\mathrm{d}x)}.$$

It remains to refer to the Muckenhoupt theorem on the boundednesss of the operator $\mathcal{M}$ in $L_\sigma(w\,\mathrm{d}x)$ for $1 < \sigma < \infty$. Inequality (12.1.37) is proved.

Duplicating the same arguments and using Theorem 12.1.5 one arrives at (12.1.38). □

12.2 Sharp Pointwise Inequalities for ∇u

12.2.1 The Case of Nonnegative Functions

Let v be a function in $\mathbb{R}^n$ and let the function $T_\omega(\nabla f;x)$ be defined by (1.3.45), where $\mathbb{R}$ is replaced by $\mathbb{R}^n$. We start with the following almost obvious multidimensional corollary of the one-dimensional estimate (1.3.47).

Theorem. *Let f be a differentiable nonnegative function on $\mathbb{R}^n$. Then for all $x \in \operatorname{supp} f$*

$$\big|\nabla f(x)\big| \le T_\omega(\nabla f;x)\psi^{-1}\bigg(\frac{f(x)}{T_\omega(\nabla f;x)}\bigg), \tag{12.2.1}$$

where ψ^{-1} is the inverse of (1.3.46). *If ω is concave, then this inequality with $x=0$ becomes an equality for*

$$f(x) = \psi(1) - x_n + \int_0^{x_n} \omega(\tau)\,\mathrm{d}\tau.$$

Proof. Let $h(t) = f(x+te_x)$, where e_x is a unit vector parallel to the gradient direction of f at the point x and $t \in \mathbb{R}$. Theorem 1.3.6 applied to h yields

$$\big|h'(t)\big| \le T_\omega(h';x)\psi^{-1}\left(\frac{h(t)}{T_\omega(h';x)}\right).$$

The estimate (12.2.1) follows if we notice that the function (1.3.51) is increasing.

The sharpness of inequality (12.2.1) is proved in the same way as in Theorem 1.3.6. □

Remark. The last theorem obviously implies the following n-dimensional generalization of (1.3.52):

$$\big|\nabla f(x)\big|^{\alpha+1} \le \left(\frac{\alpha+1}{\alpha}\right)^{\alpha}\big(f(x)\big)^{\alpha} \sup_{y\in\mathbb{R}^n} \frac{|\nabla f(x)-\nabla f(y)|}{|x-y|^{\alpha}}, \tag{12.2.2}$$

where $\alpha > 0$. The equality sign in (12.2.2) with $x=0$ is attained at

$$f(x) = \frac{x_n^{\alpha+1}+\alpha}{\alpha+1} - x_n.$$

12.2.2 Functions with Unrestricted Sign. Main Result

Let $\mathcal{M}^\diamond$ be the maximal operator defined by (12.0.4) where u is a locally integrable function on $\mathbb{R}^n$, $n \ge 1$, $B_r(x)$ is the ball $\{y \in \mathbb{R}^n : |x-y| < r\}$, and the bar stands for the mean value of the integral. Clearly, $\mathcal{M}^\diamond u(x)$ does not exceed the sharp maximal function of Fefferman and Stein [274]:

$$\mathcal{M}^\sharp u(x) = \sup_{r>0} \fint_{B_r(x)} \left| u(y) - \fint_{B_r(x)} u(z)\,\mathrm{d}z \right| \mathrm{d}y.$$

We write B_r instead of $B_r(0)$ and use the notation $|B_1| = m_n(B_1)$ and $|S^{n-1}| = s(\partial B_1)$. We introduce the mean value of the vector-valued function $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ over the sphere $\partial B_r(x)$ as follows:

$$A\mathbf{v}(x;r) = \fint_{\partial B_r(x)} \mathbf{v}(y)\,\mathrm{d}s_y, \tag{12.2.3}$$

and set

$$D_\omega(\mathbf{v};x) = \sup_{r>0} \frac{|\mathbf{v}(x) - A\mathbf{v}(x;r)|}{\omega(r)}. \tag{12.2.4}$$

In particular, for a function v of one variable we have

$$D_\omega(v;x) = \sup_{r>0} \frac{|2v(x) - v(x+r) - v(x-r)|}{2\omega(r)}.$$

In what follows, we assume that ω is a continuous nondecreasing function on $[0,\infty)$ such that $\omega(0) = 0$ and $\omega(\infty) = \infty$.

The objective of this section is the next result.

Theorem. *Let the function*

$$\Omega(t) := \int_0^1 (1 - n + n\sigma)\sigma^{n-1}\omega(\sigma t)\,\mathrm{d}\sigma \tag{12.2.5}$$

be strictly increasing on $[0,\infty)$ and let Ω^{-1} be the inverse function for Ω. Further let

$$\Psi(t) = \int_0^t \Omega^{-1}(\tau)\,\mathrm{d}\tau.$$

Then for any $u \in C^1(\mathbb{R}^n)$

$$\left|\nabla u(x)\right| \le n(n+1) D_\omega(\nabla u;x) \Psi^{-1}\left(\frac{\mathcal{M}^\diamond u(x)}{n D_\omega(\nabla u;x)} \right), \tag{12.2.6}$$

where Ψ^{-1} is the inverse function for Ψ.

Let $\omega \in C^1(0,\infty)$. Suppose the function $t\omega'(t)$ is nondecreasing on $(0,\infty)$ and that, for $n > 1$, the function $t\Omega'(t)$ is nondecreasing on $(0,\infty)$. Let R be a unique root of the equation

$$n(n+1)\Omega(t) = 1. \tag{12.2.7}$$

Inequality (12.2.6) with $x = 0$ becomes an equality for the function

$$u(x) = \begin{cases} x_n\big(1 - n\int_0^1 \sigma^{n-1}\omega(\sigma|x|)\,\mathrm{d}\sigma\big) & \textit{for } 0 \le |x| \le R, \\ 0 \quad \textit{for } |x| \ge (n+1)R, \\ \frac{nx_n}{|x|}((n+1)R - |x|)\int_0^1((n+1)\sigma - n)\sigma^{n-1}\omega(\sigma\frac{(n+1)R-|x|}{n})\,\mathrm{d}\sigma \\ \quad \textit{for } R < |x| < (n+1)R. \end{cases} \tag{12.2.8}$$

12.2.3 Proof of Inequality (12.2.6)

It suffices to prove (12.2.6) for $x = 0$. We have

$$\int_{B_1} \big(\nabla u(0) - \nabla u(y)\big)\big(1 - |y|\big)\,\mathrm{d}y \\ = \frac{1}{n(n+1)}\nabla u(0)\big|S^{n-1}\big| - \int_{B_1} u(y)\frac{y}{|y|}\,\mathrm{d}y. \tag{12.2.9}$$

Hence,

$$\frac{|B_1|}{n+1}\nabla u(0) = \int_{B_1} u(y)\frac{y}{|y|}\,\mathrm{d}y \\ + \big|S^{n-1}\big|\int_0^1 r^{n-1}(1-r)\big(\nabla u(0) - A\nabla u(0;r)\big)\,\mathrm{d}r. \tag{12.2.10}$$

After the scaling $y \to y/t$, $r \to r/t$, equality (12.2.10) becomes

$$\frac{|B_1|}{n+1}t\nabla u(0) = \frac{1}{t^n}\int_{B_t} u(y)\frac{y}{|y|}\,\mathrm{d}y \\ + \big|S^{n-1}\big|\frac{1}{t^n}\int_0^t r^{n-1}(t-r)\big(\nabla u(0) - Au(0;r)\big)\,\mathrm{d}r. \tag{12.2.11}$$

This implies

$$\big|\nabla u(0)\big| \le \frac{n+1}{t}\mathcal{M}^\diamond u(0) + \frac{n(n+1)}{t^{n+1}}D_\omega(\nabla u;0)\int_0^t r^{n-1}(t-r)\omega(r)\,\mathrm{d}r,$$

which can be written as

$$0 \le -t\big|\nabla u(0)\big| + (n+1)\mathcal{M}^\diamond u(0) + n(n+1)D_\omega(\nabla u;0)\int_0^t \Omega(\tau)\,\mathrm{d}\tau. \tag{12.2.12}$$

Since Ω is strictly increasing, it follows that the right-hand side in (12.2.12) attains its minimum value at

$$t_* = \Omega^{-1}\bigg(\frac{|\nabla u(0)|}{n(n+1)D_\omega(\nabla u;0)}\bigg). \tag{12.2.13}$$

Thus, by (12.2.5) one has

$$0 \le (n+1)\mathcal{M}^\diamond u(0) - \big|\nabla u(0)\big| \varOmega^{-1}\bigg(\frac{|\nabla u(0)|}{n(n+1)D_\omega(\nabla u; 0)}\bigg)$$
$$+ n(n+1)\, D_\omega(\nabla u; 0) \int_0^{\varOmega^{-1}(\frac{|\nabla u(0)|}{n(n+1)D_\omega(\nabla u;0)})} \varOmega(\tau)\,\mathrm{d}\tau$$
$$= (n+1)\mathcal{M}^\diamond u(0) - n(n+1)D_\omega(\nabla u; 0) \int_0^{\varOmega^{-1}(\frac{|\nabla u(0)|}{n(n+1)D_\omega(\nabla u;0)})} x\,\mathrm{d}\varOmega(x).$$

Therefore,

$$\mathcal{M}^\diamond u(0) \ge n D_\omega(\nabla u; 0) \int_0^{\frac{|\nabla u(0)|}{n(n+1)D_\omega(\nabla u;0)}} \varOmega^{-1}(\tau)\,\mathrm{d}\tau,$$

which is equivalent to (12.2.6).

12.2.4 Proof of Sharpness

Let us prove first that (12.2.7) has a unique root. Note that $\varOmega(0) = 0$ by (12.2.5). Since $t\varOmega'(t)$ is nondecreasing, one has $\varOmega(\infty) = \infty$. It remains to show that $\varOmega'(t) > 0$ for $t > 0$. To this end, we only need to check that $t\varOmega'(t)|_{t=0} = 0$. Since the function (12.2.5) can be written as

$$\varOmega(t) = \frac{n}{t^{n+1}} \int_0^t \tau^n \omega(\tau)\,\mathrm{d}\tau - \frac{n-1}{t^n} \int_0^t \tau^{n-1}\omega(\tau)\,\mathrm{d}\tau, \tag{12.2.14}$$

we see that

$$n\int_0^t \varOmega(\tau)\,\mathrm{d}\tau - \frac{1}{t^n}\int_0^t \tau^n\omega(\tau)\,\mathrm{d}\tau$$
$$= \frac{n}{t^{n-1}} \int_0^t \tau^{n-1}\omega(\tau)\,\mathrm{d}\tau - \frac{n+1}{t^n}\int_0^t \tau^n\omega(\tau)\,\mathrm{d}\tau. \tag{12.2.15}$$

Hence,

$$\frac{1}{t^n}\int_0^t \tau^n\omega(\tau)\,\mathrm{d}\tau - (n-1)\int_0^t \varOmega(\tau)\,\mathrm{d}\tau = t\varOmega(t) \tag{12.2.16}$$

and thus

$$t\varOmega'(t) = \omega(t) - \frac{n}{t^{n+1}}\int_0^t \tau^n\omega(\tau)\,\mathrm{d}\tau - n\varOmega(t). \tag{12.2.17}$$

The last relation, combined with $\omega(0) = \varOmega(0) = 0$, shows that $t\varOmega'(t)$ vanishes at $t = 0$.

Let us now prove that u defined by (12.2.8) is in $C^1(\mathbb{R}^n)$. We claim that u is continuous on the sphere $|x| = R$ together with its first partial derivatives. We use spherical coordinates to write $x_n = r\cos\theta$. Denote the function $u(x)/\cos\theta$ by $u_1(r)$ for $0 \le |x| \le R$ and by $u_2(r)$ for $R < |x| < (n+1)R$, i.e.,

$$u_1(r) = r - \frac{n}{r^{n-1}} \int_0^r t^{n-1}\omega(t)\,\mathrm{d}t \tag{12.2.18}$$

and

$$u_2(r) = n^2\left(\frac{n+1}{\kappa_r^n}\int_0^{\kappa_r} t^n\omega(t)\,\mathrm{d}t - \frac{n}{\kappa_r^{n-1}}\int_0^{\kappa_r} t^{n-1}\omega(t)\,\mathrm{d}t\right), \tag{12.2.19}$$

where

$$\kappa_r = n^{-1}\big((n+1)R - r\big). \tag{12.2.20}$$

Clearly,

$$u_2(R) = n^2\left(\frac{n+1}{R^n}\int_0^R t^n\omega(t)\,\mathrm{d}t - \frac{n}{R^{n-1}}\int_0^R t^{n-1}\omega(t)\,\mathrm{d}t\right). \tag{12.2.21}$$

By (12.2.14) we can write

$$u_2(R) = n\left((n+1)R\,\Omega(R) - \frac{1}{R^{n-1}}\int_0^R t^{n-1}\omega(t)\,\mathrm{d}t\right).$$

It follows from (12.2.7) and (12.2.18) that the right-hand side is equal to $u_1(R)$. Let us show that

$$u_2'(R) = u_1'(R). \tag{12.2.22}$$

From (12.2.18) we obtain

$$u_1'(r) = 1 + \frac{n(n-1)}{r^n}\int_0^r t^{n-1}\omega(t)\,\mathrm{d}t - n\omega(r). \tag{12.2.23}$$

By (12.2.19)

$$u_2'(r) = n^2\left(\frac{n+1}{\kappa_r^{n+1}}\int_0^{\kappa_r} t^n\omega(t)\,\mathrm{d}t - \frac{n-1}{\kappa_r^n}\int_0^{\kappa_r} t^{n-1}\omega(t)\,\mathrm{d}t - \frac{\omega(\kappa_r)}{n}\right). \tag{12.2.24}$$

Therefore,

$$u_2'(R) = n^2\left(\frac{n+1}{R^{n+1}}\int_o^R t^n\omega(t)\,\mathrm{d}t - \frac{n-1}{R^n}\int_0^R t^{n-1}\omega(t)\,\mathrm{d}t - \frac{\omega(R)}{n}\right),$$

which by (12.2.14) can be rewritten as

$$u_2'(R) = n(n+1)\Omega(R) + \frac{n(n-1)}{R^n}\int_0^R t^{n-1}\omega(t)\,\mathrm{d}t - n\omega(R).$$

Using (12.2.7) and (12.2.23) we arrive at (12.2.22). It remains to note that by (12.2.19) and (12.2.24)

$$u_2\big((n+1)R\big) = 0, \qquad u_2'\big((n+1)R\big) = 0.$$

Hence, u is in $C^1(\mathbb{R}^n)$.

Our next goal is to show that

$$\mathcal{M}^\diamond u(0) = \frac{R}{n+1} - n\int_0^R \Omega(t)\,\mathrm{d}t. \tag{12.2.25}$$

Let us find the maxima of the function

$$r \to M_r u := \frac{1}{|B_r|}\left|\int_{B_r} \frac{y}{|y|} u(y)\,\mathrm{d}y\right| \tag{12.2.26}$$

on $[0, R]$ and $[R, (n+1)R]$ separately. Recall that, for $0 \le |x| \le R$, the function u can be written as $\cos\theta\, u_1(r)$, where u_1 is defined by (12.2.18). It is clear that the function (12.2.26) is equal to

$$\frac{2|S^{n-2}|}{|B_1|r^n}\bigg(\int_0^r \rho^n\,\mathrm{d}\rho \int_0^{\pi/2} (\cos\theta)^2(\sin\theta)^{n-2}\,\mathrm{d}\theta \\ - n\int_0^r\int_0^\rho t^{n-1}\omega(t)\,\mathrm{d}t\,\mathrm{d}\rho \int_0^{\pi/2}(\cos\theta)^2(\sin\theta)^{n-2}\,\mathrm{d}\theta\bigg).$$

Since

$$2\big|S^{n-2}\big|\int_0^{\pi/2}(\cos\theta)^2(\sin\theta)^{n-2}\,\mathrm{d}\theta = n^{-1}\big|S^{n-1}\big|, \tag{12.2.27}$$

it follows that

$$\begin{aligned} M_r u &= \frac{r}{n+1} - \frac{n}{r^n}\int_0^r\int_0^\rho t^{n-1}\omega(t)\,\mathrm{d}t\,\mathrm{d}\rho \\ &= \frac{r}{n+1} - n\bigg(\frac{1}{r^{n-1}}\int_0^r \tau^{n-1}\omega(\tau)\,\mathrm{d}\tau - \frac{1}{r^n}\int_0^r \tau^n\omega(\tau)\,\mathrm{d}\tau\bigg), \end{aligned}$$

and by (12.2.16) we arrive at

$$M_r u = \frac{r}{n+1} - n\int_0^r \Omega(t)\,\mathrm{d}t. \tag{12.2.28}$$

As was proved above, $\Omega'(t) > 0$ for $t > 0$. Therefore,

$$\max_{0\le r\le R} M_r u = M_R u. \tag{12.2.29}$$

We prove that

$$\max_{R\le r\le (n+1)R} M_r u = M_R u. \tag{12.2.30}$$

By (12.2.26) one has

$$M_r u = \frac{1}{|B_1| r^n}\left(|B_1| R^n \left(\frac{R}{n+1} - n\int_0^R \Omega(t)\,\mathrm{d}t \right) + 2\big|S^{n-2}\big| \int_0^{\pi/2} (\cos\theta)^2 (\sin\theta)^{n-2}\,\mathrm{d}\theta \int_R^r u_2(\rho)\rho^{n-1}\,\mathrm{d}\rho \right).$$

In view of (12.2.27) we obtain

$$M_r u = r^{-n}\left(R^n \left(\frac{R}{n+1} - n\int_0^R \Omega(t)\,\mathrm{d}t \right) + \int_R^r u_2(\rho)\rho^{n-1}\,\mathrm{d}\rho \right).$$

Hence and by (12.2.28), to prove (12.2.30) we need to show that the function

$$G(r) := \big(r^n - R^n\big)\left(\frac{R}{n+1} - n\int_0^R \Omega(t)\,\mathrm{d}t \right) - \int_R^r u_2(\rho)\rho^{n-1}\,\mathrm{d}\rho \qquad (12.2.31)$$

is nonnegative on the interval $[R,(n+1)R]$. Clearly,

$$G'(r) = nr^{n-1}\left(\frac{R}{n+1} - n\int_0^R \Omega(t)\,\mathrm{d}t - \frac{1}{n}u_2(r) \right).$$

Note that by (12.2.19) and (12.2.14)

$$-\frac{1}{n}u_2(r) = n\left(\int_0^{\kappa_r} \Omega(t)\,\mathrm{d}t - \kappa_r \Omega(\kappa_r) \right) = -n\int_0^{\kappa_r} t\Omega'(t)\,\mathrm{d}t. \qquad (12.2.32)$$

Integrating by parts and using (12.2.7), we find

$$G'(r) = nr^{n-1}\left(n\int_0^R t\Omega'(t)\,\mathrm{d}t - n\int_0^{\kappa_r} t\Omega'(t)\,\mathrm{d}t \right) \geq 0.$$

Hence and by $G(R) = 0$, one has $G(r) \geq 0$ for $R \geq r \geq (n+1)R$. Thus, (12.2.25) holds.

Let us now justify the relation

$$\sup_{r>0} \frac{|\nabla u(0) - A\nabla u(0;r)|}{\omega(r)} = 1, \qquad (12.2.33)$$

where A is defined by (12.2.3). Let $0 \leq r \leq R$. It follows from (12.2.8) that

$$\frac{\partial u}{\partial x_n} = 1 - \frac{n}{r^n}\int_0^r t^{n-1}\omega(t)\,\mathrm{d}t - (\cos\theta)^2 \left(n\omega(r) - \frac{n^2}{r^n}\int_0^r t^{n-1}\omega(t)\,\mathrm{d}t \right), \qquad (12.2.34)$$

which together with (12.2.27), implies

$$A\frac{\partial u}{\partial x_n}(0;r) = 1 - \omega(r).$$

Hence,

$$\frac{\partial u}{\partial x_n}(0) - A\frac{\partial u}{\partial x_n}(0;r) = \omega(r).$$

Combining this fact with the formulas

$$\frac{\partial u}{\partial x_k}(0) = 0, \qquad A\frac{\partial u}{\partial x_k}(0;r) = 0, \quad k = 1, \dots, n-1,$$

we obtain

$$\frac{|\nabla u(0) - A\nabla u(0;r)|}{\omega(r)} = 1 \quad \text{for } 0 \le r \le R. \tag{12.2.35}$$

We now claim that

$$\omega(\kappa_r) \le \frac{\partial u}{\partial x_n}(0) - A\frac{\partial u}{\partial x_n}(0;r) \le \omega(r) \quad \text{for } R \le r \le (n+1)R. \tag{12.2.36}$$

Note that

$$\frac{\partial u}{\partial x_n} = \frac{1}{r}u_2(r) - (\cos\theta)^2\left(\frac{u_2(r)}{r} - u_2'(r)\right),$$

where u_2 is given by (12.2.19). In view of (12.2.27) one has

$$A\frac{\partial u}{\partial x_n}(0;r) = \frac{n-1}{n}\frac{u_2(r)}{r} + \frac{1}{n}u_2'(r). \tag{12.2.37}$$

By (12.2.19) and (12.2.24),

$$\begin{aligned} A\frac{\partial u}{\partial x_n}(0;r) &= \frac{n(n-1)}{r}\left(\frac{n+1}{\kappa_r^n}\int_0^{\kappa_r} t^n\omega(t)\,\mathrm{d}t - \frac{n}{\kappa_r^{n-1}}\int_0^{\kappa_r} t^{n-1}\omega(t)\,\mathrm{d}t\right) \\ &\quad + n\left(\frac{n+1}{\kappa_r^{n+1}}\int_0^{\kappa_r} t^n\omega(t)\,\mathrm{d}t - \frac{n-1}{\kappa_r^n}\int_0^{\kappa_r} t^{n-1}\omega(t)\,\mathrm{d}t\right) - \omega(\kappa_r). \end{aligned} \tag{12.2.38}$$

Next observe that

$$\begin{aligned} &\frac{n+1}{\kappa_r^n}\int_0^{\kappa_r} t^n\omega(t)\,\mathrm{d}t - \frac{n}{\kappa_r^{n-1}}\int_0^{\kappa_r} t^{n-1}\omega(t)\,\mathrm{d}t \\ &\quad = \int_0^{\kappa_r}\left(1 - \frac{t}{\kappa_r}\right)\left(\frac{t}{\kappa_r}\right)^{n-1} t\omega'(t)\,\mathrm{d}t \ge 0, \end{aligned}$$

because ω is nondecreasing. This, together with (12.2.38) and the inequality $r \ge \kappa_r$, yields

$$A\frac{\partial u}{\partial x_n}(0;r) \le n(n+1)\left(\frac{n}{\kappa_r^{n+1}}\int_0^{\kappa_r} t^n\omega(t)\,\mathrm{d}t - \frac{n-1}{\kappa_r^n}\int_0^{\kappa_r} t^{n-1}\omega(t)\,\mathrm{d}t\right) - \omega(\kappa_r).$$

By (12.2.14), this inequality can be written as

$$A\frac{\partial u}{\partial x_n}(0;r) \le n(n+1)\Omega(\kappa_r) - \omega(\kappa_r). \tag{12.2.39}$$

Since Ω_1 is strictly increasing and $\kappa_r < R$, it follows that

$$A\frac{\partial u}{\partial x_n}(0;r) \le n(n+1)\Omega(R) - \omega(\kappa_r).$$

We now use the identity

$$n(n+1)\Omega(R) = 1 = \frac{\partial u}{\partial x_n}(0)$$

to obtain

$$A\frac{\partial u}{\partial x_n}(0;r) \le \frac{\partial u}{\partial x_n}(0) - \omega(\kappa_r),$$

which implies the left inequality in (12.2.36).

To prove the right inequality in (12.2.36), we note that, by (12.2.15), relation (12.2.38) can be rewritten as

$$A\frac{\partial u}{\partial x_n}(0;r) = \frac{n(n-1)}{r}\left(\frac{1}{\kappa_r^n}\int_0^{\kappa_r} t^n\omega(t)\,\mathrm{d}t - n\int_0^{\kappa_r}\Omega(t)\,\mathrm{d}t\right) + \frac{n}{\kappa_r^{n+1}}\int_0^{\kappa_r} t^n\omega(t)\,\mathrm{d}t + n\Omega(\kappa_r) - \omega(\kappa_r). \tag{12.2.40}$$

Let

$$B(r) := r\omega(r) - r + rA\frac{\partial u}{\partial x_n}(0;r). \tag{12.2.41}$$

The right inequality in (12.2.36) is equivalent to the inequality $B(r) \ge 0$ for all values of r in $[R,(n+1)R]$. By (12.2.40) we have

$$B(r) := r\omega(r) - r + n(n-1)\left(\frac{1}{\kappa_r^n}\int_0^{\kappa_r} t^n\omega(t)\,\mathrm{d}t - n\int_0^{\kappa_r}\Omega(t)\,\mathrm{d}t\right) + r\left(\frac{n}{\kappa_r^{n+1}}\int_0^{\kappa_r} t^n\omega(t)\,\mathrm{d}t + n\Omega(\kappa_r) - \omega(\kappa_r)\right). \tag{12.2.42}$$

Using the relations $\kappa_R = R$ and $n(n+1)\Omega(R) = 1$, we obtain

$$B(R) = \frac{n^2}{R^n}\int_0^R t^n\omega(t)\,\mathrm{d}t - n^2(n-1)\int_0^R\Omega(t)\,\mathrm{d}t - \frac{n}{n+1}R. \tag{12.2.43}$$

We note that relation (12.2.43), together with (12.2.16) for $t = R$, gives $B(R) = 0$.

The next step is to show that $B'(r) \ge 0$ for $r \in [R,(n+1)R]$. Combining (12.2.37) and (12.2.32), we see that

$$A\frac{\partial u}{\partial x_n}(0;r) = \frac{n(n-1)}{r}\int_0^{\kappa_r} t\Omega'(t)\,\mathrm{d}t - \kappa_r\Omega'(\kappa_r),$$

which together with (12.2.41), gives

$$B(r) = r\omega(r) - r + n(n-1)\int_0^{\kappa_r} t\Omega'(t)\,\mathrm{d}t - r\kappa_r\Omega'(\kappa_r).$$

Clearly,

$$B'(r) = \big(r\omega(r)\big)' - 1 - n\kappa_r\Omega'(\kappa_r) + \frac{r}{n}\big(t\Omega'(t)\big)'\Big|_{t=\kappa_r}. \tag{12.2.44}$$

For $n = 1$, by (12.2.17) and (12.2.14), one has

$$t\Omega'(t) = \omega(t) - \frac{2}{t^2}\int_0^t \tau\omega(\tau)\,\mathrm{d}\tau = \frac{1}{t^2}\int_0^t \tau^2\omega'(\tau)\,\mathrm{d}\tau.$$

Since $t\omega'(t)$ is nondecreasing, it follows that

$$\left(t^{-2}\int_0^t \tau^2\omega'(\tau)\,\mathrm{d}\tau\right)' \ge 0.$$

Thus, $t\,\Omega'(t)$ is also nondecreasing for $n = 1$. Hence, and by assumption of the theorem, both the functions $t\,\Omega'(t)$ and $t\,\omega'(t)$ are nondecreasing for $n \ge 1$. Therefore, the last term on the right-hand side in (12.2.44) is nonnegative and $\kappa_r\Omega'(\kappa_r) \le R\Omega'(R)$ for $r \ge R$. Thus,

$$B'(r) \ge \omega(R) + R\omega'(R) - 1 - nR\Omega'(R) \tag{12.2.45}$$

for $r \in [R, (n+1)R]$. Owing to relation (12.2.17) for $t = R$, the last inequality can be rewritten as

$$B'(r) \ge R\omega'(R) - (n-1)\omega(R) + \frac{n^2}{R^{n+1}}\int_0^R t^n\omega(t)\,\mathrm{d}t - \frac{1}{n+1}. \tag{12.2.46}$$

By (12.2.14) for $t = R$, relation (12.2.46) gives

$$B'(r) \ge R\omega'(R) - (n-1)\omega(R) + \frac{n(n-1)}{R^n}\int_0^R t^{n-1}\omega(t)\,\mathrm{d}t.$$

Integrating by parts on the right-hand side, we obtain

$$B'(r) \ge R\omega'(R) - \frac{n-1}{R^n}\int_0^R t^n\omega'(t)\,\mathrm{d}t \ge \frac{1}{R^n}\int_0^R t^n\big(t\omega'(t)\big)'\,\mathrm{d}t.$$

Since the function $t\omega'(t)$ is nondecreasing, it follows that the right-hand side is nonnegative. This implies the right inequality (12.2.36), and together with (12.2.35), leads to (12.2.33).

Finally, we must show that inequality (12.2.6) with $x = 0$ becomes an equality for u given by (12.2.12). It follows from (12.2.7) and (12.2.25) that

$$\begin{aligned} n \int_0^{\frac{1}{n(n+1)}} \Omega^{-1}(\tau)\, \mathrm{d}\tau \\ &= n \int_0^R t \, \mathrm{d}\Omega(t) = n\Big(R\Omega(R) - \int_0^R \Omega(t)\, \mathrm{d}t \Big) \\ &= \frac{R}{n+1} - n \int_0^R \Omega(t)\, \mathrm{d}t = \mathcal{M}^{\diamond} u(0). \end{aligned}$$

By (12.2.33), the right-hand side of (12.2.6) is equal to

$$n(n+1)\Psi^{-1}\left(\Psi\left(\frac{1}{n(n+1)}\right)\right) = 1.$$

The proof is complete.

12.2.5 Particular Case $\omega(r) = r^{\alpha}, \alpha > 0$

Setting $\omega(r) = r^{\alpha}$ with $\alpha > 0$ in Theorem 12.2.2 and using the notation

$$\mathcal{D}_{\alpha}(\nabla u; x) = \sup_{r>0} \frac{|\nabla u(x) - A\nabla u(x;r)|}{r^{\alpha}},$$

we obtain the following corollary to Theorem 12.2.2.

Corollary 1. *Let $u \in C^1(\mathbb{R}^n)$, and let $\alpha > 0$. Then inequality*

$$\big|\nabla u(x)\big| \le C_1 \big(\mathcal{M}^{\diamond} u(x)\big)^{\frac{\alpha}{\alpha+1}} \left(\sup_{r>0} \frac{|\nabla u(x) - A\nabla u(x;r)|}{r^{\alpha}} \right)^{\frac{1}{\alpha+1}} \tag{12.2.47}$$

holds with the best constant

$$C_1 = (n+1)\frac{\alpha+1}{\alpha}\left(\frac{\alpha n}{(n+\alpha)(n+\alpha+1)}\right)^{\frac{1}{\alpha+1}}. \tag{12.2.48}$$

Inequality (12.2.47) with $x = 0$ becomes an equality for the function

$$u(x) = \begin{cases} x_n(1 - \frac{n}{n+\alpha}|x|^{\alpha}) & \text{for } 0 \le |x| \le R, \\ \frac{\alpha n^{1-\alpha}((n+1)R - |x|)^{\alpha+1}}{(n+\alpha)(n+\alpha+1)} \frac{x_n}{|x|} & \text{for } R < |x| < (n+1)R, \\ 0 & \text{for } |x| \ge (n+1)R, \end{cases}$$

where

$$R = \left(\frac{(n+\alpha)(n+\alpha+1)}{(\alpha+1)n(n+1)}\right)^{\frac{1}{\alpha}}.$$

Corollary 2. (Local version of Corollary 12.2.5.) *Let $\tilde{\mathcal{M}}^{\diamond}$ denote the modified maximal operator given by*

$$\tilde{\mathcal{M}}^{\diamond} u(x) = \sup_{0<r<1} \left| \fint_{B_r(x)} \frac{y-x}{|y-x|} u(y)\, \mathrm{d}y \right|$$

and let

$$\tilde{\mathcal{D}}_\alpha(\nabla u; x) = \sup_{0<r<1} \frac{|\nabla u(x) - A\nabla u(x;r)|}{r^\alpha}.$$

Then, for any $\alpha > 0$*, the inequality*

$$\big|\nabla u(x)\big| \le \big(C_1 \big(\tilde{\mathcal{D}}_\alpha(\nabla u; x)\big)^{\frac{1}{\alpha+1}} + C_2 \big(\tilde{\mathcal{M}}^{\diamond} u(x)\big)^{\frac{1}{\alpha+1}}\big)\big(\tilde{\mathcal{M}}^{\diamond} u(x)\big)^{\frac{\alpha}{\alpha+1}} \tag{12.2.49}$$

holds with the best constants C_1 *defined by* (12.2.48) *and* $C_2 = n+1$.

Proof. It suffices to set $x = 0$. It follows from (12.2.11) that

$$\big|\nabla u(0)\big| \le (n+1)\bigg(\tilde{\mathcal{M}}^{\diamond} u(0) t^{-1} + \frac{n}{(n+\alpha)(n+\alpha+1)} \tilde{\mathcal{D}}_\alpha(\nabla u; 0) t^\alpha\bigg).$$

The right-hand side attains its minimum value either at

$$t = \left(\frac{(n+\alpha)(n+\alpha+1)}{\alpha n} \frac{\tilde{\mathcal{M}}^{\diamond} u(0)}{\tilde{\mathcal{D}}_\alpha(\nabla u; 0)}\right)^{\frac{1}{\alpha+1}} < 1$$

or at $t = 1$. Thus we arrive at the following alternatives: either

$$\tilde{\mathcal{M}}^{\diamond} u(0) \le \frac{\alpha n}{(n+\alpha)(n+\alpha+1)} \tilde{\mathcal{D}}_\alpha(\nabla u; 0)$$

and

$$\big|\nabla u(0)\big| \le C_1 \big(\tilde{\mathcal{M}}^{\diamond} u(0)\big)^{\frac{\alpha}{\alpha+1}} \big(\tilde{\mathcal{D}}_\alpha(\nabla u; 0)\big)^{\frac{1}{\alpha+1}}, \tag{12.2.50}$$

with C_1 defined by (12.2.48), or

$$\tilde{\mathcal{M}}^{\diamond} u(0) \ge \frac{\alpha n}{(n+\alpha)(n+\alpha+1)} \tilde{\mathcal{D}}_\alpha(\nabla u; 0)$$

and

$$\big|\nabla u(0)\big| \le \left(\frac{C_1}{\alpha+1} \big(\tilde{\mathcal{D}}_\alpha(\nabla u; 0)\big)^{\frac{1}{\alpha+1}} + C_2 \big(\tilde{\mathcal{M}}^{\diamond} u(0)\big)^{\frac{1}{\alpha+1}}\right) \big(\tilde{\mathcal{M}}^{\diamond} u(0)\big)^{\frac{\alpha}{\alpha+1}}. \tag{12.2.51}$$

Inequalities (12.2.50) and (12.2.51) imply (12.2.49).

To show that the constant C_1 is sharp, we make the dilation $x \to \delta x$, $0 < \delta < 1$, in (12.2.49). Then

$$\big|\nabla u(x)\big| \le \big(C_1 \big(\mathcal{D}_\alpha(\nabla u; x)\big)^{\frac{1}{\alpha+1}} + C_2 \delta \big(\mathcal{M}^{\diamond} u(x)\big)^{\frac{1}{\alpha+1}}\big)\big(\mathcal{M}^{\diamond} u(x)\big)^{\frac{\alpha}{\alpha+1}}.$$

Passing to the limit as $\delta \to 0$, we arrive at (12.2.49) with the best constant C_1.

To prove that the constant C_2 is sharp, we set $u(x) = x_n$. Then (12.2.49) becomes

$$\big|\nabla u(x)\big| \le (n+1)\tilde{\mathcal{M}}^{\diamond} u(x). \tag{12.2.52}$$

Clearly,

$$\begin{aligned}\tilde{\mathcal{M}}^{\diamond} u(0) &= \sup_{0<r<1} \left| \fint_{B_r} \frac{x}{|x|} x_n \,\mathrm{d}x \right| = \sup_{0<r<1} \fint_{B_r} \frac{x_n^2}{|x|} \,\mathrm{d}x \\ &= \sup_{0<r<1} \frac{2n|S^{n-2}|}{|S^{n-1}|r^n} \int_0^{\pi/2} (\cos\theta)^2 (\sin\theta)^{n-2} \,\mathrm{d}\theta \int_0^r \rho^n \,\mathrm{d}\rho,\end{aligned}$$

which together with (12.2.27) implies

$$\tilde{\mathcal{M}}^{\diamond} u(0) = (n+1)^{-1}.$$

Thus inequality (12.2.52) with $x = 0$ becomes an equality. This completes the proof of the corollary. □

12.2.6 One-Dimensional Case

For $n = 1$ the operator (12.2.4) becomes

$$D_\omega(u'; x) = \sup_{r>0} \frac{|2u'(x) - u'(x+r) - u'(x-r)|}{2\omega(r)}$$

and $\mathcal{M}^{\diamond}$ is defined by

$$\mathcal{M}^{\diamond} u(x) = \sup_{r>0} \frac{1}{2r} \left| \int_{x-r}^{x+r} \operatorname{sign}(y-x) u(y) \,\mathrm{d}y \right|. \tag{12.2.53}$$

The next corollary immediately follows from Theorem 12.2.2.

Corollary 1. *Let $u \in C^1(\mathbb{R})$. Then the inequality*

$$\big|u'(x)\big| \le 2D_\omega(u'; x) \Psi^{-1}\left(\frac{\mathcal{M}^{\diamond} u}{D_\omega(u'; x)} \right) \tag{12.2.54}$$

holds, where Ψ^{-1} is the inverse function for

$$\Psi(t) = \int_0^t \Omega^{-1}(\tau) \,\mathrm{d}\tau,$$

with Ω^{-1} standing for the inverse function for

$$\Omega(t) = \int_0^1 \sigma \omega(\sigma t) \,\mathrm{d}\sigma.$$

Suppose $t\omega'(t)$ is nondecreasing on $(0, \infty)$. Then inequality (12.2.54) with $x = 0$ becomes an equality for the odd function u given on the semi-axis $x \ge 0$ by the formula

$$u(x) = \begin{cases} x(1 - \int_0^1 \omega(\sigma x)\, d\sigma) & \text{for } 0 \le x \le R, \\ (2R - x) \int_0^1 (2\sigma - 1)\omega(\sigma(2R - x))\, d\sigma & \text{for } R < x < 2R, \\ 0 & \text{for } x \ge 2R, \end{cases}$$

where R is a unique root of the equation $2\Omega(t) = 1$.

Set

$$\mathcal{T}_\omega(u'; x) = \sup_{y \in \mathbb{R}} \frac{|u'(x) - u'(y)|}{\omega(|x - y|)}$$

and note that $D_\omega(u'; x) \le \mathcal{T}_\omega(u'; x)$. Moreover, if u is odd, then $D_\omega(u'; 0) = \mathcal{T}_\omega(u'; 0)$. Therefore, Corollary 12.2.6 implies the following assertion.

Corollary 2. *Let $u \in C^1(\mathbb{R})$. Then*

$$\left|u'(x)\right| \le 2\ \mathcal{T}_\omega(u'; x)\Psi^{-1}\left(\frac{\mathcal{M}^\diamond u}{\mathcal{T}_\omega(u'; x)}\right). \tag{12.2.55}$$

Inequality (12.2.55) becomes an equality for the same function as in Corollary 12.2.6. As in Corollary 12.2.6, we here assume that $r\omega'(r)$ is nondecreasing on $(0, \infty)$.

In the special case $\omega(t) = t^\alpha$, $\alpha > 0$, Corollaries 12.2.6 and 12.2.6 can be stated as follows.

Corollary 3. *Let $u \in C^1(\mathbb{R})$, and let $\alpha > 0$. The inequality*

$$\left|u'(x)\right| \le C_1 \big(\mathcal{M}^\diamond u(x)\big)^{\frac{\alpha}{\alpha+1}} \left(\sup_{r>0} \frac{|2u'(x) - u'(x + r) - u'(x - r)|}{r^\alpha}\right)^{\frac{1}{\alpha+1}} \tag{12.2.56}$$

holds with the best constant

$$C_1 = \left(\frac{2(\alpha + 1)}{\alpha(\alpha + 2)^{\frac{1}{\alpha}}}\right)^{\frac{\alpha}{\alpha+1}}. \tag{12.2.57}$$

Inequality (12.2.56) becomes an equality for the odd function u whose values for $x \ge 0$ are given by

$$u(x) = \begin{cases} (\alpha + 1)x - x^{\alpha+1} & \text{for } 0 \le x \le (\frac{\alpha+2}{2})^{1/\alpha}, \\ \frac{\alpha}{\alpha+2}(2(\frac{\alpha+2}{2})^{1/\alpha} - x)^{\alpha+1} & \text{for } (\frac{\alpha+2}{2})^{1/\alpha} < x < 2(\frac{\alpha+2}{2})^{1/\alpha}, \\ 0 & \text{for } x \ge 2(\frac{\alpha+2}{2})^{1/\alpha}. \end{cases} \tag{12.2.58}$$

Note that the sharp estimate (12.0.5) is a particular case of (12.2.56) for $\alpha = 1$. It implies a rougher, but nevertheless sharp estimate

$$\left|u'(x)\right|^2 \le \frac{8}{3}(\mathcal{M}^\diamond u)(x)\|u''\|_{L_\infty}$$

with the equality sign for $x = 0$ provided by the function (12.2.58) with $\alpha = 1$.

Taking into account that the second difference at 0 for an odd function (12.2.58) is twice as big as the corresponding first difference, we arrive at the following assertion.

Corollary 4. *Let $u \in C^1(\mathbb{R})$, and let $\alpha > 0$. Then*

$$\big|u'(x)\big|^{\alpha+1} \le \frac{2^{\alpha+1}}{\alpha+2}\left(\frac{\alpha+1}{\alpha}\right)^{\alpha}\big(\mathcal{M}^{\diamond}u(x)\big)^{\alpha}\sup_{y\in\mathbb{R}}\frac{|u'(y)-u'(x)|}{|y-x|^{\alpha}}, \qquad (12.2.59)$$

which becomes an equality for the same function as in Corollary 12.2.6.

12.3 Pointwise Interpolation Inequalities Involving "Fractional Derivatives"

Let m be positive and a noninteger with $[m]$ and $\{m\}$ denoting its integer and fractional parts. We introduce the operator $D_{p,m}$

$$(D_{p,m}u)(x) = \left(\int_{\mathbb{R}^n}\big|\nabla_{[m]}u(x) - \nabla_{[m]}u(y)\big|^p|x-y|^{-n-p\{m\}}\,\mathrm{d}y\right)^{1/p}.$$

Sometimes, we call $D_{p,m}u$ the fractional derivative of u of order m. In Sect. 12.3.1 we derive pointwise interpolation inequalities with $D_{p,m}u$ in their right-hand sides.

12.3.1 Inequalities with Fractional Derivatives on the Right-Hand Sides

Theorem. (i) *Let k and l be integer, and let m be noninteger, $0 \le l \le k < m$. Then*

$$\big|\nabla_k u(x)\big| \le c\big((\mathcal{M}\nabla_l u)(x)\big)^{\frac{m-k}{m-l}}\big((D_{p,m}u)(x)\big)^{\frac{k-l}{m-l}} \qquad (12.3.1)$$

for almost all $x \in \mathbb{R}^n$.

(ii) *Let k and m be integer, and let l be noninteger, $0 < l < k \le m$. Then*

$$\big|\nabla_k u(x)\big| \le c\big((D_{p,l}u)(x)\big)^{\frac{m-k}{m-l}}\big((\mathcal{M}\nabla_m u)(x)\big)^{\frac{k-l}{m-l}} \qquad (12.3.2)$$

for almost all $x \in \mathbb{R}^n$.

(iii) *Let k be integer, and let l and m be noninteger, $0 < l < k < m$. Then*

$$\big|\nabla_k u(x)\big| \le c\big((D_{p,l}u)(x)\big)^{\frac{m-k}{m-l}}\big((D_{p,m}u)(x)\big)^{\frac{k-l}{m-l}} \qquad (12.3.3)$$

for almost all $x \in \mathbb{R}^n$.

Proof. (i) It suffices to prove inequality (12.3.1) for $l = 0$ and $x = 0$. Let η be a function in the ball B_1 with Lipschitz derivatives of order $m-2$, which vanishes on ∂B_1 together with all these derivatives. Also let

$$\int_{B_1} \eta(y)\,\mathrm{d}y = 1.$$

Let t be an arbitrary positive number to be chosen later. We shall use the Sobolev integral representation

$$v(0) = \sum_{|\beta|<[m]-k} t^{-n} \int_{B_t} \frac{(-y)^\beta}{\beta!} \partial^\beta v(y) \eta(y/t)\,\mathrm{d}y + (-1)^{[m]-k}\big([m]-k\big)$$
$$\times \sum_{|\alpha|=[m]-k} \int_{B_t} \frac{y^\alpha}{\alpha!} \partial^\alpha v(y) \int_{|y|/t}^\infty \eta\Big(\rho\frac{y}{|y|}\Big)\rho^{n-1}\,\mathrm{d}\rho\,\frac{\mathrm{d}y}{|y|^n} \tag{12.3.4}$$

(see Sect. 1.1.10).

By setting here $v = \partial^\gamma u$ with an arbitrary multi-index γ of order k and integrating by parts in the first integral, we arrive at the identity

$$\partial^\gamma u(0) = (-1)^k t^{-n} \int_{B_t} u(y) \sum_{|\beta|<[m]-k} \frac{1}{\beta!} \partial^{\beta+\gamma}\big(y^\beta \eta(y/t)\big)\,\mathrm{d}y$$
$$+ \sum_{|\alpha|=[m]-k} (-1)^{[m]-k}\big([m]-k\big)$$
$$\times \int_{B_t} \frac{y^\alpha}{\alpha!} \partial^{\alpha+\gamma} u(y) \int_{|y|/t}^\infty \eta\Big(\rho\frac{y}{|y|}\Big)\rho^{n-1}\,\mathrm{d}\rho\,\frac{\mathrm{d}y}{|y|^n}. \tag{12.3.5}$$

Hence, for $k < [m]$ we have

$$\big|\nabla_k u(0)\big| \le c\bigg(t^{-k}\mathcal{M}u(0) + t^{[m]-k}\big|\nabla_{[m]}u(0)\big|$$
$$+ \int_{B_t} \frac{|\nabla_{[m]}u(y) - \nabla_{[m]}u(0)|}{|y|^{n-[m]+k}}\,\mathrm{d}y\bigg). \tag{12.3.6}$$

Hölder's inequality implies

$$\int_{B_t} \frac{|\nabla_{[m]}u(y) - \nabla_{[m]}u(0)|}{|y|^{n-[m]+k}}\,\mathrm{d}y \le ct^{m-k}(D_{p,m}u)(0). \tag{12.3.7}$$

Let γ be an arbitrary multi-index of order $[m]$. The identity

$$\partial^\gamma u(0) = t^{-n}\int_{B_t} \eta\Big(\frac{y}{t}\Big)\partial^\gamma u(y)\,\mathrm{d}y + t^{-n}\int_{B_t} \eta\Big(\frac{y}{t}\Big)\big[\partial^\gamma u(0) - \partial^\gamma(y)\big]\,\mathrm{d}y$$

implies

$$\big|\nabla_{[m]}u(0)\big| \le t^{-n-[m]}\bigg|\int_{B_t} u(y)(\nabla_{[m]}\eta)\Big(\frac{y}{t}\Big)\,\mathrm{d}y\bigg|$$
$$+ t^{\{m\}}\bigg(\int_{B_t} \big|\eta(y)\big|^q |y|^{(\frac{n}{p}+\{m\})q}\,\mathrm{d}y\bigg)^{1/q}(D_{p,m}u)(0), \tag{12.3.8}$$

where $p^{-1} + q^{-1} = 1$. Combining (12.3.7)–(12.3.8) we arrive at

$$\big|\nabla_k u(0)\big| \le c\big(t^{-k}(\mathcal{M}u)(0) + t^{m-k}(D_{p,m}u)(0)\big).$$

Minimizing the right-hand side in t we complete the proof of (i).

(ii) It is sufficient to take $l \in (0,1)$ and $x = 0$. Since the function $\partial^{\beta+\gamma}(y^\beta \eta(y))$ in (12.3.5) is orthogonal to 1 in $L_2(B_1)$, it follows from (12.3.5) that

$$\big|\nabla_k u(0)\big| \le c\bigg(t^{-n-k}\int_{B_t}\big|u(y)-u(0)\big|\,\mathrm{d}y + \int_{B_t}\frac{|\nabla_m u(y)|}{|y|^{n-m+k}}\,\mathrm{d}y\bigg). \tag{12.3.9}$$

If $m - k \ge n$, the second integral does not exceed

$$t^{m-k-n}\int_{B_t}\big|\nabla_m u(y)\big|\,\mathrm{d}y.$$

In the case $m - k < n$ the same integral is estimated by (11.2.3)

$$\int_{B_t}\big|\nabla_m u(y)\big|\frac{\mathrm{d}y}{|y|^{n-m+k}} \le \frac{n}{m-k}t^{m-k}\sup_{\tau\le t}\tau^{-n}\int_{B_\tau}\big|\nabla_m u(y)\big|\,\mathrm{d}y. \tag{12.3.10}$$

Hence, and by Hölder's inequality, applied to the first integral in (12.3.9) we have

$$\big|\nabla_k u(0)\big| \le c\big(t^{-k}D_{p,l}u(0) + t^{m-k}\mathcal{M}\nabla_m(x)\big).$$

The result follows. □

(iii) By (12.3.9) with m replaced by $[m]$

$$\big|\nabla_k u(0)\big| \le c\bigg(t^{l-k}D_{p,l}u(0) + t^{[m]-k}\big|\nabla_{[m]}u(0)\big| + \int_{B_t}\frac{|\nabla_{[m]}u(y) - \nabla_{[m]}u(0)|}{|y|^{n-[m]+k}}\,\mathrm{d}y\bigg).$$

By (12.3.7) the third term on the right does not exceed

$$t^{m-k}D_{p,m}u(0).$$

Now we note that (12.3.8) implies

$$\big|\nabla_{[m]}u(0)\big| \le c\big(t^{l-[m]}D_{p,l}u(0) + t^{\{m\}}D_{p,m}u(0)\big).$$

Hence

$$\big|\nabla_k u(0)\big| \le c\big(t^{-k}D_{p,l}u(0) + t^{m-k}D_{p,m}u(0)\big).$$

The result follows.

12.3.2 Inequality with a Fractional Derivative Operator on the Left-Hand Side

Theorem. *Let* $0 < \alpha < 1$ *and let* $p \in [1, \infty)$*. Then*

$$(D_{p,\alpha}u)(x) \le c\big(\big(\mathcal{M}\big|u - u(x)\big|^p\big)(x)\big)^{(1-\alpha)/p}\big(\big(\mathcal{M}|\nabla u|^q\big)(x)\big)^{\alpha/q}, \quad (12.3.11)$$

where $q \ge \max\{1, pn/(n+p)\}$.

Proof. Let first $p \ge n/(n-1)$. Then $q = pn/(n+p)$. Let t be an arbitrary positive number to be chosen later. By Sobolev's embedding theorem

$$\begin{aligned}\Big(\int_{B_t(x)} & \frac{|u(y) - u(x)|^p}{|y-x|^{n+\alpha p}}\,\mathrm{d}y\Big)^{1/p} \\ &\le c_1\Big(\int_{B_t(x)}\Big|\nabla_y\Big(\frac{u(y)-u(x)}{|y-x|^{\alpha+n/p}}\Big)\Big|^q\,\mathrm{d}y\Big)^{1/q} \\ &\quad + c_2 t^{-1}\Big(\int_{B_t(x)}\frac{|u(y)-u(x)|^q}{|y-x|^{(\alpha+n/p)q}}\,\mathrm{d}y\Big)^{1/q}.\end{aligned}$$

The right-hand side of this inequality does not exceed

$$c_1\Big(\int_{B_t(x)}\frac{|\nabla u(y)|^q\,\mathrm{d}y}{|y-x|^{(\alpha+n/p)q}}\Big)^{1/q} + c_2\Big(\int_{B_t(x)}\frac{|u(y)-u(x)|^q}{|y-x|^{(\alpha+1+n/p)q}}\,\mathrm{d}y\Big)^{1/q},$$

which by Hardy's inequality is dominated by

$$c\Big(\int_{B_t(x)}\frac{|\nabla u(y)|^q\,\mathrm{d}y}{|y-x|^{(\alpha+n/p)q}}\Big)^{1/q}.$$

Estimating this by (11.2.3) we arrive at the inequality

$$\Big(\int_{B_t(x)}\frac{|u(y)-u(x)|^p}{|y-x|^{n+\alpha p}}\,\mathrm{d}y\Big)^{1/p} \le c t^{1-\alpha}\big(\big(\mathcal{M}|\nabla u|^q\big)(x)\big)^{1/q}. \quad (12.3.12)$$

Now let $p \in [1, n/(n-1))$ and let $\beta \in (n-1+\alpha, n)$. By Hölder's inequality with exponents $n/(n-1)p$ and $n/(n-(n-1)p)$ we have

$$\begin{aligned}\int_{B_t(x)} & \frac{|u(y)-u(x)|^p}{|y-x|^{n+\alpha p}}\,\mathrm{d}y \\ &\le \Big(\int_{B_t(x)}\Big(\frac{|u(y)-u(x)|}{|y-x|^\beta}\Big)^{\frac{n}{n-1}}\,\mathrm{d}y\Big)^{\frac{(n-1)p}{n}} \\ &\quad \times \Big(\int_{B_t(x)}\frac{\mathrm{d}y}{|y-x|^{\frac{(n+p(\alpha-\beta))n}{n-(n-1)p}}}\Big)^{1-\frac{(n-1)p}{n}}.\end{aligned}$$

The right-hand side is equal to

$$ct^{p(\beta-\alpha-n+1)}\Big(\int_{B_t(x)}\Big(\frac{|u(y)-u(x)|}{|y-x|^\beta}\Big)^{\frac{n}{n-1}}\,\mathrm{d}y\Big)^{\frac{(n-1)p}{n}},$$

which by the Gagliardo–Nirenberg inequality does not exceed

$$ct^{p(\beta-\alpha-n+1)}\Big(\int_{B_t(x)}\Big|\nabla_y\Big(\frac{u(y)-u(x)}{|y-x|^\beta}\Big)\Big|\,\mathrm{d}y+t^{-1}\int_{B_t(x)}\frac{|u(y)-u(x)|}{|y-x|^\beta}\,\mathrm{d}y\Big)^p$$
$$\le ct^{p(\beta-\alpha-n+1)}\Big(\int_{B_t(x)}\frac{|\nabla u(y)|}{|y-x|^\beta}\,\mathrm{d}y+\int_{B_t(x)}\frac{|u(y)-u(x)|}{|y-x|^{\beta+1}}\,\mathrm{d}y\Big)^p.$$

By Hardy's inequality the second integral on the right is dominated by the first one. Hence and by (11.2.3) we obtain

$$\int_{B_t(x)}\frac{|u(y)-u(x)|^p}{|y-x|^{n+\alpha p}}\,\mathrm{d}y\le ct^{p(\beta-\alpha-n+1)}\Big(\int_{B_t(x)}\frac{|\nabla u(y)|}{|y-x|^\beta}\,\mathrm{d}y\Big)^p$$
$$\le ct^{p(1-\alpha)}\big((\mathcal{M}|\nabla u|)(x)\big)^p. \tag{12.3.13}$$

Unifying this with (12.3.12), we find

$$\int_{B_t(x)}\frac{|u(y)-u(x)|^p}{|y-x|^{n+\alpha p}}\,\mathrm{d}y\le ct^{p(1-\alpha)}\big((\mathcal{M}|\nabla u|^q)(x)\big)^{p/q}, \tag{12.3.14}$$

where $q=\max\{1,pn/(n+p)\}$. For any nonnegative f and any $a>0$ there holds the inequality

$$\int_{\mathbb{R}^n\setminus B_t(x)}\frac{f(y)\,\mathrm{d}y}{|y-x|^{n+a}}\le ct^{-a}(\mathcal{M}f)(x), \tag{12.3.15}$$

which follows from the readily checked identity

$$\int_{\mathbb{R}^n\setminus B_t(x)}\frac{f(y)\,\mathrm{d}y}{|x-y|^{n+a}}$$
$$=(n+a)\int_t^\infty\int_{B_s(x)}f(y)\,\mathrm{d}y\frac{\mathrm{d}s}{s^{n+a+1}}-t^{-a-n}\int_{B_t(x)}f(y)\,\mathrm{d}y. \tag{12.3.16}$$

(see Hedberg [365]). By (12.3.15)

$$\int_{\mathbb{R}^n\setminus B_t(x)}\frac{|u(y)-u(x)|^p}{|y-x|^{n+\alpha p}}\,\mathrm{d}y\le ct^{-p\alpha}\big(\mathcal{M}\big|u-u(x)\big|^p\big)(x). \tag{12.3.17}$$

Summing up (12.3.14) and (12.3.17) we arrive at

$$\big((D_{p,\alpha}u)(x)\big)^p\le c\big(t^{p(1-\alpha)}\big((\mathcal{M}|\nabla u|^q)(x)\big)^{p/q}+t^{-p\alpha}\big(\mathcal{M}\big|u-u(x)\big|^p\big)(x)\big).$$

Minimizing the right-hand side in $t>0$ we complete the proof. □

12.3.3 Application: Weighted Gagliardo–Nirenberg-Type Inequalities for Derivatives

Proposition. *Let* $1 < q < \infty$, $1 < p < \infty$, *and let*

$$\frac{1}{s} = \frac{k}{m}\frac{1}{p} + \left(1 - \frac{k}{m}\right)\frac{1}{q}.$$

(i) *Let* k *be an integer and let* l *and* m *be noninteger,* $0 < l < k < m$. *Then for any nonnegative measurable* w

$$\|\nabla_k u\|_{L_s(w\,\mathrm{d}x)} \le c\|D_{p,l}u\|_{L_q(w\,\mathrm{d}x)}^{\frac{m-k}{m-l}} \|D_{p,m}u\|_{L_p(w\,\mathrm{d}x)}^{\frac{k-l}{m-l}}. \tag{12.3.18}$$

(ii) *Let* k *and* l *be integers and let* m *be noninteger,* $0 \le l \le k < m$. *Further let* $w \in A_q$. *Then*

$$\|\nabla_k u\|_{L_s(w\,\mathrm{d}x)} \le c\|\nabla_l u\|_{L_q(w\,\mathrm{d}x)}^{\frac{m-k}{m-l}} \|D_{p,m}u\|_{L_p(w\,\mathrm{d}x)}^{\frac{k-l}{m-l}}. \tag{12.3.19}$$

(iii) *Let* k *and* m *be integers and let* l *be noninteger,* $0 < l < k \le m$. *Further let* $w \in A_p$. *Then*

$$\|\nabla_k u\|_{L_s(w\,\mathrm{d}x)} \le c\|D_{p,l}u\|_{L_q(w\,\mathrm{d}x)}^{\frac{m-k}{m-l}} \|\nabla_m u\|_{L_p(w\,\mathrm{d}x)}^{\frac{k-l}{m-l}}. \tag{12.3.20}$$

Proof. We use the same argument as in the proof of Proposition 12.1.7. Inequality (12.3.18) follows from (12.3.3) and Hölder's inequality. To show (12.3.19) we notice that by (12.3.1) its right-hand side does not exceed

$$c\left(\int_{\mathbb{R}^n} \big(\mathcal{M}\nabla_l u(x)\big)^{s\frac{m-k}{m-l}} \big(D_{p,m}u(x)\big)^{s\frac{k-l}{m-l}} w(x)\,\mathrm{d}x\right)^{1/s},$$

which by Hölder's inequality is majorized by

$$c\|\mathcal{M}\nabla_l u\|_{L_q(w\,\mathrm{d}x)}^{\frac{m-k}{m-l}} \|D_{p,m}u\|_{L_p(w\,\mathrm{d}x)}^{\frac{k-l}{m-l}},$$

and it remains to refer to the Muckenhoupt theorem on the boundedness of the operator $\mathcal{M}$ in $L_\sigma(w\,\mathrm{d}x))$, $1 < \sigma < \infty$. Inequality (12.3.20) is proved in the same way. □

12.4 Application of (12.3.11) to Composition Operator in Fractional Sobolev Spaces

12.4.1 Introduction

In this section we use the space $\mathcal{W}_p^s$ of distributions with the finite norm

$$\|u\|_{\mathcal{W}_p^s} = \|\nabla_{[s]} u\|_{\mathcal{W}_p^{\{s\}}},$$

where $\{s\} > 0$ and

$$\|v\|_{\mathcal{W}_p^{\{s\}}} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^p}{|x-y|^{n+\{s\}p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}.$$

The application mentioned in the title of this section is as follows.

Theorem. *Let $p \geq 1$ and let s be noninteger, $1 < s < \infty$. For every complex-valued function f defined on $\mathbb{R}$ and such that $f(0) = 0$ and $f', \dots, f^{([s]+1)} \in L_\infty(\mathbb{R})$ there holds*

$$\big\|f(u)\big\|_{\mathcal{W}_p^s(\mathbb{R}^n)} \leq c \sum_{l=1}^{[s]+1} \big\|f^{(l)}\big\|_{L_\infty(\mathbb{R})} \big(\|u\|_{\mathcal{W}_p^s(\mathbb{R}^n)} + \|\nabla u\|^s_{L_{ps}(\mathbb{R}^n)}\big), \tag{12.4.1}$$

where c is a constant independent of f and u.

If additionally, $f^{([s]+1)} \in C(\mathbb{R})$ then the map

$$\mathcal{W}_p^s\big(\mathbb{R}^n\big) \cap W^1_{sp}\big(\mathbb{R}^n\big) \ni u \to f(u) \in \mathcal{W}_p^s\big(\mathbb{R}^n\big) \tag{12.4.2}$$

is continuous.

We formulate a particular case of the inequality (12.3.11), which is used in the proof of this theorem.

Corollary. *Suppose $\alpha \in (0,1)$, $p \geq 1$, and $u \in W_p^1(\mathbb{R}^n, \mathrm{loc})$. Then for almost all $x \in \mathbb{R}^n$*

$$(D_{p,\alpha} u)(x) \leq c\Big(\big(\mathcal{M}\big|u - u(x)\big|^p\big)(x)\Big)^{(1-\alpha)/p} \Big(\big(\mathcal{M}|\nabla u|^p\big)(x)\Big)^{\alpha/p}, \tag{12.4.3}$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator.

Another ingredient in the proof of the theorem is the Gagliardo–Nirenberg-type inequality

$$\|\nabla_k u\|_{L_{ps/k}(\mathbb{R}^n)} \leq c\|\nabla u\|^{\frac{s-k}{s-1}}_{L_{ps}(\mathbb{R}^n)} \|D_{p,s} u\|^{\frac{k-1}{s-1}}_{L_p(\mathbb{R}^n)}, \tag{12.4.4}$$

where $1 \leq k < s$, s is noninteger, and $p \geq 1$.

A short argument leading to (12.4.4) is by the pointwise inequality

$$\big|\nabla_k u(x)\big| \leq c\Big(\big(\mathcal{M}|\nabla u|\big)(x)\Big)^{\frac{s-k}{s-1}} \big((D_{p,s} u)(x)\big)^{\frac{k-1}{s-1}}. \tag{12.4.5}$$

In fact, one uses (12.4.5) to majorize the $L_{ps/k}$-norm of $|\nabla_k u|$ and applies Hölder's inequality with exponents

$$(s-1)k/(s-k) \quad \text{and} \quad (s-1)k/(k-1)s$$

together with the boundedness of $\mathcal{M}$ in $L_{ps}(\mathbb{R}^n)$. Inequality (12.4.5) was proved in Sect. 12.3.1.

12.4.2 Proof of Inequality (12.4.1)

Since $f(0)=0$, we have

$$\big\|f(u)\big\|_{L_p(\mathbb{R}^n)} \le \|f'\|_{L_\infty(\mathbb{R})}\|u\|_{L_p(\mathbb{R}^n)}. \tag{12.4.6}$$

By the Leibnitz rule,

$$\big\|D_{p,s}f(u)\big\|_{L_p(\mathbb{R}^n)} \le c\sum_{l=1}^{[s]} \sum_{\substack{|\alpha_1|+\ldots+|\alpha_l|=[s]\\ |\alpha_i|\ge 1}} \bigg\|D_{p,\{s\}}\bigg(f^{(l)}(u)\prod_{i=1}^{l}\partial^{\alpha_i}u\bigg)\bigg\|_{L_p(\mathbb{R}^n)}. \tag{12.4.7}$$

We continue by putting

$$v=f^{(l)}(u) \quad \text{and} \quad w=\prod_{i=1}^{l}\partial^{\alpha_i}u$$

in the obvious inequality

$$\big\|D_{p,\{s\}}(vw)\big\|_{L_p(\mathbb{R}^n)} \le \|vD_{p,\{s\}}w\|_{L_p(\mathbb{R}^n)} + \|wD_{p,\{s\}}v\|_{L_p(\mathbb{R}^n)} \tag{12.4.8}$$

and arrive at

$$\begin{aligned}\big\|D_{p,s}f(u)\big\|_{L_p(\mathbb{R}^n)} \le c\sum_{l=1}^{[s]} \sum_{\substack{|\alpha_1|+\ldots+|\alpha_l|=[s]\\ |\alpha_i|\ge 1}} \bigg(\bigg\|\prod_{i=1}^{l}\partial^{\alpha_i}u\cdot D_{p,\{s\}}f^{(l)}(u)\bigg\|_{L_p(\mathbb{R}^n)}\\ +\big\|f^{(l)}\big\|_{L^\infty(\mathbb{R})}\bigg\|D_{p,\{s\}}\bigg(\prod_{i=1}^{l}\partial^{\alpha_i}u\bigg)\bigg\|_{L_p(\mathbb{R}^n)}\bigg).\end{aligned} \tag{12.4.9}$$

We set

$$\mathcal{I}_l := \bigg\|\prod_{i=1}^{l}\partial^{\alpha_i}u\cdot D_{p,\{s\}}f^{(l)}(u)\bigg\|_{L_p(\mathbb{R}^n)}. \tag{12.4.10}$$

By inequality (12.4.3) with $f^{(l)}(u)$ in place of u,

$$\begin{aligned}D_{p,\{s\}}f^{(l)}(u)(x) &\le c\big\|f^{(l)}\big\|^{1-\{s\}}_{L_\infty(\mathbb{R})}\big(\big(\mathcal{M}\big|\nabla f^{(l)}(u)\big|^p\big)(x)\big)^{\{s\}/p}\\ &\le c\big\|f^{(l)}\big\|^{1-\{s\}}_{L_\infty(\mathbb{R})}\big\|f^{(l+1)}\big\|^{\{s\}}_{L_\infty(\mathbb{R})}\big(\big(\mathcal{M}|\nabla u|^p\big)(x)\big)^{\{s\}/p}.\end{aligned}$$

Hence, using Hölder's inequality with exponents $s/\{s\}$, $s/|\alpha_i|$, $i=1,\ldots,l$, we find

$$\begin{aligned}\mathcal{I}_l \le c\big\|f^{(l)}\big\|^{1-\{s\}}_{L_\infty(\mathbb{R})}\big\|f^{(l+1)}\big\|^{\{s\}}_{L_\infty(\mathbb{R})}\\ \times\prod_{i=1}^{l}\big\|\partial^{\alpha_i}u\big\|_{L_{\frac{ps}{|\alpha_i|}}(\mathbb{R}^n)}\big\|\mathcal{M}|\nabla u|^p\big\|^{\{s\}/p}_{L_s(\mathbb{R}^n)}.\end{aligned} \tag{12.4.11}$$

Since $\mathcal{M}$ is bounded in $L_s(\mathbb{R}^n)$, the last factor on the right is majorized by $c\|\nabla u\|^{\{s\}}_{L_{ps}(\mathbb{R}^n)}$. By (12.4.4) the product $\prod_{i=1}^l$ in (12.4.11) does not exceed

$$c\prod_{i=1}^{l}\|\nabla u\|^{\frac{s-|\alpha_i|}{s-1}}_{L_{ps}(\mathbb{R}^n)}\|D_{p,s}u\|^{\frac{|\alpha_i|-1}{s-1}}_{L_p(\mathbb{R}^n)}$$
$$= c\|\nabla u\|^{\frac{sl-[s]}{s-1}}_{L_{ps}(\mathbb{R}^n)}\|D_{p,s}u\|^{\frac{[s]-l}{s-1}}_{L_p(\mathbb{R}^n)}. \tag{12.4.12}$$

Hence and by (12.4.11)

$$\mathcal{I}_l \le c\big\|f^{(l)}\big\|^{1-\{s\}}_{L_\infty(\mathbb{R})}\big\|f^{(l+1)}\big\|^{\{s\}}_{L_\infty(\mathbb{R})}\|\nabla u\|^{\frac{s(l-1+\{s\})}{s-1}}_{L_{ps}(\mathbb{R}^n)}\|D_{p,s}u\|^{\frac{[s]-l}{s-1}}_{L_p(\mathbb{R}^n)}. \tag{12.4.13}$$

Let

$$\mathcal{J}_l := \left\|D_{p,\{s\}}\left(\prod_{i=1}^{l}\partial^{\alpha_i}u\right)\right\|_{L_p(\mathbb{R}^n)}. \tag{12.4.14}$$

Clearly,

$$\mathcal{J}_1 \le \|D_{p,s}u\|_{L_p(\mathbb{R}^n)}. \tag{12.4.15}$$

Now let $l > 1$. By (12.4.8),

$$\mathcal{J}_l \le \left\|\prod_{i=2}^{l}\partial^{\alpha_i}u\cdot D_{p,\{s\}}\partial^{\alpha_1}u\right\|_{L_p(\mathbb{R}^n)} + \left\|\partial^{\alpha_1}u\cdot D_{p,\{s\}}\left(\prod_{i=2}^{l}\partial^{\alpha_i}u\right)\right\|_{L_p(\mathbb{R}^n)}. \tag{12.4.16}$$

Applying first the inequality (12.4.3) and then Hölder's inequality with exponents

$$s/|\alpha_i|,\quad 2\le i\le l,\quad s/|\alpha_1|\big(1-\{s\}\big),\quad \text{and}\quad s/\big(1+|\alpha_1|\big)\{s\},$$

we find

$$\left\|\prod_{i=2}^{l}\partial^{\alpha_i}u\cdot D_{p,\{s\}}\partial^{\alpha_1}u\right\|_{L_p(\mathbb{R}^n)}$$
$$\le c\left\|\prod_{i=2}^{l}\partial^{\alpha_i}u\cdot\big(\mathcal{M}|\partial^{\alpha_1}u|^p\big)^{(1-\{s\})/p}\big(\mathcal{M}|\nabla\partial^{|\alpha_1}u|^p\big)^{\{s\}/p}\right\|_{L_p(\mathbb{R}^n)}$$
$$\le c\prod_{i=2}^{l}\|\partial^{\alpha_i}u\|_{L_{\frac{ps}{|\alpha_i|}}(\mathbb{R}^n)}\big\|\mathcal{M}|\partial^{\alpha_1}u|^p\big\|^{\frac{1-\{s\}}{p}}_{L_{\frac{s}{|\alpha_1|}}(\mathbb{R}^n)}\big\|\mathcal{M}|\nabla\partial^{\alpha_1}u|^p\big\|^{\frac{\{s\}}{p}}_{L_{\frac{s}{1+|\alpha_1|}}(\mathbb{R}^n)}.$$

Hence, noting that $s > 1+|\alpha_1|$ and using the boundedness of $\mathcal{M}$ in $L_q(\mathbb{R}^n)$ with $q > 1$, we obtain that the first term on the right-hand side of (12.4.16) is dominated by

$$c\prod_{i=2}^{l}\|\partial^{\alpha_i}u\|_{L_{\frac{ps}{|\alpha_i|}}(\mathbb{R}^n)}\|\partial^{\alpha_1}u\|^{1-\{s\}}_{L_{\frac{ps}{|\alpha_1|}}(\mathbb{R}^n)}\|\nabla\partial^{\alpha_1}u\|^{\{s\}}_{L_{\frac{ps}{1+|\alpha_1|}}(\mathbb{R}^n)}. \tag{12.4.17}$$

By (12.4.4) this does not exceed

$$c\|\nabla u\|^{\frac{s(l-1)}{s-1}}_{L_{ps}(\mathbb{R}^n)}\|D_{p,s}u\|^{\frac{s-l}{s-1}}_{L_p(\mathbb{R}^n)}. \tag{12.4.18}$$

We estimate the second term in the right-hand side of (12.4.16). By (12.4.3),

$$\begin{aligned}&\left\|\partial^{\alpha_1}u\cdot D_{p,\{s\}}\left(\prod_{i=2}^{l}\partial^{\alpha_i}u\right)\right\|_{L_p(\mathbb{R}^n)}\\&\quad\le c\left\|\partial^{\alpha_1}u\left(\mathcal{M}\left|\prod_{i=2}^{l}\partial^{\alpha_i}u\right|^p\right)^{(1-\{s\})/p}\left(\mathcal{M}\left|\nabla\left(\prod_{i=2}^{l}\partial^{\alpha_i}u\right)\right|^p\right)^{\{s\}/p}\right\|_{L_p(\mathbb{R}^n)},\end{aligned}$$

which by Hölder's inequality with exponents

$$s/|\alpha_1|,\quad s/\big([s]-|\alpha_1|\big)\big(1-\{s\}\big),\quad\text{and}\quad s/\big(1+[s]-|\alpha_1|\big)\{s\},$$

is majorized by

$$c\|\partial^{\alpha_1}u\|_{L_{\frac{ps}{|\alpha_1|}}(\mathbb{R}^n)}\left\|\mathcal{M}\prod_{i=2}^{l}|\partial^{\alpha_i}u|^p\right\|^{(1-\{s\})/p}_{L_{\frac{s}{[s]-|\alpha_1|}}(\mathbb{R}^n)}\left\|\mathcal{M}\left|\nabla\prod_{i=2}^{l}\partial^{\alpha_i}u\right|^p\right\|^{\{s\}/p}_{L_{\frac{s}{1+[s]-|\alpha_1|}}(\mathbb{R}^n)}.$$

Using the L_q-boundedness of $\mathcal{M}$ with

$$q=s/\big([s]-|\alpha_1|\big)\quad\text{and}\quad q=s/\big(1+[s]-|\alpha_1|\big),$$

we conclude that

$$\begin{aligned}&\left\|\partial^{\alpha_1}u\cdot D_{p,\{s\}}\left(\prod_{i=2}^{l}\partial^{\alpha_i}u\right)\right\|_{L_p(\mathbb{R}^n)}\\&\quad\le c\|\partial^{\alpha_1}u\|_{L_{\frac{ps}{|\alpha_1|}}(\mathbb{R}^n)}\left\|\prod_{i=2}^{l}\partial^{\alpha_i}u\right\|^{1-\{s\}}_{L_{\frac{ps}{[s]-|\alpha_1|}}(\mathbb{R}^n)}\left\|\nabla\prod_{i=2}^{l}\partial^{\alpha_i}u\right\|^{\{s\}}_{L_{\frac{ps}{1+[s]-|\alpha_1|}}(\mathbb{R}^n)}.\end{aligned} \tag{12.4.19}$$

By Hölder's inequality with exponents $([s]-|\alpha_1|)/|\alpha_i|$, $2\le i\le l$, the second norm on the right-hand side of (12.4.19) does not exceed

$$c\prod_{i=2}^{l}\|\partial^{\alpha_i}u\|_{L_{\frac{ps}{|\alpha_i|}}(\mathbb{R}^n)}. \tag{12.4.20}$$

Again by Hölder's inequality, now with exponents

$$\big(1+[s]-|\alpha_1|\big)/|\alpha_j|, \quad 2\le j\le l,\ j\ne i, \quad \text{and} \quad \big(1+[s]-|\alpha_1|\big)/\big(1+|\alpha_i|\big),$$

the third norm on the right-hand side of (12.4.19) is dominated by

$$\sum_{i=2}^{l}\prod_{\substack{j=2\\ j\ne i}}^{l}\big\|\partial^{\alpha_j}u\big\|_{L_{\frac{ps}{|\alpha_j|}}(\mathbb{R}^n)}\big\|\nabla\partial^{\alpha_i}u\big\|_{L_{\frac{ps}{1+|\alpha_i|}}(\mathbb{R}^n)}. \tag{12.4.21}$$

Combining (12.4.19)–(12.4.21) with (12.4.4) we find that the left-hand side of (12.4.19) does not exceed (12.4.18). Thus, (12.4.18) is a majorant for the second term in (12.4.16). Hence and by (12.4.15),

$$\mathcal{J}_l\le c\|\nabla u\|_{L_{ps}(\mathbb{R}^n)}^{\frac{s(l-1)}{s-1}}\|D_{p,s}u\|_{L_p(\mathbb{R}^n)}^{\frac{s-l}{s-1}}, \quad 1\le l\le[s]. \tag{12.4.22}$$

Inserting (12.4.13) and (12.4.22) into (12.4.9) we arrive at

$$\big\|D_{p,s}f(u)\big\|_{L_p(\mathbb{R}^n)}\le c\sum_{l=1}^{[s]+1}\big\|f^{(l)}\big\|_{L_\infty(\mathbb{R})}\big(\|D_{p,s}u\|_{L_p(\mathbb{R}^n)}+\|\nabla u\|^s_{L_{ps}(\mathbb{R}^n)}\big).$$

Hence and by (12.4.6) the proof of (12.4.1) is complete.

12.4.3 Continuity of the Map (12.4.2)

Let $u_\nu\to u$ in $\mathcal{W}_p^s(\mathbb{R}^n)\cap L^1_{sp}(\mathbb{R}^n)$. Since

$$\big\|f^{(l)}(u_\nu)-f^{(l)}(u)\big\|_{L_p(\mathbb{R}^n)}\le c\big\|f^{(l+1)}\big\|_{L_\infty(\mathbb{R})}\|u_\nu-u\|_{L_p(\mathbb{R}^n)}$$

for $l=0,\dots,[s]$, we have

$$f^{(l)}(u_\nu)\to f^{(l)}(u) \quad \text{in } L_p\big(\mathbb{R}^n\big). \tag{12.4.23}$$

We shall prove that

$$\big\|D_{p,s}\big(f(u_\nu)-f(u)\big)\big\|_{L_p(\mathbb{R}^n)}\to 0. \tag{12.4.24}$$

Let α be a multi-index of order $|\alpha|=[s]$. By the Leibnitz rule,

$$\partial^\alpha\big(f(u)-f(u_\nu)\big)=\sum_{l=1}^{[s]}\sum c(l,\alpha_1,\dots,\alpha_l)$$
$$\times\left(f^{(l)}(u)\prod_{i=1}^{l}\partial^{\alpha_i}u-f^{(l)}(u_\nu)\prod_{i=1}^{l}\partial^{\alpha_i}u_\nu\right),$$

where the second sum is taken over all l-tuples of multi-indices $\{\alpha_1, \dots, \alpha_l\}$ such that $\alpha_1 + \dots + \alpha_l = \alpha$ and $|\alpha_i| \geq 1$. We rewrite the difference

$$f^{(l)}(u) \prod_{i=1}^{l} \partial^{\alpha_i} u - f^{(l)}(u_\nu) \prod_{i=1}^{l} \partial^{\alpha_i} u_\nu \tag{12.4.25}$$

using the identity

$$\prod_{i=0}^{l} a_i - \prod_{i=0}^{l} b_i = \sum_{k=0}^{l} b_0, \dots, b_{k-1}(a_k - b_k) a_{k+1}, \dots, a_l, \tag{12.4.26}$$

where the products of either b_i or a_i are missing if $k = 0$ or $k = l$, respectively. Setting

$$a_0 = f^{(l)}(u), \qquad a_i = \partial^{\alpha_i} u, \qquad b_0 = f^{(l)}(u_\nu), \qquad b_i = \partial^{\alpha_i} u_\nu, \quad 1 \leq i \leq l,$$

in (12.4.26), we find that (12.4.25) is equal to

$$\big(f^{(l)}(u) - f^{(l)}(u_\nu)\big) \prod_{i=1}^{l} \partial^{\alpha_i} u + f^{(l)}(u_\nu) \sum_{k=1}^{l} \prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \partial^{\alpha_k} (u - u_\nu) \prod_{i=k+1}^{l} \partial^{\alpha_i} u.$$

Consequently,

$$\begin{aligned} &\big\| D_{p,\{s\}} \big(\nabla_{[s]} \big(f(u) - f(u_\nu)\big)\big) \big\|_{L_p(\mathbb{R}^n)} \\ &\quad \leq c \sum_{l=1}^{[s]} \sum_{\substack{|\alpha_1| + \dots + |\alpha_l| = [s] \\ |\alpha_i| \geq 1}} \left(\left\| D_{p,\{s\}} \left(\big(f^{(l)}(u) - f^{(l)}(u_\nu)\big) \prod_{i=1}^{l} \partial^{\alpha_i} u \right) \right\|_{L_p(\mathbb{R}^n)} \right. \\ &\qquad \left. + \sum_{k=1}^{l} \left\| D_{p,\{s\}} \left(f^{(l)}(u_\nu) \prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \partial^{\alpha_k} (u - u_\nu) \prod_{i=k+1}^{l} \partial^{\alpha_i} u \right) \right\|_{L_p(\mathbb{R}^n)} \right). \end{aligned} \tag{12.4.27}$$

By (12.4.22), (12.4.23) and the boundedness of derivatives of f we can apply the Lebesgue dominated convergence theorem to conclude that

$$\left\| \big(f^{(l)}(u) - f^{(l)}(u_\nu)\big) D_{p,\{s\}} \left(\prod_{i=1}^{l} \partial^{\alpha_i} u \right) \right\|_{L_p(\mathbb{R}^n)} \to 0. \tag{12.4.28}$$

Using (12.4.3) with u replaced by $f^{(l)}(u) - f^{(l)}(u_\nu)$ and employing Hölder's inequality with exponents $s/|\alpha_i|$, $1 \leq i \leq l$, and $s/\{s\}$, we obtain

$$\Big\| \prod_{i=1}^{l} \partial^{\alpha_i} u \cdot D_{p,\{s\}} \big(f^{(l)}(u) - f^{(l)}(u_\nu)\big) \Big\|_{L_p(\mathbb{R}^n)}$$
$$\le c \Big\| \prod_{i=1}^{l} \partial^{\alpha_i} u \cdot \big(\mathcal{M} |f^{(l)}(u) - f^{(l)}(u_\nu)|^p\big)^{\frac{1-\{s\}}{p}}$$
$$\times \big(\mathcal{M} |\nabla \big(f^{(l)}(u) - f^{(l)}(u_\nu)\big)|^p\big)^{\frac{\{s\}}{p}} \Big\|_{L_p(\mathbb{R}^n)}$$
$$\le c \|f^{(l)}\|^{1-\{s\}}_{L^\infty(\mathbb{R})} \prod_{i=1}^{l} \|\partial^{\alpha_i} u\|_{L_{\frac{ps}{|\alpha_i|}}(\mathbb{R}^n)} \big\|\mathcal{M} |\nabla \big(f^{(l)}(u) - f^{(l)}(u_\nu)\big)|^p \big\|^{\frac{\{s\}}{p}}_{L_s(\mathbb{R}^n)}.$$

The boundedness of $\mathcal{M}$ in $L_s(\mathbb{R}^n)$ implies that the left-hand side of the last inequality is dominated by

$$c\|f^{(l)}\|^{1-\{s\}}_{L_\infty(\mathbb{R})} \prod_{i=1}^{l} \|\partial^{\alpha_i} u\|_{L_{\frac{ps}{|\alpha_i|}}(\mathbb{R}^n)} \big\|\nabla \big(f^{(l)}(u) - f^{(l)}(u_\nu)\big)\big\|^{\{s\}}_{L_{ps}(\mathbb{R}^n)}. \quad (12.4.29)$$

By (12.4.4), the product $\prod_{i=1}^{l}$ has the majorant (12.4.12). Obviously,

$$\big\|\nabla \big(f^{(l)}(u) - f^{(l)}(u_\nu)\big)\big\|_{L_{ps}(\mathbb{R}^n)}$$
$$\le c\big(\|f^{(l)}\|_{L_\infty(\mathbb{R})} \|\nabla (u - u_\nu)\|_{L_{ps}(\mathbb{R}^n)}$$
$$+ \big\|\big(f^{(l+1)}(u) - f^{(l+1)}(u_\nu)\big)\nabla u\big\|_{L_{ps}(\mathbb{R}^n)}\big).$$

Hence and by Lebesgue's dominated convergence theorem

$$\Big\| \prod_{i=1}^{l} \partial^{\alpha_i} u \cdot D_{p,\{s\}} \big(f^{(l)}(u) - f^{(l)}(u_\nu)\big) \Big\|_{L_p(\mathbb{R}^n)} \to 0.$$

This together with (12.4.28) implies that the first term in brackets in the right-hand side of (12.4.27) tends to zero.

We now show that

$$\Big\| f^{(l)}(u_\nu) D_{p,\{s\}} \Big(\prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \partial^{\alpha_k}(u_\nu - u) \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u \Big) \Big\|_{L_p(\mathbb{R}^n)} \to 0 \quad (12.4.30)$$

for any $k = 1, \dots, l$. Here the products

$$\prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \quad \text{and} \quad \prod_{j=k+1}^{l} \partial^{\alpha_j} u$$

are missing for $k = 1$ and $k = l$, respectively. Clearly, (12.4.30) holds for $l = 1$. Let $l > 1$. By (12.4.8), the left-hand side in (12.4.30) is majorized by

$$\|f^{(l)}\|_{L_\infty(\mathbb{R})}\Bigg(\Bigg\|\prod_{i=1}^{k-1}\partial^{\alpha_i}u_\nu\cdot\prod_{j=k+1}^{l}\partial^{\alpha_j}u\cdot D_{p,\{s\}}\big(\partial^{\alpha_k}(u_\nu-u)\big)\Bigg\|_{L_p(\mathbb{R}^n)}$$
$$+\Bigg\|\partial^{\alpha_k}(u_\nu-u)\cdot D_{p,\{s\}}\Bigg(\prod_{i=1}^{k-1}\partial^{\alpha_i}u_\nu\cdot\prod_{j=k+1}^{l}\partial^{\alpha_j}u\Bigg)\Bigg\|_{L_p(\mathbb{R}^n)}\Bigg). \qquad (12.4.31)$$

Applying inequality (12.4.3), we find that the first term in brackets in (12.4.31) does not exceed

$$c\Bigg\|\prod_{i=1}^{k-1}\partial^{\alpha_i}u_\nu\cdot\prod_{j=k+1}^{l}\partial^{\alpha_j}u\cdot\big(\mathcal{M}\big|\partial^{\alpha_k}(u_\nu-u)\big|^p\big)^{\frac{1-\{s\}}{p}}$$
$$\times\big(\mathcal{M}\big|\nabla\partial^{\alpha_k}(u_\nu-u)\big|^p\big)^{\frac{\{s\}}{p}}\Bigg\|_{L_p(\mathbb{R}^n)}. \qquad (12.4.32)$$

Hölder's inequality with exponents

$$\frac{s}{|\alpha_i|},\quad 1\le i\le k-1,\qquad \frac{s}{|\alpha_j|},\quad k+1\le j\le l,$$
$$\frac{s}{|\alpha_k|(1-\{s\})},\qquad \frac{s}{(1+|\alpha_k|)\{s\}},$$

as well as the boundedness of $\mathcal{M}$ in $L_q(\mathbb{R}^n)$ with $q=s/|\alpha_k|$ and $q=s/(1+|\alpha_k|)$ yield that (12.4.32) is dominated by

$$c\prod_{i=1}^{k-1}\|\partial^{\alpha_i}u_\nu\|_{L_{\frac{ps}{|\alpha_i|}}(\mathbb{R}^n)}\prod_{j=k+1}^{l}\|\partial^{\alpha_j}u\|_{L_{\frac{ps}{|\alpha_j|}}(\mathbb{R}^n)}$$
$$\times\|\partial^{\alpha_k}(u_\nu-u)\|^{1-\{s\}}_{L_{\frac{ps}{|\alpha_k|}}(\mathbb{R}^n)}\|\nabla\partial^{\alpha_k}(u_\nu-u)\|^{\{s\}}_{L_{\frac{ps}{1+|\alpha_k|}}(\mathbb{R}^n)}. \qquad (12.4.33)$$

By (12.4.4) applied to each factor we see that (12.4.33) and therefore the first term in brackets in (12.4.31) tends to zero.

Making use of inequality (12.4.3) once more, we obtain that the second term in brackets in (12.4.31) is majorized by

$$c\Bigg\|\partial^{\alpha_k}(u_\nu-u)\Bigg(\mathcal{M}\Bigg|\prod_{i=1}^{k-1}\partial^{\alpha_i}u_\nu\cdot\prod_{j=k+1}^{l}\partial^{\alpha_j}u\Bigg|^p\Bigg)^{\frac{1-\{s\}}{p}}$$
$$\times\Bigg(\mathcal{M}\Bigg|\nabla\Bigg(\prod_{i=1}^{k-1}\partial^{\alpha_i}u_\nu\cdot\prod_{j=k+1}^{l}\partial^{\alpha_j}u\Bigg)\Bigg|^p\Bigg)^{\frac{\{s\}}{p}}\Bigg\|_{L_p(\mathbb{R}^n)}. \qquad (12.4.34)$$

Applying Hölder's inequality with exponents

$$\frac{s}{|\alpha_k|},\qquad \frac{s}{([s]-|\alpha_k|)(1-\{s\})},\qquad \frac{s}{(1+[s]-|\alpha_k|)\{s\}},$$

and using the L_q-boundedness of $\mathcal{M}$ for $q = s/([s] - |\alpha_k|)$ and $q = s/(1+[s]-|\alpha_k|)$, we find that (12.4.34) does not exceed

$$cN_1^{1-\{s\}} N_2^{\{s\}} \left\|\partial^{\alpha_k}(u_\nu - u)\right\|_{L_{\frac{ps}{|\alpha_k|}}(\mathbb{R}^n)}, \tag{12.4.35}$$

where

$$N_1 := \left\| \prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u \right\|_{L_{\frac{ps}{[s]-|\alpha_k|}}(\mathbb{R}^n)}$$

and

$$N_2 := \left\| \nabla \left(\prod_{i=1}^{k-1} \partial^{\alpha_i} u_\nu \cdot \prod_{j=k+1}^{l} \partial^{\alpha_j} u \right) \right\|_{L_{\frac{ps}{1+[s]-|\alpha_k|}}(\mathbb{R}^n)}.$$

By (12.4.4),

$$\left\|\partial^{\alpha_k}(u_\nu - u)\right\|_{L_{\frac{ps}{|\alpha_k|}}(\mathbb{R}^n)} \to 0.$$

It remains to show the boundedness of N_1 and N_2. Using Hölder's inequality with exponents

$$\frac{[s]-|\alpha_k|}{|\alpha_i|}, \quad 1 \le i \le k-1, \qquad \text{and} \qquad \frac{[s]-|\alpha_k|}{|\alpha_j|}, \quad k+1 \le j \le l,$$

we find

$$N_1 \le c \prod_{i=1}^{k-1} \left\|\partial^{\alpha_i} u_\nu\right\|_{L_{\frac{ps}{|\alpha_i|}}(\mathbb{R}^n)} \prod_{j=k+1}^{l} \left\|\partial^{\alpha_j} u\right\|_{L_{\frac{ps}{|\alpha_j|}}(\mathbb{R}^n)},$$

which is bounded owing to (12.4.4). Again by Hölder's inequality, now with exponents

$$\frac{1+[s]-|\alpha_k|}{|\alpha_i|}, \quad 1 \le i \le k-1, i \ne r, \qquad \frac{1+[s]-|\alpha_k|}{1+|\alpha_r|},$$

and

$$\frac{1+[s]-|\alpha_k|}{|\alpha_j|}, \quad k+1 \le j \le l, j \ne r, \qquad \frac{1+[s]-|\alpha_k|}{1+|\alpha_r|},$$

we find that N_2 is majorized by

$$\begin{aligned} c\Bigg(&\sum_{r=1}^{k-1} \prod_{\substack{i=1 \\ i \ne r}}^{k-1} \left\|\partial^{\alpha_i} u_\nu\right\|_{L_{\frac{ps}{|\alpha_i|}}(\mathbb{R}^n)} \left\|\nabla \partial^{\alpha_r} u_\nu\right\|_{L_{\frac{ps}{1+|\alpha_r|}}(\mathbb{R}^n)} \prod_{j=k+1}^{l} \left\|\partial^{\alpha_j} u\right\|_{L_{\frac{ps}{|\alpha_j|}}(\mathbb{R}^n)} \\ &+ \sum_{r=k+1}^{l} \prod_{i=1}^{k-1} \left\|\partial^{\alpha_i} u_\nu\right\|_{L_{\frac{ps}{|\alpha_i|}}(\mathbb{R}^n)} \\ &\times \prod_{\substack{j=k+1 \\ j \ne r}}^{l} \left\|\partial^{\alpha_j} u\right\|_{L_{\frac{ps}{|\alpha_j|}}(\mathbb{R}^n)} \left\|\nabla \partial^{\alpha_r} u\right\|_{L_{\frac{ps}{1+|\alpha_r|}}(\mathbb{R}^n)} \Bigg). \end{aligned}$$

By (12.4.4) every norm on the right is bounded. The proof is complete.

12.5 Comments to Chap. 12

First pointwise multiplicative inequalities for Riesz potentials are due to Hedberg: Lemma 11.2.1/1 was proved in [365] (see also [15], Sect. 3.1). A version of inequalities (11.2.4) and (11.2.5) appeared already in [364], Lemma 2.

Bojarski and Hajłasz noticed in [123], 1993, that Hedberg's inequality (11.2.4) combined with the classical Sobolev representation theorem gives the pointwise estimate

$$\big|f(x) - f(y)\big| \le \big(\big(\mathcal{M}|\nabla f|\big)(x) + \big(\mathcal{M}|\nabla f|\big)(y)\big)|x-y| \tag{12.5.1}$$

as well as its generalizations to gradients of higher order. Inequality (12.5.1) influenced the development of analysis on arbitrary metric measure spaces (see, for example, Hajłasz [337, 338, 342], Hajłasz and Koskela [344]).

Hedberg's Lemma 11.2.1/1 was generalized in Mircea and Szeptycki [609] to potentials with kernels k satisfying the homogeneity condition $k(tx) = t^{-\alpha n}k(x)$ for all $t > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. The result is the following. Let $Iu(x) = k * u(x)$ and let M be the maximal operator defined by

$$Mu(x) = \sup_{t>0} \frac{1}{\operatorname{vol}(tX)} \int_{tX} \big|u(x-y)\big| \, \mathrm{d}y,$$

where $X = \{x \in \mathbb{R}^n \setminus \{0\} : |K(x)| \ge 1\} \cup \{0\}$. Then

$$\big|Iu(x)\big| \le A\big(Mu(x)\big)^{1-(1-\alpha)p} \|u\|_{L_p}^{(1-\alpha)p}$$

for any $u \in L_p$ and almost all $x \in \mathbb{R}^n$ with the best constant

$$A = \frac{1}{(1-\alpha)p}\bigg(\frac{\alpha p}{1-(1-\alpha)p}\operatorname{vol}(X)\bigg)^{\alpha}.$$

Section 12.1. Lemma in Sect. 12.1.1 is a refinement of Stein's estimate

$$k \star u \le c\|k\|_{L_1}\mathcal{M}u$$

(see [724] Theorem 2, Chap. 3, Sect. 2.29). It was proved by Maz'ya and Shaposhnikova in [581]. Interpolation inequalities for Riesz and Bessel potentials of complex order (Theorems 12.1.2 and 12.1.5) were obtained in [581].

Section 12.2. Theorem 12.2.1 is proved in the paper by Kufner and Maz'ya [571]. Other results in this section were obtained in Maz'ya and Shaposhnikova [586] where, together with $D_\omega(\mathbf{v}, x)$, the nonlinear operator

$$\mathcal{D}_\omega(\mathbf{v}; x) = \sup_{r>0} \frac{1}{\omega(r)} \bigg|\mathbf{v}(x) - ⨍_{B_r(x)} \mathbf{v}(y)\,\mathrm{d}y\bigg|$$

was considered and a sharp analog of inequality (12.0.2) involving $\mathcal{D}_\omega$ was obtained. In the case $\omega(r) = r^\alpha$, $\alpha > 0$, one has the following result.

Proposition. *Let $u \in C^1(\mathbb{R}^n)$, and let $\alpha > 0$. Then the inequality*

$$\big|\nabla u(x)\big| \le C_2\big(\mathcal{M}^\diamond u(x)\big)^{\frac{\alpha}{\alpha+1}} \left(\sup_{r>0} r^{-\alpha}\left|\nabla u(x) - \fint_{B_r(x)} \nabla u(y)\,\mathrm{d}y\right|\right)^{\frac{1}{\alpha+1}} \tag{12.5.2}$$

holds with the best constant

$$C_2 = (n+1)\frac{\alpha+1}{\alpha}\left(\frac{\alpha}{n+\alpha+1}\right)^{\frac{1}{\alpha+1}}. \tag{12.5.3}$$

Inequality (12.5.2) *with $x = 0$ becomes an equality for the odd function given for $x \ge 0$ by the formula*

$$u(x) = \begin{cases} x_n(1-|x|^\alpha) & \text{for } 0 \le |x| \le \mathcal{R}, \\ \frac{\alpha n^{-\alpha}}{n+\alpha+1}((n+1)\mathcal{R} - |x|)^{\alpha+1}\frac{x_n}{|x|} & \text{for } \mathcal{R} < |x| < (n+1)\mathcal{R}, \\ 0 & \text{for } |x| \ge (n+1)\mathcal{R}, \end{cases}$$

where

$$\mathcal{R} = \left(\frac{n+\alpha+1}{(\alpha+1)(n+1)}\right)^{\frac{1}{\alpha}}.$$

Note that for $n = 1$ estimate (12.5.2) takes the form

$$\big|u'(x)\big| \le C_2\big(\mathcal{M}^\diamond u(x)\big)^{\frac{\alpha}{\alpha+1}} \left(\sup_{r>0} \frac{\|u'(x) - \frac{u(x+r)+u(x-r)}{2r}\|}{r^\alpha}\right)^{\frac{1}{\alpha+1}} \tag{12.5.4}$$

(compare with (12.2.56)) with the best constant

$$C_2 = \frac{2(\alpha+1)}{\alpha}\left(\frac{\alpha}{\alpha+2}\right)^{\frac{1}{\alpha+1}}. \tag{12.5.5}$$

We note that the operator

$$(\mathcal{T}_\alpha u)(x) = \sup_{y\in\mathbb{R}^n} \frac{|u(y)-u(x)|}{|y-x|^\alpha}$$

has been extensively studied (see, e.g., Carton-Lebrun and Heinig [173]).

Section 12.3. The paper by Kałamajska [411] is dedicated to some integral representation formulas for differentiable functions and pointwise interpolation inequalities on bounded domains with the operator $\mathcal{M}$ both on the right- and left-hand sides. Kałamajska proved, in particular, that if

$$\lim_{R\to\infty} R^{-k}\int_{B_{rR}(aR)} \big|u(x)\big|\,\mathrm{d}x = 0, \quad \text{where } a \in \mathbb{R}^n \text{ and } r > 0,$$

then, for any polynomial P of degree less than j,

$$\mathcal{M}\nabla_k u(x) \le \big(\mathcal{M}\big(u(x) - P(x)\big)\big)^{\frac{j-k}{j}} \big(\mathcal{M}\nabla_j u(x)\big)^{\frac{k}{j}}.$$

Interpolation inequalities for $\nabla_k u$ and $D_{p,m}u$ collected in Theorem 12.3.1 are borrowed from Maz'ya and Shaposhnikova [580]. A slightly weaker version of Theorem 12.3.2 can be found in [583].

In [582] the interpolation inequality

$$\big|\nabla_k u(x)\big| \le c\big((\mathcal{M}u)(x)\big)^{1-\frac{k}{r}} \big((S_r u)(x)\big)^{\frac{k}{r}}, \quad 0 < k < r, \{r\} > 0,$$

involving the Strichartz function

$$(S_r u)(x) = \left(\int_0^\infty \left(\int_{B_y} \big|\nabla_{[r]}u(x+h) - \nabla_{[r]}u(x)\big| \,\mathrm{d}h\right)^2 \frac{\mathrm{d}y}{y^{1+2\{r\}+2n}}\right)^{\frac{1}{2}},$$

was proved and used to obtain a description of the maximal algebra embedded in the space of multipliers between Bessel potential spaces.

Section 12.4. Inequality (12.4.1), with Bessel potential space $H^s_p(\mathbb{R}^n)$, $p \in (1,\infty)$, and $s > 0$, instead of $\mathcal{W}^s_p(\mathbb{R}^n)$, was obtained by D.R. Adams and Frazier [13].

For $p > 1$, a theorem on the boundedness and continuity of the operator (12.4.2) was proved by Brezis and Mironescu [144]. As they say, their proof is quite involved being based on a "microscopic" improvement of the Gagliardo–Nirenberg inequality, in the Triebel–Lizorkin scale, namely $\mathcal{W}^s_p \cap L_\infty \subset F^\sigma_{q,\nu}$ for every ν and on an important estimate on products of functions in the Triebel–Lizorkin spaces, due to Runst and Sickel [685]. To this Brezis and Mironescu added: "It would be interesting to find a more elementary argument which avoids this excursion into $F^s_{p,q}$ scale."

In the present section, which follows the paper by Maz'ya and Shaposhnikova [583], such an elementary argument, which includes $p = 1$, is given. The problem of the description of functions of one real variable acting by composition on spaces on the Sobolev type, dealt with here, was the subject of numerous studies: Dahlberg [218]; Marcus and Mizel [515]; Bourdaud [129–132]; Bourdaud and Meyer [135]; D.R. Adams and Frazier [13]; Bourdaud and M. Kateb [134]; Oswald [651]; Bourdaud and D. Kateb [133]; Runst and Sickel [685]; Labutin [472]; Bourdaud, Moussai, and Sickel [136]; et al.

13

A Variant of Capacity

In Sect. 10.4.1 we introduced the capacity $\text{cap}(e, S_p^l)$ of a compactum $e \subset \mathbb{R}^n$ for any one of the spaces $S_p^l = H_p^l$, h_p^l, B_p^l, and so on. In other words, we considered the set function

$$\text{cap}\big(e, S_p^l\big) = \inf\big\{\|u\|_{S_p^l}^p : u \in \mathfrak{N}(e)\big\},$$

where $\mathfrak{N}(e) = \{u \in \mathscr{D} : u \geq 1 \text{ on } e\}$.

Replacing $\mathfrak{N}(e)$ by

$$\mathfrak{M}(e, \Omega) = \big\{u \in C_0^\infty(\Omega) : u = 1 \text{ in a neighborhood of } e\big\},$$

we obtain another set function that appears to be useful in various applications. This new set function will also be called the capacity. It will be denoted by $\text{Cap}(e, S_p^l(\Omega))$.

Having in mind further applications we restrict ourselves to the capacities Cap generated by the spaces $\mathring{L}_p^l(\Omega)$ and W_p^l, $l = 1, 2, \ldots$. Various estimates for these capacities are collected in Sect. 13.1. Section 13.2 deals briefly with the so-called (p, l)-polar sets in $\mathbb{R}^n$ and in Sect. 13.3 we show the equivalence of $\text{Cap}(e, \mathring{L}_p^l)$ and $\text{cap}(e, \mathring{L}_p^l)$ for $p > 1$.

Finally, in Sect. 13.4 an example of application of Cap to the problem of removable singularities of polyharmonic functions is given.

13.1 Capacity Cap

13.1.1 Simple Properties of $\text{Cap}(e, \mathring{L}_p^l(\Omega))$

The inner and outer capacities of an arbitrary subset of the set Ω are defined by

$$\underline{\text{Cap}}\big(E, \mathring{L}_p^l(\Omega)\big) = \sup_{e \subset E} \text{Cap}\big(e, \mathring{L}_p^l(\Omega)\big),$$

V. Maz'ya, *Sobolev Spaces*,
Grundlehren der mathematischen Wissenschaften 342,
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$$\overline{\mathrm{Cap}}\big(E, \mathring{L}^l_p(\Omega)\big) = \inf_{G\supset E} \underline{\mathrm{Cap}}\big(G, \mathring{L}^l_p(\Omega)\big),$$

respectively. Here e is an arbitrary compact subset of E and G is an arbitrary open subset of Ω containing E. If the inner and outer capacities coincide, their common value is called the capacity of the set E and is denoted by $\mathrm{Cap}(E, \mathring{L}^l_p(\Omega))$. Henceforth when speaking of the (p, l)-capacity we shall mean this set function.

The definition of the capacity $\mathrm{Cap}(e, \mathring{L}^l_p(\Omega))$ immediately implies the following two properties.

Monotonicity. If $e_1 \subset e_2$ and $\Omega_1 \supset \Omega_2$, then

$$\mathrm{Cap}\big(e_1, \mathring{L}^l_p(\Omega_1)\big) \le \mathrm{Cap}\big(e_2, \mathring{L}^l_p(\Omega_2)\big).$$

Right continuity. For each $\varepsilon > 0$ there exists a neighborhood ω of the compactum e with $\bar{\omega} \subset \Omega$ such that for an arbitrary compactum e' with $e \subset e' \subset \omega$ we have

$$\mathrm{Cap}\big(e', \mathring{L}^l_p(\Omega)\big) \le \mathrm{Cap}\big(e, \mathring{L}^l_p(\Omega)\big) + \varepsilon.$$

The following three propositions establish the simplest connections between the capacities $\mathrm{Cap}(\cdot, S_1)$ and $\mathrm{Cap}(\cdot, S_2)$.

Proposition 1. *Let ω and Ω be open sets in $\mathbb{R}^n$ with $D = \operatorname{diam}\Omega < \infty$ and $\bar{\omega} \subset \Omega$. Then for any compactum $e \subset \omega$, located at the distance Δ_e from $\partial\omega$, the inequality*

$$\mathrm{Cap}\big(e, \mathring{L}^l_p(\omega)\big) \le c\Big(\frac{D}{\Delta_e}\Big)^{lp} \mathrm{Cap}\big(e, \mathring{L}^l_p(\Omega)\big)$$

is valid.

Proof. Let $u \in \mathfrak{M}(e, \Omega)$ and let $\alpha \in C_0^\infty(\omega)$, $\alpha = 1$ in a neighborhood of e and $|\nabla_i \alpha(x)| \le c\Delta_e^{-i}$, $i = 1, 2, \ldots$ (cf. Stein [724], §2, Chap. 6). Then

$$\mathrm{Cap}\big(e, \mathring{L}^l_p(\omega)\big) \le \big\|\nabla_l(\alpha u)\big\|^p_{L_p(\Omega)} \le c \sum_{j=0}^{l} \Delta_e^{(j-l)p} \big\|\nabla_j u\big\|^p_{L_p(\Omega)}.$$

It remains to make use of the Friedrichs inequality

$$\|\nabla_j u\|_{L_p(\Omega)} \le c\, D^{l-j} \|\nabla_l u\|_{L_p(\Omega)}.$$

The proof is complete. □

The following proposition can be proved in a similar way.

Proposition 2. *Let $e \subset \Omega$, $D = \operatorname{diam}\Omega$, $\Delta_e = \operatorname{dist}\{e, \partial\Omega\}$. Then*

$$c_1 \min\big\{1, \Delta_e^{lp}\big\} \le \frac{\mathrm{Cap}(e, W^l_p)}{\mathrm{Cap}(e, \mathring{L}^l_p(\Omega))} \le c_2 \max\big\{1, D^{lp}\big\},$$

where $W^l_p = W^l_p(\mathbb{R}^n)$.

An immediate corollary of the embedding $\mathring{L}_p^l(\mathbb{R}^n) \subset \mathring{L}_q^m(\mathbb{R}^n)$ for appropriate values of p, l, q, and m is the following.

Proposition 3. *Let $n > lp$, $p > 1$ or $n \geq l$, $p = 1$. If $e \subset B_r$, then*

$$\operatorname{Cap}\big(e, \mathring{L}_p^l(B_{2r})\big) \leq c \operatorname{Cap}\big(e, \mathring{L}_p^l\big), \tag{13.1.1}$$

where $c = c(n, p, l)$.

Proof. Let $u \in \mathfrak{M}(e)$, $\alpha \in \mathfrak{M}(\bar{B}_r, B_{2r})$. We have

$$\big\|\nabla_l(\alpha u)\big\|_{L_p(B_{2r})} \leq c \sum_{j=0}^{l} \|\nabla_j u\|_{L_p(B_{2r})} \leq c_1 \sum_{j=0}^{l} \|\nabla_j u\|_{L_{q_j}} \leq c_2 \|\nabla_l u\|_{L_p},$$

where $q_j = pn(n - (l - j)p)^{-1}$. □

Next we present two lower bounds for $\operatorname{Cap}(e, \mathring{L}_p^l(\Omega))$.

Proposition 4. *Let $e \subset \Omega \cap \mathbb{R}^s$, $s \leq n$. Then*

$$\operatorname{Cap}\big(e, \mathring{L}_p^l(\Omega)\big) \geq c\,(m_s e)^{p/q},$$

where $q = ps/(n - lp)$ for $p > 1$, $s > n - lp > 0$, or $s \geq n - l \geq 0$ for $p = 1$. The constant c depends only on n, p, l, and q.

For the proof it suffices to apply the inequality

$$\|u\|_{L_q(\mathbb{R}^s)} \leq c\,\|\nabla_l u\|_{L_p(\mathbb{R}^n)}$$

to an arbitrary $u \in \mathfrak{M}(e, \Omega)$. □

Proposition 5. *If Ω is an open set, e is a compactum in Ω and*

$$d_e = \min_{x \in e} \max_{y \in \partial\Omega} |x - y|,$$

then

$$\operatorname{Cap}\big(e, \mathring{L}_p^l(\Omega)\big) \geq c\, d_e^{n-p\,l}, \tag{13.1.2}$$

where $p\,l > n$, $p > 1$ or $l \geq n$, $p = 1$.

Proof. For any $u \in \mathfrak{M}(e, \Omega)$ and a certain $x \in e$ we have

$$1 = \big|u(x)\big|^p \leq c\, d_e^{pl-n} \int_{|y-x|<d_e} |\nabla_l u|^p \,\mathrm{d}y, \quad x \in e.$$

Therefore, $1 \leq c\, d_e^{l-n/p} \|\nabla_l u\|_{L_p(\Omega)}$. The result follows. □

Corollary. *Let $p\,l > n$, $p > 1$ or $l \geq n$, $p = 1$ and let x_0 be a point in B_ϱ. Then*

$$\operatorname{Cap}\big(x_0, \mathring{L}_p^l(B_{2\varrho})\big) \sim \varrho^{n-lp}. \tag{13.1.3}$$

Proof. The lower estimate for the capacity follows from Proposition 5 and the upper estimate results from substituting the function $u(x) = \eta((x - x_0)\varrho^{-1})$, where $\eta \in C_0^\infty(B_1)$, into the norm $\|\nabla_l u\|_{L_p(B_{2\varrho})}$. □

13.1.2 Capacity of a Continuum

Proposition 1. *Let $n > lp > n-1$, $p \geq 1$ and let e be a continuum with diameter d. Then*

$$\mathrm{Cap}\big(e, \mathring{L}_p^l\big) \sim d^{n-lp}. \tag{13.1.4}$$

Proof. We include e in the ball $\bar{B}_d$ with radius d and we denote the concentric ball with radius $2d$ by B_{2d}. Using the monotonicity of the capacity, we obtain

$$\mathrm{Cap}\big(e, \mathring{L}_p^l\big) \leq \mathrm{Cap}\big(\bar{B}_d, \mathring{L}_p^l\big) = c\, d^{n-lp}.$$

Let O and P be points in e with $|O-P| = d$. Let the axis Ox_n be directed from O to P. We introduce the notation

$$x = (x', x_n), \qquad x' = (x_1, \ldots, x_{n-1}), \qquad e(t) = e \cap \{x : x_n = t\},$$

$$B_{2d}^{(n-1)}(t) = B_{2d} \cap \{x : x_n = t\}, \qquad \nabla_l' = \big\{\partial^l / \partial x_1^{\alpha_1}, \ldots, \partial x_{n-1}^{\alpha_{n-1}}\big\},$$

$$\alpha_1 + \cdots + \alpha_{n-1} = l.$$

For any $u \in \mathfrak{M}(e, B_{2d})$ we have

$$\begin{aligned}\int_{B_{2d}} |\nabla_l u|^p \,\mathrm{d}x &\geq \int_0^d \mathrm{d}t \int_{B_{2d}^{(n-1)}(t)} |\nabla_l' u|^p \,\mathrm{d}x' \\ &\geq \int_0^d \mathrm{Cap}\big[e(t), \mathring{L}_p^l\big(B_{2d}^{(n-1)}(t)\big)\big]\,\mathrm{d}t.\end{aligned}$$

Since $e(t) \neq \varnothing$, $e(t) \subset \bar{B}_d$, and $p\,l > n-1$, it follows that

$$\mathrm{Cap}\big(e(t), \mathring{L}_p^l\big(B_{2d}^{(n-1)}(t)\big)\big) \geq c\, d^{n-1-lp}.$$

Minimizing $\|\nabla_l u\|_{L_p(B_{2d})}^p$ over the set $\mathfrak{M}(e, B_{2d})$, we obtain

$$\mathrm{Cap}\big(e, \mathring{L}_p^l(B_{2d})\big) \geq c\, d^{n-lp}.$$

To complete the proof it remains to use estimate (11.1.1). □

Proposition 2. *If $n = lp$, $p > 1$, then for any continuum e with diameter d, $2d < D$, the equivalence*

$$\mathrm{Cap}\big(e, \mathring{L}_p^l(B_D)\big) \sim \left(\log \frac{D}{d}\right)^{1-p} \tag{13.1.5}$$

holds. Here B_D is the open ball with radius D and with center $O \in e$.

Proof. First we derive the upper bound for the capacity. Let the function v be defined on $B_D \backslash B_d$ as follows:

$$v(x) = \left[\log \frac{D}{d}\right]^{-1} \log \frac{D}{|x|}.$$

Let α denote a function in $C^\infty[0,1]$ equal to zero near $t = 0$, to unity near $t = 1$ and such that $0 \le \alpha(t) \le 1$. Further let $u(x) = \alpha[v(x)]$ for $x \in B_D \backslash B_d$, $u(x) = 1$ in B_d and $u(x) = 0$ outside B_D. It is clear that $u \in \mathfrak{M}(B_d, B_D)$. Also, we can easily see that

$$\left|\nabla_l u(x)\right| \le c\left[\log \frac{D}{d}\right]^{-1} |x|^{-l}$$

on $B_D \backslash B_d$. This implies

$$\begin{aligned} \operatorname{Cap}\big(\bar{B}_d, \mathring{L}_p^l(B_D)\big) &\le \int_{B_D} \left|\nabla_l u(x)\right|^p \mathrm{d}x \\ &\le c\left[\log \frac{D}{d}\right]^{-p} \int_{B_D \backslash B_d} |x|^{-lp}\, \mathrm{d}x = c\left[\log \frac{D}{d}\right]^{1-p}. \end{aligned}$$

We proceed to the lower bound for the capacity. Let P and Q be points in e with $|P - Q| = d$. By (r, ω) we denote the spherical coordinates of a point in the coordinate system with origin Q, $r > 0$, $\omega \in \partial B_1(Q)$. Let u be a function in $\mathfrak{M}(e, B_{2D}(Q))$ such that

$$\int_{B_{2D}(Q)} |\nabla_l u|^p\, \mathrm{d}x \le \gamma - \varepsilon,$$

where

$$\gamma = \operatorname{Cap}\big(e, \mathring{L}_p^l\big(B_{2D}(Q)\big)\big) \le \operatorname{Cap}\big(e, \mathring{L}_p^l(B_D)\big)$$

and ε is a small positive number. We introduce the function

$$U(r) = \big\|u(r, \cdot)\big\|_{L_p(\partial B_1(Q))}.$$

Since $u = 1$ at least one point of the sphere $\{x : |x - Q| = r\}$, where $r < d$ and $p\,l > n - 1$, we have

$$\big|1 - U(r)\big| \le c\big\|u(r, \cdot) - U(r)\big\|_{W_p^l(\partial B_1(Q))}.$$

Hence

$$\left|1 - 2d^{-1} \int_{d/2}^{d} U(r)\, \mathrm{d}r\right| \le c \sum_{j=1}^{l} d^{j-1} \|\nabla_j u\|_{L_p(B_d(Q) \backslash B_{d/2}(Q))}. \tag{13.1.6}$$

Using $(l-1)p < n$, we obtain

$$\int_{B_{2D}(Q)} r^{(j-1)p} |\nabla_j u|^p\, \mathrm{d}x \le c \int_{B_{2D}(Q)} |\nabla_l u|^p\, \mathrm{d}x, \quad 1 \le j < l. \tag{13.1.7}$$

Therefore, the right-hand side in (13.1.6) does not exceed

$$c\,\|\nabla_l u\|_{L_p(B_{2D}(Q))} \le c_0(\gamma - \varepsilon).$$

If $\gamma \ge (2c_0)^{-1}$, then the required estimate follows. Let $\gamma < (2\,c_0)^{-1}$. Hence

$$\int_{d/2}^{d} U(r)\,\mathrm{d}r > d/4,$$

and $U(r_0) > \frac{1}{2}$ for some $r_0 \in (d/2, d)$. Using (13.1.7) once more, we conclude that

$$\gamma - \varepsilon \ge c \int_{\partial B_1(Q)} \mathrm{d}\omega \int_d^{2D} |r\nabla u|^p \frac{\mathrm{d}r}{r} - \varepsilon.$$

By the Hölder inequality we have

$$\begin{aligned}\gamma - \varepsilon &\ge c\int_{r_0}^{2D} \big|rU'(r)\big|^p \frac{\mathrm{d}r}{r} - \varepsilon \ge c\Big(\log\frac{2D}{d}\Big)^{1-p}\bigg|\int_{r_0}^{2D} U'(r)\,\mathrm{d}r\bigg|^p\\ &= c\Big(\log\frac{2D}{d}\Big)^{1-p} U(r_0)^p \ge 2^{-p}c\Big(\log\frac{D}{d}\Big)^{1-p}.\end{aligned}$$

The result follows. □

13.1.3 Capacity of a Bounded Cylinder

Proposition 1. *Let $C_{\delta,d}$ be the cylinder*

$$\big\{x : (x', x_n) : |x'| \le \delta,\ |x_n| \le d/2\big\},$$

where $0 < 2\delta < d$ and $Q_{2d} = \{x : |x_i| < d\}$. Then

$$\mathrm{Cap}\big(C_{\delta,d}, \mathring{L}_p^l(Q_{2d})\big) \sim \begin{cases} d\delta^{n-p\,l-1} & \text{for } n-1 > pl,\\ d(\log\frac{d}{\delta})^{1-p} & \text{for } n-1 = pl.\end{cases}$$

Proof. Let $u \in \mathfrak{M}(C_{\delta,d}, Q_{2d})$. Obviously,

$$\int_{Q_{2d}} |\nabla_l u|^p\,\mathrm{d}x \ge \int_{-d/2}^{d/2} \mathrm{d}x_n \int_{Q_{2d}^{(n-1)}} |\nabla_l' u|^p\,\mathrm{d}x', \tag{13.1.8}$$

where $\nabla_l' = \{\partial^l/\partial x_1^{\alpha_1}, \dots, \partial x_{n-1}^{\alpha_{n-1}}\}$, $\alpha_1 + \dots + \alpha_{n-1} = l$, and $Q_{2d}^{(n-1)} = \{x' : |x_i| < d, i = 1, \dots, n-1\}$. The inner integral on the right in (13.1.8) exceeds

$$\mathrm{Cap}\big(B_\delta^{(n-1)}, \mathring{L}_p^l\big(B_\varrho^{(n-1)}\big)\big),$$

where $B_\varrho^{(n-1)}$ is the $(n-1)$-dimensional ball $\{x : |x'| < \varrho\}$ and $\varrho = 2(n-1)^{1/2}d$. Hence from Propositions 13.1.2/1 and 13.1.2/2 it follows that the integral under consideration majorizes $c\,\delta^{n-p\,l-1}$ for $n-1 > p\,l$ and $c\,(\log d/\delta)^{1-p}$ for

$n-1=pl$. Minimizing the left-hand side of (13.1.8) over the set $\mathfrak{M}(C_{\delta,d},Q_{2d})$, we obtain the required lower estimate for the capacity.

We proceed to the upper bound. Let $x' \to u(x')$ be a smooth function with support in the ball $B_d^{(n-1)}$ that is equal to unity in a neighborhood of the ball $B_\delta^{(n-1)}$. Further, we introduce the function $\eta_d(x)=\eta(x/d)$, where $\eta \in \mathfrak{M}(\bar{Q}_1,Q_2)$. Since the function $x \to \eta_d(x)u(x')$ is in the class $\mathfrak{M}(C_{\delta,d},Q_{2d})$, we have

$$\begin{aligned}\operatorname{Cap}\bigl(C_{\delta,d},\mathring{L}^l_p(Q_{2d})\bigr) &\le \int_{Q_{2d}} \bigl|\nabla_l(u\eta_d)\bigr|^p\,\mathrm{d}x\\ &\le c\,d\sum_{k=0}^{l} d^{-pk}\int_{B_d^{(n-1)}} |\nabla_{l-k}u|^p\,\mathrm{d}x'\\ &\le c_1 d\int_{B_d^{(n-1)}} |\nabla_l u|^p\,\mathrm{d}x'.\end{aligned}$$

Minimizing the latter integral and using Propositions 13.1.2/1 and 13.1.2/2, we arrive at the required estimate. □

13.1.4 Sets of Zero Capacity $\operatorname{Cap}(\cdot,W^l_p)$

The definition of the capacity $\operatorname{Cap}(\cdot,\mathring{L}^l_p(\Omega))$ and Proposition 13.1.1/2 imply that $\operatorname{Cap}(e,W^l_p)=0$ if and only if there exists a bounded open set Ω containing e such that $\operatorname{Cap}(e,\mathring{L}^l_p(\Omega))=0$. The choice of Ω is not essential by Proposition 13.1.1/2.

From Corollary 13.1.1, for $lp>n$, $p>1$ and for $l\ge n$, $p=1$, we obtain that the equality $\operatorname{Cap}(e,W^l_p)=0$ is valid only if $e=\varnothing$.

Proposition 13.1.1/3 shows that in any one of the cases $n>lp$, $p>1$ or $n\ge l$, $p=1$ the equalities $\operatorname{Cap}(e,W^l_p)=0$ and $\operatorname{Cap}(e,\mathring{L}^l_p)=0$ are equivalent. Corollary 13.1.1 and Propositions 13.1.2/1 and 13.1.2/2 imply that no similar property is true for $n\le lp$, $p>1$. To be precise, $\operatorname{Cap}(e,\mathring{L}^l_p)=0$ for any compactum e provided $n\le lp$, $p>1$.

13.2 On (p,l)-Polar Sets

Let $W^{-l}_{p'}$ denote the space of linear continuous functionals $T: u\to(u,T)$ on W^l_p.

The set $E\subset\mathbb{R}^n$ is called a (p,l)-polar set if zero is the only element in $W^{-l}_{p'}$ with support in E.

Theorem 1. *The space $\mathscr{D}(\Omega)$ is dense in W^l_p if and only if $C\Omega$ is a (p,l)-polar set.*

Proof. 1. Suppose $\mathscr{D}(\Omega)$ is not dense in W_p^l. Then there exists a nonzero functional $T \in W_{p'}^{-l}$, equal to zero on $\mathscr{D}(\Omega)$, i.e., with support in $C\Omega$. (Here we make use of the following corollary of the Hahn–Banach theorem. Let M be a linear set in the Banach space B and let x_0 be an element of B situated at a positive distance from M. Then there exists a nonzero functional $T \in B^*$ such that $(x, T) = 0$ for all $x \in M$.) Consequently, $C\Omega$ is not a (p, l)-polar set.

2. Suppose $\mathscr{D}(\Omega)$ is dense in W_p^l. For any functional $T \in W_{p'}^{-l}$ with support in $C\Omega$ we have

$$(u, T) = 0$$

for all $u \in \mathscr{D}(\Omega)$. Therefore, the last equality is valid for all $u \in W_p^l$ and so $T = 0$. Thus, $C\Omega$ is a (p, l)-polar set. $\square$

Theorem 2. *The set E is (p, l)-polar if and only if $\underline{\mathrm{Cap}}(E, W_p^l) = 0$.*

Proof. 1. Let $\underline{\mathrm{Cap}}(E, W_p^l) = 0$ and $T \in W_{p'}^{-l}$, $\operatorname{supp} T \subset E$. Without loss of generality we may assume that supp T is a compactum (otherwise we could take αT with $\alpha \in \mathscr{D}$ instead of T). We take an arbitrary $\varphi \in \mathscr{D}$ and a sequence $\{u_m\}_{m \geq 1}$ of functions in $\mathscr{D}$ which equal unity in a neighborhood of supp T and tend to zero in W_p^l. Since $\varphi(1 - u_m) = 0$ in a neighborhood of T, we have $(\varphi, T) = (\varphi u_m, T)$. The right-hand side converges to zero as $m \to \infty$; hence $(\varphi, T) = 0$ for all $\varphi \in \mathscr{D}$. Since $\mathscr{D}$ is dense in W_p^l, we conclude that $T = 0$.

2. Let E be a (p, l)-polar set. Then any compactum K in E is also a (p, l)-polar set, and $\mathscr{D}(\mathbb{R}^n \backslash K)$ is dense in W_p^l. Let $v \in \mathfrak{M}(K)$. By the density of $\mathscr{D}(\mathbb{R}^n \backslash K)$ in W_p^l, there exists a sequence $v_m \in \mathscr{D}(\mathbb{R}^n \backslash K)$ that converges to v in W_p^l. Every function $v_m - v$ equals unity near K, has compact support, and $\|v_m - v\|_{W_p^l} \to 0$ as $m \to \infty$. Therefore, $\mathrm{Cap}(K, W_p^l) = 0$. The proof is complete. $\square$

Taking into account the just proved assertion we can give an equivalent formulation of Theorem 1.

Theorem 3. *The space $\mathscr{D}(\Omega)$ is dense in W_p^l if and only if*

$$\underline{\mathrm{Cap}}\big(C\Omega, W_p^l\big) = 0.$$

13.3 Equivalence of Two Capacities

We compare the capacities $\mathrm{Cap}(e, \mathring{L}_p^1)$ and $\mathrm{cap}(e, \mathring{L}_p^1)$, $p \geq 1$. Obviously, the first capacity majorizes the second. One can easily check that the converse inequality also holds. In fact, for the function $v_\varepsilon = \min\{(1 - \varepsilon)^{-1} u, 1\}$ there exists a sequence in $\mathfrak{M}(e)$ that converges to v_ε in $\mathring{L}_p^l$ for an arbitrary number $\varepsilon \in (0, 1)$ and a function $u \in \mathfrak{N}(e)$. Therefore,

$$\mathrm{Cap}\big(e, \mathring{L}_p^1\big) \le \int |\nabla v_\varepsilon|^p \,\mathrm{d}x \le (1-\varepsilon)^{-p} \int |\nabla u|^p \,\mathrm{d}x$$

and so $\mathrm{Cap}(e, \mathring{L}_p^1) \le \mathrm{cap}(e, \mathring{L}_p^1)$. Thus the capacities $\mathrm{Cap}(e, \mathring{L}_p^1)$ and $\mathrm{cap}(e, \mathring{L}_p^1)$ coincide.

Since the truncation along the level surfaces does not keep functions in the spaces $\mathring{L}_p^l$ and W_p^l for $l > 1$, the previous argument is not applicable to the proof of equivalence of the capacities Cap and cap, generated by these spaces. Nevertheless, in the present section we show that the equivalence occurs for $p > 1$.

Theorem 1. *Let $p > 1$, $n > p\,l$, $l = 1, 2, \ldots$. Then*

$$\mathrm{cap}\big(e, \mathring{L}_p^l\big) \le \mathrm{Cap}\big(e, \mathring{L}_p^l\big) \le c\,\mathrm{cap}\big(e, \mathring{L}_p^l\big) \tag{13.3.1}$$

for any compactum e.

Proof. The left inequality results from the inclusion $\mathfrak{M}(e) \subset \mathfrak{N}(e)$. We proceed to the upper bound for Cap. Let G denote a bounded open set such that $G \supset e$ and

$$\mathrm{cap}\big(\bar{G}, L_p^l\big) \le \mathrm{cap}\big(e, \mathring{L}_p^l\big) + \varepsilon.$$

Let U be the (p,l)-capacitary potential of the set $\bar{G}$ (cf. Sect. 10.4.2). By Proposition 10.4.2/2, the inequality $U \ge 1$ is valid (p,l)-quasi-everywhere in $\bar{G}$, and therefore, quasi-everywhere in some neighborhood of the compactum e. We apply the mollification with radius m^{-1}, $m = 1, 2, \ldots$, to U and multiply the result by the truncating function η_m, $\eta_m(x) = \eta(x/m)$ where $\eta \in C_0^\infty$, $\eta(0) = 1$. Thus we obtain a sequence of functions $\{U_m\}_{m \ge 1}$ in C_0^∞ such that $0 \le U_m \le c$ in $\mathbb{R}^n$, $U_m \ge 1$ in a neighborhood of e and

$$\lim_{m\to\infty} \|\nabla_l U_m\|_{L_p}^p \le \mathrm{cap}\big(e, \mathring{L}_p^l\big) + \varepsilon. \tag{13.3.2}$$

(The inequality $U_m \le c$ follows from Proposition 10.4.2/1.)

We introduce the function

$$w = 1 - \big[(1 - U_m)_+\big]^{l+1},$$

which obviously has compact support and is in C^l. Also,

$$\|\nabla_l w\|_{L_p} \le \big\|\nabla_l\big[(1 - U_m)^{l+1}\big]\big\|_{L_p}.$$

Applying (1.8.7), we obtain

$$\|\nabla_l w\|_{L_p} \le c\,\|\nabla_l U_m\|_{L_p}.$$

It remains to make use of inequality (13.3.2). □

The following theorem has a similar proof.

Theorem 2. *Let $p > 1$, $l = 1, 2, \dots$. Then*

$$\operatorname{cap}\big(e, W_p^l\big) \le \operatorname{Cap}\big(e, W_p^l\big) \le c \operatorname{cap}\big(e, W_p^l\big) \tag{13.3.3}$$

for any compactum e.

Corollary 1. *Let $p > 1$, $l = 1, 2, \dots$, and let e be a closed subset of the cube $\tilde{Q}_d = \{x : 2|x_i| \le d\}$. Then*

$$\operatorname{cap}\big(e, \mathring{L}_p^l(Q_{2d})\big) \le \operatorname{Cap}\big(e, \mathring{L}_p^l(Q_{2d})\big) \le c \operatorname{cap}\big(e, \mathring{L}_p^l(Q_{2d})\big). \tag{13.3.4}$$

Proof. It suffices to derive (13.3.4) for $d = 1$. The left inequality is trivial; the right one follows from Proposition 13.1.1/2 and Theorem 1. □

Remark. The proofs of Theorems 1 and 2 and of Corollary 1 do not change provided we replace the class $\mathfrak{M}(e, \Omega)$ in the definition of the capacity Cap by the smaller one

$$\mathfrak{P}(e, \Omega) = \big\{u \in C_0^\infty(\Omega) : u = 1 \text{ in a neighborhood of } e, 0 \le u \le 1\big\}.$$

Corollary 2. *Let $p > 1$, $l = 1, 2, \dots$, and let e_1, e_2 be compacta in $\tilde{Q}_d$. Then*

$$\operatorname{Cap}\big(e_1 \cup e_2, \mathring{L}_p^l(Q_{2d})\big) \le c_* \sum_{i=1}^{2} \operatorname{Cap}\big(e_i, \mathring{L}_p^l(Q_{2d})\big), \tag{13.3.5}$$

where c_* is a constant that depends only on n, p, and l.

Proof. Let $u_i \in \mathfrak{P}(e_i, Q_{2d})$, $i = 1, 2$, and let

$$\|\nabla_l u_i\|_{L_p}^p \le c \operatorname{cap}\big(e_i, \mathring{L}_p^l(Q_{2d})\big) + \varepsilon \tag{13.3.6}$$

(cf. preceding Remark). The function $u = u_1 + u_2$ is contained in $C_0^\infty(Q_{2d})$ and satisfies the inequality $u \ge 1$ on $e_1 \cup e_2$. Hence from Corollary 1 we obtain

$$\begin{aligned}\operatorname{Cap}\big(e_1 \cup e_2, \mathring{L}_p^l(Q_{2d})\big) &\le c \operatorname{cap}\big(e_1 \cup e_2, \mathring{L}_p^l(Q_{2d})\big)\\ &\le c\, \|\nabla_l u\|_{L_p}^p \le 2^{p-1} c \sum_{i=1}^{2} \|\nabla_l u_i\|_{L_p}^p,\end{aligned}$$

which together with (13.3.6) leads to (13.3.5). □

13.4 Removable Singularities of l-Harmonic Functions in L_2^m

Let F be a compactum in the open unit ball B. Here we show that the capacity Cap enables one to characterize removable singularity sets for l-harmonic functions in $L_2^m(B)$. For other results of such a kind see Maz'ya and Havin [569].

Proposition. *Let l and m be integers, $l \geq m \geq 0$. Every function $u \in L_2^m(B)$ satisfying the equation*

$$\Delta^l u = 0 \quad \text{on } B\backslash F$$

is a solution of the same equation on B if and only if

$$\operatorname{Cap}\big(F, \mathring{L}_2^{2l-m}(B)\big) = 0. \tag{13.4.1}$$

Proof. Sufficiency. Let (13.4.1) hold. For every $v \in \mathscr{D}(B\backslash F)$

$$\int_B \Delta^l u \cdot v \, \mathrm{d}x = 0.$$

Hence

$$\sum_{\{\alpha:|\alpha|=m\}} \frac{m!}{\alpha!} \int_B D^\alpha u \cdot D^\alpha \Delta^{l-m} v \, \mathrm{d}x = 0. \tag{13.4.2}$$

Since the norm

$$\big\|\Delta^{l-m} v\big\|_{\mathring{L}_2^m(B)} \tag{13.4.3}$$

is equivalent to the norm in $\mathring{L}_2^{2l-m}(B)$, it follows from Theorems 13.2/1 and 13.2/2 that $\mathscr{D}(B\backslash F)$ is dense in $\mathscr{D}(B)$ in the sense of the norm (13.4.3). Therefore, using the inclusion $u \in L_2^m(B)$, we obtain (13.4.2) for all $v \in \mathscr{D}(B)$, which is equivalent to $\Delta^l u = 0$ on B.

Necessity. Let u denote the function in $\mathring{L}_2^{2l-m}(B)$ providing the infimum of the functional

$$\sum_{\{\alpha:|\alpha|=2l-m\}} \frac{(2l-m)!}{\alpha!} \int_B \big(D^\alpha \varphi\big)^2 \, \mathrm{d}x$$

defined on the set of φ in $\mathfrak{M}(F, B)$. For every $v \in \mathscr{D}(B\backslash F)$

$$\sum_{\{\alpha:|\alpha|=2l-m\}} \frac{(2l-m)!}{\alpha!} \int_B D^\alpha u \cdot D^\alpha v \, \mathrm{d}x = 0. \tag{13.4.4}$$

Hence

$$\Delta^{2l-m} u = 0 \quad \text{on } B\backslash F,$$

which can be written as

$$\Delta^l \big(\Delta^{l-m} u\big) = 0 \quad \text{on } B\backslash F. \tag{13.4.5}$$

The function $\Delta^{l-m} u$ belongs to $L_2^m(B\backslash F)$. Furthermore, by (13.4.5) it is l-harmonic on $B\backslash F$ and we deduce from the necessity assumption that

$$\Delta^l \big(\Delta^{l-m} u\big) = 0 \quad \text{on } B,$$

i.e., u is $(2l-m)$-harmonic on B. Since u was chosen as a function in $\mathring{L}_2^{2l-m}(B)$, we see that $u = 0$ on B. Hence (13.4.1) holds. □

13.5 Comments to Chap. 13

Section 13.1. The polyharmonic capacity $\mathrm{Cap}(e, \mathring{L}_2^l(\Omega))$ was introduced by the author [533]. The content of this section follows, to a large extent, the author's paper [544].

Section 13.2. The definition of a $(2, l)$-polar set is borrowed from the paper by Hörmander and Lions [385]. The results of this section are due to Littman [503]. For Theorem 13.2/2 see the earlier paper by Grushin [329].

Section 13.3. The equivalence of the capacities $\mathrm{Cap}(e, \mathring{L}_p^l)$ and $\mathrm{cap}(e, \mathring{L}_p^l)$ was established by the author [539, 542] for integer l. For fractional l the equivalence of the corresponding capacities is proved by D.R. Adams and Polking [19].

Section 13.4. This application of the polyharmonic capacity to description of removable singularities of polyharmonic functions was not published.

14

Integral Inequality for Functions on a Cube

Let Q_d be an n-dimensional cube with edge length d and with sides parallel to coordinate axes. Let $p \geq 1$ and k, l be integers, $0 \leq k \leq l$. We denote a function in $W_p^l(Q_d)$, $p \geq 1$, by u.

The inequality

$$\|u\|_{L_q(Q_d)} \leq C \sum_{j=k+1}^{l} d^{j-1} \|\nabla_j u\|_{L_p(Q_d)} \tag{14.0.1}$$

with q in the same interval as in the Sobolev embedding theorem often turns out to be useful. This inequality occurs repeatedly in the following chapters. Obviously, (14.0.1) is not valid for all $u \in W_p^l(Q_d)$, but it holds provided u is subject to additional requirements.

In the present chapter we establish two-sided estimates for the best constant C in (14.0.1). In Sects. 14.1 and 14.2 we mainly consider the case of u vanishing near a compactum $e \subset \bar{Q}_d$ and $k = 0$. The existence of C is equivalent to the positiveness of the (p, l)-capacity of e. For a (p, l)-negligible set e, upper and lower bounds for C are stated in terms of this capacity. If $q \geq p$ and e is (p, l)-essential, i.e., its capacity is comparable with the capacity of the cube, then C is estimated by the so-called (p, l)-inner diameter.

In Sect. 14.3 the function u is a priori contained in an arbitrary linear subset $\mathfrak{C}$ of the space $W_p^l(Q_d)$. There we present a generalization of the basic theorem in Sect. 14.1 and give applications for particular classes $\mathfrak{C}$. In this connection we have to introduce some functions of the class $\mathfrak{C}$ that play a role similar to that of the (p, l)-capacity.

In conclusion we note that the statements as well as the proofs of results in the present chapter remain valid after replacing the cube Q_d by an arbitrary bounded Lipschitz domain with diameter d.

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14.1 Connection Between the Best Constant and Capacity (Case $k = 1$)

14.1.1 Definition of a (p, l)-Negligible Set

Definition. Let e be a compact subset of the cube $\bar{Q}_d$. In either of the cases $n \geq pl$, $p > 1$ or $n > l$, $p = 1$ we say that e is *a (p, l)-negligible subset of Q_d* if

$$\operatorname{Cap}\big(e, \mathring{L}_p^l(Q_{2d})\big) \leq \gamma d^{n-pl}, \tag{14.1.1}$$

where γ is a sufficiently small constant that depends only on n, p, and l. For the purposes of the present chapter we can take γ to be an arbitrary positive number satisfying the inequality

$$\gamma \leq 4^{-pn}. \tag{14.1.2}$$

If (14.1.1) fails, then, by definition, e is a (p, l)-*essential subset of* $\bar{Q}_d$. For $n < pl$, $p > l$ or for $n \leq l$, $p = 1$ only the empty set is called (p, l)-negligible. The collection of all (p, l)-negligible subsets of the cube $\bar{Q}_d$ will be denoted by $\mathscr{N}(Q_d)$.

14.1.2 Main Theorem

Let $\bar{u}_{Q_d}$ denote the mean value of u on the cube Q_d, i.e.,

$$\bar{u}_{Q_d} = \big[m_n(Q_d)\big]^{-1} \int_{Q_d} u \, \mathrm{d}x.$$

In what follows, by c, c_1, and c_2 we mean positive constants depending only on n, p, l, k, and q.

We introduce the seminorm

$$\boldsymbol{|} u \boldsymbol{|}_{p,l,Q_d} = \sum_{j=1}^{l} d^{j-l} \|\nabla_j u\|_{L_p(Q_d)}.$$

Theorem. *Let e be a closed subset of the cube $\bar{Q}_d$.*

1. *For all $u \in C^\infty(\bar{Q}_d)$ with $\operatorname{dist}(\operatorname{supp} u, e) > 0$ the inequality*

$$\|u\|_{L_q(Q_d)} \leq C \boldsymbol{|} u \boldsymbol{|}_{p,l,Q_d}, \tag{14.1.3}$$

where $q \in [1, pn(n-pl)^{-1}]$ for $n > pl$, $p \geq 1$, and $q \in [1, \infty)$ for $n = pl$, $p > 1$, holds. Moreover, the constant C admits the estimate

$$C^{-p} \geq c_1 d^{-np/q} \operatorname{Cap}\big(e, \mathring{L}_p^l(Q_{2d})\big). \tag{14.1.4}$$

2. *For functions $u \in C^\infty(\bar{Q}_d)$ with $\operatorname{dist}(\operatorname{supp} u, e) > 0$, let*

$$\|u\|_{L_q(Q_{d/2})} \leq C \boldsymbol{|} u \boldsymbol{|}_{p,l,Q_d}, \tag{14.1.5}$$

where $e \in \mathscr{N}(Q_d)$ and q satisfies the same conditions as in item 1. Then

$$C^{-p} \leq c_2 d^{-np/q} \operatorname{Cap}\big(e, \mathring{L}_p^l(Q_{2d})\big). \tag{14.1.6}$$

For the proof of this theorem we need the two-sided estimate for Cap.

Lemma. *Let e be a compactum in $\bar{Q}_1$. There exists a constant c such that*

$$c^{-1}\,\mathrm{Cap}\big(e, \mathring{L}^l_p(Q_2)\big) \le \inf\big\{\|1-u\|^p_{V^l_p(Q_1)} : u \in C^\infty(\bar{Q}_1),\ \mathrm{dist}(\mathrm{supp}\, u, e) > 0\big\}$$
$$\le c\,\mathrm{Cap}\big(e, \mathring{L}^l_p(Q_2)\big). \tag{14.1.7}$$

Proof. To obtain the left estimate we need the following well-known assertion (cf. Theorem 1.1.17).

There exists a linear continuous mapping

$$A : C^{k-1,1}(\bar{Q}_d) \to C^{k-1,1}(\bar{Q}_{2d}), \quad k = 1, 2, \dots,$$

such that (i) $Av = v$ on $\bar{Q}_d$, (ii) if $\mathrm{dist}(\mathrm{supp}\, v, e) > 0$, then $\mathrm{dist}(\mathrm{supp}\, Av, e) > 0$, and (iii)

$$\|\nabla_i Av\|_{L_p(Q_{2d})} \le c\|\nabla_i v\|_{L_p(Q_d)}, \quad i = 0, 1, \dots, l,\ 1 \le p \le \infty. \tag{14.1.8}$$

Let $v = A(1-u)$. Let η denote a function in $\mathscr{D}(Q_2)$ that is equal to unity in a neighborhood of the cube Q_1. Then

$$\mathrm{Cap}(e, Q_2) \le c\int_{Q_2} \big|\nabla_l(\eta v)\big|^p\, dx \le c\|v\|^p_{V^l_p(Q_2)}. \tag{14.1.9}$$

Now the left estimate in (14.1.7) follows from (14.1.8) and (14.1.9).

Next we derive the rightmost estimate in (14.1.7). Let $w \in \mathfrak{M}(e, Q_2)$. Then

$$\|w\|^p_{V^l_p(Q_1)} \le c\sum_{k=0}^{l} \|\nabla_k w\|^p_{L_p(Q_2)} \le c\|\nabla_l w\|^p_{L_p(Q_2)}.$$

Minimizing the last norm over the set $\mathfrak{M}(e, Q_2)$ we obtain

$$\|w\|^p_{V^l_p(Q_1)} \le c\,\mathrm{Cap}\big(e, \mathring{L}^l_p(Q_2)\big).$$

We complete the proof of the Lemma by minimizing the left-hand side. □

Proof of the theorem. It suffices to consider only the case $d = 1$ and then use a similarity transformation.

1. Let $N = \|u\|_{L_p(Q_1)}$. Since $\mathrm{dist}(\mathrm{supp}\, u, e) > 0$, it follows by the Lemma that

$$\mathrm{Cap}\big(e, \mathring{L}^l_p(Q_2)\big) \le c\big\|1 - N^{-1}u\big\|^p_{V^l_p(Q_1)} = cN^{-p}\,|\!|\!| u |\!|\!|^p_{p,l,Q_1} + c\big\|1 - N^{-1}u\big\|^p_{L_p(Q_1)},$$

i.e.,

$$N^p\,\mathrm{Cap}\big(e, \mathring{L}^l_p(Q_2)\big) \le c\,|\!|\!| u |\!|\!|^p_{p,l,Q_1} + c\|N - u\|^p_{L_p(Q_1)}. \tag{14.1.10}$$

Without loss of generality, we may assume that $\bar{u}_{Q_1} \geq 0$. Then

$$|N - \bar{u}_{Q_1}| = \|u\|_{L_p(Q_1)} - \|\bar{u}_{Q_1}\|_{L_p(Q_1)} \leq \|u - \bar{u}_{Q_1}\|_{L_p(Q_1)}.$$

Consequently,

$$\begin{aligned}\|N - u\|_{L_p(Q_1)} &\leq \|N - \bar{u}_{Q_1}\|_{L_p(Q_1)} + \|u - \bar{u}_{Q_1}\|_{L_p(Q_1)} \\ &\leq 2\|u - \bar{u}_{Q_1}\|_{L_p(Q_1)}.\end{aligned} \tag{14.1.11}$$

By (14.1.10), (14.1.11), and the Poincaré inequality

$$\|u - \bar{u}_{Q_1}\|_{L_p(Q_1)} \leq c\|\nabla u\|_{L_p(Q_1)}$$

we obtain

$$\operatorname{Cap}\big(e, \mathring{L}_p^l(Q_2)\big)\|u\|^p_{L_p(Q_1)} \leq c\,|\!|\, u \,|\!|^p_{p,l,Q_1}.$$

From the Sobolev embedding theorem and the preceding inequality we conclude that

$$\|u\|^p_{L_q(Q_1)} \leq c\big(|\!|\, u \,|\!|^p_{p,l,Q_1} + \|u\|^p_{L_p(Q_1)}\big) \leq c\big\{1 + \big[\operatorname{Cap}\big(e, \mathring{L}_p^l(Q_2)\big)\big]^{-1}\big\}|\!|\, u \,|\!|^p_{p,l,Q_1}.$$

Thus the first item of the theorem follows.

2. For $pl > n$, $p > 1$ or for $l \geq n$, $p = 1$ the assertion is trivial. Consider the other values of p and l. Let $\psi \in \mathfrak{M}(e, Q_2)$ be such that

$$\|\nabla_l \psi\|^p_{L_p(Q_2)} \leq \operatorname{Cap}\big(e, \mathring{L}_p^l(Q_2)\big) + \varepsilon, \tag{14.1.12}$$

and let $u = 1 - \psi$. Applying the inequality

$$\|\nabla_j \psi\|_{L_p(Q_2)} \leq c\|\nabla_l \psi\|_{L_p(Q_2)}, \quad j = 1, \dots, l-1,$$

we obtain

$$|\!|\, u \,|\!|_{p,l,Q_1} = |\!|\, \psi \,|\!|_{p,l,Q_1} \leq c\|\nabla_l \psi\|_{L_p(Q_2)}.$$

Hence from (14.1.12) it follows that

$$\|u\|_{L_p(Q_{1/2})} \leq cC\big[\operatorname{Cap}\big(e, \mathring{L}_p^l(Q_2)\big) + \varepsilon\big]^{1/p}.$$

By Hölder's inequality we have

$$1 - \bar{\psi}_{Q_{1/2}} = \bar{u}_{Q_{1/2}} \leq cC\big[\operatorname{Cap}\big(e, \mathring{L}_p^l(Q_2)\big)\big]^{1/p}. \tag{14.1.13}$$

It remains to show that the mean value of ψ on $Q_{1/2}$ is small. Noting that

$$\int_{-1}^{1} |w|\,\mathrm{d}t \leq \int_{-1}^{1} |t||w'|\,\mathrm{d}t \leq \int_{-1}^{1} |w'|\,\mathrm{d}t$$

for any function $w \in C^1[-1, 1]$ satisfying $w(-1) = w(1) = 0$, we obtain

$$\int_{Q_{1/2}} \psi \,\mathrm{d}x \le \int_{Q_2} |\psi| \,\mathrm{d}x \le \int_{Q_2} \left|\frac{\partial \psi}{\partial x_1}\right| \mathrm{d}x \le \int_{Q_2} \left|\frac{\partial^2 \psi}{\partial x_1^2}\right| \mathrm{d}x$$
$$\le \cdots \le \int_{Q_2} \left|\frac{\partial^l \psi}{\partial x_1^l}\right| \mathrm{d}x \le 2^{(p-1)n/p} \|\nabla_l \psi\|_{L_p(Q_2)}. \quad (14.1.14)$$

Therefore,

$$\bar{\psi}_{Q_{1/2}} \le 2^{(2p-1)n/p} \big[\mathrm{Cap}\big(e, \mathring{L}_p^l(Q_2)\big) + \varepsilon\big]^{1/p}.$$

This and (14.1.1) and (14.1.2) imply

$$\bar{\psi}_{Q_{1/2}} \le 2^{-n/p},$$

which together with (14.1.13) completes the proof of the second part of the theorem. □

14.1.3 Variant of Theorem 14.1.2 and Its Corollaries

In the following theorem, which will be used in Chap. 16, we prove an assertion similar to the first part of Theorem 14.1.2 and relating it to a wider class of functions. The parameters p, l, and q are the same as in Theorem 14.1.2.

Theorem. *Let e be a closed subset of $\bar{Q}_d$ and let δ be a number in the interval $(0,1)$. Then for all functions in the set*

$$\big\{u \in C^\infty(\bar{Q}_d) : \bar{u}_{Q_d} \ge 0,\ u(x) \le \delta d^{-n/p} \|u\|_{L_p(Q_d)} \text{ for all } x \in e\big\},$$

inequality (14.1.3) is valid and

$$C^{-p} \ge c(1-\delta)^p d^{-np/q} \,\mathrm{cap}\big(e, \mathring{L}_p^l(Q_{2d})\big).$$

Proof. Duplicating the proof of Lemma 14.1.2, we obtain

$$c^{-1} \,\mathrm{cap}\big(e, \mathring{L}_p^l(Q_2)\big) \le \inf\big\{\|1-u\|^p_{V_p^l(Q_1)} : u \in C^\infty(\bar{Q}_1),\ u \le 0 \text{ on } e\big\}$$
$$\le c \,\mathrm{cap}\big(e, \mathring{L}_p^l(Q_2)\big).$$

Further, we note that the inequality $1 - N^{-1}u \ge 1 - \delta$ on e implies

$$(1-\delta)^p \,\mathrm{cap}\big(e, \mathring{L}_p^l(Q_2)\big) \le c\big\|1 - N^{-1}u\big\|^p_{V_p^l(Q_2)}$$

and we follow the argument of the proof of the first part of Theorem 14.1.2. □

Corollary 1. *Let e be a closed subset of $\bar{Q}_d$. Then the inequality*

$$\|u\|^p_{L_q(Q_d)} \le c\bigg(d^{p-n+np/q} \|\nabla u\|^p_{L_p(Q_d)} + \frac{d^{np/q}}{\mathrm{cap}(e, \mathring{L}_p^l(Q_{2d}))} \|\nabla_l u\|^p_{L_p(Q_d)}\bigg) \quad (14.1.15)$$

is valid for all functions $u \in C^\infty(\bar{Q}_d)$ that vanish on e.

Proof. It suffices to put $d = 1$. Let

$$P(u) = \sum_{0 \le \beta < l} x^\beta \int_{Q_1} \varphi_\beta(y) u(y)\, \mathrm{d}y$$

be the polynomial in the generalized Poincaré inequality for the cube Q_1 (see Lemma 1.1.11). Further let $S(u) = P(u) - \int_{Q_1} \varphi_0(y) u(y)\, \mathrm{d}y$. Since all functions φ_β are orthogonal to unity for $|\beta| > 0$, then

$$\bigl|S(u)\bigr| \le c \|\nabla u\|_{L_p(Q_1)}. \tag{14.1.16}$$

It suffices to obtain (14.1.15) under the assumption

$$\|\nabla u\|_{L_p(Q_1)} \le \delta \|u\|_{L_p(Q_1)},$$

where $\delta = \delta(n, p, l)$ is a small constant. Then the function $v = u - S(u)$ satisfies the inequality

$$\bigl|v(x)\bigr| \le c\delta \|v\|_{L_p(Q_1)}$$

on e. We can assume that $\bar{v}_{Q_1} \ge 0$. Applying the theorem in the present section to the function v, we arrive at

$$\|v\|^p_{L_q(Q_1)} \le \frac{c}{\operatorname{cap}(e, \mathring{L}^l_p(Q_2))} \sum_{j=1}^{l} \|\nabla_j v\|^p_{L_p(Q_1)}.$$

Hence from Lemma 1.1.11 we obtain

$$\begin{aligned}\bigl\|u - S(u)\bigr\|^p_{L_q(Q_1)} &\le \frac{c}{\operatorname{cap}(e, \mathring{L}^l_p(Q_2))} \sum_{j=1}^{l} \bigl\|\nabla_j \bigl(u - P(u)\bigr)\bigr\|^p_{L_p(Q_1)} \\ &\le \frac{c}{\operatorname{cap}(e, \mathring{L}^l_p(Q_2))} \|\nabla_l u\|^p_{L_p(Q_1)}.\end{aligned}$$

Now a reference to (14.1.16) completes the proof. □

Corollary 14.1.3 implies the following assertion.

Corollary 2. *Let e be a closed subset of $\bar{Q}_d$. The inequality*

$$\begin{aligned}\|u\|^p_{L_q(Q_d)} \le c\biggl(& d^{pk-n+np/q} \|\nabla_k u\|^p_{L_p(Q_d)} \\ &+ \frac{d^{np/q}}{\operatorname{cap}(e, \mathring{L}^{l-k+1}_p(Q_{2d}))} \|\nabla_l u\|^p_{L_p(Q_d)}\biggr)\end{aligned} \tag{14.1.17}$$

holds for all functions $u \in C^\infty(\bar{Q}_d)$ vanishing on e along with all derivatives up to order k, $k < l$.

Proof. It suffices to derive (14.1.17) for $q = p$ and $d = 1$. By Corollary 1,

$$\|\nabla_j u\|^p_{L_p(Q_1)} \leq c\Big(\|\nabla_{j+1} u\|^p_{L_p(Q_1)} + \frac{1}{\operatorname{cap}(e, \mathring{L}^{l-j}_p(Q_1))}\|\nabla_l u\|^p_{L_p(Q_1)}\Big)$$

for $j = 0, 1, \dots, k-1$. Therefore

$$\|u\|^p_{L_p(Q_1)} \leq c\Big(\|\nabla_k u\|^p_{L_p(Q_1)} + \sum_{j=0}^{k-1} \frac{1}{\operatorname{cap}(e, L^{l-j}_p(Q_1))}\|\nabla_l u\|^p_{L_p(Q_1)}\Big).$$

Since

$$\operatorname{cap}\big(e, L^{l-j}_p(Q_1)\big) \geq \operatorname{cap}\big(e, L^{l-k+1}_p(Q_1)\big)$$

for $j = 0, \dots, k-1$, the result follows. □

Remark 1. By Corollary 13.3/1, for $p > 1$ we can replace cap by Cap in the statements of Theorem and Corollaries 1 and 2.

Remark 2. From Proposition 13.1.1/3 it follows that we can replace cap(e, $\mathring{L}^l_p(Q_{2d})$) by cap($e, \mathring{L}^l_p(\mathbb{R}^n)$) in the statements of the Theorem and Corollary 1 for $n > lp$. A similar remark applies to Corollary 2 in the case $n > p(l-k+1)$.

Remark 3. Proposition 10.4.2/2 and the properties of (p, l)-refined functions (see 10.4.4) imply that in the definition of the capacity cap(E, h^l_p) of a Borel set we can minimize the norm $\|u\|_{h^l_p}$ over all (p, l)-refined functions in h^l_p satisfying the inequality $u(x) \geq 1$ for (p, l)-quasi-every $x \in E$. Therefore, in the theorem of the present subsection, we can deal with (p, l)-refined functions in $V^l_p(Q_d)$ for which the inequality

$$u(x) \leq \delta d^{-n/p}\|u\|_{L_p(Q_d)}$$

is valid (p, l)-quasi everywhere on the Borel set $E \subset \bar{Q}_d$. Similarly, in Corollary 1 we can consider a Borel set $E \subset \bar{Q}_d$ and (p, l)-refined functions in $V^l_p(Q_d)$ equal to zero quasi-everywhere on E. The class of functions in Corollary 2 can also be enlarged if we consider the class $\mathfrak{C}^k(E)$ of refined functions $u \in V^l_p(Q_d)$ such that $D^\alpha u(x) = 0$ for $(p, l - |\alpha|)$-quasi all $x \in E$ and for all multi-indices of order $|\alpha| \leq k$.

14.2 Connection Between Best Constant and the (p, l)-Inner Diameter (Case $k = 1$)

14.2.1 Set Function $\lambda^l_{p,q}(G)$

Definition. With any open set $G \subset Q_d$ we associate the number

$$\lambda^l_{p,q}(G) = \inf \frac{|\!|\!| u |\!|\!|^p_{p,l,Q_d}}{\|u\|^p_{L_p(Q_d)}},$$

where $p \geq 1$ and the infimum is taken over all functions $u \in C^\infty(\bar{Q}_d)$ vanishing in a neighborhood of $\overline{Q_d \backslash G}$.

By Theorem 14.1.2, if $\overline{Q_d \backslash G}$ is a (p,l)-negligible subset of $\bar{Q}_d$, then

$$\lambda^l_{p,q}(G) \sim d^{-np/q}\,\mathrm{Cap}\big(\overline{Q_d \backslash G}, \mathring{L}^l_p(Q_{2d})\big).$$

This relation fails without the condition of smallness on $\mathrm{Cap}(\overline{Q_d \backslash G}, \mathring{L}^l_p(Q_{2d}))$. In fact, if G is "small," then the value $\lambda^l_{p,q}(G)$ becomes large (for instance, we can easily check that $\lambda^l_{p,q}(G) \sim \varepsilon^{n-pl-np/q}$ provided G is a cube with small edge length ε) whereas

$$\mathrm{cap}\big(\overline{Q_d \backslash G}, \mathring{L}^l_p(Q_{2d})\big) \leq c d^{n-pl}.$$

In the present section we give a description of the set function $\lambda^l_{p,q}(G)$ for $q \geq p \geq 1$ in certain new terms connected with the (p,l)-capacity under the condition that $\overline{Q_d \backslash G} \notin \mathscr{N}(Q_d)$.

14.2.2 Definition of the (p,l)-Inner Diameter

We fix the cube Q_d and we denote by $\mathfrak{Q}_\delta$ an arbitrary cube in Q_d with edge length δ and with sides parallel to those of Q_d.

Definition. Let G be an open subset of Q_d. The supremum of δ for which the set $\{\mathfrak{Q}_\delta : \overline{\mathfrak{Q}_\delta \backslash G} \in \mathscr{N}(\mathfrak{Q}_\delta)\}$ is not empty will be called the (p,l)-*inner* (*cubic*) *diameter of* G *relative to* Q_d and denoted by $D_{p,l}(G, Q_d)$. In the case $Q_d = \mathbb{R}^n$ we shall use the notation $D_{p,l}(G)$ and call it the (p,l)-*inner* (*cubic*) *diameter of* G. Obviously, $D_{p,l}(G, Q_d) = d$ provided $\overline{Q_d \backslash G}$ is a (p,l)-negligible subset of $\bar{Q}_d$.

Let $n < pl$, $p > 1$ or $n = l$, $p = 1$. By definition, for such p and l, all the sets except the empty set are (p,l)-essential. Therefore, for any open set $G \subset Q_d$, the (p,l)-inner (cubic) diameter $D_{p,l}(G, Q_d)$ coincides with the inner (cubic) diameter $D(G)$, i.e., with the supremum of edge lengths of cubes $\mathfrak{Q}_\delta$ inscribed in G.

14.2.3 Estimates for the Best Constant in (14.1.3) by the (p,l)-Inner Diameter

The following theorem contains two-sided estimates for $\lambda^l_{p,q}(G)$ for $q \geq p \geq 1$.

Theorem 1. *Let G be an open subset of Q_d such that $\overline{Q_d \backslash G}$ is a (p,l)-essential subset of $\bar{Q}_d$.*

1. *For all functions $u \in C^\infty(\bar{Q}_d)$ vanishing in a neighborhood of $\overline{Q_d \backslash G}$ inequality (14.1.3) is valid with $q \geq p \geq 1$ and*

$$C \leq c_1 \big[D_{p,l}(G, Q_d)\big]^{l-n(p^{-1}-q^{-1})}. \tag{14.2.1}$$

2. *If for all functions $u \in C^\infty(\bar{Q}_d)$ that vanish in a neighborhood of the set $\overline{Q_d \backslash G}$ inequality (14.1.3) holds, then*

$$C \ge c_2 \big[D_{p,l}(G, Q_d)\big]^{l-n(p^{-1}-q^{-1})}. \tag{14.2.2}$$

Proof. 1. Assume for the moment that $D_{p,l}(G, Q_d) < d$. We denote an arbitrary number in $(D_{p,l}(G, Q_d), d]$ by δ. The definition of the (p,l)-inner diameter implies that, for any cube $\bar{\mathfrak{Q}}_\delta$, the set $e = \overline{\mathfrak{Q}_\delta \backslash G}$ is a (p,l)-essential subset, i.e.,

$$\operatorname{cap}\big(\mathfrak{Q}_\delta \cap e, \mathring{L}_p^l(\mathfrak{Q}_{2\delta})\big) > \gamma \delta^{n-pl}. \tag{14.2.3}$$

(Here and in what follows $\mathfrak{Q}_{c\delta}$ is the open cube with edge length $c\delta$ whose center coincides with that of $\mathfrak{Q}_\delta$ and whose sides are parallel to the sides of $\mathfrak{Q}_\delta$.) In the case $D_{p,l}(G, Q_d) = d$ we put $\delta = d$. Then (14.2.3) is also valid since, by hypothesis, e is a (p,l)-essential subset of the cube $\bar{\mathfrak{Q}}_\delta = \bar{Q}_d$.

According to the first part of Theorem 14.1.2 and inequality (14.2.3), we have

$$\begin{aligned}\|u\|^p_{L_q(\mathfrak{Q}_\delta)} &\le \frac{c\delta^{np/q}}{\operatorname{cap}(\mathfrak{Q}_\delta \cap e, \mathring{L}_p^l(\mathfrak{Q}_{2\delta}))} |\!|\!| u |\!|\!|^p_{p,l,\mathfrak{Q}_\delta} \\ &\le c\delta^{lp-n(1-p/q)} |\!|\!| u |\!|\!|^p_{p,l,\mathfrak{Q}_\delta}.\end{aligned} \tag{14.2.4}$$

We construct a covering of Q_d by cubes $\mathfrak{Q}_\delta$ whose multiplicity does not exceed some number that depends only on n. Next we sum (14.2.4) over all cubes of the covering. Then

$$\|u\|^p_{L_p(Q_d)} \le c\delta^{lp} \sum_{j=1}^{l} \delta^{p(j-l)} \|\nabla_j u\|^p_{L_p(Q_d)}. \tag{14.2.5}$$

Using a well-known multiplicative inequality, we obtain

$$\|\nabla_j v\|_{L_p(Q_d)} \le c\|v\|^{1-j/l}_{L_p(Q_d)} \left(\sum_{i=0}^{l} d^{i-l} \|\nabla_i v\|_{L_p(Q_d)} \right)^{j/l} \tag{14.2.6}$$

(cf. Lemma 1.4.7). Putting $v = u - \bar{u}_{Q_d}$ in (14.2.6) and applying the Poincaré inequality

$$\|u - \bar{u}_{Q_d}\|_{L_p(Q_d)} \le cd \|\nabla u\|_{L_p(Q_d)},$$

we obtain

$$\|\nabla_j u\|_{L_p(Q_d)} \le c\|u\|^{1-j/l}_{L_p(Q_d)} |\!|\!| u |\!|\!|^{j/l}_{p,l,Q_d}.$$

Hence from (14.2.5) with $q = p$ we obtain

$$1 \le c \sum_{j=1}^{l} \left(\delta^l \frac{|\!|\!| u |\!|\!|_{p,l,Q_d}}{\|u\|_{L_p(Q_d)}} \right)^{pj/l}.$$

Therefore,

$$\|u\|_{L_p(Q_d)} \le c\delta^l \,|\!|\!|u|\!|\!|_{p,l,Q_d}, \tag{14.2.7}$$

and the first part of the theorem follows for $q = p$. Let $q > p$. Summing the inequality

$$\|u\|^p_{L_p(\mathfrak{Q}_\delta)} \le c\delta^{lp-n(1-p/q)}\big(\|\nabla_l u\|^p_{L_p(\mathfrak{Q}_d)} + \delta^{-pl}\|u\|^p_{L_p(\mathfrak{Q}_d)}\big)$$

over all cubes of the covering $\{\mathfrak{Q}_\delta\}$ and making use of the inequality $(\sum a_i)^\varepsilon \le \sum a_i^\varepsilon$, where $a_i > 0$, $0 < \varepsilon < 1$, we conclude that

$$\|u\|^p_{L_p(Q_d)} \le c\delta^{lp-n(1-p/q)}\big(\|\nabla_l u\|^p_{L_p(Q_d)} + \delta^{-l}\|u\|^p_{L_p(Q_d)}\big).$$

It remains to apply inequality (14.2.7).

2. Let $0 < \delta < D_{p,l}(G, Q_d)$ and let $\bar{\mathfrak{Q}}_\delta$ be a cube having a (p,l)-negligible intersection with $\overline{Q_d \backslash G}$. Let η denote a function in $C^\infty(\mathfrak{Q}_\delta)$ that vanishes near $\partial\mathfrak{Q}_\delta$, is equal to unity on the cube $\mathfrak{Q}_{\delta/2}$, and satisfies $|\nabla_j \eta| \le c\delta^{-j}$, $j = 1, 2, \ldots$. If v is an arbitrary function in $C^\infty(\bar{\mathfrak{Q}}_\delta)$ that vanishes near $\overline{Q_d \backslash G}$ then the function $u = \eta v$ extended by zero on the exterior of $\mathfrak{Q}_\delta$ satisfies (14.1.5) by the hypothesis of the theorem. Therefore

$$\begin{aligned}\|v\|_{L_p(\mathfrak{Q}_{\delta/2})} &\le C\sum_{j=1}^{l}\big\|\nabla_j(\eta v)\big\|_{L_p(\mathfrak{Q}_\delta)} \le cC\sum_{j=1}^{l}\sum_{k=0}^{j}\delta^{k-j}\|\nabla_k v\|_{L_p(\mathfrak{Q}_\delta)}\\ &\le cC\big(|\!|\!|v|\!|\!|_{p,l,\mathfrak{Q}_\delta} + \delta^{-l}\|v\|_{L_p(\mathfrak{Q}_\delta)}\big).\end{aligned}$$

This and the estimate

$$\|v\|_{L_p(\mathfrak{Q}_\delta)} \le c\big(\delta\|\nabla v\|_{L_p(\mathfrak{Q}_\delta)} + c\delta^{n(p^{-1}-q^{-1})}\|v\|_{L_q(\mathfrak{Q}_{\delta/2})}\big)$$

yield

$$\|v\|_{L_p(\mathfrak{Q}_{\delta/2})} \le c'C\big(|\!|\!|v|\!|\!|_{p,l,\mathfrak{Q}_\delta} + \delta^{-l+n(p^{-1}-q^{-1})}\|v\|_{L_q(\mathfrak{Q}_{\delta/2})}\big). \tag{14.2.8}$$

We may assume that $2c'C\delta^{-l+n(p^{-1}-q^{-1})} < 1$ since the reverse inequality is the required inequality (14.2.3). Then by (14.2.8)

$$\|v\|_{L_q(\mathfrak{Q}_{\delta/2})} \le 2c'C\,|\!|\!|v|\!|\!|_{p,l,\mathfrak{Q}_\delta},$$

and (14.2.2) follows from the second part of Theorem 14.1.2 applied to the cube $\mathfrak{Q}_\delta$. The proof is complete. □

In each of the cases $pl > n$, $p > 1$ and $l = n$, $p = 1$, Theorem 1 can be stated in terms of the inner diameter $D(G)$. To be precise, we formulate the following statement.

Theorem 2. *Let G be an arbitrary open subset of Q_d, $G \ne Q_d$ and let the numbers n, p, and l satisfy either of the conditions $pl > n$, $p > 1$ or $l = n$, $p = 1$. Further, let C be the best constant in* (14.1.3) *with $q \in [p, \infty)$. Then*

$$C \sim D(G)^{l-n(p^{-1}-q^{-1})}. \tag{14.2.9}$$

14.3 Estimates for the Best Constant C in the General Case

Let $\mathfrak{C}$ denote an arbitrary linear subset of the space $W_p^l(Q_d)$. Our goal is the study of the inequality

$$\|u\|_{L_q(Q_d)} \le C \sum_{j=k+1}^{l} d^{j-l} \|\nabla_j u\|_{L_p(Q_d)}, \tag{14.3.1}$$

where $u \in \mathfrak{C}$ and q, p, and l are the same as in Theorem 14.1.2. The norm on the right in (14.3.1) can be replaced by an equivalent one retaining only the summands corresponding to $j = l$ and $j = k + 1$.

14.3.1 Necessary and Sufficient Condition for Validity of the Basic Inequality

Let $\bar{\mathfrak{C}}$ be the closure of $\mathfrak{C}$ in the metric of the space $V_p^l(Q_d)$ and let $\mathbb{P}_k$ be the set of polynomials Π of degree $k \le l - 1$, normalized by

$$d^{-n} \int_{Q_d} |\Pi|^p \, \mathrm{d}x = 1. \tag{14.3.2}$$

Theorem. *The inequality* (14.3.1) *holds if and only if* $\mathbb{P}_k \cap \bar{\mathfrak{C}} = \varnothing$.

Proof. The necessity of this condition is obvious. We will prove the sufficiency.

If $\mathbb{P}_k \cap \bar{\mathfrak{C}} = \varnothing$ then in $\bar{\mathfrak{C}}$ we can introduce the norm

$$|u|_{\mathfrak{C}} = \sum_{j=k+1}^{l} d^{j-1} \|\nabla_j u\|_{L_p(Q_d)},$$

which makes it a Banach space. Let I be the identity mapping from $\bar{\mathfrak{C}}$ into $L_p(Q_d)$. Since $\bar{\mathfrak{C}} \subset V_p^l(Q_d) \subset L_q(Q_d)$, we see that I is defined on $\bar{\mathfrak{C}}$. We will show that it is closed. Let $|u_m|_{\mathfrak{C}} \to 0$ and $\|u_m - u\|_{L_q(Q_d)} \to 0$ as $m \to \infty$. Then there exists a sequence of polynomials $\{\Pi_m\}_{m \ge 1}$ of a degree not higher than k such that $u_m - \Pi_m \to 0$ in $L_q(Q_d)$. Consequently, $u = \lim \Pi_m$ in the space $V_p^l(Q_d)$ and since $\mathbb{P}_k \cap \bar{\mathfrak{C}} = \varnothing$ then $u = 0$. Thus I is closed. Now from the Banach theorem it follows that E is continuous, that is, (14.3.1) holds. The theorem is proved. □

Example. Consider the class $\mathfrak{C}^r(E)$ ($r = 0, \dots, l-1$, E is a Borel subset of $\bar{Q}_d$) of $(p, l-j)$-refined functions $u \in V_p^l(Q_d)$, $p > 1$, such that $\nabla_j u = 0$ $(p, l-j)$-quasi-everywhere on E, $j = 0, \dots, r$.

Since any sequence of (p, l)-refined functions that converges in $V_p^l(Q_d)$ contains a subsequence that converges (p, l)-quasi-everywhere (cf. Sect. 10.4.4), it follows that $\mathfrak{C}^r(E)$ is a closed subset of $V_p^l(Q_d)$.

Thus, by the Theorem, inequality (14.3.1) is valid for all $u \in \mathfrak{C}^r(E)$ if and only if $\mathbb{P}_k$ does not contain a polynomial $\varPi$ such that $\nabla_j \varPi = 0$ $(p, l-j)$-quasi-everywhere on E, $j = 0, \ldots, r$.

14.3.2 Polynomial Capacities of Function Classes

Let $\varPi$ be a polynomial in $\mathbb{P}_k$ and let

$$\operatorname{cap}\big(\mathfrak{C}, \varPi, \mathring{L}^l_p(Q_{2d})\big) = \inf \int_{Q_{2d}} |\nabla_l u|^p \, dx,$$

where the infimum is taken over all functions $u \in \mathring{L}^l_p(Q_{2d})$ such that the restriction of $u - \varPi$ to $\bar{Q}_d$ is contained in a linear subset $\mathfrak{C}$ of the space $V^l_p(Q_d)$.

With $\mathfrak{C}$ we associate l capacities

$$\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big) = \inf_{\{\varPi : \varPi \in \mathbb{P}_k\}} \operatorname{cap}\big(\mathfrak{C}, \varPi, \mathring{L}^l_p(Q_{2d})\big),$$

$k = 0, \ldots, l-1$. In other words,

$$\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big) = \inf_{\{\varPi, u\}} \int_{Q_{2d}} |\nabla_l u|^p \, dx, \tag{14.3.3}$$

where the infimum is taken over all pairs $\{\varPi, u\}$ such that $\varPi \in \mathbb{P}_k$, $u|_{\bar{Q}_d} \in \mathfrak{C}$ and $\varPi - u \in \mathring{L}^l_p(Q_{2d})$.

It is clear that $\mathrm{CAP}_k(\mathfrak{C}, \mathring{L}^l_p(Q_{2d}))$ does not increase as k increases. The following inequality holds:

$$\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big) \le c d^{n-pl}. \tag{14.3.4}$$

In fact, let $\eta \in \mathfrak{M}(\bar{Q}_1, Q_2)$ and let $\eta_d(x) = \eta(x/d)$. Since $1 \in \mathbb{P}_k$ and the restriction of the function $1 - \eta_d$ to Q_d equals zero, the pair $\{1, \eta_d\}$ is admissible for the problem (14.3.3). This implies (14.3.4).

We introduce the norm

$$\|u\|_{V^l_p(Q_d)} = \sum_{j=0}^{l} d^{j-l} \|\nabla_j u\|_{L_p(Q_d)}.$$

The next assertion is similar to Lemma 14.1.2.

Lemma. *The capacity* $\mathrm{CAP}_k(\mathfrak{C}, \mathring{L}^l_p(Q_d))$ *is equivalent to the following capacity of the class* $\mathfrak{C}$:

$$\inf \|\varPi - u\|^p_{V^l_p(Q_d)}, \tag{14.3.5}$$

where the infimum is taken over all pairs $\{\varPi, u\}$ *such that* $\varPi \in \mathbb{P}_k$, $\varPi - u \in V^l_p(Q_d)$, $u \in \mathfrak{C}$.

Proof. We have

$$\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}_p^l(Q_{2d})\big) \le \int_{Q_{2d}} \big|\nabla_l\big(\eta_d(\Pi - Au)\big)\big|^p \,\mathrm{d}x,$$

where A is the extension operator in Lemma 14.1.2. Obviously, the right-hand side does not exceed $c\|\Pi - Au\|^p_{V_p^l(Q_d)}$. From (14.1.8) it follows that $A\Pi = \Pi$. Therefore, using (14.1.8) once more, we obtain

$$\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}_p^l(Q_{2d})\big) \le c\|\Pi - u\|^p_{V_p^l(Q_d)}.$$

Minimizing the right-hand side, we arrive at the required upper bound for CAP_k.

We now prove the lower estimate. Since $(\Pi - u)|_{Q_d}$ can be extended to a function in $\mathring{L}_p^l(Q_{2d})$, the classes of admissible functions in the definitions of both capacities under consideration are simultaneously empty or nonempty. Let $\Pi \in \mathbb{P}_k$, $v \in \mathring{L}_p^l(Q_{2d})$, $(\Pi - v)|_{\bar{Q}_d} \in \mathfrak{C}$. Then the capacity (14.3.5) does not exceed

$$\|v\|_{V_p^l(Q_d)} \le c\|\nabla_l v\|_{L_p(Q_{2d})}.$$

The lemma is proved. □

14.3.3 Estimates for the Best Constant C in the Basic Inequality

From Theorem 14.3.1 it follows that (14.3.1) holds if and only if

$$\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}_p^l(Q_{2d})\big) > 0.$$

The next theorem yields two-sided estimates for the best constant C in (14.3.1) expressed in terms of the capacity $\mathrm{CAP}_k(\mathfrak{C}, \mathring{L}_p^l(Q_{2d}))$.

Theorem. 1. *If* $\mathrm{CAP}_k(\mathfrak{C}, \mathring{L}_p^l(Q_{2d})) > 0$ *then, for all* $u \in \mathfrak{C}$*, the inequality* (14.3.1) *holds with*

$$C \le c d^{n/q}\big[\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}_p^l(Q_{2d})\big)\big]^{-1/p}. \tag{14.3.6}$$

2. *If* (14.3.1) *holds for all* $u \in \mathfrak{C}$ *and if*

$$\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}_p^l(Q_{2d})\big) \le c_0 d^{n-pl},$$

where c_0 *is a small enough constant that depends only on* n, p, l, *and* k, *then*

$$C \ge c d^{n/q}\big[\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}_p^l(Q_{2d})\big)\big]^{-1/p}. \tag{14.3.7}$$

Proof. 1. Let $u \in \mathfrak{C}$ be normalized by

$$\|u\|_{L_p(Q_d)} = d^{n/p}, \tag{14.3.8}$$

and let $\varPi$ be any polynomial in $\mathbb{P}_k$. According to Lemma 14.3.2 we have

$$\big[\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big)\big]^{1/p} \le c\sum_{i=0}^{k} d^{i-l}\big\|\nabla_i(\varPi - u)\big\|_{L_p(Q_d)} + c\sum_{i=k+1}^{l} d^{i-l}\|\nabla_i u\|_{L_p(Q_d)}. \qquad (14.3.9)$$

Hence from the inequality

$$\|\nabla_i v\|_{L_p(Q_d)} \le c d^{l-i}\|\nabla_l v\|_{L_p(Q_d)} + c d^{-i}\|v\|_{L_p(Q_d)},$$

we obtain that the first sum in (14.3.9) does not exceed

$$c d^{-l}\|\varPi - u\|_{L_p(Q_d)} + c\|\nabla_l u\|_{L_p(Q_d)}.$$

Therefore

$$\big[\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big)\big]^{1/p} \le c d^{-l}\|\varPi - u\|_{L_p(Q_d)} + c\sum_{i=k+1}^{l} d^{i-l}\|\nabla_i u\|_{L_p(Q_d)}. \qquad (14.3.10)$$

For each $u \in V^l_p(Q_d)$ there exists a polynomial π of degree less than $k+1$ such that

$$\|\pi - u\|_{L_p(Q_d)} \le c' d^{k+1}\|\nabla_{k+1} u\|_{L_p(Q_d)}. \qquad (14.3.11)$$

First suppose that

$$\|\nabla_{k+1} u\|_{L_p(Q_d)} > (2c')^{-1} d^{n/p-k-1}.$$

Then, by (14.3.4) we have

$$\big[\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big)\big]^{1/p} \le c d^{k-l+1}\|\nabla_{k+1} u\|_{L_p(Q_d)}. \qquad (14.3.12)$$

Now let

$$\|\nabla_{k+1} u\|_{L_p(Q_d)} \le (2c')^{-1} d^{n/p-k-1}.$$

From (14.3.11) we obtain

$$\|\pi - u\|_{L_p(Q_d)} \le 2^{-1} d^{n/p} = 2^{-1}\|u\|_{L_p(Q_d)}$$

and consequently

$$2^{-1} d^{n/p} \le \|\pi\|_{L_p(Q_d)} \le 3\cdot 2^{-1} d^{n/p}. \qquad (14.3.13)$$

We put

$$\varPi = d^{n/p}\|\pi\|^{-1}_{L_p(Q_d)}\pi.$$

Then (14.3.13) implies

$$\|\Pi - u\|_{L_p(Q_d)} \le 2\big\|\pi - d^{n/p}\|\pi\|_{L_p(Q_d)} u\big\|_{L_p(Q_d)}.$$

Obviously, the right-hand side does not exceed

$$\begin{aligned}&2\|\pi - u\|_{L_p(Q_d)} + 2\|u\|_{L_p(Q_d)}\big|d^{n/p}\|\pi\|_{L_p(Q_d)} - 1\big|\\ &\quad = 2\|\pi - u\|_{L_p(Q_d)} + 2\big|\|\pi\|_{L_p(Q_d)} - \|u\|_{L_p(Q_d)}\big| \le 4\|\pi - u\|_{L_p(Q_d)}.\end{aligned}$$

Using (14.3.11), we obtain

$$\|\Pi - u\|_{L_p(Q_d)} \le 4c' d^{k+1}\|\nabla_{k+1} u\|_{L_p(Q_d)},$$

which together with (14.3.10) and (14.3.12) implies the estimate

$$\big[\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big)\big]^{1/p} \le c \sum_{i=k+1}^{l} d^{i-l}\|\nabla_i u\|_{L_p(Q_d)} \tag{14.3.14}$$

for all $u \in \mathfrak{C}$, normalized by (14.3.8).

From the Sobolev embedding theorem and (14.3.4) we obtain

$$\begin{aligned}\|u\|_{L_p(Q_d)} &\le cd^{k+1+n(p^{-1}-q^{-1})}\|\nabla_{k+1}u\|_{L_p(Q_d)} + cd^{n(q^{-1}-p^{-1})}\|u\|_{L_p(Q_d)}\\ &\le cd^{k-l+nq^{-1}+1}\big[\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big)\big]^{-1/p}\|\nabla_{k+1}u\|_{L_p(Q_d)}\\ &\quad + cd^{n(q^{-1}-p^{-1})}\|u\|_{L_p(Q_d)},\end{aligned}$$

which together with (14.3.14) yields (14.3.1) with the constant C satisfying (14.3.6).

2. Let ε be an arbitrary positive number, $\Pi \in \mathbb{P}_k$, $\psi \in \mathring{L}^l_p(Q_{2d})$, $(\Pi - \psi)|_{Q_d} \in \mathfrak{C}$ and let

$$\int_{Q_{2d}} |\nabla_l \psi|^p \,\mathrm{d}x \le \mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big) + \varepsilon d^{n-lp}.$$

Since the restriction of $\psi - \Pi$ to $\bar{Q}_d$ is contained in $\mathfrak{C}$, by the hypothesis of the theorem we have

$$\|\psi - \Pi\|_{L_q(Q_d)} \le C \sum_{j=k+1}^{l} d^{j-l}\|\nabla_j \psi\|_{L_p(Q_d)}. \tag{14.3.15}$$

The right-hand side does not exceed

$$cC\|\nabla_l \psi\|_{L_p(Q_d)} \le cC\big(\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big) + \varepsilon d^{n-lp}\big)^{1/p}$$

because $\psi \in \mathring{L}^l_p(Q_{2d})$. Similarly,

$$\|\psi\|_{L_q(Q_d)} \le c d^{l+n(q^{-1}-p^{-1})} \|\nabla_l \psi\|_{L_p(Q_{2d})} \le c(c_0+\varepsilon)^{1/p} d^{n/p}.$$

Thus

$$\|\psi - \Pi\|_{L_q(Q_d)} \ge \|\Pi\|_{L_q(Q_d)} - c(c_0+\varepsilon)^{1/p} d^{n/q}$$

and by (14.3.15)

$$\|\psi\|_{L_q(Q_d)} \le c(c_0+\varepsilon)^{1/p} d^{n/q} + cC\big(\mathrm{CAP}_k\big(\mathfrak{C}, \mathring{L}^l_p(Q_{2d})\big) + \varepsilon d^{n-lp}\big)^{1/p}. \quad (14.3.16)$$

Now we note that

$$\|\Pi\|_{L_p(Q_d)} \le c d^{n(p^{-1}-q^{-1})} \|\Pi\|_{L_q(Q_d)}.$$

The preceding estimate follows from the Hölder inequality for $p \le q$. In the case $p > q$ it results as follows:

$$\begin{aligned}\|\Pi\|_{L_p(Q_d)} &\le c\big(d^{k+1}\|\nabla_{k+1}\Pi\|_{L_p(Q_d)} + d^{n(p^{-1}-q^{-1})}\|\Pi\|_{L_q(Q_d)}\big)\\ &= c d^{n(p^{-1}-q^{-1})}\|\Pi\|_{L_q(Q_d)}.\end{aligned}$$

Since $\Pi \in \mathbb{P}_k$, we have $\|\Pi\|_{L_p(Q_d)} = d^{n/p}$. Therefore $\|\Pi\|_{L_q(Q_d)} \ge c d^{n/q}$. Using the smallness of the constant c_0, we arrive at (14.3.7). The theorem is proved. □

14.3.4 Class $\mathfrak{C}_0(e)$ and Capacity $\mathrm{Cap}_k(e, \mathring{L}^l_p(Q_{2d}))$

The rest of the section deals with the class

$$\mathfrak{C}_0(e) = \big\{u \in C^\infty(\bar{Q}_d) : \mathrm{dist}(\mathrm{supp}\, u, e) > 0\big\}, \quad (14.3.17)$$

where e is compact subset of the cube $\bar{Q}_d$.

We introduce the following set function:

$$\mathrm{Cap}_k\big(e, \mathring{L}^l_p(Q_{2d})\big) = \inf_{\Pi \in \mathbb{P}_k} \inf_{\{f\}} \int_{Q_{2d}} |\nabla_l f|^p \, dx, \quad (14.3.18)$$

where $p \ge 1$ and $\{f\}$ is a collection of functions in $\mathring{L}^l_p(Q_{2d})$ such that $f = \Pi$ in a neighborhood of e where $\Pi \in \mathbb{P}_k$.

Since $\mathbb{P}_0 = \{\pm 1\}$, we have

$$\mathrm{Cap}_0\big(e, \mathring{L}^l_p(Q_{2d})\big) = \mathrm{Cap}\big(e, \mathring{L}^l_p(Q_{2d})\big).$$

We show that the capacities $\mathrm{Cap}_k(e, \mathring{L}^l_p(Q_{2d}))$ and $\mathrm{CAP}_k(\mathfrak{C}_0(e), \mathring{L}^l_p(Q_{2d}))$ are equivalent.

Lemma. *The following inequalities are valid:*

$$\begin{aligned}\mathrm{CAP}_k\big(\mathfrak{C}_0(e), \mathring{L}^l_p(Q_{2d})\big) &\le \mathrm{Cap}_k\big(e, \mathring{L}^l_p(Q_{2d})\big)\\ &\le c\,\mathrm{CAP}_k\big(\mathfrak{C}_0(e), \mathring{L}^l_p(Q_{2d})\big).\end{aligned}$$

Proof. The upper estimate for the CAP_k inequality is an obvious corollary of the definitions of the two capacities.

We shall prove the lower estimate. Let $\Pi \in \mathbb{P}_k$, $u \in \mathfrak{C}_0(e)$, let A be the extension operator in Lemma 14.1.2, and let η_d be the function used in the proof of inequality (14.3.4). From property (ii) of the operator A it follows that $\eta_d(\Pi - Au)$ is contained in the class $\{f\}$ introduced in the definition of $\text{Cap}_k(e, \mathring{L}_p^l(Q_{2d}))$. Therefore

$$\text{Cap}_k\big(e, \mathring{L}_p^l(Q_{2d})\big) \le \big\|\eta_d(\Pi - Au)\big\|^p_{L_p^l(Q_{2d})} \le c\|\Pi - Au\|^p_{V_p^l(Q_{2d})}.$$

Taking into account the equality $A\Pi = \Pi$ and the estimate (14.1.8) for the function $v = \Pi - u$ we complete the proof by reference to Lemma 14.3.2. □

From Theorem 14.3.1, applied to the class $\mathfrak{C}_0(e)$, and from the preceding lemma there immediately follows an assertion that coincides with Theorem 14.1.2 for $k = 0$.

Corollary. 1. *If $\text{Cap}_k(e, \mathring{L}_p^l(Q_{2d})) > 0$, then, for all $u \in \mathfrak{C}_0(e)$, the inequality* (14.3.1) *holds and*

$$C \le c d^{n/p}\big[\text{Cap}_k\big(e, \mathring{L}_p^l(Q_{2d})\big)\big]^{-1/p}.$$

2. *If* (14.3.1) *holds for all $u \in \mathfrak{C}_0(e)$ and if*

$$\text{Cap}_k\big(e, \mathring{L}_p^l(Q_{2d})\big) \le c_0 d^{n-pl},$$

where c_0 is small enough constant that depends only on n, p, and l, then

$$C \ge c d^{n/q}\big[\text{Cap}_k\big(e, \mathring{L}_p^l(Q_{2d})\big)\big]^{-1/p}.$$

14.3.5 Lower Bound for Cap_k

We derive a lower bound for the capacity $\text{Cap}_k(e, \mathring{L}_p^l(Q_{2d}))$ by the capacity $\text{Cap}(e, \mathring{L}_p^{l-k}(Q_{2d}))$.

Proposition. *The following inequality holds:*

$$\text{Cap}_k\big(e, \mathring{L}_p^l(Q_{2d})\big) \ge c d^{-kp}\,\text{Cap}\big(e, \mathring{L}_p^{l-k}(Q_{2d})\big). \tag{14.3.19}$$

Proof. It suffices to consider the case $d = 1$. From the inequality

$$\|\nabla_l v\|_{L_p(Q_2)} \ge c\|\nabla_{l-k} v\|_{L_p(Q_2)}, \quad v \in \mathring{L}_p^l(Q_2),$$

we obtain

$$\operatorname{Cap}\big(e, \mathring{L}_p^l(Q_2)\big) \geq c\operatorname{Cap}\big(e, \mathring{L}_p^{l-k}(Q_2)\big). \tag{14.3.20}$$

Let $\Pi \in \mathbb{P}_k$ and let f be a function in $\mathring{L}_p^l(Q_{2d})$ such that $f = \Pi$ in a neighborhood of e. Obviously, the difference $\partial\Pi/\partial x_i - \partial f/\partial x_i$ is contained in $\mathfrak{C}_0(e)$ for all $i = 1, \dots, n$. For some i, let

$$\|\partial\Pi/\partial x_i\|_{L_p(Q_2)} \geq \varepsilon, \tag{14.3.21}$$

where ε is a positive number (that depends only on k, l, or n) which will be specified later. Then

$$\|\nabla_l f\|^p_{L_p(Q_2)} \geq \left\|\nabla_{l-1}\frac{\partial f}{\partial x_i}\right\|^p_{L_p(Q_2)} \geq \varepsilon^p \operatorname{Cap}_{k-1}\big(e, \mathring{L}_p^{l-1}(Q_2)\big). \tag{14.3.22}$$

If for all $i = 1, \dots, n$ the inequality (14.3.21) fails, the condition $\|\Pi\|_{L_p(Q_1)} = 1$ implies

$$\big||\Pi(x)| - 1\big| \leq c\varepsilon, \quad x \in Q_2.$$

We can take $\varepsilon = (2c)^{-1}$. Since $\Pi = f$ on e, we have $|f(x)| \geq \frac{1}{2}$, $x \in e$, and hence

$$\|\nabla_l f\|^p_{L_p(Q_2)} \geq 2^{-p}\operatorname{Cap}\big(e, \mathring{L}_p^l(Q_2)\big).$$

The preceding result and (14.3.22) yield

$$\operatorname{Cap}_k\big(e, \mathring{L}_p^l(Q_2)\big) \geq c\min\big\{\operatorname{Cap}_{k-1}\big(e, \mathring{L}_p^{l-1}(Q_2)\big), \operatorname{Cap}\big(e, \mathring{L}_p^l(Q_2)\big)\big\}.$$

Applying (14.3.20), we complete the proof. □

We present an example of a set for which

$$\operatorname{Cap}_1\big(e, \mathring{L}_2^2(Q_2)\big) = 0 \quad \text{and} \quad \operatorname{Cap}\big(e, \mathring{L}_2^2(Q_2)\big) > 0.$$

Example. Let $n = 3$, $p = 2$, $l = 2$, and let e be the center of the cube $Q_1 = \{x : |x_i| < \frac{1}{2}\}$. Since by the Sobolev embedding theorem

$$\big|u(e)\big|^2 \leq c\int_{Q_2} |\nabla_2 u|^2\,\mathrm{d}x$$

for all $u \in \mathscr{D}(Q_2)$, it follows that $\operatorname{Cap}(e, \mathring{L}_2^2(Q_2)) \geq c^{-1} > 0$.

We show that $\operatorname{Cap}_1(e, \mathring{L}_2^2(Q_2)) = 0$. Let $\Pi = 2\sqrt{3}x$. Obviously, $\Pi \in \mathbb{P}_1$. Let η_ε denote a function in $\mathfrak{M}(\bar{Q}_\varepsilon, Q_{2\varepsilon})$ such that $|\nabla_j\eta_\varepsilon| \leq c\varepsilon^{-j}$. The function $\Pi\eta_\varepsilon$ coincides with Π in a neighborhood of e. Hence

$$\operatorname{Cap}_1\big(e, \mathring{L}_2^2(Q_2)\big) \leq \int_{Q_2} \big|\nabla_2(\Pi\eta_\varepsilon)\big|^2\,\mathrm{d}x.$$

On the other hand, the last integral is $O(\varepsilon)$. Thus $\operatorname{Cap}_1(e, \mathring{L}_2^2(Q_2)) = 0$.

Remark. In connection with the previous example consider the quadratic forms

$$S_1(u,u) = \int_{Q_1} \left[\sum_{i,j=1}^{3} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 + \sum_{i,j=1}^{3} \left(\frac{\partial u}{\partial x_i} \right)^2 \right] \mathrm{d}x,$$

$$S_2(u,u) = \int_{Q_1} \sum_{i,j=1}^{3} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \mathrm{d}x,$$

defined on functions $u \in C^\infty(\bar{Q}_1)$ that vanish near the center of the cube Q_1. The forms generate the operators $\Delta^2 - \Delta$ and Δ^2 with the Neumann boundary data on ∂Q_1 and with the complementary condition $u = 0$ at the point e. Corollary 14.3.4 and the above example imply that the first operator is positive definite and that the second is not.

In general, for $p = 2$, the basic results of the present section can be reformulated as necessary and sufficient conditions for positive definiteness and as two-sided estimates for the first eigenvalue of the elliptic operator generated by the quadratic form $S(u,u)$. This form is given on a linear subset $\mathfrak{C}$ of the space $V_2^l(Q_1)$ and satisfies the "coerciveness" condition

$$c_1 \sum_{j=k+1}^{l} \|\nabla_j u\|^2_{L_2(Q_1)} \le S(u,u) \le c_2 \sum_{j=k+1}^{l} \|\nabla_j u\|^2_{L_2(Q_1)}$$

for all $u \in \mathfrak{C}$.

14.3.6 Estimates for the Best Constant in the Case of Small (p,l)-Inner Diameter

Here we show that the best constant in (14.3.1) (for $q \ge p \ge 1$ and $\mathfrak{C} = \mathfrak{C}_0(e)$) is equivalent to some power of the (p,l)-inner diameter of $\bar{Q}_d \backslash e$ provided this diameter is small.

Lemma. *Let G be an open subset of the cube Q_d such that*

$$D_{p,l}(G, Q_d) \le c_0 d, \tag{14.3.23}$$

where c_0 is a small enough constant that depends only on n, p, and l. Then, for all functions $u \in C^\infty(\bar{Q}_d)$ vanishing in a neighborhood of $\bar{Q}_d \backslash G$, the inequality

$$\|\nabla_j u\|_{L_p(Q_d)} \le c \big[D_{p,l}(G, Q_d)\big]^{l-j} \|\nabla_l u\|_{L_p(Q_d)} \tag{14.3.24}$$

holds, where $j = 0, 1, \dots, l-1$.

Proof. It suffices to assume that $d = 1$ and $l > 1$. We put $D = D_{p,l}(G, Q_1)$. Since $\delta < 1$, it follows that $\overline{Q_d \backslash G} \notin \mathscr{N}(Q_1)$. Therefore, by Theorems 14.2.3/1 and 14.2.3/2, we have

$$\|u\|_{L_p(Q_1)} \le cD^l \sum_{j=1}^{l} \|\nabla_j u\|_{L_p(Q_1)}.$$

Hence from the inequality

$$\|\nabla_j u\|_{L_p(Q_1)} \le c\big(\|\nabla_l u\|_{L_p(Q_1)} + \|u\|_{L_p(Q_1)}\big),$$

we obtain that

$$\|u\|_{L_p(Q_1)} \le cD^l \big(\|\nabla_l u\|_{L_p(Q_1)} + \|u\|_{L_p(Q_1)}\big).$$

Thus, (14.3.24) follows for $j = 0$.

To obtain the estimate for $\|\nabla_j u\|_{L_p(Q_1)}$ with $j \ge 1$ we can use the inequality

$$\|\nabla_j u\|_{L_p(Q_1)} \le c\big(\|\nabla_l u\|_{L_p(Q_1)} + \|u\|_{L_p(Q_1)}\big)^{j/l} \|u\|^{(l-j)/l}_{L_p(Q_1)}.$$

The lemma is proved. □

Theorem. *Let q be the same number as in Theorem 14.1.2 and let condition (14.3.23) hold. Then for all functions $u \in C^\infty(\bar{Q}_d)$ that vanish in a neighborhood of $\bar{Q}_d \backslash G$ the inequality*

$$\|u\|_{L_q(Q_d)} \le C \sum_{j=k+1}^{l} d^{j-1} \|\nabla_j u\|_{L_p(Q_d)}, \tag{14.3.25}$$

where $k = 0, 1, \dots, l-1$ holds. The best constant in (14.3.25) satisfies the inequalities

$$c_1 \big[D_{p,l}(G, Q_d)\big]^{l-n(p^{-1}-q^{-1})} \le C \le c_2 \big[D_{p,l}(G, Q_d)\big]^{l-n(p^{-1}-q^{-1})}. \tag{14.3.26}$$

(*In the case $n < pl$, $p > 1$ or $n = l$, $p = 1$ the value $D_{p,l}(G, Q_d)$ can be replaced by inner diameter $D(G, Q_d)$ in (14.3.25) and (14.3.26).*)

Proof. The right estimate in (14.3.26) follows from (14.2.1) and the previous lemma, and the left estimate is contained in the second part of Theorem 14.2.3/1 and in Theorem 14.2.3/2. □

Remark. The smallness of the (p, l)-inner diameter is crucial for the validity of the Theorem. In fact, let G be the cube $Q_1 \subset \mathbb{R}^3$ with center excluded. Then $d = 1$, $D(G) = \frac{1}{2}$, whereas, according to Remark 14.3.5, the inequality

$$\|u\|_{L_2(Q_1)} \le C \|\nabla_2 u\|_{L_2(Q_1)} \tag{14.3.27}$$

is not true. (This can be seen directly by insertion of the function $u(x) = x_1 \zeta(x/\varepsilon)$, where $\varepsilon > 0$, $\zeta = 0$ on $B_1(0)$, $\zeta = 1$ outside $B_2(0)$ into (14.3.27).)

14.3.7 A Logarithmic Sobolev Inequality with Application to the Uniqueness Theorem for Analytic Functions in the Class $L_p^1(U)$

From the inequality (14.1.3) we can deduce an estimate for the integral of the logarithm of the modulus of a function in L_p^1 that characterizes the smallness of the set of zeros of this function that suffices for this integral to be finite (see Maz'ya and Havin [569]).

Let E be Borel set in $\mathbb{R}^{n-1} = \{x = (x', x_n) \in \mathbb{R}^n : x_n = 0\}$ and let $\{\mathscr{B}\}$ be a collection of n-dimensional open balls with centers in $\mathbb{R}^{n-1}$. We denote the concentric ball with doubled radius by $2\mathscr{B}$. Let

$$
\begin{aligned}
&s(\mathscr{B}) = m_{n-1}\big(\mathbb{R}^{n-1} \cap \mathscr{B}\big),\\
&c(E \cap \mathscr{B}) = \operatorname{cap}\big(E \cap \mathscr{B}, \mathring{W}_2^1(2\mathscr{B})\big),\\
&S = \sum_{\{\mathscr{B}\}} s(\mathscr{B}).
\end{aligned}
$$

We denote the number of different balls $\mathscr{B}$ that contain a point x by χ.

Lemma. *Let φ be a (p,l)-refined function in the class $L_p^1(\bigcup \mathscr{B})$ that vanishes on $E \cap G$. Then*

$$
\begin{aligned}
&\frac{1}{S}\int_{\cup(\mathscr{B}\cap\mathbb{R}^{n-1})} \log\big|\varphi(x')\big|\chi(x')\,\mathrm{d}x'\\
&\quad \le \frac{1}{pS}\sum_{\{\mathscr{B}\}} s(\mathscr{B})\log\frac{s(\mathscr{B})}{c(E\cap\mathscr{B})}\\
&\qquad + \frac{1}{p}\log\bigg[\frac{c}{S}\int_{\cup\mathscr{B}} |\nabla\varphi|^p\chi\,\mathrm{d}x\bigg], \qquad (14.3.28)
\end{aligned}
$$

where $c = c(n,p)$, $p \ge 1$.

Proof. If $(\mathscr{X}, \Sigma, \mu)$ is a measure space with finite μ and f is a nonnegative function defined on $\mathscr{X}$ and measurable with respect to the σ-algebra Σ, then for any $p > 0$

$$
\exp\bigg(\frac{1}{\mu(\mathscr{X})}\int_{\mathscr{X}} \log f\,\mathrm{d}\mu\bigg) \le \bigg(\frac{1}{\mu(\mathscr{X})}\int_{\mathscr{X}} f^p\,\mathrm{d}\mu\bigg)^{1/p}. \qquad (14.3.29)
$$

Therefore

$$
\frac{1}{s(\mathscr{B})}\int_{\mathscr{B}\cap\mathbb{R}^{n-1}} \log|\varphi|\,\mathrm{d}x' \le \frac{1}{p}\log\bigg(\frac{1}{s(\mathscr{B})}\int_{\mathscr{B}\cap\mathbb{R}^{n-1}} |\varphi|^p\,\mathrm{d}x'\bigg),
$$

where the ball $\mathscr{B}$ is arbitrary. Now we note that, for $p \ge 1$,

$$
\begin{aligned}
&\frac{1}{s(\mathscr{B})}\int_{\mathscr{B}\cap\mathbb{R}^{n-1}} |\varphi|^p\,\mathrm{d}x'\\
&\quad \le c\bigg(\frac{1}{m_n(\mathscr{B})}\int_{\mathscr{B}} |\varphi|^p\,\mathrm{d}x + \frac{1}{\operatorname{cap}(\mathscr{B}, \mathring{W}_p^1(2\mathscr{B}))}\int_{\mathscr{B}} |\nabla\varphi|^p\,\mathrm{d}x\bigg)
\end{aligned}
$$

and use (14.1.3) and Remark 14.1.3/3. We have

$$\frac{1}{m_n(\mathscr{B})}\int_{\mathscr{B}}|\varphi|^p\,\mathrm{d}x \le \frac{c}{c(E\cap\mathscr{B})}\int_{\mathscr{B}}|\nabla\varphi|^p\,\mathrm{d}x.$$

Then

$$\frac{1}{s(\mathscr{B})}\int_{\mathscr{B}\cap\mathbb{R}^{n-1}}|\varphi|^p\,\mathrm{d}x' \le \frac{c}{c(E\cap\mathscr{B})}\int_{\mathscr{B}}|\nabla\varphi|^p\,\mathrm{d}x$$

and consequently,

$$\begin{aligned}&\int_{\bigcup(\mathscr{B}\cap\mathbb{R}^{n-1})}\log|\varphi|\chi\,\mathrm{d}x'\\ &\quad=\sum_{\{\mathscr{B}\}}\int_{\mathscr{B}\cap\mathbb{R}^{n-1}}\log|\varphi|\,\mathrm{d}x'\\ &\quad\le\frac{1}{p}\sum_{\{\mathscr{B}\}}\Big[s(\mathscr{B})\log\frac{S(\mathscr{B})}{c(E\cap\mathscr{B})}\Big]+\frac{1}{p}\sum_{\{\mathscr{B}\}}\Big[s(\mathscr{B})\log\Big(\frac{c}{s(\mathscr{B})}\int_{\mathscr{B}}|\nabla\varphi|^p\,\mathrm{d}x\Big)\Big].\end{aligned}$$

Applying (14.3.29) to the last sum we arrive at the required estimate. □

By virtue of (14.3.28) we can prove the uniqueness theorem for analytic functions of the class L_p^l in the unit disk U (see the paper by Havin and the author [569], where this problem is considered in detail and where a bibliography is given).

Let $\mathfrak{A}$ be the set of all functions analytic in the disk U and let $\mathscr{X}\in\mathfrak{A}$. We say that a set E, contained in the interval $(0,2\pi)$, is the uniqueness set for $\mathscr{X}$ if each function $f\in\mathscr{X}$ with $\lim_{r\to 1-0}f(re^{i\theta})=0$ for any $\theta\in E$ vanishes identically on U.

Let E be a Borel set in the interval $(0,2\pi)$ and let $\{\delta\}$ be the set of pairwise, disjoint open intervals $\delta\subset(0,2\pi)$. Denote the length of δ by $l(\delta)$, the disk with diameter δ by $\mathscr{B}$, and the concentric disk with doubled radius by $2\mathscr{B}$. We also use the notation $c(E\cap\delta)=\operatorname{cap}(E\cap\delta,\mathring{W}_p^1(2\mathscr{B}))$.

Let E satisfy the condition

$$\sum l(\delta)\log\frac{l(\delta)}{c(E\cap\delta)}=-\infty \tag{14.3.30}$$

and let an analytic function $f\in L_p^1(U)$ satisfy $\lim_{r\to 1-0}f(r\mathrm{e}^{\mathrm{i}\theta})=0$ for any $\theta\in E$.

From (14.3.28) it follows that

$$\int_0^{\infty}\log\Big(\lim_{r\to 1-0}\big|f(r\mathrm{e}^{\mathrm{i}\theta})\big|\Big)\,\mathrm{d}\theta=-\infty$$

(see Havin and Maz'ya [569]) which together with the well-known uniqueness theorem for analytic functions of the Hardy class H^1 shows that $f(z)=0$ for all $z\in U$. Thus E is the uniqueness set for $L_p^1(U)$.

Since $c(E\cap\delta)\sim 1$ for $p>2$, then in this case we can omit $c(E\cap\delta)$ in (14.3.30). For $p<2$ we can replace $\operatorname{cap}(E\cap\delta,W_p^1(\mathbb{R}^2))$ by $\operatorname{cap}(E\cap\delta)$.

14.4 Comments to Chap. 14

Section 14.1. Theorem 14.1.2 is due to the author [533] for $p = 2$ (see also Maz'ya [544] for the case $p \geq 1$). Similar results were rediscovered by Donoghue [240], Polking [665], and R.A. Adams [23].

Inequality (14.1.17) and the particular case (14.1.15) of it, which refines the first part of Theorem 14.1.2, was established by Hedberg [368] for $p > 1$ who used a different method for which the restriction $p \neq 1$ is important. In 10.1.3 these results follow from Theorem 14.1.3, which was implicitly proved in the author's paper [544]. We note that our proof is also valid for the case $p = 1$.

Inequality (14.1.17) plays an important role in the aforementioned paper by Hedberg where the well-known problem of spectral synthesis in Sobolev spaces is solved (see [369] for the history of this problem). The basic result of Hedberg runs as follows.

Let $u \in W_p^l(\mathbb{R}^n)$, $p > 1$, *and let* m *be a positive integer. Let* K *be a closed subset of* $\mathbb{R}^n$ *and* $D^\alpha u|_K = 0$ *for all* α *with* $0 \leq |\alpha| \leq l - 1$. *Then* $u \in \mathring{W}_p^l(\mathbb{R}^n \backslash K)$, *i.e., there exist functions* $u_m \in C_0^\infty(\mathbb{R}^n \backslash K)$ *such that*

$$\lim_{m \to \infty} \|u - u_m\|_{W_p^l(\mathbb{R}^n)} = 0.$$

An obvious corollary to this theorem is the following uniqueness theorem for the Dirichlet problem (see Hedberg [368]).

Let $\Omega \subset \mathbb{R}^n$ *be an open bounded set in* $\mathbb{R}^n$ *and let* u *be a solution of* $\Delta^l u = 0$ *in* Ω *in the space* $W_2^m(\mathbb{R}^n)$ *satisfying* $D^\alpha u|_{\partial\Omega} = 0$, $0 \leq |\alpha| \leq l - 1$. *Then* $u = 0$ *in* Ω.

The existence of the spectral synthesis in the Sobolev space $W_p^l(\mathbb{R}^n)$ was proved by a different method by Netrusov [634] who also solved the problem for a wide class of function spaces including Besov and Lizorkin-Triebel spaces [632]. The above-mentioned results by Netrusov with complete proofs can be found in the books by D.R. Adams and Hedberg [15] and by Hedberg and Netrusov [371].

The following characterization of the space $\mathring{W}_p^l(\Omega)$ was given by Swanson and Ziemer [736].

Theorem. *Let* l *be a positive integer, let* $1 < p < \infty$, *and let* $f \in \mathring{W}_p^l(\Omega)$. *If* $\Omega \subset \mathbb{R}^n$ *is an arbitrary open set, then* $f \in \mathring{W}_p^l(\Omega)$ *if and only if*

$$\lim_{r \to 0} \frac{1}{r^n} \int_{B(x,r) \cap \Omega} \left|\nabla_k f(y)\right| \mathrm{d}y = 0$$

for $(l-k)$*-quasi-every* $x \in \mathbb{R}^n \backslash \Omega$ *and for all integer* k, $0 \leq k \leq l - 1$.

Section 14.2. A set function similar to $\lambda_{2,2}^l$ was introduced and applied to the investigation of the uniqueness conditions for the solution of the first boundary value problem by Kondratiev [448, 449]. In these papers it is called the capacity $C^n_{l,d}$. The connection of $\lambda^l_{p,q}$ with the (p,l)-inner diameter was studied by the author [546].

Section 14.3. The results of this section (except Proposition 14.3.5 and those of Sect. 10.3.6) are borrowed from the author's paper [544].

Proposition 14.3.5 was published in the author's book [555]. This proposition together with Corollary 14.3.4 shows that the constant C in Corollary 14.3.4 satisfies the inequality

$$C \le c d^{k+n/q} \big[\mathrm{Cap}\big(e, \mathring{L}^{l-k}_p(Q_{2d})\big)\big]^{-1/p},$$

which was obtained earlier by a direct method in the paper by Hedberg [367] (compare with the stronger inequality (14.1.17) that was discussed in the comments to Sect. 14.1).

In connection with the content of the present chapter we mention the paper by Meyers [598] in which the inequality

$$\|u - Lu\|_{W^k_p(\Omega)} \le C \|\nabla_{k+1} u\|_{W^{l-k-1}_p(\Omega)}, \quad u \in W^l_p(\Omega),$$

where L is a projection mapping $W^l_p(\Omega) \to \mathscr{P}_k$ and Ω is a Lipschitz domain, is studied.

A certain family of "polynomial (p,l)-capacities" similar to $\mathrm{cap}(\mathfrak{C}_0(e), \Pi, \mathring{L}^l_p(Q_{2d}))$ was used by Bagby [61] in the study of approximation in L_p by solutions of elliptic equations. Nyström obtained a lower estimate for CAP_{l-1} by the Bessel (p,l)-capacity assuming that the set e preserves Markov's inequality [642].

We conclude this chapter by mentioning the problem of characterization of sets of uniqueness for Sobolev spaces, by considering the particular case of W^l_p, $p > 1$. By the set of uniqueness we mean a set $E \subset \mathbb{R}^n$ enjoying the following property. From the condition $u \in W^l_p$, $u(x) = 0$ for all $x \in \mathbb{R}^n \backslash E$, outside a suitable subset of zero (p,l)-capacity follows that $u = 0$.

A description of the set of uniqueness in W^l_p in the Theorem below was given by Hedberg [366]. The first result of this kind, for the space $W^{1/2}_2$ on the circle, was obtained in 1950 by Ahlfors and Beurling [31].

Theorem. *Let E be a Borel subset of $\mathbb{R}^n$ and let $c_{l,p}$ stand for the capacity generated by the space W^l_p. The following conditions are equivalent*:

(i) *E is the uniqueness set for W^l_p;*
(ii) *$c_{l,p}(G\backslash E) = c_{l,p}(G)$ for each open set G;*
(iii) *for almost all x*

$$\limsup_{\varrho \to 0} \varrho^{-n} c_{p,l}(\mathscr{B}_\varrho \backslash E) > 0.$$

If $lp > n$ then E is a uniqueness set for W^l_p if and only if it does not have inner points.

15

Embedding of the Space $\mathring{L}_p^l(\Omega)$ into Other Function Spaces

15.1 Preliminaries

If $n > pl$, $p > 1$, or $n \geq l$, $p = 1$, then by Sobolev's theorem the mapping $\mathring{L}_p^l(\Omega) \ni u \to u \in L_q(\Omega)$, where $q = pn/(n - pl)$, is continuous. We can easily show that this operator is one to one. In fact, let zero be the image of $u \in \mathring{L}_p^l(\Omega)$ in $L_q(\Omega)$ and let a sequence $\{u_m\}_{m\geq 1}$ of functions in $\mathscr{D}(\Omega)$ converge to u in $L_p^l(\Omega)$. Then for all multi-indices α with $|\alpha| = l$ and for all $\varphi \in \mathscr{D}(\Omega)$

$$\lim_{m\to\infty} \int_\Omega \varphi D^\alpha u_m \,\mathrm{d}x = \lim_{m\to\infty} (-1)^l \int_\Omega u_m D^\alpha \varphi \,\mathrm{d}x = 0.$$

Since the sequence $D^\alpha u_m$ converges in $L_p(\Omega)$, it tends to zero.

The above considerations show that each element of $\mathring{L}_p^l(\Omega)$ (for $n > p\,l$, $p > 1$ or $n \geq l$, $p = 1$) can be identified with a function in $L_q(\Omega)$ and the identity mapping

$$\mathring{L}_p^l(\Omega) \ni u \to u \in L_q(\Omega)$$

is one to one, linear, and continuous (i.e., it is a topological embedding).

If $n \leq lp$, $p > 1$ or $n < l$, $p = 1$ and if

$$\|u\|_{L_q(\Omega)} \leq C\|\nabla_l u\|_{L_p(\Omega)}$$

for some $q > 0$ with a constant C independent of $u \in \mathscr{D}(\Omega)$, then we arrive at the same conclusion. However, for these values of n, l, and p the above inequality for the norms does not hold for all domains and moreover, in general, $\mathring{L}_p^l(\Omega)$ is not necessarily embedded into the space of distributions $\mathscr{D}'(\Omega)$.

The theorems of the present chapter contain necessary and sufficient conditions for the topological embedding of $\mathring{L}_p^l(\Omega)$ into $\mathscr{D}'(\Omega)$, $L_q(\Omega, \mathrm{loc})$, and $L_q(\Omega)$ (Sects. 15.1–15.5). In Sect. 15.6 we obtain criteria for the compactness of the embedding $\mathring{L}_p^l(\Omega) \subset L_q(\Omega)$. In Sect. 15.7 results of the previous sections are applied to the study of the first boundary value problem for linear elliptic equations of order $2\,l$, and in Sect. 15.8 we consider applications to the Dirichlet and Neumann problems for quasilinear elliptic equations.

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15.2 Embedding $\overset{\circ}{L}{}^l_p(\Omega) \subset \mathscr{D}'(\Omega)$

The aim of the present section is to prove the following assertion.

Theorem. *The space* $\overset{\circ}{L}{}^l_p(\Omega)$ $(1 \le p < \infty)$ *is topologically embedded into* $\mathscr{D}'(\Omega)$ *if and only if any one of the following conditions holds*:

1. $n > pl$, $p > 1$ *or* $n \ge l$, $p = 1$;
2. $C\Omega$ *is not* $(p, n/p)$*-polar if* $n = p\,l$, $p > 1$;
3. $C\Omega$ *is not empty if* $n < p\,l$ *and* n/p *is not integer*;
4. $C\Omega$ *is not* $(p, n/p)$*-polar or it is not contained in an* $(n-1)$*-dimensional hyperplane if* $n < pl$ *and* n/p *is integer.*

Taking into account Theorem 13.2/2, we may read "*the set of zero inner* $(p, n/p)$*-capacity*" for "*the* $(p, n/p)$*-polar set*" in the last theorem.

15.2.1 Auxiliary Assertions

Lemma 1. *The space* $\overset{\circ}{L}{}^l_p(\Omega)$ *is topologically embedded into* $\mathscr{D}'(\Omega)$ *if and only if, for any* $\psi \in \mathscr{D}(\Omega)$, *the functional*

$$\mathscr{D}(\Omega) \ni u \to \int_\Omega u\psi \, \mathrm{d}x \equiv (u, \psi) \in \mathscr{D}'(\Omega) \tag{15.2.1}$$

is continuous with respect to the norm $\|\nabla_l u\|_{L_p(\Omega)}$.

Proof. 1. If $\overset{\circ}{L}{}^l_p \subset \mathscr{D}'(\Omega)$, then any sequence $u_m \in \mathscr{D}'(\Omega)$ that is a Cauchy sequence in the norm $\|\nabla_l u\|_{L_p(\Omega)}$ converges in $\mathscr{D}'(\Omega)$. Consequently, the functional (u, ψ) is continuous.

2. Let the functional (u, ψ) be continuous in the norm $\|\nabla_l u\|_{L_p(\Omega)}$. Then the mapping (15.2.1) can be continuously extended to $\overset{\circ}{L}{}^l_p(\Omega)$. It remains to show that the resulting mapping is one to one. Let u_m be a Cauchy sequence in $\overset{\circ}{L}{}^l_p(\Omega)$ that converges to zero in $\mathscr{D}'(\Omega)$. Obviously, for all $\varphi \in \mathscr{D}(\Omega)$ and all multi-indices α with $|\alpha| = l$ we have

$$\int_\Omega \varphi D^\alpha u_m \, \mathrm{d}x = (-1)^l \int_\Omega u_m D^\alpha \varphi \, \mathrm{d}x \to 0.$$

Since the sequence $D^\alpha u_m$ converges in $L_p(\Omega)$, it tends to zero. □

The just-proved lemma can be reformulated as follows.

Corollary. *The space* $\overset{\circ}{L}{}^l_p(\Omega)$ *is embedded into* $\mathscr{D}'(\Omega)$ *if and only if*

$$\big|(u, \psi)\big| \le K \|\nabla_l u\|_{L_p(\mathbb{R}^n)} \tag{15.2.2}$$

for any $\psi \in \mathscr{D}(\Omega)$ *and for all* $u \in \mathscr{D}(\Omega)$.

Lemma 2. *The space* $\overset{\circ}{L}{}^l_p$ *is embedded into* $\mathscr{D}'(\Omega)$ *if and only if for any* $\psi \in \mathscr{D}(\Omega)$ *there exists a distribution* $T \in \mathscr{D}'(\Omega)$ *supported in* $C\Omega$ *such that*

$$\bigl|(u, \psi - T)\bigr| \le K \|\nabla_l u\|_{L_p(\mathbb{R}^n)}, \quad u \in \mathscr{D}(\mathbb{R}^n). \tag{15.2.3}$$

Proof. Necessity. Let $\mathring{L}^l_p(\Omega) \subset \mathscr{D}'(\Omega)$. Then by the previous corollary inequality (15.2.2) is valid for all $u \in \mathscr{D}(\Omega)$. The space $\mathscr{D}(\Omega)$ can be identified with the subspace $\mathscr{D}(\mathbb{R}^n)$ by zero extension to $C\Omega$. By the Hahn–Banach theorem, functional (15.2.1) can be extended to a functional $u \to (u, s)$ on $\mathscr{D}(\mathbb{R}^n)$ that is continuous with respect to the norm $\|\nabla_l u\|_{L_p(\mathbb{R}^n)}$, i.e., to a functional satisfying

$$\bigl|(u, s)\bigr| \le K \|\nabla_l u\|_{L_p(\mathbb{R}^n)}, \tag{15.2.4}$$

where K does not depend on u. Since $s = \psi$ on Ω, the support of the distribution $T = \psi - s$ is contained in $C\Omega$. The estimates (15.2.3) and (15.2.4) are equivalent.

Sufficiency. Suppose (15.2.3) is valid. Then the functional $\mathscr{D}(\mathbb{R}^n) \ni u \to (u, s)$ with $s = \psi - T$ is continuous with respect to the norm $\|\nabla_l u\|_{L_p(\mathbb{R}^n)}$ on $\mathscr{D}(\mathbb{R}^n)$ and hence on $\mathscr{D}(\Omega)$ where it coincides with (15.2.1). It remains to refer to the Corollary. □

Lemma 3. *Let $n \le pl$ for $p > 1$ or $n < l$ for $p = 1$. If for any $\psi \in \mathscr{D}(\Omega)$ there exists a distribution $T \in W^{-l}_{p'}(\mathbb{R}^n)$ with compact support in $C\Omega$ such that*

$$\bigl(x^\beta, \psi - T\bigr) = 0 \tag{15.2.5}$$

for all multi-indices β with $|\beta| \le k = [l - n/p]$, then the space $\mathring{L}^l_p(\Omega)$ is embedded into $\mathscr{D}'(\Omega)$.

Proof. Let B_ϱ be the ball $\{x : |x| < \varrho\}$ containing the supports of ψ and T. We show that (15.2.3) is valid for all $u \in \mathscr{D}(\mathbb{R}^n)$. Indeed, for any polynomial P of a degree not higher than k we have

$$\bigl|(u, \psi - T)\bigr| = \bigl|(u - P, \psi - T)\bigr| = \bigl|\bigl((u - P)\eta, \psi - T\bigr)\bigr|,$$

where $\eta \in \mathscr{D}(B_{2\varrho})$, $\eta = 1$ in B_ϱ. Let $K = \|\psi - T\|_{W^{-l}_{p'}(\mathbb{R}^n)}$. Then

$$\bigl|(u, \psi - T)\bigr| \le K \bigl\|(u - P)\eta\bigr\|_{W^l_p(\mathbb{R}^n)} \le C\bigl(\|\nabla_l u\|_{L_p(\mathbb{R}^n)} + \|u - P\|_{L_p(B_{2\varrho})}\bigr).$$

Applying the inequality

$$\inf_{\mathscr{P}_k} \|u - P\|_{L_p(B_{2\varrho})} \le C \|\nabla_{k+1} u\|_{L_p(B_{2\varrho})},$$

where $\mathscr{P}_k$ is the set of all polynomials of a degree not higher than k, as well as the inequalities

$$\|\nabla_{k+1} u\|_{L_p(B_{2\varrho})} \le C \|\nabla_{k+1} u\|_{L_q(B_{2\varrho})} \le c \|\nabla_l u\|_{L_p(\mathbb{R}^n)},$$

where $q = pn[n - p(l - k - 1)]^{-1}$, we arrive at (15.2.3). The lemma is proved. □

15.2.2 Case $\Omega = \mathbb{R}^n$

Lemma. *Let $n \le lp$ for $p > 1$ or $n < l$ for $p = 1$ and let α be a multi-index of order $|\alpha| \le 1 - n/p$. Then there exists a sequence of functions $u_\nu \in \mathscr{D}(\mathbb{R}^n)$ such that $u_\nu \to x^\alpha$ in $\mathscr{D}'(\mathbb{R}^n)$ and $u_\nu \to 0$ in $\mathring{L}_p^l(\mathbb{R}^n)$.*

Proof. Let η be an infinitely differentiable function on $(0, \infty)$ that is equal to unity in a neighborhood of $[0, 1]$ and to zero in a neighborhood of $[2, \infty)$ and let $|\alpha| < l - n/p$. On the one hand, clearly, the sequence

$$u_\nu(x) = x^\alpha \eta\big(\nu^{-1}|x|\big)$$

converges to x^α in $\mathscr{D}'(\Omega)$. On the other hand,

$$\|\nabla_l u_\nu\|_{L_p(\mathbb{R}^n)} \le \text{const}\, \nu^{n/p+|\alpha|-l} \xrightarrow{\nu\to\infty} 0.$$

Let $|\alpha| = l - n/p$. We put

$$v_\nu(x) = \frac{1}{\log \nu} \log \frac{\nu^2}{|x|}$$

for $x \in B_{\nu^2} \backslash B_\nu = \{x : \nu^2 > |x| \ge \nu\}$ $(\nu > 2)$. By $\varphi(t)$ we denote a function in $C^\infty[0, 1]$ equal to zero near $t = 0$ and to unity near $t = 1$. Further, let $w_\nu(x) = \varphi[v_\nu(x)]$ for $x \in B_{\nu^2} \backslash B_\nu$, $w_\nu(x) = 1$ in B_ν and $w_\nu(x) = 0$ outside B_{ν^2}. On the one hand, it is clear that $u_\nu(x) = x^\alpha w_\nu(x)$ converges to x^α in $\mathscr{D}'(\mathbb{R}^n)$. On the other hand, since

$$\big|\nabla_j w_\nu(x)\big| \le \text{const}(\log \nu)^{-1}|x|^{-j}$$

for $j > 0$, we have

$$\begin{aligned}\|\nabla_l u_\nu\|_{L_p(\mathbb{R}^n)} &\le \text{const}(\log \nu)^{-1}\bigg(\int_{B_{\nu^2}\backslash B_\nu} |x|^{(|\alpha|-l)p}\, \mathrm{d}x\bigg)^{1/p} \\ &= \text{const}(\log \nu)^{1/p-1} \xrightarrow{\nu\to\infty} 0.\end{aligned}$$

□

Theorem. *The space $\mathring{L}_p^l(\mathbb{R}^n)$ is embedded into $\mathscr{D}'(\mathbb{R}^n)$ if and only if either $n > lp$, $p > 1$, or $n \ge l$, $p = 1$.*

The necessity follows immediately from the Lemma and the sufficiency results from the estimate

$$\|u\|_{L_{pn/(n-lp)}(\Omega)} \le c(n, l, p)\|\nabla_l u\|_{L_p(\Omega)}$$

and the proof of the one-to-one correspondence presented in Sect. 15.1. □

Corollary. *The space $\mathring{L}_p^l(\Omega)$ is embedded into $\mathscr{D}'(\Omega)$ for any open set Ω if and only if $n > lp$, $p > 1$ or $n \ge l$, $p = 1$.*

Proof. The necessity was proved in the preceding theorem. Extending any $u \in \mathscr{D}(\Omega)$ by zero to $C\Omega$, we obtain the embedding $\mathscr{D}(\Omega) \subset \mathscr{D}(\mathbb{R}^n)$ and hence the embedding $\mathscr{D}'(\mathbb{R}^n) \subset \mathscr{D}'(\Omega)$. Therefore,

$$\mathring{L}^l_p(\Omega) \subset \mathring{L}^l_p(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n) \subset \mathscr{D}'(\Omega).$$

Here and henceforth $\subset$ means a topological embedding. □

15.2.3 Case $n = p\,l$, $p > 1$

Lemma. *If a closed set E is not a (p,l)-polar set, then there exists a distribution $T \in W^{-l}_{p'}(\mathbb{R}^n)$ with compact support in E such that $(1,T) = 1$.*

Proof. Since E is not (p,l)-polar, there exists a distribution $S \in W^{-l}_{p'}(\mathbb{R}^n)$, $S \neq 0$ with support in E. Let the function $T = \varphi S$ satisfy $(\varphi, S) = 1$. Obviously, the distribution $T = \varphi S$ satisfies the requirements of the lemma. □

Theorem. *Let $n = lp$, $p > 1$. In order that $\mathring{L}^l_p(\Omega) \subset \mathscr{D}'(\Omega)$ it is necessary and sufficient that $C\Omega$ is not a $(p, n/p)$-polar set.*

Proof. Necessity. From Lemma 15.2.2 it follows that, if the function $\psi \in \mathscr{D}(\Omega)$ with $(1,\psi) \neq 0$ is arbitrary, then the inequality

$$\big|(u,\psi)\big| \le C\|\nabla_l u\|_{L_p(\mathbb{R}^n)}$$

cannot be valid for all $u \in \mathscr{D}(\mathbb{R}^n)$. Since $\mathring{L}^l_p(\Omega) \subset \mathscr{D}'(\Omega)$, there exists a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ such that supp $T \subset C\Omega$ and

$$\big|(u,\psi - T)\big| \le c\|\nabla_l u\|_{L_p(\mathbb{R}^n)}$$

for all $u \in \mathscr{D}(\mathbb{R}^n)$ (see Lemma 15.2.1/2). Obviously, $T \neq 0$. Moreover,

$$\big|(u,T)\big| \le c\|\nabla_l u\|_{L_p(\mathbb{R}^n)} + \big|(u,\psi)\big| \le c\|u\|_{W^l_p(\mathbb{R}^n)},$$

i.e., $T \in W^{-l}_{p'}(\mathbb{R}^n)$. Thus $C\Omega$ is not a $(p, n/p)$-polar set.

Sufficiency. Since $C\Omega$ is not a $(p,n/p)$-polar set, there exists a distribution $T_0 \in W^{-l}_{p'}(\mathbb{R}^n)$ with compact support in $C\Omega$ such that $(1, T_0) = 1$. Let $\psi \in \mathscr{D}(\Omega)$. We put $T = (\psi, 1)T_0$. Since $(1, \psi - T) = 0$, then, according to Lemma 15.2.1/3 (where $k = 0$), $\mathring{L}^l_p(\Omega) \subset \mathscr{D}'(\Omega)$. The theorem is proved. □

15.2.4 Case $n < p\,l$ and Noninteger n/p

Theorem. *If $n < pl$ and if n/p is noninteger then the condition $C\Omega \neq \varnothing$ is necessary and sufficient for the embedding $\mathring{L}^l_p(\Omega) \subset \mathscr{D}(\Omega)$.*

Proof. Necessity. If $C\Omega = \varnothing$, then by Theorem 15.2.2 the space $\mathring{L}_p^l(\Omega)$ is not embedded into $\mathscr{D}'(\Omega)$.

Sufficiency. Let $C\Omega \neq \varnothing$. We may assume $0 \in C\Omega$. We put

$$T = \sum_{|\alpha| \le k} (-1)^\alpha \frac{(\psi, x^\alpha)}{\alpha!} D^\alpha \delta(x),$$

where δ is the Dirac delta-function, D is the usual gradient, $k = [l - n/p]$, and $\psi \in \mathscr{D}(\Omega)$. Since $k < l - n/p$, we have $D^\alpha \delta \in W_{p'}^{-l}(\mathbb{R}^n)$ for $|\alpha| \le k$ and $T \in W_{p'}^{-l}(\mathbb{R}^n)$. Besides, obviously, $(x^\beta, \psi - T) = 0$ for $|\beta| \le k$. It remains to make use of Lemma 15.2.1/3. The theorem is proved. □

15.2.5 Case $n < pl$, $1 < p < \infty$, and Integer n/p

Theorem. *Let n/p be an integer, $n < pl$, $1 < p < \infty$. The space $\mathring{L}_p^l(\Omega)$ is embedded into $\mathscr{D}'(\Omega)$ if and only if $C\Omega$ is not a $(p, n/p)$-polar set and is not contained in a $(n-1)$-dimensional hyperplane.*

Proof. We put $k = l - n/p$.

Sufficiency. (a) Suppose the $C\Omega$ is not a $(p, n/p)$-polar set. Then by Lemma 15.2.3 there exists a distribution $T_0 \in W_{p'}^{k-l}(\mathbb{R}^n)$ with compact support in $C\Omega$ such that $(1, T_0) = 1$. We put

$$T = \sum_{|\alpha| \le k} a_\alpha (-1)^{|\alpha|} D^\alpha T_0.$$

Obviously, T is a distribution in $W_{p'}^{-l}(\mathbb{R}^n)$ with compact support in $C\Omega$.

It remains to show that given any $\psi \in \mathscr{D}(\Omega)$ we can find numbers a_α so that (15.2.5) holds for all multi-indices β with $|\beta| \le k$. In other words, the linear algebraic system

$$\sum_{|\alpha| \le k} a_\alpha \big(D^\alpha T_0, x^\beta\big) = \big(\psi, x^\beta\big), \quad |\beta| \le k,$$

must be solvable. Since $(D^\gamma T_0, 1) = 0$ for $|\gamma| > 0$, this system can be rewritten in the form

$$\sum_{\alpha \le \beta} a_\alpha \frac{\beta!}{(\beta - \alpha)!} \big(T_0, x^{\beta - \alpha}\big) = \big(\psi, x^\beta\big). \qquad (15.2.6)$$

The matrix of system (15.2.6) is triangular with nonzero elements on the main diagonal (it consists of the numbers $\beta!(T_0, 1) = \beta!$). Thus (15.2.6) is solvable.

(b) Suppose $C\Omega$ is not in an $(n-1)$-dimensional hyperplane. Suppose also that the points $0, a_1, \dots, a_n$ are situated in $C\Omega$ and are affinely independent. Further, let $\psi \in \mathscr{D}(\Omega)$. We introduce the distribution

$$T = P_0(-i\nabla)\delta(x) + \sum_{j=1}^{n} P_j(-i\nabla)\delta(x - a_j),$$

where $P_0, P_1, \dots, P_n$ are homogeneous polynomials of degrees not higher than $k-1$. Since $p(l-k+1) > n$, we have $T \in W_{p'}^{-l}(\mathbb{R}^n)$.

We choose polynomials $P_0, P_1, \dots, P_n$ so that (15.2.5) holds. Let $Q(\xi)$ and $R(\xi)$ be the sums of terms of a degree not higher than k in the Taylor expansions of the Fourier transforms $\hat{T}(\xi)$ and $\hat{\psi}(\xi)$, respectively. We denote the homogeneous part of $R(\xi)$ of degree k by $r(\xi)$. It is clear that

$$\hat{T}(\xi) = P_0(\xi) + \sum_{j=1}^{n} P_j(\xi) \exp\bigl(-i\langle a_j, \xi\rangle\bigr),$$

$$Q(\xi) = \sum_{j=0}^{n} P_j(\xi) - i\sum_{j=1}^{n} \langle a_j, \xi\rangle P_j(\xi).$$

Since the forms $\langle a_j, \xi\rangle$ are independent, we can choose $P_1, \dots, P_n$ so that

$$-i\sum_{j=1}^{n} \langle a_j, \xi\rangle P_j(\xi) = r(\xi).$$

We define the polynomial P_0 by $P_0 = -(P_1 + \cdots + P_n) + R - r$ (obviously, the degree of P_0 is less than k). Thus $Q(\xi) = R(\xi)$, which is equivalent to (15.2.5).

Necessity. Let $\mathring{L}_p^l(\Omega) \subset \mathscr{D}'(\Omega)$ and let $C\Omega \subset \mathbb{R}^{n-1}$ (for definiteness we put $\mathbb{R}^{n-1} = \{x : x_1 = 0\}$). We show that $C\Omega$ is not a $(p, n/p)$-polar set. By Lemma 15.2.1/2 for any $\psi \in \mathscr{D}(\Omega)$ there exists a distribution T with support in $C\Omega$ such that (15.2.3) holds. Since $C\Omega \subset \mathbb{R}^{n-1}$, it follows that

$$T = \sum_{j=0}^{l} \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^j \delta(x_1) \times S_j(x_2, \dots, x_n) = \sum_{j=0}^{l} T_j,$$

where $S_j \in \mathscr{D}'(\mathbb{R}^{n-1})$, supp $S_j \subset C\Omega$ (cf. L. Schwartz [695]). We show that $T_j \neq 0$ and $T_j \in W_{p'}^{-l}(\mathbb{R}^n)$. We have

$$\hat{T}(\xi) = \sum_{j=0}^{l} \xi_1^j \hat{S}_j(\xi_2, \dots, \xi_n) = \sum_{j=0}^{l} \hat{T}_j(\xi).$$

Let $\beta_0, \dots, \beta_l$ be distinct numbers in $(0, 1)$. According to the Lagrange interpolation formula there exist constants $\gamma_0, \dots, \gamma_l$ such that $a_k = \sum_{j=0}^{l} \gamma_j P(\beta_j)$ for any polynomial $P(t) = a_0 + a_1 t + \cdots + a_l t^l$. In particular, if $P(t) = t^k$, then

$$1 = \sum_{j=0}^{l} \gamma_j \beta_j^k. \tag{15.2.7}$$

For the polynomial

$$\hat{T}(t\xi_1, \xi_2, \dots, \xi_n) = \sum_{j=0}^{l} \xi_1^j \hat{S}_j(\xi_2, \dots, \xi_n) t^j,$$

we have

$$\xi_1^k \hat{S}_k(\xi_2, \dots, \xi_n) = \sum_{j=0}^{l} \gamma_j \hat{T}(\beta_j \xi_1, \xi_2, \dots, \xi_n).$$

Consequently,

$$\begin{aligned} T_k &= \left(\frac{1}{i} \frac{\partial}{\partial x_1} \right)^k \delta(x_1) \times S_k(x_2, \dots, x_n) \\ &= \sum_{j=0}^{l} \frac{\gamma_j}{\beta_j} T\left(\frac{x_1}{\beta_j}, x_2, \dots, x_n \right). \end{aligned}$$

We define the function

$$\psi_k = \sum_{j=0}^{l} \frac{\gamma_j}{\beta_j} \psi\left(\frac{x_1}{\beta_j}, x_2, \dots, x_n \right). \tag{15.2.8}$$

Inequality (15.2.3) for $\psi - T$ implies

$$\big|(u, \psi_k - T_k)\big| \le K \|\nabla_l u\|_{L_p(\mathbb{R}^n)} \tag{15.2.9}$$

for all $u \in \mathscr{D}(\mathbb{R}^n)$. We may assume that the function $\psi \in \mathscr{D}(\mathbb{R}^n)$ satisfies $(\psi, x_1^k) = 1$. This along with (15.2.7) and (15.2.8) yields $(\psi_k, x_1^k) = 1$. Suppose that $T_k = 0$. Then for all $u \in \mathscr{D}(\mathbb{R}^n)$

$$\big|(u, \psi_k)\big| \le K \|\nabla_l u\|_{L_p(\mathbb{R}^n)}.$$

Hence by this and Lemma 15.2.2 we obtain $(\psi_k, x_1^k) = 0$. Thus $T_k \neq 0$, or equivalently, $S_k \neq 0$. Since $\varphi \in \mathscr{D}(\mathbb{R}^n)$, (15.2.9) implies $T_k \in W_{p'}^{-l}(\mathbb{R}^n)$.

Next we show that $\delta(x_1) \times S_k \in W_{p'}^{k-l}(\mathbb{R}^n)$. It is well known (see, for instance, Stein [724], §4, Chap. 6) that for any collection of functions $g_j \in W_p^{l-j}(\mathbb{R}^n)$ $(j = 0, 1, \dots, l-1)$ there exists a function $\Phi \in W_p^l(\mathbb{R}^n)$ such that $\partial^j \Phi / \partial x_j = g_j$ for $x \in \mathbb{R}^{n-1}$ and

$$\|\Phi\|_{W_p^l(\mathbb{R}^n)} \le K \sum_{j=0}^{l-1} \|g_j\|_{W_p^{l-j}(\mathbb{R}^n)}.$$

Let $g_j = 0$ for $j \neq k$ and $g_k = u$, where u is an arbitrary function in $W_p^{l-k}(\mathbb{R}^n)$. Then

$$\big(u, \delta(x_1) \times S_k\big) = \left(\Phi, \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^k \delta(x_1) \times S_k\right) = (\Phi, T_k).$$

Since $T_k \in W_{p'}^{-l}(\mathbb{R}^n)$, applying the above assertion, we get

$$\big|\big(u, \delta(x_1) \times S_k\big)\big| \le K\|\Phi\|_{W_p^l(\mathbb{R}^n)} \le K\|u\|_{W_p^{l-k}(\mathbb{R}^n)}.$$

Thus $\delta(x_1)\times S_k \in W_{p'}^{-n/p}(\mathbb{R}^n)$ and $C\Omega$ is not a $(p, n/p)$-polar set. The theorem is proved. □

Thus, the proof of Theorem 15.2 is complete. □

15.3 Embedding $\mathring{L}_p^l(\Omega) \subset L_q(\Omega, \mathrm{loc})$

The following assertion shows that Theorem 15.2 contains the necessary and sufficient conditions for the embedding $\mathring{L}_p^l(\Omega) \subset L_q(\Omega, \mathrm{loc})$.

Theorem. *Let $n \le p\,l$ for $p > 1$ or $n < l$ for $p = 1$. If the space $\mathring{L}_p^l(\Omega)$ is embedded into $\mathscr{D}'(\Omega)$, then it is also embedded into $L_q(\Omega, \mathrm{loc})$ for $n = p\,l$, where q is any positive number, and into $C(\Omega)$ for $n < pl$.*

Proof. Let G be a domain in $\mathbb{R}^n$ with compact closure and smooth boundary $G \cap \Omega \ne \varnothing$. First we show that there exists a family of functions $\varphi_\alpha \in \mathscr{D}(G \cap \Omega)$, $|\alpha| = l - 1$, such that the matrix $\|(\varphi_\alpha, x^\beta)\|$ is not degenerate. Let φ be any function in $\mathscr{D}(G \cap \Omega)$ with $(\varphi, 1) \ne 0$ and let $\varphi_\alpha = D^\alpha \varphi$. Obviously, for $\alpha > \beta$ we have

$$\big(\varphi_\alpha, x^\beta\big) = (-1)^{|\alpha|}\big(\varphi, D^\alpha x^\beta\big) = 0,$$

and the matrix $\|(\varphi_\alpha, x^\beta)\|$ is triangular. Since the main diagonal terms are $(-1)^{|\alpha|}\alpha!(\varphi, 1) \ne 0$, the determinant is not zero and the existence of the functions φ_α follows.

Since $\mathring{L}_p^l(\Omega) \subset \mathscr{D}'(\Omega)$, by Corollary 15.2.1 we have

$$\big|(\varphi_\alpha, u)\big| \le K\|\nabla_l u\|_{L_p(\Omega)}, \tag{15.3.1}$$

where K is a constant independent of u. By the Sobolev embedding theorem

$$\|u\|_{L_q(G)} \le K\bigg(\|\nabla_l u\|_{L_p(G)} + \sum_{|\alpha|=l-1}\big|(\varphi_\alpha, u)\big|\bigg)$$

for all $u \in \mathscr{D}(\Omega)$, which together with (15.3.1) yields

$$\|u\|_{L_q(G)} \le K\|\nabla_l u\|_{L_p(\Omega)}.$$

In the same way as in Sect. 15.1 we can prove that the mapping $\mathring{L}_p^l(\Omega) \ni u \to u \in L_q(\Omega, \mathrm{loc})$ is one to one. The theorem is proved. □

Next we present two corollaries to Theorem 14.1.2 that complement the theorem of this section.

Corollary 1. *Let $n = p\,l$, $p > 1$, and let Q_d be a cube for which*

$$\mathrm{Cap}(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})) > 0.$$

Then, for all $u \in \mathscr{D}(\Omega)$,

$$\|u\|^p_{L_q(Q_d)} \le C \|\nabla_l u\|^p_{L_p(\Omega)}, \tag{15.3.2}$$

where $C \le c\, d^{np/q} [\mathrm{Cap}(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d}))]^{-1}$.

Proof. By Theorem 14.1.2,

$$\|u\|^p_{L_q(Q_d)} \le c\, d^{np/q} \big[\mathrm{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})\big)\big]^{-1} \| u \|^p_{p,l,Q_{2d}}. \tag{15.3.3}$$

Since

$$\|\nabla_j u\|_{L_p(Q_{2d})} \le c\, d^{\,l-j} \|\nabla_j u\|_{L_{q_j}(\Omega)} \le c\, d^{\,l-j} \|\nabla_l u\|_{L_q(\Omega)}$$

for $j \ge 1$, $q_j = pn[n - p(l-j)]^{-1}$, we have

$$\| u \|_{p,l,Q_{2d}} \le c\, \|\nabla_l u\|_{L_p(\Omega)},$$

which together with (15.3.3) yields (15.3.2). □

Corollary 2. 1. *If $p\,l > n$, n/p is an integer, and* $\mathrm{Cap}(\bar{Q}_d \backslash \Omega, \mathring{L}_p^{n/p}(Q_{2d})) > 0$, *then, for all $u \in \mathscr{D}(\Omega)$,*

$$\max_{\bar{Q}_d} |u|^p \le C \|\nabla_l u\|^p_{L_p(\Omega)}, \tag{15.3.4}$$

where $C \le c\, d^{lp-n} [\mathrm{Cap}(\bar{Q}_d \backslash \Omega, \mathring{L}_p^{n/p}(Q_{2d}))]^{-1}$.

2. *If $p\,l > n$, n/p is not an integer, and $\bar{Q}_d \backslash \Omega \neq \varnothing$, then (15.3.4) is valid for all $u \in \mathscr{D}(\Omega)$ with $C \le c\, d^{lp-n}$.*

Proof. 1. Let $kp = n$, $q > n$. Using the Sobolev theorem, we obtain

$$\max_{\bar{Q}_d} |u| \le c\, d^{\,l-k-1} \max_{\bar{Q}_d} |\nabla_{l-k-1} u| \le c\, d^{\,l-k-n/q} \|\nabla_{l-k} u\|_{L_q(Q_d)}. \tag{15.3.5}$$

By (15.3.2), with l replaced by k and u replaced by $\nabla_{l-k} u$, we have

$$\|\nabla_{l-k} u\|_{L_p(Q_d)} \le c\, d^{n/p} \big[\mathrm{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^k(Q_{2d})\big)\big]^{-1/p} \|\nabla_l u\|_{L_p(\Omega)}.$$

This and (15.3.5) imply (15.3.4).

2. Let j be an integer such that $l - p^{-1}n < j < l - p^{-1}n + 1$. Since $p(l-j) < n$, then

$$\|\nabla_j u\|_{L_q(\Omega)} \le C\|\nabla_l u\|_{L_p(\Omega)},$$

where $q = pn[n-(l-j)p]^{-1}$. The condition $p(l-j+1) > n$ is equivalent to $q > n$. Therefore

$$\max_{\bar{Q}_d} |\nabla_{j-1} u| \le c d^{1-q^{-1}n} \|\nabla_j u\|_{L_q(Q_d)} \le c d^{-p^{-1}+l-j+1} \|\nabla_l u\|_{L_p(\Omega)}.$$

Hence, from this and

$$\max_{\bar{Q}_d} |u| \le c\, d^{j-1} \max_{\bar{Q}_d} |\nabla_{j-1} u|$$

we obtain (15.3.4). The corollary is proved. □

15.4 Embedding $\mathring{L}_p^l(\Omega) \subset L_q(\Omega)$ (the Case $p \le q$)

In this section we find the necessary and sufficient conditions for the validity of the inequality

$$\|u\|_{L_q(\Omega)} \le C\|\nabla_l u\|_{L_p(\Omega)}, \tag{15.4.1}$$

where u is an arbitrary function in $\mathscr{D}(\Omega)$. The results we present here can be deduced (although only for $p > 1$) from the more general theorems proved in Section 16.2 where the space $\mathring{L}_p^l(\Omega, \nu)$ is studied. However, the separate exposition seems reasonable because of the importance of this particular case, the possibility of including $p = 1$ and the simpler statements.

15.4.1 A Condition in Terms of the (p, l)-Inner Diameter

If we put $d = \infty$ in the proofs of Theorems 14.2.3/1 and 14.2.3/2, then we arrive at the following theorem.

Theorem. *Let q satisfy any one of the conditions:*
(i) $q \in [p, np(n-p\,l)^{-1}]$ *if* $p \ge 1$, $n > pl$;
(ii) $q \in [p, \infty)$ *if* $p > 1$, $n = pl$;
(iii) $q \in [p, \infty]$ *if* $pl > n$, $p > 1$ *or* $l = n$, $p = 1$.
Then:
The inequality (15.4.1) *holds if and only if*
(α) $D_{p,l}(\Omega) < \infty$ *for* $n > pl$, $p \ge 1$ *or* $n = pl$, $p > 1$;
(β) $D(\Omega) < \infty$ *for* $n < pl$, $p > 1$ *or* $n = l$, $p = 1$.
The best constant C in (15.4.1) *satisfies*

$$C \sim \begin{cases} [D_{p,l}(\Omega)]^{l-n(p^{-1}-q^{-1})} & \textit{in case } (\alpha), \\ [D(\Omega)]^{l-n(p^{-1}-q^{-1})} & \textit{in case } (\beta). \end{cases} \tag{15.4.2}$$

15.4.2 A Condition in Terms of Capacity

By the Sobolev embedding theorem, the inequality (15.4.1) is valid for any set Ω provided $q = pn(n-pl)^{-l}$, $n > pl$, or $q = \infty$, $p = 1$, $l = n$. Therefore it remains to consider only the cases $q < pn(n-pl)^{-1}$ for $n = pl$ and $q \le \infty$ for $n < pl$, $p \ge 1$.

We present a necessary and sufficient condition for the validity of (15.4.1) stated in different terms and resulting from Theorem 14.1.2/1.

Theorem 1. *Let q be the same as in Theorem 15.4.1/1. The inequality (15.4.1) is valid if and only if*

$$\inf \operatorname{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})\big) > 0 \tag{15.4.3}$$

for some $d > 0$. Here the infimum is taken over all cubes Q_d with edge length d and with sides parallel to coordinate axes.

Proof. Sufficiency. We construct the cubic grid with edge length d. Suppose

$$d^{lp-n} \operatorname{Cap}(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})) \ge \varkappa > 0$$

for any cube of the grid. By Theorem 14.1.2, we have

$$\|u\|^p_{L_p(Q_d)} \le c\varkappa^{-1} \sum_{k=1}^{l} d^{pk} \|\nabla_k u\|^p_{L_p(Q_d)}.$$

Summing over all cubes of the grid, we obtain

$$\|u\|^p_{L_p(\Omega)} \le c\varkappa^{-1} \sum_{k=1}^{l} d^{pk} \|\nabla_k u\|^p_{L_p(\Omega)}.$$

To estimate the right-hand side we use the inequality

$$\|\nabla_k u\|_{L_p(\Omega)} \le c \|\nabla_l u\|^{k/l}_{L_p(\Omega)} \|u\|^{1-k/l}_{L_p(\Omega)}. \tag{15.4.4}$$

Then

$$\|u\|^p_{L_p(\Omega)} \le c \sum_{k=1}^{l} \big(d^{pl} \varkappa^{-l/k} \|\nabla_l u\|^p_{L_p(\Omega)}\big)^{k/l} \|u\|^{p(1-k/l)}_{L_p(\Omega)}. \tag{15.4.5}$$

Since

$$\operatorname{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})\big) \le c d^{n-pl},$$

it follows that $\varkappa \le c$ and hence the right-hand side in (15.4.5) does not exceed

$$2^{-1} \|u\|^p_{L_p(\Omega)} + c d^{pl} \varkappa^{-l} \|\nabla_l u\|^p_{L_p(\Omega)}.$$

Thus

$$\|u\|^p_{L_p(\Omega)} \le c d^{pl} \varkappa^{-l} \|\nabla_l u\|^p_{L_p(\Omega)}. \tag{15.4.6}$$

We now prove (15.4.2) for $q > p$. By Theorem 14.1.2

$$\|u\|^p_{L_p(Q_d)} \le c d^{np/q} \varkappa^{-1} \sum_{j=1}^{l} d^{pj-n} \int_{Q_d} |\nabla_j u|^p \, dx.$$

Summing over all cubes Q_d and using the inequality

$$\Big(\sum_i a_i\Big)^\varepsilon \le \sum_i a_i^\varepsilon, \quad a_i \ge 0, \ 0 < \varepsilon \le 1,$$

we obtain

$$\|u\|^p_{L_q(\Omega)} \le c\, \varkappa^{-1} d^{np/q} \sum_{j=1}^{l} d^{pj-n} \int_{\Omega} |\nabla_j u|^p \, dx.$$

Now (15.4.4) yields

$$\|u\|^p_{L_q(\Omega)} \le c\, \varkappa^{-1} d^{np/q} \sum_{j=1}^{l} d^{pj-n} \|\nabla_l u\|^{pj/l}_{L_p(\Omega)} \|u\|^{p(1-j/l)}_{L_p(\Omega)}.$$

Applying (15.4.6), we finally obtain

$$\|u\|^p_{L_p(\Omega)} \le c \varkappa^{-l} d^{np/q+pl-n} \|\nabla_l u\|^p_{L_q(\Omega)}. \tag{15.4.7}$$

Necessity. Let Q_d be an arbitrary cube with edge length d and let u be an arbitrary function in $C^\infty(\bar{Q}_d)$ such that $\operatorname{dist}(\bar{Q}_d \backslash \Omega, \operatorname{supp} u) > 0$. Replacing u to be a function η in (15.4.1), where $\eta \in \mathscr{D}(Q_d)$, $\eta = 1$ on $Q_{d/2}$, $|\nabla_j \eta| \le c\, d^{-j}$, we obtain

$$\|u\|_{L_p(Q_{d/2})} \le C \|\nabla_l (u\eta)\|_{L_p(Q_d)}.$$

Hence

$$\|u\|_{L_q(Q_{d/2})} \le c\, C \sum_{j=0}^{l} d^{j-l} \|\nabla_j u\|_{L_p(Q_d)}.$$

Applying the well-known inequality

$$\|u\|_{L_p(Q_d)} \le c d \|\nabla u\|_{L_p(Q_d)} + c \|u\|_{L_p(Q_{d/2})}$$

and the Hölder inequality we obtain

$$\|u\|_{L_q(Q_{d/2})} \le c\, C \sum_{j=1}^{l} d^{j-l} \|\nabla_j u\|_{L_p(Q_d)} + c\, C d^{-l+n(q-p)/pq} \|u\|_{L_q(Q_{d/2})}.$$

Thus for d, so large that $2c\, C\, d^{-l+n(q-p)/pq} < 1$, we have

$$\|u\|_{L_q(Q_{d/2})} \le c\, C \sum_{j=1}^{l} d^{j-1} \|\nabla_j u\|_{L_p(Q_d)}.$$

Since

$$\|u\|_{L_q(Q_d)} \le c d^{n(p-q)/pq+1} \|\nabla u\|_{L_p(Q_d)} + c\|u\|_{L_q(Q_{d/2})},$$

we obtain

$$\|u\|_{L_q(Q_d)} \le c\big(C + d^{l+n(p-q)/pq}\big) \sum_{j=1}^{l} d^{j-l} \|\nabla_j u\|_{L_p(Q_d)}.$$

By the second part of Theorem 14.1.2, we have either

$$\mathrm{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})\big) \ge \gamma d^{n-pl},$$

where γ satisfies inequality (14.1.2), or

$$\mathrm{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})\big) \ge c d^{np/q} \big(C + d^{l+n(p-q)/pq}\big)^{-p}.$$

The theorem is proved. □

Theorem 1 can be rephrased as follows.

Theorem 2. *One of the following conditions is necessary and sufficient for the validity of* (15.4.1)*,*

1. *For some* $d > 0$*,*

$$\inf_{Q_d} \mathrm{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(\mathbb{R}^n)\big) > 0 \quad \text{if } n > pl.$$

2. *For some* $d > 0$*,*

$$\inf_{Q_d} \mathrm{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})\big) > 0 \quad \text{if } n = pl, p > 1.$$

3. *The domain* Ω *does not contain arbitrarily large cubes if* (i) $n < pl$, $p > 1$, (ii) $n \le l$, $p = 1$, (iii) $n = pl$ *and* $C\Omega$ *is connected.*

Proof. Part 1 follows from Theorem 1 and Proposition 13.1.1/3, part 2 is contained in Theorem 1. For $n < pl$ the condition

$$\mathrm{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})\big) \ge c d^{n-pl}$$

is equivalent to $\bar{Q}_d \backslash \Omega \ne \varnothing$, which proves part 3 for $n \ne pl$, $p > 1$.

Let $n = pl$, $p > 1$ and let $C\Omega$ be connected. If Ω contains arbitrarily large cubes, then obviously, (15.4.3) does not hold. Suppose that cubes of an arbitrary size cannot be placed in Ω and that the number d_0 is so large that any cube Q_d with $d > d_0$ has a nonempty intersection with Ω. Then in $\mathbb{R}^n \backslash \Omega$ there exists a continuum that contains points in Q_d and in $\mathbb{R}^n \backslash Q_{2d}$. It remains to apply Proposition 13.1.2/1. This concludes the proof. □

We present an example of an unbounded domain for which the hypotheses of the Theorem are valid.

Example. In each cube $\mathscr{Q}_1^{(i)}$, $i \geq 1$, of the coordinate grid with edge length 1 we select a closed subset e_i lying in an s-dimensional plane $s > n - pl \geq 0$. Let $\inf m_s e_i \geq \text{const} > 0$.

We denote the complement of the set $\bigcup_i e_i$ by Ω. By Proposition 13.1.1/4 we have

$$\operatorname{Cap}\big(e_i, \mathring{L}_p^l(\mathscr{Q}_2^{(i)})\big) \geq \text{const} > 0$$

for any cube $\mathscr{Q}_2^{(i)}$. Hence (15.4.1) holds for Ω.

15.5 Embedding $\mathring{L}_p^l(\Omega) \subset L_q(\Omega)$ (the Case $p > q \geq 1$)

15.5.1 Definitions and Lemmas

We continue to study inequality (15.4.1). Here we obtain a necessary and sufficient condition for $q \in [1, p)$. Contrary to the case $q \geq p$ considered previously, this condition does not depend on q. It means that up to a "small error" the set Ω is the union of cubes $\mathscr{Q}^{(i)}$ with a finite multiplicity of the intersection and with edge lengths $\{d_i\}_{i \geq 1}$ satisfying

$$\sum_{i=1}^{\infty} d_i^{n+lpq/(p-q)} < \infty. \tag{15.5.1}$$

The "smallness" is described in terms of the capacity

$$\operatorname{Cap}_{l-1}\big(e, \mathring{L}_p^l(Q_{2d})\big) = \inf_{\Pi \in \mathbb{P}_{l-1}} \inf_{\{u\}} \int_{Q_{2d}} |\nabla_l u|^p \,\mathrm{d}x$$

introduced in 14.3.4. We recall that $\mathbb{P}_{l-1}$ is the set of polynomials of a degree not higher than $l-1$ normalized by the equality

$$d^{-n} \int_{Q_d} |\Pi|^p \,\mathrm{d}x = 1,$$

and $\{u\}$ is the set of functions in $\mathring{L}_p^l(Q_{2d})$, equal to polynomials $\Pi \in \mathbb{P}_{l-1}$ in a neighborhood of a compactum $e \subset \bar{Q}_d$.

We shall use the following assertion that is a particular case ($k = l-1$) of Corollary 14.3.4.

Lemma 1. 1. *Let e be a compact subset of the cube $\bar{Q}_d$ with*

$$\operatorname{Cap}_{l-1}\big(e, \mathring{L}_p^l(Q_{2d})\big) > 0.$$

Then

$$\|u\|_{L_q(Q_d)} \leq A\|\nabla_l u\|_{L_p(Q_d)}$$

for any function $u \in C^\infty(Q_d)$ *that vanishes in a neighborhood of* e. *Here* $1 \leq q \leq pn(n-pl)^{-1}$ *for* $n > pl$; $1 \leq q < \infty$ *for* $n = pl$, $1 \leq q \leq \infty$ *for* $n < pl$, *and*

$$A \leq cd^{n/q}\big[\mathrm{Cap}_{l-1}\big(e, \mathring{L}_p^l(Q_{2d})\big)\big]^{-1/p}.$$

2. *If* (15.4.1) *holds for any function* $u \in C^\infty(\bar{Q}_d)$ *that vanishes in a neighborhood of the compactum* $e \subset \bar{Q}_d$ *and if*

$$\mathrm{Cap}_{l-1}\big(e, \mathring{L}_p^l(Q_{2d})\big) \leq c_0 d^{n-pl},$$

where c_0 *is a small enough constant, then*

$$A \geq cd^{n/p}\big[\mathrm{Cap}_{l-1}\big(e, \mathring{L}_p^l(Q_{2d})\big)\big]^{-1/p}.$$

Definition 1. Let γ be a sufficiently small constant depending only on n, p, l. A compact subset e of a cube $\bar{Q}_d$ is said to be $(p, l, l-1)$-*negligible* if

$$\mathrm{Cap}_{l-1}\big(e, \mathring{L}_p^l(Q_{2d})\big) < \gamma d^{n-pl}. \tag{15.5.2}$$

Otherwise e is called $(p, l, l-1)$-*essential*.

The collection of $(p, l, l-1)$-negligible subsets of the cube $\bar{Q}_d$ is denoted by $\mathscr{N}_{l-1}(Q_d)$.

Lemma 2. *Let* $1 \leq q \leq p$ *and let* (15.4.1) *hold for any* $u \in C_0^\infty(\Omega)$. *Then there exists a constant* c *that depends only on* n, p, q, *and* l, *and is such that* $Q_d \backslash \Omega$ *is a* $(p, l, l-1)$-*essential subset of* Q_d *for any cube* Q_d *with edge length* d *satisfying*

$$d \geq cC^{pq/(n(p-q)+lpq)}.$$

Proof. Consider a function $u \in C^\infty(\bar{Q}_d)$ with $\mathrm{dist}(\mathrm{supp}\, u, \bar{Q}_d \backslash \Omega) > 0$. Let $\eta \in C_0^\infty(Q_1)$, $\eta = 1$ on $Q_{1/2}$ and let $\eta_d = \eta(x/d)$.

The insertion of the function $u\eta_d$ into (15.4.1) yields

$$\begin{aligned}\|u\|_{L_q(Q_{d/2})} \leq \|u\eta_d\|_{L_q(Q_d)} &\leq C\|\nabla_l(u\eta_d)\|_{L_p(Q_d)} \\ &\leq cC\big(\|\nabla_l u\|_{L_p(Q_d)} + d^{-l}\|u\|_{L_p(Q_d)}\big).\end{aligned}$$

Hence from the inequality

$$\|u\|_{L_p(Q_d)} \leq cd^l\|\nabla_l u\|_{L_p(Q_d)} + cd^{n(q-p)/pq}\|u\|_{L_q(Q_{d/2})} \tag{15.5.3}$$

(see, for instance, Lemma 1.1.11) we obtain

$$\|u\|_{L_q(Q_{d/2})} \leq c_0 C\big(\|\nabla_l u\|_{L_p(Q_d)} + d^{n(q-p)/pq-l}\|u\|_{L_q(Q_{d/2})}\big).$$

Consequently,

$$\|u\|_{L_q(Q_{d/2})} \leq 2c_0 C \|\nabla_l u\|_{L_p(Q_d)}$$

for $2c_0 C d^{n(q-p)/pq-l} < 1$. On the other hand, (15.5.3) and the Hölder inequality imply

$$\begin{aligned}\|u\|_{L_p(Q_d)} &\leq d^{n(p-q)/pq} \|u\|_{L_p(Q_d)} \\ &\leq c\big(d^{n(p-q)/pq+l} \|\nabla_l u\|_{L_p(Q_d)} + \|u\|_{L_q(Q_{d/2})}\big).\end{aligned}$$

Therefore

$$\|u\|_{L_q(Q_d)} \leq c_1 \big(d^{n(p-q)/pq+l} + C\big) \|\nabla_l u\|_{L_p(Q_d)}$$

for $2c_0\, C d^{n(p-q)/pq-l} < 1$. Suppose, in addition, that $d^{n(p-q)/pq+l} > C$. Then

$$\|u\|_{L_q(Q_d)} \leq 2c_1 d^{n(p-q)/pq+l} \|\nabla_l u\|_{L_p(Q_d)}. \tag{15.5.4}$$

If $\bar{Q}_d \backslash \Omega \notin \mathscr{N}_{l-1}(Q_d)$, we have nothing to prove. Otherwise, by part 2 of Lemma 1, inequality (15.5.4) implies

$$c d^{n/q} \big[\mathrm{Cap}_{l-1}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})\big)\big]^{-1/p} \leq 2c_1 d^{n(p-q)/pq+l},$$

or equivalently,

$$\mathrm{Cap}_{l-1}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})\big) \geq \left(\frac{c}{2c_1}\right)^p d^{n-pl}.$$

We may always assume that $\gamma \leq (c/(2\,c_1))^p$. Therefore $\bar{Q}_d \backslash \Omega \notin \mathscr{N}_{l-1}(Q_d)$. The proof is complete. □

Definition 2. The cube $Q_D = Q_D(x)$ with center $x \in \Omega$ is called *critical* if

$$D = \sup\big\{d : \bar{Q}_d \backslash \Omega \in \mathscr{N}_{l-1}(Q_d)\big\}.$$

Lemma 2 implies the following assertion.

Corollary. *Let $1 \leq q < p$ and let (15.4.1) hold for any $u \in C_0^\infty(\Omega)$. Then for any $x \in \Omega$ there exists a critical cube $Q_D(x)$.*

In what follows $\mathscr{Q}^{(i)}$ is an open cube with edges parallel to the coordinate axes and with edge lengths d_i, $i = 1, 2, \dots$. Further, let $c\mathscr{Q}^{(i)}$ be a concentric cube with edge length cd_i and with sides parallel to those of the cube $\mathscr{Q}^{(i)}$. Let μ denote a positive constant that depends only on n.

Definition 3. A covering $\{\mathscr{Q}^{(i)}\}_{i\geq 1}$ of a set Ω is in *the class* $C_{l,p,q}$ if:

1. $\overline{\mathscr{P}^{(i)}} \backslash \Omega \in \mathscr{N}_{l-1}(\mathscr{P}^{(i)})$, where $\mathscr{P}^{(i)} = \mu \mathscr{Q}^{(i)}$;
2. $\mathscr{P}^{(i)} \cap \mathscr{P}^{(j)} = \varnothing$ for $i \neq j$;
3. the multiplicity of the covering $\{\mathscr{Q}^{(i)}\}$ does not exceed a constant that depends only on n;
4. $\overline{\mathscr{Q}^{(i)}} \backslash \Omega \notin \mathscr{N}_{l-1}(\mathscr{Q}^{(i)})$;
5. the series (15.5.1) converges.

15.5.2 Basic Result

Theorem. *Let $1 \le q < p$. The inequality (15.4.1) holds for all $u \in \mathring{L}_p^l(\Omega)$ if and only if there exists a covering of Ω in $C_{l,p,q}$.*

Proof. Necessity. Let $x \in \Omega$, $Q_d = Q_d(x)$ and let D be the edge length of the critical cube centered at x. We put

$$g(d) = d^{pl-n} \operatorname{Cap}_{l-1}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_p^l(Q_{2d})\big).$$

Let d denote a number in the interval $[D, 2D]$ such that $g(d) \ge \gamma$ where γ is the constant in (15.5.2). Further, let M be the collection of cubes $\{Q_d\}_{x\in\Omega}$.

We show that series (15.5.1) converges for any sequence $\{\mathscr{Q}^{(i)}\}$ of disjoint cubes in M. By Lemma 15.5.1/1, given an arbitrary number $\varepsilon_i > 0$, there exists a function $v_i \in C^\infty(\overline{\mathscr{Q}^{(i)}})$ with $\operatorname{dist}(\operatorname{supp} v_i, \overline{\mathscr{Q}^{(i)}} \backslash \Omega) > 0$ such that

$$\int_{\mathscr{Q}^{(i)}} |\nabla_l v_i|^p \, dx \le \big[c d_i^{-n} \operatorname{Cap}_{l-1}\big(\overline{\mathscr{Q}^{(i)}} \backslash \Omega, \mathring{L}_p^l\big(2\mathscr{Q}^{(i)}\big)\big) + \varepsilon_i\big] \int_{\mathscr{Q}^{(i)}} |v_i|^p \, dx.$$

We assume that $\varepsilon_i = \gamma d_i^{-pl}$. Then, by (15.5.2),

$$\int_{\mathscr{Q}^{(i)}} |\nabla_l v_i|^p \, dx \le c\gamma d_i^{-pl} \int_{\mathscr{Q}^{(i)}} |v_i|^p \, dx. \tag{15.5.5}$$

Estimating the right-hand side by the inequality

$$\|v_i\|_{L_p(\mathscr{Q}^{(i)})} \le c\, d_i^l \|\nabla_l v_i\|_{L_p(\mathscr{Q}^{(i)})} + c d_i^{n(p-q)/pq} \|v_i\|_{L_p(\frac{1}{2}\mathscr{Q}^{(i)})} \tag{15.5.6}$$

(see Lemma 1.1.11) and using the smallness of the constant γ, we arrive at the estimate

$$\int_{\mathscr{Q}^{(i)}} |\nabla_l v_i|^p \, dx \le c d_i^{n(p-q)/q - pl} \|v_i\|^p_{L_q(\frac{1}{2}\mathscr{Q}^{(i)})}. \tag{15.5.7}$$

Let $\zeta_i \in \mathscr{D}(\mathscr{Q}^{(i)})$, $\zeta_i = 1$ in $\frac{1}{2}\mathscr{Q}^{(i)}$, $|\nabla_k \zeta_i| \le c d_i^{-k}, k = 1, 2, \ldots$. We introduce the function $u_i = \zeta_i v_i$. It is clear that

$$\begin{aligned}\|\nabla_l u_i\|_{L_p(\mathscr{Q}^{(i)})} &\le c \sum_{k=0}^{l} d^{k-l} \|\nabla_k v_i\|_{L_p(\mathscr{Q}^{(i)})} \\ &\le c\big(\|\nabla_l v_i\|_{L_p(\mathscr{Q}^{(i)})} + d^{-l} \|v_i\|_{L_p(\mathscr{Q}^{(i)})}\big).\end{aligned}$$

Applying (15.5.6), we obtain

$$\|\nabla_l u_i\|_{L_p(\mathscr{Q}^{(i)})} \le c \|\nabla_l v_i\|_{L_p(\mathscr{Q}^{(i)})} + c d^{n(q-p)/pq - l} \|v_i\|_{L_p(\frac{1}{2}\mathscr{Q}^{(i)})}.$$

This and (15.5.7) imply

$$\|\nabla_l u_i\|_{L_p(\mathscr{Q}^{(i)})} \le c d_i^{n(q-p)/pq - l} \|u_i\|_{L_q(\mathscr{Q}^{(i)})}. \tag{15.5.8}$$

By the hypothesis of the theorem, (15.4.1) holds for any $u \in \mathscr{D}(\Omega)$. We normalize u_i by

$$\|u_i\|_{L_q(\mathscr{Q}^{(i)})} = d_i^{n/q-pl/(q-p)} \tag{15.5.9}$$

and put $u = \sum_{i=1}^N u_i$ into (15.4.1). Then

$$\left(\sum_{i=1}^N \|u_i\|^q_{L_q(\mathscr{Q}^{(i)})}\right)^{p/q} = \left(\int_\Omega |u|^q \,\mathrm{d}x\right)^{p/q} \leq C^p \sum_{i=1}^N \int_{\mathscr{Q}^{(i)}} |\nabla_l u_i|^p \,\mathrm{d}x.$$

By (15.5.8) we have

$$\left(\sum_{i=1}^N \|u_i\|^q_{L_q(\mathscr{Q}^{(i)})}\right)^{p/q} \leq cC^p \sum_{i=1}^N d_i^{n(q-p)/q-pl} \|u_i\|^p_{L_q(\mathscr{Q}^{(i)})},$$

which together with (15.5.9) yields

$$\left(\sum_{i=1}^N d_i^{n-p\,lq/(q-p)}\right)^{(p-q)/q} \leq cC^p.$$

Thus the series (15.5.1) converges.

According to Theorem 1.2.1, there exists a sequence of cubes $\{\mathscr{Q}^{(i)}\}_{i\geq 1} \subset M$ that forms a covering of Ω of finite multiplicity with $\mu\mathscr{Q}^{(i)} \cap \mu\mathscr{Q}^{(j)} = \varnothing$, $i \neq j$. The convergence of series (15.5.1) was proved previously (the arguments should be applied to the sequence of mutually disjoint cubes $\mu\mathscr{Q}^{(i)}$). Therefore $\{\mathscr{Q}^{(i)}\}_{i\geq 1}$ is a covering in the class $C_{l,p,q}$.

Sufficiency. Let $u \in C_0^\infty(\Omega)$ and let $\{\mathscr{Q}^{(i)}\}_{i\geq 1}$ be a covering of Ω in the class $C_{l,p,q}$. Obviously,

$$\int_\Omega |u|^q \,\mathrm{d}x \leq \sum_{i\geq 1} \lambda_i^{q/p} \lambda_i^{-q/p} \int_{\mathscr{Q}^{(i)}} |u|^q \,\mathrm{d}x,$$

where $\lambda_i = d_i^{-pn/q} \operatorname{Cap}_{l-1}(\overline{\mathscr{Q}^{(i)}} \backslash \Omega, \mathring{L}_p^l(2\mathscr{Q}^{(i)}))$. Applying the Hölder inequality, we obtain

$$\int_\Omega |u|^q \,\mathrm{d}x \leq \left(\sum_{i\geq 1} \lambda_i^{q/(p-q)}\right)^{(p-q)/q} \left[\sum_{i\geq 1} \lambda_i \left(\int_{\mathscr{Q}^{(i)}} |u|^q \,\mathrm{d}x\right)^{p/q}\right]^{q/p}.$$

By Lemma 15.5.1/1,

$$c\,\lambda_i \left(\int_{\mathscr{Q}^{(i)}} |u|^q \,\mathrm{d}x\right)^{p/q} \leq \int_{\mathscr{Q}^{(i)}} |\nabla_l u|^p \,\mathrm{d}x.$$

This implies

$$\int_\Omega |u|^q \, \mathrm{d}x \le c\Big(\sum_{i\ge 1} \lambda_i^{q/(p-q)}\Big)^{(p-q)/p} \Big(\int_\Omega |\nabla_l u|^p \, \mathrm{d}x\Big)^{q/p},$$

and since $\overline{\mathscr{Q}^{(i)}}\backslash\Omega \notin \mathscr{N}(\mathscr{Q}^{(i)})$, we conclude that

$$\|u\|_{L_q(\Omega)} \le c\Big(\sum_{i\ge 1} d_i^{n+lpq/(p-q)}\Big)^{(p-q)/pq} \|\nabla_l u\|_{L_p(\Omega)}. \tag{15.5.10}$$

This completes the proof. □

In the proof of necessity we incidentally obtained the following necessary condition for the validity (15.4.1).

Proposition. *Let $\{\mathscr{Q}^{(i)}\}_{i\ge 1}$ be a sequence of disjoint cubes in Ω. Then the divergence of the series (15.5.1) is necessary for the validity of (15.4.1) with $q < p$.*

15.5.3 Embedding $\mathring{L}_p^l(\Omega) \subset L_q(\Omega)$ for an "Infinite Funnel"

Example. Consider the domain

$$\Omega = \big\{x = (x', x_n) : x' = (x_1, \dots, x_{n-1}),\ x_n > 0,\ |x'| < \varphi(x_n)\big\},$$

where φ is a bounded decreasing function.

We shall show that (15.4.1) with $p > q \ge 1$ holds if and only if

$$\int_0^\infty \big[\varphi(t)\big]^\alpha \, \mathrm{d}t < \infty, \tag{15.5.11}$$

where $\alpha = n - 1 + lp\,q/(p-q)$.

Proof. Let $\{a_i\}$ and $\{b_i\}$ be two number sequences defined as follows:

$$a_0 = 0; \qquad a_{i+1} - a_i = 2\varphi(a_i), \quad i \ge 1,$$
$$b_0 = 0; \qquad b_{i+1} - b_i = \frac{2}{\sqrt{n-1}}\varphi(b_i), \quad i \ge 1.$$

Clearly a_i, $b_i \to 0$ as $i \to \infty$, and the differences $a_{i+1} - a_i$, $b_{i+1} - b_i$ decrease.

Define two sequences of cubes as follows:

$$\mathscr{Q}_{\rm ext}^{(i)} = \big\{a_i < x_n < a_{i+1},\ 2|x_\nu| < a_{i+1} - a_i,\ 1 \le v \le n-1\big\},$$
$$\mathscr{Q}_{\rm int}^{(i)} = \big\{b_i < x_n < b_{i+1},\ 2|x_\nu| < b_{i+1} - b_i,\ 1 \le \nu \le n-1\big\}$$

(see Fig. 37). The cubes $\mathscr{Q}_{\rm ext}^{(i)}$ cover Ω. All $(n-1)$-dimensional faces of $\mathscr{Q}_{\rm ext}^{(i)}$ except for two of them are contained in $\mathbb{R}^n\backslash\Omega$ and

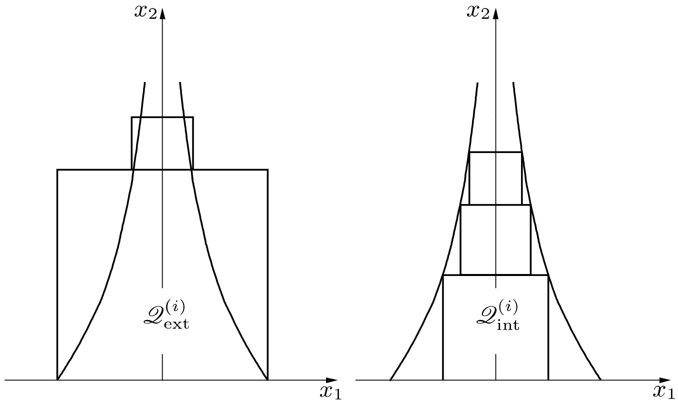

Fig. 37.

$$\operatorname{Cap}\big(\overline{\mathscr{Q}_{\text{ext}}^{(i)}}\backslash\Omega, \mathring{L}_p^1\big(2\mathscr{Q}_{\text{ext}}^{(i)}\big)\big) \geq c(b_{i+1} - b_i)^{n-p}.$$

This along with Proposition 14.3.5 implies that $\mathscr{Q}_{\text{ext}}^{(i)}\backslash\Omega$ is a $(p, l, l-1)$-essential subset of $\mathscr{Q}_{\text{ext}}^{(i)}$.

We suppose that the integral (15.5.11) converges and show the convergence of the series (15.5.1). In fact,

$$\begin{aligned}\sum_{i=0}^{\infty}(a_{i+1} - a_i)^{\alpha+1} &\leq \sum_{i=1}^{\infty}(a_{i+1} - a_i)^{\alpha}(a_i - a_{i-1}) + a_1^{\alpha+1}\\ &= \sum_{i-1}^{\infty}\big[\varphi(a_i)\big]^{\alpha}(a_i - a_{i-1}) + \big[\varphi(0)\big]^{\alpha+1}.\end{aligned}$$

Since φ does not increase, we have

$$\big[\varphi(a_i)\big]^{\alpha}(a_i - a_{i-1}) \leq \int_{a_{i-1}}^{a_i}\big[\varphi(t)\big]^{\alpha}\,\mathrm{d}t.$$

Hence

$$\sum_{i=0}^{\infty}(a_{i+1} - a_i)^{\alpha+1} \leq \big[\varphi(0)\big]^{\alpha+1} + \int_0^{\infty} 2\big[\varphi(t)\big]^{\alpha}\,\mathrm{d}t,$$

and the sufficient condition in Theorem 15.5.2 follows.

We prove the necessity of (15.5.11). Suppose that

$$\int_0^{\infty}\big[\varphi(t)\big]^{\alpha}\,\mathrm{d}t = \infty,$$

and let the series (15.5.1) converge for any sequence of disjoint cubes in Ω. By monotonicity of φ we have

$$\sum_{i=1}^{\infty}(b_i - b_{i-1})^{\alpha+1} \geq \sum_{i=1}^{\infty}(b_i - b_{i-1})^{\alpha}(b_{i+1} - b_i)$$
$$= \sum_{i=1}^{\infty}\big[\varphi(b_i)\big]^{\alpha}(b_{i+1} - b_i) \geq \sum_{i=1}^{\infty}\int_{b_i}^{b_{i+1}}\big[\varphi(t)\big]^{\alpha}\,\mathrm{d}t.$$

Consequently the series (15.5.1) diverges for the sequence of cubes $\mathscr{Q}^{(i)}_{\text{int}}$. Thus we arrived at a contradiction. It remains to apply Proposition 15.5.2.

15.6 Compactness of the Embedding $\mathring{L}^l_p(\Omega) \subset L_q(\Omega)$

In this section we obtain the necessary and sufficient conditions for the compactness of the embedding operator of $\mathring{L}^l_p(\Omega)$ into $L_q(\Omega)$ with p, $q \geq 1$.

15.6.1 Case $p \leq q$

Theorem. *The set*

$$\mathfrak{F} = \big\{u \in \mathscr{D}(\Omega) : \|\nabla_l u\|_{L_p(\Omega)} \leq 1\big\},$$

is relatively compact in $L_q(\Omega)$ if and only if one of the following conditions holds:

1. *For any $d > 0$*

$$\lim_{\varrho\to\infty}\ \inf_{Q_d\subset\mathbb{R}^n\backslash B_\varrho} \operatorname{Cap}\big(\bar{Q}_d\backslash\Omega, \mathring{L}^l_p(\Omega)\big) > kd^{n-pl} \tag{15.6.1}$$

if $n > p\,l$. Here k is a positive constant that is independent of d.

2. *For any $d > 0$*

$$\lim_{\varrho\to\infty}\ \inf_{Q_d\subset\mathbb{R}^n\backslash B_\varrho} \operatorname{Cap}\big(\bar{Q}_d\backslash\Omega, \mathring{L}^l_p(Q_{2d})\big) > k \tag{15.6.2}$$

if $n = pl$.

3. *The set Ω does not contain an infinite sequence of disjoint cubes if $pl > n$ or $pl = n$ and the set $\mathbb{R}^n\backslash\Omega$ is connected.*

Proof. Sufficiency. First we note that by Propositions 13.1.1/3, 13.1.2/1, and Corollary 13.1.1 the conditions of the theorem are equivalent to

$$\lim_{\varrho\to\infty}\ \inf_{Q_d\subset\mathbb{R}^n\backslash B_\varrho} \operatorname{Cap}\big(\bar{Q}_d\backslash\Omega, \mathring{L}^l_p(Q_{2d})\big) > kd^{n-pl} \tag{15.6.3}$$

(cf. the proof of Theorem 15.4.2/1). This and the first part of Theorem 15.4.2/1 imply (15.4.1) for all $u \in \mathscr{D}(\Omega)$ and hence the boundedness of $\mathfrak{F}$ in $W^l_p(\Omega)$.

Since any bounded subset of $W^l_p(\Omega)$ is compact in $L_q(\Omega\backslash B_\varrho)$, it suffices to prove the inequality

$$\|u\|_{L_q(\Omega\backslash B_\varrho)} \leq \varepsilon \|u\|_{W_p^l(\Omega)} \tag{15.6.4}$$

with arbitrary positive ε and sufficiently large ϱ.

Let $\eta \in C^\infty(\mathbb{R}^n)$, $\eta = 0$ in $B_{1/2}$, $\eta = 1$ outside B_1, and $\eta_\varrho(x) = \eta(x/\varrho)$. We denote by d a small number that depends on ε and which will be specified later. By (15.6.3), there exists a sufficiently large radius $\varrho(d)$ such that

$$d^{pl-n}\,\mathrm{Cap}\big(\bar{Q}_d\backslash\Omega, \mathring{L}_p^l(Q_{2d})\big) > kd^{n-pl}$$

for $\varrho > \varrho(d)$ and for all cubes $Q_d \subset \mathbb{R}^n\backslash B_{\varrho/4}$. Hence, by (15.4.1) with Ω replaced by $\Omega\backslash B_{\varrho/2}$, we have

$$\|u\eta_\varrho\|_{L_q(\Omega\backslash B_{\varrho/2})} \leq ck^{-l} d^{lp-n+np/q} \|\nabla_l(u\eta_\varrho)\|_{L_p(\Omega\backslash B_{\varrho/2})}.$$

We could choose d beforehand to satisfy

$$c\,k^{-l} d^{lp-n+np/q} < \varepsilon.$$

Then

$$\|u\|_{L_q(\Omega\backslash B_\varrho)} \leq c\varepsilon \sum_{j=0}^{l} \varrho^{j-l} \|\nabla_j u\|_{L_p(\Omega)} \leq c\varepsilon \|u\|_{W_p^l(\Omega)},$$

which completes the proof of the first part of the theorem.

Necessity. Let ε be any positive number. Suppose the set $\mathfrak{F}$ is relatively compact in $L_q(\Omega)$. Then there exists a number $\varrho = \varrho(\varepsilon)$ so large that

$$\|u\|_{L_q(\Omega)} \leq \varepsilon \|\nabla_l u\|_{L_p(\Omega)}$$

for all $u \in \mathscr{D}(\Omega\backslash\bar{B}_\varrho)$. Let Q_d denote any cube with edge length d situated outside the ball B_ϱ. In the proof of the second part of Theorem 15.4.2/1 it was shown that either

$$\mathrm{Cap}\big(\bar{Q}_d\backslash\Omega, \mathring{L}_p^l(Q_{2d})\big) \geq \gamma d^{n-pl},$$

where γ is a constant satisfying (14.1.2), or

$$\mathrm{Cap}\big(\bar{Q}_d\backslash\Omega, \mathring{L}_p^l(Q_{2d})\big) \geq cd^{np/q}\big(\varepsilon + d^{l+n(p-q)/pq}\big)^{-p}.$$

The theorem is proved. □

15.6.2 Case $p > q$

The following assertion shows that the embedding operator of $\mathring{L}_p^l(\Omega)$ into $L_q(\Omega)$ is compact and continuous simultaneously for $p > q \geq 1$.

Theorem. *The set*

$$\mathfrak{F} = \big\{u \in \mathscr{D}(\Omega) : \|\nabla_l u\|_{L_p(\Omega)} \leq 1\big\}$$

is relatively compact in $L_q(\Omega)$, $1 \le q < p$ *if and only if there exists a covering of* Ω *in the class* $C_{l,p,q}$.

Proof. The necessity follows immediately from Theorem 15.5.2. We prove the sufficiency. Let $\{\mathscr{Q}^{(i)}\}_{i \ge 1}$ be a covering of Ω in the class $C_{l,p,q}$, let d_i be the edge length of $\mathscr{Q}^{(i)}$, let ε be a positive number, and let N be an integer so large that

$$\sum_{i \ge N+1} d_i^{n+lpq/(p-q)} < \varepsilon^{pq/(p-q)}. \tag{15.6.5}$$

We denote the radius of a ball $B_\varrho = \{x : |x| < \varrho\}$ such that $B_{\varrho/4}$ contains the cubes $\mathscr{Q}^{(1)}, \dots, \mathscr{Q}^{(N)}$ by ϱ. By (15.5.10) we obtain

$$\|u\eta_\varrho\|_{L_q(\Omega \backslash B_{\varrho/2})} \le c \Big(\sum_{i \ge N+1} d_i^{n+lpq/(p-q)} \Big)^{(p-q)/pq} \big\|\nabla_l(u\eta_\varrho)\big\|_{L_p(\Omega \backslash B_{\varrho/2})},$$

where η_ϱ is the same function as in the proof of Theorem 15.6.1. This and (15.6.5) immediately imply that

$$\|u\|_{L_q(\Omega \backslash B_\varrho)} \le \varepsilon \|\nabla_l u\|_{L_p(\Omega)} + c(\varrho)\|u\|_{W_p^{l-1}(\Omega \cap (B_\varrho \backslash B_{\varrho/2}))}.$$

Now the result follows from the compactness of the embedding of $\mathring{L}_p^l(\Omega) \cap L_q(\Omega)$ into $W_p^{l-1}(\Omega \cap (B_\varrho \backslash B_{\varrho/2}))$. $\square$

15.7 Application to the Dirichlet Problem for a Strongly Elliptic Operator

Let l be a positive integer and let i, j be multi-indices of orders $|i|, |j| \le l$. Let a_{ij} be bounded measurable functions in Ω such that $a_{ij} = \overline{a_{ji}}$ for any pair (i, j). Suppose

$$\sum_{|i|=|j|=l} a_{ij}(x)\zeta_i\bar{\zeta}_j \ge \gamma \sum_{|j|=l} |\zeta_j|^2, \quad \gamma = \text{const} > 0, \tag{15.7.1}$$

for all complex numbers ζ_i, $|i| = l$, and for almost all $x \in \Omega$.

We define the quadratic form

$$\mathfrak{A}(T, T) = \int_\Omega \sum_{|i|=|j|=l} a_{ij} D^i T \overline{D^j T}\, \mathrm{d}x$$

on the space $L_2^l(\Omega)$. Obviously, the seminorms $\mathfrak{A}(T, T)^{1/2}$ and $\|\nabla_l T\|_{L_2(\Omega)}$ are equivalent.

In the following we apply the results of the previous sections to the study of the Dirichlet problem for the operator

$$Au = (-1)^l \sum_{|i|=|j|=l} D^j\big(a_{ij} D^i u\big).$$

15.7.1 Dirichlet Problem with Nonhomogeneous Boundary Data

Lemma. *Let $\mathring{L}_2^l(\Omega)$ be a subspace of $\mathscr{D}'(\Omega)$. Then any function $T \in L_2^l(\Omega)$ can be expressed in the form*

$$T = u + h, \tag{15.7.2}$$

where $u \in \mathring{L}_2^l(\Omega)$, $h \in L_2^l(\Omega)$ and $Ah = 0$ (in the sense of distributions).

Proof. We equip $\mathring{L}_2^l(\Omega)$ with the norm $[\mathfrak{A}(u, u)]^{1/2}$. Let $T = u_i + h_i$ ($i = 1, 2$) be two decompositions of the form (15.7.2). Since $A(h_1 - h_2) = 0$ and $(u_1 - u_2) \in \mathring{L}_2^l(\Omega)$, it follows that $\mathfrak{A}(u_1 - u_2, h_1 - h_2) = 0$. Consequently, $\mathfrak{A}(u_1 - u_2, u_1 - u_2) = 0$ and $u_1 = u_2$. The uniqueness of the representation (15.7.2) is proved.

The space $L_2^l(\Omega)$ becomes the Hilbert space provided we equip it with any of the inner products

$$\mathfrak{A}_N(T, G) = \mathfrak{A}(T, G) + N^{-1}(T, G)_{L_2(\omega)}, \quad N = 1, 2, \dots,$$

where ω is a nonempty open bounded set, $\bar{\omega} \subset \Omega$. Let u_N denote the projection of the function $T \in L_2^l(\Omega)$ onto $\mathring{L}_2^l(\Omega)$ in the space $L_2^l(\Omega)$ with the norm $[\mathfrak{A}_N(G, G)]^{1/2}$ (by hypothesis, $\mathring{L}_2^l(\Omega)$ is a subspace of $L_2^l(\Omega)$). Then, for any $\varphi \in \mathring{L}_2^l(\Omega)$,

$$\mathfrak{A}_N(T - u_N, \varphi) = 0. \tag{15.7.3}$$

In Sect. 15.3 we noted that the embedding $\mathring{L}_2^l(\Omega) \subset \mathscr{D}'(\Omega)$ implies the embedding $\mathring{L}_2^l(\Omega) \subset L_2(\Omega, \text{loc})$, an, hence, the estimate

$$\|u_N\|^2_{L_2(\omega)} \le C\mathfrak{A}(u_N, u_N),$$

where C is a constant that is independent of u_N. This and the obvious inequality $\mathfrak{A}(u_N, u_N) \le \mathfrak{A}_N(T, T)$ show that the sequence u_N converges weakly in $\mathring{L}_2^l(\Omega)$ and in $L_2(\Omega)$ to some $u \in \mathring{L}_2^l(\Omega)$. Passing to the limit in (15.7.3) we obtain that $h = T - u$ satisfies $\mathfrak{A}(h, \varphi) = 0$, where φ is any function in $\mathring{L}_2^l(\Omega)$. The lemma is proved. □

The representation (15.7.2) enables one to find the solution of the equation $Ah = 0$ which "has the same boundary values as T along with its derivatives of order up to $l-1$," i.e., to solve the Dirichlet problem for the equation $Ah = 0$. Therefore, the conditions for the embedding $\mathring{L}_2^l(\Omega) \subset \mathscr{D}'(\Omega)$ in Theorem 15.2 imply criteria for the solvability of the Dirichlet problem formulated in terms of the $(2, l)$-capacity. Namely, we have the following statement.

Theorem. *For any function $T \in L_2^l(\Omega)$ to be represented in the form (15.7.2) it is necessary and sufficient that any one of the following conditions be valid:* 1. $n > 2l$; 2. $C\Omega \neq \varnothing$ *for odd n, $n < 2l$;* 3. $C\Omega$ *is a set of the positive $(2, n/2)$-capacity for $n = 2l$; and* 4. $C\Omega$ *is not contained in an $(n-1)$-dimensional hyperplane or is a set of positive $(2, n/2)$-capacity for even $n < 2l$.*

15.7.2 Dirichlet Problem with Homogeneous Boundary Data

The results of Sects. 15.4 and 15.5 give conditions for the unique solvability in $\mathring{L}_2^l(\Omega)$ of the first boundary value problem for the equation $Au = f$ with $f \in L_r(\Omega)$.

We first formulate the problem. Let f be a given function in $L_{q'}(\Omega)$, $q' = q(q-1)^{-1}$, $1 < q \le \infty$. We require a distribution $T \in \mathring{L}_2^l(\Omega)$ that satisfies $AT = f$.

The following fact is well known.

Lemma. *The above Dirichlet problem is solvable for any $f \in L_{q'}(\Omega)$ if and only if*

$$\|u\|_{L_q(\Omega)} \le C\|\nabla_l u\|_{L_2(\Omega)} \tag{15.7.4}$$

for all $u \in \mathscr{D}(\Omega)$.

Proof. Sufficiency. Since

$$\big|(f,u)\big| \le \|f\|_{L_{q'}(\Omega)}\|u\|_{L_q(\Omega)} \le C\|f\|_{L_{q'}(\Omega)}\|\nabla_l u\|_{L_2(\Omega)}$$

for all $u \in \mathscr{D}(\Omega)$, the functional (f,u) defined on the linear set $\mathscr{D}(\Omega)$, which is dense in $L_2^l(\Omega)$, is bounded in $\mathring{L}_2^l(\Omega)$. Hence, by the Riesz theorem, there exists $T \in \mathring{L}_2^l(\Omega)$ such that

$$(f,u) = \mathfrak{A}(T,u)$$

for all $u \in \mathscr{D}(\Omega)$. This is equivalent to $AT = f$.

Necessity. Any $u \in \mathscr{D}(\Omega)$ with $\|u\|_{L_2^l(\Omega)} = 1$ generates the functional (v,f), defined on $L_{q'}(\Omega)$. Since to any $f \in L_{q'}(\Omega)$ there corresponds a solution Tf of the Dirichlet problem,

$$\big|(v,f)\big| \le \|\nabla_l Tf\|_{L_2(\Omega)}.$$

Consequently, the functionals (v,f) are bounded for any $f \in L_{q'}(\Omega)$. Hence the norms of (v,f) are totally bounded, which is equivalent to (15.7.4). The lemma is proved. □

Inequality (15.7.4) cannot be valid for all u with the same constant if either $q \ge 2n(n-2\,l)^{-1}$, $n > 2\,l$, or if $q = \infty$, $n = 2\,l$. On the other hand, for $q = 2n(n-2\,l)^{-1}$, $n > 2\,l$, (15.7.4) holds for an arbitrary domain Ω. The other cases were studied in Sects. 15.4 and 15.5. For $q \ge 2$, Theorem 15.4.2/1 together with the preceding lemma leads to the following statement.

Theorem 1. *The Dirichlet problem in question is solvable in $\mathring{L}_2^l(\Omega)$ for all $f \in L_{q'}(\Omega)$ $(2 \ge q' > 2n/(n+2l)$, $n \ge 2l$ and for $2 \ge q' \ge 1$, $n < 2l)$ if and only if one of the following conditions is valid:*

1. *There exists a constant $d > 0$ such that*

$$\inf_{Q_d} \operatorname{Cap}\big(\bar{Q}_d\backslash\Omega, \mathring{L}_2^l\big) > 0 \quad \textit{for } n > 2l.$$

2. *There exists a constant* $d > 0$ *such that*

$$\inf_{Q_d} \operatorname{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_2^l(Q_{2d})\big) > 0 \quad \textit{for } n = 2l.$$

3. *The domain* Ω *does not contain arbitrarily large cubes if* $n < 2l$ *or if* $n = 2l$ *and* $\mathbb{R}^n \backslash \Omega$ *is connected.*

For $q < 2$, Theorem 15.5.2 and the Lemma imply the following theorem.

Theorem 2. *The Dirichlet problem for the equation* $AT = f$ *is solvable in* $\mathring{L}_2^l(\Omega)$ *for all* $f \in L_{q'}(\Omega)$, $1 < q < 2$, *if and only if there exists a covering of the set* Ω *belonging to the class* $C_{l,2,q}$ *and having a finite multiplicity.*

15.7.3 Discreteness of the Spectrum of the Dirichlet Problem

The quadratic form $\mathfrak{A}(u,u)$ generates a selfadjoint operator A in $L_2(\Omega)$. By the well-known Rellich theorem, a necessary and sufficient condition for the discreteness of the spectrum of this operator is the compactness of the embedding $\mathring{L}_2^l(\Omega) \subset L_2(\Omega)$. So Theorem 15.6.1 implies the following criterion for the discreteness of the spectrum of A stated in terms of the $(2,l)$-capacity.

Theorem. *The spectrum of the operator* A *is discrete if and only if one of the following conditions is valid:*

1. *For any constant* $d > 0$

$$\lim_{\varrho \to \infty} \inf_{Q_d \subset \mathbb{R}^n \backslash B_\varrho} \operatorname{Cap}\big(\bar{Q}_d \backslash \Omega, L_2^l\big) > k d^{n-2l}$$

if $n > 2l$.

Here and in what follows k *is a positive number which does not exceed* d.

2. *For any* $d > 0$

$$\lim_{\varrho \to \infty} \inf_{Q_d \subset \mathbb{R}^n \backslash B_\varrho} \operatorname{Cap}\big(\bar{Q}_d \backslash \Omega, \mathring{L}_2^l(Q_{2d})\big) > k$$

if $n = 2l$.

3. *The domain* Ω *does not contain an infinite sequence of disjoint congruent cubes if* $n < 2l$, *or if* $n = 2l$ *and* $\mathbb{R}^n \backslash \Omega$ *is connected.*

15.7.4 Dirichlet Problem for a Nonselfadjoint Operator

Consider the quadratic form

$$\mathfrak{B}(u,u) = \int_\Omega \sum_{|i|,|j| \le l} a_{ij}(x) D^i u \overline{D^j u}\, dx,$$

where i, j are n-dimensional multi-indices.

We state the Dirichlet problem with homogeneous boundary data for the operator

$$Bu = \sum_{|i|,|j|\le l} (-1)^{|j|} D^j (a_{ij}(x) D^i u)$$

in the following way. Let f be a continuous functional on $\mathring{W}_2^l(\Omega)$. We require an element in $\mathring{W}_2^l(\Omega)$ such that

$$\mathfrak{B}(u,\varphi) = (\varphi, f), \tag{15.7.5}$$

where the function $\varphi \in \mathring{W}_2^l(\Omega)$ is arbitrary.

The next assertion is a particular case of a well-known theorem of Hilbert space theory (see, for instance, Lions and Magenes [500], Chap. 2, §9.1).

Lemma. *If*

$$\|\varphi\|^2_{W_2^l(\Omega)} \le C|\mathfrak{B}(\varphi,\varphi)|$$

for all $\varphi \in \mathring{W}_2^l(\Omega)$, then the Dirichlet problem (15.7.5) *is uniquely solvable.*

Let Γ denote a positive constant such that

$$\operatorname{Re} \sum_{\substack{|i|,|j|\le l,\\ |i|+|j|<2l}} a_{ij}\zeta_i\overline{\zeta_j} \ge -\Gamma \Big(\sum_{|i|\le l} |\zeta_i|^2 \Big)^{1/2} \Big(\sum_{|j|<l} |\zeta_j|^2 \Big)^{1/2}$$

for all complex numbers ζ_i, $|i| \le l$, and introduce the set function

$$\lambda_\Omega = \inf\big\{ \|\nabla_l u\|^2_{L_2(\Omega)} : \|u\|_{L_2(\Omega)} = 1, u \in \mathring{L}_2^l(\Omega) \big\}.$$

Theorem. *Let*

$$\begin{aligned} D_{2,l}(\Omega) &< c_0\gamma/\Gamma \quad \textit{if } n \ge 2l, \\ D(\Omega) &< c_0\gamma/\Gamma \quad \textit{if } n < 2l, \end{aligned}$$

where $D_{2,l}(\Omega)$ is the $(2,l)$ inner diameter of Ω, $D(\Omega)$ is the inner diameter of Ω, γ is the constant in (15.7.1)*, and c_0 is a constant that depends only on n, l. Then the Dirichlet problem* (15.7.5) *is uniquely solvable.*

Proof. Obviously,

$$\mathfrak{A}(u,u) - \operatorname{Re}\mathfrak{B}(u,u) \le \Gamma \Big(\sum_{k=0}^{l} \|\nabla_k u\|^2_{L_2(\Omega)} \Big)^{1/2} \Big(\sum_{k=0}^{l-1} \|\nabla_k u\|^2_{L_2(\Omega)} \Big)^{1/2}.$$

Using the inequality

$$\|\nabla_k u\|_{L_2(\Omega)} \le c\|\nabla_l u\|^{k/l}_{L_2(\Omega)} \|u\|^{1-k/l}_{L_2(\Omega)}, \quad u \in \mathring{L}_2^l(\Omega),$$

we obtain

$$\mathfrak{A}(u,u) - \operatorname{Re}\mathfrak{B}(u,u) \le \frac{\gamma}{2}\|\nabla_l u\|^2_{L_2(\Omega)} + \frac{c\Gamma^{2l}}{\gamma^{2l-1}}\|u\|^2_{L_2(\Omega)}.$$

Consequently,

$$\operatorname{Re}\mathfrak{B}(u,u) \ge \frac{\gamma}{4}\|\nabla u\|^2_{L_2(\Omega)} + \left(\lambda_\Omega \frac{\gamma}{4} - \frac{c\Gamma^{2l}}{\gamma^{2l-1}}\right)\|u\|^2_{L_2(\Omega)}. \tag{15.7.6}$$

It remains to note that by Theorem 15.4.1

$$\lambda_\Omega \sim \begin{cases} D_{2,l}(\Omega)^{-2l} & \text{if } n \ge 2l, \\ D(\Omega)^{-2l} & \text{if } n < 2l. \end{cases}$$

The theorem is proved.

Thus *the Dirichlet problem* (15.7.5) *is uniquely solvable for domains with small* $(2,l)$ *inner diameter for* $n \ge 2l$ *or with small inner diameter for* $n < 2l$.

15.8 Applications to the Theory of Quasilinear Elliptic Equations

In the present section we apply capacitary criteria for integral inequalities obtained in the previous and present chapters to the Dirichlet and Neumann problems for quasilinear elliptic equations of the type treated by Leray and Lions in [488]. We show in Sect. 15.8.1 that the inequality (15.4.1) is necessary and sufficient for the solvability of the Dirichlet problem with zero boundary data. It is assumed that the right-hand side of the equation is integrable with some power. Combined with the results of Sect. 15.4, this criterion provides explicit conditions for the solvability of the boundary value problem given in terms of the (p,l)-capacity. The corresponding result for linear elliptic equations was obtained in Sect. 15.7.

In Sect. 15.8.3 we prove the uniqueness theorem for the bounded solution of the Dirichlet problem vanishing outside certain compact subsets of the boundary with zero (p,l)-capacity. A similar result concerning the Neumann problem for quasilinear second-order equations is derived in Sect. 15.8.4.

Let Ω be an unbounded open subset of $\mathbb{R}^n$. Here we use the notations $\|u\|_p = \|u\|_{L_p(\Omega)}$ and $Q_d = \{x : 2|x_i| < d,\ i = 1,\dots,n\}$. When the domain of the integration is not indicated, it is assumed to be Ω. In this section we suppose that $1 < p < \infty$.

15.8.1 Solvability of the Dirichlet Problem for Quasilinear Equations in Unbounded Domains

Consider the equation

$$Au \equiv (-1)^l D^\alpha\big(a_\alpha(x, \nabla_l u)\big) = f(x), \quad x \in \Omega, \tag{15.8.1}$$

where f is integrable in Ω, α is a multi-index of order l, and $D^\alpha = \partial^l/\partial x_1^{\alpha_1}, \dots, \partial x_n^{\alpha_n}$.

Suppose that the functions a_α are continuous in all variables, but x for almost all $x \in \Omega$ and measurable with respect to x for all other variables. Besides, we assume that for any vector $v = \{v_\alpha\}$

$$a_\alpha(x, v)v_\alpha \geq |v|^p, \quad \sum_\alpha \big|a_\alpha(x, v)\big| \leq \lambda |v|^{p-1} \tag{15.8.2}$$

with some $p > 1$. Further, we impose the "monotonicity condition"

$$\big[a_\alpha(x, v) - a_\alpha(x, w)\big](v_\alpha - w_\alpha) > 0 \tag{15.8.3}$$

for $w \neq v$.

We show that the Dirichlet problem for (15.8.1) is solvable for all $f \in L_{q'}(\Omega)$ if and only if

$$\|v\|_q \leq C\|\nabla_l v\|_p \tag{15.8.4}$$

with $q \geq 1$ holds for all $v \in C_0^\infty(\Omega)$.

We say that $u \in \mathring{L}_p^l(\Omega) \cap L(\Omega, \mathrm{loc})$ is a solution of the Dirichlet problem

$$Au = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{15.8.5}$$

if

$$\int a_\alpha(x, \nabla_l u) D^\alpha \varphi \, \mathrm{d}x = \int f\varphi \, \mathrm{d}x \tag{15.8.6}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Lemma 1. *Suppose that* (15.8.4) *for some $q \geq 1$ and for all $v \in C_0^\infty(\Omega)$. Then, for any $f \in L_{q'}(\Omega)$ there exists a unique solution of* (15.8.5).

Proof. Let $\{\Omega_k\}$ be an expanding sequence of bounded open sets such that $\bar{\Omega}_k \subset \Omega_{k+1}$ and $\bigcup_k \Omega_k = \Omega$. We introduce a sequence $\{u_k\}$ of solutions of the problem

$$Au_k = 0 \quad \text{in } \Omega_k, \qquad u_k = 0 \quad \text{on } \partial\Omega_k. \tag{15.8.7}$$

Such solutions exist by the Leray–Lions theorem (see [488]). Since $u_k \in \mathring{L}_p^l(\Omega_k)$, we have

$$\int a_\alpha(x, \nabla_l u_k) D^\alpha u_k \, \mathrm{d}x = \int f u_k \, \mathrm{d}x \tag{15.8.8}$$

with u_k extended by zero outside Ω_k. Hence

$$\|\nabla_l u_k\|_p^{p-1} \leq C\|f\|_{q'}, \qquad \|u_k\|_q^{p-1} \leq C^p\|f\|_{q'}. \tag{15.8.9}$$

Let $\{v_k\}$ be a subsequence of $\{u_k\}$ weakly converging both in $\mathring{L}_p^l(\Omega)$ and in $L_q(\Omega)$. If u is the weak limit of $\{v_k\}$, it follows that

$$\lim_{k\to\infty}\int a_\alpha(x,\nabla_l u)\big(D^\alpha v_k - D^\alpha u\big)\,\mathrm{d}x = 0. \tag{15.8.10}$$

By (15.8.8),

$$\int a_\alpha(x,\nabla_l v_k)D^\alpha v_k\,\mathrm{d}x \to \int fu\,\mathrm{d}x. \tag{15.8.11}$$

Let $w_m \in C_0^\infty(\Omega)$, $w_m \to u$ in $L_p^l(\Omega)$. Since $\operatorname{supp} w_m \subset \Omega_k$ for a fixed m and sufficiently large k, we obtain

$$\int a_\alpha(x,\nabla_l v_k)D^\alpha u\,\mathrm{d}x = \int a_\alpha(x,\nabla_l v_k)D^\alpha(u-w_m)\,\mathrm{d}x + \int f\omega_m\,\mathrm{d}x.$$

This and (15.8.8) imply

$$\begin{aligned}\limsup_{k\to\infty}&\left|\int a_\alpha(x,\nabla_l v_k)D^\alpha u\,\mathrm{d}x - \int fu\,\mathrm{d}x\right|\\ &\le c\lambda\limsup_{k\to\infty}\|\nabla_l v_k\|_p^{p-1}\big\|\nabla_l(u-w_m)\big\|_p + \left|\int f(u-w_m)\mathrm{d}x\right|\\ &\le \|f\|_{q'}\big(c\lambda C\big\|\nabla_l(u-w_m)\big\|_p + \|u-w_m\|_q\big).\end{aligned}$$

Thus

$$\int a_\alpha(x,\nabla_l v_k)D^\alpha u\,\mathrm{d}x \to \int fu\,\mathrm{d}x,$$

which together with (15.8.10) and (15.8.11) gives

$$J_k = \int\big(a_\alpha(x,\nabla_l v_k) - a_\alpha(x,\nabla_l u)\big)\big(D^\alpha v_k - D^\alpha u\big)\,\mathrm{d}x \to 0.$$

Next we take a subsequence $\{w_k\}$ of the sequence $\{v_k\}$ such that

$$\lim_{k\to\infty}\big[a_\alpha\big(x,\nabla_l w_k(x)\big) - a_\alpha\big(x,\nabla_l u(x)\big)\big]\big(D^\alpha w_k(x) - D^\alpha u(x)\big) = 0 \tag{15.8.12}$$

for almost all $x\in\Omega$. Let x be a point where (15.8.12) holds, let ξ^* be the limit of $\nabla_l w_k(x)$, and $\xi = \nabla_l u(x)$. We show that $|\xi^*| < \infty$. In fact,

$$\begin{aligned}&\big(a_\alpha(x,\nabla_l w_k) - a_\alpha(x,\nabla_l u)\big)\big(D^\alpha w_k - D^\alpha u\big)\\ &\qquad\ge a_\alpha(x,\nabla_l w_k)D^\alpha w_k - c\big(|\nabla_l w_k|^{p-1} + |\nabla_l w_k| + 1\big).\end{aligned}$$

Under the assumption $\xi^* = \infty$ we find

$$a_\alpha\big(x,\nabla_l w_k(x)\big)D^\alpha w_k(x) \to \infty,$$

which contradicts (15.8.12). Since $a_\alpha(x,y)$ is continuous in y, we have $[a_\alpha(x,\xi^*) - a_\alpha(x,\xi)](\xi^*-\xi) = 0$. Therefore, $\xi^* = \xi$, that is,

$$\nabla_l w_k(x) \to \nabla_l u(x), \qquad a_\alpha\big(x, \nabla_l w_k(x)\big) \to a_\alpha\big(x, \nabla_l u(x)\big),$$

almost everywhere. Next we use the following simple property (see Leray and Lions [488]): If g_k, $g \in L_p(\Omega)$, $\|g_k\|_p \le c$ and $g_k \to g$ almost everywhere in Ω, it follows that $g_k \to g$ weakly in $L_p(\Omega)$. This implies that $a_\alpha(x, \nabla_l w_k) \to a_\alpha(x, \nabla_l u)$ weakly in $L_p(\Omega)$. Hence

$$\int a_\alpha(x, \nabla_l u) D^\alpha w \, \mathrm{d}x = \lim_{k\to\infty} \int a_\alpha(x, \nabla_l w_k) D^\alpha w \, \mathrm{d}x$$

for any $w \in C_0^\infty(\Omega)$. Taking into account that supp $w \subset \Omega_k$ for large enough k, we see that

$$\int a_\alpha(x, \nabla_l w_k) D^\alpha w \, \mathrm{d}x = \int f w \, \mathrm{d}x.$$

Thus, u is a solution of the equation $Au = f$.

Let u_1,u_2 be two solutions of the problem (15.8.5). Since $(u_1 - u_2) \in \mathring{L}^l_p(\Omega)$, we have

$$\int a_\alpha(x, \nabla_l u_i) D^\alpha (u_1 - u_2) \, \mathrm{d}x = \int f (u_1 - u_2) \, \mathrm{d}x$$

with $i = 1, 2$ and therefore,

$$\int \big[a_\alpha(x, \nabla_l u_1) - a_\alpha(x, \nabla_l u_2)\big] \big(D^\alpha u_1 - D^\alpha u_2\big) \, \mathrm{d}x = 0.$$

This and (15.8.3) imply that $u_1 = u_2$ almost everywhere in Ω. The Lemma is proved.

Now we prove a converse assertion.

Lemma 2. *If* (15.8.5) *is solvable for any* $f \in L_{q'}(\Omega)$, *then* (15.8.4) *holds for all* $v \in C_0^\infty(\Omega)$.[1]

Proof. Let $v \in C_0^\infty(\Omega)$ with $\|v\|_{\mathring{L}^l_p(\Omega)} = 1$. The functional

$$v(f) = \int f v \, \mathrm{d}x,$$

defined on $L_{q'}(\Omega)$, can be written in the form

$$v(f) = \int a_\alpha(x, \nabla_l u) D^\alpha v \, \mathrm{d}x,$$

where $u \in \mathring{L}^l_p(\Omega) \cap L(\Omega, \mathrm{loc})$. Hence

$$\big|v(f)\big| \le c\,\lambda \|\nabla_l u\|_p^{p-1},$$

and the functionals $v(f)$ are bounded for each $f \in L_p(\Omega)$. Therefore, the norms of $v(f)$ are uniformly bounded and (15.8.4) holds.

[1] Note that Lemma 2 holds without the condition (15.8.3).

Combining Lemmas 1 and 2 with Theorem 14.1.2, we arrive at the following assertion.

Theorem. *Let $q \in [p, pn/(n-pl)]$ for $n \geq pl$, and $q \in [p, \infty]$ for $n > pl$. The problem (15.8.5) is solvable for any $f \in L_{q'}(\Omega)$ if and only if Ω satisfies one of the conditions:*

1. *For certain $d > 0$ and $n > pl$*

$$\inf_{Q_d} \operatorname{Cap}\bigl(\bar{Q}_d \cap C\Omega, \mathring{L}^l_p\bigr) > 0$$

with the infimum here and below taken over all cubes Q_d with edge length d.

2. *For certain $d > 0$ and $n = pl$*

$$\inf_{Q_d} \operatorname{Cap}\bigl(\bar{Q}_d \cap C\Omega, \mathring{L}^l_p(Q_{2d})\bigr) > 0.$$

3. *The set Ω does not contain large cubes if $n < pl$ and the complement of Ω is connected if $n = pl$.*

15.8.2 A Weighted Multiplicative Inequality

The goal of this subsection is to prove the following auxiliary fact.

Proposition. *Let ω be an open subset of $\mathbb{R}^n$ and let u and z be functions in $L^l_p(\omega) \cap L_\infty(\omega)$ such that the product uz has a compact support. Then*

$$\bigl\|z^m \nabla_m u\bigr\|_{\frac{pl}{m}} \leq c\|u\|_\infty^{1-\frac{m}{l}} \bigl\|z^l \nabla_l u\bigr\|_p^{\frac{m}{l}} + c\|u\|_\infty \|\nabla z\|^m_{pl}, \tag{15.8.13}$$

where $m = 1, 2, \dots, l-1$, and c is a constant depending only on m, l, and p.

The proof is based on the next two lemmas.

Lemma 1. *Let $a_0, a_1, \dots, a_l, A$ be nonnegative numbers satisfying the inequality*

$$a_k \leq C_1\bigl(a_{k-1}^{1/2} a_k^{1/2} + a_{k-1} A\bigr) \tag{15.8.14}$$

for $k = 1, 2, \dots, l$. Then

$$a_k \leq C_2\bigl(a_0^{1/(k+1)} a_{k+1}^{\frac{k}{k+1}} + a_0 A^k\bigr) \tag{15.8.15}$$

and therefore,

$$a_k \leq C_3\bigl(a_0^{\frac{l-k}{k}} a_l^{\frac{k}{l}} + a_0 A^k\bigr). \tag{15.8.16}$$

Here C_1, C_2, and C_3 are constants depending on l.

Proof. The result follows by induction.

Lemma 2. *For all functions v and z given on $\mathbb{R}$ and such that v' and z are absolutely continuous, and the support of vz is compact, the following inequality holds*:

$$\big\langle v'|z|^{\gamma}\big\rangle_{\frac{q}{\gamma}} \le C\Big(\big\langle v''|z|^{\gamma+1}\big\rangle^{1/2}_{\frac{q}{\gamma+1}} + \big\langle v|z|^{\gamma-1}\big\rangle_{\frac{q}{\gamma+1}}\langle z'\rangle_q\Big), \tag{15.8.17}$$

where $\langle\cdot\rangle_p$ *is the norm in* $L_p(\mathbb{R})$, $\gamma \ge 1$, $q > \gamma+1$, *and* C *is a constant depending only on* γ *and* q.

Proof. It may be assumed that

$$\big\langle v''|z|^{\gamma+1}\big\rangle_{\frac{q}{\gamma+1}} < \infty.$$

We introduce the notation

$$b_0 = \int |v|^{\frac{q}{\gamma-1}}|z|^q\,\mathrm{d}x, \qquad b_1 = \int |v'|^{\frac{q}{\gamma}}|z|^q\,\mathrm{d}x,$$
$$b_2 = \int |v''|^{\frac{q}{\gamma+1}}|z|^q\,\mathrm{d}x, \qquad b = \int |z'|^q\,\mathrm{d}x.$$

(The integration is carried out over $\mathbb{R}$.)

We first examine the case $q < 2\gamma$. Let $\delta = \frac{q(2\gamma-q)}{2\gamma(\gamma-1)}$, and let ε be an arbitrary positive number. By the Hölder inequality

$$\begin{aligned} b_1 &= \int \big(|v|+\varepsilon\big)^{\delta}\frac{|v'|^{\frac{q}{\gamma}}}{(|v|+\varepsilon)^{\delta}}|z|^q\,\mathrm{d}x \\ &\le \bigg(\int \big(|v|+\varepsilon\big)^{\frac{q}{\gamma-1}}|z|^q\,\mathrm{d}x\bigg)^{1-\frac{q}{2\gamma}}\bigg(\int \frac{|v'|^2|z|^q}{(|v|+\varepsilon)^{\frac{2\gamma-q}{\gamma-1}}}\,\mathrm{d}x\bigg)^{\frac{q}{2\gamma}}. \end{aligned} \tag{15.8.18}$$

Integrating by parts, we find that the last integral is not greater than

$$c\int \big(|v|+\varepsilon\big)^{1-\frac{2\gamma-q}{\gamma-1}}|v''||z|^q\,\mathrm{d}x + c\int \big(|v|+\varepsilon\big)^{1-\frac{2\gamma-q}{\gamma-1}}|v'||z'||z|^{q-1}\,\mathrm{d}x.$$

We denote these integrals by i_1 and i_2 and estimate them with the aid of the Hölder inequality:

$$\begin{aligned} i_1 &\le \bigg(\int |v''|^{\frac{q}{\gamma+1}}|z|^q\,\mathrm{d}x\bigg)^{\frac{\gamma+1}{q}}\bigg(\int \big(|v|+\varepsilon\big)^{\frac{q}{\gamma-1}}|z|^q\,\mathrm{d}x\bigg)^{\frac{q-\gamma-1}{q}} \xrightarrow{\varepsilon\to 0} b_2^{\frac{\gamma+1}{q}}b_0^{1-\frac{\gamma}{q}}, \\ i_2 &\le \bigg(\int |v'|^{\frac{q}{\gamma}}|z|^q\,\mathrm{d}x\bigg)^{\frac{\gamma}{q}}\bigg(\int \big(|v|+\varepsilon\big)^{\frac{q}{\gamma-1}}|z|^q\,\mathrm{d}x\bigg)^{\frac{q-\gamma-1}{q}} \\ &\quad\times\bigg(\int |z'|^q\,\mathrm{d}x\bigg)^{\frac{1}{q}} \xrightarrow{\varepsilon\to 0} b_1^{\frac{\gamma}{q}}b_0^{1-\frac{\gamma+1}{q}}b^{\frac{1}{q}}, \end{aligned}$$

and these estimates combined with (15.8.18) imply the estimate (15.8.17).

Let $q \ge 2\gamma$. After integration by parts in b_1 we obtain

$$b_1 \le c\int |vv''||v'|^{\frac{q}{\gamma}-2}|z|^q\,\mathrm{d}x + c\int |v||v'|^{\frac{q}{\gamma-1}}|z'||z|^{q-1}\,\mathrm{d}x.$$

We apply the triple Hölder inequality with exponents $p_1 = \frac{q}{q-2\gamma}$, $p_2 = \frac{q}{\gamma-1}$, and $p_3 = \frac{q}{\gamma+1}$ to the first integral on the right-hand side. We estimate the second integral by means of the same inequality with exponents $p_1 = \frac{q}{q-\gamma}$, $p_2 = \frac{q}{\gamma-1}$, and $p_3 = q$. Then

$$b_1 \le c\big(b_1^{1-\frac{2\gamma}{q}} b_0^{\frac{\gamma-1}{q}} b_2^{\frac{\gamma+1}{q}} + b_1^{1-\frac{\gamma}{q}} b_o^{\frac{\gamma-1}{q}} b^{\frac{1}{q}}\big).$$

This proves the lemma. □

Now we are in a position to justify (15.8.13).

Proof of Proposition. Let $q > m+1$. By Lemma 2 we have

$$\begin{aligned}\big\|z^m \nabla_m u\big\|_{\frac{q}{m}} \le c\big\|z^{m+1}\nabla_{m+1}u\big\|^{\frac{1}{2}}_{\frac{q}{m+1}} \big\|z^{m-1}\nabla_{m-1}u\big\|^{\frac{1}{2}}_{\frac{q}{m-1}} \\ + c\big\|z^{m-1}\nabla_{m-1}u\big\|_{\frac{q}{m-1}} \|\nabla z\|_q.\end{aligned} \tag{15.8.19}$$

Hence, by Lemma 1 we arrive at (15.8.13). □

15.8.3 Uniqueness of a Solution to the Dirichlet Problem with an Exceptional Set for Equations of Arbitrary Order

Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let e be a compact subset of $\partial\Omega$. We say that a function u given on Ω belongs to the space $L_p^l(\Omega, e, \mathrm{loc})$ if, for any open set $\omega \subset \Omega$ with $\bar\omega \cap e = \varnothing$, $u \in L_p^l(\omega, \mathrm{loc})$. Let $\mathring{L}_p^l(\Omega, e, \mathrm{loc})$ be the set of functions u in $L_p^l(\Omega, e, \mathrm{loc})$ such that for any open set $\omega \subset \Omega$ with $\bar\omega \cap e = \varnothing$, u is a $W_p^l(\Omega)$-limit of a sequence of functions in $W_p^l(\Omega)$ vanishing near $\partial\omega \cap \partial\Omega$.

Lemma 1. *Let $\xi \in \mathfrak{M}(e, \mathbb{R}^n)$ and let $u \in \mathring{L}_p^l(\Omega, e, \mathrm{loc}) \cap L_\infty(\Omega)$. Then*

$$c\big\|(1-\xi)^m \nabla_m u\big\|_{\frac{pl}{m}} \le \|u\|_\infty^{1-\frac{m}{l}} \big\|(1-\xi)^l \nabla_l u\big\|_p^{\frac{m}{l}} + \|u\|_\infty \|\nabla_l \xi\|_p^{\frac{m}{l}} \tag{15.8.20}$$

with $m = 1, \ldots, l-1$. Here and elsewhere in this subsection by c we mean positive constants depending only on p, m, and l.

Proof. The result follows from (15.8.13) combined with the Gagliardo–Nirenberg inequality (1.8.1), where $j = m$ and $u = \xi$. □

We say that u is a bounded solution of the Dirichlet problem for (15.8.1) with an exceptional compact set $e \subset \partial\Omega$ if $u \in \mathring{L}_p^l(\Omega, e, \mathrm{loc}) \cap L_\infty(\Omega)$ and, for all $\varphi \in \mathring{L}_p^l(\Omega) \cap L_\infty(\Omega)$ vanishing in a neighborhood of e,

$$\int a_\alpha(x, \nabla_l u) D^\alpha \varphi \, \mathrm{d}x = \int f\varphi \, \mathrm{d}x, \quad f \in L_1(\Omega). \tag{15.8.21}$$

Lemma 2. *Let $\xi \in \mathfrak{P}(e, \mathbb{R}^n)$, where $\mathfrak{P}$ is the class of functions introduced in Remark* 13.3. *Also let u be a solution of the Dirichlet problem with an exceptional set e such that $u \in \mathring{L}_p^l(\Omega, e, \mathrm{loc}) \cap L_\infty(\Omega)$. Then*

$$c\big\|(1-\xi)^l \nabla_l u\big\|_p \le \|u\|_\infty \|\nabla_l \xi\|_p + \|u\|_\infty^{1/p} \|f\|_1^{1/p}, \tag{15.8.22}$$

where c is a positive constant independent of u and ξ.

Proof. Let $z = 1 - \xi$. Setting $\varphi = uz^{lp}$ in (15.8.21), we find

$$\begin{aligned}\int z^{lp} a_\alpha(x, \nabla_l u) D^\alpha u \, \mathrm{d}x \\ = -\int a_\alpha(x, \nabla_l u) \sum_{\alpha \ge \beta > 0} \frac{\alpha!}{(\alpha-\beta)!\beta!} D^{\alpha-\beta} u D^\beta\big(z^{lp}\big) \, \mathrm{d}x \\ + \int f z^{lp} u \, \mathrm{d}x,\end{aligned}$$

which together with (15.8.2) implies

$$c\int z^{lp} |\nabla_l u|^p \, \mathrm{d}x \le \int |\nabla_l u|^{p-1} \sum_{k=1}^{l} |\nabla_{l-k} u| \big|\nabla_k\big(z^{lp}\big)\big| \, \mathrm{d}x + \|u\|_\infty \|f\|_1.$$

Applying the Hölder inequality on the right-hand side, we obtain

$$c\big\|z^l \nabla_l u\big\|_p \le \Bigg\| \sum_{k=1}^{l} z^{l-k} |\nabla_{l-k} u| \sum_{i=1}^{k} z^{k-i} \prod_{\sum_{j=1}^{i} m_j = k} |\nabla_{m_j} \xi| \Bigg\|_p + A,$$

where $A^p = \|u\|_\infty \|f\|_1$. The first term on the right-hand side does not exceed

$$c\sum_{k=1}^{l} \big\|z^{l-k} \nabla_{l-k} u\big\|_{\frac{pl}{l-k}} \sum_{i=1}^{k} \prod_{\sum_{j=1}^{i} m_j = k} \|\nabla_{m_j} \xi\|_{\frac{lp}{m_j}}.$$

By the Gagliardo–Nirenberg inequality (1.8.1) with j replaced by m_j and u by ξ we find

$$c\big\|z^l \nabla_l u\big\|_p \le \sum_{k=1}^{l} \|\nabla_l \xi\|_p^{\frac{k}{l}} \big\|z^{l-k} \nabla_{l-k} u\big\|_{\frac{pl}{l-k}} + A.$$

Using (15.8.20) with $m = l - k$, we arrive at

$$\begin{aligned}c\big\|z^l \nabla_{l-k} u\big\|_p \le \sum_{k=1}^{l} \|\nabla_l \xi\|_p^{\frac{k}{l}} \|u\|_\infty^{\frac{k}{l}} \big\|z^l \nabla_l u\big\|_p^{1-\frac{k}{l}} \\ + \|u\|_\infty \|\nabla_l \xi\|_p + A\end{aligned}$$

$$\le \varepsilon \big\| z^l \nabla_l u \big\|_p + c_\varepsilon \|u\|_\infty \|\nabla_l \xi\|_p + A$$

for all $\varepsilon > 0$. The estimate (15.8.22) is proved. □

Lemma 3. *If e is a compact (p,l)-polar subset of $\partial\Omega$ and Π is a polynomial of degree at most $l-1$ from $\mathring{L}^l_p(\Omega, e, \mathrm{loc}) \cap L_\infty(\Omega)$, then $\Pi \equiv 0$.*

Proof. Let $\bar\Omega \subset Q_d$. Since $\mathrm{Cap}(e, \mathring{L}^l_p(Q_{2d})) = 0$, it follows that $\mathrm{cap}(e, \mathring{L}^1_p(Q_{2d})) = 0$. Let $\varepsilon > 0$ and let g be an open set such that $e \subset g \subset Q_{2d}$ and $\mathrm{Cap}(g, \mathring{L}^1_p(Q_{2d})) < \varepsilon$. We cover $\partial\Omega \backslash g$ by a finite number of open cubes q_i with $q_i \cap e = \varnothing$. By the monotonicity and semiadditivity of the $(p,1)$-capacity we have

$$\begin{aligned}
&\sum_i \mathrm{cap}\big(q_i \cap \partial\Omega, \mathring{L}^1_p(Q_{2d})\big) \\
&\quad \ge \mathrm{cap}\big((\partial\Omega)\backslash g, \mathring{L}^1_p(Q_{2d})\big) \\
&\quad \ge \mathrm{cap}\big(\partial\Omega, \mathring{L}^1_p(Q_{2d})\big) - \mathrm{cap}\big(g, \mathring{L}^1_p(Q_{2d})\big) \ge \mathrm{cap}\big(\partial\Omega, \mathring{L}^1_p(Q_{2d})\big) - \varepsilon
\end{aligned}$$

with a sufficiently small $\varepsilon > 0$. Therefore, there exists an open cube Q_δ such that

$$\bar Q_\delta \cap e = \varnothing, \quad \mathrm{cap}\big(\bar Q_\delta \cap \partial\Omega, \mathring{L}^1_p(Q_{2d})\big) > 0.$$

Let $\{u_m\}$ be a sequence of functions in $C^\infty(\bar Q_\delta)$ vanishing in a neighborhood of $\bar Q_\delta \backslash \Omega$ and converging to Π in $W^l_p(Q_\delta \cap \Omega)$. Clearly,

$$\|\nabla_l u_m\|_{L_p(Q_\delta)} = \|\nabla_l u_m\|_{L_p(Q_\delta \cap \Omega)} \to 0$$

as $m \to \infty$. By Corollary 14.3.4 and Theorem 14.1.2 we have $u_m \to 1$ in $L_p(Q_\delta \cap \Omega)$. Hence $\Pi = 0$ on $Q_d \cap \Omega$ and thus $\Pi = 0$ everywhere. □

Theorem. *The Dirichlet problem for the equation* (15.8.1) *with an exceptional (p,l)-polar set has at most one bounded solution.*

Proof. Let $\bar\Omega \supset Q_d$ and let u and v be two bounded solutions of the Dirichlet problem. Further, let $z = 1 - \xi$ with $\xi \in \mathfrak{P}(e, Q_{2d})$ (see Remark 13.3). Then

$$\int \big[a_\alpha(x, \nabla_l u) - a_\alpha(x, \nabla_l v)\big] D^\alpha \big(z^{lp}(u - v)\big)\, \mathrm{d}x = 0$$

and hence

$$\begin{aligned}
J &\stackrel{\text{def}}{=} \int z^{lp} \big[a_\alpha(x, \nabla_l u) - a_\alpha(x, \nabla_l v)\big] D^\alpha (u - v)\, \mathrm{d}x \\
&= -\int \big[a_\alpha(x, \nabla_l u) - a_\alpha(x, \nabla_l v)\big] \\
&\qquad \times \sum_{\alpha \ge \beta > 0} \frac{\alpha!}{(\alpha - \beta)!\beta!} D^{\alpha-\beta}(u - v) D^\beta \big(z^{lp}\big)\, \mathrm{d}x.
\end{aligned}$$

This implies the inequality

$$J \le c \sum_{\alpha} \left(\left\| z^{l(p-1)} a_\alpha(x, \nabla_l u) \right\|_{p'} + \left\| z^{l(p-1)} a_\alpha(x, \nabla_l v) \right\|_{p'} \right) \\ \times \sum_{k=1}^{l} \|\nabla_l \xi\|_p^{\frac{k}{l}} \left\| z^{l-k} \nabla_{l-k}(u-v) \right\|_{\frac{pl}{l-k}}.$$

By Lemma 1 we obtain

$$J \le c\left(\|z^l \nabla_l u\|_p^{p-1} + \left\|z^l \nabla_l v\right\|_p^{p-1}\right) \sum_{k=1}^{l} \|\nabla_l \xi\|_p^{\frac{k}{l}} \left[\|u-v\|_\infty^{\frac{k}{l}} \left\|z^l \nabla_l (u-v)\right\|_p^{1-\frac{k}{l}} \\ + \|u-v\|_\infty \|\nabla_l \xi\|_p^{1-\frac{k}{l}}\right].$$

Combining the condition $u,\, v \in L_\infty(\Omega)$ with Lemma 2, we see that the right-hand side of the last inequality does not exceed $F(\|\nabla_l \xi\|_p)\|\nabla_l \xi\|_p^{1/l}$, where F is a continuous function on $[0, +\infty)$. Let $\{\xi_k\}$ be a sequence of functions in $\mathfrak{P}(e, Q_{2d})$ with $\|\nabla_l \xi_k\|_{L_p(Q_{2d})} \to 0$. Then

$$\lim_{k\to\infty} \int (1-\xi_k)^{lp} \left[a_\alpha(x, \nabla_l u) - a_\alpha(x, \nabla_l v)\right] D^\alpha (u-v)\, \mathrm{d}x = 0.$$

Since $\xi_k \to 0$ in measure, it follows that

$$\int \left[a_\alpha(x, \nabla_l u) - a_\alpha(x, \nabla_l v)\right] D^\alpha (u-v)\, \mathrm{d}x = 0$$

and hence $\nabla_l(u-v) = 0$. Thus we find that $u-v$ is a polynomial of a degree at most $l-1$ and therefore, by Lemma 3, $u = v$. The theorem is proved.

15.8.4 Uniqueness of a Solution to the Neumann Problem for Quasilinear Second-Order Equation

Let Ω be an open bounded set in $\mathbb{R}^n$. Consider the second-order equation (15.8.1), that is, the equation

$$-\frac{\partial}{\partial x_i}\big(a_i(x, \nabla u)\big) = f, \quad f \in L(\Omega), \tag{15.8.23}$$

with coefficients satisfying (15.8.2) and (15.8.3).

Let e be a closed subset of $\partial\Omega$. We say that a function $u \in L_p^1(\Omega, e, \mathrm{loc})$ is a bounded solution of the Neumann problem for (15.8.23) with the exceptional set e if $u \in L_\infty(\Omega)$ and for all $\varphi \in L_p^1(\Omega) \cap L_\infty(\Omega)$ vanishing in a neighborhood of e,

$$\int a_i(x, \nabla u) \frac{\partial \varphi}{\partial x_i}\, \mathrm{d}x = \int f \varphi \, \mathrm{d}x. \tag{15.8.24}$$

Lemma. *Let $z \in C^\infty(\mathbb{R}^n)$, $z \geq 0$, and let $z = 0$ in a neighborhood of $e \subset \partial\Omega$. For any bounded solution of the Neumann problem for (15.8.23) with the exceptional set e the inequality*

$$\int z^p |\nabla u|^p \,\mathrm{d}x \leq c \operatorname{osc} u \|z\|_\infty^p \|f\|_1 + c(\operatorname{osc} u)^p \|\nabla z\|_p^p \tag{15.8.25}$$

holds.

Proof. We put

$$\varphi = z^p(u - C), \quad C = \text{const},$$

in (15.8.24). Then

$$\int z^p a_i(x, \nabla u) \frac{\partial u}{\partial x_i} \,\mathrm{d}x$$
$$\leq \|u - C\|_\infty \|z\|_\infty^p \|f\|_1 + p\|u - C\|_\infty \int z^{p-1} \left| a_i(x, \nabla u) \frac{\partial z}{\partial x_i} \right| \mathrm{d}x.$$

The result follows by (15.8.2) and Hölder's inequality.

Corollary. *Let e be a closed $(p,1)$-polar subset of $\partial\Omega$ and let u be a bounded solution of the Neumann problem for (15.8.23) with the exceptional set e. Then $u \in L_p^1(\Omega)$.*

Proof. Let $\bar{\Omega} \subset Q_d$ and let $\{\xi_k\}$ be a sequence of functions in $\mathfrak{P}(e, Q_{2d})$ with $\|\xi_k\|_{L_p(Q_{2d})} \to 0$. By (15.8.25) with $z = 1 - \xi_k$ we have

$$\limsup_{k\to\infty} \int (1 - \xi_k)^p |\nabla u|^p \,\mathrm{d}x \leq c \operatorname{osc} u \|f\|_1.$$

Since $\xi_k \to 0$ in measure, we conclude that

$$\int |\nabla u|^p \,\mathrm{d}x \leq c \operatorname{osc} u \|f\|_1. \tag{15.8.26}$$

The corollary is proved.

Theorem 1. *The difference of any two bounded solutions of the Neumann problem for (15.8.23) with an exceptional $(p,1)$-polar set is a constant.*

Proof. Let u and v be two solutions. For any z in $C^\infty(\mathbb{R}^n)$, $z \geq 0$, vanishing in a neighborhood of e, we have

$$\int \big[a_i(x, \nabla u) - a_i(x, \nabla v)\big] \frac{\partial}{\partial x_i} \big[(u - v) z^p\big] \,\mathrm{d}x = 0.$$

Hence

$$\int z^p \big[a_i(x, \nabla u) - a_i(x, \nabla v)\big] \frac{\partial}{\partial x_i} (u - v) \,\mathrm{d}x$$

$$\le p\|u-v\|_\infty \|\big[a_i(x,\nabla u)-a_i(x,\nabla v)\big]z^{p-1}\|_{p'}\|\nabla z\|_p$$
$$\le p\|u-v\|_\infty \big(\|z\nabla u\|_p^{p-1}+\|z\nabla v\|_p^{p-1}\big)\|\nabla z\|_p. \tag{15.8.27}$$

Let $\bar\Omega \supset Q_d$ and let $\{z_k\}$ be a sequence of functions in $\mathfrak{P}(e, Q_{2d})$ with $\|\nabla z_k\|_{L_p(Q_d)} \to 0$. Passing to the limit in (15.8.27) and using (15.8.26), we arrive at

$$\int \big[a_i(x,\nabla u)-a_i(x,\nabla v)\big]\frac{\partial}{\partial x_i}(u-v)\,\mathrm{d}x = 0.$$

Thus $u - v = \text{const}$.

Definition. Let e be a compact subset of $\partial\Omega$. By $\mathring{W}_p^1(\Omega, e)$ we denote the completion in $W_p^1(\Omega)$ of the set of functions in $W_p^1(\Omega)$ vanishing in a neighborhood of e.

The next theorem also concerns (15.8.23). Under an additional assumption on $\partial\Omega$ we prove the necessity of the $(p,1)$-polarity of e.

Theorem 2. *Let Ω be the image of a cube under a bi-Lipschitz mapping. If the Neumann problem for (15.8.23) with an exceptional set e has the only bounded solution $u \equiv \text{const}$, then e is a $(p,1)$-polar set.*

Proof. Let $\bar\Omega \subset Q_d$. Suppose that $\operatorname{cap}(e, \mathring{L}_p^1(Q_d)) > 0$. We introduce two compact sets e_1 and e_2 with

$$e_1 \cap e_2 = \varnothing,\ e_i \subset e,\ \operatorname{cap}\big(e_i, \mathring{L}_p^1(Q_d)\big) > 0 \quad (i = 1, 2).$$

(The existence of such sets easily follows from the semi-additivity of the $(p,1)$-capacity.) Let ω be a neighborhood of e with the properties $\bar\omega \subset Q_d$ and $e_2 \cap \bar\omega = \varnothing$ and let $h \in \mathfrak{P}(e, \omega)$. By the Leray–Lions theorem (see [488]), there exists $u \in W_p^1(\Omega)$ with $u - h \in \mathring{W}_p^1(\Omega, e_1 \cup e_2)$, such that

$$\int a_i(x,\nabla u)\frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x = 0 \tag{15.8.28}$$

for all $\varphi \in \mathring{W}_p^1(\Omega, e_1 \cup e_2)$. Since the functions in $W_p^1(\Omega)$ are absolutely continuous on almost all straight lines parallel to the coordinate axes, it can be easily shown that both $(u-1)_+$ and u_- belong to $\mathring{W}_p^1(\Omega, e_1 \cup e_2)$. Plugging them into (15.8.28) as φ and using (15.8.2), we find that $0 \le u \le 1$ almost everywhere in Ω. Together with (15.8.28) this implies that u is a bounded solution of the Neumann problem with the exceptional set $e_1 \cup e_2 \subset e$.

It remains to prove that u_1 is a constant. Since $u \in \mathring{W}_p^1(\Omega, e_2)$ and $\operatorname{cap}(e_2, \mathring{L}_p^1(Q_d)) > 0$ it follows from Theorem 14.1.2 that

$$\|u\|_p \le C\|\nabla u\|_p.$$

Hence there exists a constant C_1 independent of u such that

$$\|u\|_{W_p^1(\Omega)} \le C_1\|\nabla u\|_p.$$

The assumption that Ω is a Lipschitz image of a cube implies the existence of an extension v of u onto Q_d with the properties $v - h \in \mathring{W}_p^1(Q_d, e_1)$ and

$$\|\nabla v\|_{L_p(Q_d)} \le C_2 \|u\|_{W_p^1(\Omega)}.$$

Therefore,

$$0 < \operatorname{cap}\big(e_1, \mathring{L}_p^1(Q_d)\big) \le \|\nabla v\|^p_{L_p(Q_d)} \le (C_1 C_2)^p \|\nabla u\|_p^p$$

and $u = \text{const}$. The theorem is proved.

15.9 Comments to Chap. 15

Sections 15.1, 15.2. In [234] Deny and Lions studied the orthogonal projection method with respect to the Dirichlet problem for the Laplace operator. In particular, they gave the following description of the sets Ω satisfying $\mathring{L}_2^1(\Omega) \subset \mathscr{D}'(\Omega)$. This embedding occurs for $n \ge 3$ for arbitrary set Ω, for $n = 2$ if the 2-capacity of $C\Omega$ is positive and for $n = 1$ if $C\Omega \ne \varnothing$.

For integers l the problem of the embedding $\mathring{L}_2^l(\Omega) \subset \mathscr{D}'(\Omega)$ was solved by Hörmander and Lions [385]. Their result was formulated in terms of $(2, l)$-polarity. In the author's paper [535] it was noted that the conditions of the Hörmander and Lions theorem can be restated in terms of the l-harmonic capacity.

Our proof of Theorem 15.2 follows the paper by Khvoles and the author [570], where the next result is obtained.

Let $p > 1$, $l > 0$ and let $\mathring{h}_p^l(\Omega)$ be the completion of $\mathscr{D}(\Omega)$ with respect to the norm $\|(-\Delta)^{l/2} u\|_{L_p(\mathbb{R}^n)}$ (in particular, $\mathring{h}_p^l(\Omega) = \mathring{L}_p^l(\Omega)$ for integer l).

Theorem. *The space $\mathring{h}_p^l(\Omega)$ is embedded into $\mathscr{D}'(\Omega)$ if and only if one of the following conditions is valid:*

1. *$n > pl$;* 2. $\operatorname{cap}(C\Omega, H_p^l) > 0$, *if $n = pl$;* 3. *$C\Omega \ne \varnothing$ if $n < pl$ and $l - n/p$ is noninteger;* 4. *either* $\operatorname{cap}(C\Omega, H_p^{n/p}) > 0$ *or $C\Omega$ does not lie in a $(n-1)$-dimensional hyperplane, if $n < pl$, $l - n/p$ is noninteger.*

Sections 15.3–15.6. The results of these sections are due to the author [544, 546, 549]. A proof of Theorem 15.4.2/1, based upon polynomial capacities in Chap. 14, was proposed by Wannebo [784].

A topic close to the theme of this chapter, but not treated here, is the question of the validity of the Hardy-type inequality

$$\int_\Omega |u(x)|^p \frac{dx}{[\operatorname{dist}(x, \partial\Omega)]^p} \le c_p \int_\Omega |\nabla u(x)|^p \, dx \quad \text{for all } u \in C_0^\infty(\Omega), \quad (15.9.1)$$

and its generalizations, under minimal restrictions on Ω. Let, for brevity, $d = \operatorname{dist}(x, \partial\Omega)$. Corollary 2.3.4 shows that (15.9.1) holds if and only if for all compact subsets F of Ω

$$\int_F \frac{\mathrm{d}x}{d^p} \le \mathrm{const}\,\mathrm{cap}_p(F,\Omega). \tag{15.9.2}$$

Moreover, the best constant c_p in (15.9.1) satisfies

$$\frac{(p-1)^{p-1}}{p^p} \le \sup_F \frac{\int_F d^{-p}\,\mathrm{d}x}{\mathrm{cap}_p(F,\Omega)} \le c_p.$$

Since (15.9.1) results from the inequality

$$\int_\Omega |u| \frac{\mathrm{d}x}{d^p} \le c_0 \int_\Omega |\nabla u| \frac{\mathrm{d}x}{d^{p-1}},$$

by changing u for $|v|^{p-1}v$, the following sufficient condition for (15.9.1) is a consequence of Theorem 2.1.3. For all bounded open sets g with smooth boundaries and closures $\bar{g} \subset \Omega$, the weighted isoperimetric inequality holds:

$$\int_g \frac{\mathrm{d}x}{d^p} \le c_0 \int_{\partial g} \frac{\mathrm{d}s}{d^{p-1}},$$

and the best constant in (15.9.1) is subject to the inequality

$$c_p \le p^p \sup_g \frac{\int_g d^{-p}\,\mathrm{d}x}{\int_{\partial g} d^{1-p}\,\mathrm{d}s}.$$

A different approach to the characterization of (15.9.1) for general domains was initiated by Ancona [47] and developed by Lewis [493]. They proved (15.9.1) for the so-called uniformly p-thick domains, which means that for every $x \in C\Omega$ and for all $\varrho > 0$

$$\mathrm{cap}_p\big(\bar{B}(x,\varrho)\backslash\Omega, B(x,2\varrho)\big) \ge \mathrm{const}\,\mathrm{cap}_p\big(\bar{B}(x,\varrho), B(x,2\varrho)\big).$$

Due to Ancona ($n = 2$) and Lewis ($n \ge 2$), the uniform thickness condition is equivalent to (15.9.1) in the critical case $p = n$.

A comprehensive survey of further development of this area, together with new results, can be found in Kinnunen and Korte [424]. We only give a list of publications related to the topic: Mikkonen [602]; Wannebo [784–786]; Hajłasz [339]; Sugawa [732]; Björn, MacManus, and Shanmugalingam [106]; Calude and Pavlov [164]; Korte and Shanmugalingam [453]; Laptev and A. Sobolev [480]; et al. To this we add that in [785], Wannebo studied the generalized Hardy inequality

$$\sum_{k=0}^m \int |\nabla_k u|^p \, d^{t-(m-k)p}\,\mathrm{d}x \le c \int |\nabla_m u|^p \, d^t\,\mathrm{d}x,$$

where $u \in C_0^\infty(\Omega)$ and t is less than some small positive number. He obtained a sufficient condition on Ω formulated in terms of the polynomial capacity Cap_{m-1}.

For Hardy inequalities of type (15.9.1) of fractional order $\alpha \in (0,1)$ we refer to Rafeiro and Samko [669]. The authors show that such an inequality holds under the additional assumption

$$\inf_{x \in \partial\Omega} m_n\big(B(x,r) \setminus \Omega\big) \geq C r^n$$

if and only if a certain property of Ω (related to a property of χ_Ω to be a pointwise multiplier in the space of the Riesz potentials of order α) holds.

Sections 15.7–15.8. The presented scheme of applications of integral inequalities to boundary value problems in variational form is well known (see, for instance, Deny and Lions [234], Lions and Magenes [500], Leray and Lions [488]). The results in these sections are due to the author.

The criterion for the discreteness of the spectrum of the Dirichlet problem for the Laplace operator (Theorem 15.7.3 for $l = 1$) was found by Molchanov [610]. It was improved by the author and Shubin in [590] (see Chapter 18 of the present book). For any integer l, Theorem 15.7.3 was proved by the author [533]. Similar results for more general operators are presented in Sect. 16.5.

The unique solvability of the Dirichlet problem for the operator B (see Sect. 15.7.4) in domains with small $(2,l)$ inner diameter (which is understood in a sense different from ours) was shown by Kondratiev [448, 449] (see the Comments to Sect. 14.2). Among other applications of the results of Chap. 14 to the theory of elliptic equations we list the following: theorems of the Phragmèn–Lindelöf type for elliptic equations of arbitrary order (see Landis [476]), estimates for eigenvalues of the operator of the Dirichlet problem in an unbounded domain (see Rozenblum [683] and Otelbaev [652, 653]), necessary and sufficient conditions for the Wiener regularity of a boundary point with respect to higher-order elliptic equations (see Maz'ya [559]).

Applications of the results of Chaps. 14 and 15 to the quasilinear elliptic equations given in Sect. 15.8 are due to the author [540].

The weighted multiplicative inequality (15.8.13) was proved in Maz'ya [542].

16

Embedding $\mathring{L}_p^l(\Omega, \nu) \subset W_r^m(\Omega)$

In this chapter we denote by $\mathring{L}_p^l(\Omega, \nu)$ the completion of $\mathscr{D}(\Omega)$ with respect to the metric

$$\|\nabla_l u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega,\nu)},$$

where $p > 1$, Ω is an open set in $\mathbb{R}^n$ and ν is a measure in Ω.

For instance, in Sect. 16.2 we show that the space $\mathscr{D}(\Omega)$ equipped with the norm $\mathring{L}_p^l(\mathbb{R}^n, \nu)$ is continuously embedded into $L_p(\mathbb{R}^n)$ if and only if

$$\inf_{\{e\}} \nu(Q_d \backslash e) \geq \text{const} > 0 \tag{16.0.1}$$

for any cube Q_d with sufficiently large edge length d. Here $\{e\}$ is the collection of all subsets of the cube Q_d with small enough capacity $\text{cap}(e, \mathring{L}_p^l(Q_{2d}))$. The corresponding embedding operator is compact if and only if condition (16.0.1) holds and $\inf_{\{e\}} \nu(Q_d \backslash e)$ tends to infinity as the cube Q_d tends to infinity (Sect. 16.3).

In Sect. 16.4 we study the closability of certain embedding operators. One of the theorems in this section asserts that the identity operator defined on $\mathscr{D}(\Omega)$ and acting from $L_p(\Omega, \nu)$ into $\mathring{L}_p^l(\Omega)$ is closable if and only if the measure ν is absolutely continuous with respect to the (p, l)-capacity. In Sect. 16.5 the previously proved criteria are reformulated for $p = 2$ as necessary and sufficient conditions for the positive definiteness and for the discreteness of the spectrum of the selfadjoint elliptic operator generated by the quadratic form

$$\int_\Omega \sum_{|\alpha|=|\beta|=l} a_{\alpha\beta}(x) D^\alpha u \overline{D^\beta u}\, \mathrm{d}x + \int_\Omega |u|^2\, \mathrm{d}\nu, \quad u \in \mathscr{D}(\Omega).$$

16.1 Auxiliary Assertions

Lemma 1. *For any* $u \in C^\infty(\bar{Q}_d)$ *we have*

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Grundlehren der mathematischen Wissenschaften 342,
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$$\|u\|^p_{L_p(Q_d)} \le c\lambda^{-1} d^{pl} \| u \|^p_{p,l,Q_d} + \frac{cd^n}{\inf \nu(Q_d \backslash e)} \|u\|^p_{L_p(Q_d,\nu)}, \tag{16.1.1}$$

where ν is a measure in Q_d, the infimum is taken over all compacta $e \subset \bar{Q}_d$ with $\operatorname{cap}(e, \mathring{L}_p^l(Q_{2d})) \le \lambda d^{n-pl}$ *(λ is an arbitrary constant), and the seminorm $\| \cdot \|_{p,l,Q_d}$ is that introduced in* 14.1.2.

Proof. We assume that the average value $\bar{u}_{Q_d}$ of the function u in Q_d is nonnegative and put

$$2\tau = d^{-n/p} \|u\|_{L_p(Q_d)}, \qquad e_\tau = \{x \in \bar{Q}_d : u(x) \le \tau\}.$$

Obviously,

$$\|u\|_{L_p(Q_d)} \le \|u - \tau\|_{L_p(Q_d)} + \tau d^{n/p}$$

and hence

$$\|u\|_{L_p(Q_d)} \le 2\|u - \tau\|_{L_p(Q_d)}. \tag{16.1.2}$$

First consider the case $\operatorname{cap}(e_\tau, \mathring{L}_p^l(Q_{2d})) > \lambda d^{n-pl}$. If $\bar{u}_{Q_d} \ge \tau$, then applying Theorem 14.1.3 to the function $u - \tau$ and using (16.1.2) we deduce the estimate

$$\|u\|^p_{L_p(Q_d)} \le c\lambda^{-1} d^{pl} \| u \|^p_{p,l,Q_d}. \tag{16.1.3}$$

Further, if $\bar{u}_{Q_d} < \tau$, then by virtue of the inequality

$$\|u - \bar{u}_{Q_d}\|_{L_p(Q_d)} \le cd \|\nabla u\|_{L_p(Q_d)},$$

we obtain

$$\|u\|_{L_p(Q_d)} \le 2\big(\|u\|_{L_p(Q_d)} - \bar{u}_{Q_d} d^{n/p}\big) \le 2cd\|\nabla u\|_{L_p(Q_d)}.$$

(Here we used the fact that $\bar{u}_{Q_d} \ge 0$.) So, for $\operatorname{cap}(e_\tau, \mathring{L}_p^l(Q_{2d})) > \lambda d^{n-pl}$ the estimate (16.1.3) is valid. In the case $\operatorname{cap}(e_\tau, \mathring{L}_p^l(Q_{2d})) \le \lambda d^{n-pl}$ we have

$$\|u\|^p_{L_p(Q_d)} = 2^p d^n \tau^p \le \frac{2^p d^n}{\nu(Q_d \backslash e_\tau)} \int_{Q_d \backslash e_\tau} |u|^p \, d\nu \le \frac{2^p d^n}{\inf \nu(Q_d \backslash e)} \|u\|^p_{L_p(Q_d,\nu)}.$$

This and (16.1.3) imply (16.1.1). □

Lemma 2. *Let E be a compact subset of $\bar{Q}_d$ with*

$$\operatorname{cap}\big(E, \mathring{L}_p^l(Q_{2d})\big) < \mu d^{n-pl}, \tag{16.1.4}$$

where μ is a sufficiently small positive constant which depends only on n, p, and l. Then

$$\inf_{u \in C_0^\infty(Q_d \backslash e)} \frac{\|u\|_{\mathring{L}_p^l(\mathbb{R}^n,\nu)}}{\|u\|_{L_p(\mathbb{R}^n)}} \le c\big(d^{-l} + d^{n/p} \nu(Q_d \backslash e)^{1/p}\big). \tag{16.1.5}$$

Proof. Clearly, it suffices to consider the case $d = 1$. By Remark 13.3 there exists a $\varphi \in \mathfrak{P}(e, Q_2)$ such that $0 \le \varphi \le 1$ and

$$\|\nabla_l \varphi\|_{L_p(\mathbb{R}^n)} \le c_0 \mu^{1/p}. \tag{16.1.6}$$

Let ω be an arbitrary function in $C_0^\infty(Q_1)$ that is equal to unity on $Q_{1/2}$ and satisfies $0 \le \omega \le 1$. For the function $u = \omega(1-\varphi)$, which is obviously in $C_0^\infty(Q_1 \backslash E)$, we obtain

$$\int_{Q_1} |u|^p \, \mathrm{d}\nu \le \nu(Q_1 \backslash E), \tag{16.1.7}$$

$$\begin{aligned} \|\nabla_l u\|_{L_p(\mathbb{R}^n)} &\le c\big(\|\nabla_l \omega\|_{L_p(\mathbb{R}^n)} + \|\nabla_l(\omega\varphi)\|_{L_p(\mathbb{R}^n)}\big) \\ &\le c_1 + c_2 \|\nabla_l \varphi\|_{L_p(\mathbb{R}^n)} \le c_1 + c_0 c_2 \mu^{1/p}. \end{aligned} \tag{16.1.8}$$

We obtain the following lower bound for the norm of u in L_p:

$$\begin{aligned} \|u\|_{L_p(\mathbb{R}^n)} &\ge \|\omega\|_{L_p(\mathbb{R}^n)} - \|\varphi\|_{L_p(\mathbb{R}^n)} \\ &\ge 2^{-n/p} - \|\nabla_l \varphi\|_{L_p(\mathbb{R}^n)} \sup_{u \in C_0^\infty(Q_2)} \frac{\|u\|_{L_p(\mathbb{R}^n)}}{\|\nabla_l u\|_{L_p(\mathbb{R}^n)}}. \end{aligned}$$

This and (16.1.6) along with the smallness of μ imply $\|u\|_{L_p(\mathbb{R}^n)} \ge 2^{-1-n/p}$. Combining this estimate with (16.1.7) and (16.1.8) we arrive at (16.1.5). The lemma is proved. □

From the proof of Lemma 2 it follows that μ can be subjected to the inequality

$$\mu \le 2^{-n-p} c_0^{-p} \inf_{u \in C_0^\infty(Q_2)} \frac{\|\nabla_l u\|_{L_p(\mathbb{R}^n)}}{\|u\|^p_{L_p(\mathbb{R}^n)}}, \tag{16.1.9}$$

where c_0 is the constant in (16.1.6).

In what follows in this chapter, the subsets of Q_d that satisfy the inequality

$$\operatorname{Cap}\big(e, \mathring{L}_p^l(Q_{2d})\big) \le \gamma d^{n-pl},$$

where $n \ge pl$, $\gamma = \mu c_*^{-1}$, c_* is the constant in (13.3.5), will be called (p,l)-negligible (cf. Definition 14.1.1). As before for $pl > n$, by definition, the only (p,l)-negligible set is the empty one.

Similar to Chap. 14, the collection of all (p,l)-negligible closed subsets of the cube $\bar{Q}_d$ will be denoted by $\mathscr{N}(Q_d)$.

16.2 Continuity of the Embedding Operator $\mathring{L}_p^l(\Omega,\nu) \to W_r^m(\Omega)$

Let Ω be an arbitrary open set $\mathbb{R}^n$ and let ν be a measure in Ω. We denote by F_Ω the set of all cubes $\bar{Q}_d$ whose intersections with $\mathbb{R}^n \backslash \Omega$ are (p,l)-negligible.

We introduce the number

$$D = D_{p,l}(\nu,\Omega) = \sup_{\bar{Q}_d \in F_\Omega} \Big\{ d : d^{n-pl} \geq \inf_{e \in \mathscr{N}(Q_d)} \nu(\bar{Q}_d \backslash e) \Big\}. \tag{16.2.1}$$

Obviously, D is a nondecreasing function of the set Ω. For $\nu = 0$ the number D coincides with the (p,l) inner diameter of the set Ω, introduced in Sect. 14.2.

Theorem 1. *Let* $0 \leq m \leq l$, $p \leq r < \infty$, $l - m > n/p - n/r$. *Then*
(a) *The inequality*

$$\|u\|_{W_r^m(\mathbb{R}^n)} \leq C \|u\|_{\mathring{L}_p^l(\Omega,\nu)} \tag{16.2.2}$$

is valid for all $u \in C_0^\infty(\Omega)$ *if and only if there exist positive constants* d *and* k *such that*

$$\nu(\bar{Q}_d \backslash E) \geq k \tag{16.2.3}$$

for all cubes $\bar{Q}_d$ *in* F_Ω *and for all compacta* E *in* $\mathscr{N}(Q_d)$.
(b) *The best constant in* (16.2.2) *satisfies the estimates*

$$c^{-1} C \leq D^{l - n(1/p - 1/r)} \max\big\{ D^{-m}, 1 \big\} \leq cC. \tag{16.2.4}$$

Proof. We begin with the right inequality in (16.2.4) and with the necessity of condition (16.2.3). From the definition of D it follows that for any $\varepsilon > 0$ there exists a cube $\bar{Q}_d \in F_\Omega$ with

$$d^{n-pl} \geq \inf_{e \in \mathscr{N}(Q_d)} \nu(\bar{Q}_d \backslash e)$$

and with $D \geq d \geq D - \varepsilon$ if $D < \infty$, $d > \varepsilon^{-1}$ if $D = \infty$.

Let $e \in \mathscr{N}(Q_d)$. According to Corollary 13.3/2 the set $E = e \cup (\bar{Q}_d \backslash \Omega)$ satisfies (16.1.4). By Lemma 16.1/2 we can find a function $u \in C_0^\infty(Q_d \backslash E)$ such that

$$\begin{aligned} \|u\|_{\mathring{L}_p^l(\Omega,\nu)} &\leq c_1 \big(d^{-pl} + d^{-n} \nu(\bar{Q}_d \backslash E) \big)^{1/p} \|u\|_{L_p(Q_d)} \\ &\leq c_2 d^{-l} \|u\|_{L_p(Q_d)}. \end{aligned} \tag{16.2.5}$$

Making use of the obvious inequalities

$$\begin{aligned} \|u\|_{L_p(Q_d)} &\leq c_3 \min\big\{ d^{n/p - n/r} \|u\|_{L_r(Q_d)}, d^{n/p - n/r + m} \|\nabla_m u\|_{L_r(\mathbb{R}^n)} \big\} \\ &\leq c_4 d^{n/p - n/r} \min\big\{1, d^m\big\} \|u\|_{W_r^m(\mathbb{R}^n)}, \end{aligned}$$

from (16.2.5) we obtain

$$\|u\|_{\mathring{L}_p^l(\Omega,\nu)} \leq c_5 d^{-l + n/p - n/r} \min\big\{1, d^m\big\} \|u\|_{W_r^m(\mathbb{R}^n)}.$$

Since ε is arbitrarily small, the right inequality in (16.2.4) is proved.

If (16.2.2) holds then the right-hand side of (16.2.4) together with $l > n/p - n/r$ imply $D < \infty$. In this case, for $d = 2D$ and $k = d^{n-pl}$ we have (16.2.3).

Now we shall prove the sufficiency of (16.2.3) and the left inequality in (16.2.4). Cover $\mathbb{R}^n$ by the cubic grid $\{\bar{Q}_d\}$ where d is chosen to satisfy (16.2.3). If the cube Q_d has a (p,l)-essential intersection with $\mathbb{R}^n\backslash\Omega$ then, by Theorem 14.1.2,

$$\|u\|^p_{L_p(Q_d)} \le c d^{pl} \,|\!|\!| u |\!|\!|^p_{p,l,Q_d}.$$

On the other hand, if $\bar{Q}_d\backslash\Omega \notin \mathscr{N}(Q_d)$ then by Lemma 16.1/1

$$\|u\|^p_{L_p(Q_d)} \le c d^{pl} \,|\!|\!| u |\!|\!|^p_{p,l,Q_d} + c k^{-1} d^n \|u\|^p_{L_p(Q_d,\nu)}. \tag{16.2.6}$$

Summing over all cubes of the grid, we obtain

$$\|u\|^p_{L_p(\Omega)} \le c \sum_{j=1}^{l} d^{pj} \|\nabla_j u\|^p_{L_p(\Omega)} + c k^{-1} d^n \|u\|^p_{L_p(\Omega,\nu)}. \tag{16.2.7}$$

Applying (15.4.4) and Hölder's inequality from (16.2.7) we obtain

$$\|u\|^p_{L_p(\Omega)} \le c d^{pl} \|\nabla_l u\|^p_{L_p(\Omega)} + c k^{-1} d^n \|u\|^p_{L_p(\Omega,\nu)}. \tag{16.2.8}$$

Since the embedding operator of W_p^l into W_r^m is continuous, the sufficiency of (16.2.3) is proved.

Let

$$R = \max\bigl\{d, \bigl(d^n k^{-1}\bigr)^{1/pl}\bigr\}.$$

Then by (16.2.8) we have

$$\|u\|_{L_p} \le c R^l \|u\|_{\mathring{L}_p^l(\Omega,\nu)},$$

which implies the following estimate for the best constant C in (16.2.2):

$$C \le c \sup_{u\in\mathscr{D}} \frac{\|\nabla_m u\|_{L_r} + \|u\|_{L_r}}{\|\nabla_l u\|_{L_p} + R^{-l}\|u\|_{L_p}}.$$

Replace $u(x/R)$ by u. Then

$$\begin{aligned} C &\le c \sup_{u\in\mathscr{D}} \frac{R^{-m+n/r}\|\nabla_m u\|_{L_r} + R^{n/r}\|u\|_{L_r}}{R^{n/p-l}(\|\nabla_l u\|_{L_p} + \|u\|_{L_p})} \\ &\le c R^{l-n/p+n/r} \max\bigl\{R^{-m}, 1\bigr\} \sup_{u\in\mathscr{D}} \frac{\|u\|_{W_r^m}}{\|u\|_{W_p^l}}. \end{aligned} \tag{16.2.9}$$

We may assume that $D < \infty$. The definition of D implies the validity of (16.2.3) for $d = 2D$, $k = d^{n-pl}$. Inserting $\lambda = 2D$ into (16.2.9) we arrive at the left inequality in (16.2.4). The theorem is proved. □

Since for $pl > n$ the only negligible set is the empty set, Theorem 1 can, in this case, be restated in the following equivalent formulation without the notion of capacity.

Theorem 2. *Let $pl > n$, $0 \le m \le l$, $p \le r < \infty$, and $l - m > n/p - n/r$. Then*:

(a) *Inequality* (16.2.2) *holds for all $u \in \mathscr{D}(\Omega)$ if and only if the estimate $\nu(Q_d) > k$ is valid for some $d > 0$, $k > 0$ for all cubes Q_d with $\bar{Q}_d \subset \Omega$.*

(b) *The best constant C in* (16.2.2) *satisfies* (16.2.4) *with*

$$D = D_{p,l}(\nu,\Omega) = \sup_{\bar{Q}_d \subset \Omega} \big\{ d : d^{n-pl} \ge \nu(\bar{Q}_d) \big\}. \tag{16.2.10}$$

Part (a) of Theorem 1 can also be simplified for $pl = n$ when $\mathbb{R}^n \backslash \Omega$ is connected.

Theorem 3. *Let $pl = n$ and let $\mathbb{R}^n \backslash \Omega$ be connected. Inequality* (16.2.2) *is valid for all $u \in \mathscr{D}(\Omega)$ if and only if there exist constants $d > 0$, $k > 0$ such that* (16.2.3) *holds for all cubes Q_d with $\bar{Q}_d \subset \Omega$ and for all (p,l)-negligible compacta $F \subset \bar{Q}_d$.*

Proof. We need only establish the sufficiency since the necessity is contained in Theorem 1.

Let $\bar{Q}_d$ be a cube of the coordinate grid having a nonempty intersection with $\mathbb{R}^n \backslash \Omega$. Then Q_{2d} contains a continuum in $\mathbb{R}^n \backslash \Omega$ with a length not less than d. So according to Proposition 13.1.2/2 we have

$$\operatorname{cap}\big(\bar{Q}_{2d} \backslash \Omega, \mathring{L}_p^l(Q_{4d})\big) \ge c.$$

This and Theorem 14.1.2 imply

$$\|u\|^p_{L_p(Q_d)} \le c d^{pl} \| u \|^p_{p,l,Q_d}, \quad u \in \mathscr{D}(\Omega). \tag{16.2.11}$$

The latter estimate, applied to each cube $\bar{Q}_d$ that intersects $\mathbb{R}^n \backslash \Omega$, together with inequality (16.2.6) for cubes $\bar{Q}_d \subset \Omega$ leads to (16.2.7). The further arguments are just the same as in the proof of the sufficiency of condition (16.2.3) in Theorem 1. The theorem is proved. □

16.3 Compactness of the Embedding Operator $\mathring{L}_p^l(\Omega,\nu) \to W_r^m(\Omega)$

16.3.1 Essential Norm of the Embedding Operator

Let E be the identity mapping of the space $C_0^\infty(\Omega)$ considered as an operator from $\mathring{L}_p^l(\Omega,\nu)$ into $W_r^m(\mathbb{R}^n)$.

With E we associate its essential norm, i.e., the value

$$\varrho = \varrho_{p,l,m} = \inf_{\{T\}} \|E - T\|, \tag{16.3.1}$$

where $\{T\}$ is the set of all compact operators

$$\mathring{L}_p^l(\Omega,\nu) \to W_r^m(\mathbb{R}^n).$$

Theorem. *Let* $0 \le m < l$, $p \le r < \infty$, $l - m > n/p - n/r$. *Then*:
(a) $\varrho < \infty$ *if and only if* $D = D_{p,l}(\nu,\Omega) < \infty$;
(b) *there exists a constant* $c > 1$ *such that*

$$c^{-1}\varrho \le \overset{\infty}{D}{}^{l-n/p+n/r} \max\{\overset{\infty}{D}{}^{-m}, 1\} \le c\varrho \tag{16.3.2}$$

with

$$\overset{\infty}{D} = \overset{\infty}{D}_{p,l}(\nu,\Omega) = \lim_{N\to\infty} D_{p,l}(\nu, \Omega\backslash\bar{Q}_N), \tag{16.3.3}$$

where Q_N *is a cube with center* 0 *and edge length* N.

Proof. Part (a) follows from Theorem 16.2/1. We prove the left-hand side of (16.3.2). Let T_N be the operator of multiplication by $\eta(N^{-1}x)$, $N = 1, 2, \dots$, where $\eta \in C_0^\infty(Q_2)$ $\eta = 1$ on Q_1. By Theorem 16.2/1, for any $u \in C_0^\infty(Q_2)$ we have

$$c^{-1}\|u\|_{\mathring{L}_p^l(\Omega,\nu)} \le \|\nabla_l u\|_{L_p} + D^{-l}\|u\|_{L_p} + \|u\|_{L_p(\Omega,\nu)} \le c\|u\|_{\mathring{L}_p^l(\Omega,\nu)}. \tag{16.3.4}$$

Hence if $N \ge D$ then

$$\|T_N\|_{\mathring{L}_p^l(\Omega,\nu)\to\mathring{L}_p^l(\Omega,\nu)} \le c. \tag{16.3.5}$$

From (16.3.4) and the well-known compactness theorem it follows that the mapping $T_N : \mathring{L}_p^l(\Omega,\nu) \to W_r^m$ is compact. Applying Theorem 16.2/1 to the set $\Omega\backslash\bar{Q}_N$ we obtain for any $u \in C_0^\infty(\Omega)$ that

$$\|(E - T_N)\|_{W_r^m} \le c D_N^{l-n/p+n/r} \max\{D_N^{-m}, 1\} \|(E - T_N)u\|_{L_p^l(\Omega,\nu)},$$

where $D_N = D_{p,l}(\nu, \Omega\backslash\bar{Q}_N)$. Hence, using (16.3.5) and passing to the limit as $N \to \infty$, we arrive at the left inequality in (16.3.2).

We prove the right inequality in (16.3.2). We may assume that $\overset{\infty}{D} \ne 0$. Obviously $D_1 \ge D_2 \ge \cdots \ge \overset{\infty}{D}$. Using the same arguments as in the proof of the right inequality (16.3.4), for any large enough number N we construct a function $u_N \in C_0^\infty(\Omega\backslash\bar{Q}_N)$ with the diameter of the support not exceeding $2\overset{\infty}{D}$ and such that

$$\|u_N\|_{\mathring{L}_p^l(\Omega,\nu)} \le c\overset{\infty}{D}{}^{-l}, \qquad \|u_N\|_{L_p} = 1. \tag{16.3.6}$$

Obviously,

$$\begin{aligned}\|u_N\|_{L_p} &= \overset{\infty}{D}{}^{n/p} \|G_{\overset{\infty}{D}} u_N\|_{L_p} \\ &\le c_1 \overset{\infty}{D}{}^{n/p} \min\{\|G_{\overset{\infty}{D}} u_n\|_{L_r}, \|\nabla_m G_{\overset{\infty}{D}} u_N\|_{L_r}\},\end{aligned} \tag{16.3.7}$$

where G_a is the operator defined by $(G_a u)(x) = u(ax)$. We choose a sequence $\{N_i\}_{i\ge 1}$ so that the distance between the supports of u_{N_i} and u_{N_j} $(i \ne j)$ is more than $2\sqrt{n}\overset{\infty}{D}$. From (16.3.6) and (16.3.7) we obtain

$$\begin{aligned}1 &\le c_2 \overset{\infty}{D}{}^{n/p} \min\big\{\big\|G_{\overset{\infty}{D}}(u_{N_j} - u_{N_i})\big\|_{L_r}, \big\|\nabla_m G_{\overset{\infty}{D}}(u_{N_i} - u_{N_j})\big\|_{L_r}\big\} \\ &\le c_3 \overset{\infty}{D}{}^{n/p-n/r} \min\big\{\overset{\infty}{D}{}^m, 1\big\} \|u_{N_i} - u_{N_j}\|_{W_r^m}.\end{aligned} \tag{16.3.8}$$

We denote an arbitrary compact operator by $T : \mathring{L}_p^l(\Omega,\nu) \to W_r^m$. Passing, if necessary, to a subsequence we may assume that the sequence $\{Tu_{N_i}\}$ converges in W_r^m. Further,

$$\big\|(E-T)(u_{N_i} - u_{N_j})\big\|_{W_r^m} \ge \|u_{N_i} - u_{N_j}\|_{W_r^m} - \big\|T(u_{N_i} - u_{N_j})\big\|_{W_r^m},$$

which along with (16.3.8) shows that

$$c_5 \limsup_{N_i, N_j \to \infty} \big\|(E-T)(u_{N_i} - u_{N_j})\big\|_{W_r^m} \ge \overset{\infty}{D}{}^{l-n/p+n/r} \max\big\{\overset{\infty}{D}{}^{-m}, 1\big\}.$$

This and (16.3.6) imply

$$c_6 \|E - T\|_{\mathring{L}_p^l(\Omega,\nu) \to W_r^m} \ge \overset{\infty}{D}{}^{l-n/p+n/r} \max\big\{\overset{\infty}{D}{}^{-m}, 1\big\}.$$

The proof is complete. □

16.3.2 Criteria for Compactness

From Theorem 16.3.1 we immediately obtain the following criterion.

Theorem 1. *Let $0 \le m \le l$, $p \le r < \infty$, and $l - m > n(p^{-1} - r^{-1})$. Then the set*

$$\mathfrak{F} = \big\{u \in \mathscr{D}(\Omega) : \|u\|_{\mathring{L}_p^l(\Omega,\nu)} \le 1\big\}$$

is relatively compact in W_r^m if and only if $D = D_{p,l}(\nu, \Omega) < \infty$ and

$$\lim_{N\to\infty} D_{p,l}(\nu, \Omega \backslash \bar{Q}_N) = 0.$$

The preceding theorem admits the equivalent formulation.

Theorem 2. *Let $0 \le m \le l$, $p \le r < \infty$, $l - m > n(p^{-1} - r^{-1})$. Then the set $\mathfrak{F}$ is relatively compact in W_r^m if and only if:*

(a) *there exist positive constants d_0 and k such that inequality (16.2.3) is valid for any cube Q_{d_0} with $\bar{Q}_{d_0} \subset F_\Omega$ and for any compactum $E \in \mathscr{N}(Q_{d_0})$;*

(b) *we have*

$$\inf_{e\in\mathscr{N}(Q_d)} \nu(\bar{Q}_d\backslash e) \to \infty \tag{16.3.9}$$

as the cube Q_d, $\bar{Q}_d \subset F_\Omega$, *tends to infinity, where* d *is an arbitrary positive number.*

Proof. Necessity. Inequality (16.2.3) follows from Theorem 16.2/1. If (16.3.9) is not valid then there exists a sequence of cubes $\{\mathscr{Q}^{(i)}\}_{i\geq 1}$, with a limit point at infinity and satisfying the conditions (i) the set $\{d_i\}$ of the edge lengths of the cubes $\mathscr{Q}^{(i)}$ is bounded, (ii) the inequality

$$\inf_{e\in\mathscr{N}(\mathscr{Q}^{(i)})} \nu\big(\overline{\mathscr{Q}^{(i)}}\backslash e\big) < c_0$$

holds. This and Lemma 16.1/2 imply that we can find a sequence of functions $\{u_i\}_{i\geq 1}$ in $\mathscr{D}(\Omega)$ with

$$\operatorname{diam}\operatorname{supp} u_i \leq c, \qquad \operatorname{dist}(\operatorname{supp} u_i, \operatorname{supp} u_j) \geq c > 0, \quad i \neq j,$$
$$\|u_i\|_{\mathring{L}_p^l(\Omega,\nu)} \leq c, \qquad \|u_i\|_{W_r^m} \geq c_1, \qquad \|u_i\|_{L_p} \geq c_2 > 0.$$

Sufficiency. Let conditions (16.2.3) and (16.3.9) be valid. By Theorem 16.2/1 and (16.2.3) the value $D_{p,l}(\nu,\Omega)$ is finite. Now (16.3.9) and (16.2.8) yield

$$\lim_{N\to\infty} \sup_{u\in C_0^\infty(\Omega\backslash\bar{Q}_N)} \|u\|_{W_r^m}/\|u\|_{\mathring{L}_p^l(\Omega,\nu)} = 0.$$

This and the right inequality (16.2.4) applied to $\Omega\backslash\bar{Q}_N$ imply

$$\lim_{N\to\infty} D_{p,l}(\nu,\Omega\backslash\bar{Q}_N) = 0.$$

Hence, $\mathfrak{F}$ is relatively compact in W_r^m by virtue of Theorem 1. The proof is complete. □

Clearly, in the case $pl > n$ Theorem 2 can be stated as follows.

Theorem 3. *Let* $pl > n$, $0 \leq m < l$, $p \leq r < \infty$, *and* $l - m > n/p - n/r$. *The set* $\mathfrak{F}$ *is relatively compact in* W_r^m *if and only if the condition of Theorem 16.2/1 holds and if*

$$\nu(Q_d) \to \infty \tag{16.3.10}$$

as the cube Q_d *with* $\bar{Q}_d \subset \Omega$ *tends to infinity.*

In the case $pl = n$ under the hypothesis of the connectedness of $\mathbb{R}^n\backslash\Omega$ the statement of Theorem 2 can be simplified in the following way.

Theorem 4. *Let* $pl = n$, $0 \leq m < l$, $p \leq r < \infty$, $m < n/r$, *and suppose the complement of* Ω *is connected. Then* $\mathfrak{F}$ *is relatively compact in* W_r^m *if and* *only if the condition of Theorem 16.2/3 holds and (16.3.9) is valid as the cube Q_d, $\bar{Q}_d \subset \Omega$, tends to infinity.*

Proof. Taking into account Theorem 2 we can limit ourselves to proving sufficiency. According to the Gagliardo–Nirenberg theorem (see Theorem 1.4.7),

$$\|\nabla_k u\|_{L_r} \le c\|\nabla_l u\|^{\beta}_{L_p}\|u\|^{1-\beta}_{L_p},$$

where $0 \le k \le l$ and $\beta = l^{-1}(n/p - n/r + k)$. Hence, it suffices to obtain the relative compactness of $\mathfrak{F}$ in L_p. By Theorem 16.2/3, $\mathfrak{F}$ is bounded in $W^l_p(\mathbb{R}^n)$ and consequently it is relatively compact in $L_p(B_\varrho)$ for any $\varrho \in (0,\infty)$. It remains to show that for any ε we can find a $\varrho = \varrho(\varepsilon)$ such that

$$\|u\|_{L_p(\mathbb{R}^n\setminus B_\varrho)} \le \varepsilon. \tag{16.3.11}$$

Put $d^l = \varepsilon$ in (16.1.1) and choose $\varrho = \varrho(\varepsilon)$ so that for any cube $\bar{Q}_d \subset \Omega$ that intersects $\mathbb{R}^n\setminus B_\varrho$ we have

$$\inf_{e\in\mathscr{N}(Q_d)} \nu(\bar{Q}_d\setminus e) \ge \varepsilon^{-p} d^n.$$

Then for such cubes

$$\|u\|_{L_p(Q_d)} \le c\varepsilon\big(|\!\!|u|\!\!|_{p,l,Q_d} + \|u\|_{L_p(Q_d,\nu)}\big). \tag{16.3.12}$$

For all $\bar{Q}_d$ that intersect both $\mathbb{R}^n\setminus\Omega$ and $\mathbb{R}^n\setminus B_\varrho$ the estimate

$$\|u\|_{L_p(Q_d)} \le c\varepsilon|\!\!|u|\!\!|_{p,l,Q_d} \tag{16.3.13}$$

holds by (16.2.11). We take the pth power of (16.3.12) and of (16.3.13) and then sum them over all cubes of the coordinate grid with edge length d which have a nonempty intersection with $\mathbb{R}^n\setminus B_\varrho$. This implies (16.3.11). The theorem is proved. □

16.4 Closability of Embedding Operators

Let ν be a measure in $\mathbb{R}^n$. We call it (p,l) *absolutely continuous* if the equality $\mathrm{cap}(B, W^l_p) = 0$, where B is a Borel set, implies $\nu(B) = 0$.

For instance, by Proposition 10.4.3/2 the Hausdorff φ-measure is (p,l) absolutely continuous provided the integral (10.4.16) converges.

Let $\mathscr{E}$ be the identity operator defined on $\mathscr{D}(\Omega)$, which maps $L_p(\Omega)$ into $\mathring{L}^l_p(\Omega,\nu)$.

Theorem 1. *Let $n \ge pl$, $p > 1$. The operator $\mathscr{E}$ is closable if and only if the measure ν is (p,l) absolutely continuous.*

Proof. Sufficiency. Suppose a sequence of functions $\{u_k\}_{k\ge1}$ in $\mathscr{D}(\Omega)$ converges to zero in $L_p(\Omega)$ and that it is a Cauchy sequence in $\mathring{L}^l_p(\Omega,\nu)$. Further, let $D^\alpha u_k \to v_\alpha$ in $L_p(\Omega)$ for any multi-index α with $|\alpha| = l$. Then for all $\varphi \in \mathscr{D}(\Omega)$

$$(-1)^{|\alpha|} \int_\Omega v_\alpha \varphi \, \mathrm{d}x = \lim_{k\to\infty} \int_\Omega u_k D^\alpha \varphi \, \mathrm{d}x = 0$$

and $v_\alpha = 0$ almost everywhere in Ω. Consequently, $u_k \to 0$ in $\mathring{W}_p^l(\Omega)$. The sequence $\{u_k\}$ contains a subsequence $\{w_k\}$ that converges to zero (p,l)-quasi-everywhere (see 10.4.4). The (p,l)-absolute continuity of the measure ν implies $w_k \to 0$ ν-almost everywhere. Since $\{u_k\}$ is a Cauchy sequence in $L_p(\Omega, \nu)$ then $u_k \to 0$ in the same space.

Necessity. Suppose there exists a Borel set $B \subset \Omega$ with $\operatorname{cap}(B, W_p^l) = 0$ and $\nu(B) > 0$. Let F denote a compact subset of B satisfying $2\nu(F) > \nu(B)$. Further let $\{\omega_k\}_{k\geq 0}$ be a sequence of open sets with the properties 1. $F \subset \bar{\omega}_{k+1} \subset \omega_k \subset \Omega$, 2. $\operatorname{cap}(\bar{\omega}_k, W_p^l) \to 0$, and 3. $\nu(\omega_k) \to \nu(F)$. We introduce the capacitary measure μ_k and the capacitary Bessel potential $w_k = V_{p,l}\mu_k$ of the set $\bar{\omega}_k$ (see Proposition 10.4.2/2).

We show that $w_k \to 0$ in the space $C(e)$ where e is any compactum that does not intersect F. Let $\delta = \operatorname{dist}(e, F)$ and let k be a number satisfying $2\operatorname{dist}(e, \bar{\omega}_k) > \delta$. We set

$$g_k(y) = \left[\int G_l(y-z)\,\mathrm{d}\mu_k(z)\right]^{1/(p-1)},$$

(see the definition of the Bessel potential in 10.1.2). If $4|x-y| < \delta$ then $4|z-y| \geq \delta$ for all $z \in \bar{\omega}_k$ and consequently

$$g_k(y) \leq c(\delta)\big[\mu_k(\bar{\omega}_k)\big]^{1/(p-1)} = c(\delta)\big[\operatorname{cap}\big(\bar{\omega}_k, W_p^l\big)\big]^{1/(p-1)}.$$

Therefore,

$$\int_{4|x-y|\leq\delta} G_l(x-y) g_k(y)\,\mathrm{d}y \leq c_1(\delta)\big[\operatorname{cap}\big(\bar{\omega}_k, W_p^l\big)\big]^{1/(p-1)}.$$

On the other hand,

$$\int_{4|x-y|>\delta} G_l(x-y) g_k(y)\,\mathrm{d}y \leq c_2(\delta)\|g_k\|_{L_p} = c_2(\delta)\big[\operatorname{cap}\big(\bar{\omega}_k, W_p^l\big)\big]^{1/(p-1)}.$$

These estimates and the equality

$$w_k(x) = \int G_l(x-y) g_k(y)\,\mathrm{d}y$$

imply $w_k \to 0$ in $C(e)$.

The arguments used in the proof of Theorem 13.3/1 show that the functions $v_k = 1 - [(1-w_k)_+]^l$ satisfy

$$\|v_k\|_{W_p^l} \leq c\|w_k\|_{W_p^l}. \tag{16.4.1}$$

Moreover, it is clear that $v_k = 1$ in a neighborhood of F, $0 \le v_k \le 1$ in $\mathbb{R}^n$ and that the sequence $\{v_k\}$ converges uniformly to zero on any compactum that does not intersect F.

Let u_k denote a mollification of v_k with small enough radius. Using the equality $\lim \operatorname{cap}(\bar{\omega}_k, W_p^l) = 0$ and the estimate (16.3.9) we obtain $\|u_k\|_{W_p^l} \to 0$. Let η be any function in $\mathfrak{P}(F,\Omega)$ with support S. Clearly, $\eta u_k = 1$ in a neighborhood of F and $\|\eta u_k\|_{W_p^l} \to 0$. We show that ηu_k is a Cauchy sequence in $L_p(\Omega,\nu)$. Let ε be an arbitrary positive number and let M be so large that $\nu(\omega_M \backslash F) < \varepsilon$. We choose an integer N to satisfy

$$\big|u_k(x)\big|^p \le \varepsilon/\nu(S), \quad x \in S\backslash\omega_M$$

for $k > N$. Hence, for any $k, l > N$, we obtain

$$\int_\Omega \big|\eta(u_k - u_l)\big|^p \,\mathrm{d}\nu \le c\bigg(\int_{S\backslash\omega_M} |u_k - u_l|^p \,\mathrm{d}\nu + \int_{\omega_M\backslash F} |u_k - u_l|^p \,\mathrm{d}\nu\bigg) < c\varepsilon.$$

Therefore ηu_k is a Cauchy sequence in $L_p(\Omega,\nu)$. Since $u_k = 1$ on F then

$$\|\eta u_k\|^p_{L_p(\Omega,\nu)} \ge \nu(F) > 0.$$

The existence of such a sequence contradicts the closability of the operator $\mathscr{E}$. The theorem is proved. □

In the case $pl > n$ the operator $\mathscr{E}$ is always closable, which trivially follows from the Sobolev theorem on the embedding $W_p^l(\Omega) \subset C(\Omega)$.

Theorem 2. *The identity operator defined on $\mathscr{D}(\Omega)$, which maps $\mathring{W}_p^l(\Omega)$ into $L_p(\Omega,\nu)$, is closable if and only if the measure ν is (p,l) absolutely continuous.*

Proof. Sufficiency. Let $\{u_k\}_{k\ge1}$, $u_k \in \mathscr{D}(\Omega)$ be a Cauchy sequence in $L_p(\Omega,\nu)$ that converges to zero in $\mathring{W}_p^l(\Omega)$. Then it converges to zero in $L_p(\Omega,\nu)$ by the previous theorem.

Necessity. Suppose there exists a Borel set B with $\operatorname{cap}(B, W_p^l) = 0$ and $\nu(B) > 0$. Under this condition, in Theorem 1 we constructed a sequence of functions $u_k \in \mathscr{D}(\Omega)$, which is a Cauchy sequence in $L_p(\Omega,\nu)$ that converges to zero in $\mathring{W}_p^l(\Omega)$ and does not converge to zero in $L_p(\Omega,\nu)$. The result follows. □

Theorem 3. 1. *The identity operator defined on $\mathscr{D}(\Omega)$, which maps $\mathring{L}_p^l(\Omega)$ into $L_p(\Omega,\nu)$ $(lp < n)$, is closable if and only if the measure ν is (p,l) absolutely continuous.*

2. *The same holds for $n = pl$ provided $\mathbb{R}^n\backslash\Omega$ is a set of positive (p,l)-capacity.*

Proof. 1. *Sufficiency.* Let a sequence of functions u_k in $\mathscr{D}(\Omega)$ be a Cauchy sequence in $L_p(\Omega,\nu)$ that converges to zero in $\mathring{L}_p^l(\Omega)$. Since $lp < n$, we have the embedding $\mathring{L}^l_p(\Omega) \subset L_q(\Omega)$, $q = pn(n-pl)^{-1}$ and hence $\{u_k\}$ converges to zero in n-dimensional Lebesgue measure. Further, using the same approach as in the proof of sufficiency in Theorem 1, from $\{u_k\}$ we can select a subsequence $\{w_k\}$ that converges to zero ν almost everywhere.

Necessity follows from Theorem 2.

2. The case $n = pl$ is considered in the same way, making use of the embedding $\mathring{L}^l_p(\Omega) \subset L_q(\Omega, \text{loc})$ (see Sect. 15.3). □

16.5 Application: Positive Definiteness and Discreteness of the Spectrum of a Strongly Elliptic Operator

We define two quadratic forms

$$\mathfrak{A}(u,u) = \sum_{|\alpha|=|\beta|=l} \int_\Omega a_{\alpha\beta}(x) D^\alpha u D^\beta \bar{u}\, dx,$$

$$\mathfrak{B}(u,u) = \mathfrak{A}(u,u) + \int_\Omega |u|^2\, d\nu,$$

where $u \in \mathscr{D}(\Omega)$, $a_{\alpha\beta}$ are measurable functions and ν is a measure in Ω. Suppose

$$\varkappa_1 \|\nabla_l u\|^2_{L_2(\Omega)} \le \mathfrak{A}(u,u) \le \varkappa_2 \|\nabla_l u\|^2_{L_2(\Omega)}$$

for all $u \in \mathscr{D}(\Omega)$.

By definition the form $\mathfrak{B}$ is closable in $L_2(\Omega)$ if any Cauchy sequence $u_m \in \mathscr{D}(\Omega)$ that converges to zero in $L_2(\Omega)$ in the norm $(\mathfrak{B}(u,u))^{1/2}$ converges to zero in the norm $(\mathfrak{B}(u,u))^{1/2}$.

This condition implies the existence of the unique selfadjoint operator B in $L_2(\Omega)$ with

$$(u, Bv) = \mathfrak{B}(u,v) \quad \text{for all } u, v \in \mathscr{D}(\Omega).$$

Clearly, the closability of the form $\mathfrak{B}$ is necessary for the existence of B.

By Theorem 16.4/1, $\mathfrak{B}$ is closable in $L_2(\Omega)$ if and only if the measure ν is (p,l) absolutely continuous. We shall assume that ν has this property.

The next assertion is a particular case of Theorem 16.2/1.

Theorem 1. 1. *The operator B is positive definite if and only if*

$$\nu(Q_d \backslash F) \ge k \tag{16.5.1}$$

for all cubes $\bar{Q}_d$ having $(2,l)$-negligible intersection with $\mathbb{R}^n \backslash \Omega$ and for all $(2,l)$-negligible compacta $F \subset \bar{Q}_d$ with certain $d > 0$, $k > 0$.

2. *The lower bound Λ of the spectrum of the operator B satisfies*

$$c_1 \varkappa_1 D^{-2l} \le \Lambda \le c_2 \varkappa_2 D^{-2l}, \tag{16.5.2}$$

where c_1 and c_2 are constants that depend only on n and l and D is defined by (16.2.1) with $p = 2$.

Corollary 1. 1. *For $2l > n$ a necessary and sufficient condition for the positive definiteness of B is the inequality $\nu(Q_d) \ge k$, where d and k are certain positive constants and $\bar{Q}_d$ is any cube in Ω.*

2. *The lower bound Λ of the spectrum of B satisfies (16.5.2) with D defined by (16.2.10) for $p = 2$.*

Corollary 2. *If $2l = n$ and the set $\mathbb{R}^n \backslash \Omega$ is connected, then a necessary and sufficient condition for the positive definiteness of B is the validity of (16.5.1) for all cubes $\bar{Q}_d \subset \Omega$ and for all $(2, n/2)$-negligible compacta $F \subset \bar{Q}_d$ with certain $d > 0$ and $k > 0$.*

This corollary is a special case of Theorem 16.2/3.

Theorem 2. *The lower bound Γ of points of condensation of the spectrum of B satisfies*

$$c_1 \varkappa_1 \overset{\infty}{D}{}^{-2l} \le \Gamma \le c_2 \varkappa_2 \overset{\infty}{D}{}^{-2l}, \tag{16.5.3}$$

where $\overset{\infty}{D}$ is defined by (16.3.3) with $p = 2$.

Proof. Let ϱ be the number defined by (16.3.1) for $p = 2$. By Theorem 16.3.1 it suffices to prove that $\Gamma = \varrho^{-2}$. From Theorems 16.3.1 and 1 it follows that $\Gamma = 0$ if and only if $\varrho = \infty$. So we may suppose $\Gamma \ne 0$ and $\varrho \ne \infty$.

We introduce the family $\{E_\lambda\}$ of orthogonal projective operators that form a resolution of the identity generated by the selfadjoint operator B in $L_2(\Omega)$. Then

$$\lambda^{-1} \mathfrak{B}(u, u) \ge \big\| (E - E_\lambda) u \big\|^2_{L_2(\Omega)}. \tag{16.5.4}$$

Since the projective operator E_λ is finite-dimensional for $\lambda < \Gamma$, then (16.5.4) and the definition of ϱ imply $\Gamma \le \varrho^{-2}$.

For any ε, $0 < \varepsilon < \Gamma$, in $L_2(\Omega)$ there exists an orthogonal and normalized infinite sequence of functions $\{\varphi_i\}_{i \ge 1}$ such that $(B\varphi_i, \varphi_j) = 0$ for $i \ne j$ and $\Gamma - \varepsilon \le (B\varphi_i, \varphi_j) \le \Gamma + \varepsilon$.

Let T be an arbitrary compact operator mapping $\mathring{L}_2^l(\Omega, \nu)$ into $L_2(\Omega)$. We have that

$$\begin{aligned} &\frac{\|(E - T)(\varphi_i - \varphi_j)\|_{L_2(\Omega)}}{\|\varphi_i - \varphi_j\|_{\mathring{L}_2^l(\Omega,\nu)}} \\ &\quad \ge \frac{\|\varphi_i - \varphi_j\|_{L_2(\Omega)} - \|T(\varphi_i - \varphi_j)\|_{L_2(\Omega)}}{(B(\varphi_i - \varphi_j), \varphi_i - \varphi_j)^{1/2}} \\ &\quad = \frac{(\|\varphi_i\|^2_{L_2(\Omega)} + \|\varphi_j\|^2_{L_2(\Omega)})^{1/2} - \|T(\varphi_i - \varphi_j)\|_{L_2(\Omega)}}{[(B\varphi_i, \varphi_i) + (B\varphi_j, \varphi_j)]^{1/2}} \\ &\quad \ge (\Gamma + \varepsilon)^{-1/2} - 2^{-1/2} (\Gamma - \varepsilon)^{-1/2} \big\| T(\varphi_i - \varphi_j) \big\|_{L_2(\Omega)}. \end{aligned}$$

The upper limit of the right-hand side as i, $j \to \infty$ equals $(\Gamma + \varepsilon)^{-1/2}$. So by the arbitrariness of ε and the definition of ϱ we obtain $\varrho^2 \ge \Gamma^{-1}$. The theorem is proved. □

The preceding result implies the next theorem.

Theorem 3. *The spectrum of the operator B is discrete if and only if $\overset{\infty}{D} = 0$.*

This theorem is equivalent to the following assertion.

Theorem 4. *The spectrum of the operator B is discrete if and only if*

$$\inf_{F \in \mathscr{N}(Q_d)} \nu(\bar{Q}_d \backslash F) \to \infty \tag{16.5.5}$$

as the cube $\bar{Q}_d$ (d is an arbitrary positive number), having a $(2,l)$-negligible intersection with $\mathbb{R}^n \backslash \Omega$, tends to infinity.

For $2l > n$ this criterion can be reformulated as follows.

Theorem 5. *Let $2l > n$. The operator B has a discrete spectrum if and only if*

$$\nu(\bar{Q}_d) \to \infty$$

as the cube $\bar{Q}_d \subset \Omega$, where $d > 0$, tends to infinity.

Theorem 16.3.2/4 implies the following result.

Theorem 6. *Let $2l = n$ and let $\mathbb{R}^n \backslash \Omega$ be connected. The operator B has a discrete spectrum if and only if (16.5.5) holds as the cube $\bar{Q}_d \subset \Omega$, where $d > 0$, tends to infinity.*

16.6 Comments to Chap. 16

In 1934, K. Friedrichs [292] proved that the spectrum of the Schrödinger operator $-\Delta + V$ in $L_2(\mathbb{R}^n)$ with a locally integrable potential V is discrete provided $V(x) \to +\infty$ as $|x| \to \infty$ (see also Berezin and Shubin [84]). On the other hand, if we assume that V is semibounded in the following, then the discreteness of spectrum easily implies that for every $d > 0$

$$\int_{Q_d} V(x)\,\mathrm{d}x \to +\infty \quad \text{as } Q_d \to \infty, \tag{16.6.1}$$

where Q_d is an open cube with the edge length d and with the edges parallel to coordinate axes, $Q_d \to \infty$ means that the cube Q_d goes to infinity (with fixed d). This was first noticed by A.M. Molchanov in 1953 (see [610]) who also showed that this condition is, in fact, necessary and sufficient in the case $n = 1$, but not sufficient for $n \geq 2$. Moreover, in the same paper Molchanov discovered a modification of condition (16.6.1) that is fully equivalent to the discreteness of the spectrum in the case $n \geq 2$. In other words, he found a criterion for the compactness of the embedding $\mathring{L}^1_2(\Omega, \nu)$ into $L_2(\Omega)$, where $\nu = V\,\mathrm{d}x$. Molchanov's test states that for every $d > 0$

$$\inf_F \int_{Q_d \setminus F} V(x)\,\mathrm{d}x \to +\infty \quad \text{as } Q_d \to \infty, \tag{16.6.2}$$

where the infimum is taken over all compact subsets F of the closure $\bar{Q}_d$ which are called *negligible*. The negligibility of F in the sense of Molchanov means that

$$\operatorname{cap}(F) \le \gamma \operatorname{cap}(Q_d), \tag{16.6.3}$$

where cap is the harmonic capacity and $\gamma > 0$ is a sufficiently small constant. More precisely, Molchanov proved that we can take $\gamma = c_n$ where for $n \ge 3$

$$c_n = (4n)^{-4n}\big(\operatorname{cap}(Q_1)\big)^{-1}. \tag{16.6.4}$$

The notion of negligibility will be studied in detail in Chap. 18 following the article by Maz'ya and Shubin [590].

The results of the present chapter develop the result by Molchanov [610] for the case of higher derivatives. Such criteria were obtained for the space $\mathring{L}_2^l(\Omega,\nu)$ with $2l > n$ (when there is no need for a capacity) by Birman and Pavlov [102]. In the case $q \ge p > 1$, $l = 1, 2, \dots$, the necessary and sufficient conditions for the boundedness and compactness of the embedding operator of $\mathring{L}_p^l(\Omega,\nu)$ into $L_q(\Omega)$ were established by the author [544]. Two-sided estimates for the norm and for the essential norm of this operator are due to the author and Otelbaev [573], where the space with the norm

$$\big\|(-\Delta)^{l/2}u\big\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega,\nu)}$$

is considered (l is any positive number). The embedding theorems for more general spaces with weighted norms were obtained by Oleĭ nik and Pavlov [645], Lizorkin and Otelbaev [506], and Otelbaev [653]. The function

$$D_{p,l}(x) = \sup\Big\{d : d^{n-pl} \ge \inf_{e \in \mathscr{N}(Q_d(x))} \nu\big(Q_d(x)\setminus e\big)\Big\},$$

(compare with (16.2.1)) was used by Otelbaev [652] in the derivation of bounds for the Kolmogorov diameter of the unit ball in $\mathring{L}_p^l(\Omega,\nu)$ measured in $L_p(\Omega)$. We present his result.

The *Kolmogorov diameter* of a set M in a Banach space B is defined to be the number

$$d_k(M,B) = \begin{cases} \inf_{\{L_k\}} \sup_{f \in M} \inf_{g \in L_k} \|f-g\|_B, & k = 1, 2, \dots, \\ \sup_{f \in M} \|f\|_B, & k = 0, \end{cases}$$

where $\{L_k\}$ is the set of subspaces of B with $\dim L_k \le k$.

Theorem. (Otelbaev [652].) *Let M be the unit ball in the space $\mathring{L}_p^l(\Omega,\nu)$ and let $N(\lambda)$ be the number of Kolmogorov diameters of M in $L_p(\Omega)$ which exceed λ^{-1}, $\lambda > 0$. Then*

$$c^{-1}N(c\lambda) \le \lambda^{-n/l} m_n\bigl\{x : D_{p,l}(x) \ge \lambda^{1/l}\bigr\} \le cN\bigl(c^{-1}\lambda\bigr),$$

where c *does not depend on* Ω, ν, *and* λ.

By definition, the embedding operator of $\mathring{L}^l_p(\Omega,\nu)$ into $L_p(\Omega)$ is in *the class* l_θ if

$$\sum_{k=0}^{\infty} \bigl[d_k\bigl(M, L_p(\Omega)\bigr)\bigr]^\theta < \infty.$$

The just-stated Otelbaev theorem shows that *the embedding operator under consideration is in the class* l_θ *if and only if* $\theta l > n$ *and*

$$\int_\Omega \bigl[D_{p,l}(x)\bigr]^{l\theta - n}\,\mathrm{d}x < \infty.$$

An immediate application of these results are two-sided estimates for the eigenvalues of the Dirichlet problem for the operator $(-\Delta)^{l/2} + \nu$ and the conditions for the nuclearity of the resolvent of this operator.

In conclusion we mention the article [39] by Alekseev and Oleinik, where two-sided estimates for Kolmogorov diameters of the unit ball of $\mathring{L}^l_p(\Omega)$ in the space $L_p(\mu,\Omega)$ for $pl > n$ and arbitrary measure μ are found.

17

Approximation in Weighted Sobolev Spaces

17.1 Main Results and Applications

For $p \geq 1$ we define $L^{m,p}(\mathbb{R}^n)$ as the set of distributions u on $\mathbb{R}^n$ such that

$$\|u\|_{m,p} = \left(\sum_{k=1}^{m} \int \left| \nabla_k u(x) \right|^p \mathrm{d}x \right)^{1/p} < \infty.$$

Here $\nabla_k u$ denotes the vector $(D^\alpha u)_{|\alpha|=k}$, $\mathring{L}^{m,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the $L^{m,p}$ seminorm. Now let $\Omega \subset \mathbb{R}^n$ be an open set and let μ be a nontrivial positive Radon measure on Ω. We will study the space $H_\mu^{m,p}(\Omega)$, defined as the completion of $L_p(\mu) \cap L^{m,p}(\mathbb{R}^n) \cap C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{m,p;\mu} = \|u\|_{L_p(\mu)} + \|u\|_{m,p}.$$

The closure of $C_0^\infty(\Omega)$ in $H_\mu^{m,p}(\Omega)$ is denoted $\mathring{H}_\mu^{m,p}(\Omega)$. Note that if $p < n$ then by Sobolev's inequality the elements in $\mathring{L}^{m,p}$ can be identified with functions in L_{p^*}, where $p^* = \frac{np}{n-p}$. If a domain in the notations of a space is not indicated it is assumed to be $\mathbb{R}^n$. The elements in $\mathring{H}_\mu^{m,p}(\Omega)$ are naturally identified with elements in $\mathring{L}^{m,p}$. Note also that $L^{m,p} \subset L_p(\mathbb{R}^n, \mathrm{loc})$.

The theorem to be proved in this chapter is the following. (See Sect. 17.2 for definitions of the capacities $B_{m,p}$ and H_1^{n-m} used in the following.)

Theorem. *Let μ be a nontrivial positive Radon measure concentrated on $\Omega \subset \mathbb{R}^n$ and let $\mathcal{C}$ denote $B_{m,p}$ for $p > 1$ and H_1^{n-m} for $p = 1$. Then*

(i) $\mathring{H}_\mu^{m,p}(\Omega) = H_\mu^{m,p}(\Omega)$ *if either $p \geq n$ or $p < n$ and $\mathcal{C}(\Omega^c) = \infty$.*

Suppose now that $p < n$ and $\mathcal{C}(\Omega^c) < \infty$. Then $\mathring{H}_\mu^{m,p}(\Omega) = H_\mu^{m,p}(\Omega)$ if and only if

(ii) $\mu(\Omega) = \infty$, *when either $m \geq n$, $p = 1$ or $mp > n$, $p > 1$;*

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(iii) $\mu(F^c) = \infty$ *for every closed set* $F \subset \mathbb{R}^n$ *satisfying* $\mathcal{C}(F) < \infty$, *when either* $1 < p \le \frac{n}{m}$ *or* $m < n$, $p = 1$.

Remark. When $1 < p < n$, $mp > n$ or $1 = p < n$, $m \ge n$ the condition $\mathcal{C}(\Omega^c) < \infty$ is just a complicated way of saying that Ω^c should be bounded.

Among other results of this chapter there is the following criterion obtained in Sect. 17.3. *The identity operator defined on* $C_0^\infty(\Omega)$ *and mapping* $L_1(\Omega)$ *into the completion of* $C_0^\infty(\Omega)$ *with respect to the norm*

$$\int |\nabla_m u(x)| \, \mathrm{d}x + \int |u| \, \mathrm{d}\mu$$

is closable if and only if μ *is absolutely continuous with respect to the Hausdorff capacity* H_1^{n-m}. Similar facts for $p > 1$ were established in Sect. 16.4.

To show the usefulness of the above Theorem we give its applications to the problem of equality of the minimal and maximal Dirichlet Schrödinger forms and to the question of selfadjointness of a differential operator.

Consider the quadratic form

$$Q[u,u] = \int_\Omega |\nabla u|^2 \, \mathrm{d}x + \int_\Omega |u|^2 \, \mu(\mathrm{d}x). \tag{17.1.1}$$

The closure of Q defined on the set

$$\{u \in C^\infty(\mathbb{R}^n) : \operatorname{supp} u \subset \Omega, Q[u,u] < \infty\},$$

which may contain functions with noncompact support, will be denoted by $Q_{\max}$. Another quadratic form $Q_{\min}$ is introduced as the closure of Q defined on $C_0^\infty(\Omega)$.

The following necessary and sufficient condition for this equality results directly from the above theorem, where cap stands for the 2-capacity with respect to $\mathbb{R}^n$.

Proposition 1. (i) *If either* $n \le 2$ *or* $n > 2$ *and* $\operatorname{cap}(\mathbb{R}^n \backslash \Omega) = \infty$, *then* $Q_{\max} = Q_{\min}$.

(ii) *Suppose that* $n > 2$ *and* $\operatorname{cap}(\mathbb{R}^n \backslash \Omega) < \infty$. *Then* $Q_{\max} = Q_{\min}$ *if and only if*

$$\mu(\Omega \backslash F) = \infty \quad \textit{for every closed set } F \subset \Omega \textit{ with } \operatorname{cap}(F) < \infty. \tag{17.1.2}$$

Note that the equality $Q_{\max} = Q_{\min}$ is equivalent to (17.1.2) in the particular case $\Omega = \mathbb{R}^n$.

The next example, which illustrates a possible shape of a set with infinite capacity, was treated at the end of the proof of Theorem 2.6.2/3.

Example. Consider the domain Ω complementing the infinite funnel

$$\{x = (x', x_n) : x_n \geq 0,\ |x'| \leq f(x_n)\},$$

where f is a continuous and decreasing function on $[0, \infty)$ subject to $f(t) \leq cf(2t)$. It was shown in the proof of Theorem 2.6.2/3 that $\operatorname{cap}(\mathbb{R}^n \backslash \Omega) < \infty$ if and only if the function $f(t)^{n-2}$ for $n > 3$ and the function $(\log(t/f(t)))^{-1}$ for $n = 3$ are integrable on $(1, \infty)$.

Now we turn to another application of the Theorem concerning the question of selfadjointness.

Proposition 2. *Let $\rho \in L_1(\mathbb{R}^n, \mathrm{loc})$, $n \geq 3$, be a positive function, locally bounded away from zero, and let the operator $A = -\varrho^{-1}\Delta$ be defined on $C_0^\infty(\mathbb{R}^n)$.*

A necessary condition for A to be essentially selfadjoint in $L_2(\varrho)$ is that

$$\int_{F^c} \rho(x)\,\mathrm{d}x = \infty$$

whenever F is a closed set such that $\operatorname{cap}(F) < \infty$.

Proof. Suppose ρ does not satisfy the condition in the theorem. Then by Theorem 17.1 and the Hahn-Banach theorem there is a nonzero function $u \in H_\rho^{1,2}(\mathbb{R}^n)$ such that

$$\int u(x)v(x)\rho(x)\,\mathrm{d}x + \int \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x = 0$$

for all $v \in \mathring{H}_\rho^{1,2}(\mathbb{R}^n) \supset D(\bar{A})$, where $\bar{A}$ is the closure of A. Thus

$$(u, v + \bar{A}v)_{L_2(\rho)} = 0$$

and so $u \in D((I + A)^*) = D(A^*)$ and $u + A^*u = 0$. Now suppose $\bar{A} = A^*$. Then $u \in D(\bar{A})$ and hence

$$\int \left|u(x)\right|^2 \rho(x)\,\mathrm{d}x + \int \left|\nabla u(x)\right|^2 \mathrm{d}x = 0,$$

which implies that $u = 0$. This contradiction shows that $\bar{A}$ is not selfadjoint. □

17.2 Capacities

We will denote different constants, not depending on the essential functions or variables considered, by A. The ball with radius r and centered at x will be denoted by $B(x, r)$. If $x = 0$ we will write only $B(r)$. The annulus $B(R)\backslash B(r)$ is denoted by $A(R, r)$.

We start by listing the convolution kernels needed in the following:

Bessel kernels G_α, Riesz kernels I_α, and the truncated Bessel kernels $G_{\alpha;1}$ defined by

$$G_{\alpha;1}(x) = \theta(x)G_\alpha(x),$$

where $\theta \in C_0^\infty(B(1))^+$ is an arbitrary but fixed function such that $\theta = 1$ on $B(\frac{1}{2})$. For $\alpha \geq 1$ we define

$$K_\alpha = I_1 * G_{\alpha-1}.$$

It is easy to see, analogously to the cases with Riesz or Bessel potentials, that, for $1 < p < n$, $L_0^{m,p} = \{K_m * f : f \in L_p\}$.

Each of these kernels gives rise to the corresponding capacities as follows.

Definition. For $E \subset \mathbb{R}^n$ and $p > 1$ we define

$$\begin{aligned} B_{\alpha,p}(E) &= \inf\big\{\|f\|_p^p : f \in L_p^+,\ G_\alpha * f \geq 1 \text{ on } E\big\}, \\ R_{\alpha,p}(E) &= \inf\big\{\|f\|_p^p : f \in L_p^+,\ I_\alpha * f \geq 1 \text{ on } E\big\}, \\ C_{\alpha,p}(E) &= \inf\big\{\|f\|_p^p : f \in L_p^+,\ K_\alpha * f \geq 1 \text{ on } E\big\}, \end{aligned}$$

and

$$B_{\alpha,p;1}(E) = \inf\big\{\|f\|_p^p : f \in L_p^+,\ G_{\alpha;1} * f \geq 1 \text{ on } E\big\}.$$

It was proved by D.R. Adams [6] that $B_{\alpha,p}$ and $R_{\alpha,p}$ are finite simultaneously, although not comparable, for $1 < p < \frac{n}{\alpha}$. It is not hard to see from the proof that $C_{\alpha,p}$ and $B_{\alpha,p}$ are finite at the same time for $1 < p < n$. The set functions $B_{\alpha,p}$ and $B_{\alpha,p;1}$ are comparable. See D.R. Adams and Hedberg [15] for the proof of this fact. We need the following lemma, which, for technical convenience, is the reason for introducing the truncated Bessel kernel.

Lemma 1. *Let $p > 1$. If $F \subset \mathbb{R}^n$ is closed then*

$$B_{\alpha,p;1}(F) = \inf\big\{\|f\|_p^p : f \in L_p^+,\ C^\infty \ni G_{\alpha;1} * f \geq 1 \ \textit{on}\ F\big\}.$$

Proof. We may assume $B_{\alpha,p;1}(F)$ finite. Let $A_0 = \overline{B(1)}$ and $A_j = \overline{B(j+1)}\backslash B(j)$ for $j \geq 1$. It was proved by D.R. Adams [6] that

$$\sum_{j=1}^\infty B_{\alpha,p}(F \cap A_j) \leq AB_{\alpha,p}(F).$$

Hence

$$\sum_{j=1}^\infty B_{a,p;1}(F \cap A_j) < \infty. \tag{17.2.1}$$

Now choose $f_j \in L_+^p$ such that

$$\|f_j\|_p^p \le 2^{-j} + B_{\alpha,p;1}(F \cap A_j) \quad \text{and} \quad G_{\alpha;1} * f_j \ge 1$$

on a neighborhood of $F \cap A_j$. This can be done since $G_{\alpha;1} * f_j$ is lower semicontinuous. It is seen from the definition of $B_{\alpha,p;1}$ that we may assume that $\operatorname{supp} f_j \subset A_j'$ where we denote $E' = \{x : \operatorname{dist}(x,E) \le 1\}$. By mollification we obtain functions $g_j \in C_0^\infty(A_j'')^+$ such that

$$\|g_j\|_p^p \le A\big(2^{-j} + B_{\alpha,p;1}(F \cap A_j)\big)$$

and $G_{\alpha;1} * g_j \ge 1$ on $F \cap A_j$. Consequently, if we set $g = \sum_{j=M}^\infty g_j$ for some M to be specified later we get $G_{\alpha;1} * g \ge 1$ on $F \backslash B(M)$. Also, since the sum defining g is uniformly locally finite, it follows that $G_{\alpha;1} * g \in C^\infty$ and

$$\|g\|_p^p \le A \sum_{j=M}^\infty \|g_j\|_p^p \le A \sum_{j=M}^\infty \big(2^{-j} + B_{\alpha,p;1}(F \cap A_j)\big). \tag{17.2.2}$$

Now let $\varepsilon > 0$. By (17.2.1) and (17.2.2) we get $\|g\|_p < \varepsilon$ if M is large enough. By the same argument with lower semicontinuity and mollification we can find a function $h \in C_0^\infty(\mathbb{R}^n)^+$ such that

$$\|h\|_p \le B_{\alpha,p;1}\big(F \cap \overline{B(M)}\big)^{1/p} + \varepsilon \quad \text{and} \quad G_{\alpha;1} * h \ge 1 \quad \text{on } F \cap \overline{B(M)}.$$

Setting $f = g + h$ we obtain

$$\|f\|_p \le B_{\alpha,p;1}(F)^{1/p} + 2\varepsilon \quad \text{and} \quad C^\infty \ni G_{\alpha;1} * f \ge 1 \quad \text{on } F.$$

Since ε was arbitrary, the proof is complete. □

For $p = 1$ the appropriate capacities are Hausdorff capacities defined as follows.

Definition 3. Let $0 \le d < n$. Then, for subsets E of $\mathbb{R}^n$, we define

$$H_\rho^d(E) = \inf \sum_{i=1}^\infty r_i^d,$$

where the infimum is taken over all countable coverings $\bigcup_{i=1}^\infty B(x_i, r_i) \supset E$, with $r_i \le \rho$. For $d < 0$ we define $H_\rho^d(E) = H_\rho^0(E)$.

The following lemma is immediate except for (iii) which is a variant of the well-known Frostman lemma [293].

Lemma 2. *Let $0 < d < n$. Then:*

(i) $H_\infty^d(E) \le H_1^d(E) \le AH_\infty^d(E) + A(H_\infty^d(E))^{n/d}$. *In particular H_∞^d and H_1^d are finite at the same time.*

(ii) $\sum_{j=0}^\infty H_1^d(E \cap A_j) \le AH_1^d(E)$, *where A_j is as in the proof of Lemma 17.2.*

(iii) $H_1^d(E)$ *is comparable to* $\sup\{\mu(E) : \mu(B(x,r)) \le r^d, r \le 1, x \in \mathbb{R}^n\}$, *where* μ *is a positive measure, for Borel sets* E.

Lemma 3. *Let* m *be an integer,* $0 < m < n$*. Then for closed sets* $F \subset \mathbb{R}^n$, $H_1^{n-m}(F)$ *is comparable to*

$$\inf\bigl\{\|\varphi\|_1 + \|\nabla_m\varphi\|_1 : \varphi \in C^\infty,\ 0 \le \varphi \le 1,\ \varphi = 1 \text{ on a neighborhood of } F\bigr\}.$$

Proof. Suppose $\varphi \in C^\infty$, $\varphi = 1$ on a neighborhood of F and $\|\varphi\|_1 + \|\nabla_m\varphi\|_1 < \infty$. If μ is a positive measure supported by F then by Theorem 1.4.2/2 we have

$$\mu(F) \le \int \varphi \, d\mu \le A \sup_{x:0<r\le 1} \frac{\mu(B(x,r))}{r^{n-m}} \bigl(\|\varphi\|_1 + \|\nabla_m\varphi\|_1\bigr).$$

Taking the supremum over μ with

$$\sup_{x;0<r\le 1} \frac{\mu(B(x,r))}{r^{n-m}} \le 1,$$

we get by Lemma 2(iii) that $H_1^{n-m}(F) \le A(\|\varphi\|_1 + \|\nabla_m\varphi\|_1)$, which proves one direction of the lemma.

To prove the other direction suppose first that F is compact. Cover F by balls $B(x_i, r_i)$, $r_i \le 1$, $1 \le i \le s$, such that

$$\sum_{i=1}^{s} r_i^{n-m} \le H_1^{n-m}(F) + \varepsilon,$$

where $\varepsilon > 0$. By Lemma 3.1 of Harvey and Polking [356] there are functions $\psi_i \in C_0^\infty(B(x_i, 2r_i))$, $1 \le i \le s$, such that $|D^\alpha\psi_i| \le A_\alpha r_i^{-|\alpha|}$ and such that $\varphi = \sum_{i=1}^{s} \psi_i$ satisfies $\varphi = 1$ on a neighborhood of F. We get

$$\begin{aligned}\|\varphi\|_1 + \|\nabla_m\varphi\|_1 &\le \sum_{i=1}^{s} \int_{B(x_i,2r_i)} \bigl(|\psi_i(x)| + |\nabla_m\psi_i(x)|\bigr)\, dx \\ &\le A \sum_{i=1}^{s} \bigl(r_i^n + r_i^{n-m}\bigr) \le AH_1^{n-m}(F) + A\varepsilon.\end{aligned}$$

Since ε is arbitrary we are done in the case where F is compact.

For the general case we introduce a partition of unity $1 = \sum_{j=0}^{\infty} \zeta_j$, where $0 \le \zeta_j \le 1$, $\zeta_j = 1$ on a neighborhood of A_{2j}, $\operatorname{supp}\zeta_j \subset A'_{2j}$ and $|\nabla_k\zeta_j| \le A$ for $1 \le k \le m$. Here A_j and A'_j are as in the proof of Lemma 1. Now choose functions φ_j corresponding to the sets $F \cap A'_{2j}$ according to the construction for compact sets in a way that

$$\|\varphi_j\|_1 + \|\nabla_m\varphi_j\|_1 \le AH_1^{n-m}(F \cap A'_{2j}) + 2^{-j}\varepsilon,$$

where $\varepsilon > 0$. Letting $\varphi = \sum_{j=0}^{\infty} \zeta_j \varphi_j$ we obtain, using Leibniz's rule and Lemma 2(ii),

$$\begin{aligned}
&\|\varphi\|_1 + \|\nabla_m \varphi\|_1 \\
&\quad \le A \sum_{j=0}^{\infty} \sum_{k=0}^{m} \int_{A'_{2j}} \big|\nabla_k \varphi(x)\big| \,\mathrm{d}x \\
&\quad \le A \sum_{j=0}^{\infty} \int \big(\big|\varphi_j(x)\big| + \big|\nabla_m \varphi_j(x)\big|\big) \,\mathrm{d}x \le \sum_{j=0}^{\infty} \big(H_1^{n-m}(F \cap A'_{2j}) + 2^{-j}\varepsilon\big) \\
&\quad \le 2\varepsilon + A \sum_{j=0}^{\infty} H_1^{n-m}(F \cap A_j) \le 2\varepsilon + A H_1^{n-m}(F).
\end{aligned}$$

Since ε is arbitrary the lemma follows. □

17.3 Applications of Lemma 17.2/3

We record here some generalizations, depending on Lemma 17.2/3, to the case $p = 1$ of some results in Chap. 16. As before let μ be a positive Radon measure concentrated on $\Omega \subset \mathbb{R}^n$ and let W, X, and Y be the completions of $C_0^\infty(\Omega)$ with respect to the norms

$$\|u\|_W = \int \big|u(x)\big| \,\mathrm{d}x + \int \big|\nabla_m u(x)\big| \,\mathrm{d}x,$$

$$\|u\|_X = \int \big|u(x)\big| \,\mathrm{d}\mu + \int \big|\nabla_m u(x)\big| \,\mathrm{d}x,$$

and

$$\|u\|_Y = \int \big|\nabla_m u(x)\big| \,\mathrm{d}x,$$

respectively. Then we have the following theorems.

Theorem 1. *The identity operator defined on $C_0^\infty(\Omega)$ and mapping $L_1(\Omega)$ into X is closable if and only if μ is absolutely continuous with respect to H_1^{n-m}.*

Theorem 2. *The identity operator defined on $C_0^\infty(\Omega)$ and mapping W into $L_1(\mu)$ is closable if and only if μ is absolutely continuous with respect to H_1^{n-m}.*

Theorem 3. *Let $m \le n$. Then the identity operator defined on $C_0^\infty(\Omega)$ and mapping Y into $L_1(\mu)$ is closable if and only if μ is absolutely continuous with respect to H_1^{n-m}.*

Remark. Note that for $m \ge n$ and for $m = n$, respectively, the above condition on absolute continuity is vacuous so the operators are always closable. In the proof of Theorem 1 below "quasi-everywhere" can be read as "everywhere" in this case.

For the proofs we will need two lemmas, where a function u is called H_1^d-quasicontinuous if it is defined H_1^d-quasi-everywhere and if for every $\varepsilon > 0$ there is an open set G such that $.u|_{G^c}$ is continuous and $H_1^d(G) < \varepsilon$.

Lemma 1. *Suppose that* $u_j \in C_0^\infty(\Omega)$ *and that* $\|u_{j+1} - u_j\|_W < 4^{-j}$. *Then* u_j *converges* H_1^{n-m}*-quasi-everywhere to an* H_1^{n-m}*-quasicontinuous function.*

Proof. Let μ be a positive Radon measure such that $\mu(B(x,r)) \le r^{n-m}$ for all $x \in \mathbb{R}^n$ and all $r \le 1$. Then by Theorem 1.4.3 we have

$$\int |u_{j+1} - u_j| \,\mathrm{d}\mu \le A\, 4^{-j}.$$

By monotone convergence, $\tilde{u}(x) = \lim_{j\to\infty} u_j(x)$ exists a.e. with respect to μ. Now, let

$$F = \{x : \tilde{u}(x) \text{ is defined}\}$$

and

$$E_j = \{x \in F : |\tilde{u}(x) - u_j(x)| \ge 2^{-j}\}.$$

Then, by part (iii) of Lemma 17.2/2, $H_1^{n-m}(F^c) = 0$. Also,

$$\mu(E_j) \le A 2^j \int |\tilde{u} - u_j| \,\mathrm{d}\mu \le A 2^{-j},$$

so $H_1^{n-m}(E_j) \le A2^{-j}$. Let $F_k = \bigcup_{j-k}^\infty E_j$. Then $H_1^{n-m}(F_k) \le A2^{-k}$. Thus, given $\varepsilon > 0$, we may choose k and an open set G_k with $H_1^{n-m}(G_k) < \varepsilon$ such that $F_k \cup F^c \subset G_k$. Since $u_j \to \tilde{u}$ uniformly on G_k^c, the lemma is proved. □

Lemma 2. *Suppose that* u *is* H_1^d*-quasicontinuous and that* $E = \{x : u(x) \ne 0\}$ *is a Borel set with* $m_n(E) = 0$. *Then* $H_1^d(E) = 0$.

Proof. Suppose that $H_1^d(E) = c > 0$. There is an open set G with $H_1^d(G) < \varepsilon$ such that $u|_{G^c}$ is continuous. We do not specify ε here because the choice of it depends on a certain constant, appearing later in the proof. However, ε is a fixed positive number less than c.

Let $K \subset E \backslash G$ be a compact set such that $H_1^d(K) > c - \varepsilon$ and set $K_j = \{x : \operatorname{dist}(x, K) \le \frac{1}{j}\}$. By Lemma 17.2/2, part (iii), we can choose measures μ_j supported by K_j such that

$$\sup_{x;0<r\le 1} \frac{\mu_j(B(x,r))}{r^d} \le A$$

and $\mu_j(K_j) = H_1^d(K_j)$.

Define $\phi_j(x) = j^n \phi(jx)$, where $\phi \in C_0^\infty(B(1))$ is a function such that $0 \le \phi \le A$ and $\int \phi = 1$. Set $\nu_j = \phi_j * \mu_j$. Then we have

$$\phi_j * \mu_j(y) = \int \phi_j(y-t)\,\mathrm{d}\mu_j(t) \le Aj^n \mu_j\Big(B\Big(y, \frac{1}{j}\Big)\Big) \le Aj^{n-d}.$$

Thus, for $r \le \frac{1}{j}$

$$\frac{\nu_j(B(x,r))}{r^d} \le A(jr)^{n-d} \le A.$$

For $\frac{1}{j} \le r \le 1$ we have

$$\begin{aligned}
\frac{\nu_j(B(x,r))}{r^d} &\le \frac{1}{r^d}\int \chi_{B(x,r)}(y)\int \phi_j(y-t)\,\mathrm{d}\mu_j(t)\,\mathrm{d}y \\
&= \frac{1}{r^d}\int \chi_{B(x,r)} * \phi_j(t)\,\mathrm{d}\mu_j(t) \le \frac{1}{r^d}\int \chi_{B(x,r+\frac{1}{j})}(t)\,\mathrm{d}\mu_j(t) \\
&\le A\frac{\mu_j(B(x,r+\frac{1}{j}))}{(r+\frac{1}{j})^d} \le A.
\end{aligned}$$

Thus we obtain

$$\sup_{x;0<r\le 1} \frac{\nu_j(B(x,r))}{r^d} \le A.$$

Also,

$$\begin{aligned}
\nu_j(\mathbb{R}^n) &= \iint \phi_j(x-y)\,\mathrm{d}\mu_j(y)\,\mathrm{d}x \\
&= \iint \phi_j(x-y)\,\mathrm{d}x\,\mathrm{d}\mu_j(y) \\
&= \mu_j(K_j) = H_1^d(K_j) \ge H_1^d(K) \ge c-\varepsilon.
\end{aligned}$$

Let $K_j^* = \operatorname{supp} \nu_j$. Then, since $m_n(E) = 0$, we get

$$H_1^d\big(K_j^* \backslash E\big) \ge A^{-1}\nu_j\big(E^c\big) = A^{-1}\nu_j\big(\mathbb{R}^n\big) \ge A^{-1}(c-\varepsilon).$$

Now we are in the position to specify ε: Take any positive ε satisfying $A^{-1}(c-\varepsilon) > \varepsilon$. Then we obtain

$$H_1^d\big(K_j^* \backslash E\big) > H_1^d(G).$$

Hence there are points $x_j \in K_j^* \cap E^c \cap G^c$. We may assume that x_j converges to some point x_0. Since $x_j \in K_j^*$ there are points $y_j \in K$ such that $|x_j - y_j| < \frac{2}{j}$. Then $y_j \to x_0$ so, in particular, $x_0 \in K$. By the continuity of u on G^c we obtain that $0 = u(x_j) \to u(x_0) \ne 0$. From this contradiction we conclude that $H_1^d(E) = 0$ and the lemma is proved. □

Proof of Theorem 1. We start with the sufficiency part. Suppose $\{u_j\} \subset C_0^\infty(\Omega)$ is a Cauchy sequence in X, converging to zero in $L_1(\Omega)$. Then $D^\alpha u_j$ converges in $L_1(\Omega)$ for $|\alpha| = m$ and since obviously $D^\alpha u_j \to 0$ as distributions we get $D^\alpha u_j \to 0$ in $L_1(\Omega)$. Hence, passing to a subsequence, we may assume that u_j converges H_1^{n-m}-quasi-everywhere by Lemma 1. Also, by Lemma 1 we get that, H_1^{n-m}-quasi-everywhere, $u_j \to 0$. Thus $u_j \to 0$ a.e. with respect to the measure μ, and since $\{u_j\}$ is a Cauchy sequence in $L_1(\mu)$, we obtain $u_j \to 0$ in X.

For the necessity part suppose $F \subset \Omega$ is a compact set satisfying $H_1^{n-m}(F) = 0$ and $\mu(F) > 0$. Let $G_j \supset F$ be shrinking open sets such that $H_1^{n-m}(\bar{G}_j) \to 0$ and $\mu(G_j \backslash F) \to 0$ as $j \to \infty$. Then, by Lemma 17.2/3, we can find $\varphi_j \in C_0^\infty(\Omega)$ such that $\varphi_j = 1$ on G_j and

$$\|\varphi_j\|_1 + \|\nabla_m \varphi_j\|_1 \to 0.$$

Moreover, by construction, $\varphi_j \to 0$ uniformly outside every neighborhood of F and there is a compact set $K \subset \Omega$ such that supp $\varphi_j \subset K$ for all j. Now let $\varepsilon > 0$ and choose j so that $\mu(G_j \backslash F) < \varepsilon/4$. Then, for i and k large enough,

$$\int_\Omega |\varphi_i - \varphi_k| \, \mathrm{d}\mu \le \int_{K \backslash G_j} |\varphi_i - \varphi_k| \, \mathrm{d}\mu + 2\mu(G_j \backslash F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{\varphi_j\}$ is a Cauchy sequence in X, converging to zero in $L_1(\Omega)$. However, since

$$\|\varphi_j\|_{L_1(\mu)} \ge \mu(F) > 0,$$

we cannot have $\varphi_j \to 0$ in X. This proves the necessity part. □

The proofs of Theorems 2 and 3 follow the same lines as the proofs of the analogous facts in Sect. 16.4, making use of the proof of Theorem 1.

Using Lemma 17.2/3 we can also obtain the necessary and sufficient conditions for continuity and compactness of the embedding of X into the Sobolev space $W^{k,q}(\mathbb{R}^n)$. To state these theorems we need first some definitions. Let $m < n$. Then a set $F \subset B(x, r)$ is called $(m, 1)$-negligible if

$$H_\infty^{n-m}(F) \le \gamma r^{n-m},$$

where γ is a sufficiently small constant, depending only on m and n. For $m \ge n$ only the empty set is called $(m, 1)$-negligible.

Let $\mathscr{F}(\Omega)$ be the family of all balls $B(x, r)$ such that $B(x, r) \backslash \Omega$ is $(m, 1)$-negligible. Then we define

$$D_{1,m}(\mu, \Omega) = \sup\bigl\{r; B(x, r) \in \mathscr{F}(\Omega), \inf \mu\bigl(B(x, r) \backslash F\bigr) \le r^{n-m}\bigr\},$$

where the infimum is taken over all $(m, 1)$-negligible closed sets $F \subset B(x, r)$. We then have the following two theorems.

Theorem 4. *Let $0 \le k \le m$, $1 \le q < \infty$ and $m - k > n(1 - 1/q)$. Then*

$$\|u\|_q + \|\nabla_k u\|_q \le A \|u\|_X$$

for all $u \in C_0^\infty(\Omega)$ if and only if there are positive constants r and a such that

$$\mu\big(B(x,r)\backslash F\big) \geq a$$

for all balls $B(x,r) \in \mathscr{F}(\Omega)$ *and all* $(m,1)$*-negligible sets* $F \subset B(x,r)$*. The best constant* A *is comparable to*

$$D^{m-n(1-1/q)} \max\big\{D^{-k}, 1\big\},$$

where $D = D_{1,m}(\mu,\Omega)$*.*

Theorem 5. *Let* $0 \leq k \leq m$*,* $1 \leq q < \infty$*, and* $m-k > n(1-1/q)$*. Then* X *is compactly embedded into* $W_q^k(\mathbb{R}^n)$ *if and only if* $D_{1,m}(\mu,\Omega) < \infty$ *and*

$$\lim_{R\to\infty} D_{1,m}\big(\mu, \Omega\backslash\overline{B(R)}\big) = 0.$$

The proofs of these statements are the same as those of the corresponding theorems for $p > 1$ in Sects. 16.2–16.3, relying now on Lemma 17.2/3.

17.4 Proof of Theorem 17.1

We will need some basic results on the function spaces $L^{m,p}$ and $\mathring{L}^{m,p}$. We state first a lemma of the Hardy type (see Sect. 1.3.1).

Lemma 1.

(i) *If* $1 \leq p < n$ *and* $u \in \mathring{L}^{1,p}(\mathbb{R}^n)$ *then*

$$\int \frac{|u(x)|^p}{|x|^p}\,\mathrm{d}x \leq A \int \big|\nabla u(x)\big|^p\,\mathrm{d}x.$$

(ii) *If* $p > n$ *and* $u \in L^{1,p}(\mathbb{R}^n)$ *then*

$$\int_{B(1)^c} \frac{|u(x)|^p}{|x|^p}\,\mathrm{d}x \leq A\bigg(\int_{B(1)} \big|u(x)\big|^p\,\mathrm{d}x + \int \big|\nabla u(x)\big|^p\,\mathrm{d}x\bigg).$$

(iii) *If* $p = n$ *and* $u \in L^{1,p}(\mathbb{R}^n)$ *then*

$$\int_{B(2)^c} \frac{|u(x)|^p}{(|x|\log|x|)^p}\,\mathrm{d}x \leq A\bigg(\int_{B(2)} \big|u(x)\big|^p\,\mathrm{d}x + \int \big|\nabla u(x)\big|^p\,\mathrm{d}x\bigg).$$

Lemma 2. *Let* $1 \leq p \leq n$*. Then for each* $u \in L^{m,p}(\mathbb{R}^n)$ *there is a unique constant* c *such that* $u - c \in \mathring{L}^{m,p}(\mathbb{R}^n)$*.*

We turn now to the proof of Theorem 17.1, divided into four cases starting with the main one.

The case $1 \leq p \leq \frac{n}{m}$ *or* $p = 1$*,* $m < n$*.* We start by proving the sufficiency part. Suppose that μ satisfies the condition in the theorem. We will show then that $C_\Omega^\infty \cap H_\mu^{m,p} \subset \mathring{L}^{m,p}$. Let $u \in C_\Omega^\infty \cap H_\mu^{m,p}$ and suppose that $c \neq 0$, where $u - c \in \mathring{L}^{m,p}$. We can assume that $c > 0$. Let

$$F = \left\{ x : \left|u(x) - c\right| \geq \frac{c}{2} \right\}$$

and suppose first that $p > 1$. Then $u - c$ can be written $u - c = K_m * f$ where $f \in L_p$ and we get

$$C_{m,p}(F) \leq C_{m,p}\left(\left\{ x : K_m * |f| \geq \frac{c}{2} \right\} \right) \leq \frac{2^p}{c^p} \|f\|_p^p < \infty.$$

If $p = 1$ then in the same way as in the proof of Lemma 17.2/3

$$H_\infty^{n-m}(F) \leq A \int \left|\nabla_m \big(u(x) - c\big)\right| \mathrm{d}x < \infty.$$

On the other hand, since $|u| \geq \frac{c}{2}$ on F^c, we have

$$\mu(F^c) \leq \left(\frac{2}{c}\right)^p \int |u|^p \,\mathrm{d}\mu < \infty.$$

This contradicts the condition on μ, hence $c = 0$. It follows that $u \in \mathring{L}^{m,p}$. Note that since $\Omega^c \subset F$ we must have $c = 0$ if $C(\Omega^c) = \infty$, without using any condition on μ.

Now let $\eta_R \in C_0^\infty(B(2R))$ satisfy $0 \leq \eta_R \leq 1$, $\eta_R = 1$ on $B(R)$ and $|\nabla_k \eta_R| \leq AR^{-k}$ for $k \leq m$. Then for $R \geq 1$ we get

$$\begin{aligned}
&\|u - u\eta_R\|_{m,p}^p \\
&\quad \leq \sum_{1 \leq l+k \leq m} \int \left|\nabla_k u(x)\right|^p \left|\nabla_l (1 - \eta_R)(x)\right|^p \mathrm{d}x \\
&\quad \leq A \sum_{k=1}^m \int_{B(R)^c} \left|\nabla_k u(x)\right|^p \mathrm{d}x + A \sum_{\substack{1 \leq l+k \leq m \\ l \geq 1}} R^{-lp} \int_{A(2R,R)} \left|\nabla_k u(x)\right|^p \mathrm{d}x \\
&\quad \leq A \sum_{k=1}^m \int_{B(R)^c} \left|\nabla_k u(x)\right|^p \mathrm{d}x + AR^{-p} \int_{A(2R,R)} \left|u(x)\right|^p \mathrm{d}x.
\end{aligned}$$

Thus $\eta_R u \to u$ in $L^{m,p}$ as $R \to \infty$ by Lemma 1(i). Also

$$\int |\eta_R u - u|^p \,\mathrm{d}\mu \leq \int_{B(R)^c} |u|^p \,\mathrm{d}\mu \to 0$$

as $R \to \infty$ since $u \in H_\mu^{m,p}(\Omega)$. It follows that $u \in \mathring{H}_\mu^{m,p}(\Omega)$, i.e., $C_\Omega^\infty \cap H_\mu^{m,p}(\Omega) \subset \mathring{H}_\mu^{m,p}(\Omega)$ and hence $H_\mu^{m,p}(\Omega) = \mathring{H}_\mu^{m,p}(\Omega)$.

We now turn to the necessity part. Suppose that $H_\mu^{m,p}(\Omega) = \mathring{H}_\mu^{m,p}(\Omega)$ and that F is a closed set such that $B_{m,p}(F) < \infty$ and $\mu(F^c) < \infty$ where now

$p > 1$. Assume also that $B_{m,p}(\Omega^c) < \infty$. Then there is an open set $G \supset \Omega^c$ such that $B_{m,p}(\bar{G}) < \infty$. By Lemma 17.2/1 we can find $f \in L_p^+$ such that $G_{m;1} * f \in C^\infty$ and $G_{m;1} * f \geq 1$ on $F \cup \bar{G}$. Now let T be a smooth function on $\mathbb{R}_+$ such that $T(t) = 1$ if $t \geq 1$ and

$$\sup_{t>0} \left| t^{k-1} T^{(k)}(t) \right| < \infty \quad \text{for } 0 \leq k \leq m.$$

Then by Lemma 17.2/1, which works also for the truncated kernel, there is a function $g \in L_p$ such that $T \circ (G_{m;1} * f) = G_{m;1} * g$ and $\|g\|_p \leq A \|f\|_p$. We set $u = 1 - G_{m;1} * g$. Then $u \in C_\Omega^\infty \cap L^{m,p}$, but by Lemma 17.4, $u \notin \mathring{L}^{m,p}$ since $G_{m;1} * g \in \mathring{L}^{m,p}$. Moreover

$$\int |u|^p \, \mathrm{d}\mu = \int_{F^c} |u|^p \, \mathrm{d}\mu \leq \mu\big(F^c\big) < \infty$$

by assumption on F, so $u \in H_\mu^{m,p}(\Omega) \backslash \mathring{H}_\mu^{m,p}(\Omega)$. This contradiction shows that μ must satisfy the condition in the theorem.

For $p = 1$ we use instead $u = 1 - \varphi$, where φ is the function constructed in Lemma 17.2/3, satisfying $\varphi = 1$ on a neighborhood of $F \cup \bar{G}$. Since

$$\int \big|\varphi(x)\big| \, \mathrm{d}x + \int \big|\nabla_m \varphi(x)\big| \, \mathrm{d}x < \infty,$$

it follows easily, using the previous multiplier η_R, that $\varphi \in \mathring{L}^{m,1}$. Hence $u \notin \mathring{H}_\mu^{m,1}(\Omega)$. But

$$\int |u| \, \mathrm{d}\mu \leq \mu\big(F^c\big) < \infty \quad \text{so } u \in H_\mu^{m,1},$$

and we have again obtained a contradiction.

The case $1 < p < n$, $mp > n$ *or* $1 = p < n$, $m \geq n$. For the sufficiency we again decompose $u = v + c$ where $v \in \mathring{L}^{m,p}$. Using Sobolev's inequality

$$\sup_{B(x,1)} |v| \leq A \left(\int_{B(x,1)} \big|v(y)\big|^{p^*} \, \mathrm{d}y \right)^{1/p^*} + A \left(\int_{B(x,1)} \big|\nabla_m v(y)\big|^p \, \mathrm{d}y \right)^{1/p}$$

for every $x \in \mathbb{R}^n$, we see that $v(y) \to 0$ uniformly as $|x| \to \infty$. Thus, if $c \neq 0$, we can find R such that $|v| \leq \frac{|c|}{2}$ on $B(R)^c$. Then, if μ is not finite,

$$\int |u|^p \, \mathrm{d}\mu \geq \left(\frac{|c|}{2} \right)^p \mu\big(B(R)^c\big) = \infty,$$

and we have a contradiction. Thus $c = 0$ and we can proceed as in the first case, again using Lemma 1(i). Note that if Ω^c is unbounded it follows immediately that $c = 0$, without any condition on μ since $v = c$ on Ω^c.

For the necessity we observe that if Ω^c is bounded then there is $u \in C_\Omega^\infty$ such that $u(x) = 1$ for large $|x|$. Then $u \in L^{m,p} \backslash \mathring{L}^{m,p}$, since $u \notin L_{p^*}$. If μ is finite we get $u \in H_\mu^{m,p}(\Omega) \backslash \mathring{H}_\mu^{m,p}(\Omega)$ and we are done.

The case $p > n$ is proved as previously, now invoking Lemma 1(ii).

The case $p = n$ is proved by Lemma 1(iii), this time using the multiplier

$$\eta_R(x) = \chi\left(\frac{1}{\log R}\log\frac{R^2}{|x|}\right),$$

where χ is a smooth function satisfying $\chi(t) = 0$ for $t \le \frac{1}{4}$ and $\chi(t) = 1$ for $t \ge \frac{3}{4}$. This completes the proof. □

Remark. The same question of the density of test functions can be asked about the more general norm

$$\|u\| = \|u\|_{L_p(\mu)} + \sum_{k=l}^{m} \|\nabla_k u\|_p,$$

where $m \ge l \ge 2$. In the case $lp \ge n$ approximation is always possible. This is proved in the same way as in the present chapter, using only somewhat different Hardy inequalities. In the general case it is easy to give an implicit necessary and sufficient condition. Namely, with obvious notation, $H_\mu^{l,m,p} = \mathring{H}_\mu^{l,m,p}$ if and only if

$$u \in \mathring{H}_\mu^{l,m,p},\ P \in \mathbb{P}_{l-1}, \qquad \int |u - P|^p \,\mathrm{d}\mu < \infty \Rightarrow P = 0.$$

However, it is not clear whether this condition can be stated in a more transparent way in the spirit of Theorem 17.1, for example, in terms of the polynomial capacities dealt with in Sect. 14.3.

17.5 Comments to Chap. 17

In this chapter we follow the article by Carlsson and the author [171].

Section 17.1. The question as to when the equality $Q_{\max} = Q_{\min}$ holds was raised by Kato [416]. Simon studied a similar question concerning the magnetic Schrödinger operator in $\mathbb{R}^n$ [701]. Proposition 17.1/2 improves the condition $\int \varrho(x)\,\mathrm{d}x = \infty$, necessary for the essential selfadjointness of the operator A, which was obtained by Eidus [255].

Section 17.2. Lemma 17.2/3 was partly proved by D.R. Adams [10].

Section 17.3. Lemmas 17.3/1 and 17.3/2 were proved in Carlsson [170].

18

Spectrum of the Schrödinger Operator and the Dirichlet Laplacian

Consider the Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^n)$ with a potential V, locally integrable and semibounded below. As we mentioned in Sect. 16.6, Molchanov's criterion (16.6.2) involves the so-called negligible sets F, that is, sets of sufficiently small harmonic capacity.

In Sects. 18.2–18.3 we show that the constant c_n given by (16.6.4) can be replaced by an arbitrary constant γ, $0 < \gamma < 1$. We even establish a stronger result allowing negligibility conditions with γ depending on d and completely describe all admissible functions γ. More precisely, in the necessary condition for the discreteness of spectrum we allow arbitrary functions $\gamma : (0, +\infty) \to (0, 1)$. If $\gamma(d) = O(d^2)$ in the negligibility condition (16.6.3), then it fails to be sufficient, i.e., it may happen that it is satisfied but the spectrum is not discrete (Sect. 18.4). However, we show that in the sufficient condition we can admit arbitrary functions γ with values in $(0, 1)$, defined for $d > 0$ in a neighborhood of $d = 0$ and satisfying

$$\limsup_{d \downarrow 0} d^{-2}\gamma(d) = +\infty. \tag{18.0.1}$$

All such relations involving functions $\gamma : (0, +\infty) \to (0, 1)$, are necessary and sufficient for the discreteness of spectrum. Therefore two conditions with different functions γ are equivalent, which is far from being obvious *a priori*. This equivalence means the following striking effect: If (16.6.2) holds for very small sets F, then it also holds for sets F which almost fill the corresponding cubes.

Another important question is whether the operator $-\Delta + V$ with $V \geq 0$ is strictly positive, i.e., the spectrum is separated from zero. Unlike the discreteness of spectrum conditions, it is the large values of d that are relevant here. The following necessary and sufficient condition for the strict positivity was obtained in Sect. 16.5: There exist positive constants d and $\varkappa$ such that for all cubes Q_d

$$\inf_F \int_{Q_d \setminus F} V(x)\,\mathrm{d}x \geq \varkappa,$$

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where the infimum is taken over all compact sets $F \subset \bar{Q}_d$ that are negligible in the sense of Molchanov. We prove in Sect. 18.5 that here again an arbitrary constant $\gamma \in (0, 1)$ in the negligibility condition is admissible.

The above-mentioned results are proved in a more general context. The family of cubes Q_d is replaced by a family of arbitrary bodies homothetic to a standard bounded domain which is starshaped with respect to a ball. Instead of locally integrable potentials $V \geq 0$ we consider positive measures. We also include operators in arbitrary open subsets of $\mathbb{R}^n$ with the Dirichlet boundary conditions.

The goal of the last Sect. 18.7 is to obtain explicit lower and upper estimates of the first eigenvalue of the Laplacian with Dirichlet data formulated in terms of a capacitary inner radius.

18.1 Main Results on the Schrödinger Operator

Let $\mathbb{V}$ be a positive Radon measure in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$. We will consider the Schrödinger operator, which is formally given by an expression $-\Delta + \mathbb{V}$. It is defined in $L_2(\Omega)$ by the quadratic form

$$h_{\mathbb{V}}(u, u) = \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u|^2 \mathbb{V}(dx), \quad u \in C_0^\infty(\Omega). \tag{18.1.1}$$

For the associated operator to be well defined we need a closed quadratic form. The form above is closable in $L_2(\Omega)$ if and only if $\mathbb{V}$ is $(2, 1)$ absolutely continuous (see Sect. 16.4). In the present chapter we will always assume that this condition is satisfied. The operator, associated with the closure of the form (18.1.1), will be denoted $H_{\mathbb{V}}$.

In particular, we can consider an absolutely continuous measure $\mathbb{V}$ which has a density $V \geq 0$, $V \in L_1(\mathbb{R}^n, \mathrm{loc})$, with respect to the Lebesgue measure dx. Such a measure will be absolutely continuous with respect to the capacity as well.

Instead of the cubes Q_d a more general family of test bodies will be used. Let us start with a standard open set $\mathcal{G} \subset \mathbb{R}^n$. We assume that $\mathcal{G}$ satisfies the following conditions:

(a) $\mathcal{G}$ is bounded and starshaped with respect to an open ball $B_\rho(0)$ of radius $\rho > 0$, with the center at $0 \in \mathbb{R}^n$;
(b) $\mathrm{diam}(\mathcal{G}) = 1$.

The first condition means that $\mathcal{G}$ is starshaped with respect to every point of $B_\rho(0)$. It implies that $\mathcal{G}$ can be presented in the form

$$\mathcal{G} = \{x : x = r\omega, |\omega| = 1, 0 \leq r < r(\omega)\}, \tag{18.1.2}$$

where $\omega \mapsto r(\omega) \in (0, +\infty)$ is a Lipschitz function on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ (see Lemma 1.1.8). Condition (b) is imposed for the convenience of formulations.

For any positive $d > 0$ denote by $\mathcal{G}_d(0)$ the body $\{x \mid d^{-1}x \in \mathcal{G}\}$ which is homothetic to $\mathcal{G}$ with coefficient d and with the center of homothety at 0. We will denote by $\mathcal{G}_d$ a body that is obtained from $\mathcal{G}_d(0)$ by a parallel translation: $\mathcal{G}_d(y) = y + \mathcal{G}_d(0)$ where y is an arbitrary vector in $\mathbb{R}^n$. The notation $\mathcal{G}_d \to \infty$ means that the distance from $\mathcal{G}_d$ to 0 goes to infinity.

Definition. Let $\gamma \in (0,1)$. The *negligibility class* $\mathcal{N}_\gamma(\mathcal{G}_d; \Omega)$ consists of all compact sets $F \subset \bar{\mathcal{G}}_d$ satisfying the following conditions:

$$\bar{\mathcal{G}}_d \setminus \Omega \subset F \subset \bar{\mathcal{G}}_d \tag{18.1.3}$$

and

$$\operatorname{cap}(F) \leq \gamma \operatorname{cap}(\bar{\mathcal{G}}_d). \tag{18.1.4}$$

The notation cap will be introduced in Sect. 18.2.

Now we formulate our main result about the discreteness of spectrum.

Theorem 1. (i) (Necessity) *Let the spectrum of $H_{\mathbb{V}}$ be discrete. Then for every function $\gamma : (0, +\infty) \to (0,1)$ and every $d > 0$*

$$\inf_{F \in \mathcal{N}_{\gamma(d)}(\mathcal{G}_d, \Omega)} \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \to +\infty \quad \text{as } \mathcal{G}_d \to \infty. \tag{18.1.5}$$

(ii) (Sufficiency) *Let a function $d \mapsto \gamma(d) \in (0,1)$ be defined for $d > 0$ in a neighborhood of* 0, *and satisfy* (18.0.1). *Assume that there exists $d_0 > 0$ such that* (18.1.5) *holds for every $d \in (0, d_0)$. Then the spectrum of $H_{\mathbb{V}}$ in $L_2(\Omega)$ is discrete.*

Let us make some comments about this theorem.

Remark 1. It suffices for the discreteness of spectrum of $H_{\mathbb{V}}$ that the condition (18.1.5) holds only for a sequence of d's, i.e., $d \in \{d_1, d_2, \dots, \}$, $d_k \to 0$ and $d_k^{-2}\gamma(d_k) \to +\infty$ as $k \to +\infty$.

Remark 2. As we will see in the proof, in the sufficiency part the condition (18.1.5) can be replaced by a weaker requirement: There exist $c > 0$ and $d_0 > 0$ such that for every $d \in (0, d_0)$ there exists $R > 0$ such that

$$d^{-n} \inf_{F \in \mathcal{N}_{\gamma(d)}(\mathcal{G}_d, \Omega)} \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \geq c\, d^{-2}\gamma(d) \tag{18.1.6}$$

whenever $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \emptyset$ (i.e., for distant bodies $\mathcal{G}_d$ having nonempty intersection with Ω). Moreover, it suffices that the condition (18.1.6) is satisfied for a sequence $d = d_k$ satisfying the condition formulated in Remark 1.

Note that unlike (18.1.5), the condition (18.1.6) does not require that the left-hand side goes to $+\infty$ as $\mathcal{G}_d \to \infty$. What is actually required is that the left-hand side has a certain lower bound, depending on d for arbitrarily small $d > 0$ and distant test bodies $\mathcal{G}_d$. Nevertheless, the conditions (18.1.5) and

(18.1.6) are equivalent because each of them is equivalent to the discreteness of spectrum.

Remark 3. If we take $\gamma = \text{const} \in (0,1)$, then Theorem 1 gives Molchanov's result, but with the constant $\gamma = c_n$ replaced by an arbitrary constant $\gamma \in (0,1)$.

Remark 4. For any two functions $\gamma_1, \gamma_2 : (0,+\infty) \to (0,1)$ satisfying the requirement (18.0.1), the conditions (18.1.5) are equivalent, and so are the conditions (18.1.6), because any of these conditions is equivalent to the discreteness of spectrum.

It follows that the conditions (18.1.5) for different constants $\gamma \in (0,1)$ are equivalent. In the particular case, when the measure $\mathbb{V}$ is absolutely continuous with respect to the Lebesgue measure, we see that the conditions (16.6.2) with different constants $\gamma \in (0,1)$ are equivalent.

Remark 5. The previous results are new even for the operator $H_0 = -\Delta$ in $L_2(\Omega)$ (but for an arbitrary open set $\Omega \subset \mathbb{R}^n$ with the Dirichlet boundary conditions on $\partial\Omega$). In this case the discreteness of spectrum is completely determined by the geometry of Ω. Namely, for the discreteness of spectrum of H_0 in $L_2(\Omega)$ it is necessary and sufficient that there exists $d_0 > 0$ such that for every $d \in (0, d_0)$

$$\liminf_{\mathcal{G}_d \to \infty} \operatorname{cap}(\bar{\mathcal{G}}_d \setminus \Omega) \geq \gamma(d) \operatorname{cap}(\bar{\mathcal{G}}_d), \tag{18.1.7}$$

where $d \mapsto \gamma(d) \in (0,1)$ is a function, which is defined in a neighborhood of 0 and satisfies (18.0.1). The conditions (18.1.7) with different functions γ, satisfying the previous conditions, are equivalent. This is a nontrivial property of capacity. It is necessary for the discreteness of spectrum that (18.1.7) holds for every function $\gamma : (0,+\infty) \to (0,1)$ and every $d > 0$, but this condition may not be sufficient if γ does not satisfy (18.0.1) (see Theorem 2 below).

The following result demonstrates that the condition (18.0.1) is precise.

Theorem 2. *Assume that $\gamma(d) = O(d^2)$ as $d \to 0$. Then there exists an open set $\Omega \subset \mathbb{R}^n$ and $d_0 > 0$ such that for every $d \in (0, d_0)$ the condition (18.1.7) is satisfied, but the spectrum of $-\Delta$ in $L_2(\Omega)$ with the Dirichlet boundary conditions is not discrete.*

Now we will state our positivity result. We will say that the operator $H_{\mathbb{V}}$ is *strictly positive* if its spectrum does not contain 0. Equivalently, we can say that the spectrum is separated from 0. Since $H_{\mathbb{V}}$ is defined by the quadratic form (18.1.1), the strict positivity is equivalent to the existence of $\lambda > 0$ such that

$$h_{\mathbb{V}}(u,u) \geq \lambda \|u\|^2_{L_2(\Omega)}, \quad u \in C_0^\infty(\Omega). \tag{18.1.8}$$

Theorem 3. (i) (Necessity) *Suppose that $H_{\mathbb{V}}$ is strictly positive, so that (18.1.8) is satisfied with a constant $\lambda > 0$. Let us take an arbitrary $\gamma \in (0,1)$.*

Then there exist $d_0 > 0$ *and* $\varkappa > 0$ *such that*

$$d^{-n} \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d, \Omega)} \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \geq \varkappa \qquad (18.1.9)$$

for every $d > d_0$ *and every* $\mathcal{G}_d$.

(ii) (Sufficiency) *Assume that there exist* $d > 0$, $\varkappa > 0$, *and* $\gamma \in (0,1)$, *such that* (18.1.9) *is satisfied for every* $\mathcal{G}_d$. *Then the operator* $H_{\mathbb{V}}$ *is strictly positive.*

Instead of all bodies $\mathcal{G}_d$ *it is sufficient to take only the ones from a finite multiplicity covering* (*or tiling*) *of* $\mathbb{R}^n$.

Remark 6. Considering the Dirichlet Laplacian $H_0 = -\Delta$ in $L_2(\Omega)$ we see from Theorem 3 that for any choice of a constant $\gamma \in (0,1)$ and a standard body $\mathcal{G}$, the strict positivity of H_0 is equivalent to the following condition:

$$\exists\, d > 0, \text{ such that } \operatorname{cap}\big(\bar{\mathcal{G}}_d \cap (\mathbb{R}^n \setminus \Omega)\big) \geq \gamma \operatorname{cap}(\bar{\mathcal{G}}_d) \text{ for all } \mathcal{G}_d. \qquad (18.1.10)$$

In particular, it follows that for two different γ's these conditions are equivalent.

18.2 Discreteness of Spectrum: Necessity

In this section we prove the necessity part (i) of Theorem 18.1/1. We will start by recalling some definitions and introducing the necessary notations.

If F is a compact subset in an open set $\mathcal{D} \subset \mathbb{R}^n$, then the harmonic capacity of F with respect to $\mathcal{D}$ is defined as

$$\operatorname{cap}(F, \mathcal{D}) = \inf \Big\{ \int_{\mathbb{R}^n} \big|\nabla u(x)\big|^2 \,\mathrm{d}x : u \in C_0^{0,1}(\mathcal{D}), u|_F = 1 \Big\}, \qquad (18.2.1)$$

which is nothing but 2-cap(F, Ω).

We will write $\mathcal{B}_d$ for a ball $B_d(z)$ with unspecified center z. Let us use the notation $\operatorname{cap}(F)$ for $\operatorname{cap}(F, \mathbb{R}^n)$ if $F \subset \mathbb{R}^n$, $n \geq 3$, and for $\operatorname{cap}(F, \mathcal{B}_{2d})$ if $F \subset \bar{\mathcal{B}}_d \subset \mathbb{R}^2$, where the disks $\mathcal{B}_d$ and $\mathcal{B}_{2d}$ have the same center. The choice of these disks will usually be clear from the context, otherwise we will specify them explicitly.

Note that the infimum does not change if we restrict ourselves to the Lipschitz functions u such that $0 \leq u \leq 1$ everywhere (see Sect. 2.2.1).

We will also need the potential theoretic definition of the harmonic capacity $\operatorname{cap}(F)$ for a compact set $F \subset \bar{\mathcal{B}}_d$. For $n \geq 3$ it is as follows:

$$\operatorname{cap}(F) = \sup \Big\{ \mu(F) : \int_F \mathcal{E}(x-y)\,\mathrm{d}\mu(y) \leq 1 \quad \text{on } \mathbb{R}^n \setminus F \Big\}, \qquad (18.2.2)$$

where the supremum is taken over all positive finite Radon measures μ on F and $\mathcal{E} = \mathcal{E}_n$ is the standard fundamental solution of $-\Delta$ in $\mathbb{R}^n$, i.e.,

$$\mathcal{E}(x) = \frac{1}{(n-2)\omega_n}|x|^{2-n}, \tag{18.2.3}$$

with ω_n being the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. If $n = 2$, then

$$\operatorname{cap}(F) = \sup\left\{\mu(F) : \int_F G(x,y)\,\mathrm{d}\mu(y) \le 1 \text{ on } \mathcal{B}_{2d} \setminus F\right\}, \tag{18.2.4}$$

where G is the Green function of the Dirichlet problem for $-\Delta$ in $\mathcal{B}_{2d}$, i.e.,

$$-\Delta G(\cdot - y) = \delta(\cdot - y), \quad y \in \mathcal{B}_{2d},$$

$G(\cdot, y)|_{\partial\mathcal{B}_{2d}} = 0$ for all $y \in \mathcal{B}_{2d}$. The maximizing measure in (18.2.2) or in (18.2.4) exists and is unique. We will denote it μ_F and call it the *equilibrium measure*. Note that

$$\operatorname{cap}(F) = \mu_F(F) = \mu_F\big(\mathbb{R}^n\big) = \langle \mu_F, 1\rangle. \tag{18.2.5}$$

The corresponding potential will be denoted P_F, so

$$P_F(x) = \int_F \mathcal{E}(x-y)\,\mathrm{d}\mu_F(y), \quad x \in \mathbb{R}^n \setminus F,\ n \ge 3,$$

$$P_F(x) = \int_F G(x,y)\,\mathrm{d}\mu_F(y), \quad x \in B_{2d} \setminus F,\ n = 2.$$

We will call P_F the *equilibrium potential* or *capacitary potential.* We will extend it to F by setting $P_F(x) = 1$ for all $x \in F$. It follows from the maximum principle that $0 \le P_F \le 1$ everywhere in $\mathbb{R}^n$ if $n \ge 3$ (and in $\mathcal{B}_{2d}$ if $n = 2$).

In the case when F is the closure of an open subset with a smooth boundary, $u = P_F$ is the unique minimizer for the Dirichlet integral in (18.2.1) where we should take $\mathcal{D} = \mathbb{R}^n$ if $n \ge 3$ and $\mathcal{D} = \mathcal{B}_{2d}$ if $n = 2$. In particular,

$$\int |\nabla P_F|^2\,\mathrm{d}x = \operatorname{cap}(F), \tag{18.2.6}$$

where the integration is taken over $\mathbb{R}^n$ (or $\mathbb{R}^n \setminus F$) if $n \ge 3$ and over $\mathcal{B}_{2d}$ (or $\mathcal{B}_{2d} \setminus F$) if $n = 2$.

The following lemma provides an auxiliary estimate which is needed for the proof.

Lemma 1. *Assume that $\mathcal{G}$ has a C^∞ boundary, and P is the equilibrium potential of $\bar{\mathcal{G}}_d$. Then*

$$\int_{\partial\mathcal{G}_d} |\nabla P|^2\,\mathrm{d}s \le nL\rho^{-1}d^{-1}\operatorname{cap}(\bar{\mathcal{G}}_d), \tag{18.2.7}$$

where the gradient ∇P in the left-hand side is taken along the exterior of $\bar{\mathcal{G}}_d$, ds is the $(n-1)$-dimensional volume element on $\partial\mathcal{G}_d$. The positive constants

ρ, L are geometric characteristics of the standard body $\mathcal{G}$ (they depend on the choice of $\mathcal{G}$ only, but not on d): ρ was introduced at the beginning of Sect. 18.1, and

$$L = \left[\inf_{x\in\partial\mathcal{G}} \nu_r(x)\right]^{-1}, \tag{18.2.8}$$

where $\nu_r(x) = \frac{x}{|x|}\cdot\nu(x)$, $\nu(x)$ is the unit normal vector to $\partial\mathcal{G}$ at x which is directed to the exterior of $\bar{\mathcal{G}}$.

Proof. It suffices to consider $\mathcal{G}_d = \mathcal{G}_d(0)$. For simplicity we will write $\mathcal{G}$ instead of $\mathcal{G}_d(0)$ in this proof, until the size becomes relevant.

We will first consider the case $n \geq 3$. Note that $\Delta P = 0$ on $\mathbb{R}^n \setminus \bar{\mathcal{G}}$. Also $P = 1$ on $\bar{\mathcal{G}}$, so, in fact, $|\nabla P| = |\partial P/\partial\nu|$. Using the Green formula, we obtain

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \Delta P\cdot\frac{\partial P}{\partial r}\,\mathrm{d}x = \int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \Delta P\left(\frac{x}{|x|}\cdot\nabla P\right)\mathrm{d}x \\
&= -\int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \nabla P\cdot\nabla\left(\frac{x}{|x|}\cdot\nabla P\right)\mathrm{d}x - \int_{\partial\mathcal{G}} \frac{\partial P}{\partial\nu}\left(\frac{x}{|x|}\cdot\nabla P\right)\mathrm{d}s \\
&= -\sum_{i,j}\int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{\partial P}{\partial x_j}\cdot\frac{\partial}{\partial x_j}\left(\frac{x_i}{|x|}\cdot\frac{\partial P}{\partial x_i}\right)\mathrm{d}x - \int_{\partial\mathcal{G}} \frac{\partial P}{\partial\nu}\cdot\frac{\partial P}{\partial r}\,\mathrm{d}s \\
&= -\sum_{i,j}\int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{\partial P}{\partial x_j}\cdot\frac{\delta_{ij}}{|x|}\cdot\frac{\partial P}{\partial x_i}\,\mathrm{d}x + \sum_{i,j}\int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{x_ix_j}{|x|^3}\cdot\frac{\partial P}{\partial x_i}\cdot\frac{\partial P}{\partial x_j}\,\mathrm{d}x \\
&\quad -\sum_{i,j}\int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{x_i}{|x|}\cdot\frac{\partial P}{\partial x_j}\cdot\frac{\partial^2 P}{\partial x_i\partial x_j}\,\mathrm{d}x - \int_{\partial\mathcal{G}} \frac{\partial P}{\partial\nu}\cdot\frac{\partial P}{\partial r}\,\mathrm{d}s \\
&= -\int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{1}{|x|}|\nabla P|^2\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{1}{|x|}\left|\frac{\partial P}{\partial r}\right|^2\mathrm{d}x \\
&\quad -\frac{1}{2}\sum_i\int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{x_i}{|x|}\cdot\frac{\partial}{\partial x_i}|\nabla P|^2\,\mathrm{d}x - \int_{\partial\mathcal{G}} |\nabla P|^2\nu_r\,\mathrm{d}s.
\end{aligned}$$

Integrating by parts in the last integral over $\mathbb{R}^n\setminus\bar{\mathcal{G}}$, we see that it equals

$$\begin{aligned}
&\frac{1}{2}\sum_i\int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{\partial}{\partial x_i}\left(\frac{x_i}{|x|}\right)\cdot|\nabla P|^2\,\mathrm{d}x + \frac{1}{2}\sum_i\int_{\partial\mathcal{G}} \frac{x_i}{|x|}|\nabla P|^2\nu_i\,\mathrm{d}s \\
&\quad = \frac{n-1}{2}\int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{1}{|x|}|\nabla P|^2\,\mathrm{d}x + \frac{1}{2}\int_{\partial\mathcal{G}} |\nabla P|^2\nu_r\,\mathrm{d}s,
\end{aligned}$$

where ν_i is the ith component of ν. Returning to the calculation above, we obtain

$$\begin{aligned}
0 = \frac{n-3}{2}\int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{1}{|x|}|\nabla P|^2\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus\bar{\mathcal{G}}} \frac{1}{|x|}\left|\frac{\partial P}{\partial r}\right|^2\mathrm{d}x \\
-\frac{1}{2}\int_{\partial\mathcal{G}} |\nabla P|^2\nu_r\,\mathrm{d}s.
\end{aligned} \tag{18.2.9}$$

It follows that

$$\int_{\partial\mathcal{G}} |\nabla P|^2 \nu_r \, \mathrm{d}s \le (n-1) \int_{\mathbb{R}^n \setminus \bar{\mathcal{G}}} \frac{1}{|x|} |\nabla P|^2 \, \mathrm{d}x.$$

Recalling that $\mathcal{G} = \mathcal{G}_d(0)$, we observe that $|x|^{-1} \le (\rho d)^{-1}$. Now using (18.2.6), we obtain the desired estimate (18.2.7) for $n \ge 3$ (with $n-1$ instead of n).

Let us consider the case $n = 2$. Then, by definition, the equilibrium potential P for $\mathcal{G} = \mathcal{G}_d(0)$ is defined in the ball $B_{2d}(0)$. It satisfies $\Delta P = 0$ in $B_{2d}(0) \setminus \bar{\mathcal{G}}$ and the boundary conditions $P|_{\partial\mathcal{G}} = 1$, $P|_{\partial B_{2d}(0)} = 0$. Let us first modify the previous calculations by taking the integrals over $B_\delta(0) \setminus \bar{\mathcal{G}}$ (instead of $\mathbb{R}^n \setminus \bar{\mathcal{G}}$), where $d < \delta < 2d$. We will get additional boundary terms with the integration over $\partial B_\delta(0)$. Instead of (18.2.9) we will obtain

$$\begin{aligned} 0 = &-\frac{1}{2} \int_{B_\delta(0)\setminus\bar{\mathcal{G}}} \frac{1}{|x|} |\nabla P|^2 \, \mathrm{d}x + \int_{B_\delta(0)\setminus\bar{\mathcal{G}}} \frac{1}{|x|} \left|\frac{\partial P}{\partial r}\right|^2 \mathrm{d}x \\ &- \frac{1}{2} \int_{\partial\mathcal{G}} |\nabla P|^2 \nu_r \, \mathrm{d}s + \frac{1}{2} \int_{\partial B_\delta(0)} \left[2\left|\frac{\partial P}{\partial r}\right|^2 - |\nabla P|^2 \right] \mathrm{d}s. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\partial\mathcal{G}} |\nabla P|^2 \nu_r \, \mathrm{d}s &\le \int_{B_\delta(0)\setminus\bar{\mathcal{G}}} \frac{1}{|x|} |\nabla P|^2 \, \mathrm{d}x + \int_{\partial B_\delta(0)} \left[2\left|\frac{\partial P}{\partial r}\right|^2 - |\nabla P|^2 \right] \mathrm{d}s \\ &\le \frac{1}{\rho\, d} \int_{B_{2d}(0)\setminus\bar{\mathcal{G}}} |\nabla P|^2 \, \mathrm{d}x + \int_{\partial B_\delta(0)} |\nabla P|^2 \, \mathrm{d}s. \end{aligned}$$

Now let us integrate both sides with respect to δ over the interval $[d, 2d]$ and divide the result by d (i.e., take the average over all δ). Then the left-hand side and the first term in the right-hand side do not change, while the last term becomes d^{-1} times the volume integral with respect to the Lebesgue measure over $B_{2d}(0) \setminus B_d(0)$. By (18.2.6) the right-hand side can be estimated by $(1+\rho)(\rho d)^{-1} \operatorname{cap}(\bar{\mathcal{G}}_d)$. Since $0 < \rho \le 1$, we get the estimate (18.2.7) for $n = 2$. □

Proof of Theorem 18.1/1, part (i). (a) We use the same notations as previously. Let us fix $d > 0$, take $\mathcal{G}_d = \mathcal{G}_d(z)$, and assume that $\mathcal{G}$ has a C^∞ boundary. Let us take a compact set $F \subset \mathbb{R}^n$ with the following properties:

(i) F is the closure of an open set with a C^∞ boundary;
(ii) $\bar{\mathcal{G}}_d \setminus \Omega \Subset F \subset B_{3d/2}(z)$;
(iii) $\operatorname{cap}(F) \le \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$ with $0 < \gamma < 1$.

Let us recall that the notation $\bar{\mathcal{G}}_d \setminus \Omega \Subset F$ means that $\bar{\mathcal{G}}_d \setminus \Omega$ is contained in the interior of F. This implies that $\mathbb{V}(\bar{\mathcal{G}}_d \setminus F) < +\infty$. The inclusion $F \subset B_{3d/2}(z)$ and the inequality (iii) hold, in particular, for compact sets F which are small neighborhoods (with smooth boundaries) of negligible compact subsets of $\bar{\mathcal{G}}_d$, and it is exactly such F's that we have in mind.

We shall refer to the sets F satisfying conditions (i)–(iii) above as *regular* ones.

Let P and P_F denote the equilibrium potentials of $\bar{\mathcal{G}}_d$ and F, respectively. The equilibrium measure $\mu_{\bar{\mathcal{G}}_d}$ has its support in $\partial\mathcal{G}_d$ and has density $-\partial P/\partial\nu$ with respect to the $(n-1)$-dimensional Riemannian measure ds on $\partial\mathcal{G}_d$. So for $n \geq 3$ we have

$$P(y) = -\int_{\partial\mathcal{G}_d} \mathcal{E}(x-y)\frac{\partial P}{\partial\nu}(x)\,\mathrm{d}s_x, \quad y \in \mathbb{R}^n;$$
$$-\int_{\partial\mathcal{G}_d} \frac{\partial P}{\partial\nu}(x)\,\mathrm{d}s_x = \operatorname{cap}(\bar{\mathcal{G}}_d);$$
$$P(y) = 1 \quad \text{for all } y \in \mathcal{G}_d, \qquad 0 \leq P(y) \leq 1 \quad \text{for all } y \in \mathbb{R}^n.$$

(If $n=2$, then the same holds only with $y \in B_{2d}$ and with the fundamental solution $\mathcal{E}$ replaced by the Green function G.) It follows that

$$-\int_{\partial\mathcal{G}_d} P_F\frac{\partial P}{\partial\nu}\,\mathrm{d}s = -\int_F\int_{\partial\mathcal{G}_d} \mathcal{E}(x-y)\frac{\partial P}{\partial\nu}(x)\,\mathrm{d}s_x\,\mathrm{d}\mu_F(y) \leq \mu_F(F) = \operatorname{cap}(F).$$

Therefore,

$$\operatorname{cap}(\bar{\mathcal{G}}_d) - \operatorname{cap}(F) \leq -\int_{\partial\mathcal{G}_d} (1-P_F)\frac{\partial P}{\partial\nu}\,\mathrm{d}s,$$

and, using Lemma 1, we obtain

$$\begin{aligned}\big(\operatorname{cap}(\bar{\mathcal{G}}_d) - \operatorname{cap}(F)\big)^2 &\leq \left(\int_{\partial\mathcal{G}_d} (1-P_F)\frac{\partial P}{\partial\nu}\,\mathrm{d}s\right)^2 \\ &\leq \|1-P_F\|^2_{L^2(\partial\mathcal{G}_d)}\|\nabla P\|^2_{L^2(\partial\mathcal{G}_d)} \\ &\leq nL(\rho d)^{-1}\operatorname{cap}(\mathcal{G}_d)\|1-P_F\|^2_{L^2(\partial\mathcal{G}_d)}, \end{aligned} \qquad (18.2.10)$$

where L is defined by (18.2.8).

(b) Our next goal will be to estimate the norm $\|1-P_F\|_{L_2(\partial\mathcal{G}_d)}$ in (18.2.10) by the norm of the same function in $L_2(\mathcal{G}_d)$. We will use the polar coordinates (r,ω) as in (18.1.2), so in particular $\partial\mathcal{G}_d$ is presented as the set $\{r(\omega)\omega|\ \omega \in S^{n-1}\}$, where $r : S^{n-1} \to (0,+\infty)$ is a Lipschitz function (C^∞ as long as we assume the boundary $\partial\mathcal{G}$ to be C^∞). Assuming that $v \in C^{0,1}(\bar{\mathcal{G}}_d)$, we can write

$$\begin{aligned}\int_{\partial\mathcal{G}_d} |v|^2\,\mathrm{d}s &= \int_{S^{n-1}} |v|^2\frac{r(\omega)^{n-1}}{\nu_r}\,\mathrm{d}\omega \\ &\leq L\int_{S^{n-1}} \big|v\big(r(\omega),\omega\big)\big|^2 r(\omega)^{n-1}\,\mathrm{d}\omega, \end{aligned} \qquad (18.2.11)$$

where $d\omega$ is the standard $(n-1)$-dimensional volume element on S^{n-1}.

Using the inequality

$$\left|f(\varepsilon)\right|^2 \le 2\varepsilon \int_0^\varepsilon \left|f'(t)\right|^2 \mathrm{d}t + \frac{2}{\varepsilon}\int_0^\varepsilon \left|f(t)\right|^2 \mathrm{d}t, \quad f \in C^{0,1}\big([0,\varepsilon]\big), \varepsilon > 0,$$

we obtain

$$\begin{aligned}
&\left|v\big(r(\omega),\omega\big)\right|^2 \\
&\quad \le 2\varepsilon r(\omega)\int_{(1-\varepsilon)r(\omega)}^{r(\omega)} \left|v'_\rho(\rho,\omega)\right|^2 \mathrm{d}\rho + \frac{2}{\varepsilon r(\omega)}\int_{(1-\varepsilon)r(\omega)}^{r(\omega)} \left|v(\rho,\omega)\right|^2 \mathrm{d}\rho \\
&\quad \le \frac{2\varepsilon r(\omega)}{[(1-\varepsilon)r(\omega)]^{n-1}}\int_{(1-\varepsilon)r(\omega)}^{r(\omega)} \left|v'_\rho(\rho,\omega)\right|^2 \rho^{n-1}\,\mathrm{d}\rho \\
&\qquad + \frac{2}{\varepsilon r(\omega)[(1-\varepsilon)r(\omega)]^{n-1}}\int_{(1-\varepsilon)r(\omega)}^{r(\omega)} \left|v(\rho,\omega)\right|^2 \rho^{n-1}\,\mathrm{d}\rho.
\end{aligned}$$

It follows that the integral in the right-hand side of (18.2.11) is estimated by

$$\begin{aligned}
&\int_{S^{n-1}} \frac{2\varepsilon r(\omega)\,\mathrm{d}\omega}{(1-\varepsilon)^{n-1}}\int_{(1-\varepsilon)r(\omega)}^{r(\omega)} \left|v'_\rho(\rho,\omega)\right|^2 \rho^{n-1}\,\mathrm{d}\rho \\
&\quad + \int_{S^{n-1}} \frac{2\,\mathrm{d}\omega}{\varepsilon(1-\varepsilon)^{n-1}r(\omega)}\left|v(\rho,\omega)\right|^2 \rho^{n-1}\,\mathrm{d}\rho.
\end{aligned}$$

Taking $\varepsilon \le 1/2$, we can majorize this by

$$2^n\varepsilon d\int_{\bar{\mathcal{G}}_d} |\nabla v|^2\,\mathrm{d}x + \frac{2^n}{\varepsilon\rho d}\int_{\bar{\mathcal{G}}_d} |v|^2\,\mathrm{d}x,$$

where $\rho \in (0,1]$ is the constant from the description of $\mathcal{G}$ in Sect. 18.1. Recalling (18.2.11), we see that the resulting estimate has the form

$$\int_{\partial\mathcal{G}_d} |v|^2\,\mathrm{d}s \le 2^n L\varepsilon d\int_{\bar{\mathcal{G}}_d} |\nabla v|^2\,\mathrm{d}x + \frac{2^n L}{\varepsilon\rho d}\int_{\bar{\mathcal{G}}_d} |v|^2\,\mathrm{d}x.$$

Now, taking $v = 1 - P_F$, we obtain

$$\int_{\partial\mathcal{G}_d} (1-P_F)^2\,\mathrm{d}s \le 2^n L\varepsilon d\,\mathrm{cap}(F) + \frac{2^n L}{\varepsilon\rho d}\int_{\bar{\mathcal{G}}_d} (1-P_F)^2\,\mathrm{d}x.$$

Using this estimate in (18.2.10), we obtain

$$\begin{aligned}
&\big(\mathrm{cap}(\bar{\mathcal{G}}_d) - \mathrm{cap}(F)\big)^2 \\
&\quad \le \rho^{-1}n2^n L^2\,\mathrm{cap}(\bar{\mathcal{G}}_d)\left(\varepsilon\,\mathrm{cap}(F) + \frac{1}{\varepsilon\rho d^2}\int_{\mathcal{G}_d} (1-P_F)^2\,\mathrm{d}x\right). \qquad (18.2.12)
\end{aligned}$$

(c) Now let us consider $\mathcal{G}$, which is starshaped with respect to a ball, but does not necessarily have a C^∞ boundary. In this case we can approximate the function $r(\omega)$ (see Sect. 18.1) from above by a decreasing sequence of

C^∞ functions $r_k(\omega)$ (e.g., we can apply a standard mollifying procedure to $r(\omega)+1/k$), so that for the corresponding bodies $\mathcal{G}^{(k)}$ the constants L_k are uniformly bounded. It is clear that in this case we will also have $\rho_k \ge \rho$, and $\operatorname{cap}(\bar{\mathcal{G}}_d^{(k)}) \to \operatorname{cap}(\bar{\mathcal{G}}_d)$. So we can pass to the limit in (18.2.12) as $k \to +\infty$ and conclude that it holds for an arbitrary $\mathcal{G}$ (which is starshaped with respect to a ball). But for the moment we still retain the regularity condition on F.

(d) Let us define

$$\mathcal{L} = \left\{u : u \in C_0^\infty(\Omega), h_{\mathbb{V}}(u,u) + \|u\|^2_{L_2(\Omega)} \le 1\right\}, \tag{18.2.13}$$

where $h_{\mathbb{V}}$ is defined by (18.1.1). By a standard functional analysis argument the spectrum of $H_{\mathbb{V}}$ is discrete if and only if $\mathcal{L}$ is precompact in $L_2(\Omega)$, which, in turn, holds if and only if for every $\eta > 0$ there exists $R > 0$ such that

$$\int_{\Omega\setminus B_R(0)} |u|^2 \,\mathrm{d}x \le \eta \quad \text{for every } u \in \mathcal{L}. \tag{18.2.14}$$

Equivalently, we can write that

$$\int_{\Omega\setminus B_R(0)} |u|^2 \,\mathrm{d}x \le \eta\left[\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_\Omega |u|^2 \mathbb{V}(\mathrm{d}x)\right], \tag{18.2.15}$$

for every $u \in C_0^\infty(\Omega)$.

Therefore, it follows from the discreteness of spectrum of $H_{\mathbb{V}}$ that for every $\eta > 0$ there exists $R > 0$ such that for every $\mathcal{G}_d$ with $\bar{\mathcal{G}}_d \cap (\mathbb{R}^n \setminus B_R(0)) \ne \varnothing$ and every $u \in C_0^\infty(\mathcal{G}_d \cap \Omega)$

$$\int_{\mathcal{G}_d} |u|^2 \,\mathrm{d}x \le \eta\left(\int_{\mathcal{G}_d} |\nabla u|^2 \,\mathrm{d}x + \int_{\bar{\mathcal{G}}_d} |u|^2 \mathbb{V}(\mathrm{d}x)\right). \tag{18.2.16}$$

In other words, $\eta = \eta(\mathcal{G}_d) \to 0$ as $\mathcal{G}_d \to \infty$ for the best constant in (18.2.16). (Note that $\eta(\mathcal{G}_d)^{-1}$ is the bottom of the Dirichlet spectrum of $H_{\mathbb{V}}$ in $\mathcal{G}_d \cap \Omega$.)

Since $1 - P_F = 0$ on F (hence in a neighborhood of $\bar{\mathcal{G}}_d \setminus \Omega$), we can take $u = \chi_\sigma(1-P_F)$, where $\sigma \in (0,1)$ to be chosen later, $\chi_\sigma \in C_0^\infty(\mathcal{G}_d)$ is a cutoff function satisfying $0 \le \chi_\sigma \le 1$, $\chi_\sigma = 1$ on $\mathcal{G}_{(1-\sigma)d}$, and $|\nabla \chi_\sigma| \le Cd^{-1}$ with $C = C(\mathcal{G})$. Then, using integration by parts and the equation $\Delta P_F = 0$ on $\mathcal{G} \setminus F$, we obtain

$$\begin{aligned}\int_{\mathcal{G}_d} |\nabla u|^2 \,\mathrm{d}x &= \int_{\mathcal{G}_d} \big(|\nabla \chi_\sigma|^2 (1-P_F)^2 - \nabla\big(\chi_\sigma^2\big) \cdot (1-P_F)\nabla P_F \\ &\qquad + \chi_\sigma^2 |\nabla P_F|^2\big) \,\mathrm{d}x \\ &= \int_{\mathcal{G}_d} |\nabla \chi_\sigma|^2 (1-P_F)^2 \,\mathrm{d}x \le C^2(\sigma d)^{-2} \int_{\mathcal{G}_d} (1-P_F)^2 \,\mathrm{d}x.\end{aligned}$$

Therefore, from (18.2.16)

$$\int_{\mathcal{G}_d} |u|^2 \, dx \le \eta \Big[C^2 (\sigma d)^{-2} \int_{\mathcal{G}_d} (1 - P_F)^2 \, dx + \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \Big],$$

hence

$$\int_{\mathcal{G}_{(1-\sigma)d}} (1 - P_F)^2 \, dx \le \eta \Big[C^2 (\sigma d)^{-2} \int_{\mathcal{G}_d} (1 - P_F)^2 \, dx + \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \Big].$$

Now, applying the obvious estimate

$$\begin{aligned} \int_{\mathcal{G}_d} (1 - P_F)^2 \, dx &\le \int_{\mathcal{G}_{(1-\sigma)d}} (1 - P_F)^2 \, dx + m_n(\mathcal{G}_d \setminus \mathcal{G}_{(1-\sigma)d}) \\ &\le \int_{\mathcal{G}_{(1-\sigma)d}} (1 - P_F)^2 \, dx + C_1 \sigma d^n \end{aligned}$$

with $C_1 = C_1(\mathcal{G})$, we see that

$$\int_{\mathcal{G}_d} (1 - P_F)^2 \, dx \le \eta \Big[C^2 (\sigma d)^{-2} \int_{\bar{\mathcal{G}}_d} (1 - P_F)^2 \, dx + \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \Big] + C_1 \sigma d^n,$$

hence

$$\int_{\mathcal{G}_d} (1 - P_F)^2 \, dx \le 2\eta \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) + 2 C_1 \sigma d^n, \tag{18.2.17}$$

provided

$$\eta C^2 (\sigma d)^{-2} \le 1/2. \tag{18.2.18}$$

Returning to (18.2.12) and using (18.2.17) we obtain

$$\begin{aligned} \left(1 - \frac{\operatorname{cap}(F)}{\operatorname{cap}(\bar{\mathcal{G}}_d)} \right)^2 &\le C_2 \Big[\varepsilon + \varepsilon^{-1} d^{-n} \int_{\mathcal{G}_d} (1 - P_F)^2 \, dx \Big] \\ &\le C_2 \big[\varepsilon + 2 C_1 \sigma \varepsilon^{-1} + 2 \varepsilon^{-1} d^{-n} \eta \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \big], \end{aligned} \tag{18.2.19}$$

where $C_2 = C_2(\mathcal{G})$. Without loss of generality we will assume that $C_2 \ge 1/2$. Recalling that $\operatorname{cap}(F) \le \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$, we can replace the ratio $\operatorname{cap}(F)/\operatorname{cap}(\bar{\mathcal{G}}_d)$ in the left-hand side by γ. Now let us choose

$$\varepsilon = \frac{(1-\gamma)^2}{4 C_2}, \qquad \sigma = \frac{\varepsilon (1-\gamma)^2}{8 C_1} = \frac{(1-\gamma)^4}{32 C_1 C_2}. \tag{18.2.20}$$

Then $\varepsilon \le 1/2$ and for every fixed $\gamma \in (0,1)$ and $d > 0$ the condition (18.2.18) will be satisfied for distant bodies $\mathcal{G}_d$ because $\eta = \eta(\mathcal{G}_d) \to 0$ as $\mathcal{G}_d \to \infty$. (More precisely, there exists $R = R(\gamma, d) > 0$, such that (18.2.18) holds for every $\mathcal{G}_d$ such that $\mathcal{G}_d \cap (\mathbb{R}^n \setminus B_R(0)) \ne \emptyset$.)

If ε and σ are chosen according to (18.2.20), then (18.2.19) becomes

$$d^{-n} \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \ge (16 C_2 \eta)^{-1} (1-\gamma)^4, \tag{18.2.21}$$

which holds for distant bodies $\mathcal{G}_d$ if $\gamma \in (0,1)$ and $d > 0$ are arbitrarily fixed.

(e) Up to this moment we worked with "regular" sets F — see conditions (i)–(iii) in part (a) of this proof. Now we can get rid of the regularity requirements (i) and (ii), retaining (iii). So let us assume that F is a compact set, $\bar{\mathcal{G}}_d \setminus \Omega \subset F \subset \bar{\mathcal{G}}_d$ and $\operatorname{cap}(F) \le \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$ with $\gamma \in (0,1)$. Let us construct a sequence of compact sets $F_k \Supset F$, $k = 1, 2, \ldots$, such that every F_k is regular,

$$F_1 \Supset F_2 \Supset \ldots, \quad \text{and} \quad \bigcap_{k=1}^{\infty} F_k = F.$$

We have then $\operatorname{cap}(F_k) \to \operatorname{cap}(F)$ as $k \to +\infty$ due to the well-known continuity property of the capacity (see, e.g., Sect. 2.2.1). According to the previous steps of this proof, the inequality (18.2.21) holds for distant $\mathcal{G}_d$'s if we replace F by F_k and γ by $\gamma_k = \operatorname{cap}(F_k)/\operatorname{cap}(\bar{\mathcal{G}}_d)$. Since the measure $\mathbb{V}$ is positive, the resulting inequality will still hold if we replace $\mathbb{V}(\bar{\mathcal{G}}_d \setminus F_k)$ by $\mathbb{V}(\bar{\mathcal{G}}_d \setminus F)$. Taking limit as $k \to +\infty$, we obtain that (18.2.21) holds with $\gamma' = \operatorname{cap}(F)/\operatorname{cap}(\bar{\mathcal{G}}_d)$ instead of γ. Since $\gamma' \le \gamma$, (18.2.21) immediately follows for arbitrary compact F such that $\bar{\mathcal{G}}_d \setminus \Omega \subset F \subset \bar{\mathcal{G}}_d$ and $\operatorname{cap}(F) \le \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$ with $\gamma \in (0,1)$.

(f) Let us fix $\mathcal{G}$ and take the infimum over all negligible F's (i.e., compact sets F, such that $\bar{\mathcal{G}}_d \setminus \Omega \subset F \subset \bar{\mathcal{G}}_d$ and $\operatorname{cap}(F) \le \gamma \operatorname{cap}(\bar{\mathcal{G}}_d)$) in the right-hand side of (18.2.21). We then get for distant $\mathcal{G}_d$'s

$$d^{-n} \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d, \Omega)} \mathbb{V}(\bar{\mathcal{G}}_d \setminus F) \ge (16 C_2 \eta)^{-1} (1-\gamma)^4. \tag{18.2.22}$$

Now let us recall that the discreteness of spectrum is equivalent to the condition $\eta = \eta(\mathcal{G}_d) \to 0$ as $\mathcal{G}_d \to \infty$ (with any fixed $d > 0$). If this is the case, then it is clear from (18.2.22) that for every fixed $\gamma \in (0,1)$ and $d > 0$, the left-hand side of (18.2.22) tends to $+\infty$ as $\mathcal{G}_d \to \infty$. This concludes the proof of part (i) of Theorem 18.1/1. □

18.3 Discreteness of Spectrum: Sufficiency

In this section we will establish the sufficiency part of Theorem 18.1/1.

Let us recall the *Poincaré inequality*

$$\|u - \bar{u}\|^2_{L_2(\mathcal{G}_d)} \le A(\mathcal{G}) d^2 \int_{\mathcal{G}_d} \left|\nabla u(x)\right|^2 \mathrm{d}x, \quad u \in C^{0,1}(\mathcal{G}_d),$$

where $\mathcal{G}_d \subset \mathbb{R}^n$ was described in Sect. 18.1,

$$\bar{u} = \frac{1}{|\mathcal{G}_d|} \int_{\mathcal{G}_d} u(x)\, \mathrm{d}x$$

is the mean value of u on $\mathcal{G}_d$, $|\mathcal{G}_d|$ is the Lebesgue volume of $\mathcal{G}_d$, and $A(\mathcal{G}) > 0$ is independent of d.

The following Lemma slightly generalizes (to an arbitrary body $\mathcal{G}$) a particular case of the first part of Theorem 14.1.2 with essentially the same proof.

Lemma 1. *There exists $C(\mathcal{G}) > 0$ such that the following inequality holds for every function $u \in C^{0,1}(\bar{\mathcal{G}}_d)$ which vanishes on a compact set $F \subset \bar{\mathcal{G}}_d$ (but is not identically 0 on $\bar{\mathcal{G}}_d$):*

$$\operatorname{cap}(F) \leq \frac{C(\mathcal{G}) \int_{\mathcal{G}_d} |\nabla u(x)|^2 \, dx}{|\mathcal{G}_d|^{-1} \int_{\mathcal{G}_d} |u(x)|^2 \, dx}. \tag{18.3.1}$$

The next lemma is an obvious adaptation of Lemma 16.1 to general test bodies $\mathcal{G}_d$ (instead of cubes Q_d). In its proof Lemma 1 should be used.

Lemma 2. *Let $\mathbb{V}$ be a positive Radon measure in Ω. There exists $C_2(\mathcal{G}) > 0$ such that for every $\gamma \in (0,1)$ and $u \in C^{0,1}(\bar{\mathcal{G}}_d)$ with $u = 0$ in a neighborhood of $\bar{\mathcal{G}}_d \setminus \Omega$,*

$$\int_{\mathcal{G}_d} |u|^2 \, dx \leq \frac{C_2(\mathcal{G}) d^2}{\gamma} \int_{\mathcal{G}_d} |\nabla u|^2 \, dx + \frac{C_2(\mathcal{G}) d^n}{\mathbb{V}_\gamma(\mathcal{G}_d, \Omega)} \int_{\bar{\mathcal{G}}_d} |u|^2 \mathbb{V}(dx), \tag{18.3.2}$$

where

$$\mathbb{V}_\gamma(\mathcal{G}_d, \Omega) = \inf_{F \in \mathcal{N}_\gamma(\mathcal{G}_d, \Omega)} \mathbb{V}(\mathcal{G}_d \setminus F). \tag{18.3.3}$$

(*Here the negligibility class $\mathcal{N}_\gamma(\mathcal{G}_d, \Omega)$ was introduced in Definition 18.1.*)

Now we move to the proof of the sufficiency part in Theorem 18.1/1.

We start with the following proposition, which gives a general (albeit complicated) sufficient condition for the discreteness of spectrum.

Proposition 1. *Given an operator $H_{\mathbb{V}}$, let us assume that the following condition is satisfied: there exists $\eta_0 > 0$ such that for every $\eta \in (0, \eta_0)$ we can find $d = d(\eta) > 0$ and $R = R(\eta) > 0$, so that if $\mathcal{G}_d$ satisfies $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \varnothing$, then there exists $\gamma = \gamma(\mathcal{G}_d, \eta) \in (0,1)$ such that*

$$\gamma d^{-2} \geq \eta^{-1} \quad \text{and} \quad d^{-n} \mathbb{V}_\gamma(\mathcal{G}_d, \Omega) \geq \eta^{-1}. \tag{18.3.4}$$

Then the spectrum of $H_{\mathbb{V}}$ is discrete.

Proof. Recall that the discreteness of spectrum is equivalent to the following condition: For every $\eta > 0$ there exists $R > 0$ such that (18.2.15) holds for every $u \in C_0^\infty(\Omega)$. This will be true if we establish that for every $\eta > 0$ there exist $R > 0$ and $d > 0$ such that

$$\int_{\mathcal{G}_d} |u|^2 \, dx \leq \eta \left[\int_{\mathcal{G}_d} |\nabla u|^2 \, dx + \int_{\bar{\mathcal{G}}_d} |u|^2 \mathbb{V}(dx) \right] \tag{18.3.5}$$

for all $\mathcal{G}_d$ such that $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \varnothing$ and for all $u \in C^\infty(\bar{\mathcal{G}}_d)$, such that $u = 0$ in a neighborhood of $\bar{\mathcal{G}}_d \setminus \Omega$. Indeed, assume that (18.3.5) is true.

Let us take a covering of $\mathbb{R}^n$ by bodies $\bar{\mathcal{G}}_d$ so that it has a finite multiplicity $m = m(\mathcal{G})$ (i.e., at most m bodies $\bar{\mathcal{G}}_d$ can have a nonempty intersection). Then, taking $u \in C_0^\infty(\Omega)$ and summing up the estimates (18.3.5) over all bodies $\mathcal{G}_d$ with $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \varnothing$, we obtain (18.2.15) (hence (18.2.14)) with $m\eta$ instead of η.

Now Lemma 18.3/2 and the assumptions (18.3.4) immediately imply (18.3.5) (with η replaced by $C_2(\mathcal{G})\eta$). □

Instead of requiring that the conditions of Proposition 1 are satisfied for all $\eta \in (0, \eta_0)$, it suffices to require it for a monotone sequence $\eta_k \to +0$. We can also assume that $d(\eta_k) \to 0$ as $k \to +\infty$. Then, passing to a subsequence, we can assume that the sequence $\{d(\eta_k)\}$ is strictly decreasing. Keeping this in mind, we can replace the dependence $d = d(\eta)$ by the inverse dependence $\eta = g(d)$, so that $g(d) > 0$ and $g(d) \to 0$ as $d \to +0$ (and here we can also restrict ourselves to a sequence $d_k \to +0$). This leads to the following, essentially equivalent, but more convenient reformulation of Proposition 1.

Proposition 2. *Given an operator $H_{\mathbb{V}}$, assume that the following condition is satisfied: There exists $d_0 > 0$ such that for every $d \in (0, d_0)$ we can find $R = R(d) > 0$ and $\gamma = \gamma(d) \in (0,1)$, so that if $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \varnothing$, then*

$$d^{-2}\gamma \geq g(d)^{-1} \quad \textit{and} \quad d^{-n}\mathbb{V}_\gamma(\mathcal{G}_d, \Omega) \geq g(d)^{-1}, \tag{18.3.6}$$

where $g(d) > 0$ and $g(d) \to 0$ as $d \to +0$. Then the spectrum of $H_{\mathbb{V}}$ is discrete.

Proof of Theorem 18.1/1, part (ii). Instead of (ii) in Theorem 18.1/1 it suffices to prove the (stronger) statement formulated in Remark 18.1/4. So suppose that $\exists\, d_0 > 0$, $\exists c > 0$, $\forall d \in (0, d_0)$, $\exists R = R(d) > 0$, $\exists \gamma(d) \in (0,1)$, satisfying (18.0.1), such that (18.1.6) holds for all $\mathcal{G}_d$ with $\bar{\mathcal{G}}_d \cap (\Omega \setminus B_R(0)) \neq \varnothing$.

Since the left-hand side of (18.1.6) is exactly $d^{-n}\mathbb{V}_{\gamma(d)}(\mathcal{G}_d, \Omega)$, we see that (18.1.6) can be rewritten in the form

$$d^{-n}\mathbb{V}_\gamma(\mathcal{G}_d, \Omega) \geq c d^{-2}\gamma(d),$$

hence we can apply Proposition 18.3 with $g(d) = c^{-1}d^2\gamma(d)^{-1}$ to conclude that the spectrum of $H_{\mathbb{V}}$ is discrete. □

18.4 A Sufficiency Example

In this section we prove Theorem 18.1/2. We shall construct a domain $\Omega \subset \mathbb{R}^n$, such that the condition (18.1.7) is satisfied with $\gamma(d) = Cd^2$ (with an arbitrarily large $C > 0$), and yet the spectrum of $-\Delta$ in $L_2(\Omega)$ (with the Dirichlet boundary condition) is not discrete. This will prove Theorem 18.1/2 showing that the condition (18.0.1) is precise. We assume that $n \geq 3$.

Let us use the following notations:

- $L^{(j)}$ is the spherical layer $\{x \in \mathbb{R}^n : \log j \le |x| \le \log(j+1)\}$. Its width is $\log(j+1) - \log j$ which is $< j^{-1}$ for all j and equivalent to j^{-1} for large j.
- $\{Q_k^{(j)}\}_{k\ge 1}$ is a collection of closed cubes that form a tiling of $\mathbb{R}^n$ and have edge length $\epsilon(n)\, j^{-1}$, where $\epsilon(n)$ is a sufficiently small constant depending on n (to be adjusted later).
- $x_k^{(j)}$ is the center of $Q_k^{(j)}$.
- $\{B_k^{(j)}\}_{k\ge 1}$ is the collection of closed balls centered at $x_k^{(j)}$ with radii ρ_j given by
$$\omega_n(n-2)\,\rho_j^{n-2} = C\big(\epsilon(n)/j\big)^n,$$
where ω_n is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and C is an arbitrary constant. The last equality can be written as
$$\operatorname{cap}\big(B_k^{(j)}\big) = C\, m_n\, Q_k^{(j)}. \tag{18.4.1}$$
Among the balls $B_k^{(j)}$ we select a subcollection which consists of the balls with the additional property $B_k^{(j)} \subset L^{(j)}$. We shall refer to these balls as *selected* ones and denote selected balls by $\tilde{B}_k^{(j)}$. By an abuse of notation we do not introduce a special letter for the subscripts of the selected balls. We also denote by $\tilde{Q}_k^{(j)}$ the corresponding cubes $Q_k^{(j)}$, so that
$$\tilde{Q}_k^{(j)} = Q_k^{(j)} \supset \tilde{B}_k^{(j)}.$$
- $\Lambda^{(j)} = \bigcup_{k\ge 1} \tilde{B}_k^{(j)} \subset L^{(j)}$.
- Ω is the complement of $\bigcup_{j\ge 1} \Lambda^{(j)}$.
- $B_r(P)$ is the closed ball with radius $r \le 1$ centered at a point P. We shall make a more precise choice of r later.

Proposition 1. *The spectrum of* $-\Delta$ *in* Ω (*with the Dirichlet boundary condition*) *is not discrete.*

Proof. Let $j \ge 7$ and $P \in L^{(j)}$, i.e.,
$$\log j \le |P| \le \log(j+1).$$
Note that the ball $B_r(P)$ is a subset of the spherical layer $\bigcup_{l\ge s\ge m} L^{(s)}$ if and only if
$$\log m \le |P| - r \quad \text{and} \quad |P| + r \le \log(l+1).$$
Therefore, if
$$\log m \le \log j - r$$
and
$$\log(j+1) + r \le \log(l+1),$$
then $B_r(P) \subset \bigcup_{l\ge s\ge m} L^{(s)}$. The last two inequalities can be written as

$$m \le j\, e^{-r} \quad \text{and} \quad j+1 \le (l+1)e^{-r}. \tag{18.4.2}$$

If we take, for example,

$$m = [j/3] \quad \text{and} \quad l = 3j,$$

then, by the inequality $j \ge 7$, we deduce that

$$B_r(P) \subset \bigcup_{[j/3] \le s \le 3j} L^{(s)}. \tag{18.4.3}$$

Using (18.4.2), the definition of Ω and subadditivity of capacity, we obtain

$$\begin{aligned} \operatorname{cap}\big(B_r(P)\setminus \Omega\big) &= \operatorname{cap}\bigg(B_r(P) \cap \Big(\bigcup_{s\ge 1} \Lambda^{(s)}\Big)\bigg) \\ &\le \sum_{[j/3]\le s \le 3j} \sum_{k \ge 1} \operatorname{cap}\big(B_r(P) \cap \tilde{B}_k^{(s)}\big) \\ &\le C \sum_{[j/3]\le s\le 3j} \sum_{\{k: B_r(P)\cap \tilde{Q}_k^{(s)} \ne \emptyset\}} m_n\, \tilde{Q}_k^{(s)}. \end{aligned}$$

We see that the multiplicity of the covering of $B_r(P)$ by the cubes $\tilde{Q}_k^{(s)}$, participating in the last sum, is at most 2, provided $\epsilon(n)$ is chosen sufficiently small. Hence,

$$\operatorname{cap}\big(B_r(P)\setminus\Omega\big) \le c(n) C r^n. \tag{18.4.4}$$

On the other hand, we know that the discreteness of spectrum guarantees that for every $r > 0$

$$\liminf_{|P|\to\infty} \operatorname{cap}\big(B_r(P)\setminus\Omega\big) \ge \gamma(n)\, r^{n-2},$$

where $\gamma(n)$ is a constant depending only on n (cf. Remark 18.1/7). For sufficiently small $r > 0$ this contradicts (18.4.4). □

Proposition 2. *The domain Ω satisfies*

$$\liminf_{|P|\to\infty} \operatorname{cap}\big(B_r(P)\setminus\Omega\big) \ge \delta(n)\, C\, r^n, \tag{18.4.5}$$

where $\delta(n) > 0$ depends only on n.

Proof. Let $\mu_k^{(s)}$ be the capacitary measure on $\partial\tilde{B}_k^{(s)}$ (extended by zero to $\mathbb{R}^n \setminus \partial\tilde{B}_k^{(s)}$), and let $\epsilon_1(n)$ denote a sufficiently small constant to be chosen later. We introduce the measure

$$\mu = \epsilon_1(n) \sum_{k,s} \mu_k^{(s)},$$

where the summation here and in the following is taken over k, s which correspond to the selected balls $\tilde{B}_k^{(s)}$. Taking $P \in L^{(j)}$, let us show that

$$\int_{B_{r/2}(P)} \mathcal{E}(x-y)\,\mathrm{d}\mu(y) \le 1 \quad \text{on } \mathbb{R}^n, \tag{18.4.6}$$

where $\mathcal{E}(x)$ is given by (18.2.3). It suffices to verify (18.4.6) for $x \in B_r(P)$ because for $x \in \mathbb{R}^n \setminus B_r(P)$ this will follow from the maximum principle.

Obviously, the potential in (18.4.6) does not exceed

$$\sum_{\{s,k:\tilde{B}_k^{(s)} \cap B_{r/2}(P) \neq \varnothing\}} \epsilon_1(n) \int_{\partial \tilde{B}_k^{(s)}} \mathcal{E}(x-y)\,\mathrm{d}\mu_k^{(s)}(y).$$

We divide this sum into two parts $\sum'$ and $\sum''$, the first sum being extended over all points $x_k^{(s)}$ with the distance $\le j^{-1}$ from x. Recalling that $x \in B_r(P)$ and using (18.4.3), we see that the number of such points does not exceed a certain constant $c_1(n)$. We define the constant $\epsilon_1(n)$ by

$$\epsilon_1(n) = \big(2c_1(n)\big)^{-1}.$$

Since $\mu_k^{(s)}$ is the capacitary measure, we have

$$\sum{}' \cdots \le \epsilon_1(n)\, c_1(n) = 1/2.$$

Furthermore, by (18.4.1)

$$\begin{aligned}\sum{}'' \cdots &\le c_2(n) \sum{}'' \frac{\operatorname{cap}(\tilde{B}_k^{(s)})}{|x-x_k^{(s)}|^{n-2}} = c_2(n)\, C \sum{}'' \frac{m_n\, \tilde{Q}_k^{(s)}}{|x-x_k^{(s)}|^{n-2}} \\ &\le c_3(n)\; C \int_{B_r(P)} \frac{\mathrm{d}y}{|x-y|^{n-2}} < c_4(n)\; C\; r^2.\end{aligned}$$

We can assume that

$$r \le (2c_4(n)C)^{-1/2},$$

which implies $\sum'' \le 1/2$. Therefore (18.4.6) holds.

It follows that for large $|P|$ (i.e., for P with $|P| \ge R = R(r) > 0$), or equivalently, for large j, we have

$$\begin{aligned}\operatorname{cap}\big(B_r(P) \setminus \varOmega\big) &\ge \sum_{\{s,k:\tilde{B}_k^{(s)} \subset B_{r/2}(P)\}} \epsilon_1(n) \mu_k^{(s)}\big(\partial \tilde{B}_k^{(s)}\big) \\ &= \epsilon_1(n) \sum_{\{s,k:\tilde{B}_k^{(s)} \subset B_{r/2}(P)\}} \operatorname{cap}\big(\tilde{B}_k^{(s)}\big) \\ &= \epsilon_1(n)\; C \sum_{\{s,k:\tilde{B}_k^{(s)} \subset B_{r/2}(P)\}} m_n\, Q_k^{(s)} \ge \delta(n)\; C\; r^n.\end{aligned}$$

This ends the proof of Proposition 2, hence of Theorem 18.1/2. □

Remark. Slightly modifying the previous construction, we provide an example of an operator $H = -\Delta + V(x)$ with $V \in C^\infty(\mathbb{R}^n)$, $n \geq 3$, $V \geq 0$, such that the corresponding measure $V dx$ satisfies (18.1.5) with $\gamma(d) = Cd^2$ and an arbitrarily large $C > 0$, but the spectrum of H in $L_2(\mathbb{R}^n)$ is not discrete. So the condition (18.0.1) is precise even in the case of the Schrödinger operators with C^∞ potentials.

18.5 Positivity of $H_{\mathbb{V}}$

In this section we prove Theorem 18.1/3.

Proof of Theorem 18.1/3 (Necessity). Let us assume that the operator $H_{\mathbb{V}}$ is strictly positive. This implies that the estimate (18.2.16) holds with some $\eta > 0$ for every $\mathcal{G}_d$ (with an arbitrary $d > 0$) and every $u \in C_0^\infty(\mathcal{G}_d \cap \Omega)$. But then we can use the arguments of Sect. 18.2 which lead to (18.2.22), provided (18.2.18) is satisfied. It will be satisfied if d is chosen sufficiently large. □

Proof of Theorem 18.1/3 (Sufficiency). Let us assume that there exist $d > 0$, $\varkappa > 0$ and $\gamma \in (0, 1)$ such that for every $\mathcal{G}_d$ the estimate (18.1.9) holds. Then by Lemma 18.3/2, for every $\mathcal{G}_d$ and every $u \in C^\infty(\bar{\mathcal{G}}_d)$, such that $u = 0$ in a neighborhood of $\bar{\mathcal{G}}_d \setminus \Omega$, we have

$$\int_{\mathcal{G}_d} |u|^2 \, dx \leq \frac{C_2(\mathcal{G})d^2}{\gamma} \int_{\mathcal{G}_d} |\nabla u|^2 \, dx + \frac{C_2(\mathcal{G})d^n}{\varkappa} \int_{\bar{\mathcal{G}}_d} |u|^2 \mathbb{V}(dx).$$

Let us take a covering of $\mathbb{R}^n$ of finite multiplicity N by bodies $\bar{\mathcal{G}}_d$. It follows that for every $u \in C_0^\infty(\Omega)$

$$\int_\Omega |u|^2 \, dx \leq NC_2(\mathcal{G})d^2 \max\left\{\frac{1}{\gamma}, \frac{d^{n-2}}{\varkappa}\right\}\left(\int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u|^2 \mathbb{V}(dx)\right),$$

which proves positivity of $H_{\mathbb{V}}$. □

18.6 Structure of the Essential Spectrum of $H_{\mathbb{V}}$

Lemma. *If the spectrum of $H_{\mathbb{V}}$ is not purely discrete, the essential spectrum of $H_{\mathbb{V}}$ extends to infinity. Moreover, if 0 belongs to the essential spectrum of $H_{\mathbb{V}}$, then this spectrum coincides with $[0, \infty)$.*

Proof. Let Λ be the bottom of the essential spectrum. Then there exists a sequence of real-valued functions $\{\varphi_\nu\}_{\nu \geq 1}$ in $C_0^\infty(\Omega)$ subject to the conditions

$$\|\varphi_\nu\|_{L_2(\Omega)} = 1, \qquad \varphi_\nu \to 0 \quad \text{weakly in } L_2(\Omega), \tag{18.6.1}$$

$$\big\|(H_{\mathbb{V}}-\Lambda)\varphi_\nu\big\|_{L_2(\Omega)} \to 0. \tag{18.6.2}$$

We set

$$u_\nu = \varphi_\nu \exp\Big(i(\alpha-\Lambda)^{1/2}\sum_{k=1}^{n} x_k\Big),$$

where $\alpha > \Lambda$. We see that u_ν satisfies (18.6.1) and that

$$\big\|(H_{\mathbb{V}}-\alpha)u_\nu\big\|^2_{L_2(\Omega)} = \big\|(H_{\mathbb{V}}-\Lambda)\varphi_\nu\big\|^2_{L_2(\Omega)} + 4(\alpha-\Lambda)\Big\|\sum_{k=1}^{n}\partial\varphi_\nu/\partial x_k\Big\|^2_{L_2(\Omega)}.$$

Since the right-hand side does not exceed

$$\big\|(H_{\mathbb{V}}-\Lambda)\varphi_\nu\big\|^2_{L_2(\Omega)} + 4n(\alpha-\Lambda)Q[\varphi_\nu,\varphi_\nu],$$

we have

$$\begin{aligned}\big\|(H_{\mathbb{V}}-\alpha)u_\nu\big\|^2_{L_2(\Omega)} \le \big\|(H_{\mathbb{V}}-\Lambda)\varphi_\nu\big\|^2_{L_2(\Omega)} &+ 4n(\alpha-\Lambda)\big\|(H_{\mathbb{V}}-\Lambda)\varphi_\nu\big\|_{L_2(\Omega)}\\ &+ 4n\Lambda(\alpha-\Lambda).\end{aligned}$$

By (18.6.2)

$$\limsup_{\nu\to\infty}\big\|(H_{\mathbb{V}}-\alpha)u_\nu\big\|_{L_2(\Omega)} \le \rho(\alpha),$$

where $\rho(\alpha) = 2(n\,\Lambda(\alpha-\Lambda))^{1/2}$. It follows that any segment $[\alpha-\rho(\alpha),\alpha+\rho(\alpha)]$ contains points of the essential spectrum. If, in particular, $\Lambda = 0$ then every positive α belongs to the essential spectrum. □

In concert with this lemma the pairs $(\Omega,\mathbb{V})$ can be divided into three nonoverlapping classes.

(i) The first class includes $(\Omega,\mathbb{V})$ such that the spectrum of $H_{\mathbb{V}}$ is discrete.
(ii) The pair $(\Omega,\mathbb{V})$ belongs to the second class if the essential spectrum of $H_{\mathbb{V}}$ is unbounded and strictly positive.
(iii) Finally, $(\Omega,\mathbb{V})$ is of the third class if the essential spectrum of $H_{\mathbb{V}}$ coincides with $[0,\infty)$.

By Theorem 18.1, condition (i) is equivalent to (18.1.5) and (ii) holds if and only if (18.1.9) is valid. Finally, (iii) is equivalent to the failure of (18.1.7) by Theorem 18.1/3 and Lemma. In other words, (iii) holds if and only if

$$\liminf_{\mathcal{G}_d\to\infty}\ \inf_{F\in\mathcal{N}_{\gamma(d)}(\mathcal{G}_d,\Omega)} \mathbb{V}(\bar{\mathcal{G}}_d\setminus F) = 0$$

for every $d > 0$.

18.7 Two-Sided Estimates of the First Eigenvalue of the Dirichlet Laplacian

18.7.1 Main Result

Let us consider an open set $\Omega \subset \mathbb{R}^n$, $n > 2$ and denote the bottom of the spectrum of its minus Dirichlet Laplacian $(-\Delta)_{\rm Dir}$ by $\Lambda(\Omega)$, i.e.,

$$\Lambda(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx}. \tag{18.7.1}$$

Obviously, $\Omega' \subset \Omega$ implies $\Lambda(\Omega) \leq \Lambda(\Omega')$. In particular, if B_r is an open ball of radius r, such that $B_r \subset \Omega$, then $\Lambda(\Omega) \leq \Lambda(B_r) = C_n r^{-2}$ where $C_n = \Lambda(B_1)$. It follows that for the interior radius of Ω, which is defined as

$$r_\Omega = \sup\{r : \exists B_r \subset \Omega\},$$

we have

$$\Lambda(\Omega) \leq C_n r_\Omega^{-2}.$$

However, this estimate is not good for unbounded domains or domains with complicated boundaries. For instance, a similar estimate for $\Lambda(\Omega)$ from below does not hold for the complement Ω of any Cartesian coordinate grid in $\mathbb{R}^n$ when $\Lambda(\Omega) = 0$ and $r_\Omega < \infty$.

The way to improve this estimate is to relax (as in Sect. 15.4) the requirement for B_r to be completely inside Ω by allowing some part of B_r, which has a "small" harmonic capacity, to stick out of Ω. Namely, let us take an *arbitrary* $\gamma \in (0, 1)$ and call a compact set $F \subset \bar{B}_r$ negligible (or more precisely, γ-negligible) if

$$\operatorname{cap}(F) \leq \gamma \operatorname{cap}(\bar{B}_r). \tag{18.7.2}$$

(Here $\operatorname{cap}(F)$ denotes the Wiener (harmonic) capacity of F, $\operatorname{cap}(F; \mathring{L}_2^1(\mathbb{R}^n))$.)

Now write

$$r_{\Omega,\gamma} = \sup\{r : \exists B_r, \bar{B}_r \setminus \Omega \text{ is } \gamma\text{-negligible}\}.$$

This is the *interior capacitary radius.*

Theorem. *Let us fix* $\gamma \in (0, 1)$. *Then there exist* $c = c(\gamma, n) > 0$ *and* $C = C(\gamma, n) > 0$, *such that for every open set* $\Omega \subset \mathbb{R}^n$

$$c r_{\Omega,\gamma}^{-2} \leq \Lambda(\Omega) \leq C r_{\Omega,\gamma}^{-2}. \tag{18.7.3}$$

Explicit values of constants $c = c(\gamma, n)$ *and* $C = C(\gamma, n)$ *are provided in* (18.7.22) *and* (18.7.38), *respectively.*

Let us formulate some corollaries of this theorem.

Corollary 1. $\Lambda(\Omega) > 0$ *if and only if* $r_{\Omega,\gamma} < \infty$.

This corollary gives a necessary and sufficient condition of strict positivity of the operator $(-\Delta)_{\mathrm{Dir}}$ in Ω.

Since the condition $\Lambda(\Omega) > 0$ does not contain γ, we immediately obtain the following corollary.

Corollary 2. *Conditions $r_{\Omega,\gamma} < \infty$, taken for different γ's, are equivalent.*

Writing $F = \mathbb{R}^n \setminus \Omega$ (which can be an arbitrary closed subset in $\mathbb{R}^n$), we obtain from the previous corollary (comparing $\gamma = 0.01$ and $\gamma = 0.99$):

Corollary 3. *Let F be a closed subset in $\mathbb{R}^n$, which has the following property: There exists $r > 0$ such that*

$$\operatorname{cap}(F \cap \bar{B}_r) \geq 0.01 \operatorname{cap}(\bar{B}_r)$$

for all B_r. Then there exists $r_1 > 0$ such that

$$\operatorname{cap}(F \cap \bar{B}_{r_1}) \geq 0.99 \operatorname{cap}(\bar{B}_{r_1})$$

for all B_{r_1}.

The inequalities (18.7.3) for sufficiently small $\gamma > 0$ were established in Chaps. 14 and 15. The Theorem provides a substantial improvement, in particular, allowing Corollaries 2 and 3 and providing explicit values of the constants.

18.7.2 Lower Bound

In this section we will establish the lower bound for $\Lambda(\Omega)$ from Theorem 18.7.1, which is an easier part of this theorem. The key part of the lower bound proof is presented in the following lemma, which is a particular case of a more general Theorem 14.1.2, part 1, though without an explicit constant, which we provide to specify explicit constants in Theorem 18.7.1.

Lemma 1. *The following inequality holds for every complex-valued function $u \in C^{0,1}(\bar{B}_r)$ which vanishes on a compact set $F \subset \bar{B}_r$ (but is not identically 0 on $\bar{B}_r$):*

$$\operatorname{cap}(F) \leq \frac{C_n \int_{B_r} |\nabla u(x)|^2 \, \mathrm{d}x}{r^{-n} \int_{B_r} |u(x)|^2 \, \mathrm{d}x}, \tag{18.7.4}$$

where

$$C_n = 4\omega_n \left(1 - \frac{2}{n^2}\right). \tag{18.7.5}$$

Beginning of Proof. A. Clearly, it is sufficient to consider the ball B_r centered at 0, and real-valued functions $u \in C^{0,1}(\bar{B}_r)$. By scaling we see that it suffices to consider the case $r = 1$. (The corresponding estimate for an arbitrary $r > 0$ follows from the one with $r = 1$ with the same constant C_n.) So we need to prove the estimate

$$\int_{B_1} |u|^2 \, \mathrm{d}x \le \frac{C_n}{\mathrm{cap}(F)} \int_{B_1} |\nabla u|^2 \, \mathrm{d}x, \tag{18.7.6}$$

where F is a compact subset of $\bar{B}_1$, $u \in C^{0,1}(\bar{B}_1)$, and $u|_F = 0$.

Consider the function $U \in C^{0,1}(\mathbb{R}^n)$

$$U(x) = \begin{cases} 1 - |u(x)|, & \text{if } |x| \le 1, \\ |x|^{2-n}(1 - |u(|x|^{-2}x)|), & \text{if } |x| \ge 1, \end{cases}$$

i.e., U extends $1 - |u|$ to $\{x : |x| \ge 1\}$ as the Kelvin transform of $1 - |u|$. Clearly, $U|_F = 1$, $|\nabla U| = |\nabla u|$ almost everywhere in B_1, $U(x) = O(|x|^{2-n})$ and $|\nabla U(x)| = O(|x|^{1-n})$ as $|x| \to \infty$. It follows that U can serve as a test function in (18.2.1), i.e.,

$$\mathrm{cap}(F) \le \int_{\mathbb{R}^n} |\nabla U|^2 \, \mathrm{d}x. \tag{18.7.7}$$

Using the harmonicity of $|x|^{2-n}$ and the Green–Stokes formula, we obtain by a straightforward calculation

$$\int_{\mathbb{R}^n} |\nabla U|^2 \, \mathrm{d}x = 2 \int_{B_1} |\nabla u|^2 \, \mathrm{d}x + (n-2) \int_{\partial B_1} \big(1 - |u(\omega)|\big)^2 \, \mathrm{d}\omega, \tag{18.7.8}$$

where $d\omega$ means the area element on ∂B_1.

B. For a function v on ∂B_1 define its average

$$\bar{v} = \fint_{\partial B_1} v \, \mathrm{d}\omega = \frac{1}{\omega_n} \int_{\partial B_1} v \, \mathrm{d}\omega.$$

To continue the proof of Lemma 1, we will need the following elementary Poincaré-type trace inequality.

Lemma 2. *For any* $v \in C^{0,1}(B_1)$,

$$\int_{\partial B_1} |v - \bar{v}|^2 \, \mathrm{d}\omega \le \int_{B_1} |\nabla v|^2 \, \mathrm{d}x. \tag{18.7.9}$$

Proof of Lemma 2. It suffices to prove it for real-valued functions v. Let us expand v in spherical functions. Let

$$\{Y_{k,l} \mid l = 0, 1, \ldots, n_k, k = 0, 1, \ldots,\}$$

be an orthonormal basis in $L_2(\partial B_1)$ which consists of eigenfunctions of the (negative) Laplace–Beltrami operator Δ_ω on ∂B_1, so that the eigenfunctions $Y_{k,l} = Y_{k,l}(\omega)$ with a fixed k have the same eigenvalue $-k(k+n-2)$ (which has multiplicity n_k+1). Note that the zero eigenvalue (corresponding to $k=0$) has multiplicity 1 and $Y_{0,0} = \text{const} = \omega_n^{-1/2}$ for the corresponding eigenfunction.

Writing $x = r\omega$, where $r = |x|$, $\omega = x/|x|$, we can present v in the form

$$v(x) = v(r,\omega) = \sum_{k,l} v_{k,l}(r) Y_{k,l}(\omega). \tag{18.7.10}$$

Then

$$\int_{B_1} \big|v(x)\big|^2 \,\mathrm{d}x = \sum_{k,l} \int_0^1 \big|v_{k,l}(r)\big|^2 r^{n-1} \,\mathrm{d}r \tag{18.7.11}$$

and

$$\int_{\partial B_1} \big|v(\omega)\big|^2 \,\mathrm{d}\omega = \sum_{k,l} \big|v_{k,l}(1)\big|^2. \tag{18.7.12}$$

It follows that

$$\int_{\partial B_1} \big|v(\omega) - \bar{v}\big|^2 \,\mathrm{d}\omega = \sum_{\{k,l:k\geq 1\}} \big|v_{k,l}(1)\big|^2. \tag{18.7.13}$$

Taking into account that

$$|\nabla v|^2 = \left|\frac{\partial v}{\partial r}\right|^2 + r^{-2}|\nabla_\omega v|^2,$$

where ∇_ω means the gradient along the unit sphere with variable ω and fixed r, we also get

$$\int_{B_1} |\nabla v|^2 \,\mathrm{d}x = \sum_{k,l} \int_0^1 \left(\big|v'_{k,l}(r)\big|^2 + \frac{k(k+n-2)}{r^2}\big|v_{k,l}(r)\big|^2\right) r^{n-1} \,\mathrm{d}r. \tag{18.7.14}$$

Comparing (18.7.13) and (18.7.14), and taking into account that $k(k+n-2)$ increases with k, we see that it suffices to establish that the inequality

$$\big|g(1)\big|^2 \leq \int_0^1 \left(\big|g'(r)\big|^2 + \frac{n-1}{r^2}\big|g(r)\big|^2\right) r^{n-1} \,\mathrm{d}r$$

holds for any real-valued function $g \in C^{0,1}([0,1])$. To this end write

$$\begin{aligned} g(1)^2 &= \int_0^1 \big(r^{n-2} g^2\big)' \,\mathrm{d}r = \int_0^1 \big[2r^{n-2} g' g + (n-2) r^{n-3} g^2\big] \,\mathrm{d}r \\ &\leq \int_0^1 \big[r^{n-1} g'^2 + (n-1) r^{n-3} g^2\big] \,\mathrm{d}r = \int_0^1 \left(g'^2 + \frac{n-1}{r^2} g^2\right) r^{n-1} \,\mathrm{d}r, \end{aligned}$$

which proves Lemma 2. □

Proof of Lemma 1 (continuation). C. Let us normalize u by requiring $\overline{|u|} = 1$, i.e., average of $|u|$ over ∂B_1 equals 1. Then by Lemma 2

$$\int_{\partial B_1} (1 - |u|)^2 \, \mathrm{d}\omega \le \int_{B_1} |\nabla u|^2 \, \mathrm{d}x.$$

Combining this with (18.7.7) and (18.7.8), we obtain

$$\operatorname{cap}(F) \le n \int_{B_1} |\nabla u|^2 \, \mathrm{d}x.$$

Removing the restriction $\overline{|u|} = 1$, we can conclude that for any $u \in C^{0,1}(B_1)$

$$\left(⨍_{\partial B_1} |u| \, \mathrm{d}\omega \right)^2 \le \frac{n}{\operatorname{cap}(F)} \int_{B_1} |\nabla u|^2 \, \mathrm{d}x. \tag{18.7.15}$$

Note that for any real-valued function $v \in C^{0,1}(B_1)$

$$⨍_{\partial B_1} |v - \bar{v}|^2 \, \mathrm{d}\omega = ⨍_{\partial B_1} |v|^2 \, \mathrm{d}\omega - \bar{v}^2,$$

hence, using (18.7.9), we get

$$⨍_{\partial B_1} |v|^2 \, \mathrm{d}\omega = \bar{v}^2 + ⨍_{\partial B_1} |v - \bar{v}|^2 \, \mathrm{d}\omega \le \bar{v}^2 + \frac{1}{\omega_n} \int_{B_1} |\nabla v|^2 \, \mathrm{d}x.$$

Applying this to $v = |u|$ and using (18.7.15), we obtain

$$\int_{\partial B_1} |u|^2 \, \mathrm{d}\omega \le \left(1 + \frac{n\,\omega_n}{\operatorname{cap}(F)} \right) \int_{B_1} |\nabla u|^2 \, \mathrm{d}x. \tag{18.7.16}$$

D. Note that our goal is an estimate that is similar to (18.7.16), but with the integral over ∂B_1 in the left-hand side replaced by the integral over B_1. To this end we again use the expansion (18.7.10) of $v = |u|$ over spherical functions, and the identities (18.7.11), (18.7.12), and (18.7.14). Let us take a real-valued function $g \in C^{0,1}([0,1])$ and write

$$Q = \int_0^1 g^2(r) r^{n-1} \, \mathrm{d}r.$$

Integrating by parts, we obtain

$$Q = -\frac{2}{n} \int_0^1 g g' r^n \, \mathrm{d}r + \frac{1}{n} g^2(1).$$

Using the inequality $2ab \le \varepsilon a^2 + \varepsilon^{-1} b^2$, where $a, b \in \mathbb{R}$, $\varepsilon > 0$, and taking into account that $r \le 1$, we obtain

$$Q \leq \frac{1}{n} \int_0^1 \left(\varepsilon g^2(r) + \frac{1}{\varepsilon} g'^2(r) \right) r^{n-1} \, \mathrm{d}r + \frac{1}{n} g^2(1)$$
$$= \frac{\varepsilon}{n} Q + \frac{1}{n\varepsilon} \int_0^1 g'^2(r) r^{n-1} \, \mathrm{d}r + \frac{1}{n} g^2(1),$$

hence for any $\varepsilon \in (0, n)$

$$Q \leq \frac{1}{(n-\varepsilon)\varepsilon} \int_0^1 g'^2(r) r^{n-1} \, \mathrm{d}r + \frac{1}{n-\varepsilon} g^2(1).$$

Taking $\varepsilon = n/2$, we obtain

$$Q \leq \frac{4}{n^2} \int_0^1 g'^2(r) r^{n-1} \, \mathrm{d}r + \frac{2}{n} g^2(1). \tag{18.7.17}$$

Now we can argue as in the proof of Lemma 2, expanding $v = |u|$ over spherical harmonics $Y_{k,l}$. Then the desired inequality follows from the inequalities for the coefficients $v_{k,l} = v_{k,l}(r)$, with the strongest one corresponding to the case $k = 0$ (unlike $k = 1$ in Lemma 2). Then using the inequality (18.7.17) for $g = v_{0,0}$ we obtain

$$\int_{B_1} |u|^2 \, \mathrm{d}x \leq \frac{4}{n^2} \int_{B_1} |\nabla u|^2 \, \mathrm{d}x + \frac{2}{n} \int_{\partial B_1} |u|^2 \, \mathrm{d}\omega. \tag{18.7.18}$$

Using (18.7.16), we deduce from (18.7.18)

$$\int_{B_1} |u|^2 \, \mathrm{d}x \leq \left[\frac{4}{n^2} + \frac{2}{n} \left(1 + \frac{n\omega_n}{\mathrm{cap}(F)} \right) \right] \int_{B_1} |\nabla u|^2 \, \mathrm{d}x. \tag{18.7.19}$$

Taking into account the inequality

$$\mathrm{cap}(F) \leq \mathrm{cap}(\bar{B}_1) = (n-2)\omega_n,$$

we can estimate the constant in front of the integral in the right-hand side of (18.7.19) as follows:

$$\frac{4}{n^2} + \frac{2}{n} \left(1 + \frac{n\omega_n}{\mathrm{cap}(F)} \right) \leq \frac{4\omega_n}{\mathrm{cap}(F)} \left(1 - \frac{2}{n^2} \right),$$

which ends the proof of Lemma 1. □

The lower bound in (18.7.3) is given by the following lemma.

Lemma 3. *There exists $c = c(\gamma, n) > 0$ such that for all open sets $\Omega \subset \mathbb{R}^n$*

$$\Lambda(\Omega) \geq c \, r_{\Omega,\gamma}^{-2}. \tag{18.7.20}$$

Proof. Let us fix $\gamma \in (0,1)$ and choose any $r > r_{\Omega,\gamma}$. Then any ball $\bar{B}_r$ has a nonnegligible intersection with $\mathbb{R}^n \setminus \Omega$, i.e.,

$$\operatorname{cap}(\bar{B}_r \setminus \Omega) \geq \gamma \operatorname{cap}(\bar{B}_r).$$

Since any $u \in C_0^\infty(\Omega)$ vanishes on $\bar{B}_r \setminus \Omega$, it follows from Lemma 1 that for any such u

$$\int_{\bar{B}_r} |u|^2 \, \mathrm{d}x \leq \frac{C_n}{r^{-n} \operatorname{cap}(\bar{B}_r \setminus \Omega)} \int_{\bar{B}_r} |\nabla u|^2 \, \mathrm{d}x \leq \frac{C_n}{r^{-n} \gamma \operatorname{cap}(\bar{B}_r)} \int_{\bar{B}_r} |\nabla u|^2 \, \mathrm{d}x.$$

Taking into account that $\operatorname{cap}(\bar{B}_r) = \operatorname{cap}(\bar{B}_1) r^{n-2}$, we obtain

$$\int_{\bar{B}_r} |u|^2 \, \mathrm{d}x \leq \frac{C_n r^2}{\gamma \operatorname{cap}(\bar{B}_1)} \int_{\bar{B}_r} |\nabla u|^2 \, \mathrm{d}x.$$

Now let us choose a covering of $\mathbb{R}^n$ by balls $\bar{B}_r = \bar{B}_r^{(k)}$, $k = 1, 2, \dots$, so that the multiplicity of this covering is at most $N = N(n)$. For example, we can make

$$N(n) \leq n \log n + n \log(\log n) + 5n, \quad n \geq 2, \tag{18.7.21}$$

which holds also for the smallest multiplicity of coverings of $\mathbb{R}^n$ by translations of any convex body (see Theorem 3.2 in Rogers [679]).

Then summing up the previous estimates over all balls in this covering, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^2 \, \mathrm{d}x &\leq \sum_k \int_{\bar{B}_r^{(k)}} |u|^2 \, \mathrm{d}x \leq \frac{C_n r^2}{\gamma \operatorname{cap}(\bar{B}_1)} \sum_k \int_{\bar{B}_r^{(k)}} |\nabla u|^2 \, \mathrm{d}x \\ &\leq \frac{C_n N r^2}{\gamma \operatorname{cap}(\bar{B}_1)} \int_{\mathbb{R}^n} |\nabla u|^2 \, \mathrm{d}x.\end{aligned}$$

Recalling (18.7.1), we see that

$$\Lambda(\Omega) \geq c r^{-2},$$

with

$$c = c(\gamma, n) = \frac{\gamma \operatorname{cap}(\bar{B}_1)}{C_n N} = \frac{\gamma n^2 (n-2)}{4(n^2-2)N}. \tag{18.7.22}$$

Taking the limit as $r \downarrow r_{\Omega,\gamma}$, we obtain (18.7.20) with the same c. □

18.7.3 Upper Bound

We divide the proof of the upper bound into parts.

1. According to (18.7.1), to get an upper bound for $\Lambda(\Omega)$ it is enough to take any test function $u \in C_0^\infty(\Omega)$ and write

$$\Lambda(\Omega) \le \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\int_\Omega |u|^2 \, \mathrm{d}x}. \tag{18.7.23}$$

For simplicity of notations we shall write Λ instead of $\Lambda(\Omega)$ everywhere in this section. The inequality (18.7.23) can be written as follows:

$$\int_\Omega |u|^2 \, \mathrm{d}x \le \Lambda^{-1} \int_\Omega |\nabla u|^2 \, \mathrm{d}x. \tag{18.7.24}$$

By approximation, it suffices to take $u \in C_0^{0,1}(\Omega)$ or even $u \in \mathring{L}_2^1(\Omega)$.

In particular, choosing a ball B_r, we can take

$$u \in C_0^{0,1}(\Omega \cap B_r) = C_0^{0,1}(\Omega) \cap C_0^{0,1}(B_r). \tag{18.7.25}$$

Let us take a compact set $F \subset \bar{B}_{3r/2}$, such that F is the closure of an open set with a smooth boundary. (In this section we will call such sets *regular* subsets of $\bar{B}_{3r/2}$.) Denote by P_F its capacitary potential (see Sect. 18.2). Regularity of F implies that $P_F \in C^{0,1}(\mathbb{R}^n)$. By definition $P_F = 1$ on F, so $1 - P_F = 0$ on F. Let us also assume that

$$\operatorname{Int} F \supset \bar{B}_r \setminus \Omega,$$

where $\operatorname{Int} F$ means the set of all interior points of F. Then $1 - P_F = 0$ in a neighborhood of $\bar{B}_r \setminus \Omega$. Therefore, multiplying $1 - P_F$ by a cutoff function $\eta \in C_0^\infty(B_r)$, we will get a function $u = \eta(1 - P_F)$, satisfying the requirement (18.7.25), hence fit to be a test function in (18.7.23).

In the future we will also assume that the cutoff function $\eta \in C_0^\infty(B_r)$ has the properties

$$0 \le \eta \le 1 \quad \text{on } B_r, \qquad \eta = 1 \quad \text{on } B_{(1-\kappa)r}, \qquad |\nabla \eta| \le 2(\kappa r)^{-1} \quad \text{on } B_r,$$

where $0 < \kappa < 1$ and the balls B_r and $B_{(1-\kappa)r}$ are supposed to have the same center. Using integration by parts and the equation $\Delta P_F = 0$ on $B_r \setminus F$, we obtain for the test function $u = \eta(1 - P_F)$

$$\begin{aligned}\int_{B_r} |\nabla u|^2 \, \mathrm{d}x &= \int_{B_r} \left(|\nabla \eta|^2 (1 - P_F)^2 - \nabla\left(\eta^2\right) \cdot (1 - P_F) \nabla P_F + \eta^2 |\nabla P_F|^2\right) \mathrm{d}x \\ &= \int_{B_r} |\nabla \eta|^2 (1 - P_F)^2 \, \mathrm{d}x \le 4(\kappa r)^{-2} \int_{B_r} (1 - P_F)^2 \, \mathrm{d}x.\end{aligned}$$

Therefore, from (18.7.24) we obtain

$$\int_{B_r} |u|^2 \, \mathrm{d}x \le \Lambda^{-1} 4(\kappa r)^{-2} \int_{B_r} (1 - P_F)^2 \, \mathrm{d}x.$$

Since $0 \le P_F \le 1$, the last integral in the right-hand side is estimated by

$$m_n(B_r) = n^{-1} \omega_n r^n.$$

Therefore,

$$\int_{B_r} |u|^2 \, dx \le 4n^{-1}\omega_n \Lambda^{-1}\kappa^{-2} r^{n-2}.$$

Restricting the integral in the left-hand side to $B_{(1-\kappa)r}$, we obtain

$$\int_{B_{(1-\kappa)r}} (1-P_F)^2 \, dx \le 4n^{-1}\omega_n \Lambda^{-1}\kappa^{-2} r^{n-2}. \tag{18.7.26}$$

2. Now we need to provide an appropriate lower bound for the left-hand side of (18.7.26). To this end we first restrict the integration to the spherical layer

$$S_{r_1,r_2} = B_{r_2} \setminus B_{r_1},$$

where $0 < r_1 < r_2 < r$. In the sequel we shall take

$$r_1 = (1-2\kappa)r, \qquad r_2 = (1-\kappa)r, \tag{18.7.27}$$

where $0 < \kappa < 1/2$, though it is convenient to write some formulas in a greater generality. Let us denote the volume of the layer S_{r_1,r_2} by $|S_{r_1,r_2}|$, i.e.,

$$|S_{r_1,r_2}| = m_n S_{r_1,r_2} = n^{-1}\omega_n \big(r_2^n - r_1^n\big).$$

We also need the notation

$$⨍_{S_{r_1,r_2}} f(x)\, dx = \frac{1}{|S_{r_1,r_2}|} \int_{S_{r_1,r_2}} f(x)\, dx$$

for the average of a positive function f over S_{r_1,r_2}. In particular, restricting the integration in (18.7.26) to S_{r_1,r_2} (with r_1, r_2 as in (18.7.27)) and dividing by $|S_{r_1,r_2}|$, we obtain

$$⨍_{S_{r_1,r_2}} (1-P_F)^2 \, dx \le \frac{4\Lambda^{-1}\kappa^{-2} r^{n-2}}{r_2^n - r_1^n}.$$

Hence, by the Cauchy–Schwarz inequality,

$$\left[1 - ⨍_{S_{r_1,r_2}} P_F \, dx\right]^2 = \left[⨍_{S_{r_1,r_2}} (1-P_F)\, dx\right]^2 \le \frac{4\Lambda^{-1}\kappa^{-2} r^{n-2}}{r_2^n - r_1^n}. \tag{18.7.28}$$

To simplify the right-hand side, let us estimate $(r_2^n - r_1^n)^{-1}$ from above. We see that

$$\begin{aligned} r_2^n - r_1^n &= (r_2 - r_1)\big(r_2^{n-1} + r_2^{n-2} r_1 + \cdots + r_1^{n-1}\big) \\ &\ge n\kappa r r_1^{n-1} = n\kappa r^n (1-2\kappa)^{n-1} \ge n\kappa r^n \big[1 - 2(n-1)\kappa\big]. \end{aligned}$$

Now note that

$$\frac{1}{1-2(n-1)\kappa} \le 1 + 4(n-1)\kappa,$$

provided

$$0<\kappa\le\frac{1}{4(n-1)}. \tag{18.7.29}$$

Under this condition it follows that

$$\frac{1}{r_2^n-r_1^n}\le n^{-1}\kappa^{-1}r^{-n}\big[1+4(n-1)\kappa\big], \tag{18.7.30}$$

and (18.7.28) takes the form

$$\left[1-\fint_{S_{r_1,r_2}}P_F\,\mathrm{d}x\right]^2\le 4n^{-1}\kappa^{-3}\big[1+4(n-1)\kappa\big]\varLambda^{-1}r^{-2}. \tag{18.7.31}$$

3. For simplicity of notation and without loss of generality we may assume that the ball B_r is centered at $0\in\mathbb{R}^n$ (and so are smaller balls and spherical layers).

To provide a lower bound for the left-hand side of (18.7.31), we will give an upper bound for the average of P_F. According to the definition of P_F and notations from Sect. 18.7.1, we can write

$$\begin{aligned}\fint_{S_{r_1,r_2}}P_F\,\mathrm{d}x&=\fint_{S_{r_1,r_2}}\left(\int_F\mathcal{E}(x-y)\,\mathrm{d}\mu_F(y)\right)\mathrm{d}x\\&=\int_F\left(\fint_{S_{r_1,r_2}}\mathcal{E}(x-y)\,\mathrm{d}x\right)\mathrm{d}\mu_F(y).\end{aligned} \tag{18.7.32}$$

The inner integral on the right-hand side can be explicitly calculated as the potential of a uniformly charged spherical layer with total charge 1. The result of this calculation is $|S_{r_1,r_2}|^{-1}V_{r_1,r_2}(y)$, where

$$V_{r_1,r_2}(y)=\begin{cases}\frac{r_2^2-r_1^2}{2(n-2)}, & \text{if } |y|\le r_1,\\ -\frac{|y|^2}{2n}+\frac{r_2^2}{2(n-2)}-\frac{r_1^n}{n(n-2)|y|^{n-2}}, & \text{if } r_1\le|y|\le r_2,\\ \frac{r_2^n-r_1^n}{n(n-2)|y|^{n-2}}, & \text{if } |y|\ge r_2.\end{cases} \tag{18.7.33}$$

The function $y\mapsto V_{r_1,r_2}(y)$ belongs to $C^1(\mathbb{R}^n)$ and is spherically symmetric; it tends to 0 as $|y|\to\infty$; it is harmonic in $\mathbb{R}^n\setminus S_{r_1,r_2}$ and satisfies the equation $\varDelta V_{r_1,r_2}=-1$ in S_{r_1,r_2}. These properties uniquely define the function V_{r_1,r_2}. Differentiating it with respect to $|y|$, we easily see that it is decreasing with respect to $|y|$, hence its maximum is at $y=0$ (hence given by the first row in (18.7.33)). So we obtain, using (18.7.30),

$$\begin{aligned}\fint_{S_{r_1,r_2}}\mathcal{E}(x-y)\,\mathrm{d}x&\le|S_{r_1,r_2}|^{-1}V_{r_1,r_2}(0)=\frac{n(r_2^2-r_1^2)}{2(n-2)\omega_n(r_2^n-r_1^n)}\\&=\frac{n\kappa r(r_1+r_2)}{2(n-2)\omega_n(r_2^n-r_1^n)}\le\frac{nr^2\kappa(1-\kappa)}{(n-2)\omega_n(r_2^n-r_1^n)}\\&\le\frac{(1-\kappa)[1+4(n-1)\kappa]}{(n-2)\omega_n r^{n-2}}\le\frac{1+(4n-5)\kappa}{(n-2)\omega_n r^{n-2}}.\end{aligned}$$

Finally, using the equality $\operatorname{cap}(B_r) = (n-2)\omega_n r^{n-2}$, we obtain

$$\fint_{S_{r_1,r_2}} \mathcal{E}(x-y)\,\mathrm{d}x \le \frac{1+(4n-5)\kappa}{\operatorname{cap}(\bar{B}_r)}, \tag{18.7.34}$$

provided r_1, r_2 chosen as in (18.7.27) and (18.7.29) is satisfied.

4. Using (18.7.34) in (18.7.32) and taking into account (18.2.5), we obtain

$$\begin{aligned}\fint_{S_{r_1,r_2}} P_F(x)\,\mathrm{d}x &\le \frac{1+(4n-5)\kappa}{\operatorname{cap}(\bar{B}_r)} \int_F \mathrm{d}\mu_F(y)\\ &= \big[1+(4n-5)\kappa\big]\frac{\operatorname{cap}(F)}{\operatorname{cap}(\bar{B}_r)} \le \big[1+(4n-5)\kappa\big]\gamma, \end{aligned}\tag{18.7.35}$$

provided F is γ-negligible. (i.e., satisfies (18.7.2)). Taking into account (18.7.29), we can set

$$\kappa = \min\left\{\frac{1}{4(n-1)}, \frac{1-\gamma}{2(4n-5)\gamma}\right\}, \tag{18.7.36}$$

so that (18.7.29) is satisfied, and besides,

$$\big[1+(4n-5)\kappa\big]\gamma \le \frac{1+\gamma}{2} = 1 - \frac{1-\gamma}{2},$$

so that (18.7.35) becomes

$$\fint_{S_{r_1,r_2}} P_F(x)\,\mathrm{d}x \le 1 - \frac{1-\gamma}{2}.$$

Taking this into account in (18.7.31) and using (18.7.29), we obtain

$$\frac{(1-\gamma)^2}{4} \le 4n^{-1}\kappa^{-3}\big[1+4(n-1)\kappa\big]\Lambda^{-1}r^{-2} \le 8n^{-1}\kappa^{-3}\Lambda^{-1}r^{-2},$$

hence

$$\Lambda \le 32(1-\gamma)^{-2}\kappa^{-3}r^{-2}. \tag{18.7.37}$$

We are now ready to prove Theorem 18.7.1.

The lower bound for Λ was established in Lemma 18.7.2/3. We obtained the estimate (18.7.37) under the condition that there exist $\gamma \in (0,1)$, a ball B_r, and a regular compact set $F \subset \bar{B}_{3r/2}$ (here the balls B_r and $B_{3r/2}$ have the same center), such that F is γ-negligible and its interior includes $\bar{B}_r \setminus \Omega$. (The estimate then holds with $\kappa = \kappa(\gamma, n)$ given by (18.7.36).) It follows, in particular, that $\bar{B}_r \setminus \Omega$ is γ-negligible.

Conversely, if $\bar{B}_r \setminus \Omega$ is γ-negligible, then we can approximate it by regular compact sets F_k, $k = 1, 2, \ldots$, such that $\operatorname{Int} F_k \supset \bar{B}_r \setminus \Omega$, $\operatorname{Int} F_k \supset F_{k+1}$, and $\bar{B}_r \setminus \Omega$ is the intersection of all F_k's. Then

$$\lim_{k\to\infty} \operatorname{cap}(F_k) = \operatorname{cap}(\bar{B}_r \setminus \Omega)$$

by the continuity property of the capacity (see Sect. 2.2.1). In this case, for any $\varepsilon > 0$ the sets F_k will be $(\gamma+\varepsilon)$-negligible for sufficiently large k. It follows that the estimate (18.7.37) will hold if we only know that there exists a ball B_r such that $\bar{B}_r \setminus \Omega$ is γ-negligible. Then the estimate still holds if we replace r by the least upper bound of the radii of such balls, which is exactly the interior capacitary radius $r_{\Omega,\gamma}$. This proves the upper bound in (18.7.3) with

$$C(\gamma, n) = 32(1-\gamma)^{-2}\kappa^{-3}, \tag{18.7.38}$$

where κ is defined by (18.7.36). □

18.7.4 Comments to Chap. 18

Sections 18.1–18.5. The material is borrowed from the article by Maz'ya and Shubin [590], which improves the results of Sect. 18.4 for the particular case $p = 2$, $l = 1$.

Section 18.2. A survey of the necessary and sufficient conditions ensuring various spectral properties of the Schrödinger operator can be found in [563]. M.E. Taylor gave an alternative formulation of the discreteness of spectrum criterion formulated in terms of the so-called scattering length [748].

The necessary and sufficient conditions for the discreteness and strict positivity of magnetic Schrödinger operators with a positive scalar potential

$$-\sum_{j=1}^{n}\left(\frac{\partial}{\partial x^j} + ia_j\right)^2 + V$$

are obtained by Kondratiev, Maz'ya, and Shubin in [450] and [451].

Section 18.6. Lemma 18.6 is due to Glazman [309].

Section 18.7. We follow the article [589] by Maz'ya and Shubin which improves the results of Sect. 15.4 for the particular case $p = 2$, $l = 1$. A variant of Lemma 18.7.2/1 was obtained by Maz'ya [532] as early as in 1963.

Lieb [495] used geometric arguments to establish a lower bound for $\Lambda(\Omega)$ which is similar to (18.7.20), but with capacity replaced by the Lebesgue measure. Such lower bounds can be also deduced from Theorem 18.7.1 if we use isoperimetric inequalities between the capacity and Lebesgue measure

$$m_n F \le A_n \big(\operatorname{cap}(F)\big)^{n/(n-2)}, \tag{18.7.39}$$

with the equality for balls (see, e.g., [666] or Sects. 2.2.3 and 2.2.4 in [556]), so

$$A_n = (m_n B_1)\big[\operatorname{cap}(B_1)\big]^{-n/(n-2)} = n^{-1}(n-2)^{-n/(n-2)}\omega_n^{-2/(n-2)}.$$

Namely, let us write for any $\alpha \in (0, 1)$

$$r^{(m_n)}_{\Omega,\alpha} = \sup\bigl\{r : \exists B_r, m_n(B_r \setminus \Omega) \le \alpha m_n B_r\bigr\}.$$

Then (18.7.39) implies that

$$r^{(m_n)}_{\Omega,\alpha} \ge r_{\Omega,\gamma} \quad \text{provided} \quad \alpha = \gamma^{n/(n-2)}.$$

Therefore, we obtain for every $\alpha \in (0,1)$

$$\Lambda(\Omega) \ge c(\gamma,n)\bigl(r^{(m_n)}_{\Omega,\alpha}\bigr)^{-2}, \quad \text{where } \gamma = \alpha^{(n-2)/n}.$$

Here $c(\gamma,n)$ is given by (18.7.22). This is exactly Lieb's inequality (1.2) in [495], though with a different constant.

There are numerous results that give lower bounds for $\Lambda(\Omega)$. We will mention only a few. The Faber–Krahn inequality ([265, 462, 666]) gives a lower bound of $\Lambda(\Omega)$ in terms of the area of $\Omega \subset \mathbb{R}^2$. Under miscellaneous topological and geometric restrictions on Ω, the interior radius was shown to provide a lower bound (hence a two-sided estimate) for $\Lambda(\Omega)$ in the case $n = 2$ by Hayman [358], Osserman [647–649], M. E. Taylor [746], Croke [217], Bañuelos and Carroll [69], and also in the case $n \ge 3$ ([358, 649]). The following two-sided estimate for $\Lambda(\Omega)$ was established by the author in [531, 534] (see Sects. 2.4 and 4.3 of this book),

$$\frac{1}{4}\inf_F \frac{\operatorname{cap}(F,\Omega)}{m_n(F)} \le \Lambda(\Omega) \le \inf \frac{\operatorname{cap}(F,\Omega)}{m_n(F)},$$

where the infimum is taken over all compact sets $F \subset \Omega$.

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List of Symbols

Function Spaces

Subsets of $\mathbb{R}^n$

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Classes of Sets in $\mathbb{R}^n$

Set Functions

Functionals

Operators

Constants

Functions

Other Symbols

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