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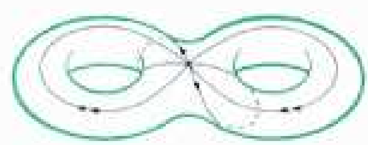

M. NAKAHARA

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Preface

This book is a considerable expansion of lectures I gave at the School of Mathematical and Physical Sciences, University of Sussex during the winter term of 1986. The audience included postgraduate students and faculty members working in particle physics, condensed matter physics and general relativity. The lectures were quite informal and I have tried to keep this informality as much as possible in this book. The proof of a theorem is given only when it is instructive and not very technical; otherwise examples will make the theorem plausible. Many figures will help the reader to obtain concrete images of the subjects.

In spite of the extensive use of the concepts of topology, differential geometry and other areas of contemporary mathematics in recent developments in theoretical physics, it is rather difficult to find a self-contained book that is easily accessible to postgraduate students in physics. This book is meant to fill the gap between highly advanced books or research papers and the many excellent introductory books. As a reader, I imagined a first-year postgraduate student in theoretical physics who has some familiarity with quantum field theory and relativity. In this book, the reader will find many examples from physics, in which topological and geometrical notions are very important. These examples are eclectic collections from particle physics, general relativity and condensed matter physics. Readers should feel free to skip examples that are out of their direct concern. However, I believe these examples should be the theoretical minima to students in theoretical physics. Mathematicians who are interested in the application of their discipline to theoretical physics will also find this book interesting.

The book is largely divided into four parts. Chapters 1 and 2 deal with the preliminary concepts in physics and mathematics respectively. In Chapter 1, a brief summary of the physics treated in this book is given. The subjects covered are path integrals, gauge theories (including monopoles and instantons), defects in condensed matter physics, general relativity, Berry's phase in quantum mechanics and strings. Most of the subjects are subsequently explained in detail from the topological and geometrical viewpoints. Chapter 2 supplements the undergraduate mathematics that the average physicist has studied. If readers are quite familiar with sets, maps and general topology, they may skip this chapter and proceed to the next.

Chapters 3 to 8 are devoted to the basics of algebraic topology and differential geometry. In Chapters 3 and 4, the idea of the classification of spaces with homology groups and homotopy groups is introduced. In Chapter 5, we define a manifold, which is one of the central concepts in modern theoretical physics. Differential forms defined there play very important roles throughout this book. Differential forms allow us to define the dual of the homology group called the de Rham cohomology group in Chapter 6. Chapter 7 deals with a manifold endowed with a metric. With the metric, we may define such geometrical concepts as connection, covariant derivative, curvature, torsion and many more. In Chapter 8. a complex manifold is defined as a special manifold on which there exists a natural complex structure.

Chapters 9 to 12 are devoted to the unification of topology and geometry. In Chapter 9, we define a fibre bundle and show that this is a natural setting for many physical phenomena. The connection defined in Chapter 7 is naturally generalised to that on fibre bundles in Chapter 10. Characteristic classes defined in Chapter 11 enable us to classify fibre bundles using various cohomology classes. Characteristic classes are particularly important in the Atiyah-Singer index theorem in Chapter 12. We do not prove this, one of the most important theorems in contemporary mathematics, but simply write down the special forms of the theorem so that we may use them in practical applications in physics.

Chapters 13 and 14 are devoted to the most fascinating applications of topology and geometry in contemporary physics. In Chapter 13, we apply the theory of fibre bundles, characteristic classes and index theorems to the study of anomalies in gauge theories. In Chapter 14, Polyakov's bosonic string theory is analysed from the geometrical point of view. We give an explicit computation of the one-loop amplitude.

I would like to express deep gratitude to my teachers, friends and students. Special thanks are due to Tetsuya Asai, David Bailin, Hiroshi Khono, David Lancaster, Sigeki Matsutani, Hiroyuki Nagashima, David Pattarini, Felix E A Pirani, Kenichi Tamano, David Waxman and David Wong. The basic concepts in Chapter 5 owe very much to the lectures by F E A Pirani at King's College, University of London. The evaluation of the string Laplacian in Chapter 14 using the Eisenstein series and the Krönecker limiting formula was suggested by T Asai. I would like to thank Euan Squires, David Bailin and Hiroshi Khono for useful comments and suggestions. David Bailin suggested that I should write this book. He also advised Professor Douglas F Brewer to include this book is his series. I would like to thank the Science and Engineering Research Council of the United Kingdom, which made my stay at Sussex possible. It is a pity that I have no secretary to thank for the beautiful typing. Word processing has been carried out by myself on two NEC PC9801 computers. Jim A Revill of Adam Hilger helped me in many ways while preparing the manuscript. His indulgence over my failure to meet deadlines is also acknowledged. Many musicians have filled my office with beautiful music during the preparation of the manuscript: I am grateful to J S Bach, Ryuichi Sakamoto, Ravi Shankar and Erik Satie. Finally I am greatly indebted to my wife Yoko, to whom this book is dedicated, for her encouragement and moral support.

MIKIO NAKAHARA
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BACKGROUND IN PHYSICS

We assume that the reader is familiar with elementary quantum field theory and elementary relativity. In the present chapter, we outline the physics which we shall be concerned with in this book. This chapter is intended to establish notations and conventions and also to give enough background of selected topics with which many students may not be very familiar. Most of the topics are subsequently analysed in detail from topological and geometrical viewpoints.

We put c (the speed of light) $= \hbar$ (Planck's constant$/2\pi$) $= k$ (Boltzmann's constant) $= 1$, unless written explicitly. We employ the **Einstein summation convention**: if the same index appears twice, once as a superscript and once as a subscript, then the index is summed over all possible values. For example, if μ runs from 1 to m, we have

$$A^{\mu}B_{\mu} = \sum_{\mu=1}^{m} A^{\mu}B_{\mu}.$$

The Euclidean metric is $g_{\mu\nu} = \delta_{\mu\nu} = \mathrm{diag}(+1, \ldots, +1)$ while the Minkowski metric is $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers, integers, real numbers and complex numbers, respectively. $\mathbb{H}$ denotes the set of quaternions. Let $(1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ be a basis such that $\boldsymbol{i}\cdot\boldsymbol{j} = -\boldsymbol{j}\cdot\boldsymbol{i} = \boldsymbol{k}$, $\boldsymbol{j}\cdot\boldsymbol{k} = -\boldsymbol{k}\cdot\boldsymbol{j} = \boldsymbol{i}$, $\boldsymbol{k}\cdot\boldsymbol{i} = -\boldsymbol{i}\cdot\boldsymbol{k} = \boldsymbol{j}$, $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$. Then

$$\mathbb{H} = \{a + \boldsymbol{i}b + \boldsymbol{j}c + \boldsymbol{k}d \,|\, a, b, c, d \in \mathbb{R}\}.$$

Note that $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ have the 2×2 matrix representations, $\boldsymbol{i} = \mathrm{i}\sigma_3$, $\boldsymbol{j} = \mathrm{i}\sigma_2$, $\boldsymbol{k} = \mathrm{i}\sigma_1$, where the σ_i are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The symbol ■ denotes the end of a proof.

1.1 Path integral and quantum field theories

Quantum field theory (QFT) has achieved great success in particle physics as well as in condensed matter physics. We cannot find any evidence against QFT when applied to metals, superconductors, superfluids, quantum electrodynamics (QED), quantum chromodynamics

(QCD), electroweak theory and grand unified theories (GUTS). So far we have not established a QFT for gravity. Superstring theory seems to be a good candidate for the *Theory of Everything* (TOE), including gravity. Although superstring theory deals with one-dimensional objects rather than particles, the basic tool to describe it is QFT. We start our exposition with a short review of the standard QFT in the path integral formalism. Relevant references are Bailin and Love (1986), Cheng and Li (1984) and Ramond (1981). Huang (1982) and Ryder (1985) contain a good introduction to topological methods in QFT. Federbush (1987) is a survey of QFT written by a mathematician.

1.1.1 Path integral formulation of quantum mechanics

Let $\hat{q}$ be a position operator in the Schrödinger picture and let $|q\rangle$ be its eigenvector with eigenvalue q:

$$\hat{q}|q\rangle = q|q\rangle. \tag{1.1}$$

$\hat{q}$ is independent of time and so is the eigenvector $|q\rangle$. A state $|\psi(t)\rangle$ satisfies the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{1.2}$$

whose formal solution is $|\psi(t)\rangle = \exp(-\mathrm{i}Ht)|\psi(0)\rangle$. If the coordinate is diagonalised, the state is represented as

$$\psi(q, t) = \langle q|\psi(t)\rangle. \tag{1.3}$$

Let $\hat{q}(t)$ be a position operator in the Heisenberg picture and let $|q, t\rangle$ be an *instantaneous* eigenvector of $\hat{q}(t)$:

$$\hat{q}(t)|q, t\rangle = q|q, t\rangle. \tag{1.4}$$

Since $\hat{q}(t)$ depends on time, $|q, t\rangle$ may not be an eigenvector of $\hat{q}(t')$ for $t' \neq t$. The dynamics of $\hat{q}(t)$ is dictated by the Heisenberg equation of motion, with the formal solution

$$\hat{q}(t) = \mathrm{e}^{\mathrm{i}Ht}\hat{q}\mathrm{e}^{-\mathrm{i}Ht} \tag{1.5}$$

from which we find

$$|q, t\rangle = \mathrm{e}^{\mathrm{i}Ht}|q\rangle. \tag{1.6}$$

The wavefunction is $\psi(q, t) = \langle q, t|\psi\rangle$.

Let us consider a process in which a particle starting at q at time t is found at q' at later time t'. By the fundamental assumption of quantum mechanics, the probability amplitude associated with this process is

$$\langle q', t'|q, t\rangle = \langle q'|\mathrm{e}^{-\mathrm{i}H(t'-t)}|q\rangle. \tag{1.7}$$

We show that this amplitude is evaluated by summing over all possible paths that connect (q, t) and (q', t'). Inserting the identity

$$\int \mathrm{d}q |q, t\rangle\langle q, t| = 1$$

into (1.7), we have

$$\langle q', t'|q, t\rangle = \int \mathrm{d}q_1 \ldots \mathrm{d}q_n \langle q', t'|q_n, t_n\rangle \ldots \langle q_2, t_2|q_1, t_1\rangle\langle q_1, t_1|q, t\rangle \tag{1.8}$$

where we have divided the interval $t' - t$ into $n + 1$ pieces,

$$t_{i+1} - t_i = \varepsilon \qquad t_0 = t \qquad t_{n+1} = t'. \tag{1.9}$$

Each inner product is

$$\langle q_{i+1}, t_{i+1}|q_i, t_i\rangle = \langle q_{i+1}|\mathrm{e}^{-\mathrm{i}H\varepsilon}|q_i\rangle \simeq \langle q_{i+1}|q_i\rangle - \mathrm{i}\varepsilon\langle q_{i+1}|H|q_i\rangle. \tag{1.10}$$

Suppose the Hamiltonian is of the form

$$H = \hat{p}^2/2m + V(\hat{q}). \tag{1.11}$$

Noting that $\langle q_{i+1}|q_i\rangle = \delta(q_{i+1} - q_i)$ and $\langle q_i|p_i\rangle = \mathrm{e}^{\mathrm{i}pq}$, we find

$$\begin{aligned}\langle q_{i+1}|H(\hat{p}, \hat{q})|q_i\rangle &\simeq \langle q_{i+1}|\hat{p}^2/2m|q_i\rangle + V\left(\frac{q_i + q_{i+1}}{2}\right)\delta(q_{i+1} - q_i)\\ &= \int \frac{\mathrm{d}p}{2\pi} H\left(p, \frac{q_i + q_{i+1}}{2}\right)\mathrm{e}^{\mathrm{i}(q_{i+1}-q_i)p}\end{aligned} \tag{1.12}$$

where use has been made of the completeness

$$\int \frac{\mathrm{d}p}{2\pi} |p, t\rangle\langle p, t| = 1.$$

Substituting this result into (1.10), we have

$$\begin{aligned}\langle q_{i+1}, t_{i+1}|q_i, t_i\rangle &\simeq \int \frac{\mathrm{d}p}{2\pi}\left[1 - \mathrm{i}\varepsilon H\left(p, \frac{q_i + q_{i+1}}{2}\right)\right]\mathrm{e}^{\mathrm{i}(q_{i+1}-q_i)p}\\ &\simeq \int \frac{\mathrm{d}p}{2\pi}\mathrm{e}^{\mathrm{i}p(q_{i+1}-q_i)}\exp\left[-\mathrm{i}\varepsilon H\left(p, \frac{q_i + q_{i+1}}{2}\right)\right].\end{aligned} \tag{1.13}$$

This becomes exact when $\varepsilon \to 0$, that is when $n \to \infty$. The amplitude (1.8) is now given by

$$\begin{aligned}\langle q', t'|q, t\rangle \simeq \lim_{n\to\infty} &\int \frac{\mathrm{d}p_0}{2\pi} \ldots \frac{\mathrm{d}p_n}{2\pi} \int \mathrm{d}q_1 \ldots \mathrm{d}q_n\\ &\times \exp\left\{\mathrm{i}\varepsilon \sum_{i=0}^{n}\left[p_i \frac{q_{i+1} - q_i}{\varepsilon} - H\left(p_i, \frac{q_i + q_{i+1}}{2}\right)\right]\right\}.\end{aligned} \tag{1.14}$$

This is symbolically written as

$$\int \mathcal{D}p\,\mathcal{D}q \exp\left(\mathrm{i}\int_t^{t'} \mathrm{d}t\,[p\dot{q} - H(p, q)]\right) \tag{1.15}$$

which is called the **path integral** for the transition amplitude. It is clear from this construction that we have summed over all paths satisfying the boundary condition.

Example 1.1 Consider a free particle with $H = \hat{p}^2/2m$. (1.14) is

$$\langle q', t'|q, t\rangle = \lim_{n\to\infty} \int \prod_{i=0}^{n} \frac{\mathrm{d}p_i}{2\pi} \int \prod_{i=1}^{n} \mathrm{d}q_i \exp\left[\mathrm{i}\sum_{i=0}^{n}\left(p_i(q_{i+1} - q_i) - \varepsilon\frac{p_i^2}{2m}\right)\right].$$

To integrate over q, we rewrite the exponent as

$$-p_0 q + p_n q' + \sum_{i=1}^{n} q_i(p_{i-1} - p_i) - \varepsilon\sum_{i=1}^{n}\frac{p_i^2}{2m}.$$

q-integrations yield an infinite product of δ-functions

$$(2\pi)^n \prod_{i=1}^{n} \delta(p_i - p_{i-1})$$

which states that the momentum is conserved at each stage of the evolution. The amplitude becomes

$$\langle q', t'|q, t\rangle = \int \frac{\mathrm{d}p_0}{2\pi} \exp\left[\mathrm{i}\left(p_0(q' - q) - \frac{p_0^2}{2m}(t' - t)\right)\right]. \tag{1.16}$$

To evaluate this amplitude, we note the formula

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \exp(-ap^2 - bp) = \frac{1}{(4\pi a)^{1/2}} \exp\left(\frac{b^2}{4a}\right) \qquad (a > 0). \tag{1.17}$$

We uncritically think that $\mathrm{i}(t' - t)/2m$ is a positive real number. (We may introduce a new variable $\tau \equiv \mathrm{i}t$ (**Wick rotation**) so that $(\tau' - \tau)/2m > 0$.) We finally have

$$\langle q', t'|q, t\rangle = \left(\frac{m}{2\pi\mathrm{i}(t' - t)}\right)^{1/2} \exp\left(\frac{\mathrm{i}m(q' - q)^2}{2(t' - t)}\right). \tag{1.18}$$

When the kinetic energy is of the form (1.11), we may execute p-integrations in (1.14). If we Wick rotate the time so that $\tau = \mathrm{i}t$ and replace $\mathrm{i}\varepsilon$ by ε, we have

$$\int \frac{\mathrm{d}p_i}{2\pi} \exp\left(\mathrm{i}p_i(q_{i+1} - q_i) - \varepsilon\frac{p_i^2}{2m}\right) = \left(\frac{m}{2\pi\varepsilon}\right)^{1/2} \exp\left(\frac{m(q_{i+1} - q_i)^2}{-2\varepsilon}\right)$$

where use has been made of (1.17). Now the amplitude is given in terms of q-integrations only,

$$\langle q', \tau'|q, \tau\rangle = \lim_{n\to\infty}\left(\frac{m}{2\pi\varepsilon}\right)^{(n+1)/2}\int\prod_{i=1}^{n}\mathrm{d}q_i$$
$$\times\exp\left\{\varepsilon\sum_{i=0}^{n}\left[-\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\varepsilon}\right)^2 - V\left(\frac{q_i+q_{i+1}}{2}\right)\right]\right\}$$
$$\equiv \mathcal{N}\int\mathcal{D}q\exp\left\{-\int_{\tau}^{\tau'}\mathrm{d}\tau\left[\left(\frac{m}{2}\right)\dot{q}^2 - V(q)\right]\right\} \tag{1.19}$$

where $\mathcal{N}$ is the normalisation factor. The exponent in (1.19) is the action $S[q(t), \dot{q}(t)]$.

The time-ordered correlation function has a path integral representation given by

$$\langle q', \tau'|T[\hat{q}(\tau_1)\hat{q}(\tau_2)]|q, \tau\rangle$$
$$= \int\mathcal{D}q\mathcal{D}p\, q(\tau_1)q(\tau_2)\exp\left(-\int_{\tau}^{\tau'}\mathrm{d}\tau[p\dot{q} - H(p, q)]\right). \tag{1.20}$$

To obtain general n-point functions, it is convenient to couple an external source $J(\tau)$ to $\hat{q}(\tau)$ and define the transition amplitude in the presence of $J(\tau)$ by

$$\langle q', \tau'|q, \tau\rangle^J \equiv \int\mathcal{D}p\mathcal{D}q\exp\left(-\int_{\tau}^{\tau'}\mathrm{d}t(p\dot{q} - H + Jq)\right). \tag{1.21}$$

We assume $\lim_{\tau\to\pm\infty}J(\tau) = 0$. By functional differentiations, we obtain the n-point function

$$\langle q', \tau'|T[\hat{q}(\tau_1)\ldots\hat{q}(\tau_n)]|q, \tau\rangle$$
$$= (-1)^n\frac{\delta^n}{\delta J(\tau_1)\ldots\delta J(\tau_n)}\langle q', \tau'|q, \tau\rangle^J\bigg|_{J=0} \tag{1.22}$$

where $\delta/\delta J(\tau)$ is defined by

$$\frac{\delta}{\delta J(\tau)}\int J(\tau')q(\tau')\,\mathrm{d}\tau' \equiv \int\delta(\tau'-\tau)q(\tau')\,\mathrm{d}\tau' = q(\tau).$$

Exercise 1.2 Prove (1.20) separately for $\tau_1 > \tau_2$ and $\tau_1 < \tau_2$. Verify that (1.22) reduces to (1.20) when $n = 2$.

Among the correlation functions, we are particularly interested in the **vacuum-to-vacuum amplitude**. Let $|0\rangle$ be the ground state: $H|0\rangle = E_0|0\rangle$. The wavefunction is

$$\psi_0(x, \tau) = \langle x|0, \tau\rangle = \langle x, \tau|0\rangle = \mathrm{e}^{-E_0\tau}\langle x|0\rangle.$$

The vacuum-to-vacuum amplitude is

$$\begin{aligned}\langle 0|T[\hat{q}(\tau_1)\ldots\hat{q}(\tau_n)]|0\rangle &= \langle 0|\int \mathrm{d}q\,\mathrm{d}q'|q',\tau'\rangle\langle q',\tau'|T[\hat{q}(\tau_1)\ldots\hat{q}(\tau_n)]|q,\tau\rangle\langle q,\tau|0\rangle \\ &= \int \mathrm{d}q\,\mathrm{d}q'\,\psi_0^*(q',\tau')\langle q',\tau'|T[\hat{q}(\tau_1)\ldots\hat{q}(\tau_n)]|q,\tau\rangle\psi_0(q,\tau).\end{aligned} \tag{1.23}$$

Define the **generating functional**

$$Z[J] \equiv \langle 0|0\rangle^J = \int \mathrm{d}q\,\mathrm{d}q'\,\psi_0^*(q',\tau')\langle q',\tau'|q,\tau\rangle^J\psi_0(q,\tau). \tag{1.24}$$

The n-point function (1.22) is given by

$$\langle 0|T[\hat{q}(\tau_1)\ldots\hat{q}(\tau_n)]|0\rangle = (-1)^n \frac{\delta^n}{\delta J(\tau_1)\ldots\delta J(\tau_n)} Z[J]\bigg|_{J=0}. \tag{1.25}$$

If we note that E_0 is the lowest eigenvalue, we find

$$\lim_{\tau_-\to-\infty} \mathrm{e}^{-E_0\tau_-}\langle q,\tau|q_1,\tau_-\rangle = \psi_0(q,\tau)\psi_0^*(q_1,0)$$

$$\lim_{\tau_+\to\infty} \mathrm{e}^{E_0\tau_+}\langle q_2,\tau_+|q',\tau'\rangle = \psi_0^*(q',\tau')\psi_0(q_2,0).$$

Inserting them into (1.24) we have

$$Z[J] = \lim_{\tau_\pm\to\pm\infty} \frac{\exp[E_0(\tau'-\tau)]}{\psi_0^*(q_1,0)\psi_0(q_2,0)}\langle q_2,\tau'|q_1,\tau\rangle^J. \tag{1.26}$$

Since q_1 and q_2 are arbitrary, we may write

$$Z[J] \propto \int \mathcal{D}p\,\mathcal{D}q \exp\left(-\int_{-\infty}^{\infty}\mathrm{d}\tau(p\dot{q} - H + Jq)\right). \tag{1.27}$$

In the case when the Hamiltonian is of the form (1.11), we have

$$Z[J] \propto \int \mathcal{D}q \exp\left(-\int_{-\infty}^{\infty}\mathrm{d}\tau(L + Jq)\right). \tag{1.28}$$

1.1.2 Scalar field theory

Let $\phi(x)$ be a real scalar field satisfying

$$\left(\Box + m^2 + \frac{\partial V(\phi)}{\partial\phi}\right)\phi = 0. \tag{1.29}$$

The Lagrangian density yielding this equation is

$$\mathcal{L} \equiv \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - V(\phi). \tag{1.30}$$

If the time is Wick rotated, the metric becomes Euclidean. The field configurations at 'time' τ and τ' are specified by

$$\phi(\boldsymbol{x}, \tau) = \varphi(\boldsymbol{x}) \qquad \phi(\boldsymbol{x}, \tau') = \varphi'(\boldsymbol{x}). \tag{1.31}$$

The path integral expression for the transition amplitude is

$$\langle \varphi'(\boldsymbol{x}), \tau' | \varphi(\boldsymbol{x}), \tau \rangle = \int \mathcal{D}\phi \mathcal{D}\pi \exp\left(-\int_\tau^{\tau'} \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x} \, [\pi\dot{\phi} - \mathcal{H}(\pi, \phi)] \right) \tag{1.32}$$

where $\mathcal{H}$ is the Hamiltonian density; $\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$.

The vacuum-to-vacuum amplitude in the presence of a source $J(x)$ is

$$Z[J] \equiv \int \mathcal{D}\phi \mathcal{D}\pi \exp\left(-\int_{-\infty}^{\infty} \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x} \, [\pi\dot{\phi} - \mathcal{H}(\pi, \phi) + J\phi] \right). \tag{1.33}$$

The integration is over all π and all ϕ that satisfy the boundary condition (1.31). For the Lagrangian density (1.30), it is easy to perform the π-integration in (1.33) to yield

$$Z[J] \propto \int \mathcal{D}\phi \exp\left(-\int_{-\infty}^{\infty} \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x} \, [\mathcal{L}(\phi, \partial_\mu\phi) + J\phi] \right). \tag{1.34}$$

It can be shown that $Z[J]$ generates the n-point functions. We are particularly interested in the *connected* n-point functions, which are generated by $W[J]$ defined by

$$Z[J] = \mathrm{e}^{-W[J]}. \tag{1.35}$$

From $W[J]$, we define the **effective action** $\Gamma[\phi_{\mathrm{cl}}]$ by

$$\Gamma[\phi_{\mathrm{cl}}] \equiv W[J] - \int \mathrm{d}t \, \mathrm{d}\boldsymbol{x} \, J\phi_{\mathrm{cl}} \tag{1.36}$$

where

$$\phi_{\mathrm{cl}} \equiv \langle \phi \rangle^J = \delta W[J]/\delta J. \tag{1.37}$$

Γ is a Legendre transform of W. $\Gamma[\phi_{\mathrm{cl}}]$ generates *one-particle irreducible* diagrams.

1.1.3 ζ-function evaluation of amplitude

Let A be a symmetric positive-definite $n \times n$ matrix. The Gaussian integral is given by

$$\int \mathrm{d}x_1 \ldots \mathrm{d}x_n \exp\left(-\sum_{i,j} x_i A_{ij} x_j \right) = \pi^{n/2} (\det A)^{-1/2}. \tag{1.38}$$

Since A is symmetric, it can be diagonalised by an element of $\mathrm{O}(n)$. If $\{\lambda_i\}$ denotes the set of eigenvalues of A, we have

$$\det A = \prod_{i=1}^{n} \lambda_i. \tag{1.39}$$

We apply these formulae to evaluate the effective action.

Consider a scalar field ϕ. The vacuum-to-vacuum amplitude in the absence of the source J is

$$Z \equiv \langle 0|0\rangle = \int \mathcal{D}\phi \exp\left[-\frac{1}{2}\int \mathrm{d}x\phi\left(-\partial_\mu\partial^\mu + m + \frac{\partial V}{\partial\phi}\right)\phi\right] \tag{1.40}$$

where the time is Wick rotated as before. Let λ_i be an eigenvalue of the operator $A = -\partial_\mu\partial^\mu + m^2 + \partial V/\partial\phi$ with the eigenfunction ψ_i. We take $\{\psi_i\}$ to be orthonormal $\int \mathrm{d}x\psi_i\psi_j = \delta_{ij}$. We make $\{\lambda_i\}$ discrete by imposing a periodic boundary condition, for example. Since any function ϕ satisfying the boundary condition is expanded in ψ_i as $\phi = \Sigma\, a^i\psi_i$, we find

$$\begin{aligned} Z &= \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int \mathrm{d}x\phi A\phi\right) = \int \prod d^i a \exp\left(-\frac{1}{2}\sum a^i a^j \int \mathrm{d}x\psi_i\lambda_j\psi_j\right) \\ &= \int \prod d^i a \exp\left(-\frac{1}{2}\sum \lambda_i (a^i)^2\right) = \left(\prod_{i\geq 1}\lambda_i\right)^{-1/2} = (\det A)^{-1/2} \end{aligned} \tag{1.41}$$

where we have dropped an irrelevant factor. Clearly (1.41) is ill-defined since λ_i is not bounded. Define the **generalised ζ-function** $\zeta_A(s)$ by

$$\zeta_A(s) \equiv \sum_{i\geq 1}\lambda_i^{-s} \tag{1.42}$$

where the summation extends over all the eigenvalues λ_i of A. For a well behaved spectrum, (1.42) may be analytically continued to the complex s-plane and is finite except at possible poles (Hawking 1977). We easily verify that

$$\prod_{i\geq 1}\lambda_i = \exp\left(\sum_{i\geq 1}\ln\lambda_i\right) = \exp\left(-\frac{\mathrm{d}}{\mathrm{d}s}\zeta_A(s)\bigg|_{s=0}\right) \tag{1.43}$$

from which we have

$$Z \propto \exp[\tfrac{1}{2}\zeta'_A(0)]. \tag{1.44}$$

If ϕ is a *complex* scalar field, we have to double the number of degrees of freedom. The Lagrangian density is

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + m^2\phi^*\phi + V(\phi^*, \phi). \tag{1.45}$$

The vacuum-to-vacuum amplitude becomes

$$Z \propto (\det A)^{-1} \propto \exp[\zeta'_A(0)]. \tag{1.46}$$

1.1.4 Dirac fields

The path integral for a Dirac field is given by integrating over anticommuting variables (**Grassmann variables**). In an n-dimensional space of real Grassmann variables we have $\{c_i, c_j\} = 0$. Integrals including real Grassmann variables are defined by

$$\int \mathrm{d}c_i = 0 \qquad \int \mathrm{d}c_i\, c_j = \delta_{ij}.$$

The derivative must satisfy

$$\left\{\frac{\partial}{\partial c_i}, c_j\right\} = \delta_{ij} \qquad \left\{\frac{\partial}{\partial c_i}, \frac{\partial}{\partial c_j}\right\} = 0.$$

For an antisymmetric matrix A, the anticommutativity of c_i yields

$$\int \mathrm{d}c_1 \ldots \mathrm{d}c_n \exp\left(-\sum_{i,j} c_i A_{ij} c_j\right) \propto \begin{cases} 0 & n \text{ odd} \\ (\det A)^{1/2} & n \text{ even.} \end{cases} \tag{1.47}$$

This can be seen by expanding the exponential and keeping terms which contain n different c_i. For example, if $n = 2$, we find

$$\int \mathrm{d}c_1\, \mathrm{d}c_2 (1 - 2A_{12}c_1c_2) = +2A_{12} \propto (\det A)^{1/2}.$$

If $\zeta_i (1 \leqslant i \leqslant n)$ are n complex Grassmann variables, the Gaussian integral is

$$\int \mathrm{d}\zeta_1 \mathrm{d}\bar{\zeta}_1 \ldots \mathrm{d}\zeta_n \mathrm{d}\bar{\zeta}_n \exp\left(-\sum_{i,j} \bar{\zeta}_i A_{ij} \zeta_j\right) = \det A \tag{1.48}$$

where the bars denote complex conjugation.

We apply these results to the Dirac field ψ. The Lagrangian for the Dirac field is

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu + m)\psi \tag{1.49}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$. (To avoid confusion, the complex conjugate of a Dirac field is denoted by ψ^*.) Variation with respect to $\bar{\psi}$ yields the Dirac equation

$$(\mathrm{i}\gamma^\mu \partial_\mu + m)\psi = 0. \tag{1.50}$$

The generating functional $Z[\eta, \bar{\eta}]$ is defined by

$$Z[\eta, \bar{\eta}] \propto \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-\int \mathrm{d}x\, \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu + m)\psi + \bar{\psi}\eta + \bar{\eta}\psi\right) \tag{1.51}$$

where $\eta(\bar{\eta})$ is the source for $\bar{\psi}(\psi)$. The generating functional of the connected n-point functions W is defined by

$$Z[\eta, \bar{\eta}] = \exp(-W[\eta, \bar{\eta}]). \tag{1.52}$$

As in the case of a scalar field, the vacuum-to-vacuum amplitude in the absence of the source is

$$Z \propto \det S \propto \exp[-\zeta_S'(0)] \tag{1.53}$$

where $S \equiv i\gamma^{\mu}\partial_{\mu} + m$. [*Remark:* We have treated the Dirac field as a complex Grassmann variable. The *real* counterpart of this is the Majorana field, which satisfies the charge conjugation symmetry and hence halves the number of degrees of freedom.]

1.2 Gauge theories

At present, physically sensible theories of fundamental interactions are based on gauge theories. The gauge principle—*physics should not depend on how we describe it*—is in harmony with the principle of general relativity. Here we give a brief summary of classical aspects of gauge theories. For further references, the reader should consult those books listed in the previous section.

1.2.1 Abelian gauge theories

The reader should be familiar with Maxwell's equations

$$\operatorname{div} \boldsymbol{B} = 0 \tag{1.54a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{curl} \boldsymbol{E} = 0 \tag{1.54b}$$

$$\operatorname{div} \boldsymbol{E} = \rho \tag{1.54c}$$

$$\frac{\partial \boldsymbol{E}}{\partial t} - \operatorname{curl} \boldsymbol{B} = -\boldsymbol{j}. \tag{1.54d}$$

The magnetic field $\boldsymbol{B}$ and the electric field $\boldsymbol{E}$ are expressed in terms of the **vector potential** $A_{\mu} = (\phi, \boldsymbol{A})$ as

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A} \qquad \boldsymbol{E} = \frac{\partial \boldsymbol{A}}{\partial t} - \operatorname{grad} \phi. \tag{1.55}$$

Maxwell's equations are invariant under the **gauge transformation**

$$A_{\mu} \to A_{\mu} + \partial_{\mu}\chi \tag{1.56}$$

where χ is a scalar function. This invariance is manifest if we define the **electromagnetic field tensor** $F_{\mu\nu}$ by

$$F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{1.57}$$

From the construction, F is invariant under (1.56). The Lagrangian of the electromagnetic fields is given by

$$\mathcal{L}_{\rm EM} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu j^\mu \tag{1.58}$$

where $j^\mu = (\rho, \boldsymbol{j})$.

Exercise 1.3 Show that (1.54*a*) and (1.54*b*) are written as

$$\partial_\xi F_{\mu\nu} + \partial_\mu F_{\nu\xi} + \partial_\nu F_{\xi\mu} = 0 \tag{1.59a}$$

while (1.54*c*) and (1.54*d*) are

$$\partial_\nu F^{\mu\nu} = j^\mu \tag{1.59b}$$

where raising and lowering of spacetime indices are carried out with the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Verify that (1.59*b*) is the Euler–Lagrange equation derived from (1.58).

Let ψ be a Dirac field with electric charge e. The free Dirac Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu + m)\psi \tag{1.60}$$

is clearly invariant under the *global* gauge transformation

$$\psi \to \mathrm{e}^{-\mathrm{i}e\alpha}\psi \qquad \bar{\psi} \to \bar{\psi}\mathrm{e}^{\mathrm{i}e\alpha} \tag{1.61}$$

where $\alpha \in \mathbb{R}$ is a constant. We elevate this symmetry to invariance under the *local* gauge transformation,

$$\psi \to \mathrm{e}^{-\mathrm{i}e\alpha(x)}\psi \qquad \bar{\psi} \to \bar{\psi}\mathrm{e}^{\mathrm{i}e\alpha(x)}. \tag{1.62}$$

The Lagrangian transforms under (1.62) as

$$\bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu + m)\psi \to \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu + e\gamma^\mu\partial_\mu\alpha + m)\psi.$$

Since the extra term $e\partial_\mu\alpha$ looks like a gauge transformation of the vector potential, we couple the gauge field A_μ with ψ so that the Lagrangian has a local gauge symmetry. We find

$$\mathcal{L} = \bar{\psi}[\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu) + m]\psi \tag{1.63}$$

is invariant under the combined gauge transformation,

$$\begin{gathered}\psi \to \psi' = \mathrm{e}^{-\mathrm{i}e\alpha(x)}\psi \\ \bar{\psi} \to \bar{\psi}' = \bar{\psi}\mathrm{e}^{\mathrm{i}e\alpha(x)} \qquad A_\mu \to A'_\mu = A_\mu - \partial_\mu\alpha(x).\end{gathered} \tag{1.64}$$

Let us introduce the **covariant derivatives**,

$$\nabla_\mu \equiv \partial_\mu - \mathrm{i}eA_\mu \qquad \nabla'_\mu \equiv \partial_\mu - \mathrm{i}eA'_\mu. \tag{1.65}$$

The reader should verify that $\nabla_\mu\psi$ transforms in a nice way,

$$\nabla'_\mu\psi' = \mathrm{e}^{-\mathrm{i}e\alpha(x)}\nabla_\mu\psi. \tag{1.66}$$

The total QED Lagrangian is

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(\mathrm{i}\gamma^{\mu}\nabla_{\mu} + m)\psi. \tag{1.67}$$

Exercise 1.4 Let $\phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$ be a complex scalar field with electric charge e. Show that the Lagrangian

$$\mathcal{L} = \eta^{\mu\nu}(\nabla_{\mu}\phi)^{\dagger}(\nabla_{\nu}\phi) + m^2\phi^{\dagger}\phi \tag{1.68}$$

is invariant under the gauge transformation

$$\phi \to \mathrm{e}^{-\mathrm{i}e\alpha(x)}\phi \qquad \phi^{\dagger} \to \phi^{\dagger}\mathrm{e}^{\mathrm{i}e\alpha(x)} \qquad A_{\mu} \to A_{\mu} - \partial_{\mu}\alpha(x). \tag{1.69}$$

1.2.2 Non-Abelian gauge theories

The gauge transformation above is a member of a U(1) group, that is a complex number of modulus 1, which happens to be an *Abelian* group. A few decades ago, Yang and Mills (1954) introduced non-Abelian gauge transformations. At that time, non-Abelian gauge theories were studied from curiosity. Nowadays, they play a central role in elementary particle physics.

Let G be a compact semi-simple Lie group such as SO(N) or SU(N). The anti-Hermitian generators $\{T_{\alpha}\}$ satisfy the commutation relations

$$[T_{\alpha}, T_{\beta}] = f_{\alpha\beta}{}^{\gamma}T_{\gamma} \tag{1.70}$$

where the numbers $f_{\alpha\beta}$ are called the **structure constants** of G. An element g of G near the unit element can be expressed as

$$g = \exp(-\theta^{\alpha}T_{\alpha}). \tag{1.71}$$

We suppose a Dirac field ψ transforms under $g \in G$ as

$$\psi \to g\psi \qquad \bar{\psi} \to \bar{\psi}g^{-1}. \tag{1.72}$$

[*Remark:* Strictly speaking, we have to specify the *representation* of G to which ψ belongs. If readers feel uneasy about (1.72), they may consider ψ is in the fundamental representation, for example.]

Consider the Lagrangian

$$\mathcal{L} = \bar{\psi}[\mathrm{i}\gamma^{\mu}(\partial_{\mu} + \mathcal{A}_{\mu}) + m]\psi \tag{1.73}$$

where the **Yang–Mills gauge field** $\mathcal{A}_{\mu}$ takes its values in the Lie algebra of G, that is, $\mathcal{A}_{\mu}$ can be expanded in terms of T_{α} as $\mathcal{A}_{\mu} = A_{\mu}{}^{\alpha}T_{\alpha}$. (Script fields are anti-Hermitian.) It is easily verified that $\mathcal{L}$ is invariant under

$$\begin{aligned} &\psi \to \psi' = g\psi \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi}g^{-1} \\ &\mathcal{A}_{\mu} \to \mathcal{A}'_{\mu} = g\mathcal{A}_{\mu}g^{-1} + g\partial_{\mu}g^{-1}. \end{aligned} \tag{1.74a}$$

The covariant derivative is defined by $\nabla_{\mu} = \partial_{\mu} + \mathcal{A}_{\mu}$ as before. $\nabla_{\mu}\psi$ transforms covariantly under the gauge transformation

$$\nabla'_\mu \psi' = g \nabla_\mu \psi. \tag{1.75}$$

The **Yang–Mills field tensor** is

$$\mathcal{F}_{\mu\nu} \equiv \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{1.76a}$$

The component $F_{\mu\nu}{}^\alpha$ is

$$F_{\mu\nu}{}^\alpha = \partial_\mu A_\nu{}^\alpha - \partial_\nu A_\mu{}^\alpha + f_{\beta\gamma}{}^\alpha A_\mu{}^\beta A_\nu{}^\gamma. \tag{1.76b}$$

If we define the **dual field tensor** $*\mathcal{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\kappa\lambda}\mathcal{F}^{\kappa\lambda}$, it satisfies the **Bianchi identity**,

$$\mathcal{D}_\mu * \mathcal{F}^{\mu\nu} \equiv \partial_\mu * \mathcal{F}^{\mu\nu} + [\mathcal{A}_\mu, *\mathcal{F}^{\mu\nu}] = 0. \tag{1.77}$$

Exercise 1.5 Show that $\mathcal{F}_{\mu\nu}$ transforms under (1.74*a*) as

$$\mathcal{F}_{\mu\nu} \to g\mathcal{F}_{\mu\nu}g^{-1}. \tag{1.74b}$$

From the exercise above, we find a gauge-invariant action

$$\mathcal{L}_{\mathrm{YM}} \equiv -\tfrac{1}{2}\,\mathrm{tr}(\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}) \tag{1.78a}$$

where the trace is over the group matrix. The component form is

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{2}F^{\mu\nu\alpha}F_{\mu\nu}{}^\beta\,\mathrm{tr}(T_\alpha T_\beta) = \tfrac{1}{4}F^{\mu\nu\alpha}F_{\mu\nu\alpha} \tag{1.78b}$$

where we have normalised $\{T_\alpha\}$ so that $\mathrm{tr}(T_\alpha T_\beta) = -\frac{1}{2}\delta_{\alpha\beta}$. The field equation derived from (1.78) is

$$\mathcal{D}_\mu \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{F}_{\mu\nu} + [\mathcal{A}_\mu, \mathcal{F}_{\mu\nu}] = 0. \tag{1.79}$$

1.2.3 Higgs fields

If the gauge symmetry is manifest in our world, there would be many observable massless vector fields. The absence of such fields, except the electromagnetic field, forces us to break the gauge symmetry. The theory is left renormalisable if the symmetry is broken *spontaneously*.

Let us consider a U(1) gauge field coupled to a complex scalar field ϕ, whose Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + (\nabla_\mu\phi)^\dagger(\nabla_\mu\phi) - \lambda(\phi^\dagger\phi - v^2). \tag{1.80}$$

The potential $V(\phi) = \lambda(\phi^\dagger\phi - v^2)$ has minima $V = 0$ at $|\phi| = v$. The Lagrangian (1.80) is invariant under the local gauge transformation

$$A_\mu \to A_\mu - \partial_\mu\alpha \qquad \phi \to \mathrm{e}^{-\mathrm{i}e\alpha}\phi \qquad \phi^\dagger \to \mathrm{e}^{\mathrm{i}e\alpha}\phi^\dagger. \tag{1.81}$$

This symmetry is spontaneously broken due to the **vacuum expectation value** (VEV) $\langle\phi\rangle$ of the **Higgs field**. We expand ϕ as

$$\phi = \frac{1}{\sqrt{2}}[v + \rho(x)]\mathrm{e}^{\mathrm{i}\alpha(x)/v} \sim \frac{1}{\sqrt{2}}[v + \rho(x) + \mathrm{i}\alpha(x)]$$

assuming $v \neq 0$. If $v \neq 0$, we may take the *unitary gauge* in which the phase of ϕ is 'gauged away' so that ϕ has only the real part,

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \rho(x)). \tag{1.82}$$

If we substitute (1.82) into (1.80) and expand in ρ, we have

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\partial_\mu\rho\partial^\mu\rho + \tfrac{1}{2}e^2 A_\mu A^\mu(v^2 + 2v\rho + \rho^2) \\ &- \tfrac{1}{4}\lambda(4v^2\rho^2 + 4v\rho^3 + \rho^4).\end{aligned} \tag{1.83}$$

The equations of motion for A_μ and ρ are

$$\partial^\nu F_{\nu\mu} + 2e^2v^2A_\mu = 0 \qquad \partial_\mu\partial^\mu\rho + 2\lambda v^2\rho = 0. \tag{1.84}$$

From the first equation, we find A_μ must satisfy the Lorentz condition $\partial_\mu A^\mu = 0$. The apparent degrees of freedom of (1.80) are 2 (photon) + 2 (complex scalar) = 4. If VEV $\neq 0$, we have 3 (massive vector) + 1 (real scalar) = 4. A_0 has a mass term with the wrong sign and so cannot be a physical degree of freedom.

1.3 Magnetic monopoles

Maxwell's equations unify electricity and magnetism. In the history of physics they should be recognised as the first attempt to unify forces in Nature. In spite of their great success, Dirac (1931) noticed that there existed an *asymmetry* in Maxwell's equations: the equation $\operatorname{div}\boldsymbol{B} = 0$ denies the existence of magnetic charges. He introduced the **magnetic monopole**, a point magnetic charge, to overcome this asymmetry.

1.3.1 Dirac monopole

Consider a monopole of strength g sitting at $\boldsymbol{r} = 0$,

$$\operatorname{div}\boldsymbol{B} = 4\pi g\delta^3(\boldsymbol{r}). \tag{1.85}$$

From $\Delta(1/r) = -4\pi\delta^3(\boldsymbol{r})$ and $\nabla(1/r) = -\boldsymbol{r}/r^3$, it follows that

$$\boldsymbol{B} = g\boldsymbol{r}/r^3. \tag{1.86}$$

The magnetic flux Φ is obtained by integrating $\boldsymbol{B}$ over a sphere of radius R so that

$$\Phi = \int_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = 4\pi g. \tag{1.87}$$

What about the vector potential which gives the monopole field (1.86)? If we define $\boldsymbol{A}^{\mathrm{N}}$ by

$$A^{\mathrm{N}}{}_x = \frac{-gy}{r(r+z)} \qquad A^{\mathrm{N}}{}_y = \frac{gx}{r(r+z)} \qquad A^{\mathrm{N}}{}_z = 0 \tag{1.88a}$$

we easily verify

$$\operatorname{curl} \boldsymbol{A}^{\mathrm{N}} = g\boldsymbol{r}/r^3 + 4\pi g\delta(x)\delta(y)\theta(-z). \tag{1.89}$$

We have $\operatorname{curl} \boldsymbol{A}^{\mathrm{N}} = \boldsymbol{B}$ except along the negative-z axis ($\theta = \pi$). The singularity along the z axis is called the **Dirac string** and reflects the poor choice of the coordinate system. If, instead, we define

$$A^{\mathrm{S}}{}_x = \frac{gy}{r(r-z)} \qquad A^{\mathrm{S}}{}_y = \frac{-gx}{r(r-z)} \qquad A^{\mathrm{S}}{}_z = 0 \tag{1.88b}$$

we have $\operatorname{curl} \boldsymbol{A}^{\mathrm{S}} = \boldsymbol{B}$ except along the positive-z axis ($\theta = \pi$) this time. The existence of a singularity is a natural consequence of (1.87). If there were a vector $\boldsymbol{A}$ such that $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$ with no singularity, we would have, from Gauss's law,

$$\Phi = \oint_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = \oint_S \operatorname{curl} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{S} = \int_V \operatorname{div}(\operatorname{curl} \boldsymbol{A})\, \mathrm{d}V = 0$$

where V is the volume inside the surface S. This problem is avoided only when we abandon the use of a *single* vector potential.

Exercise 1.6 Let us introduce the polar coordinates (r, θ, ϕ). Show that the vector potentials $\boldsymbol{A}^{\mathrm{N}}$ and $\boldsymbol{A}^{\mathrm{S}}$ are expressed as

$$\boldsymbol{A}^{\mathrm{N}}(\boldsymbol{r}) = \frac{g(1-\cos\theta)}{r\sin\theta}\,\hat{\boldsymbol{e}}_\varphi \tag{1.90a}$$

$$\boldsymbol{A}^{\mathrm{S}}(\boldsymbol{r}) = -\frac{g(1+\cos\theta)}{r\sin\theta}\,\hat{\boldsymbol{e}}_\phi \tag{1.90b}$$

where $\hat{\boldsymbol{e}}_\phi = -\sin\phi\,\hat{\boldsymbol{e}}_x + \cos\phi\,\boldsymbol{e}_y$.

1.3.2 The Wu–Yang monopole

Wu and Yang (1975) noticed that geometrical and topological structures of the Dirac monopole are best described by fibre bundles. In Chapters 9 and 10, we give an account of the Dirac monopole in terms of the fibre bundles and connections associated with them. Here we outline the idea of Wu and Yang without introducing the fibre bundle. Wu and Yang noted that we may employ more than one vector potential to describe a monopole. For example, we may avoid singularities if we adopt $\boldsymbol{A}^{\mathrm{N}}$ in the northern hemisphere and $\boldsymbol{A}^{\mathrm{S}}$ in the southern hemisphere of the sphere surrounding the monopole. These vector potentials yield the magnetic field $\boldsymbol{B} = g\boldsymbol{r}/r^3$, which is non-singular everywhere on the sphere. On the equator of the sphere, which is the boundary between the northern and southern hemispheres, $\boldsymbol{A}^{\mathrm{N}}$ and $\boldsymbol{A}^{\mathrm{S}}$ are related by the gauge transformation, $\boldsymbol{A}^{\mathrm{N}} - \boldsymbol{A}^{\mathrm{S}} = \operatorname{grad} \Lambda$. To compute this quantity, we employ the result of exercise 1.6,

$$\boldsymbol{A}^{\mathrm{N}} - \boldsymbol{A}^{\mathrm{S}} = \frac{2g}{r\sin\theta}\,\hat{\boldsymbol{e}}_\phi = \operatorname{grad}(2g\phi) \tag{1.91}$$

where use has been made of the expression

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\,\hat{\boldsymbol{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{e}}_\phi.$$

The gauge transformation function connecting $\boldsymbol{A}^{\mathrm{N}}$ and $\boldsymbol{A}^{\mathrm{S}}$ is

$$\Lambda = 2g\phi. \tag{1.92}$$

Note that Λ is ill-defined at $\theta = 0$ and $\theta = \pi$. Since we perform the gauge transformation only at $\theta = \pi/2$, these singularities do not show up in our theory. The total flux is

$$\Phi = \oint_S \operatorname{curl}\boldsymbol{A}\cdot \mathrm{d}\boldsymbol{S} = \int_{U_{\mathrm{N}}} \operatorname{curl}\boldsymbol{A}^{\mathrm{N}}\cdot\mathrm{d}\boldsymbol{S} + \int_{U_{\mathrm{S}}} \operatorname{curl}\boldsymbol{A}^{\mathrm{S}}\cdot\mathrm{d}\boldsymbol{S} \tag{1.93}$$

where U_{N} and U_{S} stand for the northern and southern hemispheres respectively. Stokes' theorem yields

$$\begin{aligned}\Phi &= \oint_{\text{equator}} \boldsymbol{A}^{\mathrm{N}}\cdot\mathrm{d}\boldsymbol{s} - \oint_{\text{equator}} \boldsymbol{A}^{\mathrm{S}}\cdot\mathrm{d}\boldsymbol{s} = \oint_{\text{equator}} (\boldsymbol{A}^{\mathrm{N}} - \boldsymbol{A}^{\mathrm{S}})\cdot\mathrm{d}\boldsymbol{s} \\ &= \oint_{\text{equator}} \operatorname{grad}(2g\phi)\cdot\mathrm{d}\boldsymbol{s} = 4g\pi \end{aligned}\tag{1.94}$$

in agreement with (1.87).

1.3.3 Charge quantisation

Consider a point particle with electric charge e and mass m moving in the field of a magnetic monopole of charge g. If the monopole is heavy enough, the Schrödinger equation of the particle is given by

$$\frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 \psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}). \tag{1.95}$$

It is easy to show that under the gauge transformation $\boldsymbol{A} \to \boldsymbol{A} + \operatorname{grad}\Lambda$, the wavefunction changes as $\psi \to \exp(\mathrm{i}e\Lambda/\hbar c)\psi$. In the present case, $\boldsymbol{A}^{\mathrm{N}}$ and $\boldsymbol{A}^{\mathrm{S}}$ differ only by the gauge transformation $\boldsymbol{A}^{\mathrm{N}} - \boldsymbol{A}^{\mathrm{S}} = \operatorname{grad}(2g\phi)$. If ψ^{N} and ψ^{S} are wavefunctions defined on U^{N} and U^{S} respectively, they are related by the phase change

$$\psi^{\mathrm{S}}(\boldsymbol{r}) = \exp\left(\frac{-\mathrm{i}e\Lambda}{\hbar c}\right)\psi^{\mathrm{N}}(\boldsymbol{r}). \tag{1.96}$$

Let us take $\theta = \pi/2$ and study the behaviour of wavefunctions as we go round the equator of the sphere from $\phi = 0$ to $\phi = 2\pi$. The wavefunction is required to be single valued, hence (1.96) forces us to take

$$\frac{2eg}{\hbar c} = n \qquad n \in \mathbb{Z}. \tag{1.97}$$

This is the celebrated **Dirac quantisation condition** for the magnetic charge; if the magnetic monopole exists, the magnetic charge takes discrete values,

$$g = \frac{\hbar c n}{2e} \qquad n \in \mathbb{Z}. \tag{1.98}$$

By the same token, if there exists a magnetic monopole somewhere in the universe, all the electric charges are quantised.

1.4 Instantons

Path integrals are well defined only in Euclidean space. The vacuum-to-vacuum amplitude is

$$Z \equiv \langle 0|0 \rangle \propto \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi, \partial_\mu \phi]} \tag{1.99}$$

where S is the Euclid action. (1.99) shows that the principal contribution to Z comes from the values of $\phi(x)$ which give the local minima of $S[\phi, \partial_\mu \phi]$. In many theories there exist a number of local minima in addition to the absolute minimum. In the case of non-Abelian gauge theories these minima are called **instantons**.

1.4.1 Introduction

Let us consider the SU(2) gauge theory defined in the four-dimensional Euclidean space $\mathbb{R}^4$. The action is

$$S = \int \mathrm{d}^4 x \mathcal{L}(x) = \int \mathrm{d}^4 x [-\tfrac{1}{2} \mathrm{tr} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}] \tag{1.100}$$

where the field strength is

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{1.101}$$

with

$$\mathcal{A}_\mu \equiv A_\mu{}^\alpha \frac{\sigma_\alpha}{2\mathrm{i}} \qquad \mathcal{F}_{\mu\nu} \equiv F_{\mu\nu}{}^\alpha \frac{\sigma_\alpha}{2\mathrm{i}}.$$

The field equation is

$$\mathcal{D}_\mu \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{F}_{\mu\nu} + [\mathcal{A}_\mu, \mathcal{F}_{\mu\nu}] = 0. \tag{1.102}$$

In the path integral only those field configurations with *finite action* contribute. Suppose $\mathcal{A}_\mu$ satisfies

$$\mathcal{A}_\mu \to \mathrm{i} g(x)^{-1} \partial_\mu g(x) \qquad \text{as } |x| \to \infty \tag{1.103}$$

where $g(x)$ is an element of SU(2). We easily find that $\mathcal{F}_{\mu\nu}$ vanishes for the $\mathcal{A}_\mu$ of (1.103). We require that on sphere S^3 of large radius, the gauge potential be given by (1.103).

Later we show that this configuration is characterised by the way in which S^3 is mapped to the gauge group SU(2). Non-trivial configurations are those that cannot be deformed continuously to a uniform configuration. They were proposed by Belavin *et al* (1975) and are called **instantons**.

1.4.2 The (anti-) self-dual solution

In general, solving a second-order differential equation is more difficult than solving a first-order one. It is nice when a second-order differential equation can be replaced by a first-order one which is equivalent to the original problem. Let us consider the inequality

$$\int \mathrm{d}^4x \, \mathrm{tr}(\mathcal{F}_{\mu\nu} \pm *\mathcal{F}_{\mu\nu})^2 \geqslant 0. \qquad (1.104)$$

Clearly (1.104) is saturated if

$$\mathcal{F}_{\mu\nu} = \pm *\mathcal{F}_{\mu\nu}. \qquad (1.105)$$

If the positive sign is chosen, $\mathcal{F}$ is said to be **self-dual** while the negative sign gives an **anti-self-dual** solution. If (1.105) is satisfied, the field equation is automatically satisfied since

$$\mathcal{D}_\mu \mathcal{F}_{\mu\nu} = \pm \mathcal{D}_\mu * \mathcal{F}_{\mu\nu} = 0 \qquad \text{(Bianchi identity)}. \qquad (1.106)$$

As we will show in §10.5, the integral

$$Q \equiv \frac{-1}{16\pi^2} \int \mathrm{d}^4x \, \mathrm{tr} \, \mathcal{F}_{\mu\nu} * \mathcal{F}^{\mu\nu} \qquad (1.107)$$

is an integer characterising the way S^3 is mapped to SU(2). If $\mathcal{F}$ is self-dual then Q is positive, and if $\mathcal{F}$ is anti-self-dual then Q is negative. From (1.104), we find (note that $*\mathcal{F}_{\mu\nu} * \mathcal{F}^{\mu\nu} = \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$)

$$\int \mathrm{tr}(2\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \pm 2\mathcal{F}_{\mu\nu} * \mathcal{F}^{\mu\nu}) \geqslant 0. \qquad (1.108)$$

From this inequality and the definition of the action, we find

$$S \geqslant 8\pi^2 |Q| \qquad (1.109)$$

where the inequality is saturated for (1.105).

Let us concentrate on the self-dual solution $\mathcal{F} = *\mathcal{F}$. We look for an instanton solution of the form

$$\mathcal{A}_\mu = \mathrm{i}f(r)g(x)^{-1}\partial_\mu g(x) \qquad (1.110)$$

where $r \equiv |x|$ and

$$f(r) \longrightarrow 1 \qquad \text{as } r \longrightarrow \infty \qquad (1.111a)$$

$$g(x) = \frac{1}{r}(x_4 - \mathrm{i}x_i\sigma_i). \tag{1.111b}$$

Substituting (1.110) into (1.105), we find that f satisfies

$$r\frac{\mathrm{d}f(r)}{\mathrm{d}r} = 2f(1-f). \tag{1.112}$$

The solution that satisfies the boundary condition (1.111a) is

$$f(r) = \frac{r^2}{r^2+\lambda^2} \tag{1.113}$$

where λ is a parameter that specifies the size of the instanton. Substituting this into (1.110) we find

$$\mathcal{A}_\mu(x) = \frac{\mathrm{i}r^2}{r^2+\lambda^2}\, g(x)^{-1}\partial_\mu g(x) \tag{1.114}$$

and the corresponding field strength

$$\mathcal{F}_{\mu\nu}(x) = \frac{4\lambda^2}{r^2+\lambda^2}\,\sigma_{\mu\nu} \tag{1.115}$$

where

$$\sigma_{ij} \equiv \frac{1}{4\mathrm{i}}[\sigma_i, \sigma_j],\ \sigma_{i0} \equiv \tfrac{1}{2}\sigma_i = -\sigma_{0i}. \tag{1.116}$$

This solution gives $Q = +1$ and $S = 8\pi^2$.

1.5 Orders in condensed matter systems

Recently topological methods have played increasingly important roles in condensed matter physics. For example, homotopy theory has been employed to classify possible forms of extended objects, such as solitons, vortices, monopoles and so on, in condensed systems. These classifications will be studied in §§4.8–4.9. Here, we briefly look at the order parameters of condensed systems that undergo phase transitions.

1.5.1 Order parameter

Let H be a Hamiltonian describing a condensed system. We assume H is invariant under a certain symmetry operation. The ground state of the system need not preserve the symmetry of H. If this is the case, we say the system undergoes spontaneous symmetry breakdown.

To illustrate this phenomenon, we consider the Heisenberg Hamiltonian

$$H = -J\sum_{(i,j)} \boldsymbol{S}_i\cdot\boldsymbol{S}_j + \boldsymbol{h}\cdot\sum_i \boldsymbol{S}_i \tag{1.117}$$

which describes N ferromagnetic Heisenberg spins $\{\boldsymbol{S}_i\}$. J is a positive constant, the summation is over the pair of the nearest-neighbour sites (i, j) and $\boldsymbol{h}$ is the external magnetic field. The partition function is $Z = \mathrm{tr}\, \mathrm{e}^{-\beta H}$, where $\beta = 1/T$ is the inverse temperature. The free energy F is given by $\exp(-\beta F) = Z$. The average magnetisation per spin is

$$\boldsymbol{m} \equiv \frac{1}{N} \sum_i \langle \boldsymbol{S}_i \rangle = \frac{1}{N\beta} \frac{\partial F}{\partial h_i} \tag{1.118}$$

where $\langle \ldots \rangle \equiv \mathrm{tr}(\ldots)\mathrm{e}^{-\beta H}/Z$. Let us consider the limit $\boldsymbol{h} \to 0$. Although H is invariant under the SO(3) rotations of all $\boldsymbol{S}_i$ in this limit, it is well known that $\boldsymbol{m}$ does not vanish for large enough β and the system does not observe the SO(3) symmetry. It is said that the system exhibits **spontaneous magnetisation** and the maximum temperature, such that $\boldsymbol{m} \neq 0$ is called the **critical temperature**. The vector $\boldsymbol{m}$ is the **order parameter** describing the phase transition between the ordered state ($\boldsymbol{m} \neq 0$) and the disordered state ($\boldsymbol{m} = 0$). The system is still symmetric under SO(2) rotations around the magnetisation axis $\boldsymbol{m}$.

What is the mechanism underlying the phase transition? The free energy is $F = \langle H \rangle - TS$, S being the entropy. At low temperature, the term TS in F may be negligible and the minimum of F is attained by minimising $\langle H \rangle$, which is realised if all $\boldsymbol{S}_i$ align in the same direction. At high temperature, on the other hand, the entropy term dominates F and the minimum of F is given by maximising S, which is attained if the directions of $\boldsymbol{S}_i$ are totally random.

If the system is in a uniform temperature, the magnitude $|\boldsymbol{m}|$ is independent of the position. Then $\boldsymbol{m}$ is specified by its direction. In the ground state, $\boldsymbol{m}$ is expected to be independent of position. It is convenient to introduce the polar coordinate (θ, ϕ) to specify the direction of $\boldsymbol{m}$. There is a one-to-one correspondence between $\boldsymbol{m}$ and a point on the sphere S^2 (2 since the sphere is two-dimensional). Suppose $\boldsymbol{m}$ varies as a function of position: $\boldsymbol{m} = \boldsymbol{m}(x)$. At each point $\boldsymbol{x}$ of the space, a point (θ, ϕ) of S^2 is assigned and we have a map $(\theta(\boldsymbol{x}), \phi(\boldsymbol{x}))$ from the space to S^2. Besides the ground state (and excited states that are described by small oscillations (spin waves) around the ground state) the system may carry various excited states that cannot be obtained from the ground state by small perturbations. What kind of excitations are possible depends on the dimension of the space and the order parameter. For example, if the space is two-dimensional, the Heisenberg ferromagnet may admit an excitation called the **Belavin–Polyakov monopole** shown in figure 1.1 (Belavin and Polyakov 1975). Note that $\boldsymbol{m}$ approaches a constant vector ($\hat{z}$ in this case) so the energy does not diverge. This condition guarantees the stability of this excitation; it is impossible to deform this configuration into the uniform one with $\boldsymbol{m}$ far from the origin kept fixed. These kinds of excitations whose stability depends on topological arguments are called **topological excitations**.

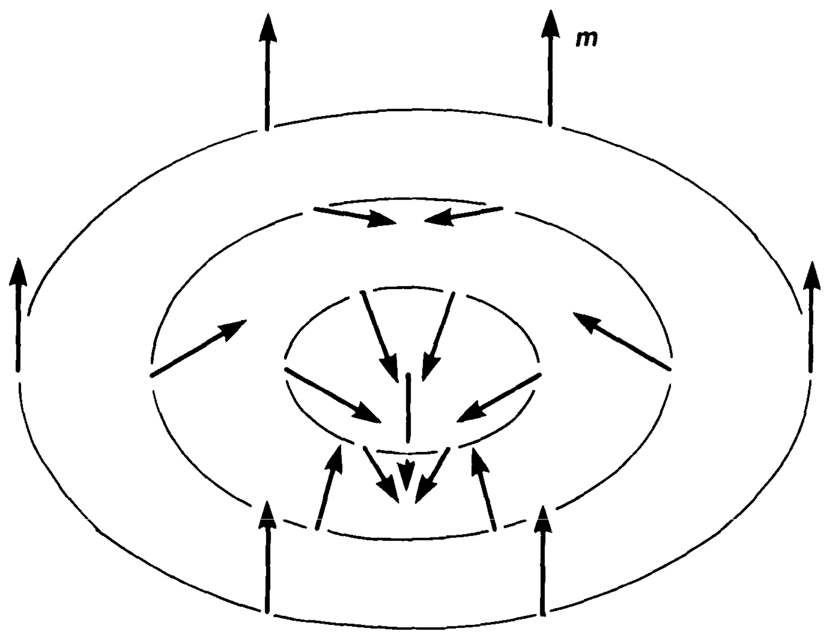


Figure 1.1 A sketch of the Belavin–Polyakov monopole. The vector $\boldsymbol{m}$ approaches $\hat{z}$ as $|\boldsymbol{x}| \to \infty$.

1.5.2 Superfluid ^{4}He and superconductors

In Bogoliubov's theory, the order parameter of superfluid ^{4}He is the expectation value

$$\langle \phi(\boldsymbol{x}) \rangle = \Psi(\boldsymbol{x}) = \Delta_0(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\alpha(\boldsymbol{x})} \tag{1.119}$$

where $\phi(\boldsymbol{x})$ is the field operator. In the operator formalism,

$$\phi(\boldsymbol{x}) \sim (\text{creation operator}) + (\text{annihilation operator})$$

from which we find the number of particles is not conserved if $\Psi(\boldsymbol{x}) \neq 0$. This is related to the spontaneous breakdown of the *global gauge symmetry*. The Hamiltonian of ^{4}He is

$$\begin{aligned} H = & \int \mathrm{d}\boldsymbol{x} \phi^{\dagger}(\boldsymbol{x}) \left(-\frac{\nabla^2}{2m} - \mu \right) \phi(\boldsymbol{x}) \\ & + \frac{1}{2} \int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y}\, \phi^{\dagger}(\boldsymbol{y}) \phi(\boldsymbol{y}) V(|\boldsymbol{x} - \boldsymbol{y}|) \phi^{\dagger}(\boldsymbol{x}) \phi(\boldsymbol{x}). \end{aligned} \tag{1.120}$$

Clearly H is invariant under the global gauge transformation

$$\phi(\boldsymbol{x}) \to \mathrm{e}^{\mathrm{i}\chi} \phi(\boldsymbol{x}). \tag{1.121}$$

The order parameter, however, transforms as

$$\Psi(\boldsymbol{x}) \to e^{i\chi}\Psi(\boldsymbol{x}) \tag{1.122}$$

and hence does not observe the symmetry of the Hamiltonian.

The phenomenological free energy describing ^{4}He is the **Ginzburg–Landau free energy**. The main contribution is the **condensation energy** given by

$$\mathcal{F}_0 = \frac{\alpha}{2!}|\Psi(\boldsymbol{x})|^2 + \frac{\beta}{4!}|\Psi(\boldsymbol{x})|^4 \tag{1.123a}$$

where $\alpha \sim \alpha_0(T - T_c)$ changes sign at the critical temperature $T_c \sim 4$ K. Figure 1.2 sketches $\mathcal{F}_0$ for $T > T_c$ and $T < T_c$. If $T > T_c$, the minimum of $\mathcal{F}_0$ is attained at $\Psi(\boldsymbol{x}) = 0$ while if $T < T_c$ at $|\Psi| = \Delta_0 \equiv [-(6\alpha/\beta)]^{1/2}$. If $\Psi(\boldsymbol{x})$ depends on $\boldsymbol{x}$, we have an additional contribution called the **gradient energy**

$$\mathcal{F}_{\text{grad}} = \tfrac{1}{2}K\overline{\nabla\Psi(\boldsymbol{x})}\cdot\nabla\Psi(\boldsymbol{x}) \tag{1.123b}$$

K being a positive constant. If the spatial variation of $\Psi(\boldsymbol{x})$ is weak enough, we may assume Δ_0 is constant (the **London limit**).

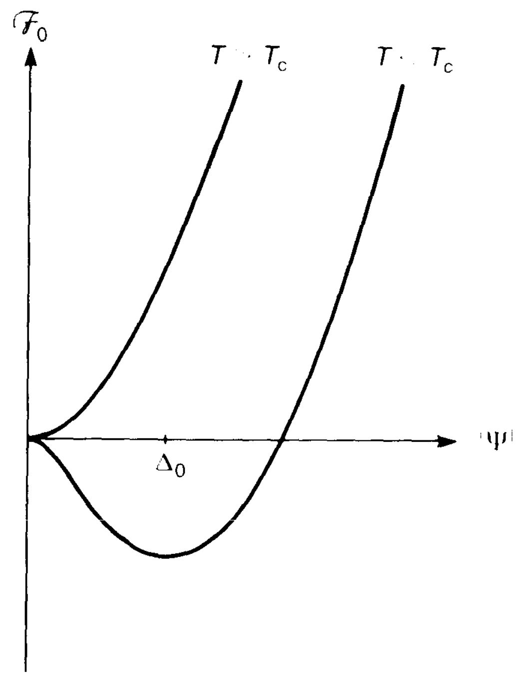


Figure 1.2 The Ginzburg–Landau free energy has a minimum at $|\Psi| = 0$ for $T > T_c$ and at $|\Psi| = \Delta_0$ for $T < T_c$.

Consider a three-dimensional medium of superfluid ^{4}He. The ground state is given by uniform $\Psi(\boldsymbol{x})$. A topological excitation called a **vortex** has a line singularity along which the order parameter is not defined. Take a slice $z = 0$ as in figure 1.3 and assume that $\Psi(\boldsymbol{x})$ is translationally invariant along the z axis. Consider a loop C centred at the origin and go round C in the counterclockwise sense. Since $\Psi(\boldsymbol{x})$ is uniquely defined, after a complete traverse of C, the phase $\varphi(\boldsymbol{x})$ of $\Psi(\boldsymbol{x})$ may change only by a multiple of 2π. If θ denotes the polar coordinate, the typical form of $\Psi(\boldsymbol{x})$ will be $\Delta_0 \exp(\mathrm{i}n\theta)$. What happens at the origin? $\mathcal{F}_{\text{grad}}$ near the origin diverges as $\mathcal{F}_{\text{grad}} \sim n^2 r^{-2} \Delta_0^2$. If $n \neq 0$, this divergence is avoided only when Δ_0 vanishes at the origin.

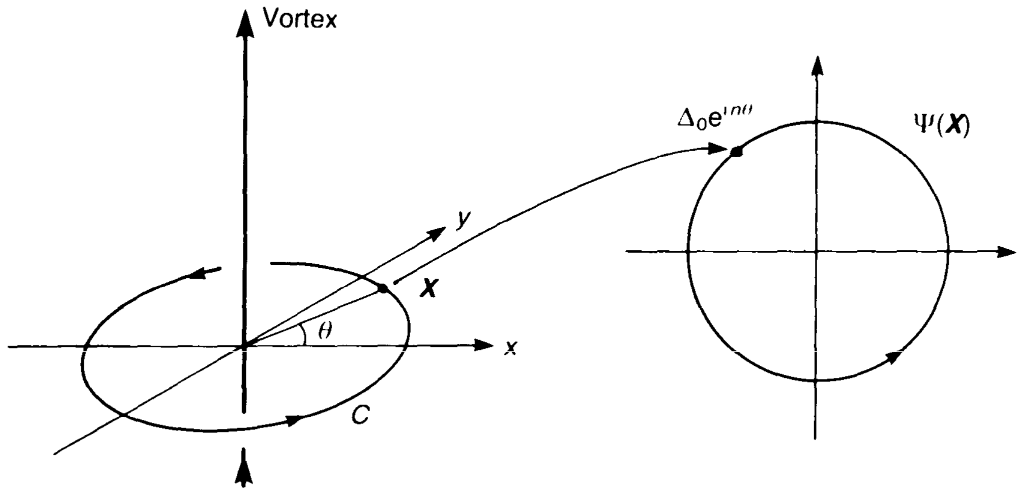


Figure 1.3 The vortex line in superfluid ^{4}He. If we complete the circuit C surrounding the vortex, the phase changes by an integral multiple of 2π.

In the BCS theory of superconductors, the order parameter is given by (Tsuneto 1982)

$$\Psi_{\alpha\beta}(\boldsymbol{x}) \equiv \langle \psi_\alpha(\boldsymbol{x})\psi_\beta(\boldsymbol{x})\rangle \tag{1.124}$$

ψ_α being the electron field operator of spin $\alpha = (\uparrow, \downarrow)$. It should be noted, however, that (1.124) is not an irreducible representation of the spin algebra. To see this, we examine the behaviour of $\Psi_{\alpha\beta}$ under a spin rotation. Consider an infinitesimal spin rotation around an axis $\boldsymbol{n}$ by an angle δ, whose matrix representation is

$$R = \mathbb{1} + \mathrm{i}\frac{\delta}{2} n^\mu \sigma_\mu,$$

σ_μ being the Pauli matrices defined at the beginning of this chapter. Since ψ_α transforms as $\psi_\alpha \to R_\alpha{}^\beta \psi_\beta$, we have

$$\begin{aligned}\Psi_{\alpha\beta} \to R_\alpha{}^{\alpha'} \Psi_{\alpha'\beta'} R_\beta{}^{\beta'} &= (R \cdot \Psi \cdot R^{\mathrm{t}})_{\alpha\beta} \\ &= \left(\Psi + \mathrm{i}\frac{\delta}{2}\, \boldsymbol{n}(\boldsymbol{\sigma}\Psi\sigma_2 - \Psi\sigma_2\boldsymbol{\sigma})\right)_{\alpha\beta}\end{aligned}$$

where we note that $\sigma_\mu{}^{\mathrm{t}} = -\sigma_2\sigma_\mu\sigma_2$. Suppose $\Psi_{\alpha\beta} \propto \mathrm{i}(\sigma_2)_{\alpha\beta}$. Then Ψ does not change under the above rotation, hence it represents the spin-singlet pairing. We write

$$\Psi_{\alpha\beta}(\boldsymbol{x}) = \Delta(\boldsymbol{x})(\mathrm{i}\sigma_2)_{\alpha\beta} = \Delta_0(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\varphi(\boldsymbol{x})}(\mathrm{i}\sigma_2)_{\alpha\beta}. \tag{1.125a}$$

If, on the other hand, we take

$$\Psi_{\alpha\beta}(\boldsymbol{x}) = \Delta^{\mu}(\boldsymbol{x})\mathrm{i}(\sigma_{\mu}\cdot\sigma_2)_{\alpha\beta} \tag{1.125b}$$

we have

$$\Psi_{\alpha\beta} \to [\Delta^{\mu} + \delta\varepsilon^{\mu\nu\lambda}n_{\nu}\Delta_{\lambda}](\mathrm{i}\sigma_{\mu}\cdot\sigma_2)_{\alpha\beta}.$$

This shows that Δ^{μ} is a vector in spin space, hence (1.125*b*) represents the spin-triplet pairing.

The order parameter of a conventional superconductor is of the form (1.125*a*) and we restrict the analysis to this case for the moment. In (1.125*a*), $\Delta(\boldsymbol{x})$ assumes the same form as $\Psi(\boldsymbol{x})$ of superfluid ^{4}He and the Ginzburg–Landau free energy is again given by (1.123). This similarity is attributed to the **Cooper pair**. In the superfluid state, a macroscopic number of ^{4}He atoms occupy the ground state (**Bose–Einstein condensation**) which then behaves like a huge molecule due to the quantum coherence. In this state creating elementary excitations requires a finite amount of energy and the flow cannot decay unless this critical energy is supplied. Since an electron is a fermion there is, at first sight, no Bose–Einstein condensation. The key observation is the Cooper pair. By the exchange of phonons, a pair of electrons feel an attractive force that barely overcomes the Coulomb repulsion. This tiny attractive force makes it possible for electrons to form a pair (in momentum space) that obeys Bose statistics. The pairs then condense to form the superfluid state of the Cooper pairs of electric charge $2e$.

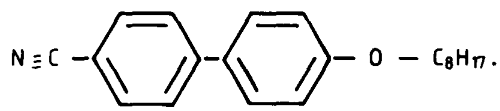
An electromagnetic field couples to the system through the minimal coupling

$$\mathcal{F}_{\text{grad}} = \tfrac{1}{2}K|(\partial_{\mu} - \mathrm{i}2eA_{\mu})\Delta(\boldsymbol{x})|^2. \tag{1.126}$$

(The term $2e$ is used since the Cooper pair carries charge $2e$.) Superconductors are roughly divided into two types according to their behaviour in applied magnetic fields. The type-I superconductor forms an intermediate state in which normal and superconducting regions coexist in strong magnetic fields. The type-II superconductor forms a vortex lattice (**Abrikosov lattice**) to confine the magnetic fields within the cores of the vortices with other regions remaining in the superconducting state. A similar vortex lattice has been observed in rotating superfluid ^{4}He in a cylinder.

1.5.3 Nematic liquid crystals

Certain organic crystals exhibit quite interesting optical properties when they are in their fluid phases. They are called **liquid crystals** and they are characterised by their optical anisotropy. Here we are interested in so-called **nematic liquid crystals**. An example of this is *octyloxy-cyanobiphenyl* whose molecular structure is



The molecule of a nematic liquid crystal is very much like a *rod* and the order parameter, called the **director**, is given by the average *direction* of the rod. Even though the molecule itself has a head and a tail, the director has an inversion symmetry; it does not make sense to distinguish the directors $\boldsymbol{n} = \rightarrow$ and $-\boldsymbol{n} = \leftarrow$. We are tempted to assign a point on S^2 to describe the director. This works except for one point. Two antipodal points $\boldsymbol{n} = (\theta, \phi)$ and $-\boldsymbol{n} = (\pi - \theta, \pi + \phi)$ represent the same state; see figure 1.4. We see later that the order parameter of the nematic liquid crystal is the **projective plane** $\mathbb{R}P^2$ which is the sphere whose antipodal points are identified.

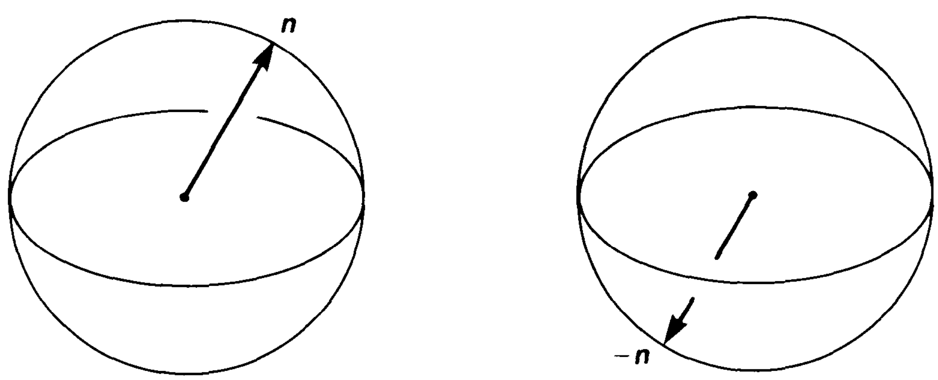


Figure 1.4 Since the director $\boldsymbol{n}$ has no head or tail, we cannot distinguish $\boldsymbol{n}$ from $-\boldsymbol{n}$.

1.5.4 Superfluid ^{3}He-A

Here comes the last and most interesting example. Before 1972 the only example of the BCS superfluid was the conventional superconductor (apart from indirect observations of superfluid neutrons in neutron stars). Figure 1.5 is the phase diagram of superfluid ^{3}He without an external magnetic field. From NMR and other observations, it turns out that the superfluid is in the spin-triplet p-wave state. Instead of the field operators (see (1.125*b*)), we define the order parameter in terms of the creation and annihilation operators. The most general form of the triplet superfluid order parameter is

$$\langle c_{\alpha,\boldsymbol{k}} c_{\beta,-\boldsymbol{k}} \rangle \propto \sum_{\mu=1}^{3} (\mathrm{i}\sigma_2\sigma_\mu)_{\alpha\beta} d_\mu(\boldsymbol{k}) \tag{1.127a}$$

where α and β are spin indices. The Cooper pair forms in the p-wave state hence $d_\mu(\boldsymbol{k})$ is proportional to $Y_{1m} \sim k_i$.

$$d_\mu(\boldsymbol{k}) = \sum_{i=1}^{3} \Delta_0 A_{\mu i} k_i. \tag{1.127b}$$

The bulk energy has several minima. The absolute minimum depends on the pressure and the temperature. We are particularly interested in the A phase in figure 1.5.

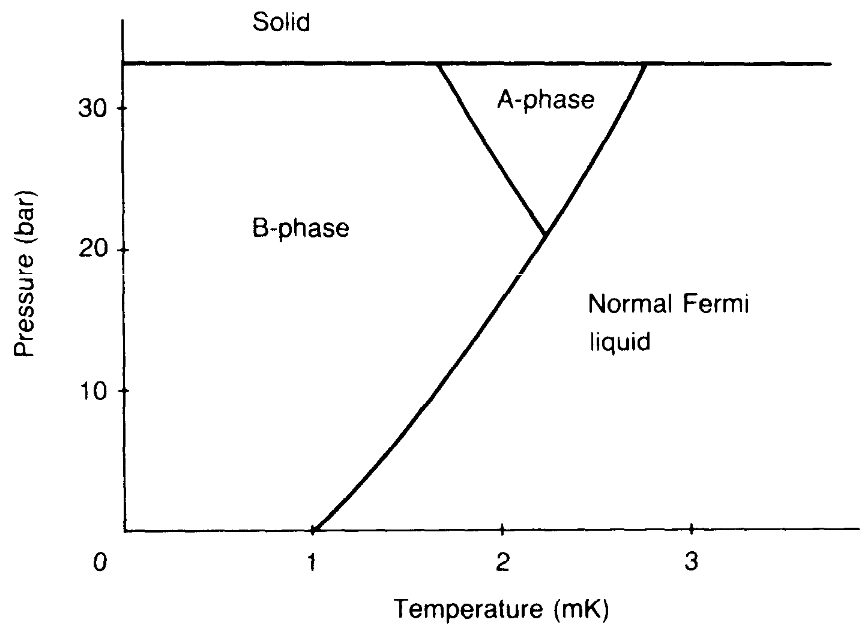


Figure 1.5 The phase diagram of superfluid ^{3}He.

The A phase order parameter takes the form

$$A_{\mu i} = d_\mu(\boldsymbol{\Delta}_1 + \mathrm{i}\boldsymbol{\Delta}_2)_i \tag{1.128}$$

where $\boldsymbol{d}$ is a unit vector along which the spin projection of the Cooper pair vanishes and $(\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2)$ is a pair of orthonormal unit vectors. $\boldsymbol{d}$ takes its value in S^2. If we define $\boldsymbol{l} \equiv \boldsymbol{\Delta}_1 \times \boldsymbol{\Delta}_2$, the triad $(\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2, \boldsymbol{l})$ forms an orthonormal frame at each point of the medium. Since any orthonormal frame can be obtained from a standard orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ by an application of a three-dimensional rotation matrix, we conclude that the order parameter of ^{3}He-A is $S^2 \times \mathrm{SO}(3)$. The vector $\boldsymbol{l}$ introduced above is the axis of the angular momentum of the Cooper pair.

Reviews on superfluid ^{3}He are found in Anderson and Brinkman (1975), Leggett (1975) and Mermin (1978).

1.6 General relativity

In classical physics, the general theory of relativity is one of the most beautiful and successful theories. There is no disagreement between the theory and astrophysical and cosmological observations such as solar system tests, gravitational radiation from pulsars, gravitational red shifts, the recently discovered gravitational lens effect and so on.

Readers not very familiar with general relativity may consult Berry (1989) or the *primer* by Price (1982).

1.6.1 The metric and curvature

A small distance on the sphere S^2 of radius r is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \tag{1.129}$$

where (θ, ϕ) is the usual polar coordinate on S^2. More generally we write a distance between a pair of points whose coordinates are $\{x^\mu\}$ and $\{x^\mu + dx^\mu\}$ as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.130}$$

For example, in the case of the sphere we have $g_{\theta\theta} = r^2$ and $g_{\phi\phi} = r^2 \sin^2 \theta$ and $g_{\theta\phi} = g_{\phi\theta} = 0$. Our convention is $g_{\mu\nu} = \delta_{\mu\nu} = \mathrm{diag}(1, 1, \ldots, 1)$ in Euclidean space and $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$ for Minkowski space. The matrix $(g_{\mu\nu})$ is called the **metric tensor**. The inverse of $(g_{\mu\nu})$ is denoted by $(g^{\mu\nu})$; $g_{\kappa\lambda}g^{\lambda\mu} = \delta_\kappa{}^\mu$, $g^{\kappa\lambda}g_{\lambda\mu} = \delta^\kappa{}_\mu$.

Intuition tells us that the Euclidean space is *flat* while a sphere is *curved*. To see how much the space or spacetime is curved we define the **Christoffel symbols**,

$$\Gamma^\kappa{}_{\lambda\mu} \equiv \frac{1}{2} g^{\kappa\nu}\left(\frac{\partial g_{\nu\mu}}{\partial x^\lambda} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu}\right). \tag{1.131}$$

Exercise 1.7 Let $ds^2 = dr^2 + r^2 d\theta^2$ be the metric in the plane $\mathbb{R}^2$. Compute $\Gamma^\kappa{}_{\lambda\mu}$. Compare this with the metric $ds^2 = dx^2 + dy^2$ for which $\Gamma^\kappa{}_{\lambda\mu} \equiv 0$.

As the exercise above shows, Γ cannot have an intrinsic meaning in the description of curvature: whether Γ vanishes or not depends on the choice of the coordinates. Classical differential geometry shows that what is intrinsic is the **Riemann curvature tensor**

$$R^\kappa{}_{\lambda\mu\nu} \equiv \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\eta{}_{\nu\lambda}\Gamma^\kappa{}_{\mu\eta} - \Gamma^\eta{}_{\mu\lambda}\Gamma^\kappa{}_{\nu\eta}. \tag{1.132}$$

$R^\kappa{}_{\lambda\mu\nu}$ vanishes if and only if the space is flat. From $R^\kappa{}_{\lambda\mu\nu}$, we define the **Ricci tensor**

$$Ric_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} \tag{1.133}$$

and the **scalar curvature**

$$\mathcal{R} \equiv g^{\mu\nu} Ric_{\mu\nu}. \tag{1.134}$$

Exercise 1.8 Compute the Riemann curvature tensor, the Ricci tensor and the scalar curvature of the sphere S^2.

1.6.2 The Einstein equation

Einstein proposed the following principles to construct the general theory of relativity

(I) **Principle of General Relativity:** All laws in physics take the same forms in any coordinate system.

(II) **Principle of Equivalence:** There exists a coordinate system in which the effect of a gravitational field vanishes locally. (An observer in a freely falling lift does not feel gravity until it crashes.)

Any theory of gravity must reduce to Newton's theory of gravity in the weak-field limit. In Newton's theory, the gravitational potential Φ satisfies the Poisson equation

$$\Delta\Phi = 4\pi G\rho \tag{1.135}$$

where ρ is the mass density. The Einstein equation generalises this classical result so that the principle of general relativity is satisfied.

In general relativity, the gravitational potential is replaced by the components of the metric tensor. Then, instead of the LHS of (1.135), we have the **Einstein tensor** defined by

$$G_{\mu\nu} \equiv Ric_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\mathcal{R}. \tag{1.136}$$

Similarly the mass density is replaced by a more general object called the **energy–momentum tensor** $T_{\mu\nu}$. The **Einstein equation** takes a very similar form to (1.135):

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{1.137}$$

The constant $8\pi G$ is chosen so that (1.137) reproduces the Newtonian result in the weak-field limit. $T_{\mu\nu}$ is obtained from the matter action by the variational principle. From Noether's theorem, $T_{\mu\nu}$ must satisfy a conservation equation of the form $\nabla_\mu T^{\mu\nu} = 0$, where ∇_μ is the covariant derivative which will be defined in Chapter 7. A similar conservation law holds for $G_{\mu\nu}$ (but not for $Ric_{\mu\nu}$). We shall see later that the LHS of (1.137) is also obtained from the variational principle (§7.10).

Exercise 1.9 Consider a metric

$$g_{00} = -1 - \frac{2\Phi}{c^2} \qquad g_{0i} = 0 \qquad g_{ij} = \delta_{ij} \qquad 1 \leqslant i, j \leqslant 3$$

and $T_{\mu\nu}$ given by $T_{00} = \rho c^2$, $T_{0i} = T_{ij} = 0$ which corresponds to dust at rest. Show that (1.137) reduces to the Poisson equation in the weak-field limit ($\Phi/c^2 \ll 1$).

1.7 Berry's phase

In quantum mechanics, we define a wavefunction up to the phase. In most cases, the phase is neglected as an irrelevant factor. Berry (1984) pointed out that if the system undergoes an adiabatic change, the phase may have observable consequences.

1.7.1 Derivation of Berry's phase

Let $H(\boldsymbol{R})$ be a Hamiltonian which depends on some parameters collectively written as $\boldsymbol{R}$. Suppose $\boldsymbol{R}$ changes adiabatically as a function of time, $\boldsymbol{R} = \boldsymbol{R}(t)$. The Schrödinger equation is

$$H(\boldsymbol{R}(t))|\psi(t)\rangle = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle. \tag{1.138}$$

We assume the system at $t = 0$ is in the nth eigenstate, $|\psi(0)\rangle = |n, \boldsymbol{R}(0)\rangle$ where

$$H(\boldsymbol{R}(0))|n, \boldsymbol{R}(0)\rangle = E_n(\boldsymbol{R}(0))|n, \boldsymbol{R}(0)\rangle. \tag{1.139}$$

What about the state $|\psi(t)\rangle$ at later time t? We assume the system is always in the nth state (adiabatic assumption).

Exercise 1.10 A naive guess of $|\psi(t)\rangle$ is

$$|\psi(t)\rangle = \exp\left(-\mathrm{i}\int_0^t \mathrm{d}s E_n(\boldsymbol{R}(s))\right)|n, \boldsymbol{R}(t)\rangle \tag{1.140}$$

where the normalised state $|n, \boldsymbol{R}(t)\rangle$ satisfies

$$H(\boldsymbol{R}(t))|n, \boldsymbol{R}(t)\rangle = E_n(\boldsymbol{R}(t))|n, \boldsymbol{R}(t)\rangle. \tag{1.141}$$

Show that (1.140) is *not* a solution of (1.138).

Since (1.140) does not satisfy the Schrödinger equation, we have to try other possibilities. Let us introduce an extra-phase $\gamma_n(t)$ in the wavefunction:

$$|\psi(t)\rangle = \exp\left(\mathrm{i}\gamma_n(t) - \mathrm{i}\int_0^t E_n(\boldsymbol{R}(s))\,\mathrm{d}s\right)|n, \boldsymbol{R}(t)\rangle. \tag{1.142}$$

Inserting (1.142) into the Schrödinger equation (1.138) we find

$$H|\psi(t)\rangle = E_n(\boldsymbol{R}(t))|\psi(t)\rangle$$

for the LHS (see (1.141)) and

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle = \left(-\,\frac{\mathrm{d}\gamma_n}{\mathrm{d}t} + E_n(\boldsymbol{R}(t))\right)|\psi(t)\rangle$$
$$+ \exp\left(\mathrm{i}\gamma_n - \mathrm{i}\int E_n(\boldsymbol{R}(s))\,\mathrm{d}s\right)\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\,|n,\,\boldsymbol{R}(t)\rangle$$

for the RHS. Equating these, we find

$$\frac{\mathrm{d}\gamma_n(t)}{\mathrm{d}t} = \mathrm{i}\left\langle n,\,\boldsymbol{R}(t)\left|\frac{\mathrm{d}}{\mathrm{d}t}\right|n,\,\boldsymbol{R}(t)\right\rangle. \tag{1.143}$$

By integrating (1.143), we have

$$\begin{aligned}\gamma_n(t) &= \mathrm{i}\int_0^t \left\langle n,\,\boldsymbol{R}(s)\left|\frac{\mathrm{d}}{\mathrm{d}s}\right|n,\,\boldsymbol{R}(s)\right\rangle \mathrm{d}s\\ &= \mathrm{i}\int_{\boldsymbol{R}(0)}^{\boldsymbol{R}(t)} \langle n,\,\boldsymbol{R}|\nabla_{\boldsymbol{R}}|n,\,\boldsymbol{R}\rangle\,\mathrm{d}\boldsymbol{R}\end{aligned} \tag{1.144}$$

where $\nabla_{\boldsymbol{R}}$ stands for the gradient in $\boldsymbol{R}$-space. Note that $\gamma_n(t)$ is real since

$$\begin{aligned}&2\mathrm{Re}\left\langle n,\,\boldsymbol{R}(s)\left|\frac{\mathrm{d}}{\mathrm{d}s}\right|n,\,\boldsymbol{R}(s)\right\rangle\\ &\quad= \left\langle n,\,\boldsymbol{R}(s)\left|\frac{\mathrm{d}}{\mathrm{d}s}\right|n,\,\boldsymbol{R}(s)\right\rangle + \left(\frac{\mathrm{d}}{\mathrm{d}s}\,\langle n,\,\boldsymbol{R}(s)|\right)|n,\,\boldsymbol{R}(s)\rangle\\ &\quad= \frac{\mathrm{d}}{\mathrm{d}s}\,\langle n,\,\boldsymbol{R}(s)|n,\,\boldsymbol{R}(s)\rangle = 0.\end{aligned}$$

Suppose the system executes a closed loop in $\boldsymbol{R}$-space; $\boldsymbol{R}(0) = \boldsymbol{R}(T)$ for some $T > 0$. We then have

$$\begin{aligned}\gamma_n(T) &= \mathrm{i}\int_0^T \left\langle n,\,\boldsymbol{R}(s)\left|\frac{\mathrm{d}}{\mathrm{d}s}\right|n,\,\boldsymbol{R}(s)\right\rangle \mathrm{d}s\\ &= \mathrm{i}\int_{\boldsymbol{R}(0)}^{\boldsymbol{R}(T)} \langle n,\,\boldsymbol{R}|\nabla_{\boldsymbol{R}}|n,\,\boldsymbol{R}\rangle\,\mathrm{d}\boldsymbol{R}.\end{aligned} \tag{1.145}$$

Since $\boldsymbol{R}(T) = \boldsymbol{R}(0)$, the last expression seems to vanish. However, the integrand is not necessarily a total derivative and $\gamma_n(T)$ may fail to vanish. The phase $\gamma_n(T)$ is called **Berry's phase** (Berry 1984). Simon (1983) pointed out the deep geometrical significance of Berry's phase, see §10.6.

1.8 String theory

QFT is occasionally called particle physics since it deals with the dynamics of particles. As far as high-energy processes whose typical energy is much smaller than the Planck energy ($\sim 10^{19}$ GeV) are concerned there is no objection to this viewpoint. However, once we try to quantise gravity in this framework, there exists an impenetrable barrier. We do not know how to renormalise the ultraviolet divergences that are ubiquitous in the QFT of gravity. In the early 1980s, physicists tried to construct a consistent theory of gravity by introducing supersymmetry. In spite of a partial improvement, the resulting supergravity could not tame the ultraviolet behaviour completely.

In the late 1960s and early 1970s, the dual resonance model was extensively studied as a candidate for a model of hadrons. In this, particles are replaced by one-dimensional objects called **strings**. Unfortunately it turned out that the theory contained tachyons (imaginary mass particles) and spin-2 particles and, moreover, it is consistent only in 26-dimensional spacetime! Due to these difficulties, the theory was abandoned and taken over by QCD. However, a small number of people noticed that the theory must contain the graviton and they thought it could be a candidate for the quantum theory of gravity.

Nowadays, supersymmetry has been built into string theory to form the **superstring theory**, which is free of tachyons and consistent in 10-dimensional spacetime. There are several candidates for consistent superstring theories. It is sometimes suggested that complete mathematical consistency will single out a unique *theory of everything* (TOE).

In this book, we study the elementary aspects of bosonic string theory in the final chapter. We also study some mathematical tools relevant for superstrings. The classical review is that of Scherk (1975). We give more references in Chapter 14.

1.8.1 The string action

The trajectory of a particle in a D-dimensional Minkowski spacetime is given by the set of D functions $X^\mu(\tau)$, $1 \leqslant \mu \leqslant D$, where τ parametrises the trajectory. A string is a one-dimensional object and its configuration is parametrised by two numbers (σ, τ), σ being spacelike and τ timelike. Its position in D-dimensional Minkowski spacetime is given by $X^\mu(\sigma, \tau)$, see figure 1.6. σ can be normalised as $0 \leqslant \sigma \leqslant \pi$. A string may be open or closed. We now seek an action that governs the dynamics of strings.

We first note that the action of a relativistic particle is the *length* of the *world line*,

$$S \equiv m \int_{s_i}^{s_f} \mathrm{d}s = m \int_{\tau_i}^{\tau_f} \mathrm{d}\tau(-\dot{X}^\mu \dot{X}_\mu)^{1/2} \tag{1.146}$$

where $\dot{X}^\mu \equiv \mathrm{d}X^\mu/\mathrm{d}\tau$. For some purposes, it is convenient to take another expression,

$$S = -\frac{1}{2}\int \mathrm{d}\tau\sqrt{g}\,(g^{-1}\dot{X}^\mu \dot{X}_\mu - m^2) \tag{1.147}$$

where the auxiliary variable $g \equiv g_{\tau\tau}$ is regarded as a metric.

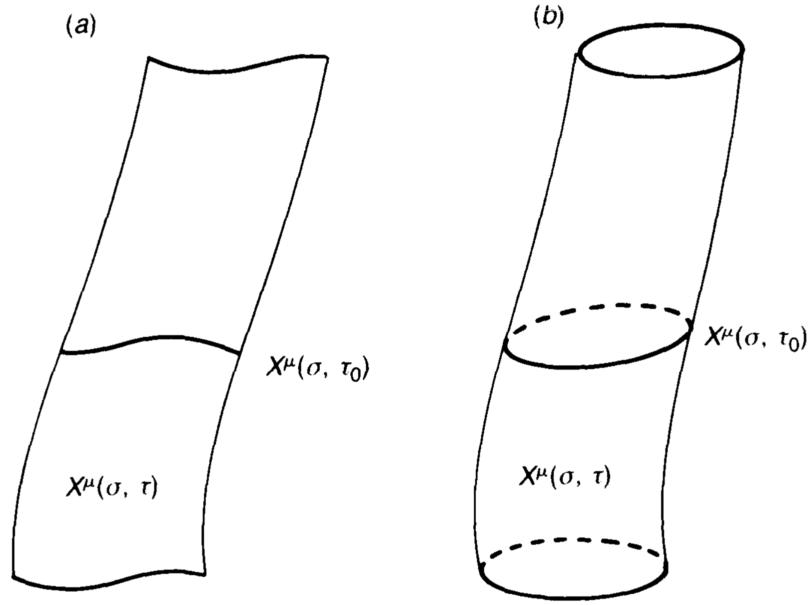


Figure 1.6 The trajectories of an open string (a) and a closed string (b). Slices of the trajectories at fixed parameter τ_0 are also shown.

Exercise 1.11 Write down the Euler–Lagrange equations derived from (1.147). Eliminate g from (1.147) making use of the equation of motion to reproduce (1.146).

What is the advantage of (1.147) over (1.146)? We first note that (1.147) makes sense even when $m^2 = 0$, while (1.146) vanishes in this case. Second, (1.147) is quadratic in X while the X-dependence of (1.146) is rather complicated.

Nambu (1970) proposed an action describing the strings, which is proportional to the *area* of the **world sheet**, the surface spanned by the trajectory of a string. Clearly this is a generalisation of the length of the world line of a particle. He proposed the **Nambu action**,

$$S = -\frac{1}{2\pi\alpha'}\int_0^\pi \mathrm{d}\sigma \int_{\tau_i}^{\tau_f} \mathrm{d}\tau [-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)]^{1/2} \tag{1.148}$$

where $\xi^0 = \tau$, $\xi^1 = \sigma$ and $\partial_\alpha X^\mu \equiv \partial X^\mu / \partial \xi^\alpha$. $\tau_i (\tau_f)$ is the initial (final) value of the parameter τ. α' is a parameter corresponding to the inverse string tension (the Regge slope).

Exercise 1.12 The action S is required to have no dimension. We take σ and τ to be dimensionless. Show that the dimension of α' is [length]^2.

Although the action provides a nice geometrical picture, it is not quadratic in X and it turned out that the quantisation of the theory was rather difficult. Let us seek an equivalent action which is easier to quantise. We proceed analogously to the case of point particles. A quadratic action for strings is called the **Polyakov action** (Polyakov 1981) and is given by

$$S = -\frac{1}{4\pi\alpha'}\int_0^{\pi} \mathrm{d}\sigma \int_{\tau_i}^{\tau_f} \mathrm{d}\tau \sqrt{-g}\, g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{1.149}$$

where $g = \det g_{\alpha\beta}$ and $g^{\alpha\beta} = (g^{-1})^{\alpha\beta}$. If the string is open, the trajectory is a sheet while if it is closed, it is a tube, see figure 1.6. In §7.10, it is shown that the action (1.149) agrees with (1.148) upon eliminating g. It should be noted though that this is true only for the Lagrangian. There is no guarantee that this remains true at the quantum level. It has been shown that the quantum theory based on the respective Lagrangians agree only for $D = 26$. The action (1.149) is invariant under

(*a*) local reparametrisation of the world sheet

$$\tau \to \tau'(\tau, \sigma) \qquad \xi \to \xi'(\tau, \sigma) \tag{1.150a}$$

(*b*) Weyl rescaling

$$g_{\alpha\beta} \to g'_{\alpha\beta} \equiv \mathrm{e}^{\phi(\sigma,\tau)} g_{\alpha\beta} \tag{1.150b}$$

(*c*) global Poincaré invariance

$$X^\mu \to X^{\mu\prime} \equiv \Lambda^\mu{}_\nu X^\nu + a^\mu \qquad \Lambda \in \mathrm{SO}(3, 1) \qquad a \in \mathbb{R}^4. \tag{1.150c}$$

These symmetries will be worked out in §7.10.

Exercise 1.13 Taking advantage of symmetries (*a*) and (*b*), it is always possible to choose $g_{\alpha\beta}$ in the form, $g_{\alpha\beta} = \eta_{\alpha\beta}$. Write down the equation of motion for X^μ to show that it obeys the equation

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\mu = 0.$$

Problems 1

1 Let $\{c_i\}$ be the Grassmann number variable in §1.1. The derivative is defined by

$$\frac{\partial c_j}{\partial c_i} = \delta_{ij} \qquad \frac{\partial a}{\partial c_i} = 0 \qquad a \in \mathbb{R}.$$

Verify that $\partial/\partial c_i = \int \mathrm{d}c_i$. Show also that

$$\mathrm{d}(ac_i) = \frac{1}{a}\,\mathrm{d}c_i \qquad a \in \mathbb{R}.$$

2 Consider a Hamiltonian of the form

$$H = \int \mathrm{d}^n x \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]$$

where $V(\phi)$ ($\geqslant 0$) is a potential. If ϕ is a time-independent classical solution, we can drop the first term and write $H[\phi] = H_1[\phi] + H_2[\phi]$, where

$$H_1[\phi] \equiv \frac{1}{2}\int \mathrm{d}^n x (\nabla\phi)^2 \qquad H_2[\phi] \equiv \int \mathrm{d}^n x V(\phi).$$

(1) Consider a scale transformation $\phi(x) \to \phi(\lambda x)$. Show that $H_i[\phi]$ transforms as

$$H_1[\phi] \to H_1^\lambda[\phi] = \lambda^{n-2} H_1[\phi] \qquad H_2[\phi] \to H_2^\lambda[\phi] = \lambda^{-n} H_2[\phi].$$

(2) Suppose ϕ satisfies the field equation. Show that

$$(2 - n)H_1[\phi] - nH_2[\phi] = 0.$$

[*Hint:* Take the λ-derivative of $H_1^\lambda[\phi] + H_2^\lambda[\phi]$ and put $\lambda = 1$.]

(3) Show that time-independent topological excitations of $H[\phi]$ exist if and only if $n = 1$ (**Derrick's Theorem**). Consider ways out of this restriction.

2

MATHEMATICAL PRELIMINARIES

In the present chapter we introduce elementary concepts in the theory of maps, vector spaces and topology. A modest knowledge of undergraduate mathematics, such as set theory, calculus, complex analysis and linear algebra is assumed.

The main purpose of this book is to study the application of the theory of manifolds to the problems in physics we outlined in the previous chapter. Vector spaces and topology are, in a sense, two extreme viewpoints of manifolds. A manifold is a space which locally looks like $\mathbb{R}^n$ (or $\mathbb{C}^n$) but not necessarily globally. As a first approximation, we may model a small part of a manifold by a Euclidean space $\mathbb{R}^n$ (or $\mathbb{C}^n$) (a small area around a point on a surface can be approximated by the tangent plane at that point); this is the viewpoint of vector space. In topology, on the other hand, we study the manifold as a whole. We want to study the properties of manifolds and classify manifolds using some sort of 'measures'. Topology usually comes with an adjective; algebraic topology, differential topology, combinatorial topology, general topology and so on. These adjectives refer to the measure we use when classifying manifolds.

2.1 Maps

2.1.1 Definitions

Let X and Y be sets. A **map** (or **mapping**) is a rule by which we assign $y \in Y$ for each $x \in X$. We write

$$f: X \to Y. \tag{2.1a}$$

If f is defined by some explicit formula, we may write

$$f: x \mapsto f(x). \tag{2.1b}$$

There may be more than two elements in X that correspond to the same $y \in Y$. A subset of X whose elements are mapped to $y \in Y$ under f is called the **inverse image** of y, denoted by $f^{-1}(y) = \{x \in X | f(x) = y\}$. The set X is called the **domain** of the map while Y is called the **range** of the map. The **image** of the map is $f(X) = \{y \in Y | y = f(x)$ for some $x \in X\} \subset Y$. The image $f(X)$ is also denoted by $\operatorname{im} f$. The reader should note that a map cannot be defined without specifying the domain and the range. Take $f(x) = \exp x$, for example. If both the domain and the range are $\mathbb{R}$, $f(x) = -1$ has no inverse image. If, however, the domain and the range are the complex plane $\mathbb{C}$, we find $f^{-1}(-1) = \{(2n+1)\pi i | n \in \mathbb{Z}\}$. The domain X and the range Y are as important as f itself in specifying a map.

Example 2.1 Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin x$. We also write $f: x \mapsto \sin x$. The domain and the range are $\mathbb{R}$ and the image $f(\mathbb{R})$ is $[-1, 1]$. The inverse image of 0 is $f^{-1}(0) = \{n\pi | n \in \mathbb{Z}\}$. Let us take the same function $f(x) = \sin x = (e^{ix} - e^{-ix})/2i$ but $f: \mathbb{C} \to \mathbb{C}$ this time. The image $f(\mathbb{C})$ is the whole complex plane $\mathbb{C}$.

Definition 2.2 If a map satisfies a certain condition it bears a special name.

(a) A map $f: X \to Y$ is called **injective** (or **one-to-one**) if $x \neq x'$ implies $f(x) \neq f(x')$.

(b) A map $f: X \to Y$ is called **surjective** (or **onto**) if for each $y \in Y$ there exists at least one element $x \in X$ such that $f(x) = y$.

(c) A map $f: X \to Y$ is called **bijective** if it is both injective and surjective.

Example 2.3 A map $f: \mathbb{R} \to \mathbb{R}$ defined by $f: x \to ax$ $(a \in \mathbb{R} - \{0\})$ is bijective. $f: \mathbb{R} \to \mathbb{R}$ defined by $f: x \mapsto x^2$ is neither injective nor surjective. $f: \mathbb{R} \to \mathbb{R}$ given by $f: x \mapsto \exp x$ is injective but not surjective.

Exercise 2.4 A map $f: \mathbb{R} \to \mathbb{R}$ defined by $f: x \mapsto \sin x$ is neither injective nor surjective. Restrict the domain and the range to make f bijective.

Example 2.5 Let M be an element of the general linear group $\mathrm{GL}(n, \mathbb{R})$ whose matrix representation is given by $n \times n$ matrices with non-vanishing determinant. Then $M: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Mx$ is bijective. If $\det M = 0$, it is neither injective nor surjective.

A **constant map** $c: X \to Y$ is defined by $c(x) = y_0$ where y_0 is a fixed element in Y and x is an arbitrary element in X. Given a map $f: X \to Y$, we may think of its **restriction** to $A \subset X$, which is denoted by $f|_A : A \to Y$. Given two maps $f: X \to Y$ and $g: Y \to Z$, the **composite map** of f and g is a map $gf: X \to Z$. A diagram of maps is called **commutative** if any composite maps between a pair of sets do not depend on how they are composed. For example, in figure 2.1, $fg = hj$ and $fg = k$ etc.

Exercise 2.6 Let $f: \mathbb{R} \to \mathbb{R}$, $f: x \mapsto x^2$ and $g: \mathbb{R} \to \mathbb{R}$, $g: x \mapsto \exp x$. What are $gf: \mathbb{R} \to \mathbb{R}$ and $fg: \mathbb{R} \to \mathbb{R}$?

$$\begin{tikzcd} X & Y \arrow[l, "g"'] \\ Z \arrow[u, "j"] & W \arrow[l, "h"] \arrow[u, "f"'] \arrow[ul, "k"'] \end{tikzcd}$$

Figure 2.1 A commutative diagram of maps.

If $A \subset X$, an **inclusion map** $i : A \to X$ is defined by $i(a) = a$ for any $a \in A$. An inclusion map is often written as $i : A \hookrightarrow X$. The **identity map** $\mathrm{id}_X : X \to X$ is a special case of an inclusion map, for which $A = X$. If $f : X \to Y$ defined by $f : x \mapsto f(x)$ is bijective, there exists an **inverse map** $f^{-1} : Y \to X$, $f^{-1} : f(x) \mapsto x$ which is also bijective. The maps f and f^{-1} satisfy $ff^{-1} = \mathrm{id}_Y$ and $f^{-1}f = \mathrm{id}_X$. Conversely, if $f : X \to Y$ and $g : Y \to X$ satisfy $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$, then f and g are bijections. This can be proved from the following exercise.

Exercise 2.7 Show that if $f : X \to Y$ and $g : Y \to X$ satisfy $gf = \mathrm{id}_X$, f is injective and g is surjective. If this is applied to $fg = \mathrm{id}_Y$ as well, we obtain the above result.

Example 2.8 Let $f : \mathbb{R} \to (0, \infty)$ be a bijection defined by $f : x \mapsto \exp x$. Then the inverse map $f^{-1} : (0, \infty) \to \mathbb{R}$ is $f^{-1} : x \mapsto \ln x$. Let $g : (-\pi/2, \pi/2) \to [-1, 1]$ be a bijection defined by $g : x \mapsto \sin x$. The inverse map is $g^{-1} : x \mapsto \sin^{-1} x$.

Exercise 2.9 The n-dimensional Euclidean group E^n is made of an n-dimensional translation $a : x \to x + a (x, a \in \mathbb{R}^n)$ and an $\mathrm{O}(n)$ rotation $R : x \to Rx$, $R \in \mathrm{O}(n)$. A general element (R, a) of E acts on x by $(R, a) : x \to Rx + a$. The product is defined by $(R_2, a_2) \times (R_1, a_1) : x \to R_2(R_1x + a_1) + a_2$, that is, $(R_2, a_2) \cdot (R_1, a_1) = (R_2R_1, R_2a_1 + a_2)$. Show that the maps a, R and (R, a) are bijections. Find their inverse maps.

Suppose certain algebraic structures (product or addition, say) are given to the sets X and Y. If $f : X \to Y$ preserves these algebraic structures, then f is called a **homomorphism**. For example, let X be endowed with a product. If f is a homomorphism, it preserves the product, $f(ab) = f(a)f(b)$. Note that ab is defined by the product rule in X, and $f(a)f(b)$ by that in Y. If a homomorphism f is bijective, f is called an **isomorphism** and X is said to be **isomorphic** to Y, denoted $X \cong Y$.

2.1.2 Equivalence relation and equivalence class

Some most important concepts in mathematics are **equivalence relations** and **equivalence classes**. Although these subjects are not directly related to maps, it is appropriate to define them at this point before we proceed further. A **relation** R defined in a set X is a *subset* of X^2. If a point $(a, b) \in X^2$ is in R, we may write aRb. For example, the relation $>$ is a subset of $\mathbb{R}^2$. If $(a, b) \in >$, then $a > b$.

Definition 2.10 An **equivalence relation** $\sim$ is a relation which satisfies the following requirements:

(i) $a \sim a$ (reflective).

(ii) If $a \sim b$, then $b \sim a$ (symmetric).

(iii) If $a \sim b$ and $b \sim c$, then $a \sim c$ (transitive).

Exercise 2.11 If an integer is divided by 2, the remainder is either 0 or 1. If two integers n and m yield the same remainder, we write $m \sim n$. Show that $\sim$ is an equivalence relation in $\mathbb{Z}$.

Given a set X and an equivalence relation $\sim$, we have a partition of X into *mutually disjoint* subsets called **equivalence classes**. A class $[a]$ is made of all the elements x in X such that $x \sim a$,

$$[a] \equiv \{x \in X | x \sim a\}. \tag{2.2}$$

$[a]$ cannot be empty since $a \sim a$. We now prove that if $[a] \cap [b] \neq \emptyset$ then $[a] = [b]$. First note that $a \sim b$. (Since $[a] \cap [b] \neq \emptyset$ there is at least one element c in $[a] \cap [b]$ that satisfies $c \sim a$ and $c \sim b$. From the transitivity, we have $a \sim b$.) Next we show that $[a] \subset [b]$. Take an arbitrary element a' in $[a]$; $a' \sim a$. Then $a \sim b$ implies $b \sim a'$, that is $a' \in [b]$. Thus we have $[a] \subset [b]$. Similarly $[a] \supset [b]$ can be shown and it follows that $[a] = [b]$. Hence two classes $[a]$ and $[b]$ satisfy either $[a] = [b]$ or $[a] \cap [b] = \emptyset$. In this way a set X is decomposed into *mutually disjoint* equivalence classes. The set of all classes is called the **quotient space** and denoted by $X/\sim$. The element a (or any element in $[a]$) is called the **representative** of a class $[a]$. In exercise 2.11, the equivalence relation $\sim$ divides integers into two classes, even integers and odd integers. We may choose the representative of the even class to be 0, and that of the odd class to be 1. We write the quotient space $\mathbb{Z}/\sim$. $\mathbb{Z}/\sim$ is isomorphic to $\mathbb{Z}_2$, the **cyclic group** of order 2, whose algebra is defined by $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$ and $1 + 1 = 0$. If all integers are divided into equivalence classes according to the remainder of division by n, the quotient space is isomorphic to $\mathbb{Z}_n$, the cyclic group of order n.

Let X be a space in our usual sense. (To be more precise, we need the notion of topological space, which will be defined in §2.3. For the time being we depend on our intuitive notion of 'space'.) Then quotient spaces may be realised as geometrical figures. For example, let x and y be two points in $\mathbb{R}$. Introduce a relation $\sim$ by: $x \sim y$ if $y = x + 2\pi n$, $n \in \mathbb{Z}$. It is easily shown that $\sim$ is an equivalence relation. The class $[x]$ is the set $\{\ldots, x - 2\pi, x, x + 2\pi, \ldots\}$. A number $x \in [0, 2\pi)$ serves as a representative of an equivalence class $[x]$, see figure 2.2(a). Note that 0 and 2π are different points in $\mathbb{R}$ but, according to the equivalence relation, these points are looked upon as the same element in $\mathbb{R}/\sim$. We arrive at the conclusion that the quotient space $\mathbb{R}/\sim$ is the circle $S^1 = \{e^{i\theta} | 0 \leq \theta < 2\pi\}$, see figure 2.2($b$). Note that a point ε is close to a point $2\pi - \varepsilon$ for infinitesimal ε. Certainly this is the case for S^1, where an angle ε is close to an angle $2\pi - \varepsilon$, but not the case for $\mathbb{R}$. The concept of closeness of points is one of the main ingredients of topology.

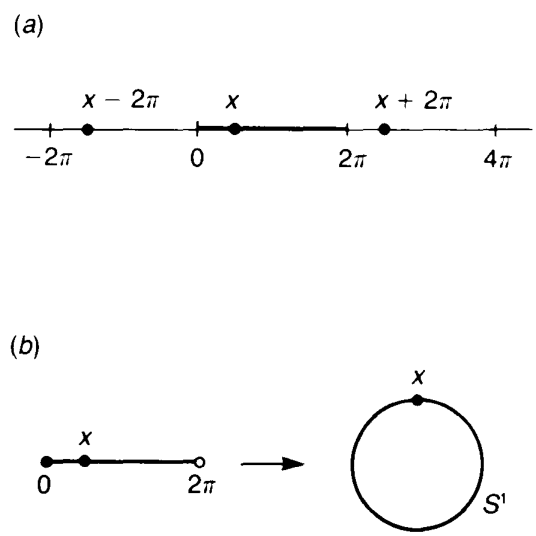


Figure 2.2 In (a) all the points $x + 2n\pi$, $n \in \mathbb{Z}$ are in the same equivalence class $[x]$. We may take $x \in [0, 2\pi)$ as a representative of $[x]$. (b) The quotient space $\mathbb{R}/\sim$ is the circle S^1.

Example 2.12

(a) Let X be a square disc $\{(x, y) \in \mathbb{R}^2 | |x| \leq 1, |y| \leq 1\}$. If we identify the points on a pair of facing edges, $(-1, y) \sim (1, y)$, for example, we obtain the cylinder, see figure 2.3(a). If we identify the points $(-1, -y) \sim (1, y)$, we find the Möbius strip, see figure 2.3(b). [*Remarks:* If readers are not familiar with the Möbius strip, they may take a strip of paper and glue up its ends after a π-twist. Because of the twist, one side of the strip has been joined to the other side, making the surface one sided. The Möbius strip is an example of a **non-orientable** surface, while the cylinder has definite sides and is said to be **orientable**. Orientability will be discussed in terms of differential forms in §5.5.]

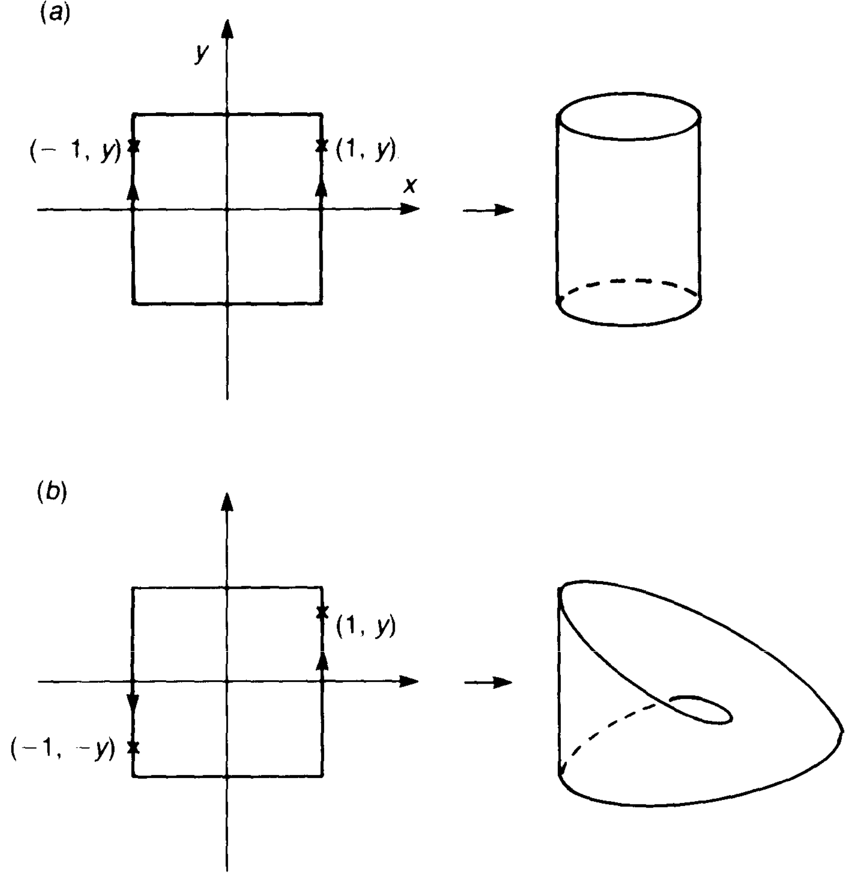


Figure 2.3 (*a*) The edges $|x| = 1$ are identified in the direction of the arrows to form a cylinder. (*b*) If the edges are identified in the opposite direction, we have a Möbius strip.

(*b*) Let (x_1, y_1) and (x_2, y_2) be two points in $\mathbb{R}^2$ and introduce an equivalence relation $\sim$ by; $(x_1, y_1) \sim (x_2, y_2)$ if $x_2 = x_1 + 2\pi n_x$ and $y_2 = y_1 + 2\pi n_y$, n_x, $n_y \in \mathbb{Z}$. Then $\sim$ is an equivalence relation. The quotient space $\mathbb{R}^2/\sim$ is the **torus** T^2 (the surface of a doughnut), see figure 2.4(*a*). Alternatively, T^2 is represented by a rectangle whose edges are identified as in figure 2.4(*b*).

(*c*) What if we identify the edges of a rectangle in other ways? Figure 2.5 gives possible identifications. The spaces obtained by these identifications are called the **Klein bottle**, figure 2.5(*a*), and the **projective plane**, figure 2.5(*b*), neither of which can be realised (or *embedded*) in the Euclidean space $\mathbb{R}^3$ without intersecting with itself. They are known to be non-orientable.

(*d*) Let us identify pairs of edges of the octagon shown in figure 2.6(*a*). The quotient space is the torus with two handles, denoted by T_2, see figure 2.6(*b*). T_g, the torus with g handles, can be obtained by a similar identification, see problem 2.1. The integer g is called the **genus** of the torus.

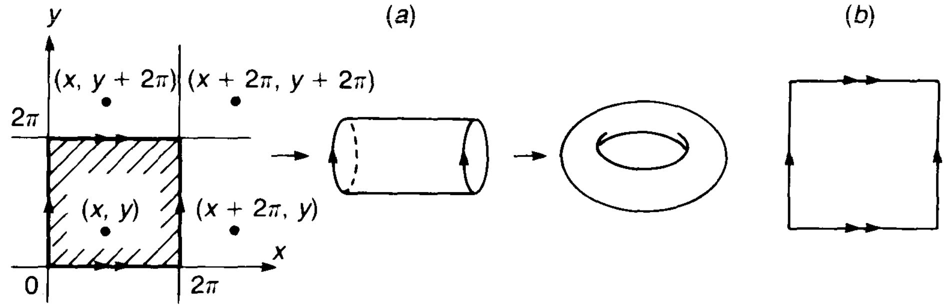


Figure 2.4 If all the points $(x + 2\pi n_x, y + 2\pi n_y)$, $n_x, n_y \in \mathbb{Z}$ are identified, the quotient space is taken to be the shaded area whose edges are identified as in (b). This space is identified with the torus T^2.

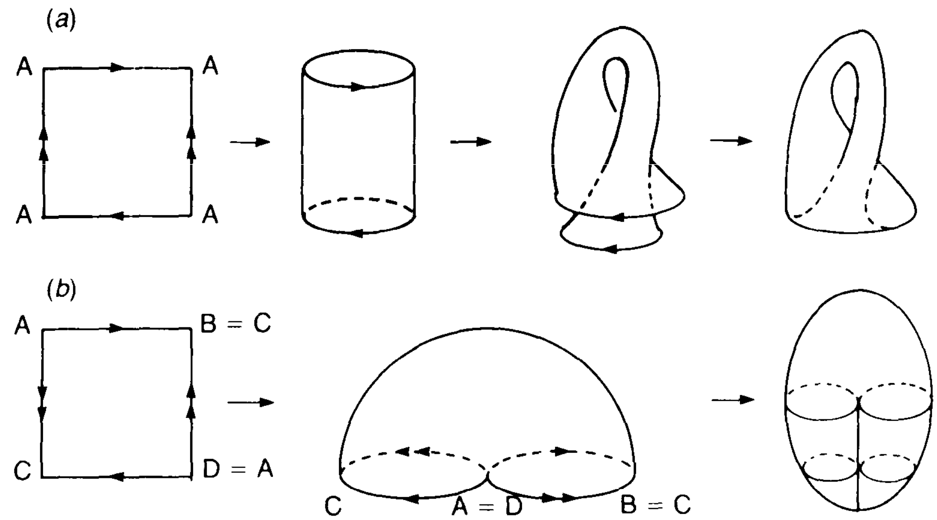


Figure 2.5 The Klein bottle (a) and the projective plane (b).

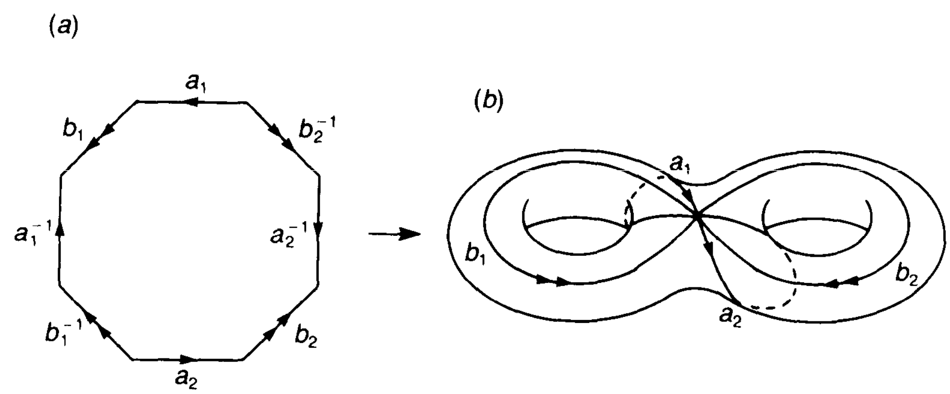


Figure 2.6 If the edges of (a) are identified a torus with two holes (genus two) is obtained.

(e) Let $D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leqslant 1\}$ be a closed disc. Identify the points on the boundary $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$; $(x_1, y_1) \sim (x_2, y_2)$ if

$x_1{}^2 + y_1{}^2 = x_2{}^2 + y_2{}^2 = 1$. Then we obtain the sphere S^2 as the quotient space $D^2/\sim$, also written as D^2/S^1, see figure 2.7. If we take an n-dimensional disc $D^n = \{(x^0, \ldots, x^n) \in \mathbb{R}^{n+1} | (x^0)^2 + \ldots + (x^n) \leqslant 1\}$ and identify the points on the surface S^{n-1}, we obtain the n-sphere S^n, namely $D^n/S^{n-1} = S^n$.

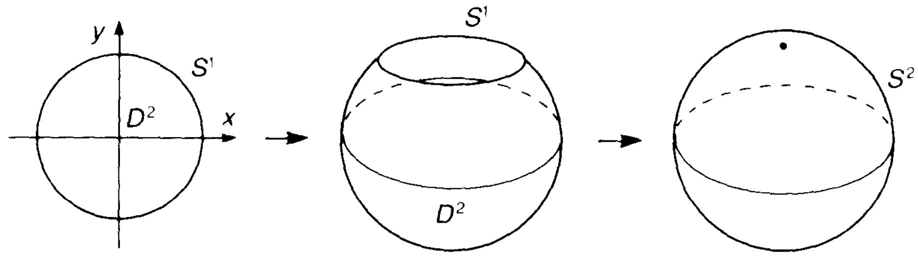


Figure 2.7 A disc D^2 whose boundary is identified is the sphere S^2.

Exercise 2.13 Let H be the upper-half complex plane $\{\tau \in \mathbb{C} | \operatorname{Im} \tau \geqslant 0\}$. Define a group

$$\mathrm{SL}(2, \mathbb{Z}) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - cb = 1 \right\}. \tag{2.3}$$

Introduce a relation $\sim$, for $\tau, \tau' \in H$, by $\tau \sim \tau'$ if there exists a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

such that

$$\tau' = (a\tau + b)/(c\tau + d). \tag{2.4}$$

Show that this is an equivalence relation. (The quotient space $H/\mathrm{SL}(2, \mathbb{Z})$ is shown in figure 8.3.)

Example 2.14 Let G be a group and H a subgroup of G. Let $g, g' \in G$ and introduce an equivalence relation $\sim$ by $g \sim g'$ if there exists $h \in H$ such that $g' = gh$. We denote the equivalence class $[g] = \{gh | h \in H\}$ by gH. The class gH is called a **(left) coset**. gH satisfies either $gH \cap g'H = \varnothing$ or $gH = g'H$. The quotient space is denoted by G/H. In general G/H is *not* a group unless H is a **normal subgroup** of G, that is, $ghg^{-1} \in H$ for any $g \in G$ and $h \in H$. If H is a normal subgroup of G, G/H is called the **quotient group**, whose group operation is given by $(gH)*(g'H) = (gg')H$, where $*$ is the product in G/H.

Exercise 2.15 Let G be a group. Two elements $a, b \in G$ are said to be **conjugate** to each other, denoted by $a \simeq b$, if there exists $g \in G$ such that $b = gag^{-1}$. Show that $\simeq$ is an equivalence relation. The equivalence class $[a] = \{gag^{-1} | g \in G\}$ is called the **conjugacy class**.

2.2 Vector spaces

2.2.1 Vectors and vector spaces

A **vector space** (or a **linear space**) V over a field K is a set in which two operations, addition and multiplication by an element of K (called a **scalar**), are defined. (In this book we are mainly interested in $K = \mathbb{R}$ and $\mathbb{C}$.) The elements (called **vectors**) of V satisfy the following axioms:

(i) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
(ii) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$.
(iii) There exists a zero vector $\boldsymbol{0}$ such that $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$.
(iv) For any $\boldsymbol{u}$, there exists $-\boldsymbol{u}$, such that $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{0}$.
(v) $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$.
(vi) $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$.
(vii) $(cd)\boldsymbol{u} = c(d\boldsymbol{u})$.
(viii) $1\boldsymbol{u} = \boldsymbol{u}$.

Here $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $c, d \in K$ and 1 is the unit element of K.

Let $\{\boldsymbol{v}_i\}$ be a set of k vectors. If the equation

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_k\boldsymbol{v}_k = 0 \tag{2.5}$$

has a non-trivial solution, $x_i \neq 0$ for some i, the set of vectors $\{\boldsymbol{v}_i\}$ is called **linearly dependent**, while if (2.5) has only a trivial solution, $x_i = 0$ for any i, $\{\boldsymbol{v}_i\}$ is said to be **linearly independent**. If at least one of the vectors is a zero vector $\boldsymbol{0}$, the set is always linearly dependent.

A set of linearly independent vectors $\{\boldsymbol{e}_i\}$ is called a **basis** of V, if any element $\boldsymbol{v} \in V$ is written *uniquely* as a linear combination of $\{\boldsymbol{e}_i\}$

$$\boldsymbol{v} = v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + \ldots + v^i\boldsymbol{e}_i + \ldots. \tag{2.6}$$

The numbers $v^i \in K$ are called the **components** of $\boldsymbol{v}$ with respect to the basis $\{\boldsymbol{e}_i\}$. If there are n elements in the basis, the **dimension** of V is n, denoted by $\dim V = n$. We usually write the n-dimensional vector space over K as $V(n, K)$ (or simply V if n and K are known from the context). We assume n is finite.

2.2.2 Linear maps, images and kernels

Given two vector spaces V and W, a map $f: V \to W$ is called a **linear map** if it satisfies $f(a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2) = a_1f(\boldsymbol{v}_1) + a_2f(\boldsymbol{v}_2)$ for any $a_1, a_2 \in K$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$. A linear map is an example of a homomorphism that preserves the vector addition and the scalar multiplication. The **image** of f is $f(V) \subset W$ and the **kernel** of f is $\{\boldsymbol{v} \in V | f(\boldsymbol{v}) = \boldsymbol{0}\}$ and denoted by $\operatorname{im} f$ and $\ker f$ respectively. $\ker f$ cannot be empty since $f(\boldsymbol{0})$ is always $\boldsymbol{0}$. If W is the field K itself, f is called a **linear function**. If f is an isomorphism, V is said to be **isomorphic** to W and vice versa, denoted by $V \cong W$. It follows that $\dim V = \dim W$. In fact all the n-dimensional vector spaces are isomorphic to K^n, and they are regarded as identical vector spaces. The isomorphism between the vector spaces is an element of $\mathrm{GL}(n, K)$

Theorem 2.16 If $f : V \to W$ is a linear map, then

$$\dim V = \dim(\ker f) + \dim(\operatorname{im} f). \tag{2.7}$$

Proof: Since f is a linear map, it is clear that $\ker f$ and $\operatorname{im} f$ are vector spaces. Let the basis of $\ker f$ be $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r\}$ and that of $\operatorname{im} f$ be $\{\boldsymbol{h}'_1, \ldots, \boldsymbol{h}'_s\}$. For each i $(1 \leq i \leq s)$, take $\boldsymbol{h}_i \in V$ such that $f(\boldsymbol{h}_i) = \boldsymbol{h}'_i$ and consider the set of vectors $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r, \boldsymbol{h}_1, \ldots, \boldsymbol{h}_s\}$.

Now we show that these vectors form a linearly independent basis of V. Take an arbitrary vector $\boldsymbol{v} \in V$. Since $f(\boldsymbol{v}) \in \operatorname{im} f$, it can be expanded as $f(\boldsymbol{v}) = c^i\boldsymbol{h}'_i = c^i f(\boldsymbol{h}_i)$. From the linearity of f, it then follows that $f(\boldsymbol{v} - c^i\boldsymbol{h}_i) = \boldsymbol{0}$, that is $\boldsymbol{v} - c^i\boldsymbol{h}_i \in \ker f$. This shows that an arbitrary vector $\boldsymbol{v}$ is a linear combination of $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r, \boldsymbol{h}_1, \ldots, \boldsymbol{h}_s\}$. Thus V is spanned by $r + s$ vectors. Next let us assume $a^i\boldsymbol{g}_i + b^i\boldsymbol{h}_i = \boldsymbol{0}$. Then $\boldsymbol{0} = f(\boldsymbol{0}) = f(a^i\boldsymbol{g}_i + b^i\boldsymbol{h}_i) = b^i f(\boldsymbol{h}_i) = b^i\boldsymbol{h}'_i$, which implies that $b^i = 0$. Then it follows from $a^i\boldsymbol{g}_i = \boldsymbol{0}$ that $a^i = 0$, and the set $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r, \boldsymbol{h}_1, \ldots, \boldsymbol{h}_s\}$ is linearly independent in V. Finally we find $\dim V = r + s = \dim(\ker f) + \dim(\operatorname{im} f)$. ■

[*Remark:* The vector space spanned by $\{\boldsymbol{h}_1, \ldots, \boldsymbol{h}_s\}$ is called the **orthogonal complement** of $\ker f$ and is denoted by $(\ker f)^\perp$.]

Exercise 2.17 Show that a linear map $f : V \to V$ is an isomorphism if and only if $\ker f = \{\boldsymbol{0}\}$.

2.2.3 Dual vector space

Let $f : V \to K$ be a linear function on a vector space $V(n, K)$ over a field K. Let $\{\boldsymbol{e}_i\}$ be a basis and take an arbitrary vector $\boldsymbol{v} = v^1\boldsymbol{e}_1 + \ldots + v^n\boldsymbol{e}_n$. From the linearity of f, we have $f(\boldsymbol{v}) = v^1 f(\boldsymbol{e}_1) + \ldots + v^n f(\boldsymbol{e}_n)$. Thus, if we know $f(\boldsymbol{e}_i)$ for all i, we know the result of the operation of f on *any* vector. It is remarkable that the set of linear functions is made into a vector space, namely a linear combination of two linear functions is also a linear function,

$$(a_1 f_1 + a_2 f_2)(\boldsymbol{v}) = a_1 f_1(\boldsymbol{v}) + a_2 f_2(\boldsymbol{v}). \tag{2.8}$$

This linear space is called the **dual vector space** to $V(n, K)$ and is denoted by $V^*(n, K)$ or simply by V^*. If $\dim V$ is finite, $\dim V^*$ is equal to $\dim V$. Let us introduce a basis $\{e^{*i}\}$ of V^*. Since e^{*i} is a linear function it is completely specified by giving $e^{*i}(\boldsymbol{e}_j)$ for all j. Let us choose the **dual basis**,

$$e^{*i}(\boldsymbol{e}_j) = \delta^i_j. \tag{2.9}$$

Any linear function f, called a **dual vector** in this context, is expanded in terms of $\{e^{*i}\}$,

$$f = f_i e^{*i}. \tag{2.10}$$

The action of f on $\boldsymbol{v}$ is interpreted as an **inner product** between a column vector and a row vector,

$$f(\boldsymbol{v}) = f_i e^{*i}(v^j \boldsymbol{e}_j) = f_i v^j e^{*i}(\boldsymbol{e}_j) = f_i v^i. \tag{2.11}$$

We sometimes use the notation, $\langle\ ,\ \rangle : V^* \times V \to K$ to denote the inner product.

Let V and W be vector spaces with a linear map $f : V \to W$ and let $g : W \to K$ be a linear function on W ($g \in W^*$). It is easy to see that the composite map gf is a linear function on V. Thus f and g give rise to an element $h \in V^*$ defined by

$$h(v) \equiv g(f(v)) \qquad v \in V. \tag{2.12}$$

Given $g \in W^*$, a map $f : V \to W$ has induced a map $h \in V^*$. Accordingly we have an induced map $f^* : W^* \to V^*$ defined by $f^* : g \mapsto h = f^*(g)$, see figure 2.8. The map h is called the **pullback** of g by f^*.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[rr, bend left, "g\cdot f"] & W \arrow[r, "g"] & K \end{tikzcd}$$

$$\begin{tikzcd} V^* & W^* \arrow[l, "f^*"'] \\ g\cdot f \arrow[u, phantom, sloped, "\in"] & g \arrow[l, mapsto] \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Figure 2.8 The pullback of a function g is a function $f{*}g = gf$.

Since $\dim V^* = \dim V$, there exists an isomorphism between V and V^*. However this isomorphism is not canonical; we have to specify an inner product in V to define an isomorphism between V and V^* and vice versa, see the next subsection. The equivalence of a vector space and its dual vector space will appear recurrently in due course.

Exercise 2.18 Suppose $\{\boldsymbol{f}_i\}$ is another basis of V and $\{f^{*i}\}$, the dual basis. In terms of the old basis, $\boldsymbol{f}_i$ is written as $\boldsymbol{f}_i = A_i{}^j \boldsymbol{e}_j$ where $A \in \mathrm{GL}(n, K)$. Show that the dual bases are related by $e^{*i} = f^{*j} A_j{}^i$.

2.2.4 *Inner product and adjoint*

Let $V = V(m, K)$ be a vector space with a basis $\{\boldsymbol{e}_i\}$ and let g be a vector space isomorphism $g : V \to V^*$, where g is an arbitrary element of $\mathrm{GL}(m, K)$. The component representation of g is

$$g : v^j \to g_{ij} v^j. \tag{2.13}$$

Once the isomorphism is given, we may define the **inner product** of two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ by

$$g(\boldsymbol{v}_1, \boldsymbol{v}_2) \equiv \langle g\boldsymbol{v}_1, \boldsymbol{v}_2 \rangle. \tag{2.14}$$

Let us assume that the field K is a real number $\mathbb{R}$, for definiteness. (2.14) has a component expression,

$$g(\boldsymbol{v}_1, \boldsymbol{v}_2) = v_1{}^i g_{ij} v_2{}^j. \tag{2.15}$$

We require that the matrix (g_{ij}) be positive definite so that the inner product $g(\boldsymbol{v}, \boldsymbol{v})$ has the meaning of the squared norm of $\boldsymbol{v}$. We also require that the metric be symmetric, $g_{ij} = g_{ji}$, so that $g(\boldsymbol{v}_1, \boldsymbol{v}_2) = g(\boldsymbol{v}_2, \boldsymbol{v}_1)$.

Next, let $W = W(n, \mathbb{R})$ be a vector space with a basis $\{\boldsymbol{f}_\alpha\}$ and a vector space isomorphism $G : W \to W^*$. Given a map $f : V \to W$, we may define the **adjoint** of f, denoted by $\tilde{f}$, by

$$G(\boldsymbol{w}, f\boldsymbol{v}) = g(\boldsymbol{v}, \tilde{f}\boldsymbol{w}) \tag{2.16a}$$

where $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$. It is easy to see that $(\tilde{f})^{\sim} = f$. The component expression of (2.16a) is

$$w^\alpha G_{\alpha\beta} f^\beta{}_i v^i = v^i g_{ij} \tilde{f}^j{}_\alpha w^\alpha \tag{2.16b}$$

where $f^\beta{}_i$ and $\tilde{f}^j{}_\alpha$ are the matrix representations of f and $\tilde{f}$ respectively. If $g_{ij} = \delta_{ij}$ and $G_{\alpha\beta} = \delta_{\alpha\beta}$, the adjoint $\tilde{f}$ reduces to the transpose f^{t} of the matrix f.

Let us show that $\dim \operatorname{im} f = \dim \operatorname{im} \tilde{f}$. Since (2.16$b$) holds for any $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$, we have $G_{\alpha\beta} f^\beta{}_i = g_{ij} \tilde{f}^j{}_\alpha$, that is,

$$\tilde{f} = g^{-1} f^{\mathrm{t}} G^{\mathrm{t}}. \tag{2.17}$$

Making use of the result of the exercise below, we obtain $\operatorname{rank} f = \operatorname{rank} \tilde{f}$, where the rank of a map is defined by that of the corresponding matrix (note that $g \in \mathrm{GL}(m, \mathbb{R})$ and $G \in \mathrm{GL}(n, \mathbb{R})$). It is obvious that $\dim \operatorname{im} f$ is the rank of a matrix representing the map f and we conclude $\dim \operatorname{im} f = \dim \operatorname{im} \tilde{f}$.

Exercise 2.19 Let $V = V(m, \mathbb{R})$ and $W = W(n, \mathbb{R})$ and let f be a matrix corresponding to a linear map from V to W. Verify that $\operatorname{rank} f = \operatorname{rank} f^{\mathrm{t}} = \operatorname{rank}(Mf^{\mathrm{t}}N)$, where $M \in \mathrm{GL}(m, \mathbb{R})$ and $N \in \mathrm{GL}(n, \mathbb{R})$.

Exercise 2.20 Let V be a vector space over $\mathbb{C}$. The inner product of two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ is defined by

$$g(\boldsymbol{v}_1, \boldsymbol{v}_2) = \bar{v}_1{}^i g_{ij} v_2{}^j \tag{2.18}$$

where $^{-}$ denotes the complex conjugate. From the positivity and symmetry of the inner product, $g(\boldsymbol{v}_1, \boldsymbol{v}_2) = \overline{g(\boldsymbol{v}_2, \boldsymbol{v}_1)}$, the vector space isomorphism $g : V \to V^*$ is required to be a positive-definite Hermitian matrix. Let $f : V \to W$ be a (complex) linear map and $G : W \to W^*$ be a vector space isomorphism. The adjoint of f is defined by $g(\boldsymbol{v}, \tilde{f}\boldsymbol{w}) = \overline{G(\boldsymbol{w}, f\boldsymbol{v})}$. Repeat the analysis above to show that
(a) $\tilde{f} = g^{-1}f^{\dagger}G^{\dagger}$, where † denotes the Hermitian conjugate, and
(b) $\dim \operatorname{im} f = \dim \operatorname{im} \tilde{f}$.

Theorem 2.21 **(toy index theorem)** Let V and W be finite-dimensional vector spaces over a field K and let $f : V \to W$ be a linear map. Then

$$\dim \ker f - \dim \ker \tilde{f} = \dim V - \dim W. \tag{2.19}$$

Proof: Theorem 2.14 tells us that

$$\dim V = \dim \ker f + \dim \operatorname{im} f$$

and, if applied to $\tilde{f} : W \to V$,

$$\dim W = \dim \ker \tilde{f} + \dim \operatorname{im} \tilde{f}.$$

We saw above that $\dim \operatorname{im} f = \dim \operatorname{im} \tilde{f}$, from which we obtain

$$\dim V - \dim \ker f = \dim W - \dim \ker \tilde{f}. \quad \blacksquare$$

Note that in (2.19), each term on the LHS depends on the details of the map f. The RHS says, however, that the *difference* of the two terms is independent of f! This is a finite-dimensional analogue of the index theorems, see Chapter 12.

2.2.5 Tensors

A dual vector is a linear object that maps a vector to a scalar. This may be generalised to multilinear objects called **tensors**, which map several vectors and dual vectors to a scalar. A tensor T of type (p, q) is a multilinear map that maps p dual vectors and q vectors to $\mathbb{R}$,

$$T : \overset{p}{\otimes} V^* \overset{q}{\otimes} V \to \mathbb{R}. \tag{2.20}$$

For example, a tensor of type $(0, 1)$ maps a vector to a real number and is identified with a dual vector. Similarly, a tensor of type $(1, 0)$ is a vector. If ω maps a dual vector and two vectors to a scalar, $\omega : V^* \times V \times V \to \mathbb{R}$, ω is of type $(1, 2)$.

The set of all tensors of type (p, q) is called the tensor space of type (p, q) and denoted by $\mathcal{T}^p_q$. The **tensor product** $\tau = \mu \otimes \nu \in \mathcal{T}^p_q \otimes \mathcal{T}^{p'}_{q'}$ is an element of $\mathcal{T}^{p+p'}_{q+q'}$ defined by

$$\tau(\omega_1, \ldots, \omega_p, \xi_1, \ldots, \xi_{p'}; u_1, \ldots, u_q, v_1, \ldots, v_{q'})$$
$$= \mu(\omega_1, \ldots, \omega_p; u_1, \ldots, u_q)\nu(\xi_1, \ldots, \xi_{p'}; v_1, \ldots, v_{q'}). \tag{2.21}$$

Another operation in a tensor space is the **contraction**, which is a map from a tensor space of type (p, q) to type $(p-1, q-1)$ defined by

$$\tau(\ldots e^{*i} \ldots; \ldots e_i \ldots) \tag{2.22}$$

where $\{e_i\}$ and $\{e^{*i}\}$ are the dual bases.

Exercise 2.22 Let V and W be vector spaces and let $f: V \to W$ be a linear map. Show that f is a tensor of type $(1, 1)$.

2.3 Topological spaces

The most general structure with which we work is a topological space. Physicists often tend to think that all the spaces they deal with are equipped with metrics. However, this is not always the case. In fact, metric spaces form a subset of manifolds and manifolds form a subset of topological spaces.

2.3.1 Definitions

Definition 2.23 Let X be any set and $\mathcal{T} = \{U_i | i \in I\}$ denote a certain collection of subsets of X. The pair $(X, \mathcal{T})$ is a **topological space** if $\mathcal{T}$ satisfies the following requirements.

(i) $\emptyset, X \in \mathcal{T}$.

(ii) If J is any (may be infinite) subcollection of I, the family $\{U_j | j \in J\}$ satisfies $\bigcup_{j\in J} U_j \in \mathcal{T}$.

(iii) If K is any *finite* subcollection of I, the family $\{U_k | k \in K\}$ satisfies $\bigcap_{k\in K} U_k \in \mathcal{T}$.

X alone is often called a topological space. The U_i are called the **open sets** and $\mathcal{T}$ is said to give a **topology** to X.

Example 2.24

(*a*) If X is a set and $\mathcal{T}$ is the collection of *all* the subsets of X, then (i) ~ (iii) are automatically satisfied. This topology is called the **discrete topology**.

(*b*) Let X be a set and $\mathcal{T} = \{\emptyset, X\}$. Clearly $\mathcal{T}$ satisfies (i) ~ (iii). This topology is called the **trivial topology**. In general the discrete topology is too stringent while the trivial topology is too trivial to give any interesting structures on X.

(*c*) Let X be the real line $\mathbb{R}$. All open intervals (a, b) and their unions define a topology called the **usual topology**. a and b may be $-\infty$ and ∞ respectively. Similarly the usual topology in $\mathbb{R}^n$ can be defined. [Take a product $(a_1, b_1) \times \ldots \times (a_n, b_n)$ and their unions . . ..]

Exercise 2.25 In definition 2.23, axioms (ii) and (iii) look somewhat unbalanced. Show that, if we allow infinite intersection in (iii), the usual topology in $\mathbb{R}$ reduces to the discrete topology (thus not very interesting).

A **metric** $d : X \times X \to \mathbb{R}$ is a function that satisfies the conditions:

(i) $d(x, y) = d(y, x)$
(ii) $d(x, y) \geqslant 0$ where the equality holds if and only if $x = y$
(iii) $d(x, y) + d(y, z) \geqslant d(x, z)$

for any $x, y, z \in X$. If X is endowed with a metric d, X is made into a topological space whose open sets are given by 'open discs',

$$U_\varepsilon(x) = \{y \in X | d(x, y) < \varepsilon\} \tag{2.23}$$

and all their possible unions. The topology $\mathcal{T}$ thus defined is called the **metric topology** determined by d. The topological space $(X, \mathcal{T})$ is called a **metric space**. [*Exercise:* Verify that a metric space $(X, \mathcal{T})$ is indeed a topological space.]

Let $(X, \mathcal{T})$ be a topological space and A be any subset of X. Then $\mathcal{T} = \{U_i\}$ induces the **relative topology** in A by $\mathcal{T}' = \{U_i \cap A | U_i \in \mathcal{T}\}$.

Example 2.26 Let $X = \mathbb{R}^{n+1}$ and take the n-sphere S^n,

$$(x^0)^2 + (x^1)^2 + \ldots + (x^n)^2 = 1. \tag{2.24}$$

A topology in S^n may be given by the relative topology induced by the usual topology on $\mathbb{R}^{n+1}$.

2.3.2 Continuous maps

Definition 2.27 Let X and Y be topological spaces. A map $f : X \to Y$ is **continuous** if the *inverse* image of an open set in Y is an open set in X.

This definition is in agreement with our intuitive notion of continuity. For instance, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} -x + 1 & x \leqslant 0 \\ -x + \frac{1}{2} & x > 0. \end{cases} \tag{2.25}$$

We take the usual topology in $\mathbb{R}$, hence any open interval (a, b) is an open set. In the usual calculus, f is said to have a discontinuity at $x = 0$.

For an open set $(3/2, 2) \subset Y$, we find $f^{-1}((3/2, 2)) = (-1, -1/2)$ which is an open set in X. If we take an open set $(1 - 1/4, 1 + 1/4) \subset Y$, however, we find $f^{-1}((1 - 1/4, 1 + 1/4)) = (-1/4, 0]$ which is not an open set in the usual topology.

Exercise 2.28 By taking a continuous function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ as an example, show that the reverse definition, '*a map f is continuous if it maps an open set in X to an open set in Y*', does not work. [*Hint:* Find where $(-\varepsilon, +\varepsilon)$ is mapped to under f.]

2.3.3 *Neighbourhoods and Hausdorff spaces*

Definition 2.29 Suppose $\mathcal{T}$ gives a topology to X. N is a **neighbourhood** of a point $x \in X$ if N is a subset of X and N contains some (at least one) open set U_i to which x belongs. (We do not require that N itself be an open set. If N happens to be an open set in $\mathcal{T}$, it is called an **open neighbourhood**.)

Example 2.30 Take $X = \mathbb{R}$ with the usual topology. $[-1, 1]$ is a neighbourhood of an arbitrary point $x \in (-1, 1)$.

Definition 2.31 A topological space $(X, \mathcal{T})$ is a **Hausdorff space** if, for an arbitrary pair of distinct points x, $x' \in X$, there always exist neighbourhoods U_x of x and $U_{x'}$ of x' such that $U_x \cap U_{x'} = \varnothing$.

Exercise 2.32 Let $X = \{\text{John, Paul, Ringo, George}\}$ and $U_0 = \varnothing$, $U_1 = \{\text{John}\}$, $U_2 = \{\text{John, Paul}\}$, $U_3 = \{\text{John, Paul, Ringo, George}\}$. Show that $\mathcal{T} = \{U_0, U_1, U_2, U_3\}$ gives a topology to X. Show also that $(X, \mathcal{T})$ is not a Hausdorff space.

Unlike the above exercise, almost all spaces that appear in physics satisfy the Hausdorff property. In the rest of the present book we always assume this is the case.

Exercise 2.33 Show that $\mathbb{R}^n$ with the usual topology is a Hausdorff space. Show also that any metric space is a Hausdorff space.

2.3.4 *Closed set*

Let $(X, \mathcal{T})$ be a topological space. A subset A of X is **closed** if its complement in X is an open set, that is $X - A \in \mathcal{T}$. According to the definition, X and $\varnothing$ are both open *and* closed. Consider a set A (either open or closed). The **closure** of A is the smallest closed set that contains A and is denoted by $\bar{A}$. The **interior** of A is the largest open subset of A and is denoted by A°. The **boundary** $b(A)$ of A is the complement of A° in $\bar{A}$; $b(A) = \bar{A} - A^\circ$. An open set is always disjoint from its boundary while a closed set always contains its boundary.

Example 2.34 Take $X = \mathbb{R}$ with the usual topology and take a pair of open intervals $(-\infty, a)$ and (b, ∞) where $a < b$. Since $(-\infty, a) \cup (b, \infty)$ is open under the usual topology, the complement $[a, b]$ is closed. Any closed interval is a closed set under the usual topology. Let $A = (a, b)$, then $\bar{A} = [a, b]$. The boundary $b(A)$ consists of two points $\{a, b\}$. The sets (a, b), $[a, b]$, $(a, b]$, and $[a, b)$ all have the same boundary, closure and interior. In $\mathbb{R}^n$, the product $[a_1, b_1] \times \ldots \times [a_n, b_n]$ is a closed set under the usual topology.

Exercise 2.35 Whether a set $A \subset X$ is open or closed depends on X. Let us take an interval $I = (0, 1)$ in the x axis. Show that I is open in the x-axis $\mathbb{R}$ while it is neither closed nor open in the xy-plane $\mathbb{R}^2$.

2.3.5 *Compactness*

Let $(X, \mathcal{T})$ be a topological space. A family $\{A_i\}$ of subsets of X is called a **covering** of X, if

$$\bigcup_{i \in I} A_i = X.$$

If all the A_i happen to be the open sets of the topology $\mathcal{T}$, the covering is called an **open covering**.

Definition 2.36 Consider a set X and all possible coverings of X. The set X is **compact** if, for every open covering $\{U_i | i \in I\}$, there exists a *finite* subset J of I such that $\{U_j | j \in J\}$ is also a covering of X.

In general, if a set is compact in $\mathbb{R}^n$, it must be bounded. What else is needed? We state the result without the proof.

Theorem 2.37 Let X be a subset of $\mathbb{R}^n$. X is compact if and only if it is *closed* and *bounded*.

Example 2.38

(a) A point is compact.

(b) Take an open interval (a, b) in $\mathbb{R}$ and choose an open covering $U_n = (a, b - 1/n)$, $n \in \mathbb{N}$. Evidently

$$\bigcup_{n \in \mathbb{N}} U_n = (a, b).$$

However, no finite subfamily of $\{U_n\}$ covers (a, b). Thus an open interval (a, b) is non-compact in conformity with theorem 2.37.

(c) S^n in example 2.26 with the relative topology is compact, since it is closed and bounded in $\mathbb{R}^{n+1}$.

The reader might not appreciate the significance of compactness from the definition and the few examples above. It should be noted, however, that some mathematical analyses as well as physics become rather simple on a compact space. For example, let us consider a system of electrons in a solid. If the solid is non-compact with infinite volume, we have to deal with quantum statistical mechanics in an infinite volume. It is known that this is mathematically quite complicated and requires knowledge of the advanced theory of Hilbert spaces. What we usually do is to confine the system in a finite volume V surrounded by hard walls so that the electron wavefunction vanishes at the walls, or to impose periodic boundary conditions on the walls, which amounts to putting the system in a torus, see example 2.12(b). In any case, the system is now put in a compact space. Then we may construct the Fock space whose excitations are labelled by *discrete* indices. Another significance of compactness in physics will be found when we study extended objects such as instantons and Belavin–Polyakov monopoles, see §§1.4 and 1.5. In field theories, we usually assume that the field approaches some asymptotic form corresponding to the vacuum (or one of the vacua) at spatial infinities. Similarly a class of order parameter distributions in which the spatial infinities have a common order parameter is an interesting class to study from various points of view as we shall see later. Since all points at infinity are mapped to a point, we have effectively compactified the non-compact space $\mathbb{R}^n$ to a compact space $S^n = \mathbb{R}^n \cup \{\infty\}$. This procedure is called the **one-point compactification**.

2.3.6 *Connectedness*

Definition 2.39

(a) A topological space X is **connected** if it cannot be written as $X = X_1 \cup X_2$ where X_1 and X_2 are both open and $X_1 \cap X_2 = \emptyset$. Otherwise X is called **disconnected**.

(b) A topological space X is called **arcwise connected** if, for any points $x, y \in X$, there exists a continuous map $f: [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. With a few pathological exceptions, arcwise connectedness is practically equivalent to connectedness.

(c) A loop in a topological space X is a continuous map $f: [0, 1] \to X$ such that $f(0) = f(1)$. If any loop in X can be continuously shrunk to a point, X is called **simply connected**.

Example 2.40

(a) The real line $\mathbb{R}$ is arcwise connected while $\mathbb{R} - \{\mathbf{0}\}$ is not. $\mathbb{R}^n$ ($n \geqslant 2$) is arcwise connected and so is $\mathbb{R}^n - \{\mathbf{0}\}$.

(b) S^n is arcwise connected. The circle S^1 is not simply connected. If $n \geqslant 2$, S^n is simply connected. The n-dimensional torus

$$T^n = \underbrace{S^1 \times S^1 \times \ldots \times S^1}_{n} \ (n \geqslant 2)$$

is arcwise connected but not simply connected.

(c) $\mathbb{R}^2 - \mathbb{R}$ is not arcwise connected. $\mathbb{R}^2 - \{\mathbf{0}\}$ is arcwise connected, but not simply connected. $\mathbb{R}^3 - \{\mathbf{0}\}$ is arcwise connected and simply connected.

2.4 Homeomorphisms and topological invariants

2.4.1 Homeomorphisms

As we mentioned at the beginning of this chapter, the main purpose of topology is to classify spaces. Suppose we have several figures and ask ourselves which are equal and which are different. Since we have not defined what is meant by *equal* or *different*, we may say 'they are all different from each other' or 'they are all the same figures'. Some of the definitions of equivalence are too stringent and some are too loose to produce any sensible classification of the figures or spaces. For example, in elementary geometry, the equivalence of figures is given by congruence, which turns out to be too stringent for our purpose. In topology, we define two figures to be equivalent if it is possible to deform one figure to the other by *continuous deformation*. Namely we introduce the equivalence relation under which geometrical objects are classified according to whether it is possible to deform one object to the other by continuous deformation. To be more mathematical, we need to introduce the following notion of homeomorphism.

Definition 2.41 Let X_1 and X_2 be topological spaces. A map $f: X_1 \to X_2$ is a **homeomorphism** if it is continuous and has an inverse $f^{-1}: X_2 \to X_1$ which is also continuous. If there exists a homeomorphism between X_1 and X_2, X_1 is said to be **homeomorphic** to X_2 and vice versa.

In other words, X_1 is homeomorphic to X_2 if there exist maps $f: X_1 \to X_2$ and $g: X_2 \to X_1$ such that $fg = \mathrm{id}_{X_2}$ and $gf = \mathrm{id}_{X_1}$. It is easy to show that a homeomorphism is an equivalence relation. Reflectivity follows from the choice $f = \mathrm{id}_X$, while symmetry follows since if $f: X_1 \to X_2$ is a homeomorphism so is $f^{-1}: X_2 \to X_1$. Transitivity follows since, if $f: X_1 \to X_2$ and $g: X_2 \to X_3$ are homeomorphisms so is $gf: X_1 \to X_3$. Now we divide all topological spaces into equivalence classes according to whether it is possible to deform one space to the other by a homeomorphism. Intuitively speaking, we suppose the topological spaces are made out of ideal rubber which we can deform at our will. Two topological spaces are homeomorphic to each other if we can deform one to the other *continuously*, that is, without tearing them apart or pasting.

Figure 2.9 shows some examples of homeomorphisms. It seems impossible to deform the left figure in figure 2.9(b) into the right one by continuous deformation. However, this is an artefact of the embedding of these objects in $\mathbb{R}^3$. In fact they are continuously deformable in $\mathbb{R}^4$. [Consider why.] To distinguish one from the other, we have to embed them in S^3, say, and compare the complements of these objects in S^3. This approach is, however, out of the scope of the present book and we will content ourselves with homeomorphisms.

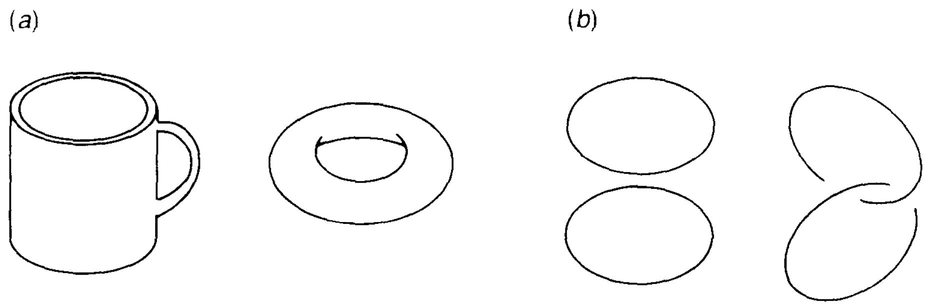


Figure 2.9 (*a*) A coffee cup is homeomorphic to a doughnut. (*b*) The linked rings are homeomorphic to the separated rings.

2.4.2 *Topological invariants*

Now our main question is '*how can we characterise the equivalence classes of homeomorphism?*' In fact we do not know the complete answer to this question yet. Instead, we have a rather modest statement, that is, if two spaces have different '**topological invariants**', they are not homeomorphic to each other. Here topological invariants are those quantities which are conserved under homeomorphisms. A topological invariant may be a number such as the number of connected components of the space, an algebraic structure such as a group or a ring which is constructed out of the space, or something like connectedness, compactness, or the Hausdorff property. (Although it seems to be intuitively clear that these are topological invariants, we have to prove that they indeed are. We omit the proofs. An interested reader may consult any text book on topology.) If we knew the complete set of topological invariants we could specify the equivalence class by giving these invariants. However, so far we know a partial set of topological invariants, which means that even if all the known topological invariants of two topological spaces coincide, they may not be homeomorphic to each other. Instead, what we can say at most is: *if two topological spaces have different topological invariants they cannot be homeomorphic to each other*.

Example 2.42

(*a*) A closed line $[-1, 1]$ is not homeomorphic to an open line $(-1, 1)$, since $[-1, 1]$ is compact while $(-1, 1)$ is not.

(*b*) A circle S^1 is not homeomorphic to $\mathbb{R}$, since S^1 is compact in $\mathbb{R}^2$ while $\mathbb{R}$ is not.

(*c*) A parabola ($y = x^2$) is not homeomorphic to a hyperbola ($x^2 - y^2 = 1$) although they are both non-compact. A parabola is (arcwise) connected while a hyperbola is not.

(*d*) A circle S^1 is not homeomorphic to an interval $[-1, 1]$, although they are both compact and (arcwise) connected. $[-1, 1]$ is simply connected while S^1 is not. Alternatively $S^1 - \{p\}$, p being any point in S^1, is connected while $[-1, 1] - \{0\}$ is not, which is more evidence against their equivalence.

(*e*) Surprisingly, an interval without the end points is homeomorphic to a line $\mathbb{R}$. To see this, let us take $X = (-\pi/2, \pi/2)$ and $Y = \mathbb{R}$ and let $f : X \to Y$ be $f(x) = \tan x$. Since $\tan x$ is one-to-one on X and has an inverse, $\tan^{-1} x$, which is one-to-one on $\mathbb{R}$, this is indeed a homeomorphism. Thus, *boundedness* is not a topological invariant.

(*f*) An open disc $D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ is homeomorphic to $\mathbb{R}^2$. A homeomorphism $f : D^2 \to \mathbb{R}^2$ may be

$$f(x, y) = \left(\frac{x}{(1 - x^2 - y^2)^{1/2}}, \frac{y}{(1 - x^2 - y^2)^{1/2}}\right) \qquad (2.26a)$$

while the inverse $f^{-1} : \mathbb{R}^2 \to \mathrm{D}^2$ is

$$f^{-1}(x, y) = \left(\frac{x}{(1 + x^2 + y^2)^{1/2}}, \frac{y}{(1 + x^2 + y^2)^{1/2}}\right). \qquad (2.26b)$$

The reader should verify that $ff^{-1} = \mathrm{id}_{\mathbb{R}^2}$, and $f^{-1}f = \mathrm{id}_{D^2}$. As we saw in example 2.12(*e*), a closed disc whose boundary S^1 corresponds to a point is homeomorphic to S^2. If we take this point away, we have an open disc. The present analysis shows that this open disc is homeomorphic to $\mathbb{R}^2$. By reversing the order of arguments, we find that if we add a point (infinity) to $\mathbb{R}^2$, we obtain a compact space S^2. This procedure is the one-point compactification $S^2 = \mathbb{R}^2 \cup \{\infty\}$ introduced in the previous section. We similarly have $S^n = \mathbb{R}^n \cup \{\infty\}$.

(*g*) A circle $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ is homeomorphic to a square $I^2 = \{(x, y) \in \mathbb{R}^2 | (|x| = 1, |y| \leq 1), \quad (|x| \leq 1, |y| = 1)\}$. A homeomorphism $f : I^2 \to S^1$ may be given by

$$f(x, y) = (x/r, y/r) \qquad r = (x^2 + y^2)^{1/2}. \qquad (2.27)$$

Since r cannot vanish, (2.27) is invertible.

Exercise 2.43 Find a homeomorphism between a circle $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ and an ellipse $E = \{(x, y) \in \mathbb{R}^2 | (x/a)^2 + (y/b)^2 = 1\}$.

2.4.3 *Homotopy type*

An equivalence class which is somewhat coarser than homeomorphism but which is still quite useful, is 'of the **same homotopy type**'. We relax the conditions in definition 2.41 so that the continuous functions f or g need not have inverses. For example, take $X = (0, 1)$ and $Y = \{0\}$ and let $f : X \to Y$, $f(x) = 0$ and $g : Y \to X$, $g(0) = \frac{1}{2}$. Then $fg = \mathrm{id}_Y$, while $gf \neq \mathrm{id}_X$. This shows that an open interval $(0, 1)$ is of the same homotopy type as a point $\{0\}$, although it is not homeomorphic to $\{0\}$. We have more on this topic in §4.2.

Example 2.44

(*a*) S^1 is of the same homotopy type as a cylinder, since a cylinder is a direct product $S^1 \times \mathbb{R}$ and we can shrink $\mathbb{R}$ to a point at each point of S^1. By the same reason, the Möbius strip is of the same homotopy type as S^1.

(*b*) A disc $D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ is of the same homotopy type as a point. $D^2 - \{(0, 0)\}$ is of the same homotopy type as S^1. Similarly $\mathbb{R}^2 - \{\mathbf{0}\}$ is of the same homotopy type as S^1 and $\mathbb{R}^3 - \{\mathbf{0}\}$ as S^2.

2.4.4 Euler characteristic: an example

The Euler characteristic is one of the most useful topological invariants. Moreover we find the prototype of the algebraic approach to topology in it. To avoid unnecessary complication, we restrict ourselves to points, lines and surfaces in $\mathbb{R}^3$. A **polyhedron** is a geometrical object surrounded by faces. The boundary of two faces is an edge and two edges meet at a vertex. We extend the definition of a polyhedron a bit to include polygons and the boundaries of polygons, lines or points. We call the faces, edges and vertices of a polyhedron **simplexes**. Note that the boundary of two simplexes is either empty or another simplex. (For example, the boundary of two faces is an edge.) Formal definitions of a simplex and a polyhedron in a general number of dimensions will be given in Chapter 3. We are now ready to define the Euler characteristic of a figure in $\mathbb{R}^3$.

Definition 2.45 Let X be a subset of $\mathbb{R}^3$, which is homeomorphic to a polyhedron K. Then the **Euler characteristic** $\chi(X)$ of X is defined by

$$\chi(X) = (\text{number of vertices in } K) - (\text{number of edges in } K) + (\text{number of faces in } K). \tag{2.28}$$

The reader might wonder if $\chi(X)$ depends on the polyhedron K or not. The following theorem due to Poincaré and Alexander guarantees that it is in fact independent of the polyhedron K.

Theorem 2.46 (**Poincaré–Alexander**) The Euler characteristic $\chi(X)$ is independent of the polyhedron K as far as K is homeomorphic to X.

Examples are in order. The Euler characteristic of a point is $\chi(\cdot) = 1$ by definition. The Euler characteristic of a line is $\chi(\text{——}) = 2 - 1 = 1$, since a line has two vertices and an edge. For a triangular disc, we find $\chi(\text{triangle}) = 3 - 3 + 1 = 1$. An example which is a bit non-trivial is the Euler characteristic of S^1. The simplest polyhedron which is homeomorphic to S^1 is made of three edges of a triangle. Then $\chi(S^1) = 3 - 3 = 0$. Similarly, the sphere S^2 is homeomorphic to the surface of a tetrahedron, hence $\chi(S^2) = 4 - 6 + 4 = 2$. It is easily seen that S^2 is also homeomorphic to the surface of a cube. Using a cube to calculate the Euler characteristic of S^2, we have $\chi(S^2) = 8 - 12 + 6 = 2$, in accord with theorem 2.46. Historically this is the conclusion of **Euler's theorem:** if K is any polyhedron homeomorphic to S^2, with v vertices, e edges, and f two-dimensional faces, then $v - e + f = 2$.

Example 2.47 Let us calculate the Euler characteristic of the torus T^2. Figure 2.10(a) is an example of a polyhedron which is homeomorphic to T^2. From this polyhedron, we find $\chi(T^2) = 16 - 32 + 16 = 0$. As we saw in example 2.12(b), T^2 is equivalent to a rectangle whose edges are identified; see figure 2.4. Taking care of this identification, we find an example of a polyhedron made of rectangular faces as in figure 2.10(b), from which we also have $\chi(T^2) = 0$. This approach is quite useful when the figure cannot be realised (embedded) in $\mathbb{R}^3$. For example, the Klein bottle (figure 2.5(a)) cannot be realised in $\mathbb{R}^3$ without intersecting with itself. From the rectangle of figure 2.5(a), we find $\chi(\text{Klein bottle}) = 0$. Similarly, we have $\chi(\text{projective plane}) = 1$.

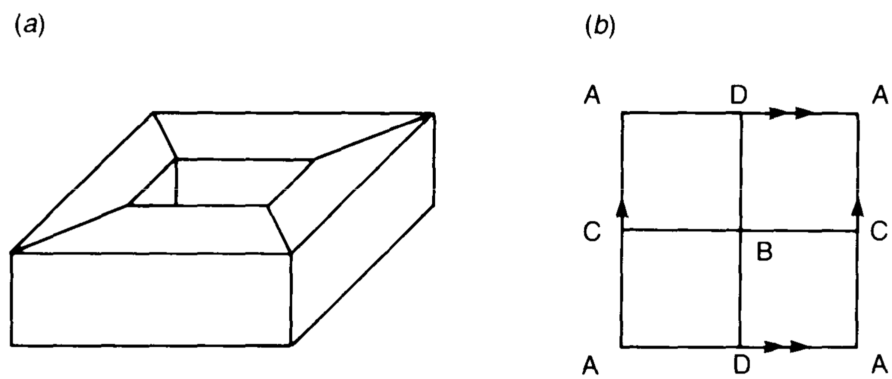


Figure 2.10 Example of a polyhedron which is homeomorphic to a torus.

Exercise 2.48

(a) Show that $\chi(\text{Möbius strip}) = 0$.

(b) Show that $\chi(T_2) = -2$, where T_2 is the torus with two handles (see example 2.12). The reader may either construct a polyhedron homeomorphic to T_2 or make use of the octagon in figure 2.6(*a*). We show below that $\chi(T_g) = 2 - 2g$.

The **connected sum** $X \# Y$ of two surfaces X and Y is a surface obtained by removing a small disc from each of X and Y and connecting the resulting holes with a cylinder; see figure 2.11. Let X be an arbitrary surface. Then it is easy to see that

$$S^2 \# X = X \tag{2.29}$$

since S^2 and the cylinder may be deformed so that they fill in the hole on X; see figure 2.11(*a*). If we take a connected sum of two tori we get (figure 2.11(*b*))

$$T^2 \# T^2 = T_2. \tag{2.30}$$

Similarly T_g may be given by the connected sum of g tori,

$$\underbrace{T^2 \# T^2 \# \ldots \# T^2}_{\ldots\, g \text{ factors} \,\ldots} = T_g. \tag{2.31}$$

The connected sum may be used as a trick to calculate an Euler characteristic of a complicated surface from those of known surfaces. Let us prove the following theorem.

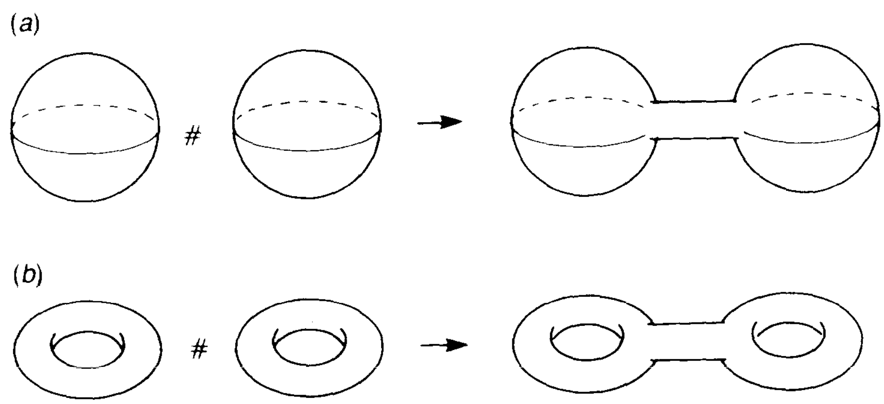


Figure 2.11 The connected sum. (*a*) $S^2 \# S^2 \cong S^2$, (*b*) $T^2 \# T^2 \cong T_2$.

Theorem 2.49 Let X and Y be two surfaces. Then the Euler characteristic of the connected sum $X \# Y$ is given by

$$\chi(X \# Y) = \chi(X) + \chi(Y) - 2. \tag{2.32}$$

Proof: Take polyhedra K_X and K_Y homeomorphic to X and Y, respectively. We assume, without loss of generality, that each of K_X and K_Y has a triangle in it. Remove the triangles from them and connect the resulting holes with a trigonal cylinder. Then the number of vertices does not change while the number of edges increases by three. Since we have removed two faces and added three faces, the number of faces increases by $-2 + 3 = 1$. Thus, the change of the Euler characteristic is $0 - 3 + 1 = -2$. ∎

From the theorem above and the equality $\chi(T^2) = 0$, we obtain $\chi(T_2) = 0 + 0 - 2 = -2$ and $\chi(T_g) = g \times 0 - 2(g - 1) = 2 - 2g$, cf exercise 2.48($b$).

The significance of the Euler characteristic is that it is a topological invariant, which is calculated relatively easily. We accept, without proof, the following theorem.

Theorem 2.50 Let X and Y be two figures in $\mathbb{R}^3$. If X is homeomorphic to Y, then $\chi(X) = \chi(Y)$. In other words, if $\chi(X) \neq \chi(Y)$, X cannot be homeomorphic to Y.

Example 2.51

(a) S^1 is not homeomorphic to S^2, since $\chi(S^1) = 0$ and $\chi(S^2) = 2$.

(b) Two figures, which are not homeomorphic to each other, may have the same Euler characteristic. A point $(\cdot)$ is not homeomorphic to a line (——) but $\chi(\cdot) = \chi(\text{——}) = 1$. This is a general consequence of the following fact: *if a figure X is of the same homotopy type as a figure Y, then $\chi(X) = \chi(Y)$.*

The reader might have noticed that the Euler characteristic is different from other topological invariants such as compactness or connectedness in character. Compactness and connectedness are geometrical properties of a figure or a space while the Euler characteristic is an *integer* $\chi(X) \in \mathbb{Z}$. Note that $\mathbb{Z}$ is an algebraic object rather than a geometrical one. Since the work of Euler, many mathematicians have worked out the relation between geometry and algebra and elaborated this idea, in this century, to establish combinatorial topology and algebraic topology. We may compute the Euler characteristic of a smooth surface by the celebrated Gauss–Bonnet theorem, which relates the integral of the Gauss curvature of the surface with the Euler characteristic calculated from the corresponding polyhedron. We will give the generalised form of the Gauss–Bonnet theorem in Chapter 12.

Problems 2

1 Show that the $4g$-gon in figure 2.12(a), with the boundary identified, represents the torus of genus g of figure 2.12(b). The reader may use (2.31).

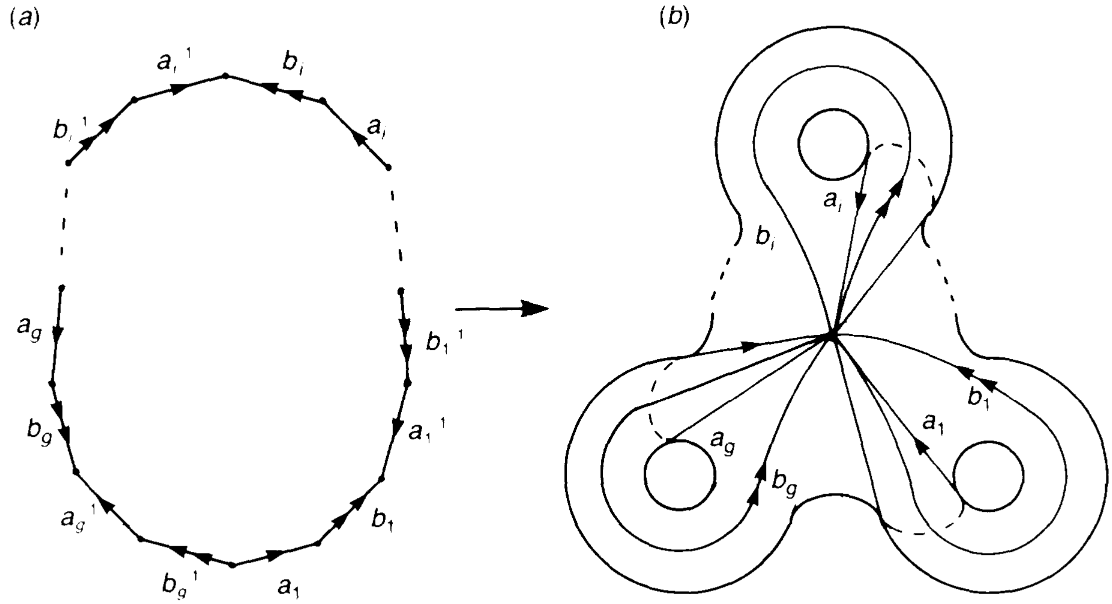


Figure 2.12 The polygon (*a*) whose edges are identified is the torus T_g with genus g.

2 Let $X = \{1, 1/2, \ldots, 1/n, \ldots\}$ be a subset of $\mathbb{R}$. Show that X is not closed in $\mathbb{R}$. Show that $Y = \{1, 1/2, \ldots, 1/n, \ldots, 0\}$ is closed in $\mathbb{R}$, hence compact.

3 Show that there are only five regular polyhedra: a tetrahedron, a hexahedron, an octahedron, a dodecahedron and an icosahedron. [*Hint:* Use Euler's theorem.]

3

HOMOLOGY GROUPS

Among the topological invariants the Euler characteristic is a quantity readily computable by the 'polyhedronisation' of space. The homology groups are *refinements*, so to speak, of the Euler characteristic. Moreover, we can easily read off the Euler characteristic from the homology groups. Let us look at figure 3.1. In figure 3.1(*a*), the interior is included but not in figure 3.1(*b*). How do we characterise this difference? An obvious observation is that the three edges of figure 3.1(*a*) form a boundary of the interior while the edges of figure 3.1(*b*) do not (the interior is *not* a part of figure 3.1(*b*)). Clearly the edges in both cases form a closed path (loop), having no boundary. In other words, the existence of a loop that is not a boundary of some area implies the existence of a hole within the loop. This is our guiding principle in classifying spaces here: *find a region without boundaries, which is not itself a boundary of some region*. This principle is mathematically elaborated into the theory of homology groups.

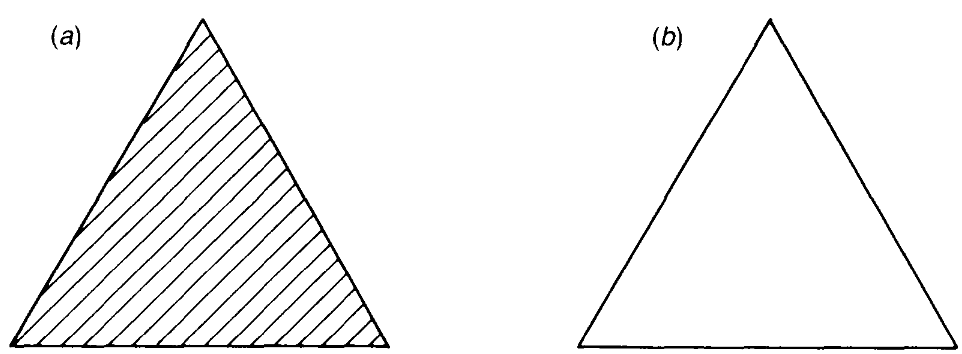


Figure 3.1 (*a*) is a solid triangle while (*b*) is the edges of a triangle without an interior.

Our exposition follows Armstrong (1983), Croom (1978) and Nash and Sen (1983). An introduction to group theory is found in Fraleigh (1976).

3.1 Abelian groups

The mathematical structures underlying homology groups are *finitely generated Abelian groups*. Throughout this chapter, the group operation is denoted by + since all the groups considered here are Abelian (commutative). The unit element is denoted by 0.

3.1.1 *Elementary group theory*

Let G_1 and G_2 be Abelian groups. A map $f: G_1 \to G_2$ is said to be a **homomorphism** if

$$f(x + y) = f(x) + f(y) \tag{3.1}$$

for any $x, y \in G_1$. If f is also a *bijection*, f is called an **isomorphism**. If there exists an isomorphism $f: G_1 \to G_2$, G_1 is said to be **isomorphic** to G_2, denoted by $G_1 \cong G_2$. For example, a map $f: \mathbb{Z} \to \mathbb{Z}_2 = \{0, 1\}$ defined by

$$f(2n) = 0 \qquad f(2n + 1) = 1$$

is a homomorphism. Indeed

$$f(2n + 2m) = f(2(m + n)) = 0 = 0 + 0 = f(2n) + f(2m)$$

$$f(2n + 1 + 2m + 1) = f(2(n + m + 1)) = 0 = 1 + 1$$
$$= f(2n + 1) + f(2m + 1)$$

$$f(2n + 1 + 2m) = f(2(m + n) + 1) = 1 = 1 + 0 = f(2n + 1) + f(2m).$$

A subset $H \subset G$ is a subgroup if it is a group with respect to the group operation of G. For example,

$$k\mathbb{Z} \equiv \{kn | n \in \mathbb{Z}\} \qquad k \in \mathbb{N}$$

is a subgroup of $\mathbb{Z}$, while $\mathbb{Z}_2 = \{0, 1\}$ is not.

Let H be a subgroup of G. We say $x, y \in G$ are equivalent if

$$x - y \in H \tag{3.2}$$

and write $x \sim y$. Clearly $\sim$ is an equivalence relation. The equivalence class to which x belongs is denoted by $[x]$. Let G/H be the quotient space. The group operation + in G naturally induces the group operation + in G/H by

$$[x] + [y] = [x + y]. \tag{3.3}$$

Note that + on the LHS is an operation in G/H while + on the RHS is that in G. The operation in G/H should be independent of the choice of representatives. In fact, if $[x'] = [x]$, $[y'] = [y]$, then $x - x' = h$, $y - y' = g$ for some $h, g \in H$ and we find

$$x' + y' = x + y - (h + g) \in [x + y].$$

Furthermore G/H becomes a group with this operation, since H is a normal subgroup of G; see example 2.14. The unit element of G/H is $[0] = [h]$, $h \in H$. If $H = G$, $0 - x \in G$ for any $x \in G$ and G/G has just one element $[0]$. If $H = \{0\}$, G/H is G itself since $x - y = 0$ if and only if $x = y$.

Example 3.1 Let us work out the quotient group $\mathbb{Z}/2\mathbb{Z}$. For even numbers we have $2n - 2m = 2(n - m) \in 2\mathbb{Z}$ and $[2m] = [2n]$. For odd numbers $(2n + 1) - (2m + 1) = 2(n - m) \in 2\mathbb{Z}$ and $[2m + 1] = [2n + 1]$. Even numbers and odd numbers do not belong to the same equivalence class since $2n - (2m + 1) \notin 2\mathbb{Z}$. Thus it follows that

$$\mathbb{Z}/2\mathbb{Z} = \{[0], [1]\}. \tag{3.4}$$

If we define an isomorphism $\varphi : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}_2$ by $\varphi([0]) = 0$ and $\varphi([1]) = 1$, we find $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$. For general $k \in \mathbb{N}$, we have

$$\mathbb{Z}/k\mathbb{Z} \cong \mathbb{Z}_k. \tag{3.5}$$

Lemma 3.2 Let $f : G_1 \to G_2$ be a homomorphism. Then

(*a*) $\ker f = \{x | x \in G_1, f(x) = 0\}$ is a subgroup of G_1

(*b*) $\operatorname{im} f = \{x | x \in f(G_1) \subset G_2\}$ is a subgroup of G_2.

Proof: (*a*) Let x, $y \in \ker f$. Then $x + y \in \ker f$ since $f(x + y) = f(x) + f(y) = 0 + 0 = 0$. $0 \in \ker f$ for $f(0) = f(0) + f(0)$. We also have $-x \in \ker f$ since $f(0) = f(x - x) = f(x) + f(-x) = 0$.

(*b*) Let $y_1 = f(x_1)$, $y_2 = f(x_2) \in \operatorname{im} f$ where $x_1, x_2 \in G_1$. Since f is a homomorphism we have $y_1 + y_2 = f(x_1) + f(x_2) = f(x_1 + x_2) \in \operatorname{im} f$. Clearly $0 \in \operatorname{im} f$ since $f(0) = 0$. If $y = f(x)$, $-y \in \operatorname{im} f$ since $0 = f(x - x) = f(x) + f(-x)$ implies $f(-x) = -y$. ■

Theorem 3.3 **(fundamental theorem of homomorphism)** Let $f : G_1 \to G_2$ be a homomorphism. Then

$$G_1/\ker f \cong \operatorname{im} f. \tag{3.6}$$

Proof: Both sides are groups according to lemma 3.2. Define a map $\varphi : G_1/\ker f \to \operatorname{im} f$ by $\varphi([x]) = f(x)$. This map is well defined since for $x' \in [x]$, there exists $h \in \ker f$ such that $x' = x + h$ and $f(x') = f(x + h) = f(x) + f(h) = f(x)$. Now we show that φ is an isomorphism. Firstly, φ is a homomorphism,

$$\begin{aligned}\varphi([x] + [y]) &= \varphi([x + y]) = f(x + y)\\ &= f(x) + f(y) = \varphi([x]) + \varphi([y]).\end{aligned}$$

Secondly, φ is one-to-one; if $\varphi([x]) = \varphi([y])$, then $f(x) = f(y)$ or $f(x) - f(y) = f(x - y) = 0$. This shows that $x - y \in \ker f$ and $[x] = [y]$. Finally, φ is onto; if $y \in \operatorname{im} f$, there exists $x \in G_1$ such that $f(x) = y = \varphi([x])$. ■

Example 3.4 Let $f : \mathbb{Z} \to \mathbb{Z}_2$ be defined by $f(2n) = 0$, $f(2n+1) = 1$. Then $\ker f = 2\mathbb{Z}$ and $\operatorname{im} f = \mathbb{Z}_2$ are groups. Theorem 3.3 states that $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$, in agreement with example 3.1.

3.1.2 Finitely generated Abelian groups and free Abelian groups

Let x be an element of a group G. For $n \in \mathbb{Z}$, nx denotes

$$\underbrace{x + x + \ldots + x}_{n} \qquad \text{if } n > 0$$

and

$$\underbrace{(-x) + \ldots + (-x)}_{|n|} \qquad \text{if } n < 0.$$

If $n = 0$, we put $0x = 0$. Take r elements $x_1, \ldots, x_r$ of G. The elements of G of the form

$$n_1x_1 + \ldots + n_rx_r \qquad (n_i \in \mathbb{Z},\ 1 \leqslant i \leqslant r) \tag{3.7}$$

form a subgroup of G, which we denote H. H is called a subgroup of G **generated** by the **generators** $x_1, \ldots, x_r$. If G itself is generated by *finite* elements $x_1, \ldots, x_r$, G is said to be **finitely generated**. If $n_1x_1 + \ldots + n_rx_r = 0$ is satisfied only when $n_1 = \ldots = n_r = 0$, $x_1, \ldots, x_r$ are said to be **linearly independent**.

Definition 3.5 If G is finitely generated by r *linearly independent* elements, G is called a **free Abelian group** of **rank** r.

Example 3.6 $\mathbb{Z}$ is a free Abelian group of rank 1 finitely generated by 1 (or -1). Let $\mathbb{Z} \oplus \mathbb{Z}$ be the set of pairs $\{(i, j) | i, j \in \mathbb{Z}\}$. It is a free Abelian group of rank 2 finitely generated by generators $(1, 0)$ and $(0, 1)$. More generally

$$\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{r}$$

is a free Abelian group of rank r. $\mathbb{Z}_2 = \{0, 1\}$ is finitely generated by 1 but is *not* free since 1 is not linearly independent (note $1 + 1 = 0$).

3.1.3 Cyclic groups

If G is generated by one element x, $G = \{0, \pm x, \pm 2x, \ldots\}$, G is called a **cyclic group**. If $nx \neq 0$ for any $n \in \mathbb{Z} - \{0\}$, it is an **infinite cyclic group** while if $nx = 0$ for some $n \in \mathbb{Z} - \{0\}$, a **finite cyclic group**. Let G be a cyclic group generated by x and let $f : \mathbb{Z} \to G$ be a homomorphism defined by $f(i) = ix$. f maps $\mathbb{Z}$ onto G but not necessarily one-to-one.

From theorem 3.3, we have $G \cong \mathbb{Z}/\ker f$. Let N be the smallest positive integer such that $Nx = 0$. Clearly

$$\ker f = \{0, \pm N, \pm 2N, \ldots\} = N\mathbb{Z} \tag{3.8}$$

and we have

$$G \cong \mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}_N. \tag{3.9}$$

If G is an infinite cyclic group, then $\ker f = \{0\}$ and $G \cong \mathbb{Z}$. Any infinite cyclic group is isomorphic to $\mathbb{Z}$ while a finite cyclic group is isomorphic to some $\mathbb{Z}_N$.

We will need the following lemma and theorem in due course. We first state the lemma without proof.

Lemma 3.7 Let G be a free Abelian group of rank r and let H $(\neq \varnothing)$ be a subgroup of G. We may always choose p generators $x_1, \ldots, x_p$ out of r generators of G so that $k_1x_1, \ldots, k_px_p$ generate H. Thus $H \cong k_1\mathbb{Z} \oplus \ldots \oplus k_p\mathbb{Z}$ and H is of rank p.

Theorem 3.8 **(fundamental theorem of finitely generated Abelian groups)** Let G be a finitely generated Abelian group (not necessarily free) with m generators. Then G is isomorphic to the direct sum of cyclic groups,

$$G \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{r} \oplus \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_p} \tag{3.10}$$

where $m = r + p$. The number r is called the **rank** of G.

Proof: Let G be generated by m elements $x_1, \ldots, x_m$ and let

$$f : \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m} \to G$$

be a surjective homomorphism,

$$f(n_1, \ldots, n_m) = n_1x_1 + \ldots + n_mx_m.$$

Theorem 3.3 states that

$$\underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m}/\ker f \cong G.$$

Since $\ker f$ is a subgroup of

$$\underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m}$$

lemma 3.7 claims that if we take $x_1, \ldots, x_m$ properly, we have

$$\ker f \cong k_1\mathbb{Z} \oplus \ldots \oplus k_p\mathbb{Z}.$$

We finally obtain

$$G \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m} / \ker f \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m} / k_1\mathbb{Z} \oplus \ldots \oplus k_p\mathbb{Z}$$

$$\cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{m-p} \oplus \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_p}. \blacksquare$$

3.2 Simplexes and simplicial complexes

Let us recall how the Euler characteristic of a surface is calculated. We first construct a polyhedron homeomorphic to the given surface, then count the numbers of vertices, edges and faces. The Euler characteristic of the polyhedron, and hence of the surface, is then given by (2.28). We abstract this procedure so that we may represent each part of a figure by some *standard* object. We take triangles and their analogues in other dimensions, called simplexes, as the standard objects. By this standardisation, it becomes possible to assign to each figure Abelian group structures.

3.2.1 Simplexes

Simplexes are building blocks of a polyhedron. A 0-simplex $\langle p_0 \rangle$ is a point, or a vertex, and a 1-simplex $\langle p_0 p_1 \rangle$ is a line, or an edge. A 2-simplex $\langle p_0 p_1 p_2 \rangle$ is defined to be a triangle with its interior included and a 3-simplex $\langle p_0 p_1 p_2 p_3 \rangle$ is a solid tetrahedron (figure 3.2). It is common to denote a 0-simplex without the bracket; $\langle p_0 \rangle$ may be written as p_0. It is easy to continue this construction to any r-simplex $\langle p_0 p_1 \ldots p_r \rangle$. Note that for an r-simplex to represent an r-dimensional object, the vertices p_i must be *geometrically independent*, that is, no $(r-1)$-dimensional hyperplane contains all the $r+1$ points. Let $p_0, \ldots, p_r$ be points geometrically independent in $\mathbb{R}^m$ where $m \geqslant r$. The r-simplex $\sigma_r = \langle p_0 \ldots p_r \rangle$ is expressed as

$$\sigma_r = \left\{ x \in \mathbb{R}^m \middle| x = \sum_{i=0}^{r} c_i p_i, \ c_i \geqslant 0, \ \sum_{i=0}^{r} c_i = 1 \right\}. \tag{3.11}$$

$(c_0, \ldots, c_r)$ is called the **barycentric coordinate** of x. Since σ_r is a bounded closed subset of $\mathbb{R}^m$, it is compact.

Let q be an integer with $0 \leqslant q \leqslant r$. If we choose $q+1$ points $p_{i_0}, \ldots, p_{i_q}$ out of $p_0, \ldots, p_r$, these $q+1$ points define a q-simplex $\sigma_q = \langle p_{i_0} \ldots p_{i_q} \rangle$, which is called a ***q*-face** of σ_r. If σ_q is a face of σ_r, we write $\sigma_q \leqslant \sigma_r$. If $\sigma_q \neq \sigma_r$, we say σ_q is a **proper face** of σ_r, denoted as $\sigma_q < \sigma_r$. Figure 3.3 shows a 0-face p_0 and a 2-face $\langle p_1 p_2 p_3 \rangle$ of a

3-simplex $\langle p_0p_1p_2p_3\rangle$. There are one 3-face, four 2-faces, six 1-faces and four 0-faces. The reader should verify that the number of q-faces in an r-simplex is $\binom{r+1}{q+1}$. A 0-simplex is defined to have no proper faces.

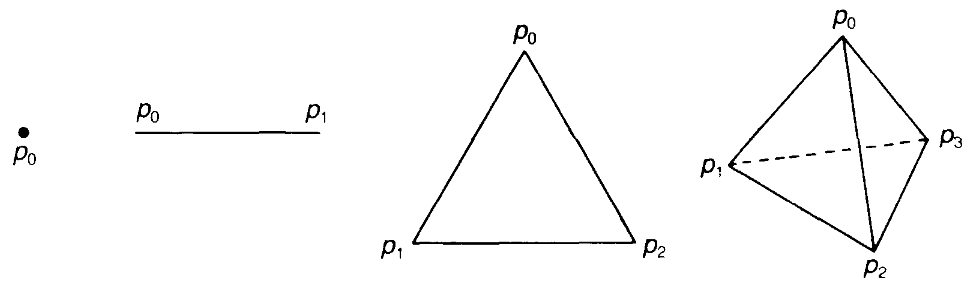


Figure 3.2 0-, 1-, 2- and 3-simplexes.

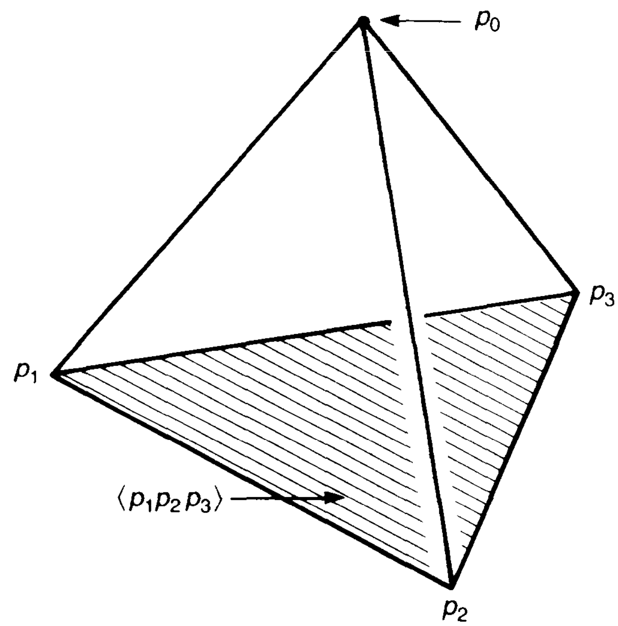


Figure 3.3 A 0-face p_0 and a 2-face $\langle p_1, p_2, p_3\rangle$ of a 3-simplex $\langle p_0p_1p_2p_3\rangle$.

3.2.2 Simplicial complexes and polyhedra

Let K be a set of finite number of simplexes in $\mathbb{R}^m$. If these simplexes are *nicely* fitted together, K is called a **simplicial complex**. By 'nicely' we mean that:

(i) An arbitrary face of a simplex of K belongs to K, that is, if $\sigma \in K$ and $\sigma' \leqslant \sigma$ then $\sigma' \in K$.

(ii) If σ and σ' are two simplexes of K, the intersection $\sigma \cap \sigma'$ is either empty or a *face* of σ and σ', that is, if $\sigma, \sigma' \in K$ then either $\sigma \cap \sigma' = \varnothing$ or $\sigma \cap \sigma' \leqslant \sigma$ and $\sigma \cap \sigma' \leqslant \sigma'$.

For example, figure 3.4(a) is a simplicial complex but figure 3.4(b) is not. The dimension of a simplicial complex K is defined to be the maximum dimension of simplexes in K.

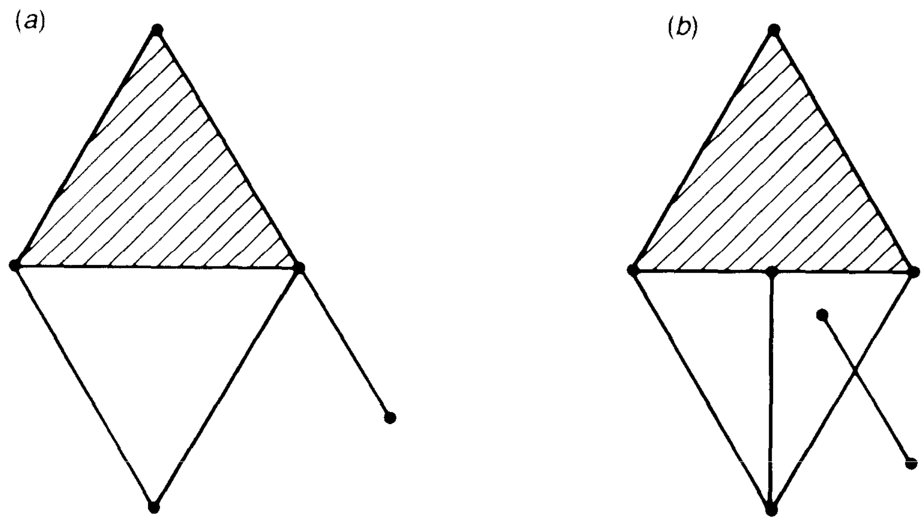


Figure 3.4 (a) is a simplicial complex but (b) is not.

Example 3.9 Let σ_r be an r-simplex and $K = \{\sigma' | \sigma' \leqslant \sigma_r\}$ be the set of faces of σ_r. K is an r-dimensional simplicial complex. For example, take $\sigma_3 = \langle p_0 p_1 p_2 p_3 \rangle$ (figure 3.3). Then

$$\begin{aligned} K = \{ & p_0, p_1, p_2, p_3, \langle p_0 p_1 \rangle, \langle p_0 p_2 \rangle, \langle p_0 p_3 \rangle, \\ & \langle p_1 p_2 \rangle, \langle p_1 p_3 \rangle, \langle p_2 p_3 \rangle, \langle p_0 p_1 p_2 \rangle, \langle p_0 p_1 p_3 \rangle, \\ & \langle p_0 p_2 p_3 \rangle, \langle p_1 p_2 p_3 \rangle, \langle p_0 p_1 p_2 p_3 \rangle \}. \end{aligned} \tag{3.12}$$

A simplicial complex K is a *set* whose elements are simplexes. If each simplex is regarded as a subset of $\mathbb{R}^m$ ($m \geqslant \dim K$), the union of all the simplexes becomes a subset of $\mathbb{R}^m$. This subset is called the **polyhedron** $|K|$ of a simplicial complex K. The dimension of $|K|$ as a subset of $\mathbb{R}^m$ is the same as that of K; $\dim |K| = \dim K$.

Let X be a topological space. If there exists a simplicial complex K and a homeomorphism $f : |K| \to X$, X is said to be **triangulable** and the pair (K, f) is called a **triangulation** of X. Given a topological space X, its triangulation is far from unique. We will be concerned with triangulable spaces only.

Example 3.10 Figure 3.5(a) is a triangulation of a cylinder $S^1 \times [0, 1]$. The reader might think that somewhat economical choices exist, figure 3.5(b), for example. This is, however, *not* a triangulation since, for $\sigma_2 = \langle p_0 p_1 p_2 \rangle$ and $\sigma_2' = \langle p_2 p_3 p_0 \rangle$, we find $\sigma_2 \cap \sigma_2' = \langle p_0 \rangle \cup \langle p_2 \rangle$, which is neither empty nor a simplex.

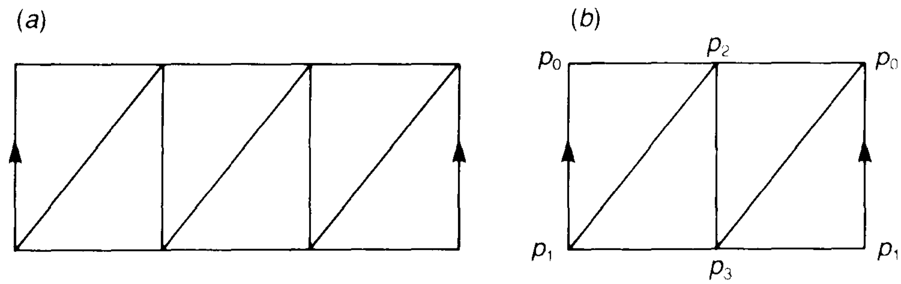


Figure 3.5 (*a*) is a triangulation of a cylinder while (*b*) is not.

3.3 Homology groups of simplicial complexes

3.3.1 Oriented simplexes

We may assign *orientations* to an r-simplex for $r \geqslant 1$. Instead of $\langle \ldots \rangle$ for an unoriented simplex, we will use $(\ldots)$ to denote an oriented simplex. The symbol σ_r is used to denote both types of simplexes. An oriented 1-simplex $\sigma_1 = (p_0 p_1)$ is a directed line segment traversed in the direction $p_0 \rightarrow p_1$ (figure 3.6(*a*)). Now $(p_0 p_1)$ should be distinguished from $(p_1 p_0)$. We require that

$$(p_0 p_1) = -(p_1 p_0). \tag{3.13}$$

Here '$-$' in front of $(p_1 p_0)$ should be understood in the sense of a finitely generated Abelian group. In fact, $(p_1 p_0)$ is regarded as the *inverse* of $(p_0 p_1)$. Going from p_0 to p_1 followed by going from p_1 to p_0 means going nowhere, $(p_0 p_1) + (p_1 p_0) = 0$, hence $-(p_1 p_0) = (p_0 p_1)$.

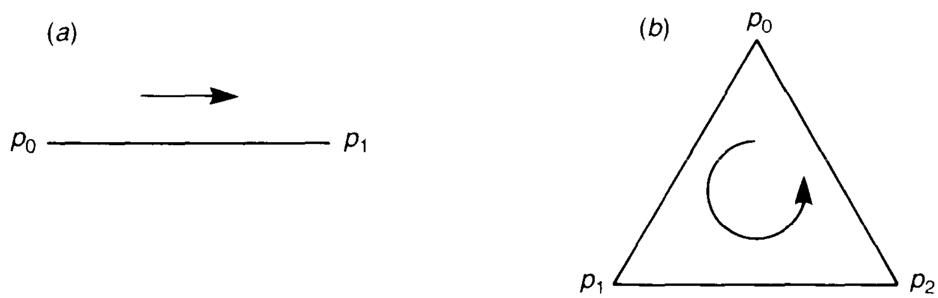


Figure 3.6 An oriented 1-simplex (*a*) and an oriented 2-simplex (*b*).

Similarly, an oriented 2-simplex $\sigma_2 = (p_0 p_1 p_2)$ is a triangular region $p_0 p_1 p_2$ with a prescribed orientation along the edges (figure 3.6(*b*)). Observe that the orientation given by $p_0 p_1 p_2$ is the same as that given by $p_2 p_0 p_1$ or $p_1 p_2 p_0$, but opposite to $p_0 p_2 p_1$, $p_2 p_1 p_0$ or $p_1 p_0 p_2$. We require that

$$(p_0p_1p_2) = (p_2p_0p_1) = (p_1p_2p_0)$$
$$= -(p_0p_2p_1) = -(p_2p_1p_0) = -(p_1p_0p_2).$$

Let P be a permutation of 0, 1, 2;

$$P = \begin{pmatrix} 0 & 1 & 2 \\ i & j & k \end{pmatrix}.$$

The above relations are summarised as

$$(p_ip_jp_k) = \operatorname{sgn}(P)(p_0p_1p_2)$$

where $\operatorname{sgn}(P) = +1\ (-1)$ if P is an even (odd) permutation.

An oriented 3-simplex $\sigma_3 = (p_0p_1p_2p_3)$ is an ordered sequence of four vertices of a tetrahedron. Let

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 \\ i & j & k & l \end{pmatrix}$$

be a permutation. We define

$$(p_ip_jp_kp_l) = \operatorname{sgn}(P)(p_0p_1p_2p_3).$$

It is now easy to construct an oriented r-simplex for any $r \geqslant 1$. The formal definition goes as follows. Take $r + 1$ geometrically independent points $p_0, p_1, \ldots, p_r$ in $\mathbb{R}^m$. Let $\{p_{i_0}, p_{i_1}, \ldots, p_{i_r}\}$ be a sequence of points obtained by a permutation of the points $p_0, \ldots, p_r$. We define $\{p_0, \ldots, p_r\}$ and $\{p_{i_0}, \ldots, p_{i_r}\}$ to be equivalent if

$$P \equiv \begin{pmatrix} 0 & 1 & \ldots & r \\ i_0 & i_1 & \ldots & i_r \end{pmatrix}$$

is an even permutation. Clearly this is an equivalence relation, the equivalence class of which is called an **oriented *r*-simplex**. There are two equivalence classes, one consists of even permutations of $p_0, \ldots, p_r$, the other of odd permutations. The equivalence class (oriented r-simplex) which contains $\{p_0, \ldots, p_r\}$ is denoted by $\sigma_r = (p_0p_1 \ldots p_r)$, while the other is denoted by $-\sigma_r = -(p_0p_1 \ldots p_r)$. In other words,

$$(p_{i_0}p_{i_1} \ldots p_{i_r}) = \operatorname{sgn}(P)(p_0p_1 \ldots p_r). \tag{3.14}$$

For $r = 0$, we formally define an oriented 0-simplex to be just a point $\sigma_0 = p_0$.

3.3.2 Chain group, cycle group and boundary group

Let $K = \{\sigma_\alpha\}$ be an n-dimensional simplicial complex. We regard the simplexes σ_α in K as oriented simplexes and denote them by the same symbols σ_α as remarked before.

Definition 3.11 The **r-chain group** $C_r(K)$ of a simplicial complex K is a free Abelian group generated by the oriented r-simplexes of K. If $r > \dim K$, $C_r(K)$ is defined to be 0. An element of $C_r(K)$ is called an **r-chain**.

Let there be I_r r-simplexes in K. We denote them by $\sigma_{r,i}$ $(1 \leqslant i \leqslant I_r)$. Then $c \in C_r(K)$ is expressed as

$$c = \sum_{i=1}^{I_r} c_i \sigma_{r,i} \qquad c_i \in \mathbb{Z}. \tag{3.15}$$

The integers c_i are called the coefficients of c. The group structure is given as follows. The addition of two r-chains, $c = \Sigma_i c_i \sigma_{r,i}$ and $c' = \Sigma_i c_i' \sigma_{r,i}$ is

$$c + c' = \sum_i (c_i + c_i')\sigma_{r,i}. \tag{3.16}$$

The unit element is $0 = \Sigma_i 0 \cdot \sigma_{r,i}$ while the inverse element of c is $-c = \Sigma_i (-c_i)\sigma_{r,i}$. [*Remark:* An oppositely oriented r-simplex $-\sigma_r$ is identified with $(-1)\sigma_r \in C_r(K)$.] Thus $C_r(K)$ is a free Abelian group of rank I_r,

$$C_r(K) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{I_r}. \tag{3.17}$$

Before we define the cycle group and the boundary group, we need to introduce the boundary operator. Let us denote the boundary of an r-simplex σ_r by $\partial_r \sigma_r$. ∂_r should be understood as an *operator* acting on σ_r to produce its boundary. This point of view will be elaborated later. Let us look at the boundaries of lower-dimensional simplexes. Since a 0-simplex has no boundary, we define

$$\partial_0 p_0 = 0. \tag{3.18}$$

For a 1-simplex $(p_0 p_1)$, we define

$$\partial_1 (p_0 p_1) = p_1 - p_0. \tag{3.19}$$

The reader might wonder about the appearance of a minus sign in front of p_0. This is again related to the orientation. The following examples will clarify this point. In figure 3.7(a), an oriented 1-simplex $(p_0 p_2)$ is divided into two, $(p_0 p_1)$ and $(p_1 p_2)$. We agree that the boundary of $(p_0 p_2)$ is $p_0 \cup p_2$ and so should be that of $(p_0 p_1) + (p_1 p_2)$. If $\partial_1 (p_0 p_2)$ were defined to be $p_0 + p_2$, we would have $\partial_1 (p_0 p_1) + \partial_1 (p_1 p_2) = p_0 + p_1 + p_1 + p_2$. This is not desirable since p_1 is a *fictitious* boundary. If, instead, we take $\partial_1 (p_0 p_2) = p_2 - p_0$, we will have $\partial_1 (p_0 p_1) + \partial_1 (p_1 p_2) = p_1 - p_0 + p_2 - p_1 = p_2 - p_0$ as expected. The next example is the triangle of figure 3.7(b). It is the sum of three oriented 1-simplexes, $(p_0p_1) + (p_1p_2) + (p_2p_0)$. We agree that it has no boundary. If we insisted on the rule $\partial_1(p_0p_1) = p_0 + p_1$, we would have

$$\partial_1(p_0p_1) + \partial_1(p_1p_2) + \partial_1(p_2p_0) = p_0 + p_1 + p_1 + p_2 + p_2 + p_0$$

which contradicts our intuition. If, on the other hand, we take $\partial_1(p_0p_1) = p_1 - p_0$, we have

$$\partial_1(p_0p_1) + \partial_1\,(p_1p_2) + \partial_1(p_2p_0) = p_1 - p_0 + p_2 - p_1 + p_0 - p_2 = 0$$

as expected. Hence we put a plus sign if the first vertex is omitted and a minus sign if the second is omitted. We employ this fact to define the boundary of a general r-simplex.

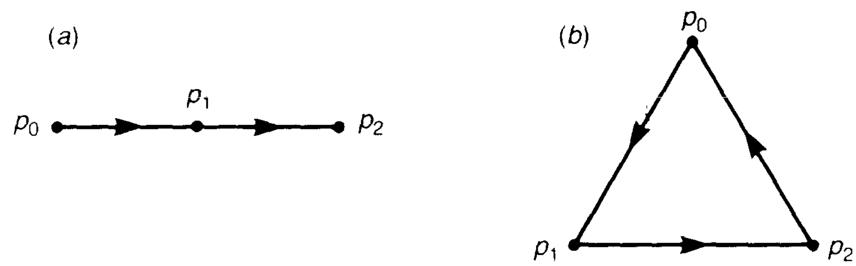


Figure 3.7 (a) An oriented 1-simplex with a fictitious boundary p_1. (b) A simplicial complex without a boundary.

Let $\sigma_r = (p_0 \ldots p_r)(r > 0)$ be an oriented r-simplex. The **boundary** $\partial_r\sigma_r$ of σ_r is an $(r-1)$-chain defined by

$$\partial_r\sigma_r \equiv \sum_{i=0}^{r}(-1)^i(p_0p_1 \ldots \hat{p}_i \ldots p_r) \tag{3.20}$$

where the point p_i under $\hat{}$ is omitted. For example,

$$\partial_2(p_0p_1p_2) = (p_1p_2) - (p_0p_2) + (p_0p_1)$$

$$\partial_3(p_0p_1p_2p_3) = (p_1p_2p_3) - (p_0p_2p_3) + (p_0p_1p_3) - (p_0p_1p_2).$$

If $r = 0$, we define $\partial_0\sigma_0 = 0$.

∂_r acts linearly on an element $c = \Sigma_i c_i\sigma_{r,i}$ of $C_r(K)$,

$$\partial_r c = \sum_i c_i\partial_r\sigma_{r,i}. \tag{3.21}$$

The RHS of (3.21) is an element of $C_{r-1}(K)$. ∂_r defines a map

$$\partial_r : C_r(K) \to C_{r-1}(K). \tag{3.22}$$

∂_r is called the **boundary operator**. It is easy to see that the boundary operator is a homomorphism.

Let K be an n-dimensional simplicial complex. There exists a sequence of free Abelian groups and homomorphisms,

$$0 \xrightarrow{i} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0 \tag{3.23}$$

where $i : 0 \hookrightarrow C_n(K)$ is an inclusion map (0 is regarded as the unit element of $C_n(K)$). This sequence is called the **chain complex** associated with K and is denoted by $C(K)$. It is interesting to study the *image* and *kernel* of the homomorphisms ∂_r.

Definition 3.12 If $c \in C_r(K)$ satisfies

$$\partial_r c = 0 \tag{3.24}$$

c is called an ***r*-cycle**. The set of r-cycles $Z_r(K)$ is a subgroup of $C_r(K)$ and is called the ***r*-cycle group**. Note that $Z_r(K) = \ker \partial_r$. [*Remark:* If $r = 0$, $\partial_0 c$ vanishes identically and $Z_0(K) = C_0(K)$.]

Definition 3.13 Let K be an n-dimensional simplicial complex and let $c \in C_r(K)$. If there exists an element $d \in C_{r+1}(K)$ such that

$$c = \partial_{r+1} d \tag{3.25}$$

then c is called an ***r*-boundary**. The set of r-boundaries $B_r(K)$ is a subgroup of $C_r(K)$ and is called the ***r*-boundary group**. Note that $B_r(K) = \operatorname{im} \partial_{r+1}$. [*Remark:* $B_n(K)$ is defined to be 0.]

From lemma 3.2, it follows that $Z_r(K)$ and $B_r(K)$ are subgroups of $C_r(K)$. We now prove an important relation between $Z_r(K)$ and $B_r(K)$, which is crucial in the definition of homology groups.

Lemma 3.14 The composite map $\partial_r \cdot \partial_{r+1} : C_{r+1}(K) \to C_{r-1}(K)$ is a zero map; that is, $\partial_r(\partial_{r+1} c) = 0$ for any $c \in C_{r+1}(K)$.

Proof: Since ∂_r is a linear operator on $C_r(K)$, it is sufficient to prove the identity $\partial_r \cdot \partial_{r+1} = 0$ for the generators of $C_{r+1}(K)$. If $r = 0$, $\partial_0 \cdot \partial_1 = 0$ since ∂_0 is a zero operator. Let us assume $r > 0$. Take $\sigma = (p_0 \ldots p_r p_{r+1}) \in C_{r+1}(K)$. We find

$$\begin{aligned}
\partial_r(\partial_{r+1}\sigma) &= \partial_r \sum_{i=0}^{r+1} (-1)^i (p_0 \ldots \hat{p}_i \ldots p_{r+1}) \\
&= \sum_{i=0}^{r+1} (-1)^i \partial_r (p_0 \ldots \hat{p}_i \ldots p_{r+1}) \\
&= \sum_{i=0}^{r+1} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j (p_0 \ldots \hat{p}_j \ldots \hat{p}_i \ldots p_{r+1}) \right. \\
&\quad \left. + \sum_{j=i+1}^{r+1} (-1)^{j-1} (p_0 \ldots \hat{p}_i \ldots \hat{p}_j \ldots p_{r+1}) \right)
\end{aligned}$$

$$= \sum_{j<i} (-1)^{i+j} (p_0 \ldots \hat{p}_j \ldots \hat{p}_i \ldots p_{r+1})$$

$$- \sum_{j>i} (-1)^{i+j} (p_0 \ldots \hat{p}_i \ldots \hat{p}_j \ldots p_{r+1})$$

$$= 0. \blacksquare$$

Theorem 3.15 Let $Z_r(K)$ and $B_r(K)$ be the r-cycle group and the r-boundary group of $C_r(K)$, then

$$B_r(K) \subset Z_r(K) \quad (\subset C_r(K)). \tag{3.26}$$

Proof: This is trivial from lemma 3.14. Any element c of $B_r(K)$ is written as $c = \partial_{r+1} d$ for some $d \in C_{r+1}(K)$. Then we find $\partial_r c = \partial_r(\partial_{r+1} d) = 0$, that is, $c \in Z_r(K)$. This implies $Z_r(K) \supset B_r(K)$. ■

What are the geometrical pictures of r-cycles and r-boundaries? With our definitions, ∂_r picks up the boundary of an r-chain. If c is an r-cycle, $\partial_r c = 0$ tells us that c has no boundary. If $c = \partial_{r+1} d$ is an r-boundary, c is the boundary of d whose dimension is higher than c by one. Our intuition tells us that *a boundary has no boundary*, hence $Z_r(K) \supset B_r(K)$. Those elements of $Z_r(K)$ that are *not* boundaries play the central role in this chapter.

3.3.3 *Homology groups*

So far we have defined three groups $C_r(K)$, $Z_r(K)$ and $B_r(K)$ associated with a simplicial complex K. How are they related to topological properties of K or to the topological space whose triangulation is K? Is it possible for $C_r(K)$ to express any property which is conserved under homeomorphism? We all know that the edges of a triangle and those of a square are homeomorphic to each other. What about their chain groups? For example, the 1-chain group associated with a triangle is

$$C_1(K_1) = \{i(p_0p_1) + j(p_1p_2) + k(p_2p_0) | i, j, k \in \mathbb{Z}\}$$
$$\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

while that associated with a square is

$$C_1(K_2) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Clearly $C_1(K_1)$ is not isomorphic to $C_1(K_2)$, hence $C_r(K)$ cannot be a candidate of a topological invariant. The same is true for $Z_r(K)$ and $B_r(K)$. It turns out that the homology groups defined below provide the desired topological invariants.

Definition 3.16 Let K be an n-dimensional simplicial complex. The ***r*th homology group** $H_r(K)$, $0 \leqslant r \leqslant n$, associated with K is defined by

$$H_r(K) \equiv Z_r(K)/B_r(K). \tag{3.27}$$

[*Remarks:* If necessary, we define $H_r(K) = 0$ for $r > n$ or $r < 0$. If we want to stress that the group structure is defined with integer coefficients, we write $H_r(K; \mathbb{Z})$. We may also define the homology groups with $\mathbb{R}$-coefficients, $H_r(K; \mathbb{R})$ or those with $\mathbb{Z}_2$-coefficients, $H_r(K; \mathbb{Z}_2)$.]

Since $B_r(K)$ is a subgroup of $Z_r(K)$, $H_r(K)$ is well defined. $H_r(K)$ is the set of equivalence classes of r-cycles,

$$H_r(K) = \{[z] | z \in Z_r(K)\} \tag{3.28}$$

where each equivalence class $[z]$ is called a **homology class**. Two r-cycles z and z' are in the same equivalence class if and only if $z - z' \in B_r(K)$, in which case z is said to be **homologous** to z' and denoted by $z \sim z'$ or $[z] = [z']$. Geometrically $z - z'$ is a boundary of some space. By definition, any boundary $b \in B_r(K)$ is homologous to 0 since $b - 0 \in B_r(K)$. We accept the following theorem without proof.

Theorem 3.17 Homology groups are topological invariants. Let X be homeomorphic to Y and let (K, f) and (L, g) be triangulations of X and Y respectively. Then we have

$$H_r(K) \cong H_r(L) \qquad r = 0, 1, 2, \ldots. \tag{3.29}$$

In particular, if (K, f) and (L, g) are two triangulations of X, then

$$H_r(K) \cong H_r(L) \qquad r = 0, 1, 2, \ldots. \tag{3.30}$$

Accordingly it makes sense to talk of homology groups of a topological space X which is not necessarily a simplicial complex but which is triangulable. For an arbitrary triangulation (K, f), $H_r(X)$ is defined to be

$$H_r(X) \equiv H_r(K) \qquad r = 0, 1, 2, \ldots. \tag{3.31}$$

Theorem 3.17 tells us that this is independent of the choice of the triangulation (K, f).

Example 3.18 Let $K = \{p_0\}$. The 0-chain is $C_0(K) = \{ip_0 | i \in \mathbb{Z}\} \cong \mathbb{Z}$. Clearly $Z_0(K) = C_0(K)$ and $B_0(K) = 0$ ($\partial_0 p_0 = 0$ and p_0 cannot be a boundary of anything). Thus

$$H_0(K) \equiv Z_0(K)/B_0(K) = C_0(K) \cong \mathbb{Z}. \tag{3.32}$$

Exercise 3.19 Let $K = \{p_0, p_1\}$ be a simplicial complex consisting of two 0-simplexes. Show that

$$H_r(K) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & (r = 0) \\ \{0\} & (r \neq 0). \end{cases} \tag{3.33}$$

Example 3.20 Let $K = \{p_0, p_1, (p_0p_1)\}$. We have

$$C_0(K) = \{ip_0 + jp_1 | i, j \in \mathbb{Z}\}$$
$$C_1(K) = \{k(p_0p_1) | k \in \mathbb{Z}\}.$$

Since (p_0p_1) is not a boundary of any simplex in K, $B_1(K) = 0$, and

$$H_1(K) = Z_1(K)/B_1(K) = Z_1(K).$$

If $z = m(p_0p_1) \in Z_1(K)$, it satisfies

$$\partial_1 z = m\partial_1(p_0p_1) = m\{p_1 - p_0\} = mp_1 - mp_0 = 0.$$

Thus m has to vanish and $Z_1(K) = 0$, hence

$$H_1(K) \cong \{0\}. \tag{3.34}$$

As for $H_0(K)$, we have $Z_0(K) = C_0(K) = \{ip_0 + jp_1\}$ and

$$B_0(K) = \operatorname{im}\partial_1 = \{\partial_1 i(p_0, p_1) | i \in \mathbb{Z}\} = \{i(p_1 - p_0) | i \in \mathbb{Z}\}.$$

Define a surjective (onto) homomorphism $f : Z_0(K) \to \mathbb{Z}$ by

$$f(ip_0 + jp_1) = i + j.$$

Then we find

$$\ker f = f^{-1}(0) = B_0(K).$$

Theorem 3.3 states that $Z_0(K)/\ker f \cong \operatorname{im} f = \mathbb{Z}$, or

$$H_0(K) = Z_0(K)/B_0(K) \cong \mathbb{Z}. \tag{3.35}$$

Example 3.21 Let $K = \{p_0, p_1, p_2, (p_0p_1), (p_1p_2), (p_2p_1)\}$, see figure 3.7($b$). This is a triangulation of S^1. Since there are no 2-simplexes in K, we have $B_1(K) = 0$ and $H_1(K) = Z_1(K)/B_1(K) = Z_1(K)$. Let $z = i(p_0p_1) + j(p_1p_2) + k(p_2p_0) \in Z_1(K)$ where $i, j, k \in \mathbb{Z}$. We require that

$$\begin{aligned}\partial_1 z &= i(p_1 - p_0) + j(p_2 - p_1) + k(p_0 - p_2) \\ &= (k - i)p_0 + (i - j)p_1 + (j - k)p_2 = 0.\end{aligned}$$

This is satisfied only when $i = j = k$. Thus we find

$$Z_1(K) = \{i\{(p_0p_1) + (p_1p_2) + (p_2p_0)\} | i \in \mathbb{Z}\}.$$

This shows that $Z_1(K)$ is isomorphic to $\mathbb{Z}$ and

$$H_1(K) = Z_1(K) \cong \mathbb{Z}. \tag{3.36}$$

Let us compute $H_0(K)$. We have $Z_0(K) = C_0(K)$ and

$$\begin{aligned}B_0(K) &= \{\partial_1[l(p_0p_1) + m(p_1p_2) + n(p_2p_0)] | l, m, n \in \mathbb{Z}\} \\ &= \{(n - l)p_0 + (l - m)p_1 + (m - n)p_2 | l, m, n \in \mathbb{Z}\}.\end{aligned}$$

Define a surjective homomorphism $f : Z_0(K) \to \mathbb{Z}$ by

$$f(ip_0 + jp_1 + kp_2) = i + j + k.$$

We verify that

$$\ker f = f^{-1}(0) = B_0(K).$$

From theorem 3.3 we find $Z_0(K)/\ker f \cong \operatorname{im} f = \mathbb{Z}$, or

$$H_0(K) = Z_0(K)/B_0(K) \cong \mathbb{Z}. \tag{3.37}$$

K is a triangulation of a circle S^1, and (3.36) and (3.37) are the homology groups of S^1.

Exercise 3.22 Let $K = \{p_0, p_1, p_2, p_3, (p_0p_1), (p_1p_2), (p_2p_3), (p_3p_0)\}$ be a simplicial complex whose polyhedron is a square. Verify that the homology groups are the same as those of example 3.21 above.

Example 3.23 Let $K = \{p_0, p_1, p_2, (p_0p_1), (p_1p_2), (p_2p_0), (p_0p_1p_2)\}$, see figure 3.6($b$). Since the structure of 0-simplexes and 1-simplexes are the same as that of example 3.21, we have

$$H_0(K) \cong \mathbb{Z}. \tag{3.38}$$

Let us compute $H_1(K) = Z_1(K)/B_1(K)$. From the previous example, we have

$$Z_1(K) = \{i\{(p_0p_1) + (p_1p_2) + (p_2p_0)\}|i \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Let $c = m(p_0p_1p_2) \in C_2(K)$. If $b = \partial_2 c \in B_1(K)$, we have

$$\begin{aligned} b &= m\{(p_1p_2) - (p_0p_2) + (p_0p_1)\} \\ &= m\{(p_0p_1) + (p_1p_2) + (p_2p_0)\}, \; m \in \mathbb{Z}. \end{aligned}$$

This shows that $Z_1(K) \cong B_1(K)$, hence

$$H_1(K) = Z_1(K)/B_1(K) \cong \{0\}. \tag{3.39}$$

Since there are no 3-simplexes in K, we have $B_2(K) = 0$. Then $H_2(K) = Z_2(K)/B_2(K) = Z_2(K)$. Let $z = m(p_0p_1p_2) \in Z_2(K)$. Since $\partial_2 z = m\{(p_1p_2) - (p_0p_2) + (p_0p_1)\} = 0$, m must vanish. Hence $Z_1(K) = 0$ and we have

$$H_2(K) \cong \{0\}. \tag{3.40}$$

Exercise 3.24 Let $K = \{p_0, p_1, p_2, p_3, (p_0p_1), (p_0p_2), (p_0p_3), (p_1, p_2), (p_1p_3), (p_2p_3), (p_0p_1p_2), (p_0p_1p_3), (p_0p_2p_3), (p_1p_2p_3)\}$ be a simplicial complex whose polyhedron is the *surface* of a tetrahedron. Verify that

$$H_0(K) \cong \mathbb{Z} \qquad H_1(K) \cong 0 \qquad H_2(K) \cong \mathbb{Z}. \tag{3.41}$$

K is a triangulation of the sphere S^2 and (3.41) gives the homology groups of S^2.

3.3.4 Computation of $H_0(K)$

Examples 3.18, 3.20, 3.21, 3.22, 3.23 and exercise 3.24 share the same 0th homology group, $H_0(K) \cong \mathbb{Z}$. What is common to these simplicial complexes? We have the following answer.

Theorem 3.25 Let K be a *connected* simplicial complex. Then

$$H_0(K) \cong \mathbb{Z}. \tag{3.42}$$

Proof: Since K is connected, for any pair of 0-simplexes p_i and p_j, there exists a sequence of 1-simplexes (p_ip_k), (p_kp_l), ..., (p_mp_j) such that $\partial_1((p_ip_k) + (p_kp_l) + \ldots + (p_mp_j)) = p_j - p_i$. Then it follows that p_i is homologous to p_j, namely $[p_i] = [p_j]$. Thus any 0-simplex in K is homologous to p_1 say. Suppose

$$z = \sum_i^{I_0} n_ip_i \in Z_0(K)$$

where I_0 is the number of 0-simplexes in K. Then the homology class $[z]$ is generated by a single point,

$$[z] = \left[\sum_i n_ip_i\right] = \sum_i n_i[p_i] = \sum_i n_i[p_1].$$

It is clear that $[z] = 0$, namely $z \in B_0(K)$, if $\sum n_i = 0$.

Let $\sigma_j = (p_{j,1}p_{j,2})$ $(1 \leqslant j \leqslant I_1)$ be 1-simplexes in K, I_1 being the number of 1-simplexes in K, then

$$\begin{aligned} B_0(K) &= \operatorname{im}\partial_1 \\ &= \{\partial_1[n_1\sigma_1 + \ldots + n_{I_1}\sigma_{I_1}] \mid n_1, \ldots, n_{I_1} \in \mathbb{Z}\} \\ &= \{n_1(p_{1,2} - p_{1,1}) + \ldots + n_{I_1}(p_{I_1,2} - p_{I_1,1}) \mid n_1, \ldots, n_{I_1} \in \mathbb{Z}\}. \end{aligned}$$

Note that n_j $(1 \leqslant j \leqslant I_1)$ always appears as a pair $+n_j$ and $-n_j$ in an element of $B_0(K)$. Thus if

$$z = \sum_j n_jp_j \in B_0(K) \qquad \text{then } \sum_j n_j = 0.$$

Now we have proved for a connected complex K that $z = \sum n_ip_i \in B_0(K)$ if and only if $\sum n_i = 0$.

Define a surjective homomorphism $f: Z_0(K) \to \mathbb{Z}$ by

$$f(n_1p_1 + \ldots + n_{I_0}p_{I_0}) = \sum_{i=1}^{I_0} n_i.$$

We then have $\ker f = f^{-1}(0) = B_0(K)$. From theorem 3.3, it follows that $H_0(K) = Z_0(K)/B_0(K) = Z_0(K)/\ker f \cong \operatorname{im} f = \mathbb{Z}$. ■

3.3.5 More homology computations

Example 3.26 This and the next example deal with homology groups of non-orientable spaces. Figure 3.8 is a triangulation of the Möbius strip. Clearly $B_2(K) = 0$. Let us take a cycle $z \in Z_2(K)$,

$$\begin{aligned} z = {} & i(p_0p_1p_2) + j(p_2p_1p_4) + k(p_2p_4p_3) \\ & + l(p_3p_4p_5) + m(p_3p_5p_1) + n(p_1p_5p_0). \end{aligned}$$

z satisfies

$$\begin{aligned} \partial_1 z = {} & i[(p_1p_2) - (p_0p_2) + (p_0p_1)] \\ & + j[(p_1p_4) - (p_2p_4) + (p_2p_1)] \\ & + k[(p_4p_3) - (p_2p_3) + (p_2p_4)] \\ & + l[(p_4p_5) - (p_3p_5) + (p_3p_4)] \\ & + m[(p_5p_1) - (p_3p_1) + (p_3p_5)] \\ & + n[(p_5p_0) - (p_1p_0) + (p_1p_5)] = 0. \end{aligned}$$

Since each of (p_0p_2), (p_1p_4), (p_2p_3), (p_4p_5), (p_3p_1) and (p_5p_0) appears once and only once in ∂z, all the coefficients must vanish, $i = j = k = l = m = n = 0$. Thus $Z_2(K) = 0$ and

$$H_2(K) = Z_2(K)/B_2(K) \cong \{0\}. \tag{3.43}$$

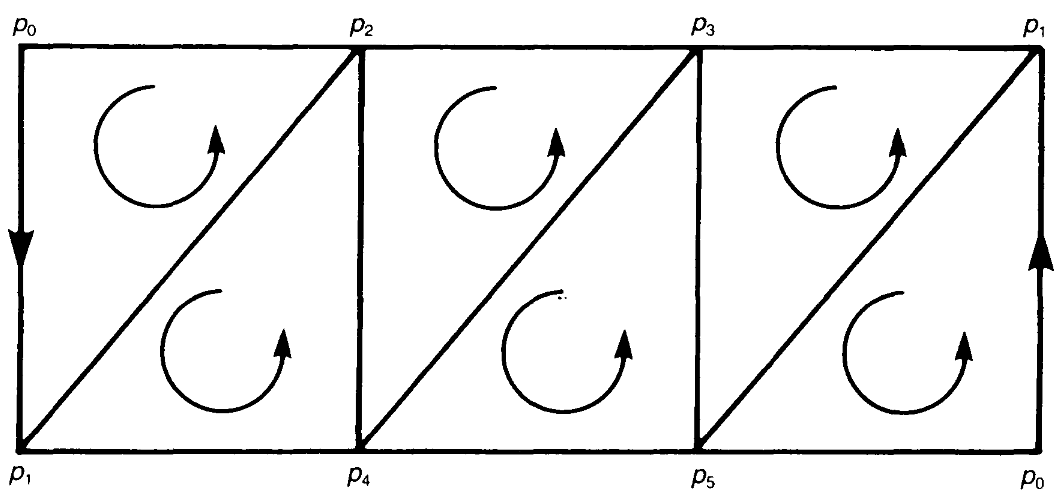


Figure 3.8 A triangulation of the Möbius strip.

To find $H_1(K)$, we use our intuition rather than doing tedious computations. Let us find the loops which make complete circuits. One such loop is

$$z = (p_0p_1) + (p_1p_4) + (p_4p_5) + (p_5p_0).$$

Then all the other complete circuits are homologous to multiples of z. For example, let us take

$$z' = (p_1p_2) + (p_2p_3) + (p_3p_5) + (p_5p_1).$$

We find that $z \sim z'$ since

$$z - z' = \partial_2\{(p_2p_1p_4) + (p_2p_4p_3) + (p_3p_4p_5) + (p_1p_5p_0)\}.$$

If, on the other hand, we take

$$z'' = (p_1p_4) + (p_4p_5) + (p_5p_0) + (p_0p_2) + (p_2p_3) + (p_3p_1)$$

we find that $z'' \sim 2z$ since

$$\begin{aligned}2z - z'' &= 2(p_0p_1) + (p_1p_4) + (p_4p_5) + (p_5p_0) - (p_0p_2)\\ &\quad - (p_2p_3) - (p_3p_1)\\ &= \partial_2\{(p_0p_1p_2) + (p_1p_4p_2) + (p_2p_4p_3) + (p_3p_4p_5)\\ &\quad + (p_3p_5p_1) + (p_0p_1p_5)\}.\end{aligned}$$

We easily verify that all the closed circuits are homologous to nz, $n \in \mathbb{Z}$. $H_1(K)$ is generated by just one element z,

$$H_1(K) = \{i[z] | i \in \mathbb{Z}\} \cong \mathbb{Z}. \tag{3.44}$$

Since K is connected, it follows from theorem 3.25 that $H_0(K) = \{i[p_a] | i \in \mathbb{Z}\} \cong \mathbb{Z}$, p_a being any 0-simplex of K.

Example 3.27 In §1.5 and example 2.12(c), we defined the projective plane $\mathbb{R}P^2$ as the sphere S^2 whose antipodal points are identified. As a coset space, we may take the hemisphere (or the disc D^2) whose opposite points on the boundary S^1 are identified, see figure 2.5(b). Figure 3.9 is a triangulation of the projective plane. Clearly $B_2(K) = 0$. Take a cycle $z \in Z_2(K)$,

$$\begin{aligned}z &= m_1(p_0p_1p_2) + m_2(p_0p_4p_1) + m_3(p_0p_5p_4)\\ &\quad + m_4(p_0p_3p_5) + m_5(p_0p_2p_3) + m_6(p_2p_4p_3)\\ &\quad + m_7(p_2p_5p_4) + m_8(p_2p_1p_5) + m_9(p_1p_3p_5) + m_{10}(p_1p_4p_3).\end{aligned}$$

The boundary of z is

$$\begin{aligned}\partial_2 z &= m_1\{(p_1p_2) - (p_0p_2) + (p_0p_1)\}\\ &\quad + m_2\{(p_4p_1) - (p_0p_1) + (p_0p_4)\}\\ &\quad + m_3\{(p_5p_4) - (p_0p_4) + (p_0p_5)\}\\ &\quad + m_4\{(p_3p_5) - (p_0p_5) + (p_0p_3)\}\\ &\quad + m_5\{(p_2p_3) - (p_0p_3) + (p_0p_2)\}\\ &\quad + m_6\{(p_4p_3) - (p_2p_3) + (p_2p_4)\}\end{aligned}$$

$$+ m_7\{(p_5p_4) - (p_2p_4) + (p_2p_5)\}$$
$$+ m_8\{(p_1p_5) - (p_2p_5) + (p_2p_1)\}$$
$$+ m_9\{(p_3p_5) - (p_1p_5) + (p_1p_3)\}$$
$$+ m_{10}\{(p_4p_3) - (p_1p_3) + (p_1p_4)\} = 0.$$

Let us look at the coefficient of each 1-simplex. For example, we have $(m_1 - m_2)(p_0p_1)$, hence $m_1 - m_2 = 0$. Similarly

$$m_1 - m_2 = 0,\ -m_1 + m_5 = 0,\ m_4 - m_5 = 0,\ m_2 - m_3 = 0,$$
$$m_1 - m_8 = 0,\ m_9 - m_{10} = 0,\ -m_2 + m_{10} = 0,\ m_5 - m_6 = 0,$$
$$m_6 - m_7 = 0,\ m_6 + m_{10} = 0.$$

These conditions are satisfied if and only if $m_i = 0$, $1 \leqslant i \leqslant 10$. This means that the cycle group $Z_2(K)$ is trivial and we have

$$H_2(K) = Z_2(K)/B_2(K) \cong \{0\}. \tag{3.45}$$

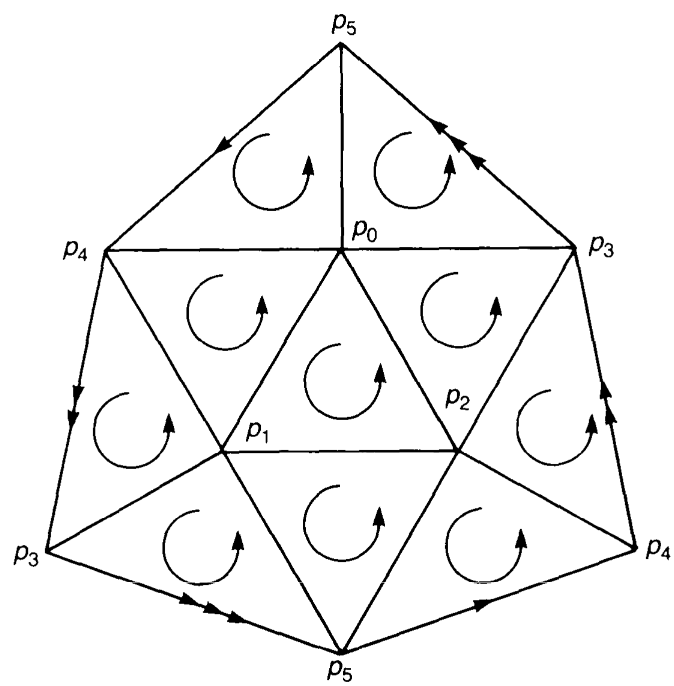


Figure 3.9 A triangulation of the projective plane.

Before we calculate $H_1(K)$, we examine $H_2(K)$ from a slightly different viewpoint. Let us add all the 2-simplexes in K with the same coefficient,

$$z \equiv \sum_{i=1}^{10} m\sigma_{2,i} \qquad m \in \mathbb{Z}.$$

Observe that each 1-simplex of K is a common face of exactly two 2-simplexes. As a consequence, the boundary of z is

$$\partial_2 z = 2m(p_3p_5) + 2m(p_5p_4) + 2m(p_4p_3). \tag{3.46}$$

Thus if $z \in Z_2(K)$, m must vanish and we find $Z_2(K) = 0$ as before. This observation remarkably simplifies the computation of $H_1(K)$. Note that any 1-cycle is homologous to a multiple of

$$z = (p_3p_5) + (p_5p_4) + (p_4p_3)$$

cf example 3.26. Furthermore, (3.46) shows that an even multiple of z is a boundary of a 2-chain. Thus z is a cycle and $z + z$ is homologous to 0. Hence we find that

$$H_1(K) = \{[z] | [z] + [z] \sim [0]\} \cong \mathbb{Z}_2. \tag{3.47}$$

This example shows that a homology group is not necessarily free Abelian but may have the full structure of a finitely generated Abelian group. Since K is connected, we have $H_0(K) \cong \mathbb{Z}$.

It is interesting to compare example 3.27 with the following examples. In these examples, we shall use the intuition developed in this section on boundaries and cycles to obtain results rather than giving straightforward and tedious computations.

Example 3.28 Let us consider the torus T^2. A formal derivation of the homology groups of T^2 is left as an exercise to the reader: see Fraleigh (1976), for example. This is an appropriate place to recall the intuitive meaning of the homology groups. The rth homology group is generated by those boundaryless r-chains that are not, by themselves, boundaries of some $(r + 1)$-chains. For example, the surface of the torus has no boundary but it is not a boundary of some 3-chain. Thus $H_2(T^2)$ is freely generated by one generator, the surface itself, $H_2(T^2) \cong \mathbb{Z}$. Let us look at $H_1(T^2)$ next. Clearly the loops a and b of figure 3.10 have no boundaries but are not boundaries of some 2-chains. Take another loop a'. a' is homologous to a since $a' - a$ bounds the shaded area of figure 3.10. Hence $H_1(T^2)$ is freely generated by a and b and $H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. Since T^2 is connected, we have $H_0(T^2) \cong \mathbb{Z}$.

Now it is easy to extend our analysis to the torus T_g of genus g. Since T_g has no boundary and there are no 3-simplexes, the surface T_g itself freely generates $H_2(T_g) \cong \mathbb{Z}$. $H_1(T_g)$ is generated by those loops which are not boundaries of some area. Figure 3.11 shows the standard choice for the generators. We find

$$\begin{aligned} H_1(T_g) &= \{i_1[a_1] + j_1[b_1] + \ldots + i_g[a_g] + j_g[b_g]\} \\ &\cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{2g}. \end{aligned} \tag{3.48}$$

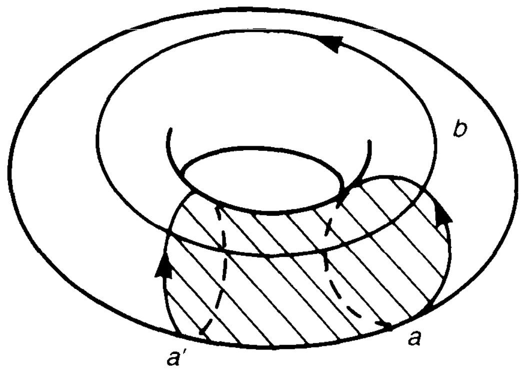


Figure 3.10 a' is homologous to a but b is not. a and b generate $H_1(T^2)$.

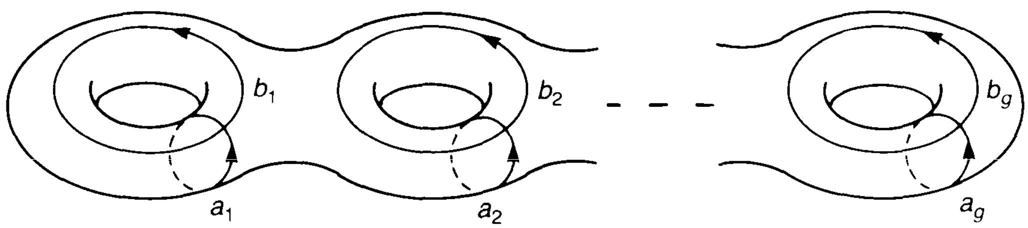


Figure 3.11 a_i and b_i $(1 \leq i \leq g)$ generate $H_1(T^2)$.

Since T_g is connected, $H_0(T_g) \cong \mathbb{Z}$. Observe that a_i (b_i) is homologous to the edge a_i (b_i) of figure 2.12. The $2g$ curves $\{a_i, b_i\}$ are called the **canonical system of curves** on T_g.

Example 3.29 Figure 3.12 is a triangulation of the Klein bottle. Computations of the homology groups are much the same as that of the projective plane. Since $B_2(K) = 0$, we have $H_2(K) = Z_2(K)$. Let $z \in Z_2(K)$. If z is a combination of all the 2-simplexes of K with the same coefficient, $z = \sum m\sigma_{2,i}$, the inner 1-simplexes cancel out to leave only the outer 1-simplexes

$$\partial_2 z = -2ma$$

where $a = (p_0p_1) + (p_1p_2) + (p_2p_0)$. For ∂z to be 0, m must vanish and we have

$$H_2(K) = Z_2(K) \cong \{0\}. \tag{3.49}$$

To compute $H_1(K)$ we first note, from our experience with the torus, that every 1-cycle is homologous to $ia + jb$ for some $i, j \in \mathbb{Z}$. For a 2-chain to have a boundary consisting of a and b only, all the 2-simplexes in K must be added with the same coefficient. As a result, for such a 2-chain $z = \sum m\sigma_{2,i}$, we have $\partial z = 2ma$. This shows that $2ma \sim 0$. Thus $H_1(K)$ is generated by two cycles a and b such that $a + a = 0$, namely

$$H_1(K) = \{i[a] + j[b] | i, j \in \mathbb{Z}\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}. \tag{3.50}$$

Since K is connected, $H_0(K) \cong \mathbb{Z}$.

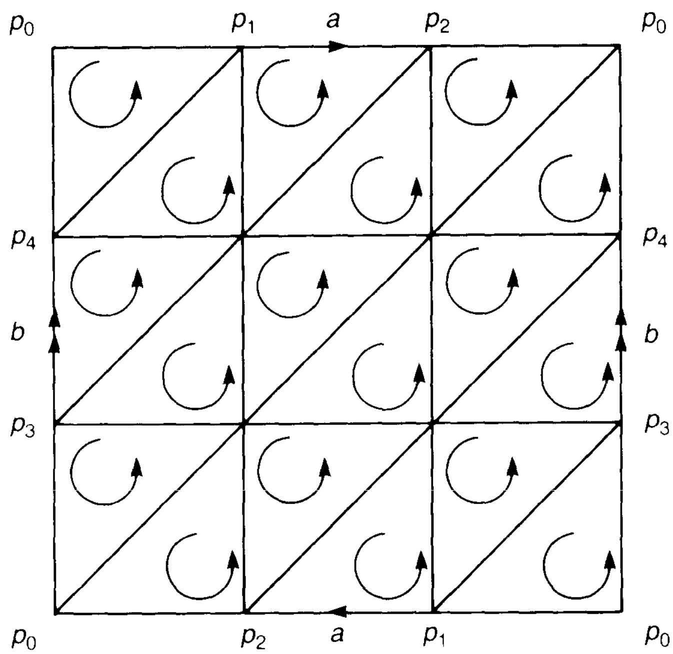


Figure 3.12 A triangulation of the Klein bottle.

3.4 General properties of homology groups

3.4.1 Connectedness and homology groups

Let $K = \{p_0\}$ and $L = \{p_0, p_1\}$. From example 3.18 and exercise 3.19, we have $H_0(K) = \mathbb{Z}$ and $H_0(L) = \mathbb{Z} \oplus \mathbb{Z}$. More generally, we have the following theorem.

Theorem 3.30 Let K be a disjoint union of N connected components, $K = K_1 \cup K_2 \cup \ldots \cup K_N$ where $K_i \cap K_j = \emptyset$. Then

$$H_r(K) = H_r(K_1) \oplus H_r(K_2) \oplus \ldots \oplus H_r(K_N). \tag{3.51}$$

Proof: We first note that an r-chain group is consistently separated into a direct sum of N r-chain subgroups. Let

$$C_r(K) = \left\{ \sum_{i=1}^{I_r} c_i \sigma_{r,i} | c_i \in \mathbb{Z} \right\}$$

where I_r is the number of linearly independent r-simplexes in K. It is always possible to rearrange σ_i so that those r-simplexes in K_1 come first, those in K_2 next and so on. Then $C_r(K)$ is separated into a direct sum of subgroups,

$$C_r(K) = C_r(K_1) \oplus \ldots \oplus C_r(K_N).$$

This separation is also carried out for $Z_r(K)$ and $B_r(K)$ as

$$Z_r(K) = Z_r(K_1) \oplus \ldots \oplus Z_r(K_N)$$
$$B_r(K) = B_r(K_1) \oplus \ldots \oplus B_r(K_N).$$

We now define the homology groups of each component K_i by

$$H_r(K_i) = Z_r(K_i)/B_r(K_i).$$

This is well defined since $Z_r(K_i) \subset B_r(K_i)$. Finally we have

$$\begin{aligned} H_r(K) &= Z_r(K)/B_r(K) \\ &= Z_r(K_1) \oplus \ldots \oplus Z_r(K_N)/B_r(K_1) \oplus \ldots \oplus B_r(K_N) \\ &= \{Z_r(K_1)/B_r(K_1)\} \oplus \ldots \oplus \{Z_r(K_N)/B_r(K_N)\} \\ &= H_r(K_1) \oplus \ldots \oplus H_r(K_N). \quad \blacksquare \end{aligned}$$

Corollary 3.31

(*a*) Let K be a disjoint union of N connected components, $K_1, \ldots, K_N$. Then it follows that

$$H_0(K) \cong \underbrace{\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{N \text{ factors}}. \tag{3.52}$$

(*b*) If $H_0(K) \cong \mathbb{Z}$, K is connected. [Together with theorem 3.25 we conclude that $H_0(K) \cong \mathbb{Z}$ if and only if K is connected.]

3.4.2 Structure of homology groups

$Z_r(K)$ and $B_r(K)$ are free Abelian groups since they are subgroups of a free Abelian group $C_r(K)$. It does not mean that $H_r(K) = Z_r(K)/B_r(K)$ is also free Abelian. In fact, according to theorem 3.8, the most general form of $H_r(K)$ is

$$H_r(K) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \ldots \oplus \mathbb{Z}}_{f} \oplus \mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_p}. \tag{3.53}$$

It is clear from our experience that the number of generators of $H_r(K)$ counts the number of $(r + 1)$-dimensional holes in $|K|$. The first f factors form a free Abelian group of rank f and the second p factors are called the **torsion subgroup** of $H_r(K)$. For example, the projective plane has $H_1(K) \cong \mathbb{Z}_2$ and the Klein bottle has $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. In a sense, the torsion subgroup detects the 'twisting' in the polyhedron $|K|$.

We now clarify why the homology groups with $\mathbb{Z}$-coefficients are preferable to those with $\mathbb{Z}_2$- or $\mathbb{R}$-coefficients. Since $\mathbb{Z}_2$ has no non-trivial subgroups, the torsion subgroup can never be recognised. Similarly, if $\mathbb{R}$-coefficients are employed, we cannot see the torsion subgroup either, since $\mathbb{R}/m\mathbb{R} \cong 0$ for any $m \in \mathbb{Z} - \{0\}$. [For any a, $b \in \mathbb{R}$, there exists a number $c \in \mathbb{R}$ such that $a - b = mc$.] If $H_r(K; \mathbb{Z})$ is given by (3.53), $H_r(K; \mathbb{R})$ is

$$H_r(K; \mathbb{R}) \cong \underbrace{\mathbb{R} \oplus \mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{f}. \tag{3.54}$$

3.4.3 *Betti numbers and the Euler–Poincaré theorem*

Definition 3.32 Let K be a simplicial complex. The **rth Betti number** $b_r(K)$ is defined by

$$b_r(K) \equiv \dim H_r(K; \mathbb{R}). \tag{3.55}$$

In other words, $b_r(K)$ is the rank of the free Abelian part of $H_r(K; \mathbb{Z})$.

For example, the Betti numbers of the torus T^2 are (see example 3.28)

$$b_0(K) = 1 \qquad b_1(K) = 2 \qquad b_2(K) = 1$$

and those of the sphere S^2 are (exercise 3.24)

$$b_0(K) = 1 \qquad b_1(K) = 0 \qquad b_2(K) = 1.$$

The following theorem relates the Euler characteristic to the Betti numbers.

Theorem 3.33 **(the Euler–Poincaré theorem)** Let K be an n-dimensional simplicial complex and let I_r be the number of r-simplexes in K. Then

$$\chi(K) \equiv \sum_{r=0}^{n} (-1)^r I_r = \sum_{r=0}^{n} (-1)^r b_r(K). \tag{3.56}$$

[*Remark:* The first equality *defines* the Euler characteristic of a general polyhedron $|K|$. Note that this is the generalisation of the Euler characteristic defined for surfaces in §2.4.]

Proof: Consider the boundary homomorphism,

$$\partial_r : C_r(K; \mathbb{R}) \to C_{r-1}(K; \mathbb{R})$$

where $C_{-1}(K; \mathbb{R})$ is defined to be 0. Since both $C_{r-1}(K; \mathbb{R})$ and $C_r(K; \mathbb{R})$ are vector spaces, theorem 2.16 can be applied to yield

$$\begin{aligned} I_r &= \dim C_r(K; \mathbb{R}) = \dim(\ker \partial_r) + \dim(\operatorname{im} \partial_r) \\ &= \dim Z_r(K; \mathbb{R}) + \dim B_{r-1}(K; \mathbb{R}) \end{aligned}$$

where $B_{-1}(K)$ is defined to be trivial. We also have

$$b_r(K) = \dim H_r(K; \mathbb{R}) = \dim[Z_r(K; \mathbb{R})/B_r(K; \mathbb{R})]$$
$$= \dim Z_r(K; \mathbb{R}) - \dim B_r(K; \mathbb{R}).$$

From these relations, we obtain

$$\chi(K) = \sum_{r=0}^{n}(-1)^r I_r = \sum_{r=0}^{n}(-1)^r[\dim Z_r(K; \mathbb{R}) + \dim B_{r-1}(K; \mathbb{R})]$$

$$= \sum_{r=0}^{n}[(-1)^r \dim Z_r(K; \mathbb{R}) - (-1)^r \dim B_r(K; \mathbb{R})]$$

$$= \sum_{r=0}^{n}(-1)^r b_r(K). \quad \blacksquare$$

Since the Betti numbers are topological invariants, $\chi(K)$ is also conserved under a homeomorphism. In particular, if $f : |K| \to X$ and $g : |K'| \to X$ are triangulations of X, we have $\chi(K) = \chi(K')$. Thus it makes sense to define the Euler characteristic of X by $\chi(K)$ for any triangulation (K, f) of X.

Problems 3

1 The most general orientable two-dimensional surface is a two-sphere with h handles and q holes. Compute the homology groups and the Euler characteristic of this surface.

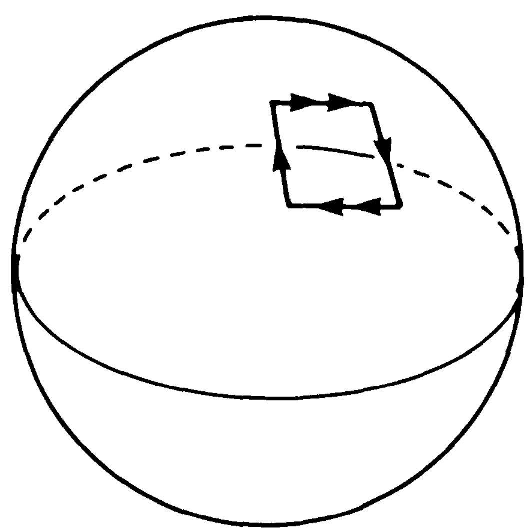


Figure 3.13 A hole in S^2, whose edges are identified as shown. We may consider S^2 with q such holes.

2 Consider a sphere with a hole and identify the edges of the hole as shown in figure 3.13. The surface we obtained was simply the projective plane $\mathbb{R}P^2$. More generally, consider a sphere with q such 'cross caps' and compute the homology groups and the Euler characteristic of this surface.

4

HOMOTOPY GROUPS

The idea of homology groups in the previous chapter was to assign a group structure to cycles that are not boundaries. In homotopy groups, on the other hand, we are interested in continuous deformation of maps one to another. Let X and Y be topological spaces and let $\mathcal{F}$ be the set of continuous maps from X to Y. We introduce an equivalence relation, called 'homotopic to', in $\mathcal{F}$ by which two maps $f, g \in \mathcal{F}$ are identified if the image $f(X)$ is continuously deformed to $g(X)$ in Y. We choose X to be some *standard* topological spaces whose structures are well known. For example, we may take the n-sphere S^n as the standard space and study all the maps from S^n to Y to see how these maps are classified according to homotopic equivalence. This is the basic idea of homotopy groups.

We will restrict ourselves to an elementary study of homotopy groups, which is sufficient for the later discussion. Nash and Sen (1983) and Croom (1978) complement this chapter.

4.1 Fundamental groups

4.1.1 Basic ideas

Let us look at figure 4.1. One disc has a hole in it, the other does not. What characterises the difference between these two discs? We note that any loop in figure 4.1(b) can be continuously shrunk to a point. On the contrary, the loop α in figure 4.1(a) cannot be shrunk to a point due to the existence of a hole in it. Some loops in figure 4.1(a) may be shrunk to a point while others cannot. We say a loop α is homotopic to β if α can be obtained from β by a *continuous* deformation. For example, any loop in Y is homotopic to a point. It turns out that 'homotopic to' is an equivalence relation, the equivalence class of which is called the homotopy class. In figure 4.1, there is only one homotopy class associated with Y. In X, each homotopy class is characterised by $n \in \mathbb{Z}$, n being the number of times the loop encircles the hole; $n < 0$ if it winds clockwise, $n > 0$ if counterclockwise, $n = 0$ if the loop does not wind round the hole. Moreover, $\mathbb{Z}$ is an additive group and the group operation (addition) has a geometrical meaning; $n + m$ corresponds to going round the hole first n times and then m times. The set of homotopy classes is endowed with a group structure called the fundamental group.

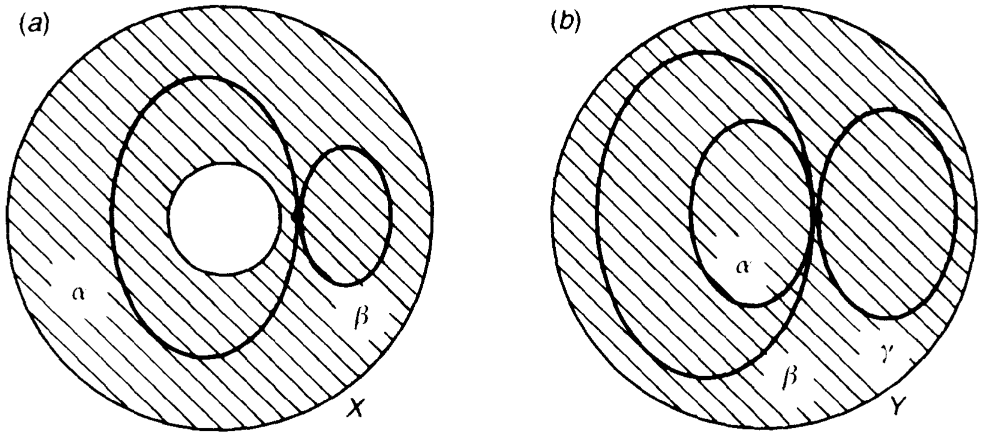


Figure 4.1 A disc with a hole (a) and without a hole (b). The hole in (a) prevents the loop α from shrinking to a point.

4.1.2 Paths and loops

Definition 4.1 Let X be a topological space and let $I = [0, 1]$. A continuous map $\alpha : I \to X$ is called a **path** with an initial point x_0 and an end point x_1 if $\alpha(0) = x_0$ and $\alpha(1) = x_1$. If $\alpha(0) = \alpha(1) = x_0$, the path is called a **loop** with **base point** x_0 (or a loop at x_0).

For $x \in X$, a **constant path** $c_x : I \to X$ is defined by $c_x(s) = x$, $s \in I$. A constant path is also a constant loop since $c_x(0) = c_x(1) = x$. The set of paths or loops in a topological space X may be endowed with an algebraic structure as follows.

Definition 4.2 Let α, $\beta : I \to X$ be paths such that $\alpha(1) = \beta(0)$. The **product** of α and β, denoted by $\alpha * \beta$, is a path in X defined by

$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & 0 \leqslant s \leqslant \frac{1}{2} \\ \beta(2s - 1) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases} \tag{4.1}$$

see figure 4.2. Since $\alpha(1) = \beta(0)$, $\alpha * \beta$ is a continuous map from I to X. [Geometrically, $\alpha * \beta$ corresponds to traversing the image $\alpha(I)$, in the first half, then followed by $\beta(I)$ in the remaining half. Note that the *velocity* is doubled.]

Definition 4.3 Let $\alpha : I \to X$ be a path from x_0 to x_1. The **inverse path** α^{-1} of α is defined by

$$\alpha^{-1}(s) \equiv \alpha(1 - s). \tag{4.2}$$

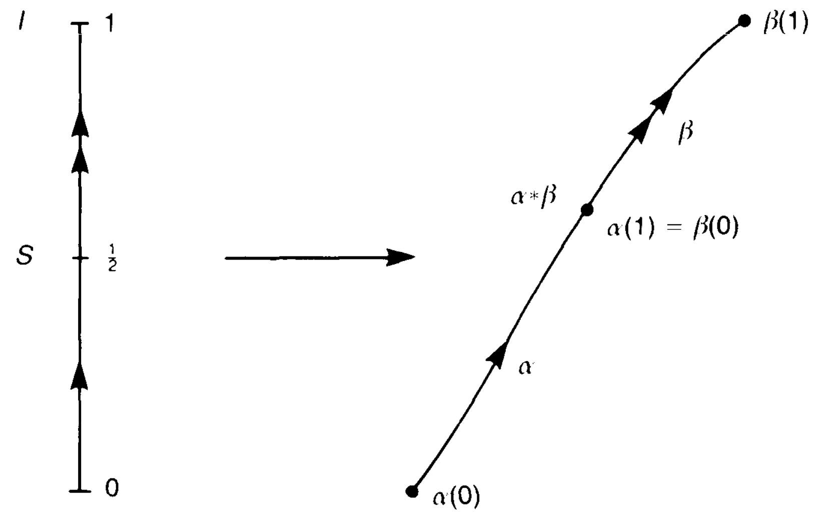


Figure 4.2 The product $\alpha*\beta$ of paths α and β with a common end point.

[The inverse path α^{-1} corresponds to traversing the image of α in the opposite direction from x_1 to x_0.]

Since a loop is a special path for which the initial point and end point agree, the product of loops and the inverse of a loop are defined in exactly the same way. It seems that a constant map c_x is the unit element. However, it is not; $\alpha*\alpha^{-1}$ is not equal to c_x! We need a concept of homotopy to define a group operation in the space of loops.

4.1.3 Homotopy

The algebraic structure of loops introduced above is not so useful as it is. For example, the constant path is not exactly the unit element. We want to classify the paths and loops according to a neat equivalence relation so that the equivalence classes admit a group structure. It turns out that if we identify paths or loops that can be deformed continuously one into another, the equivalence classes form a group. Since we are primarily interested in loops, most definitions and theorems are given for loops. However, it should be kept in mind that many statements are also applied to paths with proper modifications.

Definition 4.4 Let $\alpha, \beta : I \to X$ be loops at x_0. They are said to be **homotopic**, written as $\alpha \sim \beta$, if there exists a *continuous* map $F : I \times I \to X$ such that

$$F(s, 0) = \alpha(s), \; F(s, 1) = \beta(s) \qquad \text{for all } s \in I \qquad (4.3a)$$

$$F(0, t) = F(1, t) = x_0 \qquad \text{for all } t \in I. \qquad (4.3b)$$

The connecting map F is called a **homotopy** between α and β.

It is helpful to represent a homotopy as figure 4.3(a). The vertical edges of the square $I \times I$ are mapped to x_0. The lower edge is $\alpha(s)$ while the upper edge is $\beta(s)$. In the space X, the image is continuously deformed as in figure 4.3(b).

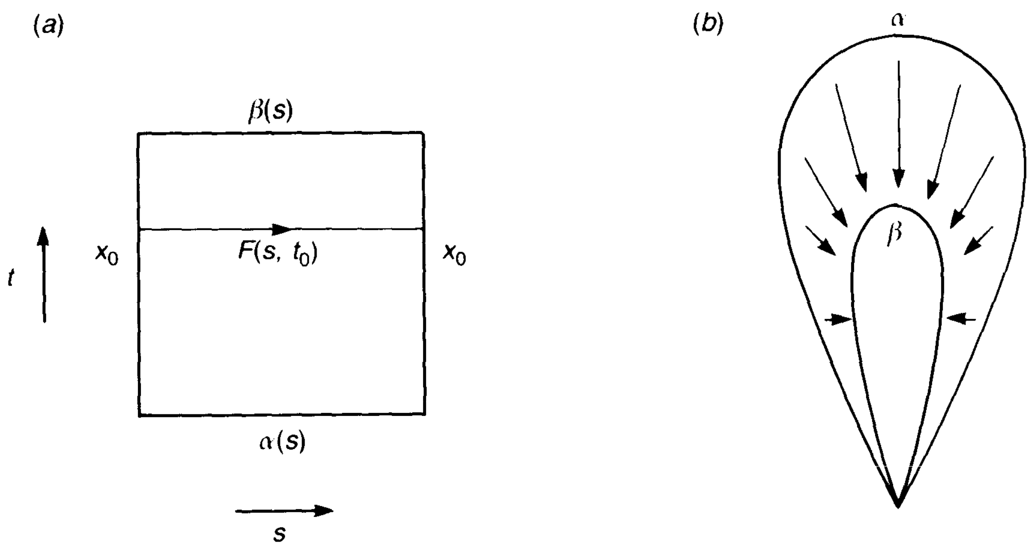


Figure 4.3 (a) The square represents a homotopy F interpolating the loops α and β. (b) In real space the image of α is continuously deformed to the image of β.

Proposition 4.5 The relation '$\alpha \sim \beta$' is an equivalence relation.

Proof: Reflectivity: $\alpha \sim \alpha$. The homotopy may be given by $F(s, t) \equiv \alpha(s)$ for any $t \in I$.
Symmetry: Let $\alpha \sim \beta$ with the homotopy $F(s, t)$ such that $F(s, 0) = \alpha(s)$, $F(s, 1) = \beta(s)$. Then $\beta \sim \alpha$, where the homotopy is given by $F(s, 1 - t)$.
Transitivity: Let $\alpha \sim \beta$ and $\beta \sim \gamma$. Then $\alpha \sim \gamma$. If $F(s, t)$ is a homotopy between α and β and $G(s, t)$ is a homotopy between β and γ, a homotopy between α and γ may be (figure 4.4)

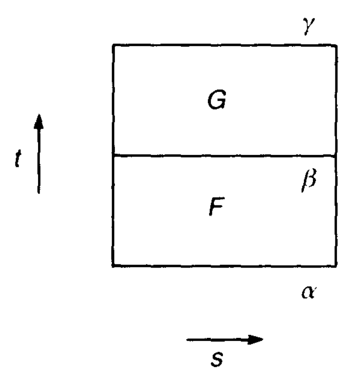


Figure 4.4 A homotopy H between α and γ via β.

$$H(s, t) = \begin{cases} F(s, 2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ G(s, 2t - 1) & \frac{1}{2} \leqslant t \leqslant 1. \ \blacksquare \end{cases}$$

4.1.4 Fundamental groups

The equivalence class of loops is denoted by $[\alpha]$ and is called the **homotopy class** of α. The product between loops naturally defines the product in the set of homotopy classes of loops.

Definition 4.6 Let X be a topological space. The set of homotopy classes of loops at $x_0 \in X$ is denoted by $\pi_1(X, x_0)$ and is called the **fundamental group** (or the **first homotopy group**) of X at x_0. The product of homotopy classes $[\alpha]$ and $[\beta]$ is defined by

$$[\alpha]*[\beta] \equiv [\alpha*\beta]. \tag{4.4}$$

Lemma 4.7 The product of homotopy classes is independent of the representative, that is, if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\alpha*\beta \sim \alpha'*\beta'$.

Proof: Let $F(s, t)$ be a homotopy between α and α' and $G(s, t)$ be a homotopy between β and β'. Then

$$H(s, t) \quad \begin{cases} F(2s, t) & 0 \leqslant s \leqslant \frac{1}{2} \\ G(2s - 1, t) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

is a homotopy between $\alpha*\beta$ and $\alpha'*\beta'$, hence $\alpha*\beta \sim \alpha'*\beta'$ and $[\alpha]*[\beta]$ is well defined. ■

Theorem 4.8 The fundamental group is a group. Namely, if α, β, ... are loops at $x \in X$, the following group properties are satisfied.

(1) $([\alpha]*[\beta])*[\gamma] = [\alpha]*([\beta]*[\gamma])$
(2) $[\alpha]*[c_x] = [\alpha]$ and $[c_x]*[\alpha] = [\alpha]$ (unit element)
(3) $[\alpha]*[\alpha^{-1}] = [c_x]$, hence $[\alpha]^{-1} = [\alpha^{-1}]$ (inverse).

Proof: (1) Let $F(s, t)$ be a homotopy between $(\alpha*\beta)*\gamma$ and $\alpha*(\beta*\gamma)$. It may be given by (figure 4.5(a))

$$F(s, t) = \begin{cases} \alpha[4s/(1+t)] & 0 \leqslant s \leqslant (1+t)/4 \\ \beta(4s - t - 1) & (1+t)/4 \leqslant s \leqslant (2+t)/4 \\ \gamma[(4s - t - 2)/(2 - t)] & (2+t)/4 \leqslant s \leqslant 1. \end{cases}$$

Thus we may simply write $[\alpha*\beta*\gamma]$ to denote $[(\alpha*\beta)*\gamma]$ or $[\alpha*(\beta*\gamma)]$.

(2) Define a homotopy $F(s, t)$ by (figure 4.5(b))

$$F(s, t) = \begin{cases} \alpha(2s/(1+t)) & 0 \leqslant s \leqslant (t+1)/2 \\ x & (t+1)/2 \leqslant s \leqslant 1. \end{cases}$$

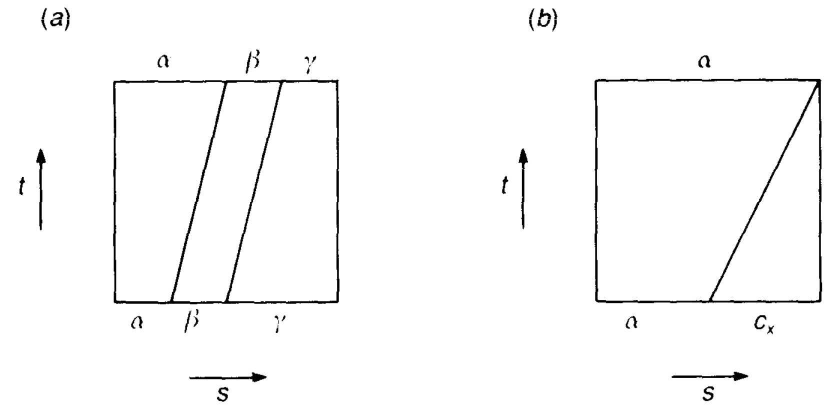


Figure 4.5 (*a*) A homotopy between $(\alpha*\beta)*\gamma$ and $\alpha*(\beta*\gamma)$. (*b*) A homotopy between $\alpha*c_x$ and α.

Clearly this is a homotopy between $\alpha*c_x$ and α. Similarly a homotopy between $c_x*\alpha$ and α is given by

$$F'(s, t) = \begin{cases} x & 0 \leqslant s \leqslant (1-t)/2 \\ \alpha[(2s-1+t)/(1+t)] & (1-t)/2 \leqslant s \leqslant 1. \end{cases}$$

This shows that $[\alpha]*[c_x] = [\alpha] = [c_x]*[\alpha]$.

(3) Define a map $F : I \times I \to X$ by

$$F(s, t) = \begin{cases} \alpha[2s(1-t)] & 0 \leqslant s \leqslant \frac{1}{2} \\ \alpha[2(1-s)(1-t)] & \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

Clearly $F(s, 0) = \alpha*\alpha^{-1}$ and $F(s, 1) = c_x$, hence

$$[\alpha*\alpha^{-1}] = [\alpha]*[\alpha^{-1}] = [c_x].$$

This shows that $[\alpha^{-1}] = [\alpha]^{-1}$. ■

In summary, $\pi_1(X, x)$ is a group whose unit element is the homotopy class of the constant loop c_x. The product $[\alpha]*[\beta]$ is well defined and satisfies the group axioms. The inverse of $[\alpha]$ is $[\alpha]^{-1} \equiv [\alpha^{-1}]$. In the next section we study the general properties of fundamental groups, which simplify the actual computations.

4.2 General properties of fundamental groups

4.2.1 Arcwise connectedness and fundamental groups

In §2.3 we defined a topological space X to be arcwise connected if, for any $x_0, x_1 \in X$, there exists a path α such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$.

Theorem 4.9 Let X be an arcwise-connected topological space and let $x_0, x_1 \in X$. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Proof: Let $\eta : I \to X$ be a path such that $\eta(0) = x_0$ and $\eta(1) = x_1$. If α is a loop at x_0, then $\eta^{-1}*\alpha*\eta$ is a loop at x_1 (figure 4.6). Given an element $[\alpha] \in \pi_1(X, x_0)$, the above correspondence induces a unique element $[\alpha'] = [\eta^{-1}*\alpha*\eta] \in \pi_1(X, x_1)$. We denote this map by $P_\eta : \pi_1(X, x_0) \to \pi_1(X, x_1)$ so that $[\alpha'] = P_\eta([\alpha])$.

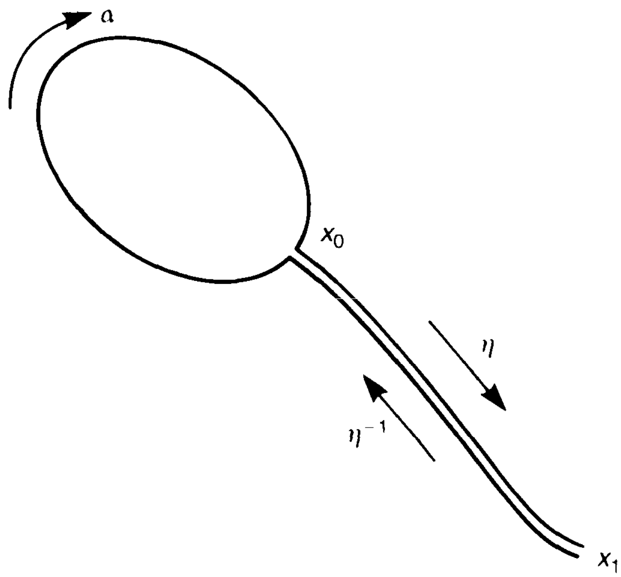


Figure 4.6 From a loop α at x_0, a loop $\eta^{-1}*\alpha*\eta$ at x_1 is constructed.

We show that P_η is an isomorphism. Firstly, P_η is a *homomorphism*, since for $[\alpha], [\beta] \in \pi_1(X, x_0)$ we have

$$\begin{aligned} P_\eta([\alpha]*[\beta]) &= [\eta^{-1}]*[\alpha]*[\beta]*[\eta] \\ &= [\eta^{-1}]*[\alpha]*[\eta]*[\eta^{-1}]*[\beta]*[\eta] \\ &= P_\eta([\alpha])*P_\eta([\beta]). \end{aligned}$$

To show that P_η is *bijective*, we introduce the inverse of P_η. Define a map $P_\eta^{-1} : \pi_1(X, x_1) \to \pi_1(X, x_0)$ whose action on $[\alpha']$ is $P_\eta^{-1}([\alpha']) = [\eta*\alpha'*\eta^{-1}]$. Clearly P_η^{-1} is the inverse of P_η since

$$P_\eta^{-1}P_\eta([\alpha]) = P_\eta^{-1}([\eta^{-1}*\alpha*\eta]) = [\eta*\eta^{-1}*\alpha*\eta*\eta^{-1}] = [\alpha].$$

Thus $P_\eta^{-1}P_\eta = \mathrm{id}_{\pi_1(X,x_0)}$. From the symmetry, we have $P_\eta P_\eta^{-1} = \mathrm{id}_{\pi_1(X,x_1)}$. We find from exercise 2.7 that P_η is one-to-one and onto. ■

Accordingly, if X is arcwise connected, we do not need to specify the base point since $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for any $x_0, x_1 \in X$, and we may simply write $\pi_1(X)$.

Exercise 4.10

(1) Let η and ζ be paths from x_0 to x_1 such that $\eta \sim \zeta$. Show that $P_\eta = P_\zeta$.

(2) Let η and ζ be paths such that $\eta(1) = \zeta(0)$. Show that $P_{\eta * \zeta} = P_\zeta \cdot P_\eta$.

4.2.2 Homotopic invariance of fundamental groups

The homotopic equivalence of paths and loops is easily generalised to arbitrary maps. Let $f, g : X \to Y$ be continuous maps. If there exists a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, f is said to be **homotopic** to g, denoted by $f \sim g$. The map F is called a **homotopy** between f and g.

Definition 4.11 Let X and Y be topological spaces. X and Y are of the same **homotopy type**, written as $X \simeq Y$, if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $fg \sim \mathrm{id}_Y$ and $gf \sim \mathrm{id}_X$. f is called the **homotopy equivalence** and g, its **homotopy inverse**. [*Remark:* If X is homeomorphic to Y, X and Y are of the same homotopy type but the converse is not necessarily true. For example, a point $\{p\}$ and the real line $\mathbb{R}$ are of the same homotopy type but $\{p\}$ is not homeomorphic to $\mathbb{R}$.]

Proposition 4.12 'Of the same homotopy type' is an equivalence relation in the set of topological spaces.

Proof: Reflectivity: $X \simeq X$ where id_X is a homotopy equivalence.
Symmetry: Let $X \simeq Y$ with the homotopy equivalence $f : X \to Y$. Then $Y \simeq X$, the homotopy equivalence being the homotopy inverse of f.
Transitivity: Let $X \simeq Y$ and $Y \simeq Z$. Suppose $f : X \to Y$, $g : Y \to Z$ are homotopy equivalences and $f' : Y \to X$, $g' : Z \to Y$, their homotopy inverses. Then

$$(gf)(f'g') = g(ff')g' \sim g\,\mathrm{id}_Y g' = gg' \sim \mathrm{id}_Z$$
$$(f'g')(gf) = f'(g'g)f \sim f'\mathrm{id}_Y f = f'f \sim \mathrm{id}_X$$

from which it follows $X \simeq Z$. ■

One of the most remarkable properties of the fundamental groups is that two topological spaces of the same homotopy type have the same fundamental group.

Theorem 4.13 Let X and Y be topological spaces of the same homotopy type. If $f : X \to Y$ is a homotopy equivalence, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, f(x_0))$.

The following corollary follows directly from theorem 4.13.

Corollary 4.14 A fundamental group is invariant under homeomorphisms, and hence is a topological invariant.

In this sense, we must admit that fundamental groups classify topological spaces in a less strict manner than homeomorphisms. What we claim at most is that if topological spaces X and Y have different fundamental groups, X cannot be homeomorphic to Y. Note, however, that the homotopy groups including the fundamental groups have many applications to physics as we shall see in due course. We should stress that the main usage of the homotopy groups in physics is not to classify spaces but to classify maps or field configurations.

It is rather difficult to appreciate what is meant by 'of the same homotopy type' for an arbitrary pair of X and Y. In practice, however, it often happens that Y is a subspace of X. We then claim that $X \simeq Y$ if Y is obtained by a *continuous deformation* of X.

Definition 4.15 Let R $(\neq \emptyset)$ be a subspace of X. If there exists a continuous map $f : X \to R$ such that $f|_R = \mathrm{id}_R$, R is called a **retract** of X and f a **retraction**.

Note that the whole of X is mapped onto R *keeping points in R fixed*. Figure 4.7 is an example of a retract and retraction.

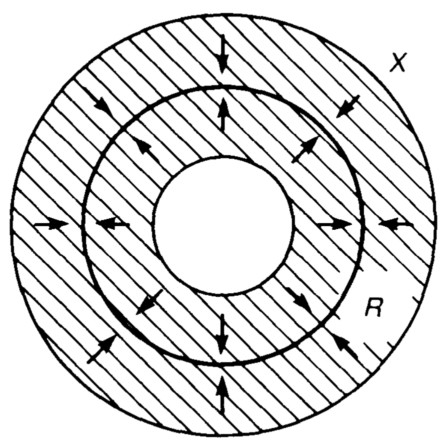


Figure 4.7 The circle R is a retract of the annulus X. The arrows depict the action of the retraction.

Definition 4.16 Let R be a subspace of X. If there exists a continuous map $H : X \times I \to X$ such that

$$H(x, 0) = x \qquad H(x, 1) \in R \qquad \text{for any } x \in X \quad (4.5a)$$

$$H(x, t) = x \qquad \text{for any } t \in I \text{ and any } x \in R \quad (4.5b)$$

R is said to be a **deformation retract** of X. Note that H is a homotopy between id_X and a retraction $f : X \to R$, which leaves all the points in R fixed during deformation.

A retract is not necessarily a deformation retract. In figure 4.8, the circle R is a retract of X but not a deformation retract, since the hole in X is an obstruction to continuous deformation of id_X to the retraction.

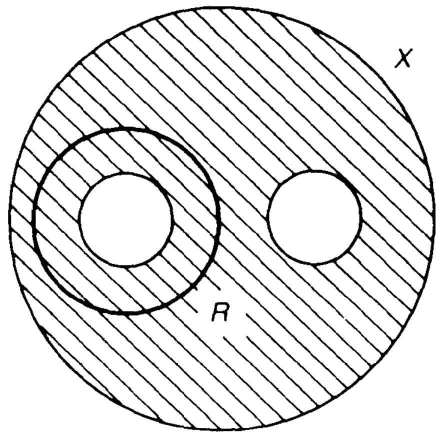


Figure 4.8 The circle R is not a deformation retract of X.

Since X and R are of the same homotopy type, we have

$$\pi_1(X, a) \cong \pi_1(R, a) \qquad a \in R. \tag{4.6}$$

Example 4.17 Let X be the unit circle and Y be the annulus,

$$X = \{e^{i\theta} | 0 \leqslant \theta < 2\pi\}$$
$$Y = \{re^{i\theta} | 0 \leqslant \theta < 2\pi, \tfrac{1}{2} \leqslant r \leqslant \tfrac{2}{3}\}$$

see figure 4.7. Define $f: X \hookrightarrow Y$ by $f(e^{i\theta}) = e^{i\theta}$ and $g: Y \to X$ by $g(re^{i\theta}) = e^{i\theta}$. Then $fg: re^{i\theta} \mapsto e^{i\theta}$ and $gf: e^{i\theta} \mapsto e^{i\theta}$. Observe that $fg \sim \mathrm{id}_Y$ and $gf = \mathrm{id}_X \sim \mathrm{id}_X$. There exists a homotopy

$$H(re^{i\theta}, t) = \{1 + (r-1)(1-t)\}e^{i\theta}$$

which interpolates between id_X and fg, keeping the points on X fixed. Hence X is a deformation retract of Y. As for the fundamental groups we have $\pi_1(X, a) \cong \pi_1(Y, a)$ where $a \in X$.

Definition 4.18 If a point $a \in X$ is a deformation retract of X, X is said to be **contractible**.

Let $c_a : X \to \{a\}$ be a constant map. If X is contractible, there exists a homotopy $H: X \times I \to X$ such that $H(x, 0) = c_a(x) = a$ and $H(x, 1) = \mathrm{id}_X(x) = x$ for any $x \in X$ and, moreover, $H(a, t) = a$ for any $t \in I$. The homotopy H is called the **contraction**.

Example 4.19 $X = \mathbb{R}^n$ is contractible to the origin 0. In fact, if we define $H: \mathbb{R}^n \times I \to \mathbb{R}^n$ by $H(x, t) = tx$, we have (i) $H(x, 0) = 0$ and

$H(x, 1) = x$ for any $x \in X$ and (ii) $H(0, t) = 0$ for any $t \in I$. Now it is clear that any convex subset of $\mathbb{R}^n$ is contractible.

Exercise 4.20 Let $D^2 = \{(x, y)|x^2 + y^2 \leqslant 1\}$. Show that the unit circle S^1 is a deformation retract of $D^2 - \{0\}$. Show also that the unit sphere S^n is a deformation retract of $D^{n+1} - \{0\}$.

Theorem 4.21 The fundamental group of a contractible space X is trivial, $\pi_1(X, x_0) \cong \{e\}$. In particular, the fundamental group of $\mathbb{R}^n$ is trivial, $\pi_1(\mathbb{R}^n, x_0) \cong \{e\}$.

Proof: A contractible space has the same fundamental group as a point $\{p\}$ and a point has a trivial fundamental group. ■

If an arcwise-connected space X has a trivial fundamental group, X is said to be **simply connected**.

4.3 Examples of fundamental groups

In general, there does not exist a routine procedure to compute the fundamental groups. However, in certain cases, they are obtained by relatively simple considerations. Here we look at the fundamental groups of the circle S^1 and related spaces.

Let us express S^1 as $\{z \in \mathbb{C}||z| = 1\}$. Define a map $p : \mathbb{R} \to S^1$ by $p : x \mapsto \exp(ix)$. Under p, the point $0 \in \mathbb{R}$ is mapped to $1 \in S^1$, which is taken to be the base point. We imagine that $\mathbb{R}$ wraps around S^1 under p, see figure 4.9. If $x, y \in \mathbb{R}$ satisfies $x - y = 2\pi m$ ($m \in \mathbb{Z}$), they are mapped to the same point in S^1. Then we write $x \sim y$. This is an equivalence relation and the equivalence class $[x] = \{y|x - y = 2\pi m$ for some $m \in \mathbb{Z}\}$ is identified with a point $\exp(ix) \in S^1$. It then follows that $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. Let $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ be a map such that $\widetilde{f}(0) = 0$ and $\widetilde{f}(x + 2\pi) \sim \widetilde{f}(x)$. It is obvious that $\widetilde{f}(x + 2\pi) = f(x) + 2n\pi$ for any $x \in \mathbb{R}$, where n is a fixed integer. If $x \sim y$ ($x - y = 2\pi m$), we have

$$\begin{aligned} \widetilde{f}(x) - \widetilde{f}(y) &= \widetilde{f}(y + 2\pi m) - \widetilde{f}(y) \\ &= \widetilde{f}(y) + 2\pi mn - \widetilde{f}(y) = 2\pi mn \end{aligned}$$

hence $\widetilde{f}(x) \sim \widetilde{f}(y)$. Accordingly $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ uniquely defines a continuous map $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ by $f([x]) = p\cdot\widetilde{f}(x)$; see figure 4.9. Note that f keeps the base point $1 \in S^1$ fixed. Conversely given a map $f : S^1 \to S^1$, which leaves $1 \in S^1$ fixed, we may define a map $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{f}(0) = 0$ and $\widetilde{f}(x + 2\pi) = \widetilde{f}(x) + 2\pi n$.

In summary, there is a one-to-one correspondence between the set of maps from S^1 to S^1 with $f(1) = 1$ and the set of maps from $\mathbb{R}$ to $\mathbb{R}$ such that $\tilde{f}(0) = 0$ and $\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2\pi n$. The integer n is called the **degree** of f and is denoted by $\deg(f)$. While x encircles S^1 once, $f(x)$ encircles S^1 n times.

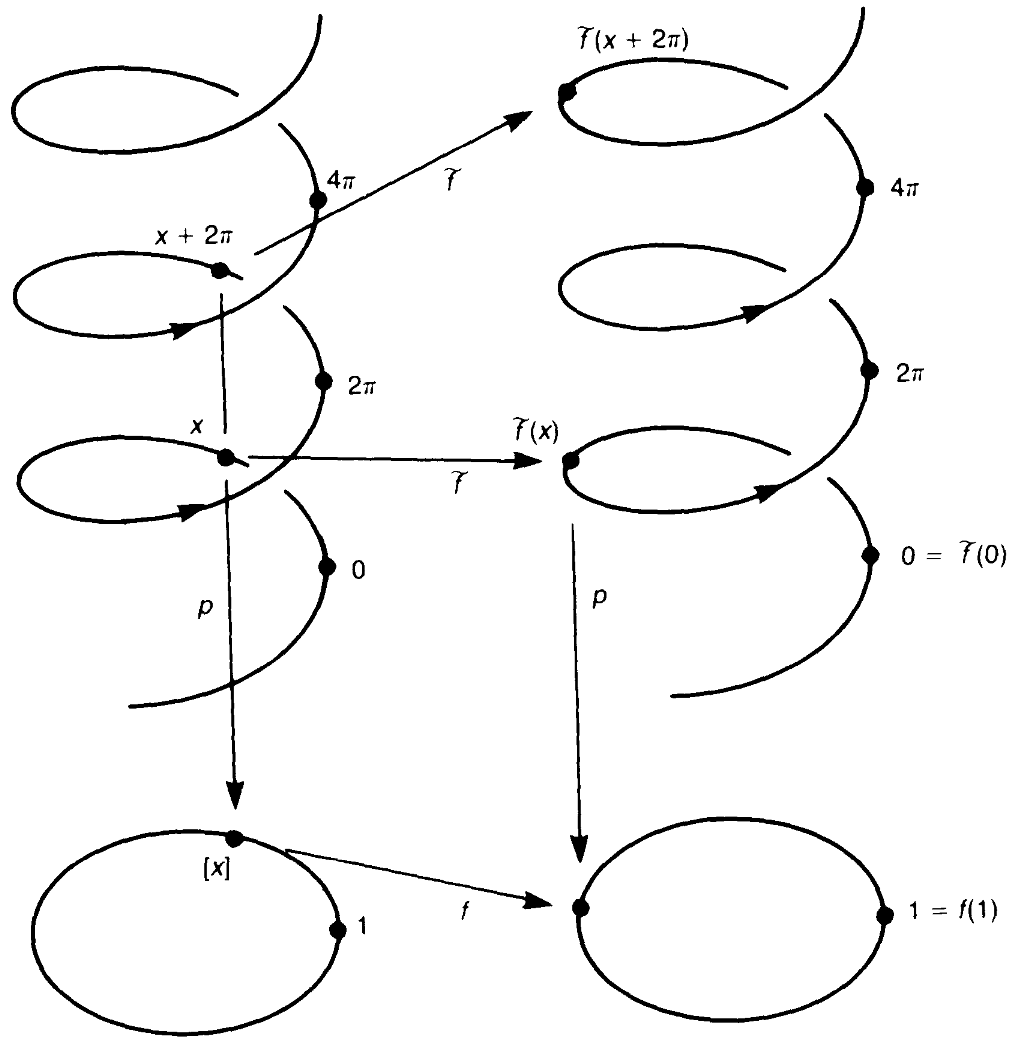


Figure 4.9 $p : \mathbb{R} \to S^1$ defined by $x \mapsto \exp(ix)$ projects $x + 2m\pi$ to the same point on S^1. $\tilde{f}:\mathbb{R} \to \mathbb{R}$, such that $\tilde{f}(0) = 0$ and $\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2n\pi$ for fixed n, defines a map $f : S^1 \to S^1$. The integer n specifies the homotopy class to which f belongs.

Lemma 4.22

(1) Let $f, g : S^1 \to S^1$ such that $f(0) = g(0) = 1$. Then $\deg(f) = \deg(g)$ if and only if f is homotopic to g.

(2) For any $n \in \mathbb{Z}$, there exists a map $f : S^1 \to S^1$ such that $\deg(f) = n$.

Proof: (1) Let $\deg(f) = \deg(g)$ and $\tilde{f}, \tilde{g} : \mathbb{R} \to \mathbb{R}$ be the corresponding maps. Then $\tilde{F}(x, t) \equiv t\tilde{f}(x) + (1 - t)\tilde{g}(x)$ is a homotopy between

$\widetilde{f}(x)$ and $\widetilde{g}(x)$. It is easy to verify that $F \equiv p\widetilde{F}$ is a homotopy between f and g. Conversely, if $f \sim g : S^1 \to S^1$, there exists a homotopy $F : S^1 \times I \to S^1$ such that $F(1, t) = 1$ for any $t \in I$. The corresponding homotopy $\widetilde{F} : \mathbb{R} \times I \to \mathbb{R}$ between $\widetilde{f}$ and $\widetilde{g}$ satisfies $\widetilde{F}(x + 2\pi, t) = \widetilde{F}(x, t) + 2n\pi$ for some $n \in \mathbb{Z}$. Thus $\deg(f) = \deg(g)$.

(2) $\widetilde{f} : x \mapsto nx$ induces a map $f : S^1 \to S^1$ with $\deg(f) = n$. ■

Lemma 4.22 tells us that by assigning an integer $\deg(f)$ to a map $f : S^1 \to S^1$ such that $f(1) = 1$, there is a bijection between $\pi_1(S^1, 1)$ and $\mathbb{Z}$. Moreover, this is an isomorphism. In fact, for $f, g : S^1 \to S^1$, $f{*}g$, defined as a product of loops, satisfies $\deg(f{*}g) = \deg(f) + \deg(g)$. [Let $\widetilde{f}(x + 2\pi) = \widetilde{f}(x) + 2\pi n$ and $\widetilde{g}(x + 2\pi) = \widetilde{g}(x) + 2\pi m$. Then $f{*}g(x + 2\pi) = f{*}g(x) + 2\pi(m + n)$. Note that $*$ is not a composite of maps but a *product* of paths.] We have finally proved the following theorem.

Theorem 4.23 The fundamental group of S^1 is isomorphic to $\mathbb{Z}$.

$$\pi_1(S^1) \cong \mathbb{Z}. \tag{4.7}$$

[Since S^1 is arcwise connected, we may drop the base point.]

Although the proof of the theorem is not too obvious, the statement itself is easily understood even by children. Suppose we encircle a cylinder with an elastic band. If it encircles the cylinder n times, the configuration cannot be continuously deformed to that with m $(\neq n)$ encirclements. If an elastic band encircles a cylinder first n times and then m times, it encircles the cylinder $n + m$ times in total.

4.3.1 Fundamental group of torus

Theorem 4.24 Let X and Y be arcwise-connected topological spaces. Then $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \oplus \pi_1(Y, y_0)$.

Proof: Define projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. If α is a loop in $X \times Y$ at (x_0, y_0), $\alpha_1 \equiv p_1 \cdot \alpha$ is a loop in X at x_0 and $\alpha_2 \equiv p_2 \cdot \alpha$ is a loop in Y at y_0. Conversely any pair of loops α_1 of X at x_0 and α_2 of Y at y_0 determines a unique loop $\alpha = (\alpha_1, \alpha_2)$ of $X \times Y$ at (x_0, y_0). Define a homomorphism $\varphi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$ by

$$\varphi([\alpha]) = ([\alpha_1], [\alpha_2]).$$

By construction φ has an inverse, hence is the required isomorphism and $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$. ■

Example 4.25

(1) Let $T^2 = S^1 \times S^1$ be a torus. Then

$$\pi_1(T^2) \cong \pi_1(S^1) \oplus \pi_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{4.8a}$$

Similarly for the n-dimensional torus

$$T^n = \underbrace{S^1 \times S^1 \times \ldots \times S^1}_{n}$$

we have

$$\pi_1(T^n) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{n}. \tag{4.8b}$$

(2) Let $X = S^1 \times \mathbb{R}$ be a cylinder. Since $\pi_1(\mathbb{R}) \cong \{e\}$, we have

$$\pi_1(X) \cong \mathbb{Z} \oplus \{e\} \cong \mathbb{Z}. \tag{4.9}$$

4.4 Fundamental groups of polyhedra

The computation of fundamental groups in the previous section was, in a sense, *ad hoc* and we certainly need a more systematic way of computing the fundamental groups. Fortunately if a space X is triangulable, we can compute the fundamental group of the polyhedron $|K|$, and hence that of X by a routine procedure. Let us start with some aspects of group theories.

4.4.1 Free groups and relations

The free groups that we define here are not necessarily Abelian and we employ multiplicative notation for the group operation. A subset $X = \{x_j\}$ of a group G is called a **free set of generators** of G if any element $g \in G - \{e\}$ is *uniquely* written as

$$g = x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \tag{4.10}$$

where n is finite and $i_k \in \mathbb{Z}$. We assume no adjacent x_j are equal; $x_j \neq x_{j+1}$. If $i_j = 1$, x_j^1 is simply written as x_j. If $i_j = 0$, the term x_j^0 should be dropped from g. For example, $g = a^3 b^{-2} c b^3$ is acceptable but $h = a^3 a^{-2} c b^0$ is not. If each element is to be written uniquely, h must be *reduced* to $h = ac$. If G has a free set of generators, it is called a **free group**.

Conversely, given a set X, we can construct a free group G whose free set of generators is X. Let us call each element of X a **letter**. The product

$$w = x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \tag{4.11}$$

is called a **word**, where $x_j \in X$ and $n_j \in \mathbb{Z}$. If $n_j \neq 0$ and $x_j \neq x_{j+1}$, the word is called a **reduced word**. It is always possible to reduce a word by finite steps. For example,

$$a^{-2}b^{-3}b^{3}a^{4}b^{3}c^{-2}c^{4} = a^{-2}b^{0}a^{4}b^{3}c^{2} = a^{2}b^{3}c^{2}.$$

A word with no letters is called an **empty word** and denoted by 1. For example, it is obtained by reducing $w = a^{0}$.

A product of words is defined by simply juxtaposing two words. Note that a product of reduced words may not be reduced, but it is always possible to reduce it. For example, if $v = a^{2}c^{-3}b^{2}$ and $w = b^{-2}c^{2}b^{3}$, the product vw is reduced as

$$vw = a^{2}c^{-3}b^{2}b^{-2}c^{2}b^{3} = a^{2}c^{-3}c^{2}b^{3} = a^{2}c^{-1}b^{3}.$$

Thus the set of all reduced words form a well defined free group called the free group generated by X, denoted by $F[X]$. The multiplication is the juxtaposition of two words followed by reduction, the unit element is the empty word and the inverse of

$$w = x_1{}^{i_1}x_2{}^{i_2}\ldots x_n{}^{i_n}$$

is

$$w^{-1} = x_n{}^{-i_n}\ldots x_2{}^{-i_2}x_1{}^{-i_1}.$$

Exercise 4.26 Let $X = \{a\}$. Show that the free group generated by X is isomorphic to $\mathbb{Z}$.

In general, an arbitrary group G is specified by the generators and certain *constraints* that these must satisfy. If $\{x_k\}$ is the set of generators, the constraints are most commonly written as

$$r = x_{k_1}{}^{i_1}x_{k_2}{}^{i_2}\ldots x_{k_n}{}^{i_n} = 1 \tag{4.12}$$

and are called **relations**. For example, the cyclic group of order n generated by x (in multiplicative notation) satisfies a relation $x^{n} = 1$.

More formally, let G be a group which is generated by $X = \{x_k\}$. Any element $g \in G$ is written as $g = x_1{}^{i_1}x_2{}^{i_2}\ldots x_n{}^{i_n}$, where we do *not* require that the expression be unique (G is not necessarily free). For example, we have $x^{i} = x^{n+i}$ in $\mathbb{Z}_n$. Let $F[X]$ be the free group generated by X. Then there is a natural homomorphism φ from $F[X]$ *onto* G defined by

$$x_1{}^{i_1}x_2{}^{i_2}\ldots x_n{}^{i_n} \xrightarrow{\varphi} x_1{}^{i_1}x_2{}^{i_2}\ldots x_n{}^{i_n} \in G. \tag{4.13}$$

Note that this is not an isomorphism since the LHS is not unique. φ is onto since X generates both $F[X]$ and G. Although $F[X]$ is not isomorphic to G, $F[X]/\ker\varphi$ is (see theorem 3.3),

$$F[X]/\ker\varphi \cong G. \tag{4.14}$$

In this sense, the set of generators X and $\ker\varphi$ completely determine the group G. [$\ker\varphi$ is a normal subgroup. Lemma 3.2 claims that $\ker\varphi$ is a subgroup of $F[X]$. Let $r \in \ker\varphi$, that is, $r \in F[X]$ and $\varphi(r) = 1$. For any element $x \in F[X]$, we have $\varphi(x^{-1}rx) = \varphi(x^{-1})\varphi(r)\varphi(x) = \varphi(x)^{-1}\varphi(x) = 1$, hence $x^{-1}rx \in \ker\varphi$.]

In this way, a group G generated by X is specified by the relations. The juxtaposition of generators and relations

$$(x_1, \ldots, x_p : r_1, \ldots, r_q) \tag{4.15}$$

is called the **presentation** of G. For example, $\mathbb{Z}_n = (x : x^n)$ and $\mathbb{Z} = (x : \varnothing)$.

Example 4.27 Let $\mathbb{Z} \oplus \mathbb{Z} = \{x^n y^m | n, m \in \mathbb{Z}\}$ be a free *Abelian* group generated by $X = \{x, y\}$. Then we have $xy = yx$. Since $xyx^{-1}y^{-1} = 1$, we have a relation $r = xyx^{-1}y^{-1}$. The presentation of $\mathbb{Z} \oplus \mathbb{Z}$ is $(x, y : xyx^{-1}y^{-1})$.

4.4.2 Calculating fundamental groups of polyhedra

We shall be sketchy here to avoid getting into the technical details. We shall follow Armstrong (1983); the interested reader should consult this book or any textbook on algebraic topology. As noted in the previous chapter, a polyhedron $|K|$ is a nice approximation of a given topological space X within a homeomorphism. Since fundamental groups are topological invariants, we have $\pi_1(X) = \pi_1(|K|)$. We assume X is an arcwise-connected space and drop the base point. Accordingly, if we have a systematic way of computing $\pi_1(|K|)$, we can also find $\pi_1(X)$.

We first define the *edge group* of a simplicial complex, which corresponds to the fundamental group of a topological space, then introduce a convenient way of computing it. Let $f : |K| \to X$ be a triangulation of a topological space X. If we note that an element of the fundamental group of X can be represented by loops in X, we expect that similar loops must exist in $|K|$ as well. Since any loop in $|K|$ is made up of 1-simplexes, we look at the set of all 1-simplexes in $|K|$, which can be endowed with a group structure called the edge group of K.

An **edge path** in a simplicial complex K is a sequence $v_0 v_1 \ldots v_k$ of vertices of $|K|$, in which the consecutive pair $v_i v_{i+1}$ is a 0- or 1-simplex of K. [For technical reasons, we allow the possibility $v_i = v_{i+1}$, in which case the relevant simplex is a 0-simplex $v_i = v_{i+1}$.] If $v_0 = v_k$ $(=v)$, the edge path is called an **edge loop** at v. We classify these loops into equivalence classes according to some equivalence relation. We define two edge loops α and β to be equivalent if one is obtained from the other by repeating the following operations a finite number of times.

(1) If the vertices u, v and w span a 2-simplex in K, the edge path uvw may be replaced by uw and vice versa; see figure 4.10(a).

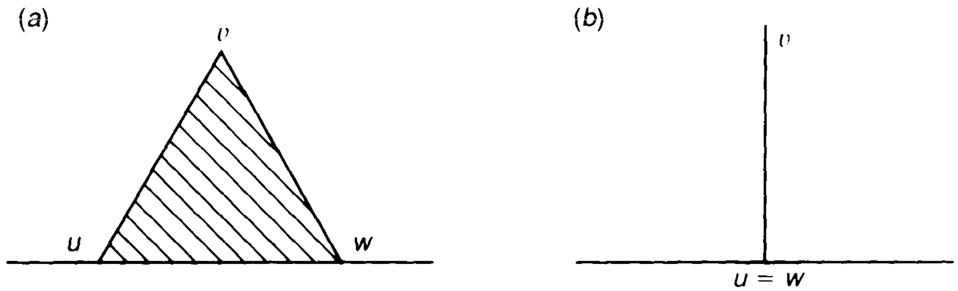


Figure 4.10 Possible deformations of the edge loops. In (a), uvw is replaced by uw. In (b), uvu is replaced by u.

(2) As a special case, if $u = w$ in (1), the edge path uvw corresponds to traversing along uv first then reversing backwards from v to $w = u$. This edge path uvu may be replaced by a 0-simplex u and vice versa, see figure 4.10(b).

Let us denote the equivalence class of edge loops at v, to which $vv_1 \ldots v_{k-1}v$ belongs by $\{vv_1 \ldots v_{k-1}v\}$. The set of equivalence classes forms a group under the product operation defined by

$$\{vu_1 \ldots u_{k-1}v\} * \{vv_1 \ldots v_{l-1}v\} = \{vu_1 \ldots u_{k-1}vv_1 \ldots v_{l-1}v\}. \quad (4.16)$$

The unit element is an equivalence class $\{v\}$ while the inverse of $\{vv_1 \ldots v_{k-1}v\}$ is $\{vv_{k-1} \ldots v_1v\}$. This group is called the **edge group** of K at v and denoted by $E(K; v)$.

Theorem 4.28 $E(K; v)$ is isomorphic to $\pi_1(|K|; v)$.

The proof is found in Armstrong (1983), for example. This isomorphism $\varphi : E(K; v) \to \pi_1(|K|; v)$ is given by identifying an edge loop in K with a loop in $|K|$. To find $E(K; v)$, we need to read off the generators and relations. Let L be a simplicial subcomplex of K, such that

(a) L contains *all the vertices* (0-simplexes) of K;
(b) the polyhedron $|L|$ is *arcwise connected* and *simply connected*.

Given an arcwise-connected simplicial complex K, there always exists a subcomplex L that satisfies the conditions above. A one-dimensional simplicial complex that is arcwise connected and simply connected is called a **tree**. A tree T_{M} is called the **maximal tree** of K if it is not a proper subset of other trees.

Lemma 4.29 A maximal tree T_{M} contains all the vertices of K and hence satisfies conditions (a) and (b) above.

Proof: Suppose T_{M} does not contain some vertex w. Since K is arcwise connected, there is a 1-simplex vw in K such that $v \in T_{\mathrm{M}}$ and $w \notin T_{\mathrm{M}}$. $T_{\mathrm{M}} \cup \{vw\} \cup \{w\}$ is a one-dimensional subcomplex of K which is arcwise connected, simply connected and contains T_{M}, which contradicts the assumption. ■

Suppose we have somehow obtained the subcomplex L. Since $|L|$ is simply connected, the edge loops in $|L|$ do not contribute to $E(K; v)$. Thus we can effectively ignore the simplexes in L in our calculations. Let v_0 $(=v)$, $v_1, \ldots, v_n$ be the vertices of K. Assign an 'object' g_{ij} for each ordered pair of vertices v_i, v_j if $\langle v_i v_j \rangle$ is a 1-simplex of K. Let $G(K; L)$ be a group that is generated by all g_{ij}. What about the relations? We have the following.

(1) Since we ignore those simplexes in L, we assign $g_{ij} = 1$ if $\langle v_i v_j \rangle \in L$.

(2) If $\langle v_i v_j v_k \rangle$ is a 2-simplex of K, there are no non-trivial loops around $v_i v_j v_k$ and we have the relation $g_{ij} g_{jk} g_{ki} = 1$.

The generators $\{g_{ij}\}$ and the set of relations completely determine the group $G(K; L)$.

Theorem 4.30 $G(K; L)$ is isomorphic to $E(K; v) \cong \pi_1(|K|; v)$.

In fact, we can be more efficient than is apparent. For example, g_{ii} should be set equal to 1 since g_{ii} corresponds to the vertex v_i which is an element of L. Moreover, from $g_{ij} g_{ji} = g_{ii} = 1$, we have $g_{ij} = g_{ji}^{-1}$. Therefore we only need to introduce those generators g_{ij} for each pair of vertices v_i, v_j such that $\langle v_i v_j \rangle \in K - L$ and $i < j$. Since there are no generators g_{ij} such that $\langle v_i v_j \rangle \in L$, we can ignore the first type of relations. If $\langle v_i v_j v_k \rangle$ is a 2-simplex of $K - L$ such that $i < j < k$, the corresponding relation is *uniquely* given by $g_{ij} g_{jk} = g_{ik}$ since we are only concerned with simplexes $\langle v_i v_j \rangle$ such that $i < j$.

To summarise, the rules of the game are as follows.

(1) First, find a triangulation $f : |K| \to X$.

(2) Find the subcomplex L that is arcwise connected, simply connected and contains all the vertices of K.

(3) Assign a generator g_{ij} to each 1-simplex $\langle v_i v_j \rangle$ of $K - L$, for which $i < j$.

(4) Impose a relation $g_{ij} g_{jk} = g_{ik}$ if there is a 2-simplex $\langle v_i v_j v_k \rangle$ such that $i < j < k$. If two of the vertices v_i, v_j and v_k form a 1-simplex of L, the corresponding generator should be set equal to 1.

(5) Now $\pi_1(X)$ is isomorphic to $G(K; L)$ which is a group generated by $\{g_{ij}\}$ with the relations obtained in (4).

Let us work out several examples.

Example 4.31 From our construction, it should be clear that $E(K; v)$ and $G(K; L)$ involve only the 0-, 1- and 2-simplexes of K. Accordingly if $K(2)$ denotes a **2-skeleton** of K, which is defined to be the set of all 0-, 1- and 2-simplexes in K, we should have

$$\pi_1(|K|) \cong \pi_1(|K(2)|). \tag{4.17}$$

This is quite useful in actual computations. For example, a 3-simplex and its boundary have the same 2-skeleton. A 3-simplex is a polyhedron $|K|$ of the solid ball D^3, while its boundary $|L|$ is a polyhedron of the sphere S^2. Since D^3 is contractible, $\pi_1(|K|) \cong \{e\}$. From (4.17) we find $\pi_1(S^2) \cong \pi_1(|K|) \cong \{e\}$. In general, for $n \geqslant 2$, the $(n+1)$-simplex σ_{n+1} and the boundary of σ_{n+1} has the same 2-skeleton. If we note that σ_{n+1} is contractible and the boundary of σ_{n+1} is a polyhedron of S^n, we find the formula

$$\pi_1(S^n) \cong \{e\} \qquad n \geqslant 2. \tag{4.18}$$

Example 4.32 Let $K \equiv \{v_1, v_2, v_3, \langle v_1 v_2 \rangle, \langle v_1 v_3 \rangle, \langle v_2 v_3 \rangle\}$ be a simplicial complex of a circle S^1. We take v_1 as the base point. A maximal tree may be $L = \{v_1, v_2, v_3, \langle v_1 v_2 \rangle, \langle v_1 v_3 \rangle\}$. There is only one generator g_{23}. Since there are no 2-simplexes in K, the relation is empty. Hence

$$\pi_1(S^1) \cong G(K; L) = (g_{23} : \varnothing) \cong \mathbb{Z} \tag{4.19}$$

in agreement with theorem 4.23.

Example 4.33 An ***n*-bouquet** is defined by the one-point union of n circles. For example, figure 4.11 is a triangulation of a 3-bouquet. Take the common point v as the base point. The bold lines in figure 4.11 form a maximal tree L. The generators of $G(K; L)$ are g_{12}, g_{34} and g_{56}.

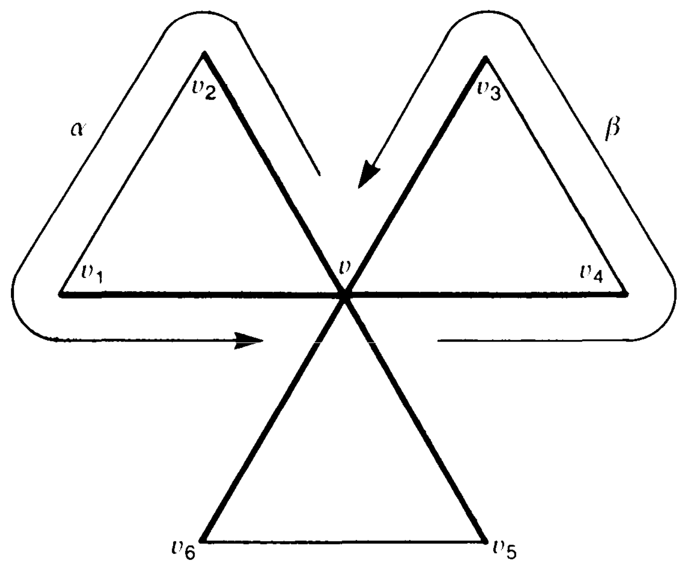


Figure 4.11 A triangulation of a 3-bouquet. The bold lines denote the maximal tree L.

There are no relations and we find

$$\pi_1(n\text{-bouquet}) = G(K; L) = (x, y, z : \varnothing). \tag{4.20}$$

Note that this is a free group but not free *Abelian*. The non-commutativity can be shown as follows. Consider loops α and β at v encircling different holes. Obviously the product $\alpha * \beta * \alpha^{-1}$ cannot be continuously deformed into β, hence $[\alpha]*[\beta]*[\alpha]^{-1} \neq [\beta]$, or

$$[\alpha]*[\beta] \neq [\beta]*[\alpha]. \tag{4.21}$$

In general, an n-bouquet has n generators $g_{12}, \ldots, g_{2n-1\,2n}$ and the fundamental group is isomorphic to the free group with n generators with no relations.

Example 4.34 Let D^2 be a two-dimensional disc. A triangulation $|K|$ of D^2 is given by a triangle with its interior included. Clearly K itself may be L and $K - L$ is empty. Thus we find $\pi_1(|K|) \cong \{e\}$.

Example 4.35 Figure 4.12 is a triangulation of the torus T^2. The shaded area is chosen to be the subcomplex L. [Verify that it contains all the vertices and is both arcwise and simply connected.] There are eleven generators with ten relations. Let us take $x = g_{02}$ and $y = g_{04}$ and write down the relations

(*a*) $\underset{x}{g_{02}}\underset{1}{g_{27}} = g_{07} \to g_{07} = x$

(*b*) $\underset{1}{g_{03}}g_{37} = \underset{x}{g_{07}} \to g_{37} = x$

(*c*) $\underset{x}{g_{37}}\underset{1}{g_{78}} = g_{38} \to g_{38} = x$

(*d*) $\underset{1}{g_{34}}g_{48} = \underset{x}{g_{38}} \to g_{48} = x$

(*e*) $g_{24}\underset{x}{g_{48}} = g_{28} \to g_{24}x = g_{28}$

(*f*) $\underset{x}{g_{02}}g_{24} = \underset{y}{g_{04}} \to xg_{24} = y$

(*g*) $\underset{y}{g_{04}}\underset{1}{g_{46}} = g_{06} \to g_{06} = y$

(*h*) $\underset{1}{g_{01}}g_{16} = \underset{y}{g_{06}} \to g_{16} = y$

(*i*) $\underset{y}{g_{16}}\underset{1}{g_{68}} = g_{18} \to g_{18} = y$

(*j*) $\underset{1}{g_{12}}g_{28} = \underset{y}{g_{18}} \to g_{28} = y.$

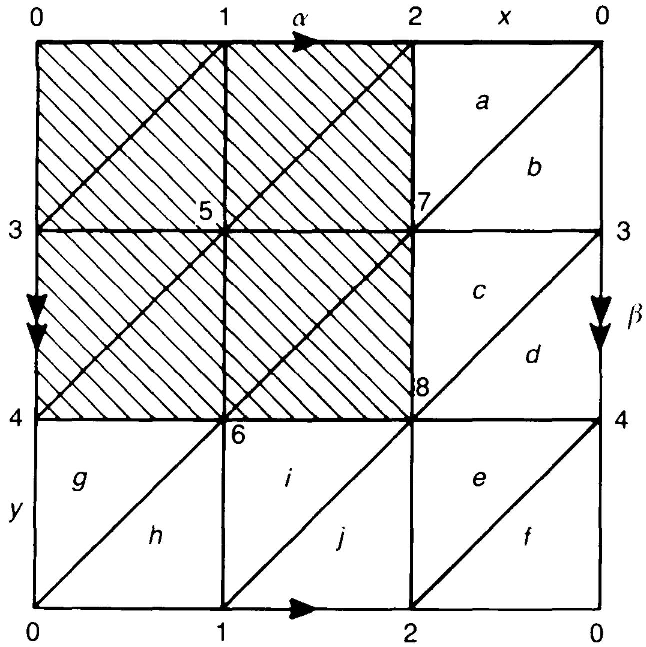


Figure 4.12 A triangulation of the torus.

From (e) and (f), it follows that $x^{-1}yx = g_{28}$. We finally have

$$g_{02} = g_{07} = g_{37} = g_{38} = g_{48} = x$$
$$g_{04} = g_{06} = g_{16} = g_{18} = g_{28} = y$$
$$g_{24} = x^{-1}y$$

with a relation $x^{-1}yx = y$ or

$$xyx^{-1}y^{-1} = 1. \tag{4.22}$$

This shows that $G(K; L)$ is generated by two *commutative* generators (note $xy = yx$), hence (cf example 4.27)

$$G(K; L) = (x, y: xyx^{-1}y^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z} \tag{4.23}$$

in agreement with (4.8a).

We have the following intuitive picture. Consider loops $\alpha = 0 \to 1 \to 2 \to 0$ and $\beta = 0 \to 3 \to 4 \to 0$. The loop α is identified with $x = g_{02}$ since $g_{12} = g_{01} = 1$ and β with $y = g_{04}$. They generate $\pi_1(T^2)$ since α and β are independent non-trivial loops. In terms of these, the relation is written as

$$\alpha * \beta * \alpha^{-1} * \beta^{-1} \sim c_v \tag{4.24}$$

where c_v is a constant loop at v, see figure 4.13.

More generally let T_g be the torus with genus g. As we have shown in problem 2.1, T_g is expressed as a subset of $\mathbb{R}^2$ with proper identifications at the boundary. The fundamental group of T_g is generated by $2g$

loops α_i, β_i $(1 \leq i \leq g)$. Similarly to (4.24), we verify that

$$\prod_{i=1}^{g} (\alpha_i * \beta_i * \alpha_i^{-1} * \beta_i^{-1}) \sim c_v. \tag{4.25}$$

If we denote the generators corresponding to α_i by x_i and β_i by y_i, there is only one relation among them,

$$\prod_{i=1}^{g} (x_i y_i x_i^{-1} y_i^{-1}) = 1. \tag{4.26}$$

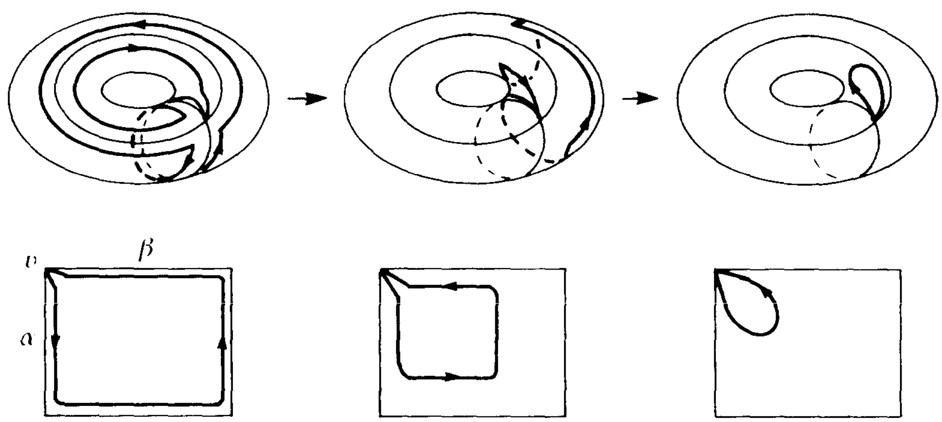


Figure 4.13 The loops α and β satisfy the relation $\alpha * \beta * \alpha^{-1} * \beta^{-1} \sim c_v$.

Exercise 4.36 Figure 4.14 is a triangulation of the Klein bottle. The shaded area is the subcomplex L. There are eleven generators and ten

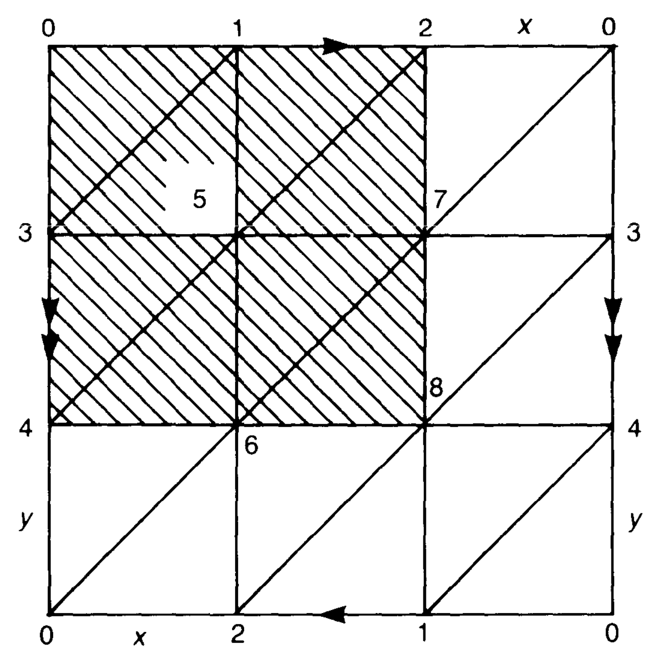


Figure 4.14 A triangulation of the Klein bottle.

relations. Take $x = g_{02}$ and $y = g_{04}$ and write down the relations for 2-simplexes to show that

$$\pi_1(\text{Klein bottle}) \cong (x, y: xyxy^{-1}). \tag{4.27}$$

Example 4.37 Figure 4.15 is a triangulation of the projective plane $\mathbb{R}P^2$. The shaded area is the subcomplex L. There are seven generators and six relations. Let us take $x = g_{23}$ and write down the relations

(*a*) $\underset{x}{g_{23}}\underset{1}{g_{34}} = g_{24} \to g_{24} = x$

(*b*) $\underset{x}{g_{24}}\underset{1}{g_{46}} = g_{26} \to g_{26} = x$

(*c*) $\underset{1}{g_{12}}\underset{x}{g_{26}} = g_{16} \to g_{16} = x$

(*d*) $\underset{1}{g_{13}}g_{36} = \underset{x}{g_{16}} \to g_{36} = x$

(*e*) $g_{35}\underset{1}{g_{56}} = \underset{x}{g_{36}} \to g_{35} = x$

(*f*) $\underset{x}{g_{23}}\underset{x}{g_{35}} = \underset{1}{g_{25}} \to x^2 = 1.$

Hence we find

$$\pi_1(\mathbb{R}P^2) \cong (x : x^2) \cong \mathbb{Z}_2. \tag{4.28}$$

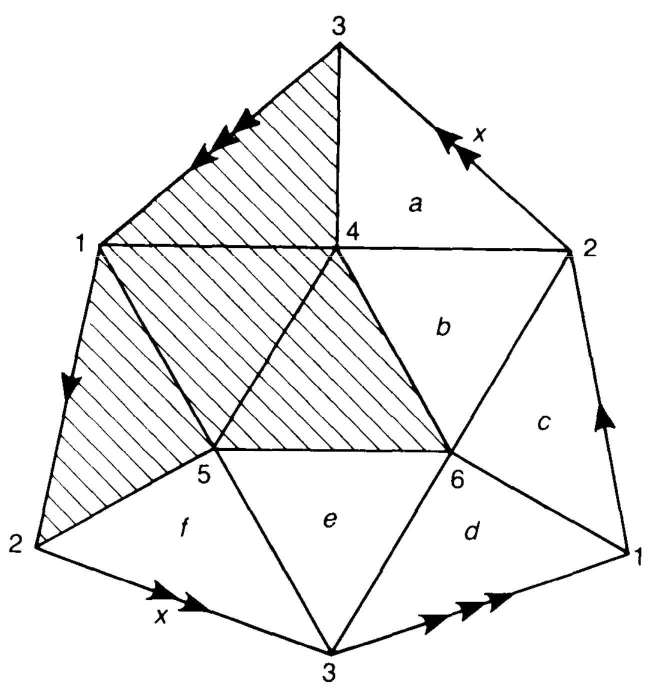


Figure 4.15 A triangulation of the projective plane.

Intuitively, the appearance of a cyclic group is understood as follows. Figure 4.16(a) is a schematic picture of $\mathbb{R}P^2$. Take loops α and β. It is easy to see that α is continuously deformed to a point, and hence is a trivial element of $\pi_1(\mathbb{R}P^2)$. Since diametrically opposite points are identified in $\mathbb{R}P^2$, β is actually a closed loop. Since it cannot be shrunk to a point, it is a non-trivial element of π_1. What about the product? $\beta * \beta$ is a loop which traverses from P to Q ~ P twice. It can be read off from figure 4.16(b) that $\beta * \beta$ is continuously shrunk to a point, and thus belongs to the trivial class. This shows that the generator x, corresponding to the homotopy class of the loop β, satisfies the relation $x^2 = 1$, which verifies our result.

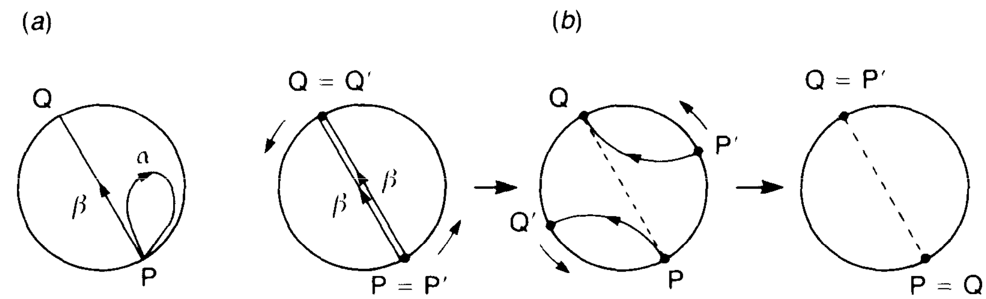


Figure 4.16 (a) α is a trivial loop while the loop β cannot be shrunk to a point. (b) $\beta * \beta$ is continuously shrunk to a point.

The same pictures can be used to show that

$$\pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2 \tag{4.29}$$

where $\mathbb{R}P^3$ is identified as S^3 with diametrically opposite points identified, $\mathbb{R}P^3 = S^3/(x \sim -x)$. If we take the hemisphere as the representative, $\mathbb{R}P^3$ can be expressed as a solid ball D^3 with diametrically opposite points on the surface identified. If the discs D^2 in figure 4.16 are interpreted as solid balls D^3, the same pictures verify (4.29).

Exercise 4.38 A triangulation of the Möbius strip is given by figure 3.8. Find the maximal tree and show that

$$\pi_1(\text{Möbius strip}) \cong \mathbb{Z}. \tag{4.30}$$

[*Note:* Of course the Möbius strip is of the same homotopy type as S^1, hence (4.30) is trivial. The reader is asked to obtain this result through routine procedures.]

4.4.3 Relations between $H_1(K)$ and $\pi_1(|K|)$

The reader might have noticed that there is a certain similarity between the first homology group $H_1(K)$ and the fundamental group $\pi_1(|K|)$. For example, the fundamental groups of many spaces (circle, disc, n-spheres, torus and many more) are identical with the corresponding first homology group. In some cases, however, they are different; H_1 (2-bouquet) $\cong \mathbb{Z} \oplus \mathbb{Z}$ and π_1(2-bouquet) $\cong (x, y : \varnothing)$, for example. Note that H_1(2-bouquet) is a free *Abelian* group while π_1(2-bouquet) is a free group. The following theorem relates $\pi_1(|K|)$ to $H_1(K)$.

Theorem 4.39 Let K be a connected simplicial complex. Then $H_1(K)$ is isomorphic to $\pi_1(|K|)/F$, where F is the commutator subgroup (see below) of $\pi_1(|K|)$.

Let G be a group whose presentation is $(x_i : r_m)$. The **commutator subgroup** F of G is a group generated by the elements of the form $x_i x_j x_i^{-1} x_j^{-1}$. Thus G/F is a group generated by $\{x_i\}$ with the set of relations $\{r_m\}$ *and* $\{x_i x_j x_i^{-1} x_j^{-1}\}$. The theorem states that if $\pi_1(|K|) \cong (x_i : r_m)$, then $H_1(K) \cong (x_i : r_m, x_i x_j x_i^{-1} x_j^{-1})$. For example, from π_1(2-bouquet) $= (x, y : \varnothing)$, we find

$$\pi_1(\text{2-bouquet})/F \cong (x, y : xyx^{-1}y^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$$

which is isomorphic to H_1(2-bouquet).

The proof of theorem 4.39 is found in Greenberg and Harper (1981) and also outlined in Croom (1978).

Example 4.40 From π_1(Klein bottle) $\cong (x, y : xyxy^{-1})$, we have

$$\pi_1(\text{Klein bottle})/F \cong (x, y : xyxy^{-1}, xyx^{-1}y^{-1}).$$

Two relations are replaced by $x^2 = 1$ and $xyx^{-1}y^{-1} = 1$ to yield

$$\begin{aligned}\pi_1(\text{Klein bottle})/F &\cong (x, y : xyx^{-1}y^{-1}, x^2)\\ &\cong \mathbb{Z} \oplus \mathbb{Z}_2 \cong H_1(\text{Klein bottle})\end{aligned}$$

where the factor $\mathbb{Z}$ is generated by y and $\mathbb{Z}_2$ by x.

Corollary 4.41 Let X be a connected topological space. Then $\pi_1(X)$ is isomorphic to $H_1(X)$ if and only if $\pi_1(X)$ is commutative. In particular if $\pi_1(X)$ is generated by one generator, $\pi_1(X)$ is always isomorphic to $H_1(X)$. [Use theorem 4.39.]

Corollary 4.42 If X and Y are of the same homotopy type, their first homology groups are identical; $H_1(X) = H_1(Y)$. [Use theorems 4.39 and 4.13.]

4.5 Higher homotopy groups

The fundamental group classifies the homotopy classes of loops in a topological space X. There are many ways to assign other groups to X.

For example, we may classify homotopy classes of the spheres in X or homotopy classes of the tori in X. It turns out that the homotopy classes of the sphere S^n $(n \geqslant 2)$ form a group similar to the fundamental group.

4.5.1 Definitions

Let I^n $(n \geqslant 1)$ denote the unit n-cube $I \times \ldots \times I$,

$$I^n \equiv \{(s_1, \ldots, s_n) | 0 \leqslant s_i \leqslant 1 \ (1 \leqslant i \leqslant n)\}. \tag{4.31}$$

The boundary ∂I^n is the geometrical boundary of I^n,

$$\partial I^n \equiv \{(s_1, \ldots, s_n) \in I^n | \text{ some } s_i = 0 \text{ or } 1\}. \tag{4.32}$$

We recall that in the fundamental group, the boundary ∂I of $I = [0, 1]$ is mapped to the base point x_0. Similarly we assume here that we shall be concerned with continuous maps $\alpha : I^n \to X$, which map the boundary ∂I^n to a point $x_0 \in X$. Since the boundary is mapped to a single point x_0, we have effectively obtained S^n from I^n; cf figure 2.7. If $I^n/\partial I^n$ denotes the cube I^n whose boundary ∂I^n is shrunk to a point, we have $I^n/\partial I^n \cong S^n$. The *map* α is called an ***n*-loop** at x_0. A straightforward generalisation of definition 4.4 is as follows.

Definition 4.43 Let X be a topological space and $\alpha, \beta : I^n \to X$ be n-loops at $x_0 \in X$. α is **homotopic** to β, denoted by $\alpha \sim \beta$, if there exists a continuous map $F : I^n \times I \to X$ such that

$$F(s_1, \ldots, s_n, 0) = \alpha(s_1, \ldots, s_n) \tag{4.33a}$$

$$F(s_1, \ldots, s_n, 1) = \beta(s_1, \ldots, s_n) \tag{4.33b}$$

$$F(s_1, \ldots, s_n, t) = x_0 \text{ for } (s_1, \ldots, s_n) \in \partial I^n, \ t \in I. \tag{4.33c}$$

F is called a **homotopy** between α and β.

Exercise 4.44 Show that $\alpha \sim \beta$ is an equivalence relation. The equivalence class to which α belongs is called the **homotopy class** of α and is denoted by $[\alpha]$.

Let us define the group operations. The product $\alpha * \beta$ of n-loops α and β is defined by

$$\alpha * \beta(s_1, s_2, \ldots, s_n) = \begin{cases} \alpha(2s_1, s_2, \ldots, s_n) & 0 \leqslant s_1 \leqslant \frac{1}{2} \\ \beta(2s_1 - 1, s_2, \ldots, s_n) & \frac{1}{2} \leqslant s_1 \leqslant 1. \end{cases} \tag{4.34}$$

The product $\alpha * \beta$ looks like figure 4.17(a) in X. It is helpful to express it as figure 4.17(b). If we define α^{-1} by

$$\alpha^{-1}(s_1, s_2, \ldots, s_n) \equiv \alpha(1 - s_1, s_2, \ldots, s_n) \tag{4.35}$$

it satisfies

$$\alpha^{-1}*\alpha(s_1, \ldots, s_n) \sim \alpha*\alpha^{-1}(s_1, \ldots, s_n) \sim c_{x_0}(s_1, \ldots, s_n) \quad (4.36)$$

where c_{x_0} is a constant n-loop at $x_0 \in X$, $c_{x_0} : (s_1, \ldots, s_n) \mapsto x_0$. Verify that both $\alpha*\beta$ and α^{-1} are n-loops at x_0.

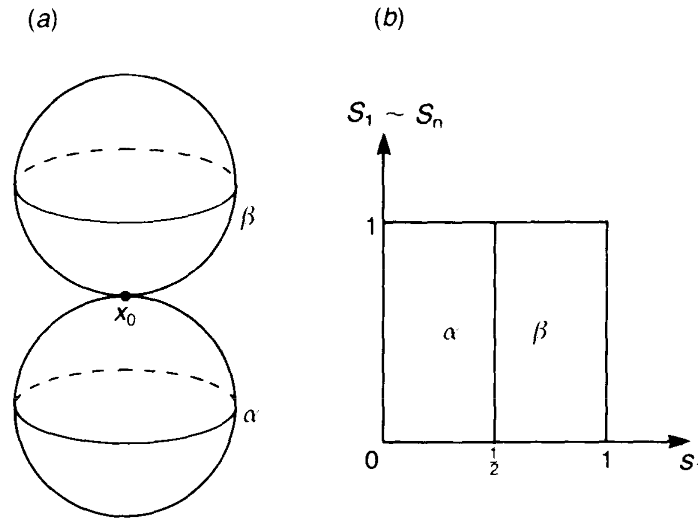


Figure 4.17 A product $\alpha*\beta$ of n-loops α and β.

Definition 4.45 Let X be a topological space. The set of homotopy classes of n-loops ($n \geqslant 1$) at $x_0 \in X$ is denoted by $\pi_n(X, x_0)$ and called the ***n*th homotopy group** at x_0. $\pi_n(X, x_0)$ is called the *higher* homotopy group if $n \geqslant 2$.

The product $\alpha*\beta$ defined above naturally induces a product of homotopy classes defined by

$$[\alpha]*[\beta] \equiv [\alpha*\beta] \quad (4.37)$$

where α and β are n-loops at x_0. The following exercises verify that this product is well defined and satisfies the group axioms.

Exercise 4.46 Show that the product of n-loops defined by (4.37) is independent of the representatives; cf lemma 4.7.

Exercise 4.47 Show that the nth homotopy group is a group. To prove this, the following facts may be verified; cf theorem 4.8.

(1) $([\alpha]*[\beta])*[\gamma] = [\alpha]*([\beta]*[\gamma])$.
(2) $[\alpha]*[c_x] = [c_x]*[\alpha] = [\alpha]$.
(3) $[\alpha]*[\alpha^{-1}] = [c_x]$, which defines the inverse $[\alpha]^{-1} = [\alpha^{-1}]$.

We have excluded $\pi_0(X, x_0)$ so far. Let us classify maps from I^0 to X.

We note $I^0 = \{0\}$ and $\partial I^0 = \emptyset$. Let α, $\beta : \{0\} \to X$ be such that $\alpha(0) = x$ and $\beta(0) = y$. We define $\alpha \sim \beta$ if there exists a continuous map $F : \{0\} \times I \to X$ such that $F(0, 0) = x$ and $F(0, 1) = y$. This shows that $\alpha \sim \beta$ if and only if x and y are connected by a curve in X, namely they are in the same (arcwise-) connected component. Clearly this equivalence relation is independent of x_0 and we simply denote the 0th homology group by $\pi_0(X)$. $\pi_0(X)$ is not a group and denotes the number of (arcwise-) connected components of X.

4.6 General properties of higher homotopy groups

4.6.1 Abelian nature of higher homotopy groups

Higher homotopy groups are always Abelian; for any n-loops α and β at $x_0 \in X$, $[\alpha]$ and $[\beta]$ satisfy

$$[\alpha]*[\beta] = [\beta]*[\alpha]. \tag{4.38}$$

To verify this assertion let us observe figure 4.18. Clearly the deformation is homotopic at each step of the sequence. This shows that $\alpha*\beta \sim \beta*\alpha$, namely $[\alpha]*[\beta] = [\beta]*[\alpha]$.

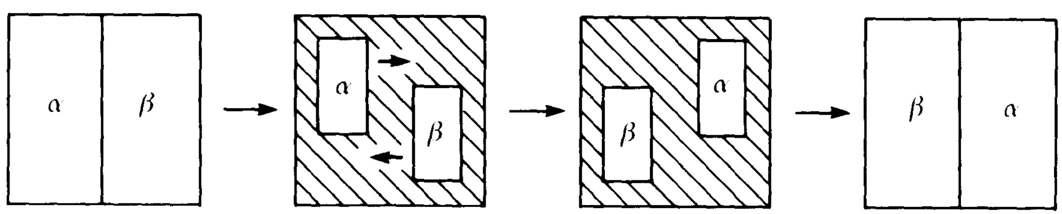


Figure 4.18 Higher homotopy groups are always commutative, $\alpha*\beta \sim \beta*\alpha$.

4.6.2 Arcwise connectedness and higher homotopy groups

If a topological space X is arcwise connected, $\pi_n(X, x_0)$ is isomorphic to $\pi_n(X, x_1)$ for any pair x_0, $x_1 \in X$. The proof is parallel to that of theorem 4.9. Accordingly, if X is arcwise connected, the base point need not be specified.

4.6.3 Homotopy invariance of higher homotopy groups

Let X and Y be topological spaces of the same homotopy type; see definition 4.11. If $f : X \to Y$ is a homotopy equivalence, the homotopy group $\pi_n(X, x_0)$ is isomorphic to $\pi_n(Y, f(x_0))$; cf theorem 4.13. Topological invariance of higher homotopy groups is the direct consequence of this fact. In particular, if X is contractible, the homotopy groups are trivial: $\pi_n(X, x_0) \cong \{e\}$, $n \geqslant 1$.

4.6.4 *Higher homotopy groups of a product space*

Let X and Y be arcwise connected topological spaces. Then

$$\pi_n(X \times Y) \cong \pi_n(X) \oplus \pi_n(Y) \tag{4.39}$$

cf theorem 4.24.

4.6.5 *Universal covering spaces and higher homotopy groups*

There are several cases in which the homotopy groups of one space are given by the known homotopy groups of the other space. There is a remarkable property between the higher homotopy groups of a topological space and its *universal covering space*.

Definition 4.48 Let X and $\widetilde{X}$ be connected topological spaces. The pair $(\widetilde{X}, p)$, or simply $\widetilde{X}$, is called the **covering space** of X if there exists a continuous map $p : \widetilde{X} \to X$ such that

(1) p is surjective (onto)

(2) for each $x \in X$, there exists a connected open set $U \subset X$ containing x, such that $p^{-1}(U)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U by p.

In particular, if $\widetilde{X}$ is simply connected, $(\widetilde{X}, p)$ is called the **universal covering space** of X. [*Remarks:* Certain groups are known to be topological spaces. They are called **topological groups**. For example $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$ are topological groups. If X and $\widetilde{X}$ in the definition above happen to be topological groups and $p : \widetilde{X} \to X$ to be a group homomorphism, the (universal) covering space is called the **(universal) covering group**.]

For example, $\mathbb{R}$ is the universal covering space of S^1, see §4.3. Since S^1 is identified with $\mathrm{U}(1)$, $\mathbb{R}$ is a universal covering group of $\mathrm{U}(1)$ if $\mathbb{R}$ is regarded as an additive group. The map $p : \mathbb{R} \to \mathrm{U}(1)$ may be $p : x \to \mathrm{e}^{\mathrm{i}2\pi x}$. Clearly p is surjective and if $U = \{\mathrm{e}^{\mathrm{i}2\pi x} | x \in (x_0 - 0.1, x_0 + 0.1)\}$, then

$$p^{-1}(U) = \bigcup_{n \in \mathbb{Z}} (x_0 - 0.1 + n, x_0 + 0.1 + n)$$

which is a disjoint union of open sets of $\mathbb{R}$. It is easy to show that p is also a homomorphism with respect to addition in $\mathbb{R}$ and multiplication in $\mathrm{U}(1)$. Hence $(\mathbb{R}, p)$ is the universal covering group of $\mathrm{U}(1) = S^1$.

Theorem 4.49 Let $(\widetilde{X}, p)$ be the universal covering space of a connected topological space X. If $x_0 \in X$ and $\widetilde{x}_0 \in \widetilde{X}$ are base points such that $p(\tilde{x}_0) = x_0$, the induced homomorphism

$$p_* : \pi_n(\widetilde{X}, \tilde{x}_0) \to \pi_n(X, x_0) \tag{4.40}$$

is an isomorphism for $n \geq 2$. [*Warning:* This theorem cannot be applied if $n = 1$; $\pi_1(\mathbb{R}) \cong \{e\}$ while $\pi_1(S^1) \cong \mathbb{Z}$.]

The proof is given in Croom (1978). For example, we have $\pi_n(\mathbb{R}) \cong \{e\}$ since $\mathbb{R}$ is contractible. Then we find

$$\pi_n(S^1) \cong \pi_n(\mathrm{U}(1)) \cong \{e\} \qquad n \geq 2. \tag{4.41}$$

Example 4.50 Let $S^n = \{x \in \mathbb{R}^{n+1} |\ |x|^2 = 1\}$. The real projective space $\mathbb{R}P^n$ is obtained from S^n by identifying the pair of antipodal points $(x, -x)$. It is easy to see that S^n is a covering space of $\mathbb{R}P^n$ for $n \geq 2$. Since $\pi_1(S^n) \cong \{e\}$ for $n \geq 2$, S^n is the universal covering space of $\mathbb{R}P^n$ and we have

$$\pi_n(\mathbb{R}P^m) \cong \pi_n(S^m). \tag{4.42}$$

It is interesting to note that $\mathbb{R}P^3$ is identified with SO(3). To see this let us specify an element of SO(3) by a rotation about an axis $\boldsymbol{n}$ by an angle θ $(0 \leq \theta \leq \pi)$ and assign a 'vector' $\boldsymbol{\Omega} \equiv \theta\boldsymbol{n}$ to this element. $\boldsymbol{\Omega}$ takes its value in the disc D^3 of radius π. Moreover, $\pi\boldsymbol{n}$ and $-\pi\boldsymbol{n}$ express the same rotation and should be identified. Thus the space to which $\boldsymbol{\Omega}$ belongs is a disc D^3 whose antipodal points on the surface S^2 are identified. On the other hand, we may express $\mathbb{R}P^3$ as the northern hemisphere D^3 of S^3, whose antipodal points on the boundary S^2 are identified. This shows that $\mathbb{R}P^3$ is identified with SO(3).

It is also interesting to see that S^3 is identified with SU(2). First note that any element $g \in \mathrm{SU}(2)$ is written as

$$g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \qquad |a|^2 + |b|^2 = 1. \tag{4.43}$$

If we write $a = u + \mathrm{i}v$ and $b = x + \mathrm{i}y$, this becomes S^3,

$$u^2 + v^2 + x^2 + y^2 = 1.$$

Collecting these results, we find

$$\pi_n(\mathrm{SO}(3)) = \pi_n(\mathbb{R}P^3) = \pi_n(S^3) = \pi_n(\mathrm{SU}(2)) \qquad n \geq 2. \tag{4.44}$$

More generally, the universal covering group SPIN(n) of SO(n) is called the **spin group**. For small n, they are

$$\mathrm{SPIN}(3) = \mathrm{SU}(2) \tag{4.45a}$$

$$\mathrm{SPIN}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{4.45b}$$

$$\mathrm{SPIN}(5) = \mathrm{USp}(4) \tag{4.45c}$$

$$\mathrm{SPIN}(6) = \mathrm{SU}(4). \tag{4.45d}$$

4.7 Examples of higher homotopy groups

In general, there are no algorithms to compute higher homotopy groups $\pi_n(X)$. An *ad hoc* method is required for each topological space for $n \geqslant 2$. Here, we study several examples in which higher homotopy groups may be obtained by intuitive arguments. We also collect useful results in table 4.1.

Example 4.51 If we note that $\pi_n(X, x_0)$ is the set of the homotopy classes of n-loops S^n in X, we immediately find

$$\pi_n(S^n, x_0) = \mathbb{Z} \qquad n \geqslant 1. \tag{4.46}$$

If α maps S^n onto a point $x_0 \in S^n$, $[\alpha]$ is the unit element $0 \in \mathbb{Z}$. Since both $I^n/\partial I^n$ and S^n are orientable, we may assign orientations to them. If α maps $I^n/\partial I^n$ homeomorphically to S^n in the same sense of orientation, then $[\alpha]$ is assigned an element $1 \in \mathbb{Z}$. If a homeomorphism α maps $I^n/\partial I^n$ onto S^n in an orientation of opposite sense, $[\alpha]$ corresponds to an element -1. For example, let $n = 2$. Since $I^2/\partial I^2 \cong S^2$, the point in I^2 can be expressed by the polar coordinate (θ, ϕ), see figure 4.19. Similarly, $X = S^2$ can be expressed by the polar coordinate (θ', ϕ'). Let $\alpha : (\theta, \phi) \to (\theta', \phi')$ be a 2-loop in X. If $\theta' = \theta$ and $\phi' = \phi$, the point (θ', ϕ') sweeps S^2 once while the point (θ, ϕ) scans I^2 once in the same orientation. This 2-loop belongs to the class $+1$ of $\pi_2(S^2, x_0)$. If $\alpha : (\theta, \phi) \to (\theta', \phi')$ is given by $\theta' = \theta$ and $\phi' = 2\phi$, the point (θ', ϕ') sweeps S^2 twice while (θ, ϕ) scans I^2 once. This 2-loop belongs to the class $2 \in \pi_2(S^2, x_0)$. In general, the map $(\theta, \phi) \mapsto (\theta, k\phi)$, $k \in \mathbb{Z}$, corresponds to the class k of $\pi_2(S^2, x_0)$. A similar argument verifies (4.46) for general $n \geqslant 2$.

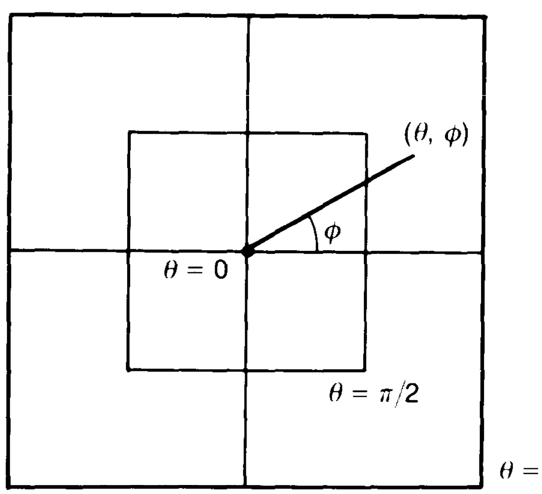


Figure 4.19 A point in I^2 may be expressed by polar coordinates (θ, ϕ).

Example 4.52 Noting that S^n is a universal covering space of $\mathbb{R}P^n$ for $n \geqslant 2$, we find

$$\pi_n(\mathbb{R}P^n) \cong \pi_n(S^n) \cong \mathbb{Z} \qquad n \geqslant 2. \tag{4.47}$$

[Of course this happens to be true for $n = 1$, since $\mathbb{R}P^1 = S^1$.] For example, we have $\pi_2(\mathbb{R}P^2) \cong \pi_2(S^2) \cong \mathbb{Z}$.

Since $\mathrm{SU}(2) \cong S^3$ is the universal covering group of $\mathrm{SO}(3) \cong \mathbb{R}P^3$, it follows from theorem 4.49 that (see also (4.44))

$$\pi_3(\mathrm{SO}(3)) \cong \pi_3(\mathrm{SU}(2)) \cong \pi_3(S^3) \cong \mathbb{Z}. \tag{4.48}$$

Shankar's monopoles in superfluid ^{3}He-A correspond to non-trivial elements of the above homotopy classes, see §4.9. $\pi_3(\mathrm{SU}(2))$ is also employed in the classification of instantons in example 9.12.

Table 4.1 Useful homotopy groups.

	π_1	π_2	π_3	π_4	π_5	π_6
SO(3)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
SO(4)	$\mathbb{Z}_2$	0	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}_2+\mathbb{Z}_2$	$\mathbb{Z}_2+\mathbb{Z}_2$	$\mathbb{Z}_{12}+\mathbb{Z}_{12}$
SO(5)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
SO(6)	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
SO(n) $n > 6$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
U(1)	$\mathbb{Z}$	0	0	0	0	0
SU(2)	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
SU(3)	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$
SU(n) $n > 3$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
G_2	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$
F_4	0	0	$\mathbb{Z}$	0	0	0
E_6	0	0	$\mathbb{Z}$	0	0	0
E_7	0	0	$\mathbb{Z}$	0	0	0
E_8	0	0	$\mathbb{Z}$	0	0	0

In summary, we have table 4.1. In this table, other useful homotopy groups are also listed. We comment on several interesting facts.

(*a*) Since SPIN(4) = SU(2) × SU(2) is the universal covering group of SO(4), we have $\pi_n(\mathrm{SO}(4)) = \pi_n(\mathrm{SU}(2)) \oplus \pi_n(\mathrm{SU}(2))$ for $n \geqslant 2$.

(*b*) There exists a map J called the ***J*-homomorphism** $J : \pi_k(\mathrm{SO}(n)) \to \pi_{k+n}(S^n)$, see Whitehead (1978). In particular, if $k = 1$, the homomorphism is known to be an isomorphism and we have $\pi_1(\mathrm{SO}(n)) \cong \pi_{n+1}(S^n)$. For example, we find

$$\pi_1(\mathrm{SO}(2)) \cong \pi_3(S^2) \cong \mathbb{Z}$$
$$\pi_1(\mathrm{SO}(3)) \cong \pi_4(S^3) \cong \pi_4(\mathrm{SU}(2)) \cong \pi_4(\mathrm{SO}(3)) \cong \mathbb{Z}_2.$$

(c) The **Bott periodicity theorem** states that

$$\pi_k(\mathrm{U}(n)) \cong \pi_k(\mathrm{SU}(n)) \cong \begin{cases} \{e\} \text{ if } k \text{ is even} \\ \mathbb{Z} \text{ if } k \text{ is odd} \end{cases} \tag{4.49}$$

for $n \geqslant (k+1)/2$. Similarly

$$\pi_k(\mathrm{O}(n)) \cong \pi_k(\mathrm{SO}(n)) \cong \begin{cases} \{e\} \text{ if } k = 2, 4, 5, 6 \pmod 8 \\ \mathbb{Z}_2 \text{ if } k = 0, 1 \pmod 8 \\ \mathbb{Z} \text{ if } k = 3, 7 \pmod 8 \end{cases} \tag{4.50}$$

for $n \geqslant k+2$. Similar periodicity holds for simplectic groups which we shall not give here. Many more will be found in Appendix A, table 6 of Ito (1987).

4.8 Defects in nematic liquid crystals

4.8.1 General consideration

In the present section and the next, we study applications of homotopy groups to the classification of defects in ordered media. The analysis of this section and the next is based on Toulouse and Kléman (1976), Mermin (1979) and Mineev (1980).

We first introduce the notion of the *order parameter space*. As we saw in §1.5, when a condensed matter system undergoes a phase transition, the symmetry of the system is reduced. This reduction is described by the order parameter. For definiteness, let us consider the three-dimensional medium of a superconductor. The order parameter takes the form $\psi(x) = \Delta_0(x)\mathrm{e}^{\mathrm{i}\varphi(x)}$. Let us consider a homogeneous system under uniform external conditions (temperature, pressure etc). The amplitude Δ_0 is uniquely fixed by minimising the condensation free energy. Note that there are still a large number of degrees of freedom left. ψ may take any value in the circle $S^1 \cong \mathrm{U}(1)$ determined by the phase $\mathrm{e}^{\mathrm{i}\varphi}$. In this way, a uniform system takes its value in a certain region M called the **order parameter space**. For a superconductor, $M = \mathrm{U}(1)$. For the Heisenberg spin system, $M = S^2$. The nematic liquid crystal has $M = \mathbb{R}P^2$ while $M = S^2 \times \mathrm{SO}(3)$ for the superfluid ^{3}He-A.

If the system is in an inhomogeneous state, the gradient free energy cannot be negligible and ψ may not be in M. If the characteristic size of the variation of the order parameter is much larger than the coherence length, however, we may still assume that the order parameter takes its value in M, where the value is a function of position this time. If this is the case, there may be points, lines, or surfaces in the medium on which the order parameter is not uniquely defined. They are called the **defects**. We have **point defects (monopoles)**, **line defects (vortices)** and **surface defects (domain walls)** according to their dimensionalities. These defects are classified by the homotopy groups.

To be more mathematical, let X be a space which is filled with the medium under consideration. The order parameter is a classical field $\psi(x)$, which is also regarded as a *map* $\psi : X \to M$. Suppose there is a defect in the medium. For concreteness, we consider a line defect in the three-dimensional medium of a superconductor. Imagine a circle S^1 which encircles the line defect. If each part of S^1 is far from the line defect, much further than the coherence length ξ, we may assume the order parameter along S^1 takes its value in the order parameter space $M = \mathrm{U}(1)$, see figure 4.20. This is how the fundamental group comes into the problem; we talk of loops in a topological space U(1). The map $S^1 \to \mathrm{U}(1)$ is classified by the homotopy classes. Take a point $r_0 \in S^1$ and require that r_0 be mapped to $x_0 \in M$. By noting that $\pi_1(\mathrm{U}(1), x_0) \cong \mathbb{Z}$, we may assign an integer to the line defect. This integer is called the **winding number** since it counts how many times the image of S^1 winds the space U(1). If two line defects have the same winding number, one can be continuously deformed to the other. If two line defects A and B merge together, the new line defect belongs to the homotopy class of the product of the homotopy classes to which A and B belonged before coalescence. Since the group operation in $\mathbb{Z}$ is an addition, the new winding number is a sum of the old winding numbers. A uniform distribution of the order parameter corresponds to the constant map $\psi(x) = x_0 \in M$, which belongs to the unit element $0 \in \mathbb{Z}$. If two line defects of opposite winding numbers merge together, the new line defect can be continuously deformed into the defect-free configuration.

What about the other homotopy groups? We first consider the dimensionality of the defect and the sphere S^n which *surrounds* it. For example, consider a point defect in a three-dimensional medium. It can be surrounded by S^2 and the defect is classified by $\pi_2(M, x_0)$. If M has many components, $\pi_0(M)$ is non-trivial. Let us consider a three-dimensional Ising model for which $M = \{\downarrow\} \cup \{\uparrow\}$. Then there is a domain wall on which the order parameter is not defined. For example, if $S = \uparrow$ for $x < 0$ and $S = \downarrow$ for $x > 0$, there is a domain wall in the yz plane at $x = 0$. In general, an m-dimensional defect in a d-dimensional medium is classified by the homotopy group $\pi_n(M, x_0)$ where

$$n = d - m - 1. \tag{4.51}$$

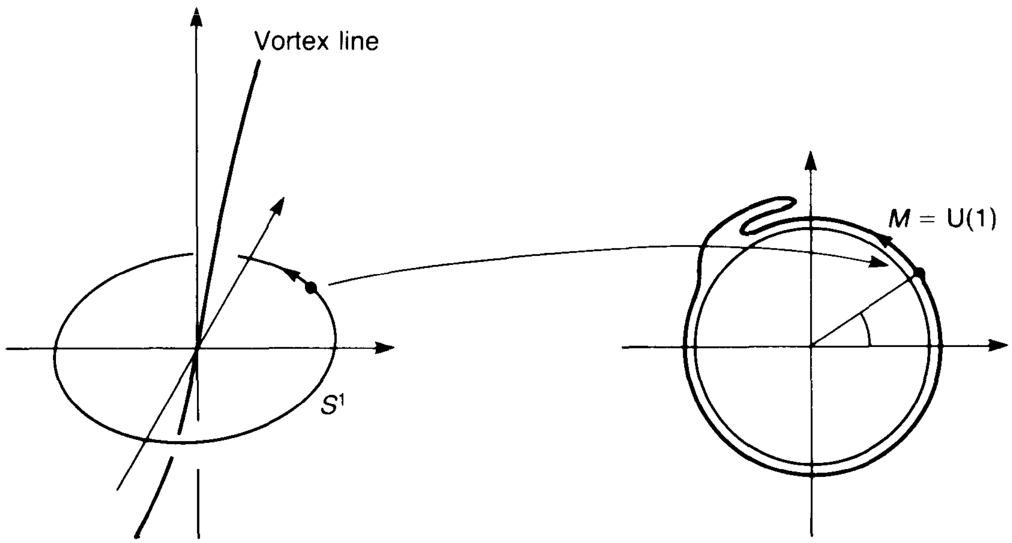


Figure 4.20 A circle S^1 surrounding a line defect (vortex) is mapped to $\mathrm{U}(1) \cong S^1$.

In the case of the Ising model above, $d = 3$, $m = 2$; hence $n = 0$.

4.8.2 Line defects in nematic liquid crystals

From example 4.37 we have $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2 = \{0, 1\}$. There exist two kinds of line defects in nematic liquid crystals; one can be continuously deformed into a uniform configuration while the other cannot. The latter represents a stable vortex, which is sketched in figure 4.21. The reader should observe how the circle α is mapped to $\mathbb{R}P^2$.

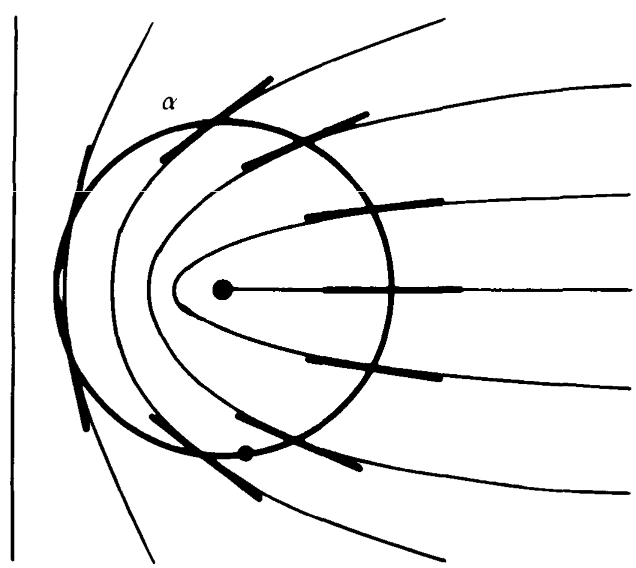


Figure 4.21 A vortex in a nematic liquid crystal, which corresponds to the non-trivial element of $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$.

Exercise 4.53 Show that the line 'defect' in figure 4.22 is fictitious, namely the singularity at the centre may be eliminated by a continuous deformation of directors with directors at the boundary fixed. This corresponds to the operation $1 + 1 = 0$.

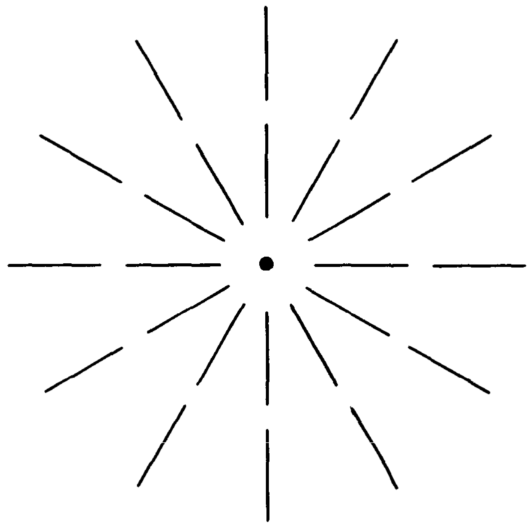


Figure 4.22 A line defect which may be continuously deformed into a uniform configuration.

4.8.3 Point defects in nematic liquid crystals

From example 4.52, we have $\pi_2(\mathbb{R}P^2) \cong \mathbb{Z}$. Accordingly there are stable point defects in the nematic liquid crystal. Figure 4.23 shows the point defects that belong to the class $1 \in \mathbb{Z}$.

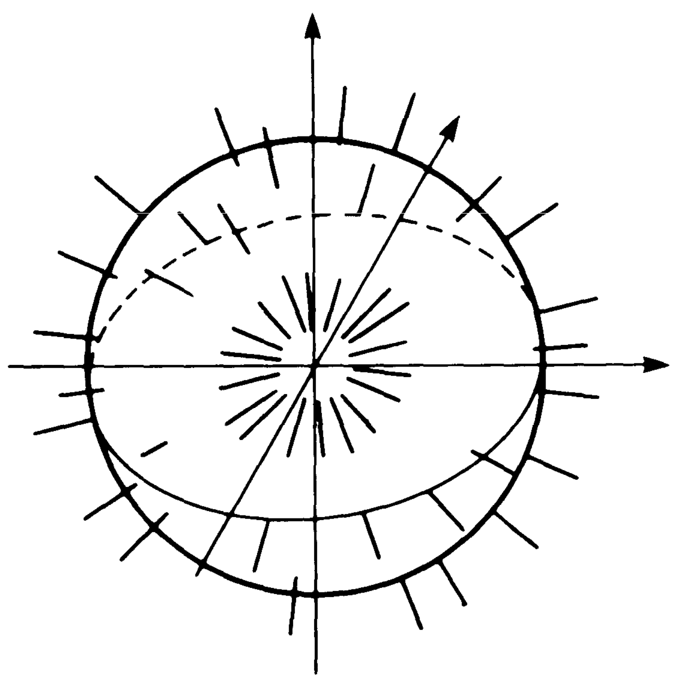


Figure 4.23 A point defect in a nematic liquid crystal.

It is interesting to point out that a line defect and a point defect may be combined into a **ring defect**, which is specified by both $\pi_1(\mathbb{R}P^2)$ *and* $\pi_2(\mathbb{R}P^2)$, see Mineev (1980). If the ring defect is observed from far away, it looks like a point defect, while its local structure along the ring is specified by $\pi_1(\mathbb{R}P^2)$. Figure 4.24 is an example of such a ring defect. The circle α classifies $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$ while the sphere β classifies $\pi_2(\mathbb{R}P^2) = \mathbb{Z}$.

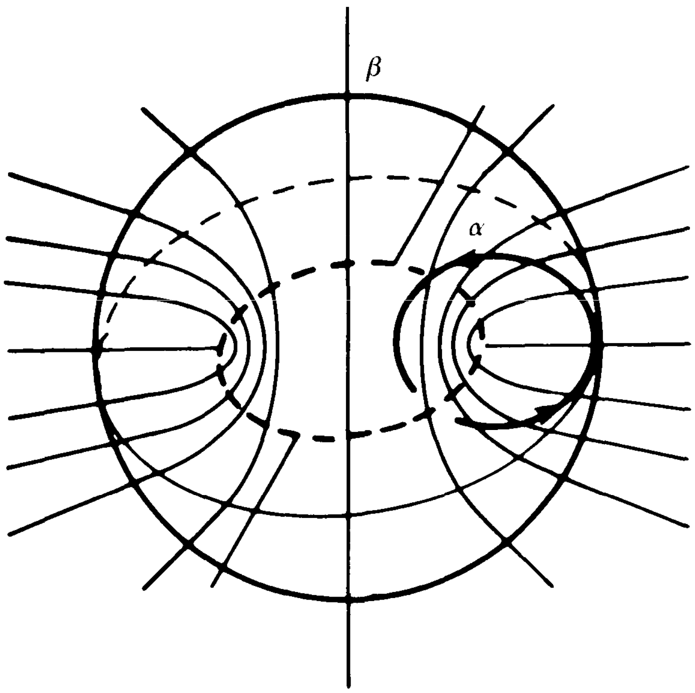


Figure 4.24 A ring defect in a nematic liquid crystal. The loop α classifies $\pi_1(\mathbb{R}P^2)$ while the sphere (2-loop) β classifies $\pi_2(\mathbb{R}P^2)$.

4.9 Textures in superfluid ^{3}He-A

4.9.1 Introduction

For simplicity, we neglect the variation of the $\hat{\boldsymbol{d}}$-vector. [In fact, $\hat{\boldsymbol{d}}$ is locked along $\hat{\boldsymbol{l}}$ due to the dipole force.] The order parameter assumes the form

$$A_i = \Delta_0(\hat{\boldsymbol{\Delta}}_1 + \mathrm{i}\hat{\boldsymbol{\Delta}}_2)_i \tag{4.52}$$

where $\hat{\boldsymbol{\Delta}}_1$, $\hat{\boldsymbol{\Delta}}_2$ and $\hat{\boldsymbol{l}} \equiv \hat{\boldsymbol{\Delta}}_1 \times \hat{\boldsymbol{\Delta}}_2$ form an orthonormal frame at each point of the medium. Let us take a standard orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. The frame $(\hat{\boldsymbol{\Delta}}_1, \hat{\boldsymbol{\Delta}}_2, \hat{\boldsymbol{l}})$ is obtained by applying an element $g \in \mathrm{SO}(3)$ to the standard frame (see §1.5.),

$$g : (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \mapsto (\hat{\boldsymbol{\Delta}}_1, \hat{\boldsymbol{\Delta}}_2, \hat{\boldsymbol{l}}). \tag{4.53}$$

Since g depends on the coordinate x, the configuration $(\hat{\mathbf{\Delta}}_1(x), \hat{\mathbf{\Delta}}_2(x), \hat{l}(x))$ defines a map $\psi : X \to \mathrm{SO}(3)$ as $x \mapsto g(x)$. The map ψ is called the **texture** of a superfluid ^{3}He. The relevant homotopy groups to classify defects in superfluid ^{3}He-A are $\pi_n(\mathrm{SO}(3))$.

If a container is filled with ^{3}He-A, the boundary poses certain conditions on the texture. The vector $\hat{l}$ is understood as the direction of the angular momentum of the Cooper pair. The pair should rotate in the plane parallel to the boundary wall, thus $\hat{l}$ should be perpendicular to the wall. [*Remark:* If the wall is *diffuse*, the orbital motion of Cooper pairs is disturbed and there is a depression in the amplitude of the order parameter in the vicinity of the wall. We assume for simplicity that the wall is *specularly smooth* so that Cooper pairs may execute orbital motion with no disturbance.] There are several kinds of free energies and the texture is determined by solving the Euler–Lagrange equation derived from the total free energy under given boundary conditions.

4.9.2 Line defects and non-singular vortices in ^{3}He-A

The fundamental group of $\mathrm{SO}(3) \cong \mathbb{R}P^3$ is $\pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2 \cong \{0, 1\}$. Textures which belong to class 0 can be continuously deformed into the uniform configuration. Configurations in class 1 are called **disgyrations**, and have been analysed by Maki and Tsuneto (1977) and Buchholtz and Fetter (1977). Figure 4.25 describes these disgyrations in their lowest free energy configurations.

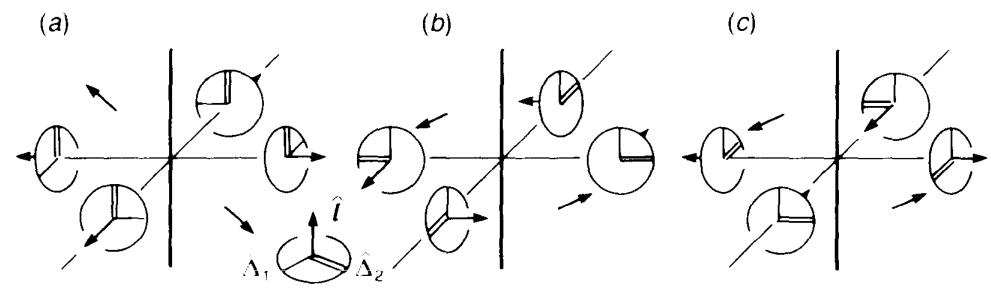


Figure 4.25 Disgyrations in ^{3}He-A.

A remarkable property of $\mathbb{Z}_2$ is the addition $1 + 1 = 0$; the coalescence of two disgyrations produces a trivial texture. By merging two disgyrations, we may construct a texture that looks like a vortex of double vorticity (homotopy class '2') without a singular core; see figure 4.26(a). It is easy to verify that the image of the loop α traverses $\mathbb{R}P^3$ twice while that of the small loop β may be shrunk to a point. This texture is called the **Anderson–Toulouse vortex** (Anderson and Toulouse (1977)). Mermin and Ho (1976) pointed out that if the medium is in a cylinder, the boundary imposes the condition $\hat{l} \perp$ (boundary) and the vortex is cut at the surface, see figure 4.26(b) (the **Mermin–Ho vortex**).

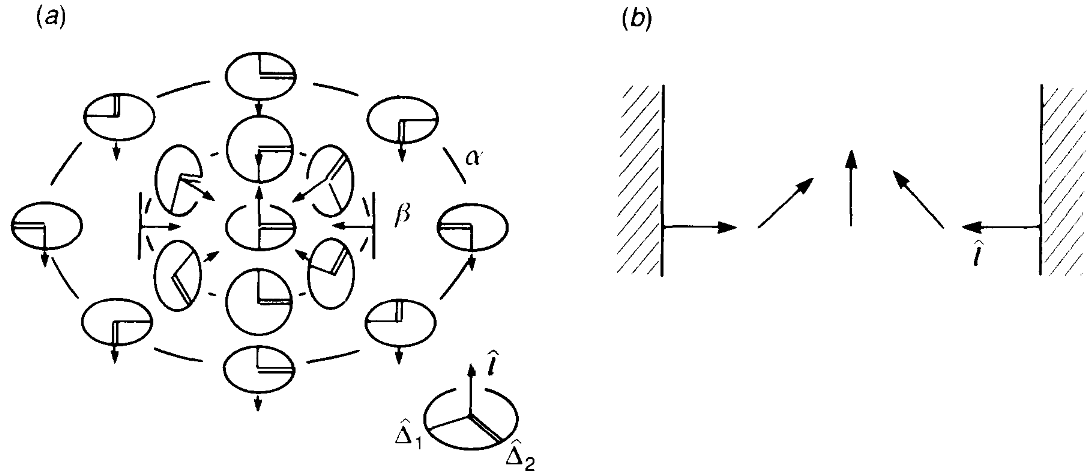


Figure 4.26 The Anderson–Toulouse vortex (a) and the Mermin–Ho vortex (b). In (b) the boundary forces $\hat{l}$ to be perpendicular to the wall.

Since $\pi_2(\mathbb{R}P^3) \cong \{e\}$, there are no point defects in ^{3}He-A. However, $\pi_3(\mathbb{R}P^3) \cong \mathbb{Z}$ introduces a new type of point-like structure called the Shankar monopole, which we will study next.

4.9.3 Shankar monopole in ^{3}He-A

Shankar (1977) pointed out that there exists a point-like singularity-free object in ^{3}He-A. Consider an infinite medium of ^{3}He-A. We assume the medium is asymptotically uniform, that is, $(\hat{\boldsymbol{\Delta}}_1(x), \hat{\boldsymbol{\Delta}}_2(x), \hat{\boldsymbol{l}}(x))$ approaches a standard orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ as $|x| \to \infty$. Since all the points far from the origin are mapped to a single point, we have compactified $\mathbb{R}^3$ to S^3. Then the texture is classified according to $\pi_3(\mathbb{R}P^3) \cong \mathbb{Z}$. Let us specify an element of SO(3) by a 'vector' $\boldsymbol{\Omega} = \theta \boldsymbol{n}$ in $\mathbb{R}P^3$ as before (example 4.50). Shankar (1977) proposed a texture,

$$\boldsymbol{\Omega}(\boldsymbol{r}) = (\boldsymbol{r}/r)\cdot f(r) \tag{4.54a}$$

where $f(r)$ is a monotonically decreasing function such that

$$f(r) = \begin{cases} 2\pi \text{ at } r = 0 \\ \\ 0 \text{ at } r = \infty. \end{cases} \tag{4.54b}$$

We formally extend the radius of $\mathbb{R}P^3$ to 2π and define the rotation angle modulo 2π. This texture is called the **Shankar monopole**, see figure 4.27(a). At first sight it appears that there is a singularity at the origin. Note, however, that the length of $\mathbf{\Omega}$ is 2π there and it is equivalent to the unit element of SO(3). Figure 4.27(b) describes the triad field. Since $\mathbf{\Omega}(\boldsymbol{r}) = 0$ as $r \to \infty$ irrespective of the direction, the space $\mathbb{R}^3$ is compactified to S^3. As we scan the whole space, $\mathbf{\Omega}(\boldsymbol{r})$ sweeps SO(3) twice and this texture corresponds to class 1 of $\pi_3(\mathrm{SO}(3)) = \mathbb{Z}$.

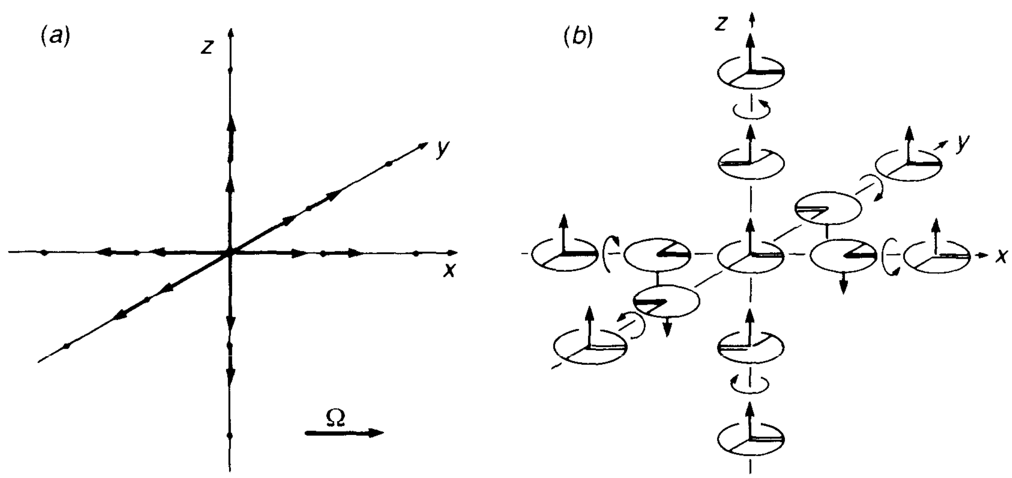


Figure 4.27 The Shankar monopole. (a) shows the 'vectors' $\mathbf{\Omega}(\boldsymbol{r})$ and (b) shows the triad $(\hat{\mathbf{\Delta}}_1, \hat{\mathbf{\Delta}}_2, \hat{l})$. Note that as $|\boldsymbol{r}| \to \infty$ the triad approaches the same configuration.

Exercise 4.54 Sketch the Shankar monopole which belongs to the class -1 of $\pi_3(\mathbb{R}P^3)$. [You cannot simply reverse the arrows in figure 4.27.]

Exercise 4.55 Let us consider classical Heisenberg spins defined in $\mathbb{R}^2$, see §1.5. Suppose spins take the asymptotic value

$$\boldsymbol{n}(x) \to \boldsymbol{e}_z \qquad |x| \geqslant L \tag{4.55}$$

for the total energy to be finite, see figure 1.1. Show that the extended objects in this system are classified by $\pi_2(S^2)$. Sketch examples of spin configurations for the classes -1 and $+2$.

Problems 4

1 Show that the n-sphere S^n is a deformation retract of punctured Euclidean space $\mathbb{R}^{n+1} - \{0\}$. Find a retraction.

2 Let D^2 be the two-dimensional closed disc and $S^1 = \partial D^2$ be its boundary. Let $f : D^2 \to D^2$ be a smooth map. Suppose f has no fixed points, namely $f(p) \neq p$ for *any* $p \in D^2$. Consider a semiline starting at p through $f(p)$ (if $p \neq f(p)$ this semiline is always well defined). The

line crosses the boundary at some point $q \in S^1$. Then define $\tilde{f} : D^2 \to S^1$ by $\tilde{f}(p) = q$. Use $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2) \cong \{0\}$ to show that such $\tilde{f}$ does not exist and hence that f has fixed points. [*Hint:* Show that if such $\tilde{f}$ existed, D^2 and S^1 would be of the same homotopy type.] This is the two-dimensional version of the **Brouwer fixed-point theorem**.

3 Construct a map $f : S^3 \to S^2$ which belongs to the elements 0 and 1 of $\pi_3(S^2) \cong \mathbb{Z}$. See also example 9.13.

5

MANIFOLDS

Manifolds are generalisations of our familiar ideas about curves and surfaces to arbitrary dimensional objects. A curve in the three-dimensional Euclidean space is parametrised locally by a single number t as $(x(t), y(t), z(t))$, while two numbers u and v parametrise a surface as $(x(u, v), y(u, v), z(u, v))$. A curve and a surface are considered *locally* homeomorphic to $\mathbb{R}$ and $\mathbb{R}^2$, respectively. A manifold in general is a topological space which is homeomorphic to $\mathbb{R}^m$ *locally*; it may be different from $\mathbb{R}^m$ *globally*. The local homeomorphism enables us to give each point in a manifold a set of m numbers called the (local) coordinate. If a manifold is not homeomorphic to $\mathbb{R}^m$ globally, we have to introduce several local coordinates. Then it is possible that a single point has two or more coordinates. We require that the transition from one coordinate to the other be *smooth*. As we will see below this enables us to develop the usual calculus on a manifold. Just as topology is based on continuity, so the theory of manifolds is based on *smoothness*.

Useful references on this subject are Crampin and Pirani (1986), Matsushima (1972), Schutz (1980) and Warner (1983). Chapter 2 and Appendices B and C of Wald (1984) are also recommended. Flanders (1963) is a beautiful introduction to differential forms. Sattinger and Weaver (1986) deals with Lie groups and Lie algebras and contains many applications to problems in physics.

5.1 Manifolds

5.1.1 Heuristic introduction

To clarify the above points, consider the usual sphere of unit radius in $\mathbb{R}^3$. We parametrise the surface of S^2, among other possibilities, by two coordinate systems—polar coordinates and stereographic coordinates. Polar coordinates θ and ϕ are usually defined by (figure 5.1)

$$x = \sin\theta\cos\phi \qquad y = \sin\theta\sin\phi \qquad z = \cos\theta. \tag{5.1}$$

In elementary geometry, ϕ runs from 0 to 2π and θ from 0 to π. They may be inverted on the sphere to yield

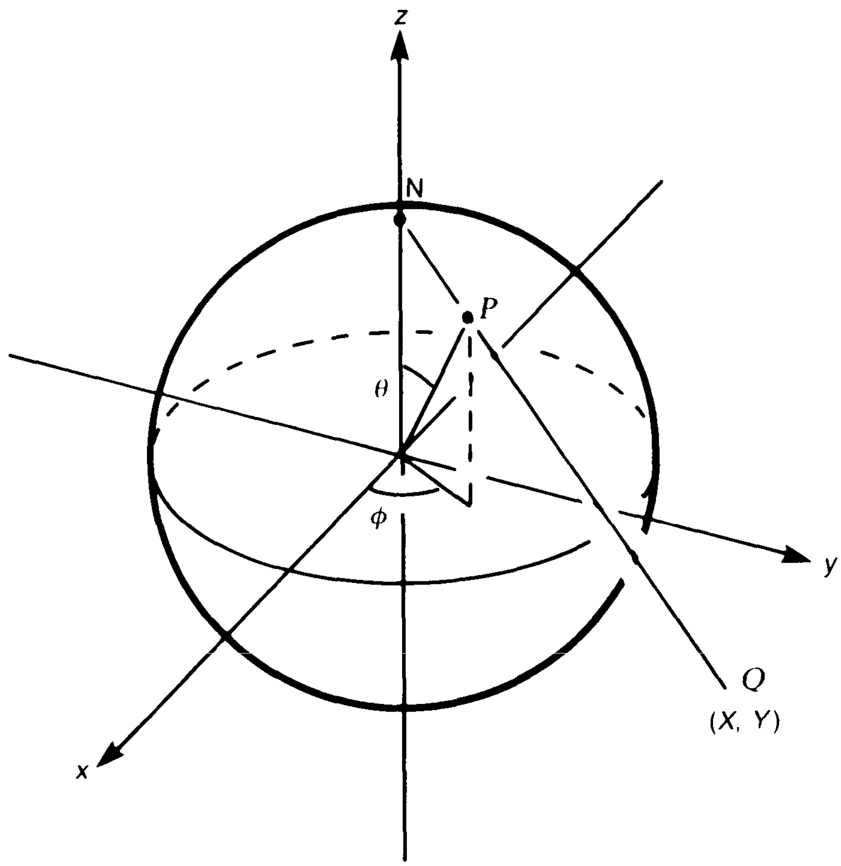


Figure 5.1 Polar coordinates (θ, ϕ) and stereographic coordinates (X, Y) of a point P on the sphere S^2.

$$\theta = \tan^{-1}\left(\frac{(x^2 + y^2)^{1/2}}{z}\right) \qquad \phi = \tan^{-1}(y/x). \tag{5.2}$$

Stereographic coordinates, on the other hand, are defined by the projection from the North Pole onto the equatorial plane as in figure 5.1. First, join the North Pole $(0, 0, 1)$ to the point $P(x, y, z)$ on the sphere and then continue in a straight line to the equatorial plane $z = 0$ to intersect at $Q(X, Y, 0)$. Then X and Y are the stereographic coordinates of P. We find

$$X = \frac{x}{1 - z} \qquad Y = \frac{y}{1 - z}. \tag{5.3}$$

Two coordinate systems are related as

$$X = \cot \tfrac{1}{2}\theta \cos \phi \qquad Y = \cot \tfrac{1}{2}\theta \sin \phi. \tag{5.4}$$

Of course, other systems, polar coordinates with different polar axes, or projections from different points on S^2, could be used. The coordinates on the sphere may be kept arbitrary until some specific calculation is to be carried out. [The longitude is historically measured from Greenwich. However, there is no reason why it cannot be measured from New York or Kyoto.] This arbitrariness of the coordinate choice underlies the theory of manifolds: *all coordinate systems are equally good*. It is also in harmony with the basic principle of physics: *a physical system behaves in the same way whatever coordinates we use to describe it*.

Another point which can be seen from this example is that *no coordinate system may be usable everywhere at once*. Let us look at the polar coordinates on S^2. Take the equator ($\theta = \frac{1}{2}\pi$) for definiteness. If we let ϕ range from 0 to 2π, then it changes continuously as we go round the equator until we get all the way to $\phi = 2\pi$. There the ϕ-coordinate has a discontinuity from 2π to 0 and nearby points have quite different ϕ-values. Alternatively we could continue ϕ through 2π. Then we will encounter another difficulty: at each point we must have infinitely many ϕ-values, different from one another by an integral multiple of 2π. A further difficulty arises at the poles, where ϕ is not determined at all. [An explorer on the Pole is in a state of *timelessness* since time is defined by the longitude.] Stereographic coordinates also have difficulties at the North Pole or at any projection point that is not projected to a point on the equatorial plane; and nearby points close to the Pole have widely different stereographic coordinates.

Thus we cannot label the points on the sphere with a single coordinate system so that both of the following conditions are satisfied.

(i) Nearby points always have nearby coordinates.
(ii) Every point has unique coordinates.

Note, however, that there are infinitely many ways to introduce coordinates that satisfy the above requirements on a *part* of S^2. We may take advantage of this fact to define coordinates on S^2; introduce two or more overlapping coordinate systems, each covering a part of the sphere whose points are to be labelled so that the following conditions hold.

(i′) Nearby points have nearby coordinates in at least one coordinate system.
(ii′) Every point has unique coordinates in each system that contains it.

For example, we may introduce two stereographic coordinates on S^2, one a projection from the North Pole, the other from the South Pole. Are these conditions (i′) and (ii′) enough to develop sensible theories of the manifold? In fact, we need an extra condition on the coordinate systems.

(iii) If two coordinate systems overlap, they are related to each other in a sufficiently smooth way.

Without this condition, a differentiable function in one coordinate system may not be differentiable in the other system.

5.1.2 Definitions

Definition 5.1 M is an **m-dimensional differentiable manifold** if

(i) M is a topological space
(ii) M is provided with a family of pairs $\{(U_i, \varphi_i)\}$
(iii) $\{U_i\}$ is a family of open sets which covers M, that is, $\cup_i U_i = M$. φ_i is a homeomorphism from U_i onto an open subset U'_i of $\mathbb{R}^m$ (figure 5.2)
(iv) Given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \varphi_i \varphi_j^{-1}$ from $\varphi_j(U_i \cap U_j)$ to $\varphi_i(U_i \cap U_j)$ is infinitely differentiable.

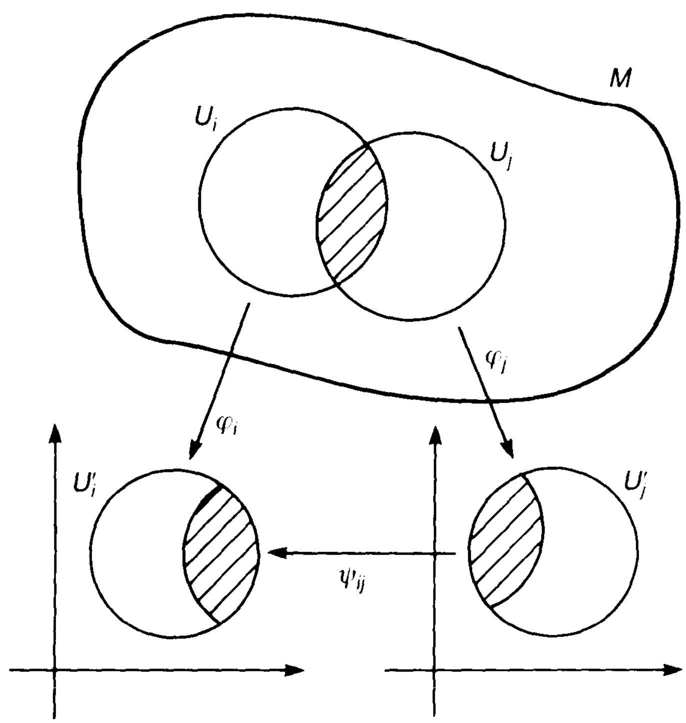


Figure 5.2 A homeomorphism φ_i maps U_i onto an open subset $U'_i \subset \mathbb{R}^m$, providing coordinates to a point $p \in U_i$. If $U_i \cap U_j \neq \emptyset$, the transition from one coordinate system to the other is smooth.

The pair (U_i, φ_i) is called a **chart** while the whole family $\{(U_i, \varphi_i)\}$ is called, for obvious reasons, an **atlas**. The subset U_i is called the **coordinate neighbourhood** while φ_i is the **coordinate function**, or simply the **coordinate**. φ_i is represented by m functions $\{x^1(p), \ldots, x^m(p)\}$. The set $\{x^\mu(p)\}$ is also called the **coordinate**. A point $p \in M$ exists *independently* of its coordinates; it is up to *us* how to assign coordinates to a point. We sometimes employ the rather sloppy notation x to denote a point whose coordinates are $\{x^1, \ldots, x^m\}$, unless several coordinate systems are in use. From (ii) and (iii) above, M is locally Euclidean. In each coordinate neighbourhood U_i, M looks like an open subset of $\mathbb{R}^m$ whose element is $\{x^1, \ldots, x^m\}$. Note that we do not require that M be $\mathbb{R}^m$ *globally*. We are living on the earth whose surface is S^2, which does not look like $\mathbb{R}^2$ globally. However, it looks like an open subset of $\mathbb{R}^2$ *locally*. Who can tell that we live on the sphere by just looking at a map of London, which of course looks like a part of $\mathbb{R}^2$?

If U_i and U_j overlap, two coordinate systems are assigned to a point in $U_i \cap U_j$. The axiom (iv) asserts that the transition from one coordinate system to the other be *smooth* (C^∞). The map φ_i assigns m coordinate values x^μ $(1 \leq \mu \leq m)$ to a point $p \in U_i \cap U_j$, while φ_j assigns y^μ $(1 \leq \mu \leq m)$ to the same point and the transition from y to x, $x^\mu = x^\mu(y)$, is given by m functions of m variables. The coordinate transformation functions $x^\mu = x^\mu(y)$ are the explicit form of the map $\psi_{ji} = \varphi_j \varphi_i^{-1}$. Thus the differentiability has been defined in the usual sense of calculus: the coordinate transformation is differentiable if each function $x^\mu(y)$ is differentiable with respect to each y^ν. We may restrict ourselves to the differentiability up to kth order (C^k). However, this does not bring about any interesting conclusions. We simply require, instead, that the coordinate transformations be infinitely differentiable, that is, of class C^∞. Now coordinates have been assigned to M in such a way that if we move over M in whatever fashion, the coordinates we use vary in a smooth manner.

If the union of two atlases $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ is again an atlas, these two atlases are said to be **compatible**. The compatibility is an equivalence relation, the equivalence class of which is called the **differentiable structure**. It is also said that mutually compatible atlases define the same differentiable structure on M.

Before we give examples, we briefly comment on manifolds *with boundaries*. So far, we have assumed that the coordinate neighbourhood U_i is homeomorphic to an open set of $\mathbb{R}^n$. In some applications, however, this turns out to be too restrictive and we need to relax this condition. If a topological space M is covered by a family of open sets $\{U_i\}$ each of which is homeomorphic to an open set of $H^m \equiv \{(x^1, \ldots, x^m) \in \mathbb{R}^m | x^m \geq 0\}$, M is said to be a **manifold with a boundary**, see figure 5.3. The set of points which are mapped to points with $x^m = 0$ is called the **boundary** of M, denoted by ∂M. The coordinates of ∂M may be given by $m - 1$ numbers $(x^1, \ldots, x^{m-1}, 0)$. Now we have to be careful when we define the smoothness. The map $\psi_{ij} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is defined on an open set of H^m in general, and ψ_{ij} is said to be smooth if it is C^∞ in an open set of $\mathbb{R}^m$

which contains $\varphi_j(U_i \cap U_j)$. Readers are encouraged to use their imagination since our definition is in harmony with our intuitive notions about boundaries. For example, the boundary of the solid ball B^3 is the sphere S^2 and the boundary of the sphere is an empty set.

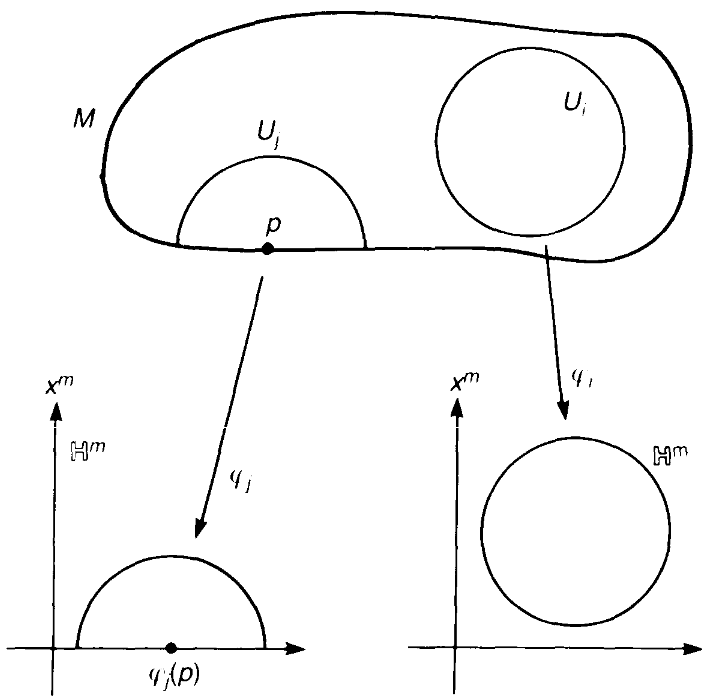


Figure 5.3 A manifold with a boundary. The point p is on the boundary.

5.1.3 Examples

We now give several examples to develop our ideas about manifolds. They are also of great relevance to physics.

Example 5.2 The Euclidean space $\mathbb{R}^m$ is the most trivial example, where a single chart covers the whole space and φ may be the identity map.

Example 5.3 Let $m = 1$ and require that M be connected. There are only two manifolds possible: a real line $\mathbb{R}$ and the circle S^1. Let us work out an atlas of S^1. For concreteness take the circle $x^2 + y^2 = 1$ in the xy plane. We need at least two charts. We may take them as in figure 5.4. Define $\varphi_1^{-1} : (0, 2\pi) \to S^1$ by

$$\varphi_1^{-1} : \theta \mapsto (\cos\theta, \sin\theta) \tag{5.5a}$$

whose image is $S^1 - \{(1, 0)\}$. Define also $\varphi_2^{-1} : (-\pi, +\pi) \to S^1$ by

$$\varphi_2^{-1} : \theta \mapsto (\cos\theta, \sin\theta) \tag{5.5b}$$

whose image is $S^1 - \{(-1, 0)\}$. Clearly φ_1^{-1} and φ_2^{-1} are invertible and all the maps φ_1, φ_2, φ_1^{-1} and φ_2^{-1} are continuous. Thus φ_1 and φ_2 are homeomorphisms. Verify that the maps $\psi_{12} = \varphi_1\varphi_2^{-1}$ and $\psi_{21} = \varphi_2\varphi_1^{-1}$ are smooth.

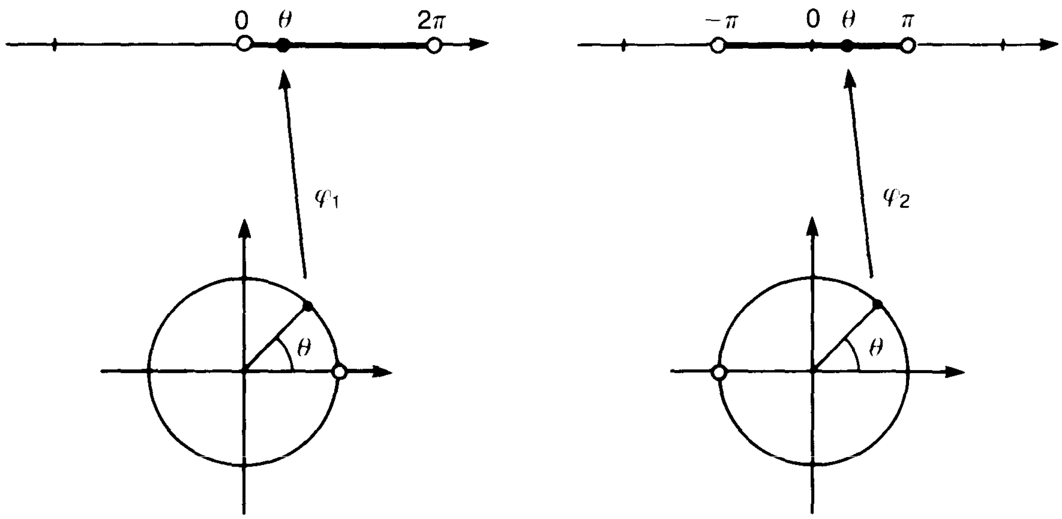


Figure 5.4 Two charts of a circle S^1.

Example 5.4 The n-dimensional sphere S^n is a differentiable manifold. It is realised in $\mathbb{R}^{n+1}$ as

$$\sum_{i=0}^{n} (x^i)^2 = 1. \tag{5.6}$$

Let us introduce the coordinate neighbourhoods

$$U_{i+} \equiv \{(x^0, x^1, \ldots, x^n) \in S^n | x^i > 0\} \tag{5.7a}$$

$$U_{i-} \equiv \{(x^0, x^1, \ldots, x^n) \in S^n | x^i < 0\}. \tag{5.7b}$$

Define the coordinate map $\varphi_{i+} : U_{i+} \to \mathbb{R}^n$ by

$$\varphi_{i+}(x^0, \ldots, x^n) = (x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) \tag{5.8a}$$

and $\varphi_{i-} : U_{i-} \to \mathbb{R}^n$ by

$$\varphi_{i-}(x^0, \ldots, x^n) = (x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) \tag{5.8b}$$

Note that the domains of φ_{i+} and φ_{i-} are different. $\varphi_{i\pm}$ are the projections of the hemispheres $U_{i\pm}$ to the plane $x^i = 0$. The transition functions are easily obtained from (5.8). Take S^2 as an example. The coordinate neighbourhoods are $U_{x\pm}$, $U_{y\pm}$ and $U_{z\pm}$. The transition function $\psi_{y-x+} \equiv \varphi_{y-}\varphi_{x+}^{-1}$ is given by

$$\psi_{y-x+} : (y, z) \mapsto (-(1 - y^2 - z^2)^{1/2}, z) \tag{5.9}$$

which is infinitely differentiable on $U_{x+} \cap U_{y-}$.

Exercise 5.5 At the beginning of this chapter, we introduced the stereographic coordinates on S^2. We may equally define the stereographic coordinates projected from points other than the North Pole. For example, the stereographic coordinates (U, V) of a point in $S^2 - \{\text{South Pole}\}$ projected from the South Pole and (X, Y) for a point in $S^2 - \{\text{North Pole}\}$ projected from the North Pole are shown in figure 5.5. Show that the transition functions between (U, V) and (X, Y) are C^∞ and that they define a differentiable structure on M. See also example 8.2.

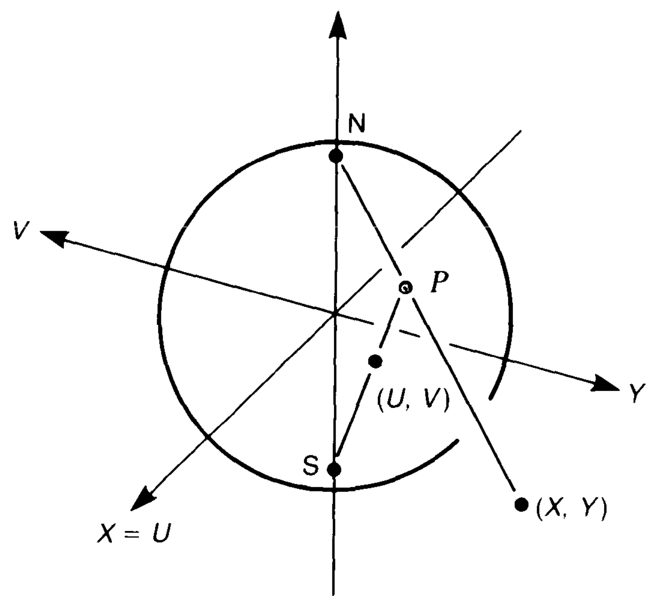


Figure 5.5 Two stereographic coordinate systems on S^2. P may be projected from the North Pole N giving (X, Y) or from the South Pole S giving (U, V).

Example 5.6 The real projective space $\mathbb{R}P^n$ is the set of lines through the origin in $\mathbb{R}^{n+1}$. If $x = (x^0, \ldots, x^n) \neq 0$, x defines a line through the origin. Note that $y \in \mathbb{R}^{n+1}$ defines the same line as x if there exists a real number $a \neq 0$ such that $y = ax$. Introduce an equivalence relation $\sim$ by $x \sim y$ if there exists $a \in \mathbb{R} - \{0\}$ such that $y = ax$. Then $\mathbb{R}P^n = (\mathbb{R}^{n+1} - \{\mathbf{0}\})/\sim$. The $n + 1$ numbers $x^0, x^1, \ldots, x^n$ are called the **homogeneous coordinates**. The homogeneous coordinates cannot be a good coordinate system, since $\mathbb{R}P^n$ is an n-dimensional manifold (an $(n + 1)$-dimensional space with a one-dimensional degree of freedom killed). The charts are defined as follows. First we take the coordinate neighbourhood U_i as the set of lines with $x^i \neq 0$, and then introduce the **inhomogeneous coordinates** on U_i by

$$\xi^j_{(i)} = x^j/x^i. \tag{5.10}$$

The inhomogeneous coordinates

$$\xi_{(i)} = (\xi^0_{(i)}, \xi^1_{(i)}, \ldots, \xi^{i-1}_{(i)}, \xi^{i+1}_{(i)}, \ldots, \xi^n_{(i)})$$

with $\xi^i_{(i)} = 1$ omitted, are well defined on U_i since $x^i \neq 0$, and furthermore they are independent of the choice of the representative of the equivalence class since $x^j/x^i = y^j/y^i$ if $y = ax$. $\xi_{(i)}$ gives the coordinate map $\varphi_i : U_i \to \mathbb{R}^n$, that is

$$\varphi_i : (x^0, \ldots, x^n) \mapsto (x^0/x^i, \ldots, x^{i-1}/x^i, x^{i+1}/x^i, \ldots, x^n/x^i)$$

where $x^i/x^i = 1$ is omitted. For $x = (x^0, x^1, \ldots, x^n) \in U_i \cap U_j$ we assign two inhomogeneous coordinates, $\xi^k_{(i)} = x^k/x^i$ and $\xi^k_{(j)} = x^k/x^j$. The coordinate transformation $\psi_{ij} = \varphi_i\varphi_j^{-1}$ is

$$\psi_{ij} : \xi^k_{(j)} \mapsto \xi^k_{(i)} = (x^j/x^i)\xi^k_{(j)}. \tag{5.11}$$

Thus ψ_{ij} is a multiplication by x^j/x^i.

In example 4.50, we defined $\mathbb{R}P^n$ as the sphere S^n with antipodal points identified. This picture is in conformity with the definition here. As a representative of the equivalence class $[x]$, we may take points $|x| = 1$ on a line through the origin. These are points on the unit sphere. Since there are two points on the intersection of a line with S^n, we have to take one of them consistently, that is nearby lines are represented by nearby points in S^n. This amounts to taking the hemisphere. Note, however, that the antipodal points on the boundary (the equator of S^n) are identified by definition, $(x^0, \ldots, x^n) \sim -(x^0, \ldots, x^n)$. This 'hemisphere' is homeomorphic to the ball D^n with antipodal points on the boundary S^{n-1} identified.

Example 5.7 A straightforward generalisation of $\mathbb{R}P^n$ is the **Grassmannian manifold**. An element of $\mathbb{R}P^n$ is a one-dimensional subspace in $\mathbb{R}^{n+1}$. The Grassmannian manifold $G_{k,n}(\mathbb{R})$ is the set of k-dimensional planes in $\mathbb{R}^n$. Note that $G_{1,n+1}(\mathbb{R})$ is nothing but $\mathbb{R}P^n$. The manifold structure of $G_{k,n}(\mathbb{R})$ is defined in a manner similar to that of $\mathbb{R}P^n$.

Let $M_{k,n}(\mathbb{R})$ be the set of $k \times n$ matrices of rank k $(k \leq n)$. Take $A = (a_{ij}) \in M_{k,n}(\mathbb{R})$ and define k vectors $\boldsymbol{a}_i$ $(1 \leq i \leq k)$ in $\mathbb{R}^n$ by $\boldsymbol{a}_i = (a_{ij})$. Since $\operatorname{rank} A = k$, k vectors $\boldsymbol{a}_i$ are linearly independent and span a k-dimensional plane in $\mathbb{R}^n$. Note, however, that there are infinitely many matrices in $M_{k,n}(\mathbb{R})$ that yield the same k-plane. Take $g \in \mathrm{GL}(k, \mathbb{R})$ and consider a matrix $\bar{A} = gA \in M_{k,n}(\mathbb{R})$. $\bar{A}$ defines the same k-plane as A, since g simply rotates the basis within the k-plane. Introduce an equivalence relation $\sim$ by $\bar{A} \sim A$ if there exists $g \in \mathrm{GL}(k, \mathbb{R})$ such that $\bar{A} = gA$. We identify $G_{k,n}(\mathbb{R})$ with the coset space $M_{k,n}(\mathbb{R})/\mathrm{GL}(k, \mathbb{R})$.

Let us find the charts of $G_{k,n}(\mathbb{R})$. Take $A \in M_{k,n}(\mathbb{R})$ and let $\{A_1, \ldots, A_l\}$, $l = \binom{n}{k}$, be the collection of all $k \times k$ minors of A. Since rank $A = k$, there exists some A_α $(1 \leqslant \alpha \leqslant l)$ such that $\det A \neq 0$. For example, let us assume the minor A_1 made of the first k columns has non-vanishing determinant,

$$A = (A_1, \widetilde{A}_1) \tag{5.12}$$

where $\widetilde{A}_1$ is a $k \times (n-k)$ matrix. Let us take the representative of the class to which A belongs to be

$$A_1^{-1} \cdot A = (\mathbb{1}_k, A_1^{-1} \cdot \widetilde{A}_1) \tag{5.13}$$

where $\mathbb{1}_k$ is the $k \times k$ unit matrix. Note that A_1^{-1} always exists since $\det A_1 \neq 0$. Thus the real degrees of freedom are given by the entries of the $k \times (n-k)$ matrix $A_1^{-1} \cdot \widetilde{A}_1$. We denote this subset of $G_{k,n}(\mathbb{R})$ by U_1. U_1 is a coordinate neighbourhood whose coordinates are given by $k \cdot (n-k)$ entries of $A_1^{-1} \cdot \widetilde{A}_1$. Since U_1 is homeomorphic to $\mathbb{R}^{k(n-k)}$ we have

$$\dim G_{k,n}(\mathbb{R}) = k \cdot (n-k). \tag{5.14}$$

In the case that $\det A_\alpha \neq 0$, where A_α is composed of the columns $(i_1, i_2, \ldots, i_k)$, we multiply A_α^{-1} to obtain the representative

$$\begin{array}{cc} \text{column} \rightarrow & \begin{array}{ccccccccc} & i_1 & & i_2 & \ldots & i_k & \end{array} \\ A_\alpha^{-1} \cdot A = & \begin{pmatrix} \ldots & 1 & \ldots & 0 & \ldots\ldots & 0 & \ldots \\ \ldots & 0 & \ldots & 1 & \ldots\ldots & 0 & \ldots \\ \ldots & \cdot & \ldots & \cdot & \ldots\ldots & \cdot & \ldots \\ \ldots & 0 & \ldots & 0 & \ldots\ldots & 1 & \ldots \end{pmatrix} \end{array} \tag{5.15}$$

where the entries not written explicitly form a $k \times (n-k)$ matrix. We denote this subset of $M_{k,n}(\mathbb{R})$ with $\det A_\alpha \neq 0$ by U_α. The entries of the $k \times (n-k)$ matrix are the coordinates of U_α.

The relation between the projective space and the Grassmannian manifold is evident. An element of $M_{1,n+1}(\mathbb{R})$ is a vector $A = (x^0, x^1, \ldots, x^n)$. Since the αth minor A_α of A is a number x^α, the condition $\det A_\alpha \neq 0$ becomes $x^\alpha \neq 0$. The representative (5.15) is just the inhomogeneous coordinate

$$(x^\alpha)^{-1} \cdot (x^0, x^1, \ldots, x^\alpha, \ldots, x^n)$$
$$= (x^0/x^\alpha, x^1/x^\alpha, \ldots, x^\alpha/x^\alpha = 1, \ldots, x^n/x^\alpha).$$

Let M be an m-dimensional manifold with an atlas $\{(U_i, \varphi_i)\}$ and N be an n-dimensional manifold with $\{(V_i, \psi_i)\}$. A **product manifold** $M \times N$ is an $(m+n)$-dimensional manifold whose atlas is $\{(U_i \times V_j, (\varphi_i, \varphi_j))\}$. A point in $M \times N$ is written as (p, q), $p \in M$, $q \in N$, and the coordinate function (φ_i, ψ_j) acts on (p, q) to yield $(\varphi_i(p), \psi_j(p)) \in \mathbb{R}^{m+n}$. The reader should verify that a product manifold indeed satisfies the axioms of definition 5.1.

Example 5.8 The torus T^2 is a product manifold of two circles, $T^2 = S^1 \times S^1$. If we denote the polar angle of each circle as $\theta_i \bmod 2\pi$ $(i = 1, 2)$, the coordinates of T^2 are (θ_1, θ_2). Since each S^1 is embedded in $\mathbb{R}^2$, T^2 may be embedded in $\mathbb{R}^4$. We often imagine T^2 as the surface of a doughnut in $\mathbb{R}^3$, in which case, however, we inevitably have to introduce bending of the surface. This is an extrinsic feature brought about by the 'embedding'. When we say 'a torus is a flat manifold', we refer to the flat surface embedded in $\mathbb{R}^4$. See definition 5.17 for further details.

We may also consider a direct product of n circles,

$$T^n = \underbrace{S^1 \times S^1 \times \ldots \times S^1}_{n}.$$

Clearly T^n is an n-dimensional manifold with the coordinates $(\theta_1, \theta_2, \ldots, \theta_n) \bmod 2\pi$. This may be regarded as an n-cube whose opposite faces are identified.

5.2 The calculus on manifolds

The significance of differentiable manifolds resides in the fact that we may use the usual calculus developed in $\mathbb{R}^n$. Smoothness of the coordinate transformations ensures that the calculus is independent of the coordinates chosen.

5.2.1 Differentiable maps

Let $f: M \to N$ be a map from an m-dimensional manifold M to an n-dimensional manifold N. A point $p \in M$ is mapped to a point $f(p) \in N$, namely $f: p \mapsto f(p)$, see figure 5.6. Take a chart (U, φ) on M and (V, ψ) on N, where $p \in U$ and $f(p) \in V$. Then f has the following coordinate presentation:

$$\psi f \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n. \tag{5.16}$$

If we write $\varphi(p) = \{x^\mu\}$ and $\psi(f(p)) = \{y^\alpha\}$, $\psi \cdot f \cdot \varphi^{-1}$ is just the usual vector-valued function $y = \psi f \varphi^{-1}(x)$ of m variables. We sometimes use (in fact, abuse!) the notation $y = f(x)$ or $y^\alpha = f^\alpha(x^\mu)$, when we know which coordinate systems on M and N are in use. If $y = \psi f \varphi^{-1}(x)$, or simply $y^\alpha = f^\alpha(x^\mu)$, is C^∞ with respect to each x^μ, f is said to be **differentiable** at p or at $x = \varphi(p)$. Differentiable functions are also said to be **smooth**. Note that we required infinite (C^∞) differentiability, in harmony with the smoothness of the transition functions ψ_{ij}.

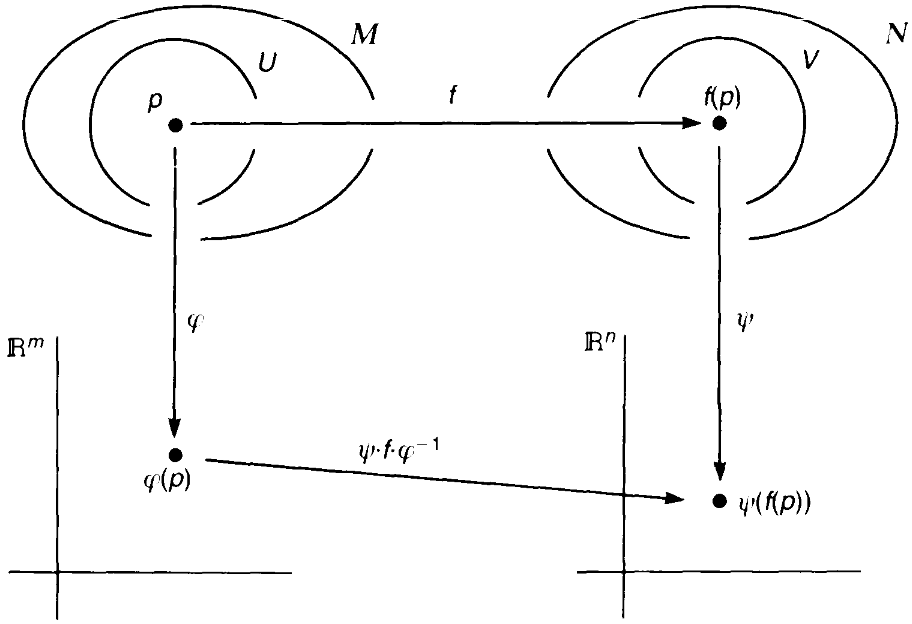


Figure 5.6 A map $f : M \to N$ has a coordinate presentation $\psi f \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$.

The differentiability of f is independent of the coordinate system. Consider two overlapping charts (U_1, φ_1) and (U_2, φ_2). Take a point $p \in U_1 \cap U_2$, whose coordinates by φ_1 are $\{x_1^\mu\}$ while those by φ_2 are $\{x_2^\nu\}$. When expressed in terms of $\{x_1^\mu\}$, f takes the form $\psi f \varphi_1^{-1}$, while in $\{x_2^\nu\}$, $\psi f \varphi_2^{-1} = \psi f \varphi_1^{-1}(\varphi_1 \varphi_2^{-1})$. By definition, $\psi_{12} = \varphi_1 \varphi_2^{-1}$ is C^∞. In the simpler expressions, they correspond to $y = f(x_1)$ and $y = f(x_1(x_2))$. It is clear that if $f(x_1)$ is C^∞ with respect to x_1^μ and $x_1(x_2)$ is C^∞ with respect to x_2^ν, then $y = f(x_1(x_2))$ is also C^∞ with respect to x_2^ν.

Exercise 5.9 Show that the differentiability of f is also independent of the chart in N.

Definition 5.10 If $\psi f \varphi^{-1}$ above is invertible (that is, there exists a map $\varphi f^{-1} \psi^{-1}$) and both $y = \psi f \varphi^{-1}(x)$ and $x = \varphi f^{-1} \psi^{-1}(y)$ are C^∞, f is called a **diffeomorphism** and M is said to be **diffeomorphic** to N and vice versa, denoted by $M \equiv N$.

Clearly $\dim M = \dim N$ if $M \equiv N$. In Chapter 2, we noted that homeomorphisms classify spaces according to whether it is possible to deform one space to the other *continuously*. Diffeomorphisms classify spaces into equivalence classes according to whether it is possible to deform one space to the other *smoothly*. Two diffeomorphic spaces are regarded as the same manifold. Clearly a diffeomorphism is a homeomorphism. What about the converse? Is a homeomorphism a diffeomorphism? In the previous section, we defined the differentiable structure as an equivalence class of atlases. Is it possible for a topological space to carry many differentiable structures? It is rather difficult to give examples of 'diffeomorphically inequivalent homeomorphisms' since it is known that this is possible only in higher-dimensional spaces ($\dim M \geqslant 4$). Before 1956, it was believed that a topological space admits only one differentiable structure. However, Milnor (1956) pointed out that S^7 admits 28 differentiable structures. A recent striking discovery in mathematics is that $\mathbb{R}^4$ admits an infinite number of differentiable structures. Interested readers should consult Donaldson (1983) and Freed and Uhlenbeck (1984). Here we assume that a manifold admits a unique differentiable structure, for simplicity.

The set of diffeomorphisms $f: M \to M$ is a group denoted by $\mathrm{Diff}(M)$. Take a point p in a chart (U, φ) such that $\varphi(p) = x^\mu(p)$. Under $f \in \mathrm{Diff}(M)$, p is mapped to $f(p)$ whose coordinates are $\varphi(f(p)) = y^\mu(f(p))$ (we have assumed $f(p) \in U$). Clearly y is a differentiable function of x; this is an *active* point of view to the coordinate transformation. On the other hand, if (U, φ) and (V, ψ) are overlapping charts, we have two coordinate values $x^\mu = \varphi(p)$ and $y^\mu = \psi(p)$ for a point $p \in U \cap V$. The map $x \mapsto y$ is differentiable by the assumed smoothness of the manifold; this reparametrisation is a *passive* point of view to the coordinate transformation. We also denote the group of reparametrisations by $\mathrm{Diff}(M)$.

Now we look at special classes of mappings, namely **curves** and **functions**. An open curve in an m-dimensional manifold M is a map $c: (a, b) \to M$, where (a, b) is an open interval such that $a < 0 < b$. We assume that the curve does not intersect with itself (figure 5.7). The

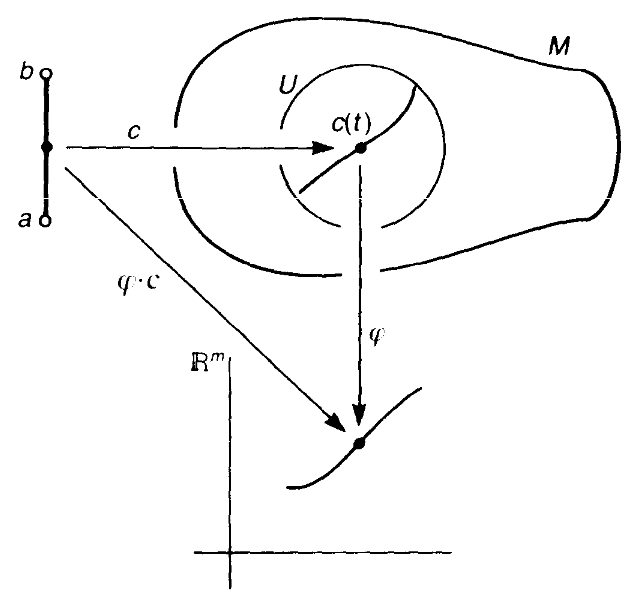


Figure 5.7 A curve c in M and its coordinate presentation φc.

number a (b) may be $-\infty$ ($+\infty$) and we have included 0 in the interval for later convenience. If a curve is closed, it is regarded as a map $c : S^1 \to M$. In both cases, c is locally a map from an open interval to M. On a chart (U, φ), a curve $c(t)$ has the coordinate presentation $x = \varphi c : \mathbb{R} \to \mathbb{R}^m$.

A **function** f on M is a smooth map from M to $\mathbb{R}$, see figure 5.8. On a chart (U, φ), the coordinate presentation of f is given by $f\varphi^{-1} : \mathbb{R}^m \to \mathbb{R}$ which is a real-valued function of m variables. We denote the set of smooth functions on M by $\mathcal{F}(M)$.

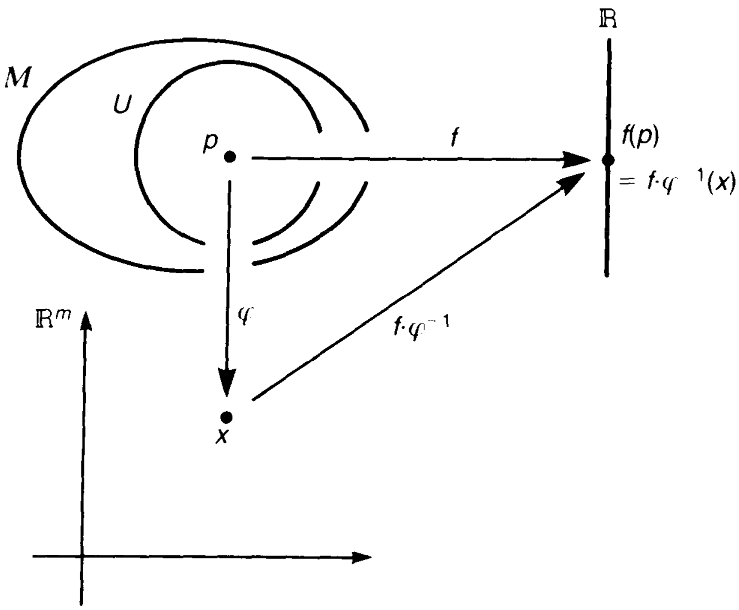


Figure 5.8 A function $f : M \to \mathbb{R}$ and its coordinate presentation $f\varphi^{-1}$.

5.2.2 Vectors

Now that we have defined maps on a manifold, we are ready to define other geometrical objects: vectors, dual vectors and tensors. In general, an elementary picture of a vector as an arrow connecting a point and the origin does not work in a manifold. [Where is the origin? What is a *straight* arrow? How do we define a straight arrow that connects London and Los Angeles on the *surface* of the Earth?] On a manifold, a vector is defined to be a **tangent vector** to a curve in M.

To begin with, let us look at a tangent line to a curve in the xy plane. If the curve is differentiable, we may approximate the curve in the vicinity of x_0 by

$$y - y(x_0) = a(x - x_0) \tag{5.17}$$

where $a = \mathrm{d}y/\mathrm{d}x|_{x=x_0}$. The tangent vectors on a manifold M generalise the tangent line above. To define a tangent vector we need a curve $c : (a, b) \to M$ and a function $f : M \to \mathbb{R}$, where (a, b) is an open interval containing $t = 0$, see figure 5.9. We define the tangent vector at $c(0)$ as a *directional derivative* of a function $f(c(t))$ along the curve $c(t)$ at $t = 0$. The rate of change of $f(c(t))$ at $t = 0$ along the curve is

$$\left.\frac{\mathrm{d}f(c(t))}{\mathrm{d}t}\right|_{t=0}. \tag{5.18}$$

In terms of the local coordinate, this becomes

$$(\partial f/\partial x^\mu)(\mathrm{d}x^\mu(c(t))/\mathrm{d}t)|_{t=0}. \tag{5.19}$$

[Note the abuse of the notation! $\partial f/\partial x^\mu$ means $\partial(f\varphi^{-1}(x))/\partial x^\mu$.] In other words, $\mathrm{d}f(c(t))/\mathrm{d}t$ at $t = 0$ is obtained by applying the differential operator X to f, where

$$X = X^\mu(\partial/\partial x^\mu) \qquad (X^\mu = \mathrm{d}x^\mu(c(t))/\mathrm{d}t|_{t=0}) \tag{5.20}$$

that is,

$$\mathrm{d}f(c(t))/\mathrm{d}t|_{t=0} = X^\mu(\partial f/\partial x^\mu) \equiv X[f]. \tag{5.21}$$

Here the last equality defines $X[f]$. It is $X = X^\mu\partial/\partial x^\mu$ which we now define as the **tangent vector** to M at $p = c(0)$ along the direction given by the curve $c(t)$.

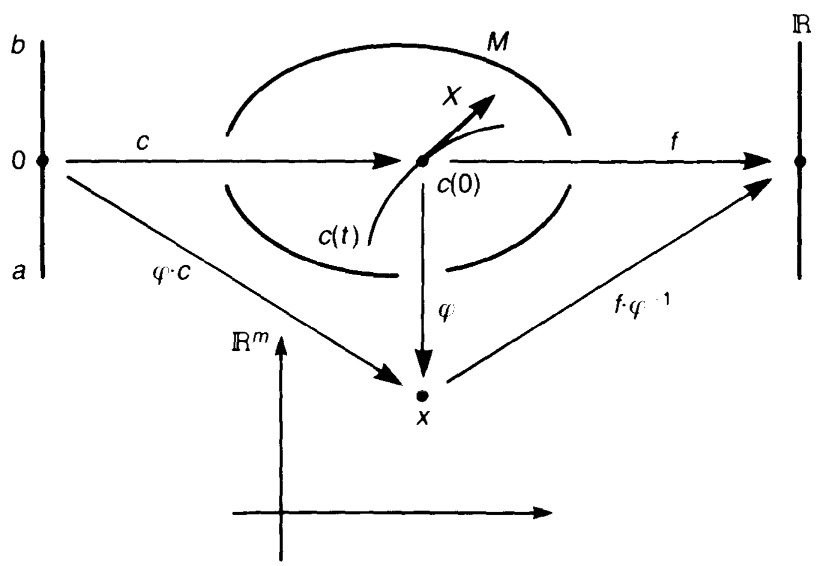


Figure 5.9 A curve c and a function f define a tangent vector along the curve in terms of the directional derivative.

Example 5.11 If X is applied to the coordinate functions $\varphi(c(t)) = x^\mu(t)$, we have

$$X[x^\mu] = (\mathrm{d}x^\nu/\mathrm{d}t)(\partial x^\mu/\partial x^\nu) = \mathrm{d}x^\mu(t)/\mathrm{d}t|_{t=0}$$

which is the μth component of the velocity vector if t is understood as time.

To be more precise, we introduce an equivalence class of curves in M. If two curves $c_1(t)$ and $c_2(t)$ satisfy

(i) $c_1(0) = c_2(0) = p$
(ii) $\mathrm{d}x^\mu(c_1(t))/\mathrm{d}t|_{t=0} = \mathrm{d}x^\mu(c_2(t))/\mathrm{d}t|_{t=0}$

$c_1(t)$ and $c_2(t)$ yield the same differential operator X at p, in which case we define $c_1(t) \sim c_2(t)$. Clearly $\sim$ is an equivalence relation and defines the equivalence classes. We identify the *tangent vector* X with the *equivalence class of curves*.

$$[c(t)] = \left\{ \tilde{c}(t) | \tilde{c}(0) = c(0) \quad \text{and} \quad \left.\frac{\mathrm{d}x^\mu(\tilde{c}(t))}{\mathrm{d}t}\right|_{t=0} = \text{and} \left.\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0} \right\} \tag{5.22}$$

rather than a curve itself.

All the equivalence classes of curves at $p \in M$, namely all the tangent vectors at p, form a vector space called the **tangent space** of M at p, denoted by T_pM. To analyse T_pM, we may use the theory of vector spaces developed in §2.2. Evidently, $e_\mu = \partial/\partial x^\mu$ $(1 \leqslant \mu \leqslant m)$ are the basis vectors of T_pM, see (5.20), and $\dim T_pM = \dim M$. The basis $\{e_\mu\}$ is called the **coordinate basis**. If a vector $V \in T_pM$ is written as $V = V^\mu e_\mu$, the numbers V^μ are called the components of V with respect to e_μ. By construction, it is obvious that a vector X exists without specifying the coordinate, see (5.21). The assignment of the coordinate is simply for our convenience. This coordinate independence of a vector enables us to find the transformation property of the *components* of the vector. Let $p \in U_i \cap U_j$, and $x = \varphi_i(p)$ and $y = \varphi_j(p)$. We have two expressions for $X \in T_pM$,

$$X = X^\mu \partial/\partial x^\mu = \tilde{X}^\mu \partial/\partial y^\mu.$$

This shows that X^μ and $\tilde{X}^\mu$ are related as

$$\tilde{X}^\mu = X^\nu \partial y^\mu/\partial x^\nu. \tag{5.23}$$

Note again that the components of the vector transform in such a way that the vector itself is left invariant.

The basis of T_pM need not be $\{e_\mu\}$, and we may think of the linear combinations $\hat{e}_i \equiv A_i{}^\mu e_\mu$, where $A = (A_i{}^\mu) \in \mathrm{GL}(m, \mathbb{R})$. The basis $\{\hat{e}_i\}$ is known as the **non-coordinate basis**.

5.2.3 One-forms

Since T_pM is a vector space, there exists a dual vector space to T_pM, whose element is a linear function from T_pM to $\mathbb{R}$, see §2.2. The dual space is called the **cotangent space** at p, denoted by T_p^*M. An element $\omega : T_pM \to \mathbb{R}$ of T_p^*M is called a **dual vector**, **cotangent vector** or, in the context of differential forms, a **one-form**. The simplest example of a one-form is the differential $\mathrm{d}f$ of a function $f \in \mathcal{F}(M)$. The action of a vector V on f is $V[f] = V^\mu(\partial f/\partial x^\mu) \in \mathbb{R}$. Then the action of $\mathrm{d}f \in T_p^*M$ on $V \in T_pM$ is defined by

$$\langle \mathrm{d}f, V\rangle \equiv V[f] = V^\mu(\partial f/\partial x^\mu) \in \mathbb{R}. \tag{5.24}$$

Clearly $\langle \mathrm{d}f, V\rangle$ is $\mathbb{R}$-linear in both V and f.

Noting that $\mathrm{d}f$ is expressed in terms of the coordinate $x = \varphi(p)$ as $\mathrm{d}f = (\partial f/\partial x^\mu)\,\mathrm{d}x^\mu$, it is natural to regard $\{\mathrm{d}x^\mu\}$ as a basis of T_p^*M. Moreover, this is a dual basis, since

$$\langle \mathrm{d}x^\nu, \partial/\partial x^\mu\rangle = \partial x^\nu/\partial x^\mu = \delta^\nu_\mu. \tag{5.25}$$

An arbitrary one-form ω is written as

$$\omega = \omega_\mu\,\mathrm{d}x^\mu \tag{5.26}$$

where the ω_μ are the components of ω. Take a vector $V = V^\mu\partial/\partial x^\mu$ and a one-form $\omega = \omega_\mu\,\mathrm{d}x^\mu$. The **inner product** $\langle\ ,\ \rangle : T_p^*M \otimes T_pM \to \mathbb{R}$ is defined by

$$\langle \omega, V\rangle = \omega_\mu V^\nu\langle \mathrm{d}x^\mu, \partial/\partial x^\nu\rangle = \omega_\mu V^\nu\delta^\mu_\nu = \omega_\mu V^\mu. \tag{5.27}$$

Note that the inner product is defined between a vector and a dual vector and not between two vectors or two dual vectors.

Since ω is defined without reference to any coordinate system, for a point $p \in U_i \cap U_j$, we have

$$\omega = \omega_\mu\,\mathrm{d}x^\mu = \widetilde{\omega}_\nu\,\mathrm{d}y^\nu$$

where $x = \varphi_i(p)$ and $y = \varphi_j(p)$. From $\mathrm{d}y^\nu = (\partial y^\nu/\partial x^\mu)\,\mathrm{d}x^\mu$, we have

$$\widetilde{\omega}_\nu = \omega_\mu\partial x^\mu/\partial y^\nu. \tag{5.28}$$

5.2.4 Tensors

A **tensor** of type (q, r) is a multilinear object which maps q elements of T_p^*M and r elements of T_pM to a real number. $\mathcal{T}^q_{r,p}(M)$ denotes the set of type (q, r) tensors at $p \in M$. An element of $\mathcal{T}^q_{r,p}(M)$ is written in terms of the bases above as

$$T = T^{\mu_1\ldots\mu_q}{}_{\nu_1\ldots\nu_r}\frac{\partial}{\partial x^{\mu_1}}\cdots\frac{\partial}{\partial x^{\mu_q}}\,\mathrm{d}x^{\nu_1}\ldots\mathrm{d}x^{\nu_r}. \tag{5.29}$$

Clearly this is a linear function from

$$\bigotimes^q T_p^*M \bigotimes^r T_pM$$

to $\mathbb{R}$. Let $V_i = V_i{}^{\mu}\partial/\partial x^{\mu}$ ($1 \leqslant i \leqslant r$) and $\omega_i = \omega_{i\mu}\mathrm{d}x^{\mu}$ ($1 \leqslant i \leqslant q$). The action of T on them yields a number

$$T(\omega_1, \ldots, \omega_q; V_1, \ldots, V_r) = T^{\mu_1\ldots\mu_q}{}_{\nu_1\ldots\nu_r}\omega_{1\mu_1}\ldots\omega_{q\mu_q}V_1{}^{\nu_1}\ldots V_r{}^{\nu_r}.$$

In the present notation, the inner product is $\langle\omega, X\rangle = \omega(X)$.

5.2.5 *Tensor fields*

If a vector is assigned *smoothly* to each point of M, it is called a **vector field** over M. In other words, V is a vector field if $V[f] \in \mathcal{F}(M)$ for any $f \in \mathcal{F}(M)$. Clearly each component of a vector field is a smooth function from M to $\mathbb{R}$. $\mathcal{X}(M)$ denotes the set of the vector fields on M. A vector field X at $p \in M$ is denoted by $X|_p$, which is an element of T_pM. Similarly we define a **tensor field** of type (q, r) by a smooth assignment of an element of $\mathcal{T}^q_{r,p}(M)$ at each point $p \in M$. The set of the tensor fields of type (q, r) on M is denoted by $\mathcal{T}^q_r(M)$. For example, $\mathcal{T}^0_1(M)$ is the set of the dual vector fields, which is also denoted by $\Omega^1(M)$ in the context of differential forms, see §5.4. Similarly $\mathcal{T}^0_0(M) = \mathcal{F}(M)$ is denoted by $\Omega^0(M)$ in the same context.

5.2.6 *Induced maps*

A smooth map $f : M \to N$ naturally induces a map f_* called the **differential map** (figure 5.10),

$$f_* : T_pM \to T_{f(p)}M. \tag{5.30}$$

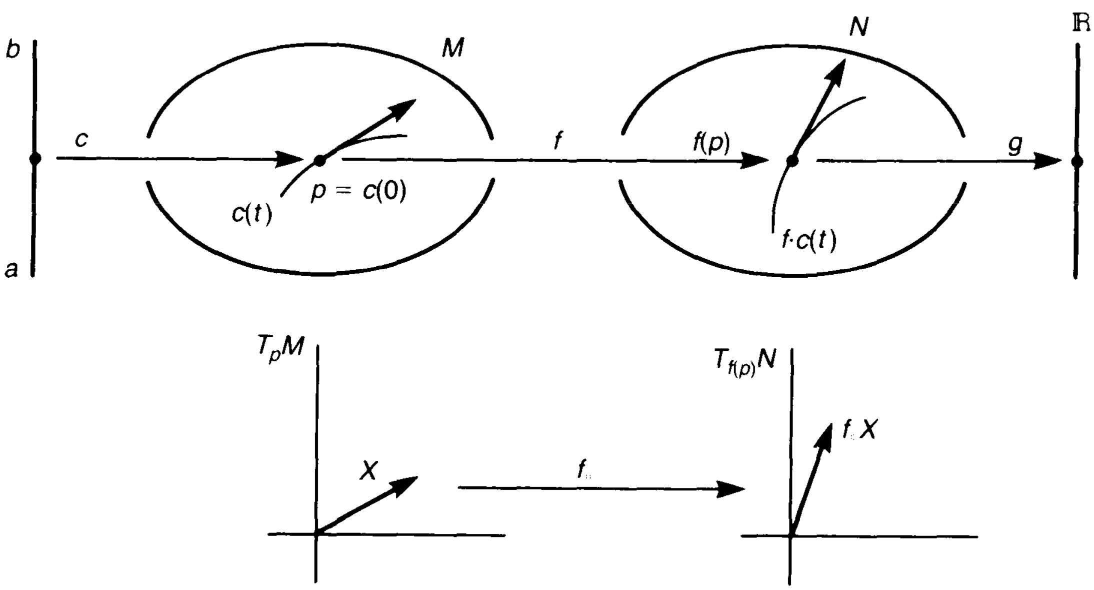


Figure 5.10 A map $f : M \to N$ induces the differential map $f_* : T_pM \to T_{f(p)}N$.

The explicit form of f_* is obtained by the definition of a tangent vector as a directional derivative along a curve. If $g \in \mathcal{F}(N)$, then $gf \in \mathcal{F}(M)$. A vector $V \in T_pM$ acts on gf to give a number $V[gf]$. Now we define $f_*V \in T_{f(p)}N$ by

$$(f_*V)[g] \equiv V[gf] \tag{5.31}$$

or, in terms of charts (U, φ) on M and (V, ψ) on N,

$$(f_*V)[g\psi^{-1}(y)] \equiv V[gf\varphi^{-1}(x)] \tag{5.32}$$

where $x = \varphi(p)$ and $y = \psi(f(p))$. Let $V = V^\mu \partial/\partial x^\mu$ and $f_*V = W^\alpha \partial/\partial y^\alpha$. Then (5.32) yields

$$W^\alpha \frac{\partial}{\partial y^\alpha}[g\psi^{-1}(y)] = V^\mu \frac{\partial}{\partial x^\mu}[gf\varphi^{-1}(x)].$$

If we take $g = y^\alpha$, we obtain the relation between W^α and V^μ,

$$W^\alpha = V^\mu \frac{\partial}{\partial x^\mu} y^\alpha(x). \tag{5.33}$$

Note that the matrix $(\partial y^\alpha/\partial x^\mu)$ is nothing but the Jacobian of the map $f: M \to N$. The differential map f_* is naturally extended to tensors of type $(q, 0)$, $f_* : \mathcal{T}^q_{0,p}(M) \to \mathcal{T}^q_{0,f(p)}(N)$.

Example 5.12 Let (x^1, x^2) and (y^1, y^2, y^3) be the coordinates in M and N, respectively, and let $V = a\partial/\partial x^1 + b\partial/\partial x^2$ be a tangent vector at (x^1, x^2). Let $f: M \to N$ be a map whose coordinate presentation is $y = (x^1, x^2, [1 - (x^1)^2 - (x^2)^2]^{1/2})$. Then

$$f_*V = V^\mu \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial}{\partial y^\mu} = a\frac{\partial}{\partial y^1} + b\frac{\partial}{\partial y^2} - \left(a\frac{y^1}{y^3} + b\frac{y^2}{y^3}\right)\frac{\partial}{\partial y^3}.$$

Exercise 5.13 Let $f: M \to N$ and $g: N \to P$. Show that the differential map of the composite map $gf: M \to P$ is

$$(gf)_* = g_*f_*. \tag{5.34}$$

A map $f: M \to N$ also induces a map

$$f^* : T^*_{f(p)}N \to T^*_pM. \tag{5.35}$$

Note that f_* goes in the same direction as f, while f^* goes backward, hence the name **pullback**, see §2.2. If we take $V \in T_pM$ and $\omega \in T^*_{f(p)}N$, the pullback of ω by f^* is defined by

$$\langle f^*\omega, V\rangle = \langle \omega, f_*V\rangle. \tag{5.36}$$

The pullback f^* naturally extends to tensors of type $(0, r)$, $f^* : \mathcal{T}^0_{r,f(p)}(N) \to \mathcal{T}^0_{r,p}(M)$. The component expression of f^* is given by the Jacobian matrix $(\partial y^\alpha/\partial x^\mu)$, see exercise 5.14.

Exercise 5.14 Let $f: M \to N$ be a smooth map. Show that for $\omega = \omega_\alpha \, \mathrm{d}y^\alpha \in T^*_{f(p)}N$, the induced one-form $f^*\omega = \xi_\mu \, \mathrm{d}x^\mu \in T^*_pM$ is

$$\xi_\mu = \omega_\alpha \partial y^\alpha / \partial x^\mu. \tag{5.37}$$

Exercise 5.15 Let f and g be as in exercise 5.13. Show that the pullback of the composite map gf is

$$(gf)^* = f^* g^*. \tag{5.38}$$

There is no natural extension of the induced map for a tensor of mixed type. The extension is only possible if $f: M \to N$ is a diffeomorphism, where the Jacobian of f^{-1} is also defined.

Exercise 5.16 Let

$$T^\mu{}_\nu \frac{\partial}{\partial x^\mu} \otimes \mathrm{d}x^\nu$$

be a tensor field of type (1, 1) on M and let $f: M \to N$ be a diffeomorphism. Show that the induced tensor on N is

$$f_*\left(T^\mu{}_\nu \frac{\partial}{\partial x^\mu} \otimes \mathrm{d}x^\nu\right) = T^\mu{}_\nu \left(\frac{\partial y^\alpha}{\partial x^\mu}\right) \left(\frac{\partial x^\nu}{\partial y^\beta}\right) \frac{\partial}{\partial y^\alpha} \otimes \mathrm{d}y^\beta.$$

5.2.7 Submanifolds

Before we close this section, we define a submanifold of a manifold. The meaning of embedding is also clarified here.

Definition 5.17 **(immersion, submanifold, embedding)** Let $f: M \to N$ be a smooth map and let $\dim M \leqslant \dim N$.

(a) The map f is called an **immersion** of M into N if $f_*: T_pM \to T_{f(p)}N$ is an injection (one-to-one), that is $\operatorname{rank} f_* = \dim M$.

(b) The map f is called an **embedding** if f is an injection and an immersion. The image $f(M)$ is called a **submanifold** of N. [In practice, $f(M)$ thus defined is diffeomorphic to M.]

If f is an immersion, f_* maps T_pM isomorphically to an m-dimensional vector subspace of $T_{f(p)}N$ since $\operatorname{rank} f_* = \dim M$. From theorem 2.16, we also find $\ker f_* = \{0\}$. If f is an embedding, M is diffeomorphic to $f(M)$. Examples will clarify these rather technical points. Consider a map $f: S^1 \to \mathbb{R}^2$ in figure 5.11(a). It is an immersion since a one-dimensional tangent space of S^1 is mapped by f_* to a subspace of $T_{f(p)}\mathbb{R}^2$. The image $f(S^1)$ is not a submanifold of $\mathbb{R}^2$ since f is not an injection. The map $g: S^1 \to \mathbb{R}^2$ in figure 5.11(b) is an embedding and $g(S^1)$ is a submanifold of $\mathbb{R}^2$. Clearly, an embedding is an immersion although the converse is not necessarily true. In the previous section, we occasionally mentioned the embedding of S^n into $\mathbb{R}^{n+1}$. Now this meaning is clear; if S^n is embedded by $f : S^n \to \mathbb{R}^{n+1}$, then S^n is diffeomorphic to $f(S^n)$.

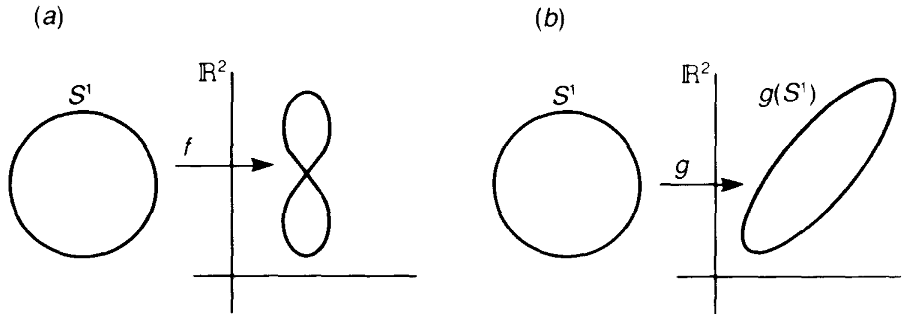


Figure 5.11 (*a*) An immersion f which is not an embedding. (*b*) An embedding g and the submanifold $g(S^1)$.

5.3 Flows and Lie derivatives

Let X be a vector field in M. An **integral curve** $x(t)$ of X is a curve in M, whose tangent vector at $x(t)$ is $X|_x$. Given a chart (U, φ) this means

$$\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t} = X^\mu(x(t)) \tag{5.39}$$

where $x^\mu(t)$ is the μth component of $\varphi(x(t))$ and $X = X^\mu \partial/\partial x^\mu$. Note the abuse of the notation: x is used to denote a point in M as well as its coordinates. [For later convenience we assume the point $x(0)$ is included in U.] Put in another way, finding the integral curve of a vector field X is equivalent to solving the autonomous system of ordinary differential equations (ODE) (5.39). The initial condition $x_0^\mu = x^\mu(0)$ corresponds to the coordinates of an integral curve at $t = 0$. The existence and uniqueness theorem of ODE guarantees that there is a unique solution to (5.39), at least locally, with the initial data x_0^μ. It may happen that the integral curve is defined only on a subset of $\mathbb{R}$, in which case we have to pay attention so that the parameter t does not exceed the given interval. In the following we assume that t is maximally extended. It is known that if M is a compact manifold, the integral curve exists for all $t \in \mathbb{R}$.

Let $\sigma(t, x_0)$ be an integral curve of X which passes a point x_0 at $t = 0$ and denote the coordinate by $\sigma^\mu(t, x_0)$. (5.39) then becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^\mu(t, x_0) = X^\mu(\sigma(t, x_0)) \tag{5.40a}$$

with the initial condition

$$\sigma^{\mu}(0, x_0) = x_0^{\mu}. \tag{5.40b}$$

The map $\sigma : \mathbb{R} \times M \to M$ is called a **flow** generated by $X \in \mathscr{X}(M)$. A flow satisfies the rule

$$\sigma(t, \sigma(s, x_0)) = \sigma(t + s, x_0) \tag{5.41}$$

for any s, $t \in \mathbb{R}$ such that both sides of (5.41) make sense. This can be seen from the uniqueness of ODE. In fact, we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^{\mu}(t, \sigma(s, x_0)) = X^{\mu}(\sigma(t, \sigma(s, x_0)))$$

$$\sigma(0, \sigma(s, x_0)) = \sigma(s, x_0)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma^{\mu}(t + s, x_0) = \frac{\mathrm{d}}{\mathrm{d}(t + s)}\sigma^{\mu}(t + s, x_0) = X^{\mu}(\sigma(t + s, x_0))$$

$$\sigma(0 + s, x_0) = \sigma(s, x_0).$$

Thus both sides of (5.41) satisfy the same ODE and the same initial condition. From the uniqueness of the solution, they should be the same. We have obtained the following theorem.

Theorem 5.18 For any point $x \in M$, there exists a differentiable map $\sigma : \mathbb{R} \times M \to M$ such that

(i) $\sigma(0, x) = x$
(ii) $t \mapsto \sigma(t, x)$ is a solution of (5.40)
(iii) $\sigma(t, \sigma(s, x)) = \sigma(t + s, x)$.

[*Note:* We denote the initial point by x instead of x_0 to emphasise that σ is a map $\mathbb{R} \times M \to M$.]

We may imagine a flow as a (steady) stream flow. If a particle is observed at a point x at $t = 0$, it will be found at $\sigma(t, x)$ at later time t.

Example 5.19 Let $M = \mathbb{R}^2$, and let $X((x, y)) = -y\partial/\partial x + x\partial/\partial y$ be a vector field in M. It is easy to verify that

$$\sigma(t, (x, y)) = (x\cos t - y\sin t, x\sin t + y\cos t)$$

is a flow generated by X. The flow through (x, y) is a circle whose centre is at the origin. Clearly, $\sigma(t, (x, y)) = (x, y)$, if $t = 2n\pi$, $n \in \mathbb{Z}$. If $(x, y) = (0, 0)$, the flow stays at $(0, 0)$.

Exercise 5.20 Let $M = \mathbb{R}^2$, and let $X = y\partial/\partial x + x\partial/\partial y$ be a vector field in M. Find the flow generated by X.

5.3.1 One-parameter group of transformations

For fixed $t \in \mathbb{R}$, a flow $\sigma(t, x)$ is a diffeomorphism from M to M, denoted by $\sigma_t : M \to M$. It is important to note that σ_t is made into a *commutative group* by the following rules.

(i) $\sigma_t(\sigma_s(x)) = \sigma_{t+s}(x)$, that is $\sigma_t \cdot \sigma_s = \sigma_{t+s}$
(ii) σ_0 = the identity map (= unit element)
(iii) $\sigma_{-t} = (\sigma_t)^{-1}$.

This group is called the **one-parameter group of transformations**. The group *locally* looks like the additive group $\mathbb{R}$, although it may not be isomorphic to $\mathbb{R}$ globally. In fact, in example 5.19 above, $\sigma_{2\pi n+t}$ is the same map as σ_t, and we find that the one-parameter group is isomorphic to SO(2), the multiplicative group of 2×2 real matrices of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

or U(1), the multiplicative group of complex numbers of unit modulus $e^{i\theta}$.

Under the action of σ_ε with an infinitesimal ε, we find from (5.40) that a point x whose coordinate is x^μ is mapped to

$$\sigma_\varepsilon^\mu(x) = \sigma^\mu(\varepsilon, x) = x^\mu + \varepsilon X^\mu(x). \tag{5.42}$$

The vector field X is called, in this context, the **infinitesimal generator** of the transformation σ_t.

Given a vector field X, the corresponding flow σ is often referred to as the **exponentiation** of X and is denoted by

$$\sigma^\mu(t, x) = \exp(tX)x^\mu. \tag{5.43}$$

The name 'exponentiation' is justified as we shall see now. Let us take a parameter t and evaluate the coordinate of a point which is separated from the initial point $x = \sigma(0, x)$ by the parameter distance t along the flow σ. The coordinate corresponding to the point $\sigma(t, x)$ is

$$\begin{aligned}
\sigma^\mu(t, x) &= x^\mu + t \frac{\mathrm{d}}{\mathrm{d}s} \sigma^\mu(s, x)\bigg|_{s=0} \\
&\quad + \frac{t^2}{2!}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^2 \sigma^\mu(s, x)\bigg|_{s=0} + \ldots \\
&= \left[1 + t\frac{\mathrm{d}}{\mathrm{d}s} + \frac{t^2}{2!}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^2 + \ldots\right] \sigma^\mu(s, x)\bigg|_{s=0} \\
&\equiv \exp\left(t\frac{\mathrm{d}}{\mathrm{d}s}\right) \sigma^\mu(s, x)\bigg|_{s=0}.
\end{aligned} \tag{5.44}$$

The last expression can also be written as $\sigma^\mu(t, x) = \exp(tX)x^\mu$, as in (5.43). The flow σ satisfies the following exponential properties.

(i) $\sigma(0, x) = x = \exp(0X)x$ (5.45a)

$$\text{(ii)}\quad \frac{\mathrm{d}\sigma(t, x)}{\mathrm{d}t} = X \exp(tX)x = \frac{\mathrm{d}}{\mathrm{d}t}[\exp(tX)x] \tag{5.45b}$$

$$\begin{aligned}\text{(iii)}\quad \sigma(t, \sigma(s, x)) &= \sigma(t, \exp(sX)x) = \exp(tX)\exp(sX)x \\ &= \exp\{(t+s)X\}x = \sigma(t+s, x).\end{aligned} \tag{5.45c}$$

5.3.2 Lie derivatives

Let $\sigma(t, x)$ and $\tau(t, x)$ be two flows generated by the vector fields X and Y,

$$\frac{\mathrm{d}\sigma^{\mu}(s, x)}{\mathrm{d}s} = X^{\mu}(\sigma(s, x)) \tag{5.46a}$$

$$\frac{\mathrm{d}\tau^{\mu}(t, x)}{\mathrm{d}t} = Y^{\mu}(\tau(t, x)). \tag{5.46b}$$

Let us evaluate the change of the vector field Y along $\sigma(s, x)$. To do this, we have to compare the vector Y at a point x with that at a nearby point $x' = \sigma_{\varepsilon}(x)$, see figure 5.12. However, we cannot simply take the difference between the components of Y at two points since they belong to different tangent spaces T_xM and $T_{\sigma_\varepsilon(x)}M$; the naive difference between vectors at different points is ill-defined. To define a sensible derivative, we first map $Y|_{\sigma_\varepsilon(x)}$ to T_xM by $(\sigma_{-\varepsilon})_* : T_{\sigma_\varepsilon(x)}M \to T_xM$, after which we take a difference between two vectors $(\sigma_{-\varepsilon})_* Y|_{\sigma_\varepsilon(x)}$ and $Y|_x$, both of which are vectors in T_xM. The **Lie derivative** of a vector field Y along the flow σ of X is defined by

$$\mathcal{L}_X Y \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[(\sigma_{-\varepsilon})_* Y|_{\sigma_\varepsilon(x)} - Y|_x]. \tag{5.47}$$

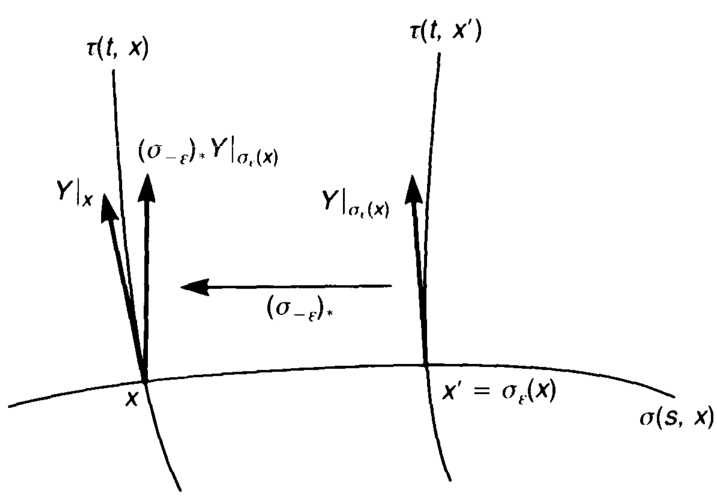


Figure 5.12 To compare a vector $Y|_x$ with $Y|_{\sigma_\varepsilon(x)}$, the latter must be transported back to x by the differential map $(\sigma_{-\varepsilon})_*$.

Exercise 5.21 Show that $\mathcal{L}_X Y$ is also written as

$$\mathcal{L}_X Y = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [Y|_x - (\sigma_\varepsilon)_* Y|_{\sigma_{-\varepsilon}(x)}]$$
$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [Y|_{\sigma_\varepsilon(x)} - (\sigma_\varepsilon)_* Y|_x].$$

Let (U, φ) be a chart with the coordinates x and let $X = X^\mu \partial/\partial x^\mu$ and $Y = Y^\mu \partial/\partial x^\mu$ be vector fields defined on U. Then $\sigma_\varepsilon(x)$ has the coordinates $x^\mu + \varepsilon X^\mu(x)$ and

$$Y|_{\sigma_\varepsilon(x)} = Y^\mu(x^\nu + \varepsilon X^\nu(x))\, e_\mu|_{x+\varepsilon X}$$
$$\simeq [Y^\mu(x) + \varepsilon X^\nu(x)\partial_\nu Y^\mu(x)]\, e_\mu|_{x+\varepsilon X}$$

where $\{e_\mu\} = \{\partial/\partial x^\mu\}$ is the coordinate basis and $\partial_\nu \equiv \partial/\partial x^\nu$. If we map this vector defined at $\sigma_\varepsilon(x)$ to x by $(\sigma_{-\varepsilon})_*$, we have

$$[Y^\mu(x) + \varepsilon X^\lambda(x)\partial_\lambda Y^\mu(x)]\partial_\mu[x^\nu - \varepsilon X^\nu(x)]e_\nu|_x$$
$$= [Y^\mu(x) + \varepsilon X^\lambda(x)\partial_\lambda Y^\mu(x)][\delta_\mu^\nu - \varepsilon\partial_\mu X^\nu(x)]e_\nu|_x$$
$$= Y^\mu(x)e_\mu|_x + \varepsilon[X^\mu(x)\partial_\mu Y^\nu(x) - Y^\mu(x)\partial_\mu X^\nu(x)]e_\nu|_x + \mathrm{O}(\varepsilon^2). \quad (5.48)$$

From (5.47) and (5.48), we find

$$\mathcal{L}_X Y = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)e_\nu. \quad (5.49a)$$

Exercise 5.22 Let $X = X^\mu \partial/\partial x^\mu$ and $Y = Y^\mu \partial/\partial x^\mu$ be vector fields in M. Define the **Lie bracket** $[X, Y]$ by

$$[X, Y]f = X[Y[f]] - Y[X[f]] \quad (5.50)$$

where $f \in \mathcal{F}(M)$. Show that $[X, Y]$ is a vector field given by

$$(X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)e_\nu.$$

This exercise shows that the Lie derivative of Y along X is

$$\mathcal{L}_X Y = [X, Y]. \quad (5.49b)$$

[*Note:* Neither XY nor YX is a vector field since they are second-order derivatives. The combination $[X, Y]$ is, however, a first-order derivative and indeed a vector field.]

Exercise 5.23 Show that the Lie bracket satisfies
(a) bilinearity

$$[X, c_1 Y_1 + c_2 Y_2] = c_1[X, Y_1] + c_2[X, Y_2]$$
$$[c_1 X_1 + c_2 X_2, Y] = c_1[X_1, Y] + c_2[X_2, Y]$$

for any constants c_1 and c_2

(*b*) skew-symmetry

$$[X, Y] = -[Y, X]$$

(*c*) the Jacobi identity

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0.$$

Exercise 5.24

(*a*) Let $X, Y \in \mathcal{X}(M)$ and $f \in \mathcal{F}(M)$. Show that

$$\mathcal{L}_{fX}Y = f[X, Y] - Y[f]M \tag{5.51a}$$

$$\mathcal{L}_X(fY) = f[X, Y] + X[f]Y. \tag{5.51b}$$

(*b*) Let $X, Y \in \mathcal{X}(M)$ and $f : M \to N$. Show that

$$f_*[X, Y] = [f_*X, f_*Y]. \tag{5.52}$$

Geometrically the Lie bracket shows non-commutability of two flows. This is easily observed from the following consideration. Let $\sigma(s, x)$ and $\tau(t, x)$ be two flows generated by vector fields X and Y, as above, see figure 5.13. If we move by a small parameter distance ε along the flow σ first, then by δ along τ, we shall be at the point whose coordinates are

$$\begin{aligned}
\tau^\mu(\delta, \sigma(\varepsilon, x)) &\simeq \tau^\mu(\delta, x^\nu + \varepsilon X^\nu(x)) \\
&\simeq x^\mu + \varepsilon X^\mu(x) + \delta Y^\mu(x^\nu + \varepsilon X^\nu(x)) \\
&\simeq x^\mu + \varepsilon X^\mu(x) + \delta Y^\mu(x) + \varepsilon\delta X^\nu(x)\partial_\nu Y^\mu(x).
\end{aligned}$$

If, on the other hand, we move by δ along τ first, then by ε along σ, we will be at the point,

$$\begin{aligned}
\sigma^\mu(\varepsilon, \tau(\delta, x)) &\simeq \sigma^\mu(\varepsilon, x^\nu + \delta Y^\nu(x)) \\
&\simeq x^\mu + \delta Y^\mu(\mathrm{x}) + \varepsilon X^\mu(x^\nu + \delta Y^\nu(x)) \\
&\simeq x^\mu + \delta Y^\mu(x) + \varepsilon X^\mu(x) + \varepsilon\delta Y^\nu(x)\partial_\nu X^\mu(x).
\end{aligned}$$

The difference between the coordinates of these two points is proportional to the Lie bracket,

$$\tau^\mu(\delta, \sigma(\varepsilon, x)) - \sigma^\mu(\varepsilon, \tau(\delta, x)) = \varepsilon\delta[X, Y]^\mu.$$

The Lie bracket of X and Y measures the failure of the closure of the parallelogram in figure 5.13. It is easy to see $\mathcal{L}_X Y = [X, Y] = 0$ if and only if

$$\sigma(s, \tau(t, x)) = \tau(t, \sigma(s, x)). \tag{5.53}$$

We may also define the Lie derivative of a one-form $\omega \in \Omega^1(M)$ along $X \in \mathcal{X}(M)$ by

$$\mathcal{L}_X\omega \equiv \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}((\sigma_\varepsilon)^*\omega|_{\sigma_\varepsilon(x)} - \omega|_x). \tag{5.54}$$

where $\omega|_x \in T_x^*M$ is ω at x. Put $\omega = \omega_\mu \, \mathrm{d}x^\mu$. Repeating a similar analysis to the above, we obtain

$$(\sigma_\varepsilon)^*\omega|_{\sigma_\varepsilon(x)} = \omega_\mu(x)\,\mathrm{d}x^\mu + \varepsilon[X^\nu(x)\partial_\nu\omega_\mu(x) + \partial_\mu X^\nu(x)\omega_\nu(x)]\,\mathrm{d}x^\mu$$

which leads to

$$\mathcal{L}_X\omega = (X^\nu\partial_\nu\omega_\mu + \omega_\nu\partial_\mu X^\nu)\,\mathrm{d}x^\mu. \tag{5.55}$$

Clearly $\mathcal{L}_X\omega \in \Omega^1(M)$, since it is a difference of two one-forms at the same point x.

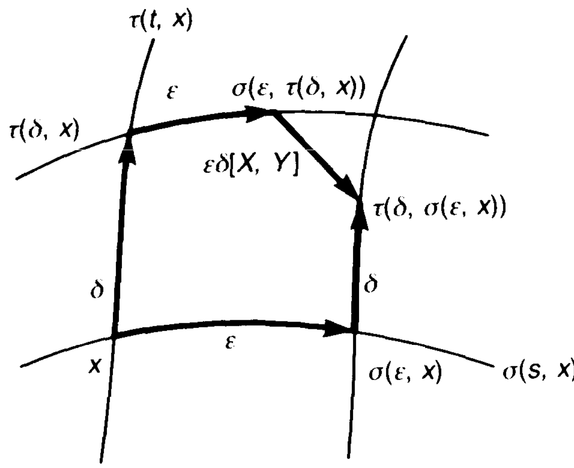


Figure 5.13 A Lie bracket $[X, Y]$ measures the failure of the closure of the parallelogram.

The Lie derivative of $f \in \mathcal{F}(M)$ along a flow σ_s generated by a vector field X is

$$\begin{aligned}\mathcal{L}_X f &\equiv \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[f(\sigma_\varepsilon(x)) - f(x)] \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[f(x^\mu + \varepsilon X^\mu(x)) - f(x^\mu)] \\ &= X^\mu(x)\frac{\partial f}{\partial x^\mu} = X[f]\end{aligned} \tag{5.56}$$

which is the usual directional derivative along X.

The Lie derivative of a general tensor is obtained from the following proposition.

Proposition 5.25 The Lie derivative satisfies

(*a*) $\mathcal{L}_X(t_1 + t_2) = \mathcal{L}_X t_1 + \mathcal{L}_X t_2$ (5.57*a*) where t_1 and t_2 are tensor fields of the same type.

$$(b)\ \mathcal{L}_X(t_1 \otimes t_2) = (\mathcal{L}_X t_1) \otimes t_2 + t_1 \otimes (\mathcal{L}_X t_2) \tag{5.57b}$$

where t_1 and t_2 are tensor fields of arbitrary types.

Proof: (a) is obvious. Rather than giving the general proof of (b), which is full of indices, we give an example whose extension to more general cases is trivial. Take $Y \in \mathcal{X}(M)$ and $\omega \in \Omega^1(M)$ and construct the tensor product $Y \otimes \omega$. Then $(Y \otimes \omega)|_{\sigma_\varepsilon(x)}$ is mapped to a tensor at x by the action of $(\sigma_{-\varepsilon})_* \otimes (\sigma_\varepsilon)^*$,

$$[(\sigma_{-\varepsilon})_* \otimes (\sigma_\varepsilon)^*](Y \otimes \omega)|_{\sigma_\varepsilon(x)} = [(\sigma_{-\varepsilon})_* Y \otimes (\sigma_\varepsilon)^*\omega]|_x.$$

Then there follows (the Leibnitz rule)

$$\begin{aligned}
\mathcal{L}_X(Y \otimes \omega) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\{(\sigma_{-\varepsilon})_* Y \otimes (\sigma_\varepsilon)^*\omega\}|_x - (Y \otimes \omega)|_x] \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[(\sigma_{-\varepsilon})_* Y \otimes \{(\sigma_\varepsilon)^*\omega - \omega\} + \{(\sigma_{-\varepsilon})_* Y - Y\} \otimes \omega] \\
&= Y \otimes (\mathcal{L}_X \omega) + (\mathcal{L}_X Y) \otimes \omega.
\end{aligned}$$

Extensions to more general cases are obvious. ■

This proposition enables us to calculate the Lie derivative of a general tensor field. For example, let $t = t_\mu{}^\nu \mathrm{d}x^\mu \otimes e_\nu \in \mathcal{T}^1_1(M)$. Proposition 5.25 gives

$$\mathcal{L}_X t = X[t_\mu{}^\nu]\mathrm{d}x^\mu \otimes e_\nu + t_\mu{}^\nu(\mathcal{L}_X \mathrm{d}x^\mu) \otimes e_\nu + t_\mu{}^\nu \mathrm{d}x^\mu \otimes (\mathcal{L}_X e_\nu).$$

Exercise 5.26 Let t be a tensor field. Show that

$$\mathcal{L}_{[X,Y]}t = \mathcal{L}_X\mathcal{L}_Y t - \mathcal{L}_Y\mathcal{L}_X t. \tag{5.58}$$

5.4 Differential forms

Before we define differential forms, we examine the symmetry property of tensors. The symmetry operation on a tensor $\omega \in \mathcal{T}^0_{r,p}(M)$ is defined by

$$P\omega(V_1, \ldots, V_r) \equiv \omega(V_{P(1)}, \ldots, V_{P(r)}) \tag{5.59}$$

where $V_i \in T_pM$ and P is an element of S_r, the **symmetric group** of order r. Take the coordinate basis $\{e_\mu\} = \{\partial/\partial x^\mu\}$. The component of ω in this basis is

$$\omega(e_{\mu_1}, e_{\mu_2}, \ldots, e_{\mu_r}) \equiv \omega_{\mu_1\mu_2\ldots\mu_r}.$$

The component of $P\omega$ is obtained from (5.59) as

$$P\omega(e_{\mu_1}, e_{\mu_2}, \ldots, e_{\mu_r}) = \omega_{\mu_{P(1)}\mu_{P(2)}\ldots\mu_{P(r)}}.$$

For a general tensor of type (q, r), the symmetry operations are defined for q indices and r indices separately.

For $\omega \in \mathcal{T}^0_{r,p}(M)$, the **symmetriser** $\mathcal{S}$ is defined by

$$\mathcal{S}\omega = \frac{1}{r!} \sum_{P \in S_r} P\omega \tag{5.60}$$

while the **antisymmetriser** $\mathcal{A}$ is

$$\mathcal{A}\omega = \frac{1}{r!} \sum_{P \in S_r} \operatorname{sgn}(P) P\omega \tag{5.61}$$

where $\operatorname{sgn}(P) = +1$ for even permutations and -1 for odd permutations. $\mathcal{S}\omega$ is *totally symmetric* (that is, $P\mathcal{S}\omega = \mathcal{S}\omega$ for any $P \in S_r$) and $\mathcal{A}\omega$ is *totally antisymmetric* ($P\mathcal{A}\omega = \operatorname{sgn}(P)\mathcal{A}\omega$).

5.4.1 *Definitions*

Definition 5.27 A **differential form** of order r, or an ***r*-form**, is a totally antisymmetric tensor of type $(0, r)$.

Let us define the **wedge product** $\wedge$ of r one-forms by the totally antisymmetric tensor product

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} = \sum_{P \in S_r} \operatorname{sgn}(P)\, \mathrm{d}x^{\mu_{P(1)}} \otimes \mathrm{d}x^{\mu_{P(2)}} \otimes \ldots \otimes \mathrm{d}x^{\mu_{P(r)}}. \tag{5.62}$$

For example,

$$\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} = \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} - \mathrm{d}x^{\nu} \otimes \mathrm{d}x^{\mu}$$

$$\begin{aligned}\mathrm{d}x^{\lambda} \wedge \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} = {} & \mathrm{d}x^{\lambda} \otimes \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} + \mathrm{d}x^{\nu} \otimes \mathrm{d}x^{\lambda} \otimes \mathrm{d}x^{\mu} \\ & + \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} \otimes \mathrm{d}x^{\lambda} - \mathrm{d}x^{\lambda} \otimes \mathrm{d}x^{\nu} \otimes \mathrm{d}x^{\mu} \\ & - \mathrm{d}x^{\nu} \otimes \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\lambda} - \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\lambda} \otimes \mathrm{d}x^{\nu}.\end{aligned}$$

It is readily verified that the wedge product satisfies the following.

(i) $\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} = 0$ if some index μ_i appears at least twice.
(ii) $\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} = \operatorname{sgn}(P)\,\mathrm{d}x^{\mu_{P(1)}} \wedge \ldots \wedge \mathrm{d}x^{\mu_{P(r)}}$.
(iii) $\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}$ is linear in each $\mathrm{d}x^{\mu_i}$.

If we denote the vector space of r-forms at $p \in M$ by $\Omega^r_p(M)$, the set of r-forms (5.62) forms a basis of $\Omega^r_p(M)$ and an element $\omega \in \Omega^r_p(M)$ is expanded as

$$\omega = \frac{1}{r!}\, \omega_{\mu_1\mu_2\ldots\mu_r} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \tag{5.63}$$

where $\omega_{\mu_1\mu_2\ldots\mu_r}$ are taken *totally antisymmetric*, reflecting the antisymmetry of the basis. For example, the components of any second-rank tensor $\omega_{\mu\nu}$ are decomposed into the symmetric part $\sigma_{\mu\nu}$ and the antisymmetric part $\alpha_{\mu\nu}$:

$$\sigma_{\mu\nu} = \omega_{(\mu\nu)} \equiv \tfrac{1}{2}(\omega_{\mu\nu} + \omega_{\nu\mu}) \tag{5.64a}$$

$$\alpha_{\mu\nu} = \omega_{[\mu\nu]} \equiv \tfrac{1}{2}(\omega_{\mu\nu} - \omega_{\nu\mu}). \tag{5.64b}$$

Observe that $\sigma_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 0$, while $\alpha_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \omega_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$.

Since there are $\binom{m}{r}$ choices of the set $(\mu_1, \mu_2, \ldots, \mu_r)$ out of $(1, 2, \ldots, m)$ in (5.62), the dimension of the vector space $\Omega_p^r(M)$ is

$$\binom{m}{r} = \frac{m!}{(m-r)!r!}.$$

For later convenience we define $\Omega_p^0(M) = \mathbb{R}$. Clearly $\Omega_p^1(M) = T_p^*M$. If r in (5.62) exceeds m, it vanishes identically since some index appears at least twice in the antisymmetrised summation. The equality $\binom{m}{r} = \binom{m}{m-r}$ implies $\dim \Omega_p^r(M) = \dim \Omega_p^{m-r}(M)$. Since $\Omega_p^r(M)$ is a vector space, $\Omega_p^r(M)$ is isomorphic to $\Omega_p^{m-r}(M)$ (see §2.2).

Define the **exterior product** of a q-form and an r-form, $\wedge : \Omega_p^q(M) \times \Omega_p^r(M) \to \Omega_p^{q+r}(M)$ by a trivial extension. Let $\omega \in \Omega_p^q(M)$ and $\xi \in \Omega_p^r(M)$, for example. The action of the $(q+r)$-form $\omega \wedge \xi$ on $q + r$ vectors is defined by

$$(\omega \wedge \xi)(V_1, \ldots, V_{q+r}) = \frac{1}{q!r!}\sum_{P \in S_{q+r}} \mathrm{sgn}(P)\omega(V_{P(1)}, \ldots, V_{P(q)})\xi(V_{P(q+1)}, \ldots, V_{P(q+r)}) \tag{5.65}$$

where $V_i \in T_pM$. If $q + r > m$, $\omega \wedge \xi$ vanishes identically. With this product, we define an algebra

$$\Omega_p^*(M) \equiv \Omega_p^0(M) \oplus \Omega_p^1(M) \oplus \ldots \oplus \Omega_p^m(M). \tag{5.66}$$

$\Omega_p^*(M)$ is the space of all differential forms at p and is closed under the exterior product.

Exercise 5.28 Take the Cartesian coordinates (x, y) in $\mathbb{R}^2$. The two-form $\mathrm{d}x \wedge \mathrm{d}y$ is the oriented area element (the vector product in elementary vector algebra). Show that, in polar coordinates, this becomes $r\,\mathrm{d}r \wedge \mathrm{d}\theta$.

Exercise 5.29 Let $\xi \in \Omega^q(M)$, $\eta \in \Omega^r(M)$ and $\omega \in \Omega^s(M)$. Show that

$$(a)\quad \xi \wedge \xi = 0 \qquad \text{if } q \text{ is odd} \tag{5.67a}$$

$$(b)\quad \xi \wedge \eta = (-1)^{qr}\eta \wedge \xi \tag{5.67b}$$

$$(c)\quad (\xi \wedge \eta) \wedge \omega = \xi \wedge (\eta \wedge \omega). \tag{5.67c}$$

We may assign an r-form smoothly at each point on a manifold M. We denote the space of smooth r-forms on M by $\Omega^r(M)$. We also define $\Omega^0(M)$ to be the algebra of smooth functions, $\mathcal{F}(M)$. In summary we have the following table.

r-forms	Basis	Dimension
$\Omega^0(M) = \mathcal{F}(M)$:	$\{1\}$	1
$\Omega^1(M) = T^*M$:	$\{\mathrm{d}x^\mu\}$	m
$\Omega^2(M)$:	$\{\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2}\}$	$m(m-1)/2$
$\Omega^3(M)$:	$\{\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \mathrm{d}x^{\mu_3}\}$	$m(m-1)(m-2)/6$
$\vdots$	$\vdots$	$\vdots$
$\Omega^m(M)$:	$\{\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^m\}$	1.

5.4.2 Exterior derivatives

Definition 5.30 The **exterior derivative** d_r is a map $\Omega^r(M) \to \Omega^{r+1}(M)$ whose action on an r-form

$$\omega = \frac{1}{r!}\, \omega_{\mu_1 \ldots \mu_r} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}$$

is defined by

$$\mathrm{d}_r\omega = \frac{1}{r!}\left(\frac{\partial}{\partial x^\nu}\, \omega_{\mu_1 \ldots \mu_r}\right) \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}. \tag{5.68}$$

It is common to drop the subscript r and write simply d. The wedge product automatically antisymmetrises the coefficient.

Example 5.31 The r-forms in three-dimensional space are

(i) $\omega_0 = f(x, y, z)$

(ii) $\omega_1 = \omega_x(x, y, z)\,\mathrm{d}x + \omega_y(x, y, z)\,\mathrm{d}y + \omega_z(x, y, z)\,\mathrm{d}z$

(iii) $\omega_2 = \omega_{xy}(x, y, z)\,\mathrm{d}x \wedge \mathrm{d}y + \omega_{yz}(x, y, z)\,\mathrm{d}y \wedge \mathrm{d}z + \omega_{zx}(x, y, z)\,\mathrm{d}z \wedge \mathrm{d}x$

(iv) $\omega_3 = \omega_{xyz}(x, y, z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

If we define an *axial vector* α^μ by $\varepsilon^{\mu\nu\lambda}\omega_{\nu\lambda}$, a two-form may be regarded as a 'vector'. The **Levi-Civita symbol** $\varepsilon^{\mu\nu\lambda}$ is defined by $\varepsilon^{P(1)P(2)P(3)} = \mathrm{sgn}(P)$ and provides the isomorphism between $\mathcal{X}(M)$ and $\Omega^2(M)$. [Note that both of these are of dimension three.]

The action of d is

(i) $\mathrm{d}\omega_0 = \dfrac{\partial f}{\partial x}\mathrm{d}x + \dfrac{\partial f}{\partial y}\mathrm{d}y + \dfrac{\partial f}{\partial z}\mathrm{d}z$

(ii) $\mathrm{d}\omega_1 = (\partial\omega_y/\partial x - \partial\omega_x/\partial y)\,\mathrm{d}x \wedge \mathrm{d}y + (\partial\omega_z/\partial y - \partial\omega_y/\partial z)\,\mathrm{d}y \wedge \mathrm{d}z + (\partial\omega_x/\partial z - \partial\omega_z/\partial x)\,\mathrm{d}z \wedge \mathrm{d}x$

(iii) $\mathrm{d}\omega_2 = (\partial\omega_{yz}/\partial x + \partial\omega_{zx}/\partial y + \partial\omega_{xy}/\partial z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$

(iv) $\mathrm{d}\omega_3 = 0$.

Hence the action of d on ω_0 is identified with 'grad', on ω_1 with 'rot', and on ω_2 with 'div' in the usual vector calculus.

Exercise 5.32 Let $\xi \in \Omega^q(M)$ and $\omega \in \Omega^r(M)$. Show that

$$\mathrm{d}(\xi \wedge \omega) = (\mathrm{d}\xi) \wedge \omega + (-1)^q \xi \wedge (\mathrm{d}\omega). \tag{5.69}$$

A useful expression for the exterior derivative is obtained as follows. Let us take $X = X^\mu \partial/\partial x^\mu$, $Y = Y^\nu \partial/\partial x^\nu \in \mathcal{X}(M)$ and $\omega = \omega_\mu \mathrm{d}x^\mu \in \Omega^1(M)$. It is easy to see that the combination

$$X[\omega(Y)] - Y[\omega(X)] - \omega([X, Y]) = (\partial\omega_\mu/\partial x^\nu)(X^\nu Y^\mu - X^\mu Y^\nu)$$

is equal to $\mathrm{d}\omega(X, Y)$, and we have the coordinate-free expression

$$\mathrm{d}\omega(X, Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([X, Y]). \tag{5.70}$$

For an r-form $\omega \in \Omega^r(M)$, this becomes

$$\mathrm{d}\omega(X_1, \ldots, X_{r+1}) = \sum_{i=1}^{r} (-1)^{i+1} X_i \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{r+1})$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{r+1}) \tag{5.71}$$

where the entry below $\hat{}$ has been omitted. As an exercise, the reader should verify (5.71) explicitly for $r = 2$.

We now prove an important formula

$$\mathrm{d}^2 = 0 \text{ (or } \mathrm{d}_{r+1}\mathrm{d}_r = 0). \tag{5.72}$$

Take

$$\omega = \frac{1}{r!}\, \omega_{\mu_1 \ldots \mu_r} \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \in \Omega^r(M).$$

The action of d^2 on ω is

$$\mathrm{d}^2\omega = \frac{1}{r!} \frac{\partial^2 \omega_{\mu_1 \ldots \mu_r}}{\partial x^\lambda \partial x^\nu} \mathrm{d}x^\lambda \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}.$$

This vanishes identically since $\partial^2\omega_{\mu_1 \ldots \mu_r}/\partial x^\lambda \partial x^\nu$ is symmetric with respect to λ and ν while $\mathrm{d}x^\lambda \wedge \mathrm{d}x^\nu$ is antisymmetric.

Example 5.33 It is known that the electromagnetic potential $A = (\phi, \boldsymbol{A})$ is a one-form, $A = A_\mu \mathrm{d}x^\mu$ (see Chapter 10). The electromagnetic tensor is defined by $F = \mathrm{d}A$ and has the components

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{5.73}$$

where

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial}{\partial x^0}\boldsymbol{A} \qquad \text{and } \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

as usual. Two Maxwell's equations, $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ and $\partial\boldsymbol{B}/\partial t = -\boldsymbol{\nabla} \times \boldsymbol{E}$ follow from the identity $\mathrm{d}F = \mathrm{d}(\mathrm{d}A) = 0$, which is known as the **Bianchi identity**.

A map $f : M \to N$ induces the pullback $f^* : T^*_{f(p)}N \to T^*_pM$ and f^* is naturally extended to tensors of type $(0, r)$; see §5.2. Since an r-form is a tensor of type $(0, r)$, this applies as well. Let $\omega \in \Omega^r(N)$ and let f be a map $M \to N$. At each point $f(p) \in N$, f induces the pullback $f^* : \Omega^r_{f(p)}(N) \to \Omega^r_p(M)$ by

$$(f^*\omega)(X_1, \ldots, X_r) \equiv \omega(f_*X_1, \ldots, f_*X_r) \tag{5.74}$$

where $X_i \in T_pM$ and f_* is the differential map $T_pM \to T_{f(p)}N$.

Exercise 5.34 Let $\xi, \omega \in \Omega^r(N)$ and let $f : M \to N$. Show that

$$\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega) \tag{5.75}$$

$$f^*(\xi \wedge \omega) = (f^*\xi) \wedge (f^*\omega). \tag{5.76}$$

The exterior derivative d_r induces the sequence

$$0 \xrightarrow{i} \Omega^0(M) \xrightarrow{\mathrm{d}_0} \Omega^1(M) \xrightarrow{\mathrm{d}_1} \cdots \xrightarrow{\mathrm{d}_{m-2}} \Omega^{m-1}(M) \xrightarrow{\mathrm{d}_{m-1}} \Omega^m(M) \xrightarrow{\mathrm{d}_m} 0 \tag{5.77}$$

where i is the inclusion map $0 \hookrightarrow \Omega^0(M)$. This sequence is called the **de Rham complex**. Since $\mathrm{d}^2 = 0$, we have $\mathrm{im}\,\mathrm{d}_r \subset \ker \mathrm{d}_{r+1}$. [Take $\omega \in \Omega^r(M)$. Then $\mathrm{d}_r\omega \in \mathrm{im}\,\mathrm{d}_r$ and $\mathrm{d}_{r+1}(\mathrm{d}_r\omega) = 0$ imply $\mathrm{d}_r\omega \in \ker \mathrm{d}_{r+1}$.] An element of $\ker \mathrm{d}_r$ is called a **closed *r*-form**, while an element of $\mathrm{im}\,\mathrm{d}_{r-1}$ is called an **exact *r*-form**. Namely, $\omega \in \Omega^r(M)$ is closed if $\mathrm{d}\omega = 0$ and exact if there exists an $(r-1)$-form ψ such that $\omega = \mathrm{d}\psi$. The quotient space $\ker \mathrm{d}_r/\mathrm{im}\,\mathrm{d}_{r-1}$ is called the rth **de Rham cohomology group** which is made into the dual space of the homology group; see Chapter 6.

5.4.3 Interior product and Lie derivative of forms

Another important operation is the **interior product** $\mathrm{i}_X : \Omega^r(M) \to \Omega^{r-1}(M)$, where $X \in \mathcal{X}(M)$. For $\omega \in \Omega^r(M)$, we define

$$i_X\omega(X_1, \ldots, X_{r-1}) \equiv \omega(X, X_1, \ldots, X_{r-1}). \tag{5.78}$$

For $X = X^\mu \partial/\partial x^\mu$ and $\omega = (1/r!)\omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$, we have

$$\begin{aligned} i_X\omega &= \frac{1}{(r-1)!} X^\nu \omega_{\nu\mu_2 \ldots \mu_r} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r} \\ &= \frac{1}{r!}\sum_{s=1}^{r} X^{\mu_s} \omega_{\mu_1 \ldots \mu_s \ldots \mu_r} (-1)^{s-1} dx^{\mu_1} \wedge \ldots \wedge \widehat{dx^{\mu_s}} \wedge \ldots \wedge dx^{\mu_r} \end{aligned} \tag{5.79}$$

where the entry below ^ has been omitted. For example, let (x, y, z) be the coordinates of $\mathbb{R}^3$. Then

$$i_{e_x}(dx \wedge dy) = dy,\ i_{e_x}(dy \wedge dz) = 0,\ i_{e_x}(dz \wedge dx) = -dz.$$

The Lie derivative of a form is most neatly written with the interior product. Let $\omega = \omega_\mu dx^\mu$ be a one-form. Consider the combination

$$\begin{aligned} (di_X + i_X d)\omega &= d(X^\mu \omega_\mu) + i_X[\tfrac{1}{2}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) dx^\mu \wedge dx^\nu] \\ &= (\omega_\mu \partial_\nu X^\mu + X^\mu \partial_\nu \omega_\mu) dx^\nu + X^\mu(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) dx^\nu \\ &= (\omega_\mu \partial_\nu X^\mu + X^\mu \partial_\mu \omega_\nu) dx^\nu. \end{aligned}$$

Comparing this with (5.55), we find

$$\mathcal{L}_X\omega = (di_X + i_X d)\omega. \tag{5.80}$$

For a general r-form $\omega = (1/r!)\omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$, we have

$$\begin{aligned} \mathcal{L}_X\omega &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}((\sigma_\varepsilon)^*\omega|_{\sigma_\varepsilon(x)} - \omega|_x) \\ &= X^\nu \frac{1}{r!} \partial_\nu \omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r} \\ &\quad + \sum_{s=1}^{r} \partial_{\mu_s} X^\nu \frac{1}{r!} \omega_{\mu_1 \ldots \underset{\nu}{\overset{s}{\downarrow}} \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}. \end{aligned} \tag{5.81}$$

We also have

$$\begin{aligned} &(di_X + i_X d)\omega \\ &= \frac{1}{r!}\sum_{s=1}^{r} [\partial_\nu X^{\mu_s} \omega_{\mu_1 \ldots \mu_s \ldots \mu_r} + X^{\mu_s} \partial_\nu \omega_{\mu_1 \ldots \mu_s \ldots \mu_r}] \\ &\quad \times (-1)^{s-1} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge \widehat{dx^{\mu_s}} \wedge \ldots \wedge dx^{\mu_r} \\ &\quad + \frac{1}{r!}\Bigg[X^\nu \partial_\nu \omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r} \\ &\quad + \sum_{s=1}^{r} X^{\mu_s} \partial_\nu \omega_{\mu_1 \ldots \mu_s \ldots \mu_r} (-1)^s dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge \widehat{dx^{\mu_s}} \wedge \ldots \wedge dx^{\mu_r}\Bigg] \end{aligned}$$

$$= \frac{1}{r!}\sum_{s=1}^{r} \partial_\nu X^{\mu_s}\omega_{\mu_1\ldots\mu_s\ldots\mu_r}(-1)^{s-1}\,\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}\hat{x}^{\mu_s} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}$$

$$+ \frac{1}{r!} X^\nu \partial_\nu \omega_{\mu_1\ldots\mu_r}\,\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}.$$

If we interchange the roles of μ_s and ν in the first term of the last expression and compare it with (5.81), we verify

$$(\mathrm{d}\mathrm{i}_X + \mathrm{i}_X\mathrm{d})\omega = \mathcal{L}_X\omega \tag{5.82}$$

for any r-form ω.

Exercise 5.35 Let $X, Y \in \mathcal{X}(M)$ and $\omega \in \Omega^r(M)$. Show that

$$\mathrm{i}_{[X,Y]}\omega = X(\mathrm{i}_Y\omega) - Y(\mathrm{i}_X\omega). \tag{5.83}$$

Show also that i_X is an antiderivation,

$$\mathrm{i}_X(\omega \wedge \eta) = \mathrm{i}_X\omega \wedge \eta + (-1)^r\omega \wedge \mathrm{i}_X\eta \tag{5.84}$$

and nilpotent,

$$\mathrm{i}_X^2 = 0. \tag{5.85}$$

Use the nilpotency to prove

$$\mathcal{L}_X\mathrm{i}_X\omega = \mathrm{i}_X\mathcal{L}_X\omega. \tag{5.86}$$

Exercise 5.36 Let $t \in \mathcal{T}^n_m(M)$. Show that

$$(\mathcal{L}_X t)^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} = X^\lambda\partial_\lambda t^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} + \sum_{s=1}^{n}\partial_{\nu_s}X^\lambda t^{\mu_1\ldots\ldots\mu_n}_{\nu_1\ldots\lambda\ldots\nu_m} - \sum_{s=1}^{n}\partial_\lambda X^{\mu_s} t^{\mu_1\ldots\lambda\ldots\mu_n}_{\nu_1\ldots\ldots\nu_m}. \tag{5.87}$$

5.5 Integration of differential forms

5.5.1 Orientation

An integration of a differential form over a manifold M is defined only when M is 'orientable'. So we first define an **orientation** of a manifold. Let M be a connected m-dimensional differentiable manifold. At a point $p \in M$, the tangent space T_pM is spanned by the basis $\{e_\mu\} = \{\partial/\partial x^\mu\}$, where x^μ is the local coordinate on the chart U_i to which p belongs. Let U_j be another chart such that $U_i \cap U_j \neq \emptyset$ with the local coordinates y^α. If $p \in U_i \cap U_j$, T_pM is spanned by either $\{e_\mu\}$ or $\{\tilde{e}_\alpha\} = \{\partial/\partial y^\alpha\}$. The basis changes as

$$\tilde{e}_\alpha = (\partial x^\mu/\partial y^\alpha)e_\mu. \tag{5.88}$$

If $J = \det(\partial x^\mu/\partial y^\alpha) > 0$ on $U_i \cap U_j$, $\{e_\mu\}$ and $\{\tilde{e}_\alpha\}$ are said to define the *same orientation* on $U_i \cap U_j$ and if $J < 0$, the *opposite orientation*.

Definition 5.37 Let M be a connected manifold covered by $\{U_i\}$. M is **orientable** if, for any overlapping charts U_i and U_j, there exist local coordinates $\{x^\mu\}$ for U_i and $\{y^\alpha\}$ for U_j such that $J = \det(\partial x^\mu/\partial y^\alpha) > 0$.

If M is non-orientable, J cannot be positive in all intersections of charts. For example, the Möbius strip in figure 5.14(a) is non-orientable since we have to choose J to be negative in the intersection B.

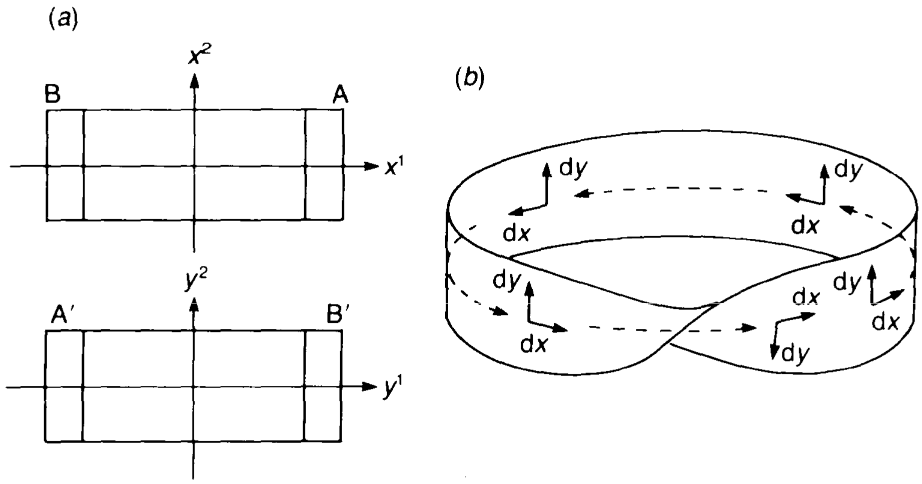


Figure 5.14 (a) The Möbius strip is obtained by twisting the part B′ of the second strip by π before pasting A with A′ and B with B′. The coordinate change on B is $y^1 = x^1$, $y^2 = -x^2$ and the Jacobian is -1. (b) Basis frames on the Möbius strip.

If an m-dimensional manifold M is orientable, there exists an m-form ω which vanishes nowhere. This m-form ω is called a **volume element**, which plays the role of a measure when we integrate a function $f \in \mathcal{F}(M)$ over M. Two volume elements ω and ω' are said to be *equivalent* if there exists a strictly positive function $h \in \mathcal{F}(M)$ such that $\omega = h\omega'$. A negative-definite function $h' \in \mathcal{F}(M)$ gives an inequivalent orientation to M. Thus any orientable manifold admits *two* inequivalent orientations, one of which is called **right-handed** the other **left-handed**. Take an m-form

$$\omega = h(p)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \tag{5.89}$$

with a positive-definite $h(p)$ on a chart (U, φ) whose coordinate is $x = \varphi(p)$. If M is orientable, we may extend ω throughout M such that the component h is positive definite on any chart U_i. If M is orientable, this ω is a volume element. Note that this positivity of h is independent of the choice of the coordinates. In fact, let $p \in U_i \cap U_j \neq 0$ and let x^μ and y^α be the coordinates of U_i and U_j, respectively. Then (5.89) becomes

$$\omega = h(p)\frac{\partial x^1}{\partial y^{\mu_1}}\,\mathrm{d}y^{\mu_1} \wedge \ldots \wedge \frac{\partial x^m}{\partial y^{\mu_m}}\,\mathrm{d}y^{\mu_m}$$

$$= h(p)\det\left(\frac{\partial x^\mu}{\partial y^\nu}\right)\mathrm{d}y^1 \wedge \ldots \wedge \mathrm{d}y^m. \qquad (5.90)$$

The determinant in (5.90) is the Jacobian of the coordinate transformation and must be positive by assumed orientability. If M is non-orientable, ω with a positive-definite component cannot be defined on M. Let us look at figure 5.14 again. If we circumnavigate the strip along the direction shown in the figure, $\omega = \mathrm{d}x \wedge \mathrm{d}y$ changes the signature $\mathrm{d}x \wedge \mathrm{d}y \to -\mathrm{d}x \wedge \mathrm{d}y$ when we come back to the starting point. Hence ω cannot be defined uniquely on M.

5.5.2 *Integration of forms*

Now we are ready to define an integration of a function $f: M \to \mathbb{R}$ over an orientable manifold M. Take a volume element ω. In a coordinate neighbourhood U_i with the coordinates x, we define the integration of an m-form $f\omega$ by

$$\int_{U_i} f\omega \equiv \int_{\varphi_i(U_i)} f(\varphi_i^{-1}(x))h(\varphi_i^{-1}(x))\,\mathrm{d}x^1 \ldots \mathrm{d}x^m. \qquad (5.91)$$

The RHS is an ordinary multiple integration of a function of m variables. Once the integral of f over U_i is defined, the integral of f over the whole of M is given with the help of the '*partition of unity*' defined below.

Definition 5.38 Take an open covering $\{U_i\}$ of M such that each point of M is covered by a finite number of U_i. [If this is always possible, M is called **paracompact**, which we assume to be the case.] If a family of differentiable functions $\varepsilon_i(p)$ satisfies

(i) $0 \leq \varepsilon_i(p) \leq 1$
(ii) $\varepsilon_i(p) = 0$ if $p \notin U_i$
(iii) $\varepsilon_1(p) + \varepsilon_2(p) + \ldots = 1$ for any point $p \in M$

the family $\{\varepsilon_i(p)\}$ is called a **partition of unity** subordinate to the covering $\{U_i\}$.

From the condition (iii) above, it follows that

$$f(p) = \sum_i f(p)\varepsilon_i(p) = \sum_i f_i(p) \qquad (5.92)$$

where $f_i(p) \equiv f(p)\varepsilon_i(p)$ vanishes outside U_i by (ii). Hence, given a point $p \in M$, assumed paracompactness ensures there are only finite terms in the summation over i in (5.92). For each $f_i(p)$, we may define the integral over U_i according to (5.91). Finally the integral of f on M is given by

$$\int_M f\omega \equiv \sum_i \int_{U_i} f_i \omega. \tag{5.93}$$

Although a different atlas $\{(V_i, \psi_i)\}$ gives different coordinates and a different partition of unity, the integral defined by (5.93) remains the same.

Example 5.39 Let us take the atlas of S^1 defined in example 5.3. Let $U_1 = S^1 - \{(1,0)\}$, $U_2 = S^1 - \{(-1,0)\}$ and $\varepsilon_1(\theta) = \sin^2(\theta/2)$ and $\varepsilon_2(\theta) = \cos^2(\theta/2)$. The reader should verify that $\{\varepsilon_i(\theta)\}$ is a partition of unity subordinate to $\{U_i\}$. Let us integrate a function $f = \cos^2\theta$, for example. [Of course we know

$$\int_0^{2\pi} \mathrm{d}\theta \cos^2\theta = \pi$$

but let us use the partition of unity.] We have

$$\begin{aligned}\int_{S^1} \mathrm{d}\theta\cos^2\theta &= \int_0^{2\pi} \mathrm{d}\theta \sin^2(\theta/2)\cos^2\theta + \int_{-\pi}^{\pi} \mathrm{d}\theta\cos^2(\theta/2)\cos^2\theta \\ &= \tfrac{1}{2}\pi + \tfrac{1}{2}\pi = \pi.\end{aligned}$$

So far, we have left h arbitrary provided it is strictly positive. The reader might be tempted to choose h to be unity. However, as we found in (5.90), h is multiplied by the Jacobian under the change of coordinates and there is no canonical way to single out the component h; unity in one coordinate might not be unity in the other. The situation changes if the manifold is endowed with a metric, as we will see in Chapter 7.

5.6 Lie groups and Lie algebras

A Lie group is a manifold on which the group manipulations, *product* and *inverse*, are defined. Lie groups play an extremely important role in the theory of fibre bundles and also find vast applications in physics. Here we will work out the geometrical aspects of Lie groups and Lie algebras.

5.6.1 Lie groups

Definition 5.40 A **Lie group** G is a differentiable manifold which is endowed with a group structure such that the group operations

(i) $\cdot : G \times G \to G$ by $(g_1, g_2) \mapsto g_1 \cdot g_2$
(ii) $^{-1} : G \to G$ by $g \mapsto g^{-1}$

are differentiable. [*Remark:* It can be shown that G has a unique *analytic* structure with which the product and the inverse operations are written as convergent power series.]

The *unit element* of a Lie group is written as e. The dimension of a Lie group G is defined to be the dimension of G as a manifold. The product symbol may be omitted and $g_1 \cdot g_2$ is usually written as $g_1 g_2$. For example, let $\mathbb{R}^* \equiv \mathbb{R} - \{0\}$. Take three elements x, y, $z \in \mathbb{R}^*$ such that $xy = z$. Obviously if we multiply a number close to x by a number close to y, we have a number close to z. Similarly, an inverse of a number close to x is close to $1/x$. In fact we can differentiate these maps with respect to the relevant arguments and $\mathbb{R}^*$ is made into a Lie group with these group operations. If the product is commutative, namely $g_1 g_2 = g_2 g_1$, we often use the additive symbol + instead of the product symbol.

Exercise 5.41

(a) Show that $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$ is a Lie group with respect to multiplication.

(b) Show that $\mathbb{R}$ is a Lie group with respect to addition.

(c) Show that $\mathbb{R}^2$ is a Lie group with respect to addition defined by $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

Example 5.42 Let S^1 be the unit circle on the complex plane,

$$S^1 = \{e^{i\theta} | \theta \in \mathbb{R}(\text{mod}\, 2\pi)\}.$$

The group operations defined by $e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}$ and $(e^{i\theta})^{-1} = e^{-i\theta}$ are differentiable and S^1 is made into a Lie group, which we call $\mathrm{U}(1)$. It is easy to see that the group operations are the same as those in exercise 5.41(b) modulo 2π.

Of particular interest in physical applications are the **matrix groups**, which are subgroups of general linear groups $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$. The product of elements is simply the matrix multiplication and the inverse is given by the matrix inverse. The coordinates of $\mathrm{GL}(n, \mathbb{R})$ are given by n^2 entries of $M = \{x_{ij}\}$. $\mathrm{GL}(n, \mathbb{R})$ is a non-compact manifold of real dimension n^2.

Interesting subgroups of $\mathrm{GL}(n, \mathbb{R})$ are the **orthogonal group** $\mathrm{O}(n)$, the **special linear group** $\mathrm{SL}(n, \mathbb{R})$ and the **special orthogonal group** $\mathrm{SO}(n)$:

$$\mathrm{O}(n) = \{M \in \mathrm{GL}(n, \mathbb{R}) | MM^t = M^t M = \mathbb{1}\}$$

$$\mathrm{SL}(n, \mathbb{R}) = \{M \in \mathrm{GL}(n, \mathbb{R}) | \det M = 1\}$$

$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$$

where ^{t} denotes the transpose of a matrix. In special relativity, we are familiar with the Lorentz group

$$O(1, 3) = \{M \in GL(4, \mathbb{R}) | M\eta M^t = \eta\}$$

where η is the Minkowski metric, $\eta = \mathrm{diag}(-1, 1, 1, 1)$. Extension to higher-dimensional spacetime is trivial.

Exercise 5.43 Show that the group O(1, 3) is non-compact and has four connected components according to the sign of the determinant and the sign of the (0, 0) entry. The component that contains the unit matrix is denoted by $O_+^\uparrow(1, 3)$.

The group $GL(n, \mathbb{C})$ is the set of non-singular linear transformations in $\mathbb{C}^n$, which are represented by $n \times n$ non-singular matrices with complex entries. The **unitary group** $U(n)$, the **special linear group** $SL(n, \mathbb{C})$ and the **special unitary group** $SU(n)$ are defined by

$$U(n) = \{M \in GL(n, \mathbb{C}) | MM^\dagger = M^\dagger M = \mathbb{1}\}$$

$$SL(n, \mathbb{C}) = \{M \in GL(n, \mathbb{C}) | \det M = 1\}$$

$$SU(n) = U(n) \cap SL(n, \mathbb{C})$$

where † is the Hermitian conjugate.

So far we have just mentioned that the matrix groups are subgroups of a Lie group $GL(n, \mathbb{R})$ (or $GL(n, \mathbb{C})$). The following theorem guarantees that they are Lie subgroups, that is, these subgroups are Lie groups by themselves. We accept this important (and difficult to prove) theorem without proof.

Theorem 5.44 Every closed subgroup H of a Lie group G is a Lie subgroup.

For example, $O(n)$, $SL(n, \mathbb{R})$ and $SO(n)$ are Lie subgroups of $GL(n, \mathbb{R})$. To see why $SL(n, \mathbb{R})$ is a closed subgroup, consider a map $f : GL(n, \mathbb{R}) \to \mathbb{R}$ defined by $A \mapsto \det A$. Obviously f is a *continuous* map and $f^{-1}(1) = SL(n, \mathbb{R})$. A point $\{1\}$ is a closed subset of $\mathbb{R}$, hence $f^{-1}(1)$ is closed in $GL(n, \mathbb{R})$. Then theorem 5.44 states that $SL(n, \mathbb{R})$ is a Lie subgroup. The reader should verify that $O(n)$ and $SO(n)$ are also Lie subgroups of $GL(n, \mathbb{R})$.

Let G be a Lie group and H a Lie subgroup of G. Define an equivalence relation $\sim$ by $g \sim g'$ if there exists an element $h \in H$ such that $g' = gh$. An equivalence class $[g]$ is a set $\{gh | h \in H\}$. The coset space G/H is a manifold (not necessarily a Lie group) with $\dim G/H = \dim G - \dim H$. G/H is a Lie group if H is a **normal subgroup** of G, that is, if $ghg^{-1} \in H$ for any $g \in G$ and $h \in H$. In fact take equivalence classes $[g]$, $[g'] \in G/H$ and construct the product $[g][g']$. If the group structure is well defined in G/H, the product must be independent of the choice of the representatives. Let gh and $g'h'$ be the representatives of $[g]$ and $[g']$ respectively. Then $ghg'h' =$

$gg'h''h' \in [gg']$ where the equality follows since there exists $h'' \in H$ such that $hg' = g'h''$. It is left as an exercise to the reader to show that $[g]^{-1}$ is also a well defined operation and $[g]^{-1} = [g^{-1}]$.

5.6.2 Lie algebras

Definition 5.45 Let a and g be elements of a Lie group G. The **right-translation** $R_a : G \to G$ and the **left-translation** $L_a : G \to G$ of g by a are defined by

$$R_a g = ga \tag{5.94a}$$

$$L_a g = ag. \tag{5.94b}$$

By definition, R_a and L_a are diffeomorphisms from G to G. Hence the maps $L_a : G \to G$ and $R_a : G \to G$ induce $L_{a*} : T_gG \to T_{ag}G$ and $R_{a*} : T_gG \to T_{ga}G$; see §5.2. Since these translations give equivalent theories, we are concerned mainly with the left-translation in the following. The analysis based on the right-translation can be carried out in a similar manner.

Given a Lie group G, there exists a special class of vector fields characterised by an *invariance* under group action. [On the usual manifold there is no canonical way of discriminating some vector fields from the others.]

Definition 5.46 Let X be a vector field on a Lie group G. X is said to be a **left-invariant vector field** if $L_{a*}X|_g = X|_{ag}$.

Exercise 5.47 Verify that a left-invariant vector field X satisfies

$$L_{a*}X|_g = X^\mu(g)\,\frac{\partial x^\nu(ag)}{\partial x^\mu(g)}\,\frac{\partial}{\partial x^\nu}\bigg|_{ag} = X^\nu(ag)\,\frac{\partial}{\partial x^\nu}\bigg|_{ag} \tag{5.95}$$

where $x^\mu(g)$ and $x^\mu(ag)$ are coordinates of g and ag.

A vector $V \in T_eG$ defines a unique left-invariant vector field X_V throughout G by

$$X_V|_g = L_{g*}V \qquad g \in G. \tag{5.96}$$

In fact, we verify from (5.34) that $X_V|_{ag} = L_{ag*}V = (L_aL_g)_*V = L_{a*}L_{g*}V = L_{a*}X_V|_g$. Conversely, a left-invariant vector field X defines a unique vector $V = X|_e \in T_eG$. Let us denote the set of left-invariant vector fields on G by $\mathfrak{g}$. The map $T_eG \to \mathfrak{g}$ defined by $V \mapsto X_V$ is an isomorphism, and it follows that the set of left-invariant vector fields is a vector space isomorphic to T_eG. In particular, $\dim \mathfrak{g} = \dim G$.

Since $\mathfrak{g}$ is a set of vector fields, it is a subset of $\mathcal{X}(G)$ and the Lie bracket defined in §5.3 is also defined on $\mathfrak{g}$. We show that $\mathfrak{g}$ is closed under the Lie bracket. Take two points g and $ag = L_a g$ in G. If we apply L_{a*} to the Lie bracket $[X, Y]$ of $X, Y \in \mathfrak{g}$, we have

$$L_{a*}[X, Y]|_g = [L_{a*}X|_g, L_{a*}Y|_g] = [X, Y]|_{ag} \tag{5.97}$$

where the left-invariances of X and Y and (5.52) have been used. Thus $[X, Y] \in \mathfrak{g}$, that is $\mathfrak{g}$ is closed under the Lie bracket.

It is instructive to work out the left-invariant vector field of $\mathrm{GL}(n, \mathbb{R})$. The coordinates of $\mathrm{GL}(n, \mathbb{R})$ are given by n^2 entries x^{ij} of the matrix. The unit element is $e = 1_n = (\delta^{ij})$. Let $g = \{x^{ij}(g)\}$ and $a = \{x^{ij}(a)\}$ be elements of $\mathrm{GL}(n, \mathbb{R})$. The left-translation is

$$L_a g = ag = \sum x^{ik}(a)x^{kj}(g).$$

Take a vector $V = \Sigma V^{ij}\partial/\partial x^{ij}|_e \in T_eG$, where the V^{ij} are the entries of V. The left-invariant vector field generated by V is

$$\begin{aligned} X_V|_g = L_{g*}V &= \sum_{ijklm} V^{ij} \frac{\partial}{\partial x^{ij}}\bigg|_e x^{kl}(g)x^{lm}(e) \frac{\partial}{\partial x^{km}}\bigg|_g \\ &= \sum V^{ij}x^{kl}(g)\delta_i{}^l\delta_j{}^m \frac{\partial}{\partial x^{km}}\bigg|_g \\ &= \sum x^{ki}(g)V^{ij} \frac{\partial}{\partial x^{kj}}\bigg|_g = \sum (gV)^{kj} \frac{\partial}{\partial x^{kj}}\bigg|_g \end{aligned} \tag{5.98}$$

where gV is the usual matrix multiplication of g and V. The vector X_V is often abbreviated as gV since it gives the components of the vector.

The Lie bracket of X_V and X_W generated by $V = V^{ij}\partial/\partial x^{ij}|_e$ and $W = W^{ij}\partial/\partial x^{ij}|_e$ is

$$\begin{aligned} [X_V, X_W]|_g &= \sum x^{ki}(g)V^{ij} \frac{\partial}{\partial x^{kj}}\bigg|_g x^{ca}(g)W^{ab} \frac{\partial}{\partial x^{cb}}\bigg|_g - (V \leftrightarrow W) \\ &= \sum x^{ij}(g)[V^{jk}W^{kl} - W^{jk}V^{kl}] \frac{\partial}{\partial x^{il}}\bigg|_g \\ &= \sum (g[V, W])^{ij} \frac{\partial}{\partial x^{ij}}\bigg|_g. \end{aligned} \tag{5.99}$$

Clearly, (5.98) and (5.99) remain true for any matrix group and we establish that

$$L_{g*}V = gV \tag{5.100}$$

$$[X_V, X_W]|_g = L_{g*}[V, W] = g[V, W]. \tag{5.101}$$

Now a Lie algebra is defined as the set of left-invariant vector fields $\mathfrak{g}$ with the Lie bracket.

Definition 5.48 The set of left-invariant vector fields $\mathfrak{g}$ with the Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called the **Lie algebra** of a Lie group G.

We denote the Lie algebra of a Lie group by the corresponding lower-case German gothic letter. For example $\mathfrak{so}(n)$ is the Lie algebra of $\mathrm{SO}(n)$

Example 5.49

(a) Take $G = \mathbb{R}$ as in example 5.41(b). If we define the left translation L_a by $x \mapsto a + x$, the left-invariant vector field is given by $X = \partial/\partial x$. In fact,

$$L_{a*}X|_x = \frac{\partial(a + x)}{\partial x}\frac{\partial}{\partial(a + x)} = \frac{\partial}{\partial(x + a)} = X|_{x+a}.$$

Clearly this is the only left-invariant vector field on $\mathbb{R}$. We also find that $X = \partial/\partial\theta$ is the unique left-invariant vector field on $G = \mathrm{SO}(2) = \{e^{i\theta}|0 \leqslant \theta \leqslant 2\pi\}$. Thus the Lie groups $\mathbb{R}$ and $\mathrm{SO}(2)$ share the common Lie algebra.

(b) Let $\mathfrak{gl}(n, \mathbb{R})$ be the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ and $c : (-\varepsilon, \varepsilon) \to \mathrm{GL}(n, \mathbb{R})$ be a curve with $c(0) = \mathbb{1}$. The curve is approximated by $c(s) = \mathbb{1} + sA + \mathrm{O}(s^2)$ near $s = 0$, where A is an $n \times n$ matrix of real entries. Note that for small enough s, $\det c(s)$ cannot vanish and $c(s)$ is indeed in $\mathrm{GL}(n, \mathbb{R})$. The tangent vector to $c(s)$ at $\mathbb{1}$ is $c'(s)|_{s=0} = A$. This shows that $\mathfrak{gl}(n, \mathbb{R})$ is the set of $n \times n$ matrices. Clearly $\dim \mathfrak{gl}(n, \mathbb{R}) = n^2 = \dim \mathrm{GL}(n, \mathbb{R})$.

Subgroups of $\mathrm{GL}(n, \mathbb{R})$ are more interesting.

(c) Let us find the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ of $\mathrm{SL}(n, \mathbb{R})$. Following the prescription above, we approximate a curve through $\mathbb{1}$ by $c(s) = \mathbb{1} + sA + \mathrm{O}(s^2)$. The tangent vector to $c(s)$ at $\mathbb{1}$ is $c'(s)|_{s=0} = A$. Now, for the curve $c(s)$ to be in $\mathrm{SL}(n, \mathbb{R})$, $c(s)$ has to satisfy $\det c(s) = 1 + s \operatorname{tr} A = 1$, namely $\operatorname{tr} A = 0$. Thus $\mathfrak{sl}(n, \mathbb{R})$ is the set of $n \times n$ *traceless* matrices and $\dim \mathfrak{sl}(n, \mathbb{R}) = n^2 - 1$.

(d) Let $c(s) = \mathbb{1} + sA + \mathrm{O}(s^2)$ be a curve in $\mathrm{SO}(n)$ through $\mathbb{1}$. Since $c(s)$ is a curve in $\mathrm{SO}(n)$, it satisfies $c(s)^t c(s) = \mathbb{1}$. Differentiating this identity, we obtain $c'(s)^t c(s) + c(s)^t c'(s) = 0$. At $s = 0$, this becomes $A^t + A = 0$. Hence $\mathfrak{so}(n)$ is the set of *skew-symmetric* matrices. Since we are interested only in the vicinity of the unit element, the Lie algebra of $\mathrm{O}(n)$ is the same as that of $\mathrm{SO}(n)$: $\mathfrak{o}(n) = \mathfrak{so}(n)$. It is easy to see that $\dim \mathfrak{o}(n) = \dim \mathfrak{so}(n) = n(n - 1)/2$.

(e) A similar analysis can be carried out for matrix groups of $\mathrm{GL}(n, \mathbb{C})$. $\mathfrak{gl}(n, \mathbb{C})$ is the set of $n \times n$ matrices with complex entries and $\dim \mathfrak{gl}(n, \mathbb{C}) = 2n^2$ (the dimension here is a real dimension). $\mathfrak{sl}(n, \mathbb{C})$ is the set of traceless matrices with real dimension $2(n^2 - 1)$. To find $\mathfrak{u}(n)$, we consider a curve $c(s) = \mathbb{1} + sA + \mathrm{O}(s^2)$ in $\mathrm{U}(n)$. Since $c(s)^\dagger c(s) = 1$, we have $c'(s)^\dagger c(s) + c(s)^\dagger c'(s) = 0$. At $s = 0$, we have

$A^{\dagger} + A = 0$. Hence $\mathfrak{u}(n)$ is the set of *skew-Hermitian* matrices with $\dim \mathfrak{u}(n) = n^2$. $\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C})$ is the set of traceless skew-Hermitian matrices with $\dim \mathfrak{su}(n) = n^2 - 1$.

Exercise 5.50 Let

$$c(s) = \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be a curve in SO(3). Find the tangent vector to this curve at $\mathbb{1}$.

5.6.3 *The one-parameter subgroup*

A vector field $X \in \mathcal{X}(M)$ generates a flow in M (§5.3). Here we are interested in the flow generated by a left-invariant vector field.

Definition 5.51 A curve $\phi : \mathbb{R} \to G$ is called a **one-parameter subgroup** of G if it satisfies the condition

$$\phi(t)\phi(s) = \phi(t + s). \tag{5.102}$$

It is easy to see that $\phi(0) = e$ and $\phi(t)^{-1} = \phi(-t)$. Note that the curve ϕ thus defined is a homomorphism from $\mathbb{R}$ to G. Although G may be non-Abelian, a one-parameter subgroup *is* an Abelian subgroup: $\phi(t)\phi(s) = \phi(t + s) = \phi(s + t) = \phi(s)\phi(t)$.

Given a one-parameter subgroup $\phi : \mathbb{R} \to G$, there exists a vector field X, such that

$$\frac{\mathrm{d}\phi^{\mu}(t)}{\mathrm{d}t} = X^{\mu}(\phi(t)). \tag{5.103}$$

We now show that the vector field X is left invariant. First note that the vector field $\mathrm{d}/\mathrm{d}t$ is left invariant on $\mathbb{R}$, see example 5.49(a). Thus we have

$$(L_t)_* \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t. \tag{5.104}$$

Next, we apply the induced map $\phi_* : T_t\mathbb{R} \to T_{\phi(t)}G$ on the vectors $\mathrm{d}/\mathrm{d}t|_0$ and $\mathrm{d}/\mathrm{d}t|_t$,

$$\phi_* \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}\phi^{\mu}(t)}{\mathrm{d}t}\right|_0 \left.\frac{\partial}{\partial g^{\mu}}\right|_e = X|_e \tag{5.105a}$$

$$\phi_* \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t = \left.\frac{\mathrm{d}\phi^{\mu}(t)}{\mathrm{d}t}\right|_t \left.\frac{\partial}{\partial g^{\mu}}\right|_g = X|_g \tag{5.105b}$$

where we put $\phi(t) = g$. From (5.104) and (5.105b), we have

$$(\phi L_t)_* \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 = \phi_* L_{t*} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 = X|_g. \tag{5.106a}$$

It follows from the commutativity $\phi L_t = L_g \phi$ that $\phi_* L_{t*} = L_{g*}\phi_*$. Then (5.106$a$) becomes

$$\phi_* L_{t*} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 = L_{g*}\phi_* \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 = L_{g*}X|_e. \tag{5.106b}$$

From (5.106), we conclude that

$$L_{g*}X|_e = X|_g. \tag{5.107}$$

Thus, given a flow $\phi(t)$, there exists an associated left-invariant vector field $X \in \mathfrak{g}$.

Conversely, a left-invariant vector field X defines a one-parameter group of transformations $\sigma(t, g)$ such that $\mathrm{d}\sigma(t, g)/\mathrm{d}t = X$ and $\sigma(0, g) = g$. If we define $\phi : \mathbb{R} \to G$ by $\phi(t) \equiv \sigma(t, e)$, the curve $\phi(t)$ becomes a one-parameter subgroup of G. To prove this, we have to show $\phi(s + t) = \phi(s)\phi(t)$. By definition, σ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\sigma(t, \sigma(s, e)) = X(\sigma(t, \sigma(s, e))). \tag{5.108}$$

[We have omitted the coordinate indices for notational simplicity. If readers feel uneasy, they may supplement the indices as in (5.103).] If the parameter s is fixed, $\bar{\sigma}(t, \phi(s)) \equiv \phi(s)\cdot\phi(t)$ is a curve $\mathbb{R} \to G$ at $\phi(s)\cdot\phi(0) = \phi(s)$. Clearly σ and $\bar{\sigma}$ satisfy the same initial condition,

$$\sigma(0, \sigma(s, e)) = \bar{\sigma}(0, \phi(s)) = \phi(s). \tag{5.109}$$

$\bar{\sigma}$ also satisfies the same differential equation as σ,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\,\bar{\sigma}(t, \phi(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\,\phi(s)\phi(t) = (L_{\phi(s)})_* \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \\ &= (L_{\phi(s)})_* X(\phi(t)) \\ &= X(\phi(s)\phi(t)) \qquad \text{(left invariance)} \\ &= X(\bar{\sigma}(t, \phi(s))). \end{aligned} \tag{5.110}$$

From the uniqueness theorem of ODE, we conclude that $\sigma(t + s, e) = \sigma(t, \sigma(s, e)) = \bar{\sigma}(t, \phi(s)) = \phi(t)\phi(s)$, that is,

$$\phi(s + t) = \phi(s)\phi(t). \tag{5.111}$$

We have found that there is a one-to-one correspondence between a one-parameter subgroup of G and a left-invariant vector field. This correspondence becomes manifest if we define the exponential map as follows.

Definition 5.52 Let G be a Lie group and $V \in T_eG$. The exponential map $\exp : T_eG \to G$ is defined by

$$\exp V \equiv \phi_V(1) \tag{5.112}$$

where ϕ_V is a one-parameter subgroup of G generated by the left-invariant vector field $X_V|_g = L_{g*}V$.

Proposition 5.53 Let $V \in T_eG$ and let $t \in \mathbb{R}$. Then

$$\exp(tV) = \phi_V(t) \tag{5.113}$$

where $\phi_V(t)$ is a one-parameter subgroup generated by $X_V = L_{g*}V$.

Proof: Let $a \neq 0$ be a constant. Then $\phi_V(at)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_V(at)\bigg|_{t=0} = a\frac{\mathrm{d}}{\mathrm{d}t}\phi_V(t)\bigg|_{t=0} = aV$$

which shows that $\phi_V(at)$ is a one-parameter subgroup generated by $L_{g*}aV$. The left-invariant vector field $L_{g*}aV$ also generates $\phi_{aV}(t)$ and, from the uniqueness of the solution, we find that $\phi_V(at) = \phi_{aV}(t)$. From definition 5.52, we have

$$\exp(aV) = \phi_{aV}(1) = \phi_V(a).$$

The proof is completed if a is replaced by t. ■

For a matrix group, the exponential map is given by the exponential of a matrix. Take $G = \mathrm{GL}(n, \mathbb{R})$ and $A \in \mathfrak{gl}(n, \mathbb{R})$. Let us define a one-parameter subgroup $\phi_A : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$ by

$$\phi_A(t) = \exp(tA) = \mathbb{1} + tA + \ldots + t^nA^n/n! + \ldots. \tag{5.114}$$

In fact, $\phi_A(t) \in \mathrm{GL}(n, \mathbb{R})$ since $[\phi_A(t)]^{-1} = \phi_A(-t)$ exists. It is also easy to see $\phi_A(t)\phi_A(s) = \phi_A(t + s)$. Now the exponential map is given by

$$\phi_A(1) = \exp(A) = \mathbb{1} + A + A^2/2! + \ldots. \tag{5.115}$$

The curve $g\exp(tA)$ is a flow through $g \in G$. We find that

$$\frac{\mathrm{d}}{\mathrm{d}t}g\exp(tA)|_{t=0} = L_{g*}A = X_A|_g$$

where X_A is a left-invariant vector field generated by A. From (5.100), we find, for a matrix group G, that

$$L_{g*}A = X_A|_g = gA. \tag{5.116}$$

The curve $g\exp(tA)$ defines a map $\sigma_t : G \to G$ by $\sigma_t(g) \equiv g\exp(tA)$ which is also expressed as a right translation,

$$\sigma_t = R_{\exp(tA)}. \tag{5.117}$$

5.6.4 *Frames and structure equation*

Let the set of n vectors $\{V_1, V_2, \ldots, V_n\}$ be a basis of T_eG where $n = \dim G$. [We assume throughout this book that n is finite.] The basis defines the set of n linearly independent left-invariant vector fields $\{X_1, \ldots, X_n\}$ at each point g in G by $X_\mu|_g = L_{g*}V_\mu$. Note that the set $\{X_\mu\}$ is a frame of basis defined throughout G. Since $[X_\mu, X_\nu]|_g$ is again an element of $\mathfrak{g}$ at g, it can be expanded in terms of $\{X_\mu\}$ as

$$[X_\mu, X_\nu] = c_{\mu\nu}{}^\lambda X_\lambda \tag{5.118}$$

where $c_{\mu\nu}{}^\lambda$ are called the **structure constants** of the Lie group G. If G is a matrix group, the LHS of (5.118) at $g = e$ is precisely the commutator of matrices V_μ and V_ν; see (5.101). We show that the $c_{\mu\nu}{}^\lambda$ are indeed constants independent of g. Let $c_{\mu\nu}{}^\lambda(e)$ be the structure constants at the unit element. If L_{g*} is applied to the Lie bracket $[X_\mu, X_\nu]|_e = c_{\mu\nu}{}^\lambda(e)X_\lambda|_e$, we have

$$[X_\mu, X_\nu]|_g = c_{\mu\nu}{}^\lambda(e)X_\lambda|_g$$

which shows the g-independence of the structure constants. In a sense, the structure constants determine a Lie group completely (Lie's theorem).

Exercise 5.54 Show that the structure constants satisfy

(*a*) *Skew-symmetry*

$$c_{\mu\nu}{}^\lambda = -c_{\nu\mu}{}^\lambda \tag{5.119}$$

(*b*) *Jacobi identity*

$$c_{\mu\nu}{}^\tau c_{\tau\rho}{}^\lambda + c_{\rho\mu}{}^\tau c_{\tau\nu}{}^\lambda + c_{\nu\rho}{}^\tau c_{\tau\mu}{}^\lambda = 0. \tag{5.120}$$

Let us introduce a dual basis to $\{X_\mu\}$ and denote it by $\{\theta^\mu\}$; $\langle\theta^\mu, X_\nu\rangle = \delta^\mu{}_\nu$. $\{\theta^\mu\}$ is a basis for the left-invariant one-forms. We will show that the dual basis satisfies **Maurer–Cartan's structure equation**,

$$\mathrm{d}\theta^\mu = -\tfrac{1}{2}c_{\nu\lambda}{}^\mu\theta^\nu \wedge \theta^\lambda. \tag{5.121}$$

This can be seen by making use of (5.70):

$$\begin{aligned}\mathrm{d}\theta^\mu(X_\nu, X_\lambda) &= X_\nu[\theta^\mu(X_\lambda)] - X_\lambda[\theta^\mu(X_\nu)] - \theta^\mu([X_\nu, X_\lambda]) \\ &= X_\nu[\delta^\mu_\lambda] - X_\lambda[\delta^\mu_\nu] - \theta^\mu(c_{\nu\lambda}{}^\kappa X_\kappa) = -c_{\nu\lambda}{}^\mu\end{aligned}$$

which proves (5.121).

We define a Lie-algebra-valued one-form $\theta : T_gG \to T_eG$ by

$$\theta : X \mapsto (L_{g^{-1}})_*X = (L_g)_*^{-1}X \qquad X \in T_gG. \tag{5.122}$$

θ is called the **canonical one-form** or **Maurer–Cartan form** on G.

Theorem 5.55

(*a*) The canonical one-form θ is expanded as

$$\theta = V_\mu \otimes \theta^\mu \tag{5.123}$$

where $\{V_\mu\}$ is the basis of T_eG and $\{\theta^\mu\}$ the dual basis of T_g^*G.

(*b*) The canonical one-form θ satisfies

$$d\theta + \tfrac{1}{2}[\theta \wedge \theta] = 0 \tag{5.124}$$

where $d\theta \equiv V_\mu \otimes d\theta^\mu$ and

$$[\theta \wedge \theta] \equiv [V_\mu, V_\nu] \otimes \theta^\mu \wedge \theta^\nu. \tag{5.125}$$

Proof: (*a*) Take any vector $Y = Y^\mu X_\mu \in T_g G$, where $\{X_\mu\}$ is the set of frame vectors generated by $\{V_\mu\}$; $X_\mu|_g = L_{g*} V_\mu$. From (5.122), we find

$$\theta(Y) = Y^\mu \theta(X_\mu) = Y^\mu L_{g*}{}^{-1}[L_{g*} V_\mu] = Y^\mu V_\mu.$$

On the other hand

$$(V_\mu \otimes \theta^\mu)(Y) = Y^\nu V_\mu \theta^\mu(X_\nu) = Y^\nu V_\mu \delta^\mu_\nu = Y^\mu V_\mu.$$

Since Y is arbitrary, we have $\theta = V_\mu \otimes \theta^\mu$.

(*b*) We use the Maurer–Cartan structure equation (5.121):

$$d\theta + \tfrac{1}{2}[\theta \wedge \theta] = -\tfrac{1}{2} V_\mu \otimes c_{\nu\lambda}{}^\mu \theta^\nu \wedge \theta^\lambda + \tfrac{1}{2} c_{\nu\lambda}{}^\mu V_\mu \otimes \theta^\nu \wedge \theta^\lambda = 0$$

where the $c_{\nu\lambda}{}^\mu$ are the structure constants of G. ■

5.7 The action of Lie groups on manifolds

In physics, a Lie group often appears as the set of transformations acting on a manifold. For example, SO(3) is the group of rotations in $\mathbb{R}^3$, while the Poincaré group is the set of transformations acting on the Minkowski spacetime. To study more general cases, we abstract the action of a Lie group G on a manifold M. We have already encountered this interaction between a group and geometry. In §5.3, we defined a flow in a manifold M as a map $\sigma : \mathbb{R} \times M \to M$, in which $\mathbb{R}$ acts as an additive group. We abstract this idea as follows.

5.7.1 Definitions

Definition 5.56 Let G be a Lie group and M be a manifold. The **action** of G on M is a differentiable map $\sigma : G \times M \to M$ which satisfies the conditions

(i) $\sigma(e, p) = p$ for any $p \in M$ (5.126*a*)

(ii) $\sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p)$. (5.126*b*)

[*Remark:* We often use the notation gp instead of $\sigma(g, p)$. The second condition in this notation is $g_1(g_2 p) = (g_1 g_2)p$.]

Example 5.57

(*a*) A flow is an action of $\mathbb{R}$ on a manifold M. If a flow is periodic with a period T, it may be regarded as an action of U(1) or SO(2) on

M. Given a periodic flow $\sigma(t, x)$ with period T, we construct a new action $\bar{\sigma}(\exp(2\pi i t/T), x) \equiv \sigma(t, x)$ whose group G is U(1).

(b) Let $M \in \mathrm{GL}(n, \mathbb{R})$ and let $x \in \mathbb{R}^n$. The action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ is defined by the usual matrix action on a vector,

$$\sigma(M, x) = M \cdot x. \tag{5.127}$$

The action of the subgroups of $\mathrm{GL}(n, \mathbb{R})$ are defined similarly. They may also act on a smaller space. For example, $\mathrm{O}(n)$ acts on $S^{n-1}(r)$, an $(n-1)$-sphere of radius r,

$$\sigma : \mathrm{O}(n) \times S^{n-1}(r) \to S^{n-1}(r). \tag{5.128}$$

(c) It is known that $\mathrm{SL}(2, \mathbb{C})$ acts on a four-dimensional Minkowski space M_4 in a special manner. For $x = (x^0, x^1, x^2, x^3) \in M_4$, define a Hermitian matrix,

$$X(x) \equiv x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{5.129}$$

where $\sigma_\mu = (I, \sigma_1, \sigma_2, \sigma_3)$, σ_i $(1 \leqslant i \leqslant 3)$ being the Pauli matrices. Conversely, given a Hermitian matrix X, we can define a unique vector $(x^\mu) \in M_4$ by

$$x^\mu = \tfrac{1}{2}\mathrm{tr}(\sigma_\mu X) \tag{5.130}$$

where tr is over the 2×2 matrix indices. Thus there is an isomorphism between M_4 and the set of 2×2 Hermitian matrices. It is interesting to note that $\det X(x) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = -X^t \eta X = -$(Minkowski norm)^2. Accordingly

$$\det X(x) \begin{cases} > 0 & \text{if } x \text{ is a timelike vector} \\ = 0 & \text{if } x \text{ is on the light cone} \\ < 0 & \text{if } x \text{ is a spacelike vector.} \end{cases}$$

Take $A \in \mathrm{SL}(2, \mathbb{C})$ and define an action of $\mathrm{SL}(2, \mathbb{C})$ on M_4 by

$$\sigma(A, x) \equiv AX(x)A^\dagger. \tag{5.131}$$

The reader should verify that this action in fact satisfies the axioms of definition 5.56. The action of $\mathrm{SL}(2, \mathbb{C})$ on M_4 represents the Lorentz transformation $\mathrm{O}(1, 3)$. First we note that the action preserves the Minkowski norm,

$$\det \sigma(A, x) = \det AX(x)A^\dagger = \det X(x)$$

since $\det A = \det A^\dagger = 1$. Moreover, there is a homomorphism $\varphi : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{O}(1, 3)$ since

$$A(BXB^\dagger)A^\dagger = (AB)X(AB)^\dagger.$$

However this homomorphism cannot be one-to-one, since $A \in \mathrm{SL}(2, \mathbb{C})$ and $-A$ give the same element of $\mathrm{O}(1, 3)$; see (5.131). We verify (exercise 5.58) that the following matrix is an explicit form of a rotation about the unit vector $\hat{\boldsymbol{n}}$ by an angle θ,

$$A = \exp\left(-\mathrm{i}\,\frac{\theta}{2}\,(\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma})\right) = \cos\frac{\theta}{2}\,\mathbb{1} - \mathrm{i}(\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma})\sin\frac{\theta}{2}. \qquad (5.132a)$$

The appearance of $\theta/2$ ensures that the homomorphism between $\mathrm{SL}(2, \mathbb{C})$ and the $\mathrm{O}(3)$ subgroup of $\mathrm{O}(1, 3)$ is indeed two-to-one. In fact, rotations about $\hat{\boldsymbol{n}}$ by θ and by $2\pi + \theta$ should be the same $\mathrm{O}(3)$ rotation, but $A(2\pi + \theta) = -A(\theta)$ in $\mathrm{SL}(2, \mathbb{C})$. This leads to the existence of spinors. [See Misner *et al* (1973) and Wald (1984).] A boost along the direction $\hat{\boldsymbol{n}}$ with the velocity $v = \tanh\alpha$ is given by

$$A = \exp\left(\frac{\alpha}{2}\,(\boldsymbol{n}\cdot\boldsymbol{\sigma})\right) = \cosh\frac{\alpha}{2}\,\mathbb{1} + (\boldsymbol{n}\cdot\boldsymbol{\sigma})\sinh\frac{\alpha}{2}. \qquad (5.132b)$$

We show that φ maps $\mathrm{SL}(2, \mathbb{C})$ onto the proper orthochronous Lorentz group $\mathrm{O}_+^{\uparrow}(1, 3) = \{\Lambda \in \mathrm{O}(1, 3) | \det\Lambda = +1, \Lambda_{00} > 0\}$. Take any

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

and suppose $x^\mu = (1, 0, 0, 0)$ is mapped to x'^μ. If we write $\varphi(A) = \Lambda$, we have

$$\begin{aligned} x'^0 &= \tfrac{1}{2}\mathrm{tr}(AXA^\dagger) = \tfrac{1}{2}\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \\ &= \tfrac{1}{2}(|a|^2 + |b|^2 + |c|^2 + |d|^2) > 0 \end{aligned}$$

hence $\Lambda_{00} > 0$. To show $\det\Lambda = +1$ we note that any element of $\mathrm{SL}(2, \mathbb{C})$ may be written as

$$\begin{aligned} A &= \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\alpha} \end{pmatrix}\begin{pmatrix} \cos\beta & \sin\beta\,\mathrm{e}^{\mathrm{i}\gamma} \\ -\sin\beta\,\mathrm{e}^{-\mathrm{i}\gamma} & \cos\beta \end{pmatrix} B \\ &= \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\alpha/2} \end{pmatrix}^2 \begin{pmatrix} \cos(\beta/2) & \sin(\beta/2)\,\mathrm{e}^{\mathrm{i}\gamma} \\ -\sin(\beta/2)\,\mathrm{e}^{-\mathrm{i}\gamma} & \cos(\beta/2) \end{pmatrix}^2 B \\ &\equiv M^2N^2B_0{}^2 \end{aligned}$$

where $B \equiv B_0^2$ is a positive-definite matrix. This shows that $\varphi(A)$ is positive definite:

$$\det\varphi(A) = (\det\varphi(M))^2(\det\varphi(N))^2(\det\varphi(B_0))^2 > 0.$$

Now we have established that $\varphi(\mathrm{SL}(2, \mathbb{C})) \subset \mathrm{O}_+^{\uparrow}(1, 3)$. (5.132) shows that for any element of $\mathrm{O}_+^{\uparrow}(1, 3)$, there is a corresponding matrix $A \in \mathrm{SL}(2, \mathbb{C})$, hence φ is onto. Thus we have established that

$$\varphi(\mathrm{SL}(2, \mathbb{C})) = \mathrm{O}_+^\uparrow(1, 3). \tag{5.133}$$

It can be shown that SL(2, $\mathbb{C}$) is simply connected and is the universal covering group SPIN(1, 3) of $\mathrm{O}_+^\uparrow(1, 3)$, see §4.6.

Exercise 5.58 Verify by explicit calculations that

(a)

$$A = \begin{pmatrix} \exp(-\mathrm{i}\theta/2) & 0 \\ 0 & \exp(\mathrm{i}\theta/2) \end{pmatrix}$$

represents a rotation about the z axis by θ.

(b)

$$A = \begin{pmatrix} \cosh(\alpha/2) + \sinh(\alpha/2) & 0 \\ 0 & \cosh(\alpha/2) - \sinh(\alpha/2) \end{pmatrix}$$

represents a boost along the z axis with the velocity $v = \tanh \alpha$.

Definition 5.59 Let G be a Lie group that acts on a manifold M by $\sigma : G \times M \to M$. The action σ is said to be

(a) **transitive** if, for any p_1, $p_2 \in M$, there exists an element $g \in G$ such that $\sigma(g, p_1) = p_2$;

(b) **free** if every non-trivial element $g \neq e$ of G has no fixed points in M, that is if there exists an element $p \in M$ such that $\sigma(g, p) = p$, then g must be the unit element e;

(c) **effective** if the unit element $e \in G$ is the unique element that defines the trivial action on M, i.e., if $\sigma(g, p) = p$ for *all* $p \in M$, then g must be the unit element e.

Exercise 5.60 Show that the right translation $R : (a, g) \mapsto R_a g$ and left translation $L : (a, g) \mapsto L_a g$ of a Lie group are free and transitive.

5.7.2 Orbits and isotropy groups

Given a point $p \in M$, the action of G on p takes p to various points in M. The **orbit** of p under the action σ is the subset of M defined by

$$Gp = \{\sigma(g, p) | g \in G\}. \tag{5.134}$$

If the action of G on M is transitive, the orbit of any $p \in M$ is M itself. Clearly the action of G on any orbit Gp is transitive.

Example 5.61

(a) A flow σ generated by a vector field $X = -y\partial/\partial x + x\partial/\partial y$ is periodic with period 2π, see example 5.19. The action $\sigma : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(t, (x, y)) \to \sigma(t, (x, y))$ is not effective since $\sigma(2\pi n, (x, y)) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$. For the same reason, this flow is not free either. The orbit through $(x, y) \neq (0, 0)$ is a circle S^1 centred at the origin.

(*b*) The action of $O(n)$ on $\mathbb{R}^n$ is not transitive since if $|x| \neq |x'|$, no element of $O(n)$ takes x to x'. On the other hand, the action of $O(n)$ on S^{n-1} is obviously transitive. The orbit through x is the sphere S^{n-1} of radius $|x|$. Accordingly, given an action $\sigma : O(n) \times \mathbb{R}^3 \to \mathbb{R}^3$, the orbits divide $\mathbb{R}^3$ into mutually disjoint spheres of different radii. Introduce a relation $\sim$ by $x \sim y$ if $y = \sigma(g, x)$ for some $g \in G$. It is easily verified that $\sim$ is an equivalence relation. The equivalence class $[x]$ is an orbit through x. The coset space $\mathbb{R}^3/O(n)$ is $[0, \infty)$ since each equivalence class is parametrised by the radius.

Definition 5.62 Let G be a Lie group that acts on a manifold M. The **isotropy group** of $p \in M$ is a subgroup of G defined by

$$H(p) = \{g \in G | \sigma(g, p) = p\}. \tag{5.135}$$

$H(p)$ is also called the **little group** or **stabiliser** of p.

It is easy to see that $H(p)$ is indeed a subgroup. Let $g_1, g_2 \in H(p)$, then $g_1 g_2 \in H(p)$ since $\sigma(g_1 g_2, p) = \sigma(g_1, \sigma(g_2, p)) = \sigma(g_1, p) = p$. Clearly $e \in H(p)$ since $\sigma(e, p) = p$ by definition. If $g \in H(p)$, then $g^{-1} \in H(p)$ since $p = \sigma(e, p) = \sigma(g^{-1}g, p) = \sigma(g^{-1}, \sigma(g, p)) = \sigma(g^{-1}, p)$.

Exercise 5.63 Suppose a Lie group G acts on a manifold M freely. Show that $H(p) = \{e\}$ for any $p \in M$.

Theorem 5.64 Let G be a Lie group which acts on a manifold M. Then the isotropy group $H(p)$ for any $p \in M$ is a Lie subgroup.

Proof: For fixed $p \in M$, we define a map $\varphi_p : G \to M$ by $\varphi_p(g) \equiv gp$. Then $H(p)$ is the inverse image $\varphi_p^{-1}(p)$ of a *point* p, and hence a closed set. The group properties have been shown already. It follows from theorem 5.44 that $H(p)$ is a Lie subgroup. ■

For example, let $M = \mathbb{R}^3$ and $G = SO(3)$ and take a point $p = (0, 0, 1) \in \mathbb{R}^3$. $H(p)$ is the set of rotations about the z axis, which is isomorphic to $SO(2)$.

Let G be a Lie group and H any subgroup of G. The coset space G/H admits a differentiable structure and G/H becomes a manifold, called a **homogeneous space**. Note that $\dim G/H = \dim G - \dim H$. Let G be a Lie group which acts on a manifold M transitively and let $H(p)$ be an isotropy group of $p \in M$. $H(p)$ is a Lie subgroup and the coset space $G/H(p)$ is a homogeneous space. In fact, if G, $H(p)$ and M satisfy certain technical requirements (for example, $G/H(p)$ be compact) it can be shown that $G/H(p)$ is homeomorphic to M, see example 5.65.

Example 5.65

(*a*) Let $G = SO(3)$ be a group acting on $\mathbb{R}^3$ and $H = SO(2)$ be the isotropy group of $x \in \mathbb{R}^3$. SO(3) acts on S^2 transitively and we have $\mathrm{SO}(3)/\mathrm{SO}(2) \cong S^2$. What is the intuitive picture of this? Let $g' = gh$ where $g, g' \in G$ and $h \in H$. Since H is the set of rotations in a plane, g and g' must be rotations about the common axis. Then the equivalence class $[g]$ is specified by the polar angles (θ, ϕ). Thus we again find that $G/H = S^2$. Since SO(2) is not a normal subgroup of SO(3), S^2 does not admit a group structure.

It is easy to generalise this result to higher-dimensional rotation groups and we have the useful result

$$\mathrm{SO}(n+1)/\mathrm{SO}(n) = S^n. \tag{5.136a}$$

$\mathrm{O}(n+1)$ also acts on S^n transitively and we have

$$\mathrm{O}(n+1)/\mathrm{O}(n) = S^n. \tag{5.136b}$$

Similar relations hold for $\mathrm{U}(n)$ and $\mathrm{SU}(n)$:

$$\mathrm{U}(n+1)/\mathrm{U}(n) = \mathrm{SU}(n+1)/\mathrm{SU}(n) = S^{2n+1}. \tag{5.137}$$

(b) The group $\mathrm{O}(n+1)$ acts on $\mathbb{R}P^n$ transitively from the left. Note first that $\mathrm{O}(n+1)$ acts on $\mathbb{R}^{n+1}$ in the usual manner and preserves the equivalence relation employed to define $\mathbb{R}P^n$ (see example 5.6). In fact, take $x, x' \in \mathbb{R}^{n+1}$ and $g \in \mathrm{O}(n+1)$. If $x \sim x'$ (that is if $x' = ax$ for some $a \in \mathbb{R} - \{0\}$), then it follows that $gx \sim gx'$ ($gx' = agx$). Accordingly this action of $\mathrm{O}(n+1)$ on $\mathbb{R}^{n+1}$ induces the natural action of $\mathrm{O}(n+1)$ on $\mathbb{R}P^n$. Clearly this action is transitive on $\mathbb{R}\mathrm{P}^n$. (Look at two representatives with the same norm.) If we take a point p in $\mathbb{R}P^n$, which corresponds to a point $(1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$, the isotropy group $H(p)$ is

$$H(p) = \begin{pmatrix} \pm 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & & & \\ \vdots & & \mathrm{O}(n) & & \\ 0 & & & & \end{pmatrix} = \mathrm{O}(1) \times \mathrm{O}(n) \tag{5.138}$$

where $\mathrm{O}(1)$ is the set $\{-1, +1\} = \mathbb{Z}_2$. Now we find that

$$\mathrm{O}(n+1)/[\mathrm{O}(1) \times \mathrm{O}(n)] \cong S^n/\mathbb{Z}_2 \cong \mathbb{R}P^n. \tag{5.139}$$

(c) The above result is easily generalised to the Grassmannian manifolds: $G_{k,n}(\mathbb{R}) = \mathrm{O}(n)/[\mathrm{O}(k) \times \mathrm{O}(n-k)]$. We first show that $\mathrm{O}(n)$ acts on $G_{k,n}(\mathbb{R})$ transitively. Let A be an element of $G_{k,n}(\mathbb{R})$, then A is a k-dimensional plane in $\mathbb{R}^n$. Define an $n \times n$ matrix P_A which projects a vector $v \in \mathbb{R}^n$ to the plane A. Let us introduce an orthonormal basis $\{e_1, \ldots, e_n\}$ in $\mathbb{R}^n$ and another orthonormal basis $\{f_1, \ldots, f_k\}$ in the plane A, where the orthonormality is defined with respect to the Euclidean metric in $\mathbb{R}^n$. In terms of $\{e_i\}$, f_a is expanded as $f_a = \Sigma_i f_{ai} e_i$ and the projected vector is

$$
\begin{aligned}
P_A v &= (vf_1)f_1 + \ldots + (vf_k)f_k \\
&= \sum_{i,j} (v_i f_{1i} f_{1j} + \ldots + v_i f_{ki} f_{kj}) e_j = \sum_{i,a,j} v_i f_{ai} f_{aj} e_j .
\end{aligned}
$$

Thus P_A is represented by a matrix

$$
(P_A)_{ij} = \sum f_{ai} f_{aj} . \tag{5.140}
$$

Note that $P_A^2 = P_A$, $P_A^t = P_A$ and $\operatorname{tr} P_A = k$. [The last relation holds since it is always possible to choose a coordinate system such that

$$
P_A = \operatorname{diag}(\underbrace{1, 1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k}).
$$

This guarantees that A is indeed a k-dimensional plane.] Conversely any matrix P that satisfies these three conditions determines a unique k-dimensional plane in $\mathbb{R}^n$, that is a unique element of $G_{k,n}(\mathbb{R})$.

We now show that $O(n)$ acts on $G_{k,n}(\mathbb{R})$ transitively. Take $A \in G_{k,n}(\mathbb{R})$ and $g \in O(n)$ and construct $P_B \equiv gP_A g^{-1}$. The matrix P_B determines an element $B \in G_{k,n}(\mathbb{R})$ since $P_B^2 = P_B$, $P_B^t = P_B$ and $\operatorname{tr} P_B = k$. Let us denote this action by $B = \sigma(g, A)$. Clearly this action is transitive since given a standard k-dimensional basis of A, $\{f_1, \ldots, f_k\}$ for example, any k-dimensional basis $\{\tilde{f}_1, \ldots, \tilde{f}_k\}$ can be reached by an action of $O(n)$ on this basis.

Let us take a special plane C_0 which is spanned by the standard basis $\{f_1, \ldots, f_k\}$. Then an element of the isotropy group $H(C_0)$ is of the form

$$
M = \begin{array}{cc} k & n-k \\ \multicolumn{2}{c}{\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}} \end{array} \begin{array}{l} \\ k \\ n-k \end{array} \tag{5.141}
$$

where $g_1 \in O(k)$. Since $M \in O(n)$, an $(n-k) \times (n-k)$ matrix g_2 must be an element of $O(n-k)$. Thus the isotropy group is isomorphic to $O(k) \times O(n-k)$. Finally we verified that

$$
G_{k,n}(\mathbb{R}) \cong O(n)/[O(k) \times O(n-k)]. \tag{5.142}
$$

The dimension of $G_{k,n}(\mathbb{R})$ is obtained from the general formula as

$$
\begin{aligned}
\dim G_{k,n}(\mathbb{R}) &= \dim O(n) - \dim[O(k) \times O(n-k)] \\
&= \tfrac{1}{2}n(n-1) - [\tfrac{1}{2}k(k-1) + \tfrac{1}{2}(n-k)(n-k-1)] \\
&= k(n-k)
\end{aligned} \tag{5.143}
$$

in agreement with the result of example 5.7. (5.142) also shows that the Grassmannian manifold is compact.

5.7.3 Induced vector fields

Let G be a Lie group which acts on M as $(g, x) \mapsto gx$. A left-invariant vector field X_V generated by $V \in T_eG$ naturally induces a vector field in M. Define a flow in M by

$$\sigma(t, x) = \exp(tV)x. \tag{5.144}$$

$\sigma(t, x)$ is a one-parameter group of transformations and define a vector field called the **induced vector field** denoted by $V^\#$,

$$V^\#|_x = \frac{\mathrm{d}}{\mathrm{d}t}\exp(tV)x|_{t=0}. \tag{5.145}$$

Thus we have obtained a map $\# : T_eG \to \mathcal{X}(M)$ given by $V \mapsto V^\#$.

Exercise 5.66 The Lie group $G = \mathrm{SO}(2)$ acts on $M = \mathbb{R}^2$ in the usual way. Let

$$V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

be an element of $\mathfrak{so}(2)$.

(*a*) Show that

$$\exp(tV) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and find the induced flow through

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

(*b*) Show that $V^\#|_{\boldsymbol{x}} = -y\partial/\partial x + x\partial/\partial y$.

Example 5.67 Let us take $G = \mathrm{SO}(3)$ and $M = \mathbb{R}^3$. The basis vectors of T_eG are generated by rotations about the x, y and z axes. We denote them by X_x, X_y and X_z, respectively (see example 5.50),

$$X_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad X_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$X_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Repeating a similar analysis to that above, we obtain the corresponding induced vectors,

$$X_x^\# = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \qquad X_y^\# = -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}$$

$$X_z^\# = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

5.7.4 The adjoint representation

A Lie group G acts on G itself in a special way.

Definition 5.68 Take any $a \in G$ and define a homomorphism $\mathrm{ad}_a : G \to G$ by the conjugation,

$$\mathrm{ad}_a : g \mapsto aga^{-1}. \tag{5.146}$$

This homomorphism is called the **adjoint representation** of G.

Exercise 5.69 Show that ad_a is a homomorphism. Define a map $\sigma : G \times G \to G$ by $\sigma(a, g) \equiv \mathrm{ad}_a g$. Show that $\sigma(a, g)$ is an action of G on itself.

Noting that $\mathrm{ad}_a e = e$, we restrict the induced map $\mathrm{ad}_{a*} : T_g G \to T_{\mathrm{ad}_a g} G$ to $g = e$,

$$\mathrm{Ad}_a : T_e G \to T_e G \tag{5.147}$$

where $\mathrm{Ad}_a \equiv \mathrm{ad}_{a*}|_{T_eG}$. If we identify T_eG with the Lie algebra $\mathfrak{g}$, we have obtained a map $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ called the **adjoint map** of G. Since $\mathrm{ad}_{a*}\mathrm{ad}_{b*} = \mathrm{ad}_{ab*}$, it follows that $\mathrm{Ad}_a \mathrm{Ad}_b = \mathrm{Ad}_{ab}$. Similarly, $\mathrm{Ad}_{a^{-1}} = {\mathrm{Ad}_a}^{-1}$ follows from $\mathrm{ad}_{a^{-1}*}\mathrm{ad}_{a*}|_{T_eG} = \mathrm{id}_{T_eG}$.

If G is a matrix group, the adjoint representation becomes a simple matrix operation. Let $g \in G$ and $X_V \in \mathfrak{g}$, and let $\sigma_V = \exp(tV)$ be a one-parameter subgroup generated by $V \in T_eG$. Then ad_g acting on $\sigma_V(t)$ yields $g\exp(tV)g^{-1} = \exp(tgVg^{-1})$. As for Ad_g we have $\mathrm{Ad}_g : V \mapsto gVg^{-1}$ since

$$\begin{aligned} \mathrm{Ad}_g V &= \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{ad}_g \exp(tV)]|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\exp(tgVg^{-1})|_{t=0} = gVg^{-1}. \end{aligned} \tag{5.148}$$

Problems 5

1 The Stiefel manifold $V(m, r)$ is the set of orthonormal vectors $\{\boldsymbol{e}_i\}$ $(1 \leqslant i \leqslant r)$ in $\mathbb{R}^n$ $(r \leqslant m)$. We may express an element A of $V(m, r)$ by an $n \times r$ matrix $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r)$. Show that $\mathrm{SO}(m)$ acts transitively on $V(m, r)$. Let

$$A_0 \equiv \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

be an element of $V(m, r)$. Show that the isotropy group of A_0 is $\mathrm{SO}(m-r)$. Verify that $V(m, r) = \mathrm{SO}(m)/\mathrm{SO}(m-r)$ and $\dim V(m, r) = [r(r-1)]/2 + r(m-r)$. [*Remark:* The Stiefel manifold is, in a sense, a generalisation of a sphere. Observe that $V(m, l) = S^{m-1}$.]

2 Let M be the Minkowski four-spacetime. Define the action of a linear operator $* : \Omega^r(M) \to \Omega^{4-r}(M)$ by

$r = 0$; $*1 = -\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$

$r = 1$; $*\mathrm{d}x^i = -\mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^0$ $\qquad *\mathrm{d}x^0 = -\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$

$r = 2$; $*\mathrm{d}x^i \wedge \mathrm{d}x^j = \mathrm{d}x^k \wedge \mathrm{d}x^0$ $\qquad *\mathrm{d}x^i \wedge \mathrm{d}x^0 = -\mathrm{d}x^j \wedge \mathrm{d}x^k$

$r = 3$; $*\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = -\mathrm{d}x^0$ $\qquad *\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^0 = -\mathrm{d}x^k$

$r = 4$; $*\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 = 1$

where (i, j, k) is an even permutation of $(1, 2, 3)$. The vector potential A and the electromagnetic tensor F are defined as in example 5.22. $J = J_\mu \mathrm{d}x^\mu \equiv \rho\, \mathrm{d}x^0 + j_k\, \mathrm{d}x^k$ is the current one-form.

(a) Write down the equation $\mathrm{d}{*F} = *J$ and verify that it reduces to two of the Maxwell equations $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho$ and $\boldsymbol{\nabla} \times \boldsymbol{B} - \partial\boldsymbol{E}/\partial t = \boldsymbol{j}$.

(b) Show that the identity $0 = \mathrm{d}(\mathrm{d}{*F}) = \mathrm{d}{*J}$ reduces to the charge conservation equation

$$\partial_\mu J^\mu = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0.$$

(c) Show that the Lorentz condition $\partial_\mu A^\mu = 0$ is expressed as $\mathrm{d}{*A} = 0$.

6

DE RHAM COHOMOLOGY GROUPS

In Chapter 3, we defined the homology groups of topological spaces. If a topological space M is a manifold, we may define the *dual* of the homology groups out of differential forms defined on M. The dual groups are called the de Rham cohomology groups. Besides physicists' familiarity with differential forms, cohomology groups have several advantages over homology groups.

We follow closely Nash and Sen (1983) and Flanders (1963). Bott and Tu (1982) contains more advanced topics.

6.1 Stokes' theorem

One of the main tools in the study of de Rham cohomology groups is Stokes' theorem with which most physicists are familiar from electromagnetism. Gauss' theorem and Stokes' theorem are treated in a unified manner here.

6.1.1 Preliminary consideration

Let us define an integration of an r-form over an r-simplex in a Euclidean space. To do this, we need first to define the **standard** $\boldsymbol{n}$**-simplex** $\bar{\sigma}_r = (p_0, p_1, \ldots, p_r)$ in $\mathbb{R}^r$ where

$$\begin{aligned} p_0 &= (0, 0, \ldots, 0) \\ p_1 &= (1, 0, \ldots, 0) \\ &\cdots\cdots\cdots\cdots \\ p_r &= (0, 0, \ldots, 1) \end{aligned}$$

see figure 6.1. If $\{x^\mu\}$ is a coordinate of $\mathbb{R}^r$, $\bar{\sigma}_r$ is given by

$$\bar{\sigma}_r = \left\{ (x^1, \ldots, x^r) \in \mathbb{R}^r | x^\mu \geqslant 0 \qquad \sum_{\mu=1}^{r} x^\mu \leqslant 1 \right\}. \tag{6.1}$$

An r-form ω (the volume element) in $\mathbb{R}^r$ is written as

$$\omega = a(x)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^r.$$

We define the integration of ω over $\bar{\sigma}_r$ by

$$\int_{\bar{\sigma}_r} \omega \equiv \int_{\bar{\sigma}_r} a(x)\, \mathrm{d}x^1\, \mathrm{d}x^2 \ldots \mathrm{d}x^r \tag{6.2}$$

where the RHS is the usual r-fold integration. For example, if $r = 2$ and $\omega = \mathrm{d}x \wedge \mathrm{d}y$, we have

$$\int_{\bar{\sigma}_2} \omega = \int_{\bar{\sigma}_2} \mathrm{d}x\, \mathrm{d}y = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y = \tfrac{1}{2}.$$

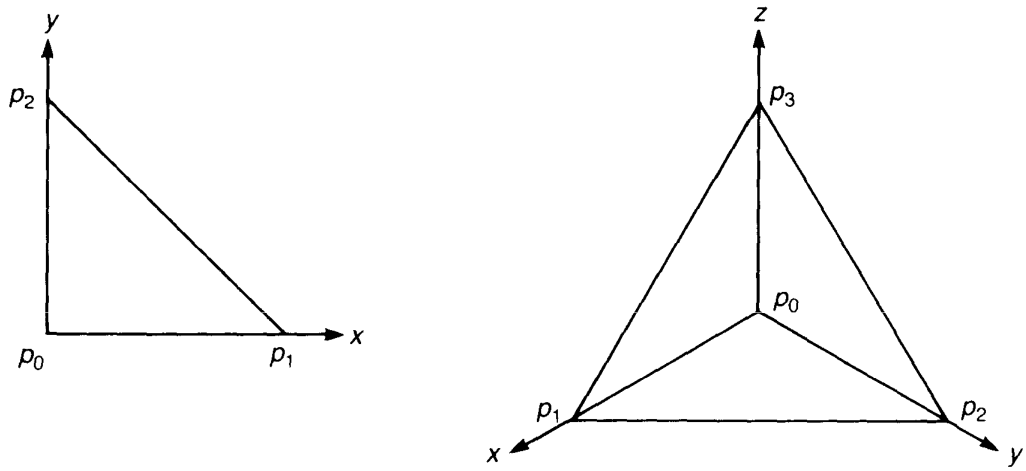


Figure 6.1 The standard 2-simplex (p_0, p_1, p_2) and the standard 3-simplex (p_0, p_1, p_2, p_3).

Next we define an r-chain, an r-cycle and an r-boundary in an m-dimensional manifold M. Let σ_r be an r-simplex in $\mathbb{R}^r$ and let $f : \sigma_r \to M$ be a smooth map. [To avoid the subtlety associated with the differentiability of f at the boundary of σ_r, f may be defined over an open subset U of $\mathbb{R}^r$, which contains σ_r.] Here we assume f is not required to have an inverse. For example, im f may be a point in M. We denote the image of σ_r in M by s_r and call it a **(singular) r-simplex** in M. These simplexes are called singular since they do not provide a triangulation of M and, moreover, *geometrical independence* of points makes no sense in a manifold (see §3.2). If $\{s_{r,i}\}$ is the set of r-simplexes in M, we define an **r-chain** in M by a formal sum of $\{s_{r,i}\}$ with $\mathbb{R}$-coefficients

$$c = \sum_i a_i s_{r,i} \qquad a_i \in \mathbb{R}. \tag{6.3}$$

In the following, we are concerned with $\mathbb{R}$-coefficients only and we omit the explicit quotation of $\mathbb{R}$. The r-chains in M form the **chain group** $C_r(M)$. Under $f : \sigma_r \to M$, the boundary $\partial\sigma_r$ is also mapped to a subset of M. Clearly, $\partial s_r \equiv f(\partial\sigma_r)$ is a set of $(r-1)$-simplexes in M and called the boundary of s_r. ∂s_r corresponds to the geometrical boundary of s_r

with an induced orientation defined in §3.3. We have a map

$$\partial : C_r(M) \to C_{r-1}(M). \tag{6.4}$$

The result of §3.3 tells us that ∂ is nilpotent; $\partial^2 = 0$.

Cycles and boundaries are defined in exactly the same way as in §3.3 (note, however, that $\mathbb{Z}$ is replaced by $\mathbb{R}$). If c_r is an ***r*-cycle**, $\partial c_r = 0$ while if c_r is an ***r*-boundary**, there exists an $(r+1)$-chain c_{r+1} such that $c_r = \partial c_{r+1}$. The **boundary group** $B_r(M)$ is the set of r-boundaries and the **cycle group** $Z_r(M)$ is the set of r-cycles. There are infinitely many singular simplexes which make up $C_r(M)$, $B_r(M)$ and $Z_r(M)$. It follows from $\partial^2 = 0$ that $Z_r(M) \supset B_r(M)$; cf theorem 3.15. The **singular homology group** is defined by

$$H_r(M) \equiv Z_r(M)/B_r(M). \tag{6.5}$$

With mild topological assumptions, the singular homology group is isomorphic to the corresponding simplicial homology group with $\mathbb{R}$-coefficients and we employ the same symbol to denote both of them.

Now we are ready to define an integration of an r-form ω over an r-chain in M. We first define an integration of ω on an r-simplex s_r of M by

$$\int_{s_r} \omega = \int_{\bar{\sigma}_r} f^*\omega \tag{6.6}$$

where $f : \bar{\sigma}_r \to M$ is a smooth map such that $s_r = f(\bar{\sigma}_r)$. Since $f^*\omega$ is an r-form in $\mathbb{R}^r$, the RHS is the usual r-fold integral. For a general r-chain $c = \Sigma a_i s_{r,i} \in C_r(M)$, we define

$$\int_c \omega = \sum a_i \int_{s_{r,i}} \omega. \tag{6.7}$$

6.1.2 Stokes' theorem

Theorem 6.1 Let $\omega \in \Omega^{r-1}(M)$ and $c \in C_r(M)$. Then

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega. \tag{6.8}$$

Proof: Since c is a linear combination of r-simplexes, it suffices to prove (6.8) for an r-simplex s_r in M. Let $f : \bar{\sigma}_r \to M$ be a map such that $f(\bar{\sigma}_r) = s_r$. Then

$$\int_{s_r} \mathrm{d}\omega = \int_{\bar{\sigma}_r} f^*(\mathrm{d}\omega) = \int_{\bar{\sigma}_r} \mathrm{d}(f^*\omega)$$

where (5.75) has been used. We also have

$$\int_{\partial s_r} \omega = \int_{\partial \bar{\sigma}_r} f^*\omega.$$

Note that $f^*\omega$ is an $(r-1)$-form in $\mathbb{R}^r$. Thus to prove Stokes' theorem

$$\int_{s_r} \mathrm{d}\omega = \int_{\partial s_r} \omega \tag{6.9a}$$

we may simply prove an alternative formula

$$\int_{\bar{\sigma}_r} \mathrm{d}\psi = \int_{\partial\bar{\sigma}_r} \psi \tag{6.9b}$$

for an $(r-1)$-form ψ in $\mathbb{R}^r$. The most general form of ψ is

$$\psi = \sum a_\mu(x)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{\mu-1} \wedge \mathrm{d}x^{\mu+1} \wedge \ldots \wedge \mathrm{d}x^r.$$

Since an integration is distributive, it suffices to prove (6.9b) for $\psi = a(x)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{r-1}$. We note that

$$\begin{aligned}\mathrm{d}\psi &= \frac{\partial a}{\partial x^r}\mathrm{d}x^r \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{r-1}\\ &= (-1)^{r-1}\frac{\partial a}{\partial x^r}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{r-1} \wedge \mathrm{d}x^r.\end{aligned}$$

By direct computation, we find, from (6.2), that

$$\begin{aligned}\int_{\bar{\sigma}_r} \mathrm{d}\psi &= (-1)^{r-1}\int_{\bar{\sigma}_r} \frac{\partial a}{\partial x^r}\,\mathrm{d}x^1 \ldots \mathrm{d}x^{r-1}\,\mathrm{d}x^r\\ &= (-1)^{r-1}\int_{x^\mu \geqslant 0, \sum_{\mu=1}^{r-1} x^\mu \leqslant 1} \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1} \int_0^{1-\sum_{\mu=1}^{r-1} x^\mu} \frac{\partial a}{\partial x^r}\,\mathrm{d}x^r\\ &= (-1)^{r-1}\int \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1}\Big[a\Big(x^1, \ldots, x^{r-1}, 1 - \sum x^\mu\Big)\\ &\quad - a(x^1, \ldots, x^{r-1}, 0)\Big].\end{aligned}$$

For the boundary of $\bar{\sigma}_r$, we have

$$\begin{aligned}\partial\bar{\sigma}_r = (p_1, p_2, \ldots, p_r) &- (p_0, p_2, \ldots, p_r)\\ &+ \ldots + (-1)^r(p_0, p_1, \ldots, p_{r-1}).\end{aligned}$$

Note that $\psi = a(x)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{r-1}$ vanishes when one of $x^1, \ldots, x^{r-1}$ is constant. Then it follows that

$$\int_{(p_0, p_2, \ldots, p_r)} \psi = 0$$

since $x^1 \equiv 0$ on $(p_0, p_2, \ldots, p_r)$. In fact most of the faces of $\partial\bar{\sigma}_r$ do not contribute to the RHS of (6.9b) and we are left with

$$\int_{\partial\bar{\sigma}_r} \psi = \int_{(p_1, p_2, \ldots, p_r)} \psi + (-1)^r \int_{(p_0, p_1, \ldots, p_{r-1})} \psi.$$

Since $(p_0, p_1, \ldots, p_{r-1})$ is the standard $(r-1)$-simplex $(x^\mu \geqslant 0, \Sigma_{\mu=1}^{r-1} x^\mu \leqslant 1)$, on which $x^r = 0$, the second term is

$$(-1)^r \int_{(p_0, p_1, \ldots, p_{r-1})} \psi = (-1)^r \int_{\bar{\sigma}_{r-1}} a(x^1, \ldots, x^{r-1}, 0)\, \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1}.$$

The first term is

$$\int_{(p_1, p_2, \ldots, p_r)} \psi = \int_{(p_1, \ldots, p_{r-1}, p_0)} a\left(x^1, \ldots, x^{r-1}, 1 - \sum_{\mu=1}^{r-1} x^\mu\right) \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1}$$

$$= (-1)^{r-1} \int_{\bar{\sigma}_{r-1}} a\left(x^1, \ldots, x^{r-1}, 1 - \sum_{\mu=1}^{r-1} x^\mu\right) \mathrm{d}x^1 \ldots \mathrm{d}x^{r-1}$$

where the integral domain $(p_1, \ldots, p_r)$ has been projected along x^r to the $(p_1, \ldots, p_{r-1}, p_0)$-plane, preserving the orientation. Collecting these results, we have (6.9*b*). ■ [The reader is advised to verify the proof above for $m = 3$ using figure 6.1.]

Exercise 6.2 Let $M = \mathbb{R}^3$ and $\omega = a\,\mathrm{d}x + b\,\mathrm{d}y + c\,\mathrm{d}z$. Show that Stokes' theorem is written as

$$\int_S \operatorname{curl} \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S} = \oint_C \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{s} \qquad \text{(Stokes' theorem)},$$

where $\boldsymbol{\omega} = (a, b, c)$ and C is the boundary of a surface S. Similarly for $\psi = \frac{1}{2}\psi_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, show that

$$\int_V \operatorname{div} \boldsymbol{\psi}\, \mathrm{d}V = \oint_S \boldsymbol{\psi} \cdot \mathrm{d}\boldsymbol{S} \qquad \text{(Gauss' theorem)}$$

where $\psi^\lambda = \varepsilon^{\lambda\mu\nu}\psi_{\mu\nu}$ and S is the boundary of a volume V.

6.2 de Rham cohomology groups

6.2.1 Definitions

Definition 6.3 Let M be an m-dimensional differentiable manifold. The set of closed r-forms is called the rth **cocycle group**, denoted by $Z^r(M)$. The set of exact r-forms is called the rth **coboundary group**, denoted by $B^r(M)$. These are vector spaces with $\mathbb{R}$-coefficients. Since $\mathrm{d}^2 = 0$, we find $Z^r(M) \supset B^r(M)$.

Exercise 6.4 Show that:
(*a*) if $\omega \in Z^r(M)$ and $\psi \in Z^s(M)$, then $\omega \wedge \psi \in Z^{r+s}(M)$,
(*b*) if $\omega \in Z^r(M)$ and $\psi \in B^s(M)$, then $\omega \wedge \psi \in B^{r+s}(M)$,
(*c*) if $\omega \in B^r(M)$ and $\psi \in B^s(M)$, then $\omega \wedge \psi \in B^{r+s}(M)$.

Definition 6.5 The rth **de Rham cohomology group** is defined by

$$H^r(M; \mathbb{R}) \equiv Z^r(M)/B^r(M). \qquad (6.10)$$

If $r \leqslant -1$ or $r \geqslant m+1$, $H^r(M; \mathbb{R})$ may be defined to be trivial. In the following, we omit the explicit quotation of $\mathbb{R}$-coefficients.

Let $\omega \in Z^r(M)$. Then $[\omega] \in H^r(M)$ is the equivalence class $\{\omega' \in Z^r(M) | \, \omega' = \omega + \mathrm{d}\psi, \ \psi \in \Omega^{r-1}(M)\}$. Two forms which differ by an exact form are called **cohomologous**. We will see later that $H^r(M)$ is isomorphic to $H_r(M)$. The following examples will clarify the idea of de Rham cohomology groups.

Example 6.6 When $r = 0$, $B^0(M)$ has no meaning since there is no (-1)-form. We define $\Omega^{-1}(M)$ to be empty, hence $B^0(M) = 0$. Then $H^0(M) = Z^0(M) = \{f \in \Omega^0(M) = \mathcal{F}(M) | \, \mathrm{d}f = 0\}$. If M is connected, the condition $\mathrm{d}f = 0$ is satisfied if and only if f is constant over M. Hence $H^0(M)$ is isomorphic to the vector space $\mathbb{R}$,

$$H^0(M) = \mathbb{R}. \tag{6.11}$$

If M has n connected components, $\mathrm{d}f = 0$ is satisfied if and only if f is constant on each connected component, hence it is specified by n real numbers,

$$H^0(M) \simeq \underbrace{\mathbb{R} \oplus \mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{n}. \tag{6.12}$$

Example 6.7 Let $M = \mathbb{R}$. From above, we have $H^0(\mathbb{R}) = \mathbb{R}$. Let us find $H^1(\mathbb{R})$ next. Let x be a coordinate of $\mathbb{R}$. Since $\dim \mathbb{R} = 1$, any one-form $\omega \in \Omega^1(\mathbb{R})$ is closed, $\mathrm{d}\omega = 0$. Let $\omega = f\,\mathrm{d}x$, where $f \in \mathcal{F}(\mathbb{R})$. Define a function $F(x)$ by

$$F(x) = \int_0^x f(s)\,\mathrm{d}s \in \mathcal{F}(\mathbb{R}) = \Omega^0(\mathbb{R}).$$

Since $\mathrm{d}F(x)/\mathrm{d}x = f(x)$, ω is an exact form,

$$\omega = f\,\mathrm{d}x = (\mathrm{d}F/\mathrm{d}x)\,\mathrm{d}x = \mathrm{d}F.$$

Thus any one-form is *closed* as well as *exact*. We have established

$$H^1(\mathbb{R}) = 0. \tag{6.13}$$

Example 6.8 Let $S^1 = \{\mathrm{e}^{\mathrm{i}\theta} | 0 \leqslant \theta < 2\pi\}$. Since S^1 is connected, we have $H^0(S^1) = \mathbb{R}$. We compute $H^1(S^1)$ next. Let $\omega = f(\theta)\,\mathrm{d}\theta \in \Omega^1(S^1)$. Is it possible to write $\omega = \mathrm{d}F$ for some $F \in \mathcal{F}(S^1)$? Let us repeat the analysis of the previous example. If $\omega = \mathrm{d}F$, then $F \in \mathcal{F}(S^1)$ must be given by

$$F(\theta) = \int_0^\theta f(\theta')\,\mathrm{d}\theta'.$$

For F to be defined uniquely on S^1, F must satisfy the periodicity $F(2\pi) = F(0)$ $(=0)$. Namely F must satisfy

$$F(2\pi) = \int_0^{2\pi} f(\theta')\,\mathrm{d}\theta' = 0.$$

If we define a map $\lambda : \Omega^1(S^1) \to \mathbb{R}$ by

$$\lambda : \omega = f\,\mathrm{d}\theta \mapsto \int_0^{2\pi} f(\theta')\,\mathrm{d}\theta' \tag{6.14}$$

then $B^1(S^1)$ is identified with $\ker\lambda$. Now we have (theorem 3.3),

$$H^1(S^1) = \Omega^1(S^1)/\ker\lambda = \mathrm{im}\,\lambda = \mathbb{R}. \tag{6.15}$$

This is also obtained from the following consideration. Let ω and ω' be closed forms that are not exact. Although $\omega - \omega'$ is not exact in general, we can show that there exists a number $a \in \mathbb{R}$ such that $\omega' - a\omega$ is exact. In fact, if we put

$$a = \int_0^{2\pi} \omega' \bigg/ \int_0^{2\pi} \omega$$

we have

$$\int_0^{2\pi} (\omega' - a\omega) = 0.$$

This shows that, given a closed form ω which is not exact, any closed form ω' is cohomologous to $a\omega$ for some $a \in \mathbb{R}$. Thus each cohomology class is specified by a real number a, hence $H^1(S^1) = \mathbb{R}$.

Exercise 6.9 Let $M = \mathbb{R}^2 - \{0\}$. Define a one-form ω by

$$\omega = \frac{-y}{x^2 + y^2}\mathrm{d}x + \frac{x}{x^2 + y^2}\mathrm{d}y. \tag{6.16}$$

(a) Show that ω is closed.

(b) Define a 'function' $F(x, y) = \tan^{-1}(y/x)$. Show that $\omega = \mathrm{d}F$. Is ω exact?

6.2.2 *Duality of $H_r(M)$ and $H^r(M)$; de Rham's theorem*

As the name itself suggests, the *co*homology group is a dual space of the homology group. The duality is provided by Stokes' theorem. We first define the inner product of an r-form and an r-chain in M. Let M be an m-dimensional manifold and let $C_r(M)$ be the chain group of M. Take $c \in C_r(M)$ and $\omega \in \Omega^r(M)$ where $1 \leqslant r \leqslant m$. Define an inner product $(\ ,\ ) : C_r(M) \times \Omega^r(M) \to \mathbb{R}$ by

$$c, \omega \mapsto (c, \omega) \equiv \int_c \omega. \tag{6.17}$$

Clearly, (c, ω) is linear in both c and ω and $(\ , \omega)$ may be regarded as a linear map acting on c and vice versa,

$$(c_1 + c_2, \omega) = \int_{c_1+c_2} \omega = \int_{c_1} \omega + \int_{c_2} \omega \tag{6.18a}$$

$$(c, \omega_1 + \omega_2) = \int_c (\omega_1 + \omega_2) = \int_c \omega_1 + \int_c \omega_2. \tag{6.18b}$$

Now Stokes' theorem takes a compact form:

$$(c, \mathrm{d}\omega) = (\partial c, \omega). \tag{6.19}$$

In this sense, the exterior derivative operator d is the adjoint of the boundary operator ∂ and vice versa.

Exercise 6.10 Let (i) $c \in B_r(M)$, $\omega \in Z^r(M)$ or (ii) $c \in Z_r(M)$, $\omega \in B^r(M)$. Show, in both cases, that $(c, \omega) = 0$.

The inner product (,) naturally induces an inner product Λ between the elements of $H_r(M)$ and $H^r(M)$. We now show that $H_r(M)$ is the dual of $H^r(M)$. Let $[c] \in H_r(M)$ and $[\omega] \in H^r(M)$ and define an inner product $\Lambda : H_r(M) \times H^r(M) \to \mathbb{R}$ by

$$\Lambda([c], [\omega]) \equiv (c, \omega) = \int_c \omega. \tag{6.20}$$

This is well defined since (6.20) is independent of the choice of the representatives. In fact, if we take $c + \partial c'$, $c' \in C_{r+1}(M)$, we have, from Stokes' theorem,

$$(c + \partial c', \omega) = (c, \omega) + (c', \mathrm{d}\omega) = (c, \omega)$$

where $\mathrm{d}\omega = 0$ has been used. Similarly, for $\omega + \mathrm{d}\psi$, $\psi \in \Omega^{r-1}(M)$,

$$(c, \omega + \mathrm{d}\psi) = (c, \omega) + (\partial c, \psi) = (c, \omega)$$

since $\partial c = 0$. Note that $\Lambda(\ , [\omega])$ is a linear map $H_r(M) \to \mathbb{R}$, and $\Lambda([c], \ )$ is a linear map $H^r(M) \to \mathbb{R}$. To prove the duality of $H_r(M)$ and $H^r(M)$, we have to show that $\Lambda(\ , [\omega])$ has the maximal rank, that is, $\dim H_r(M) = \dim H^r(M)$. We accept the following theorem due to de Rham without the proof which is highly non-trivial.

Theorem 6.11 **(de Rham's theorem)** If M is a compact manifold, $H_r(M)$ and $H^r(M)$ are finite-dimensional. Moreover the map

$$\Lambda : H_r(M) \times H^r(M) \to \mathbb{R}$$

is bilinear and non-degenerate. Thus $H^r(M)$ is the dual vector space of $H_r(M)$.

A **period** of a closed r-form ω over a cycle c is defined by $(c, \omega) = \int_c \omega$. Exercise 6.10 shows that the period vanishes if ω is exact or if c is a boundary. The following corollary is easily derived from de Rham's theorem.

Corollary 6.12 Let M be a compact manifold and let k be the rth Betti number (see §3.4). Let $c_1, c_2, \ldots, c_k$ be properly chosen elements of $Z_r(M)$ such that $[c_i] \neq [c_j]$.

(a) A closed r-form ψ is exact if and only if

$$\int_{c_i} \psi = 0 \; (1 \leqslant i \leqslant k). \tag{6.21}$$

(b) For any set of real numbers $b_1, b_2, \ldots, b_k$, there exists a closed r-form ω such that

$$\int_{c_i} \omega = b_i \; (1 \leqslant i \leqslant k). \tag{6.22}$$

Proof: (a) de Rham's theorem states that the bilinear form $\Lambda\,([c],[\omega])$ is non-degenerate. Hence if $\Lambda\,([c_i], \;)$ is regarded as a linear map acting on $H^r(M)$, the kernel consists of the trivial element, the cohomology class of exact forms. Accordingly, ψ is an exact form.

(b) de Rham's theorem ensures that corresponding to the homology basis $\{[c_i]\}$, we may choose the dual basis $\{[\omega_i]\}$ of $H^r(M)$ such that

$$\Lambda([c_i], [\omega_j]) = \int_{c_i} \omega_j = \delta_{ij}. \tag{6.23}$$

If we define $\omega \equiv \Sigma_{i=1}^k b_i \omega_i$, the closed r-form ω satisfies

$$\int_{c_i} \omega = b_i$$

as claimed. ■

For example, we observe the duality of the following groups.

(a) $H^0(M) = H_0(M) = \underbrace{\mathbb{R} \oplus \ldots \oplus \mathbb{R}}_{n}$

if M has n connected components;

(b) $H^1(S^1) = H_1(S^1) = \mathbb{R}$.

Since $H^r(M)$ is isomorphic to $H_r(M)$, we find

$$b^r(M) \equiv \dim H^r(M) = \dim H_r(M) = b_r(M) \tag{6.24}$$

where $b_r(M)$ is the Betti number of M. The Euler characteristic is now written as

$$\chi(M) = \sum_{r=1}^{m} (-1)^r b^r(M). \tag{6.25}$$

This is quite an interesting formula; the LHS is purely *topological* while the RHS is given by an *analytic* condition (note that $\mathrm{d}\omega = 0$ is a set of partial differential equations). We will frequently encounter this interplay between topology and analysis.

In summary, we have the chain complex $C(M)$ and the de Rham complex $\Omega^*(M)$,

$$\begin{aligned} &\leftarrow C_{r-1}(M) \xleftarrow{\partial_r} C_r(M) \xleftarrow{\partial_{r+1}} C_{r+1}(M) \leftarrow \\ &\rightarrow \Omega_{r-1}(\Omega) \xrightarrow{d_r} \Omega^r(M) \xrightarrow{d_{r+1}} \Omega_{r+1}(M) \rightarrow \end{aligned} \tag{6.26}$$

for which the rth homology group is defined by

$$H_r(M) = Z_r(M)/B_r(M) = \ker \partial_r / \operatorname{im} \partial_{r+1}$$

and the rth de Rham cohomology group is defined by

$$H^r(M) = Z^r(M)/B^r(M) = \ker d_{r+1}/\operatorname{im} d_r.$$

6.3 Poincaré's lemma

An exact form is always closed but the converse is not necessarily true. However, the following theorem provides the situation in which the converse is also true.

Theorem 6.13 (**Poincaré's lemma**) If a coordinate neighbourhood U of a manifold M is contractible to a point $p_0 \in M$, any closed r-form on U is also exact.

Proof: We assume U is *smoothly* contractible to p_0, that is, there exists a smooth map $F : U \times I \to U$ such that

$$F(x, 0) = x, \ F(x, 1) = p_0 \text{ for } x \in U.$$

Let us consider an r-form $\eta \in \Omega^r(U \times I)$,

$$\begin{aligned} \eta = {} & a_{i_1 \ldots i_r}(x, t)\, dx^{i_1} \wedge \ldots \wedge dx^{i_r} \\ & + b_{j_1 \ldots j_{r-1}}(x, t)\, dt \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_{r-1}} \end{aligned} \tag{6.27}$$

where x is the coordinate of U and t of I. Define a map $P : \Omega^r(U \times I) \to \Omega^{r-1}(U)$ by

$$P\eta \equiv \left(\int_0^1 ds\, b_{j_1 \ldots j_{r-1}}(x, s) \right) dx^{j_1} \wedge \ldots \wedge dx^{j_{r-1}}. \tag{6.28}$$

Next, define a map $f_t : U \to U \times I$ by $f_t(x) = (x, t)$. The pullback of the first term of (6.27) by f_t^* is an element of $\Omega^r(U)$,

$$f_t{}^*\eta = a_{i_1 \ldots i_r}(x, t)\, dx^{i_1} \wedge \ldots \wedge dx^{i_r} \in \Omega^r(U). \tag{6.29}$$

We now prove the following indentity,

$$d(P\eta) + P(d\eta) = f_1{}^*\eta - f_0{}^*\eta. \tag{6.30}$$

Each term of the LHS is calculated to be

$$\begin{aligned}
\mathrm{d}P\eta &= \mathrm{d}\left(\int_0^1 \mathrm{d}s\, b_{j_1 \ldots j_{r-1}}\right) \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_{r-1}} \\
&= \int_0^1 \mathrm{d}s(\partial b_{j_1 \ldots j_{r-1}}/\partial x^{j_r})\, \mathrm{d}x^{j_r} \wedge \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_{r-1}} \\
P\mathrm{d}\eta &= P\Big((\partial a_{i_1 \ldots i_r}/\partial x^{i_{r+1}})\, \mathrm{d}x^{i_{r+1}} \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r} \\
&\quad + (\partial a_{i_1 \ldots i_r}/\partial t)\, \mathrm{d}t \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r} \\
&\quad + (\partial b_{j_1 \ldots j_{r-1}}/\partial x^{j_r})\, \mathrm{d}x^{j_r} \wedge \mathrm{d}t \wedge \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_{r-1}}\Big) \\
&= \left(\int_0^1 \mathrm{d}s(\partial a_{i_1 \ldots i_r}/\partial s)\right) \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r} \\
&\quad - \left(\int_0^1 \mathrm{d}s(\partial b_{j_1 \ldots j_{r-1}}/\partial x^{j_r})\right) \mathrm{d}x^{j_r} \wedge \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_{r-1}}.
\end{aligned}$$

Collecting these results, we have

$$\begin{aligned}
\mathrm{d}P\eta + P\mathrm{d}\eta &= \left(\int_0^1 \mathrm{d}s(\partial a_{i_1 \ldots i_r}/\partial s)\right) \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r} \\
&= [a_{i_1 \ldots i_r}(x, 1) - a_{i_1 \ldots i_r}(x, 0)]\, \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r} \\
&= f_1{}^*\eta - f_0{}^*\eta.
\end{aligned}$$

Poincaré's lemma readily follows from (6.30). Let ω be a closed r-form on a contractible chart U. We will show that ω is written as an exact form,

$$\omega = \mathrm{d}(-PF^*\omega) \tag{6.31}$$

F being the smooth contraction map. In fact if η in (6.30) is replaced by $F^*\omega \in \Omega^r(U \times I)$ we have

$$\begin{aligned}
\mathrm{d}PF^*\omega + P\mathrm{d}F^*\omega &= f_1{}^*F^*\omega - f_0{}^*F^*\omega \\
&= (Ff_1)^*\omega - (Ff_0)^*\omega
\end{aligned} \tag{6.32}$$

where use has been made of the relation $(fg)^* = g^* \cdot f^*$. Clearly $Ff_1 : U \to U$ is a constant map $x \mapsto p_0$, hence $(Ff_1)^* = 0$. On the other hand, $Ff_0 = \mathrm{id}_U$, hence $(Ff_0)^* : \Omega^r(U) \to \Omega^r(U)$ is the identity map. Thus, the RHS of (6.32) is simply $-\omega$. The second term of the LHS vanishes since ω is closed; $\mathrm{d}F^*\omega = F^*\mathrm{d}\omega = 0$, where use has been made of (5.75). Finally, (6.32) becomes $\omega = -\mathrm{d}PF^*\omega$, which proves the theorem. ■

Any closed form is exact at least locally. The de Rham cohomology group is regarded as an obstruction to the *global* exactness of closed forms.

Example 6.14 Since $\mathbb{R}^n$ is contractible, we have

$$H^r(\mathbb{R}^n) = 0 \qquad 1 \leqslant r \leqslant n. \tag{6.33}$$

Note, however, that $H^0(\mathbb{R}^n) = \mathbb{R}$.

6.4 Structure of de Rham cohomology groups

de Rham cohomology groups exhibit quite an interesting structure that is very difficult or even impossible to appreciate with homology groups.

6.4.1 Poincaré duality

Let M be a compact m-dimensional manifold and let $\omega \in H^r(M)$ and $\eta \in H^{m-r}(M)$. Noting that $\omega \wedge \eta$ is a volume element, we define an inner product $\langle\ ,\ \rangle$: $H^r(M) \times H^{m-r}(M) \to \mathbb{R}$ by

$$\langle \omega, \eta \rangle \equiv \int_M \omega \wedge \eta. \tag{6.34}$$

The inner product is bilinear. Moreover, it is non-singular, that is, if $\omega \neq 0$ or $\eta \neq 0$, $\langle \omega, \eta \rangle$ cannot vanish identically. Thus (6.34) defines the duality of $H^r(M)$ and $H^{m-r}(M)$,

$$H^r(M) \cong H^{m-r}(M) \tag{6.35}$$

called the **Poincaré duality**. Accordingly, the Betti numbers have a symmetry

$$b_r = b_{m-r}. \tag{6.36}$$

It follows from (6.36) that the Euler characteristic of an odd-dimensional space vanishes,

$$\begin{aligned}\chi(M) &= \sum(-1)^r b_r = \frac{1}{2}\left[\sum(-1)^r b_r + \sum(-1)^{m-r} b_{m-r}\right] \\ &= \frac{1}{2}\left[\sum(-1)^r b_r - \sum(-1)^{-r} b_r\right] = 0.\end{aligned} \tag{6.37}$$

6.4.2 Cohomology rings

Let $[\omega] \in H^q(M)$ and $[\eta] \in H^r(M)$. Define a product of $[\omega]$ and $[\eta]$ by

$$[\omega] \wedge [\eta] \equiv [\omega \wedge \eta]. \tag{6.38}$$

It follows from exercise 6.4 that $\omega \wedge \eta$ is closed, hence $[\omega \wedge \eta]$ is an element of $H^{q+r}(M)$. Moreover, $[\omega \wedge \eta]$ is independent of the choice of the representatives of $[\omega]$ and $[\eta]$. For example, if we take $\omega' = \omega + \mathrm{d}\psi$ instead of ω, we have

$$[\omega'] \wedge [\eta] \equiv [(\omega + \mathrm{d}\psi) \wedge \eta] = [\omega \wedge \eta + \mathrm{d}(\psi \wedge \eta)] = [\omega \wedge \eta].$$

Thus the product $\wedge$ is a well defined map $H^q(M) \times H^r(M) \to H^{q+r}(M)$.

The **cohomology ring** $H^*(M)$ is defined by the direct sum,

$$H^*(M) \equiv \bigoplus_{r=1}^{m} H^r(M). \tag{6.39}$$

The product is provided by the exterior product defined above,

$$\wedge : H^*(M) \times H^*(M) \to H^*(M). \tag{6.40}$$

The addition is the formal sum of two elements of $H^*(M)$. One of the superiorities of cohomology groups over homology groups resides here. Products of chains are not well defined and homology groups cannot have a ring structure.

6.4.3 The Künneth formula

Let M be a product of two manifolds $M = M_1 \times M_2$. Let $\{\omega_i^p\}$ $(1 \leqslant i \leqslant b^p(M_1))$ be a basis of $H^p(M_1)$ and $\{\eta_i^p\}$ be that of $H^p(M_2)$. Clearly $\omega_i^p \wedge \eta_j^{r-p}$ $(0 \leqslant p \leqslant r)$ is a closed r-form in M. We show that it is not exact. If it were exact, it would be written as

$$\omega_i^p \wedge \eta_j^{r-p} = \mathrm{d}(\alpha^{p-1} \wedge \beta^{r-p} + \gamma^p \wedge \delta^{r-p-1}) \tag{6.41}$$

for some $\alpha^{p-1} \in \Omega^{p-1}(M_1)$, $\beta^{r-p} \in \Omega^{r-p}(M_2)$, $\gamma^p \in \Omega^p(M_1)$ and $\delta^{r-p-1} \in \Omega^{r-p-1}(M_2)$. [If $p = 0$, we put $\alpha^{p-1} = 0$.] By executing the exterior derivative in (6.41), we have

$$\begin{aligned}\omega_i^p \wedge \eta_j^{r-p} = {} & \mathrm{d}\alpha^{p-1} \wedge \beta^{r-p} + (-1)^{p-1}\alpha^{p-1} \wedge \mathrm{d}\beta^{r-p} + \mathrm{d}\gamma^p \wedge \delta^{r-p-1} \\ & + (-1)^p\gamma^p \wedge \mathrm{d}\delta^{r-p-1}.\end{aligned}$$

By comparing the LHS with the RHS, we find $\alpha^{p-1} = \delta^{r-p-1} = 0$, hence $\omega_i^p \wedge \eta_j^{r-p} = 0$ contradicting our assumption. Thus $\omega_i^p \wedge \eta_j^{r-p}$ is an element of $H^r(M)$. Conversely any element of $H^r(M)$ can be decomposed into a sum of a product of the elements of $H^p(M_1)$ and $H^{r-p}(M_2)$ for $0 \leqslant p \leqslant r$. Now we have obtained the **Künneth formula**

$$H^r(M) = \bigoplus_{p+q=r} [H^p(M_1) \otimes H^q(M_2)]. \tag{6.42}$$

In terms of the Betti numbers, we have

$$b^r(M) = \sum_{p+q=r} b^p(M_1)b^q(M_2). \tag{6.43}$$

The Künneth formula also gives a relation between the cohomology rings of the respective manifolds,

$$H^*(M) = \sum_{r=1}^{m} H^r(M) = \sum_{r=1}^{m} \bigoplus_{p+q=r} H^p(M_1) \otimes H^q(M_2)$$

$$= \sum_p H^p(M_1) \otimes \sum_q H^q(M_2) = H^*(M_1) \otimes H^*(M_2). \tag{6.44}$$

Exercise 6.15 Let $M = M_1 \times M_2$. Show that

$$\chi(M) = \chi(M_1) \cdot \chi(M_2). \tag{6.45}$$

Example 6.16 Let $T^2 = S^1 \times S^1$ be the torus. Since $H^0(S^1) = \mathbb{R}$ and $H^1(S^1) = \mathbb{R}$, we have

$$H^0(T^2) = \mathbb{R} \otimes \mathbb{R} = \mathbb{R} \tag{6.46a}$$

$$H^1(T^2) = (\mathbb{R} \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes \mathbb{R}) = \mathbb{R} \oplus \mathbb{R} \tag{6.46b}$$

$$H^2(T^2) = \mathbb{R} \otimes \mathbb{R} = \mathbb{R}. \tag{6.46c}$$

Observe the duality $H^0(T^2) = H^2(T^2)$. [*Remark:* $\mathbb{R} \otimes \mathbb{R}$ is the *tensor product* and should not be confused with the direct product. Clearly the product of two real numbers is a real number.] Let us parametrise the coordinate of T^2 as (θ_1, θ_2) where θ_i is the coordinate of S^1. $H^r(T^2)$ are generated by the following forms:

$$r = 0;\ \omega_0 = c_0 \qquad c_0 \in \mathbb{R} \tag{6.47a}$$

$$r = 1;\ \omega_1 = c_1\, d\theta_1 + c_1'\, d\theta_2 \qquad c_1, c_1' \in \mathbb{R} \tag{6.47b}$$

$$r = 2;\ \omega_2 = c_2\, d\theta_1 \wedge d\theta_2 \qquad c_2 \in \mathbb{R}. \tag{6.47c}$$

Although the one-form $d\theta_i$ looks like an exact form, there is no *function* θ_i which is defined *uniquely* on S^1. Since $\chi(S^1) = 0$, we have $\chi(T^2) = 0$.

The de Rham cohomology groups of

$$T^n = \underbrace{S^1 \times \ldots \times S^1}_{n}$$

are obtained similarly. $H^r(T^n)$ is generated by r-forms of the form

$$d\theta^{i_1} \wedge d\theta^{i_2} \wedge \ldots \wedge d\theta^{i_r} \tag{6.48}$$

where $i_1 < i_2 < \ldots < i_r$ are chosen from $1, \ldots, n$. Clearly

$$b^r = \dim H^r(T^n) = \binom{n}{r}. \tag{6.49}$$

The Euler characteristic is directly obtained from (6.49) as

$$\chi(T^n) = \sum (-1)^r \binom{n}{r} = (1 - 1)^n = 0. \tag{6.50}$$

6.4.4 Pullback of de Rham cohomology groups

Let $f: M \to N$ be a smooth map. (5.75) shows that the pullback f^* maps closed forms to closed forms and exact forms to exact forms. Accordingly, we may define a pullback of the cohomology groups $f^*: H^r(N) \to H^r(M)$ by

$$f^*[\omega] = [f^*\omega] \qquad [\omega] \in H^r(N). \tag{6.51}$$

The pullback f^* preserves the ring structure of $H^*(M)$. In fact if $[\omega] \in H^p(M)$ and $[\eta] \in H^q(M)$, we find

$$\begin{aligned} f^*([\omega] \wedge [\eta]) &= f^*[\omega \wedge \eta] = [f^*(\omega \wedge \eta)] \\ &= [f^*\omega \wedge f^*\eta] = [f^*\omega] \wedge [f^*\eta]. \end{aligned} \tag{6.52}$$

6.4.5 Homotopy and $H^1(M)$

Let $f, g: M \to N$ be smooth maps. We assume f and g are homotopic to each other, that is, there exists a *smooth map* $F: M \times I \to N$ such that $F(p, 0) = f(p)$ and $F(p, 1) = g(p)$. We now prove that $f^*: H^r(N) \to H^r(M)$ is equal to $g^*: H^r(N) \to H^r(M)$.

Lemma 6.17 Let f^* and g^* be defined as above. If $\omega \in \Omega^r(N)$ is a closed form, the difference of the pullback images is exact,

$$f^*\omega - g^*\omega = \mathrm{d}\psi \qquad \psi \in \Omega^{r-1}(M). \tag{6.53}$$

Proof: We first note that

$$f = Ff_0 \qquad g = Ff_1$$

where $f_t: M \to M \times I$ $(p \mapsto (p, t))$ has been defined in theorem 6.13. The LHS of (6.53) is

$$\begin{aligned} (Ff_0)^*\omega - (Ff_1)^*\omega &= f_0{}^*F^*\omega - f_1{}^*F^*\omega \\ &= -[\mathrm{d}P(F^*\omega) + P\mathrm{d}(F^*\omega)] = -\mathrm{d}PF^*\omega \end{aligned}$$

where (6.32) has been used. This shows that $f^*\omega - g^*\omega = \mathrm{d}[-PF^*\omega]$. ■

Now it is easy to see that $f^* = g^*$ as the pullback maps $H^r(N) \to H^r(M)$. In fact, from the lemma above,

$$[f^*\omega - g^*\omega] = [f^*\omega] - [g^*\omega] = [\mathrm{d}\psi] = 0.$$

We have established the following theorem.

Theorem 6.18 Let $f, g: M \to N$ be maps which are homotopic to each other. Then the pullback maps f^* and g^* of the de Rham cohomology groups $H^r(N) \to H^r(M)$ are identical.

Let M be a simply connected manifold, namely $\pi_1(M) = 0$. Since $H_1(M) = \pi_1(M)$ modulo the commutator subgroup (theorem 4.39), it follows that $H_1(M)$ is also trivial. In terms of the de Rham cohomology group this can be expressed as follows.

Theorem 6.19 Let M be a simply connected manifold. Then its first de Rham cohomology group is trivial.

Proof: Let ω be a closed one-form on M. It is clear that if $\omega = \mathrm{d}f$, then a function f must be of the form

$$f(p) = \int_{p_0}^{p} \omega \tag{6.54}$$

$p_0 \in M$ being a fixed point.

We first prove that an integral of a closed form along a loop vanishes. Let $\alpha : I \to M$ be a loop at $p \in M$ and let $c_p : I \to M$ $(t \mapsto p)$ be a constant loop. Since M is simply connected, there exists a homotopy $F(s, t)$ such that $F(s, 0) = \alpha(s)$ and $F(s, 1) = c_p(s)$. We assume $F : I \times I \to M$ is smooth. Define the integral of a one-form ω over $\alpha(I)$ by

$$\int_{\alpha(I)} \omega = \int_{S^1} \alpha^* \omega \tag{6.55}$$

where we have taken the integral domain in the RHS to be S^1 since $I = [0, 1]$ in the LHS is compactified to S^1. From lemma 6.17, we have, for a *closed* one-form ω,

$$\alpha^* \omega - c_p{}^* \omega = \mathrm{d}g \tag{6.56}$$

where $g = -PF^*\omega$. The pullback $c_p{}^*\omega$ vanishes since c_p is a constant map. Then (6.55) vanishes since ∂S^1 is empty,

$$\int_{S^1} \alpha^* \omega = \int_{S^1} \mathrm{d}g = \int_{\partial S^1} g = 0. \tag{6.57}$$

Let β and γ be two paths connecting p_0 and p. According to (6.57), integrals of ω along β and along γ coincide,

$$\int_{\beta(I)} \omega = \int_{\gamma(I)} \omega.$$

This shows that (6.54) is indeed well defined, hence ω is exact. ∎

Example 6.20 The n-sphere S^n $(n \geqslant 2)$ is simply connected, hence

$$H^1(S^n) = 0 \qquad n \geqslant 2. \tag{6.58}$$

From the Poincaré duality, we find

$$H^0(S^n) = H^n(S^n) = \mathbb{R}. \tag{6.59}$$

It can be shown that

$$H^r(S^n) = 0 \qquad 1 \leqslant r \leqslant n - 1. \tag{6.60}$$

$H^n(S^n)$ is generated by the volume element Ω. Since there are no $(n + 1)$-forms on S^n, every n-form is closed. Ω cannot be exact since if $\Omega = \mathrm{d}\psi$, we would have

$$\int_{S^n} \Omega = \int_{S^n} \mathrm{d}\psi = \int_{\partial S^n} \psi = 0.$$

The Euler characteristic is

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 0 & n \text{ is odd.} \\ 2 & n \text{ is even.} \end{cases} \tag{6.61}$$

Example 6.21 Take S^2 embedded in $\mathbb{R}^3$ and define

$$\Omega = \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi \tag{6.62}$$

where (θ, ϕ) is the usual polar coordinate. Verify that Ω is closed. We may *formally* write Ω as

$$\Omega = -\mathrm{d}(\cos\theta) \wedge \mathrm{d}\phi = -\mathrm{d}(\cos\theta \, \mathrm{d}\phi).$$

Note, however, that Ω is *not* exact.

7

RIEMANNIAN GEOMETRY

A manifold is a topological space which locally looks like $\mathbb{R}^n$. Calculus on a manifold is assured by the existence of smooth coordinate systems. A manifold may carry a further structure if it is endowed with a metric tensor, which is a natural generalisation of the inner product between two vectors in $\mathbb{R}^n$ to an arbitrary manifold. With this new structure, we define an inner product between two vectors in a tangent space T_pM. We may also compare a vector at a point $p \in M$ with another vector at a different point $p' \in M$ with the help of the 'connection'.

There are many books about Riemannian manifolds. Those which are accessible to physicists are Choquet-Bruhat *et al* (1982), Dodson and Poston (1977) and Hicks (1965). Lightman *et al* (1975) and Chapter 3 of Wald (1984) are also recommended.

7.1 Riemannian manifolds and pseudo-Riemannian manifolds

7.1.1 Metric tensors

In elementary geometry, the inner product between two vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ is defined by $\boldsymbol{U}\cdot\boldsymbol{V} = \sum_{i=1}^{m} U_i V_i$ where U_i and V_i are the components of the vectors in $\mathbb{R}^m$. On a manifold, an inner product is defined at *each* tangent space T_pM.

Definition 7.1 Let M be a differentiable manifold. A **Riemannian metric** g on M is a type $(0, 2)$ tensor field on M which satisfies the following axioms at each point $p \in M$:

(i) $g_p(U, V) = g_p(V, U)$,
(ii) $g_p(U, U) \geqslant 0$ where the equality holds only when $U = 0$.

Here $U, V \in T_pM$ and $g_p = g|_p$. In short, g_p is a symmetric positive-definite bilinear form.

g is a **pseudo-Riemannian metric** if it satisfies (i) and (ii′) if $g_p(U, V) = 0$ for any $U \in T_pM$, then $V = 0$.

In Chapter 5, we have defined the inner product between a vector $V \in T_pM$ and a dual vector $\omega \in T_p^*M$ as a map $\langle\ ,\ \rangle\colon T_p^*M \times T_pM \to \mathbb{R}$. If there exists a metric g, we define an inner product between two vectors $U, V \in T_pM$ by $g_p(U, V)$. Since g_p is a map

$T_pM \otimes T_pM \to \mathbb{R}$, we may define a linear map $g_p(U, \ \ ) : T_pM \to \mathbb{R}$ by $V \mapsto g_p(U, V)$. Then $g_p(U, \ \ )$ is identified with a one-form $\omega_U \in T_p^*M$. Similarly $\omega \in T_p^*M$ induces $V_\omega \in T_pM$ by $\langle \omega, U \rangle = g(V_\omega, U)$. Thus the metric g_p gives rise to an isomorphism between T_pM and T_p^*M.

Let (U, φ) be a chart in M and $\{x^\mu\}$ the coordinates. Since $g \in \mathcal{T}^2_0(M)$, it is expanded in terms of $\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ as

$$g_p = g_{\mu\nu}(p)\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{7.1a}$$

It is easily checked that

$$g_{\mu\nu}(p) = g_p(\partial/\partial x^\mu, \partial/\partial x^\nu) = g_{\nu\mu}(p). \tag{7.1b}$$

We usually omit p in $g_{\mu\nu}$ unless it may cause confusion. It is common to regard $(g_{\mu\nu})$ as a matrix whose (μ, ν)th entry is $g_{\mu\nu}$. Since $(g_{\mu\nu})$ has the maximal rank, it has an inverse denoted by $(g^{\mu\nu})$ according to the tradition: $g_{\mu\nu}g^{\nu\lambda} = g^{\lambda\nu}g_{\nu\mu} = \delta^\lambda_\mu$. The determinant $\det(g_{\mu\nu})$ is denoted by g. Clearly $\det(g^{\mu\nu}) = g^{-1}$. The isomorphism between T_pM and T_p^*M is now expressed as

$$\omega_\mu = g_{\mu\nu}U^\nu \qquad U^\mu = g^{\mu\nu}\omega_\nu. \tag{7.2}$$

From (7.1) we recover the 'old-fashioned' definition of the metric as an infinitesimal distance squared. Take an infinitesimal displacement $\mathrm{d}x^\mu\,\partial/\partial x^\mu \in T_pM$ and plug it into g,

$$\begin{aligned} \mathrm{d}s^2 &= g(\mathrm{d}x^\mu\,\partial/\partial x^\mu, \mathrm{d}x^\nu\,\partial/\partial x^\nu) = \mathrm{d}x^\mu\,\mathrm{d}x^\nu\,g(\partial/\partial x^\mu, \partial/\partial x^\nu) \\ &= g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu. \end{aligned} \tag{7.3}$$

We also call the quantity $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ a metric, although in a strict sense the metric is a *tensor* $g = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$.

Since $(g_{\mu\nu})$ is a symmetric matrix, the eigenvalues are real. If g is Riemannian, all the eigenvalues are strictly positive and if g is pseudo-Riemannian, some of them may be negative. If there are i positive and j negative eigenvalues, the pair (i, j) is called the **index** of the metric. If $j = 1$, the metric is called a **Lorentz metric**. Once a metric is diagonalised by an appropriate orthogonal matrix, it is easy to reduce all the diagonal elements to ± 1 by a suitable scaling of the basis vectors with positive numbers. If we start with a Riemannian metric we end up with the **Euclidean metric** $\delta = \mathrm{diag}(1, \ldots, 1)$ and if we start with a Lorentz metric, the **Minkowski metric** $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$.

In case (M, g) is Lorentzian, the elements of T_pM are divided into three classes as follows,

$$\begin{aligned} &\text{(i) } g(U, U) > 0 \to U \text{ is } \textbf{spacelike}, \\ &\text{(ii) } g(U, U) = 0 \to U \text{ is } \textbf{lightlike} \text{ (or } \textbf{null}\text{)}, \\ &\text{(iii) } g(U, U) < 0 \to U \text{ is } \textbf{timelike}. \end{aligned} \tag{7.4}$$

Exercise 7.2 Diagonalise the metric

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to show it reduces to the Minkowski metric. The frame on which the metric takes this form is known as the **light cone frame**. Let $\{e_0, e_1, e_2, e_3\}$ be the basis of the Minkowski frame in which the metric is $g_{\mu\nu} = \eta_{\mu\nu}$. Show that $\{e_+, e_-, e_2, e_3\}$ are the basis vectors in the light cone frame, where $e_\pm \equiv (e_1 \pm e_0)/\sqrt{2}$. Let $V = (V^+, V^-, V^2, V^3)$ be components of a vector V. Find the components of the corresponding one-form.

If a smooth manifold M admits a Riemannian metric g, the pair (M, g) is called a **Riemannian manifold**. If g is a pseudo-Riemannian metric, (M, g) is called a **pseudo-Riemannian manifold**. If g is Lorentzian, (M, g) is called a **Lorentz manifold**. Lorentz manifolds are of special interest in the theory of relativity. For example, an m-dimensional Euclidean space $(\mathbb{R}^m, \delta)$ is a Riemannian manifold and an m-dimensional Minkowski space $(\mathbb{R}^m, \eta)$ is a Lorentz manifold.

7.1.2 Induced metric

Let M be an m-dimensional submanifold of an n-dimensional Riemannian manifold N with the metric g_N. If $f: M \to N$ is the embedding which induces the submanifold structure of M (see §5.2), the pullback map f^* induces the natural metric $g_M = f^* g_N$ on M. The components of g_M are given by

$$g_{M\mu\nu}(x) = g_{N\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} \tag{7.5}$$

where f^α denote the coordinates of $f(x)$. For example, consider the metric of the unit sphere embedded in $(\mathbb{R}^3, \delta)$. Let (θ, ϕ) be the polar coordinates of S^2 and define f by the usual inclusion,

$$f: (\theta, \phi) \mapsto (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

from which we obtain the induced metric

$$\begin{aligned} g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu &= \delta_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \\ &= \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta\,\mathrm{d}\phi \otimes \mathrm{d}\phi. \end{aligned} \tag{7.6}$$

Exercise 7.3 Let $f: T^2 \to \mathbb{R}^3$ be an embedding of the torus into $(\mathbb{R}^3, \delta)$ defined by

$$f: (\theta, \phi) \mapsto ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)$$

where $R > r$. Show that the induced metric on T^2 is

$$g = r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta + (R + r\cos\theta)^2 \mathrm{d}\phi \otimes \mathrm{d}\phi. \tag{7.7}$$

When a manifold N is pseudo-Riemannian, its submanifold $f: M \to N$ need not have a metric f^*g_N. The tensor f^*g_N is a metric only when it has a fixed index on M.

7.2 Parallel transport, connection and covariant derivative

A vector X is a directional derivative acting on $f \in \mathcal{F}(M)$ as $X: f \mapsto X[f]$. However, there is no directional derivative acting on a tensor field of type (p, q), which arises naturally from the differentiable structure of M. [Note that the Lie derivative $\mathcal{L}_V X = [V, X]$ is not a directional derivative since it depends on the *derivative* of V.] What we need is an extra structure called the **connection**, which specifies how tensors are transported along a curve.

7.2.1 Heuristic introduction

We first give a heuristic approach to parallel transport and covariant derivatives. As we have noted several times, two vectors defined at different points cannot be compared naively with each other. Let us see how the derivative of a vector field in a Euclidean space $\mathbb{R}^m$ is defined. The derivative of a vector field $\boldsymbol{V} = V^\mu \boldsymbol{e}_\mu$ with respect to x^ν has the μth component

$$\frac{\partial V^\mu}{\partial x^\nu} = \lim_{\Delta x \to 0} \frac{V^\mu(\ldots, x^\nu + \Delta x^\nu, \ldots) - V^\mu(\ldots, x^\nu, \ldots)}{\Delta x^\nu}.$$

The first term in the numerator of the LHS is defined at $x + \Delta x = (x^1, \ldots, x^\nu + \Delta x^\nu, \ldots, x^m)$, while the second term is defined at $x = (x^\mu)$. To subtract $V^\mu(x)$ from $V^\mu(x + \Delta x)$, we have to transport $V^\mu(x)$ to $x + \Delta x$ *without change* and compute the difference. This transport of a vector is called a **parallel transport**. We have implicitly assumed that $V|_x$ parallel transported to $x + \Delta x$ has the same component $V^\mu(x)$. On the other hand, there is no natural way to parallel transport a vector in a manifold and we have to specify *how it is parallel transported* from one point to the other. Let $\widetilde{V}|_{x+\Delta x}$ denote a vector $V|_x$

parallel transported to $x + \Delta x$. We demand that the components $\widetilde{V}^\mu$ satisfy

$$\widetilde{V}^\mu(x + \Delta x) - V^\mu(x) \propto \Delta x \tag{7.8a}$$

$$(\widetilde{V^\mu + W^\mu})(x + \Delta x) = \widetilde{V}^\mu(x + \Delta x) + \widetilde{W}^\mu(x + \Delta x). \tag{7.8b}$$

These conditions are satisfied if we take

$$\widetilde{V}^\mu(x + \Delta x) = V^\mu(x) - V^\lambda(x)\Gamma^\mu{}_{\nu\lambda}(x)\Delta x^\nu. \tag{7.9}$$

The covariant derivative of V with respect to x^ν is defined by

$$\lim_{\Delta x^\nu \to 0} \frac{V^\mu(x + \Delta x) - \widetilde{V}^\mu(x + \Delta x)}{\Delta x^\nu} \frac{\partial}{\partial x^\mu} = \left(\frac{\partial V^\mu}{\partial x^\nu} + V^\lambda \Gamma^\mu{}_{\nu\lambda}\right) \frac{\partial}{\partial x^\mu}. \tag{7.10}$$

This quantity is a vector at $x + \Delta x$ since it is a difference of two vectors $V|_{x+\Delta x}$ and $\widetilde{V}_{x+\Delta x}$ defined at the *same* point $x + \Delta x$. There are many distinct rules of parallel transport possible, one for each choice of Γ. If the manifold is endowed with a metric, there exists a preferred choice of Γ, called the Levi-Civita connection, see example 7.4 and §7.4.

Example 7.4 Let us work out a simple example: two-dimensional Euclidean space $(\mathbb{R}^2, \delta)$. We define parallel transportation according to the usual sense in elementary geometry. In the Cartesian coordinate system (x, y), all the components of Γ vanish since $\widetilde{V}^\mu(x + \Delta x, y + \Delta y) = V^\mu(x, y)$ for any Δx and Δy. Next we take the polar coordinates (r, ϕ). If $(r, \phi) \mapsto (r\cos\phi, r\sin\phi)$ is regarded as an embedding, we find the induced metric,

$$g = \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\phi \otimes \mathrm{d}\phi. \tag{7.11}$$

Let $V = V^r \partial/\partial r + V^\phi \partial/\partial\phi$ be a vector defined at (r, ϕ). If we parallel transport this vector to $(r + \Delta r, \phi)$, we have a new vector $\widetilde{V} = \widetilde{V}^r \partial/\partial r|_{(r+\Delta r,\phi)} + \widetilde{V}^\phi \partial/\partial\phi|_{(r+\Delta r,\phi)}$ (figure 7.1(a)). Note that $V^r = V\cos\theta$ and $V^\phi = V(\sin\theta/r)$, where $V = [g(V, V)]^{1/2}$ and θ is the angle between V and $\partial/\partial r$. Then we have $\widetilde{V}^r = V^r$ and

$$\widetilde{V}^\phi = \frac{r}{r + \Delta r} V^\phi \simeq V^\phi - \frac{\Delta r}{r} V^\phi.$$

By comparing these components with (7.9), we easily find that

$$\Gamma^r{}_{rr} = 0 \qquad \Gamma^r{}_{r\phi} = 0 \qquad \Gamma^\phi{}_{rr} = 0 \qquad \Gamma^\phi{}_{r\phi} = \frac{1}{r}. \tag{7.12a}$$

Similarly if V is parallel transported to $(r, \phi + \Delta\phi)$, it becomes

$$\widetilde{V} = \widetilde{V}^r \partial/\partial r|_{(r,\phi+\Delta\phi)} + \widetilde{V}^\varphi \partial/\partial\phi|_{(r,\varphi+\Delta\phi)}$$

where

$$\widetilde{V}^r = V\cos(\theta - \Delta\phi) \simeq V\cos\theta + V\sin\theta\,\Delta\phi = V^r + V^\varphi r\Delta\phi$$

and

$$\widetilde{V}^\varphi = V\frac{\sin(\theta - \Delta\phi)}{r} \simeq V\frac{\sin\theta}{r} - V\cos\theta\frac{\Delta\phi}{r} = V^\varphi - V^r\frac{\Delta\phi}{r}$$

(figure 7.1(b)). Then we find

$$\Gamma^r{}_{\varphi r} = 0 \qquad \Gamma^r{}_{\varphi\varphi} = -r \qquad \Gamma^\varphi{}_{\varphi r} = \frac{1}{r} \qquad \Gamma^\varphi{}_{\varphi\varphi} = 0. \qquad (7.12b)$$

Note that the Γ satisfy the symmetry $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$. It is also implicitly assumed that the norm of a vector is invariant under parallel transport. A rule of parallel transport which satisfies these two conditions is called a **Levi-Civita connection**, see §7.4. Our intuitive approach leads us to the formal definition of the affine connection.

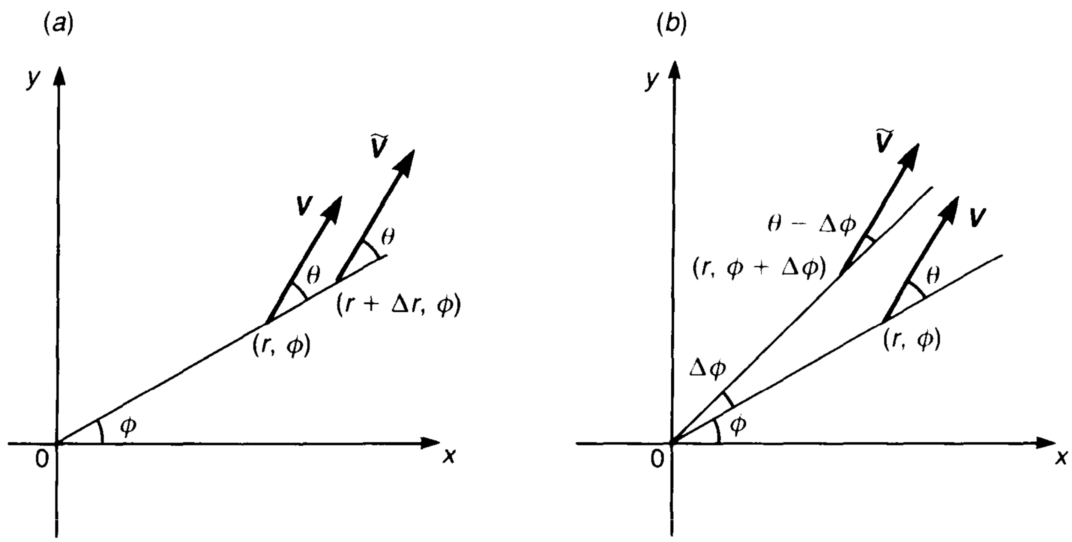


Figure 7.1 $\widetilde{V}$ is a vector V parallel transported to: (a), $(r + \Delta r, \phi)$; and (b), $(r, \phi + \Delta\phi)$.

7.2.2 *Affine connections*

Definition 7.5 An **affine connection** ∇ is a map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, or $(X, Y) \mapsto \nabla_X Y$, which satisfies the following conditions:

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z \tag{7.13a}$$

$$\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z \tag{7.13b}$$

$$\nabla_{(fX)} Y = f\nabla_X Y \tag{7.13c}$$

$$\nabla_X(fY) = X[f]Y + f\nabla_X Y \tag{7.13d}$$

where $f \in \mathcal{F}(M)$ and $X, Y, Z \in \mathcal{X}(M)$.

Take a chart (U, φ) with the coordinate $x = \varphi(p)$ on M, and define m^3 functions called the **connection coefficients** $\Gamma^{\mu}{}_{\nu\lambda}$ by

$$\nabla_\nu e_\mu \equiv \nabla_{e_\nu} e_\mu = e_\lambda \Gamma^{\lambda}{}_{\nu\mu} \tag{7.14}$$

where $\{e_\mu\} = \{\partial/\partial x^\mu\}$ is the coordinate basis in T_pM. The connection coefficients specify how the basis vectors change from point to point. Once the action of ∇ on the basis vectors is defined, we can calculate the action of ∇ on any vectors. Let $V = V^\mu e_\mu$ and $W = W^\nu e_\nu$ be elements of $\mathcal{X}(M)$. Then,

$$\begin{aligned}\nabla_V W &= V^\mu \nabla_{e_\mu}(W^\nu e_\nu) = V^\mu(e_\mu[W^\nu]e_\nu + W^\nu \nabla_{e_\mu} e_\nu)\\ &= V^\mu(\partial W^\lambda/\partial x^\mu + W^\nu \Gamma^{\lambda}{}_{\mu\nu})e_\lambda.\end{aligned} \tag{7.15}$$

Note that this definition of the connection coefficient is in agreement with the previous heuristic result (7.10). By definition, ∇ maps two vectors V and W to a new vector given by the RHS of (7.15), whose λth component is $V^\mu \nabla_\mu W^\lambda$ where

$$\nabla_\mu W^\lambda \equiv \partial W^\lambda/\partial x^\mu + \Gamma^{\lambda}{}_{\mu\nu} W^\nu. \tag{7.16}$$

Note that $\nabla_\mu W^\lambda$ is the λth component of a vector $\nabla_\mu W = \nabla_\mu W^\lambda e_\lambda$, and should not be confused with the covariant derivative of a *component* W^λ. $\nabla_V W$ is independent of the derivative of V, unlike the Lie derivative $\mathcal{L}_V W = [V, W]$. In this sense, the covariant derivative is a proper generalisation of the directional derivative of functions to tensors.

7.2.3 Parallel transport and geodesics

Given a curve in a manifold M, we may define the parallel transport of a vector along the curve. Let $c : (a, b) \rightarrow M$ be a curve in M. For simplicity, we assume the image is covered by a single chart (U, φ) whose coordinate is $x = \varphi(p)$. Let X be a vector field defined (at least) along $c(t)$,

$$X|_{c(t)} = X^\mu(c(t)) e_\mu|_{c(t)} \tag{7.17}$$

where $e_\mu = \partial/\partial x^\mu$. If X satisfies the condition

$$\nabla_V X = 0 \qquad \text{for any } t \in (a, b) \tag{7.18a}$$

X is said to be **parallel transported** along $c(t)$ where $V = \mathrm{d}/\mathrm{d}t = (\mathrm{d}x^\mu(c(t))/\mathrm{d}t)e_\mu|_{c(t)}$ is the tangent vector to $c(t)$. The condition (7.18a) is written in terms of the components as

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}t} + \Gamma^\mu{}_{\nu\lambda}\frac{\mathrm{d}x^\nu(c(t))}{\mathrm{d}t}X^\lambda = 0. \tag{7.18b}$$

If the tangent vector $V(t)$ itself is parallel transported along $c(t)$, namely if

$$\nabla_V V = 0 \tag{7.19a}$$

the curve $c(t)$ is called a **geodesic**. Geodesics are, in a sense, the *straightest possible curves* in a Riemannian manifold. In components, the geodesic equation (7.19a) becomes

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}t^2} + \Gamma^\mu{}_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\frac{\mathrm{d}x^\lambda}{\mathrm{d}t} = 0 \tag{7.19b}$$

where $\{x^\mu\}$ are the coordinates of $c(t)$. We might say that (7.19a) is too strong to be the condition for the straightest possible curve, and instead require a weaker condition

$$\nabla_V V = fV \tag{7.20}$$

where $f \in \mathcal{F}(M)$. 'Change of V is parallel to V' is also a feature of a straight line. However, under the reparametrisation $t \to t'$, the component of the tangent vector changes as

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} \to \frac{\mathrm{d}t}{\mathrm{d}t'}\cdot\frac{\mathrm{d}x^\mu}{\mathrm{d}t}$$

and (7.20) reduces to (7.19a) if t' satisfies

$$\frac{\mathrm{d}^2t'}{\mathrm{d}t^2} = f\frac{\mathrm{d}t'}{\mathrm{d}t}.$$

Thus it is always possible to reparametrise the curve so that the geodesic equation takes the form (7.19a).

Exercise 7.6 Show that (7.19b) is left invariant under the affine reparametrisation $t \to at + b (a, b \in \mathbb{R})$.

7.2.4 The covariant derivative of tensor fields

Since ∇_X has the meaning of a derivative, it is natural to define the covariant derivative of $f \in \mathcal{F}(M)$ by the ordinary directional derivative,

$$\nabla_X f = X[f]. \tag{7.21}$$

Then (7.13d) looks exactly like the Leibnitz rule,

$$\nabla_X(fY) = (\nabla_X f)Y + f\nabla_X Y. \tag{7.13d'}$$

We require that this be true for any product of tensors,

$$\nabla_X(T_1 \otimes T_2) = (\nabla_X T_1) \otimes T_2 + T_1 \otimes (\nabla_X T_2) \tag{7.22}$$

where T_1 and T_2 are tensor fields of arbitrary types. (7.22) is also true when some of the indices are contracted. With these requirements, we compute the covariant derivative of a one-form $\omega \in \Omega^1(M)$. Since $\langle \omega, Y\rangle \in \mathcal{F}(M)$ for $Y \in \mathcal{X}(M)$, we should have

$$X[\langle \omega, Y\rangle] = \nabla_X[\langle \omega, Y\rangle] = \langle \nabla_X \omega, Y\rangle + \langle \omega, \nabla_X Y\rangle.$$

Writing down both sides in terms of the components we find

$$(\nabla_X \omega)_\nu = X^\mu \partial_\mu \omega_\nu - X^\mu \Gamma^\lambda{}_{\mu\nu}\omega_\lambda. \tag{7.23}$$

In particular, for $X = e_\mu$, we have,

$$(\nabla_\mu \omega)_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu}\omega_\lambda. \tag{7.24}$$

If $\omega = \mathrm{d}x^\nu$, we have (cf (7.14))

$$\nabla_\mu \mathrm{d}x^\nu = -\Gamma^\nu{}_{\mu\lambda}\,\mathrm{d}x^\lambda. \tag{7.25}$$

It is easy to generalise these results as

$$\begin{aligned}\nabla_\nu t^{\lambda_1,\ldots,\lambda_p}_{\mu_1,\ldots,\mu_q} = {} & \partial_\nu t^{\lambda_1,\ldots,\lambda_p}_{\mu_1,\ldots,\mu_q} + \Gamma^{\lambda_1}{}_{\nu\kappa} t^{\kappa,\lambda_2,\ldots,\lambda_p}_{\mu_1,\ldots,\mu_q} + \ldots \\ & + \Gamma^{\lambda_p}{}_{\nu\kappa} t^{\lambda_1,\ldots,\lambda_{p-1},\kappa}_{\mu_1,\ldots,\mu_q} - \Gamma^\kappa{}_{\nu\mu_1} t^{\lambda_1,\ldots,\lambda_p}_{\kappa,\mu_2,\ldots,\mu_q} - \ldots \\ & - \Gamma^\kappa{}_{\nu\mu_q} t^{\lambda_1,\ldots,\lambda_p}_{\mu_1,\ldots,\mu_{q-1},\kappa}.\end{aligned} \tag{7.26}$$

Exercise 7.7 Let g be a metric tensor. Verify that

$$(\nabla_\nu g)_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^\kappa{}_{\nu\lambda} g_{\kappa\mu} - \Gamma^\kappa{}_{\nu\mu} g_{\lambda\kappa}. \tag{7.27}$$

7.2.5 The transformation properties of connection coefficients

Introduce another chart (V, ψ) such that $U \cap V \neq \varnothing$, whose coordinates are $y = \psi(p)$. Let $\{e_\mu\} = \{\partial/\partial x^\mu\}$ and $\{f_\alpha\} = \{\partial/\partial y^\alpha\}$ be bases of the respective coordinates. Denote the connection coefficients with respect to the y-coordinates by $\widetilde{\Gamma}^\alpha{}_{\beta\gamma}$. The basis vector f_α satisfies

$$\nabla_{f_\alpha} f_\beta = \widetilde{\Gamma}^\gamma{}_{\alpha\beta} f_\gamma. \tag{7.28}$$

If we write $f_\alpha = (\partial x^\mu / \partial y^\alpha) e_\mu$, the LHS becomes

$$\nabla_{f_\alpha} f_\beta = \nabla_{f_\alpha}\left(\frac{\partial x^\mu}{\partial y^\beta} e_\mu\right) = \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta} e_\mu + \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial x^\mu}{\partial y^\beta}\nabla_{e_\lambda} e_\mu$$
$$= \left(\frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta} + \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial x^\mu}{\partial y^\beta}\Gamma^\nu{}_{\lambda\mu}\right) e_\nu .$$

Since the RHS of (7.28) is equal to $\widetilde{\Gamma}^\gamma{}_{\alpha\beta}(\partial x^\nu/\partial y^\gamma) e_\nu$, the connection coefficients must transform as

$$\widetilde{\Gamma}^\gamma{}_{\alpha\beta} = \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial x^\mu}{\partial y^\beta}\frac{\partial y^\gamma}{\partial x^\nu}\Gamma^\nu{}_{\lambda\mu} + \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta}\frac{\partial y^\gamma}{\partial x^\nu}. \tag{7.29}$$

The reader should verify that this transformation rule indeed makes $\nabla_X Y$ a vector, namely

$$\widetilde{X}^\alpha(\widetilde{\partial}_\alpha \widetilde{Y}^\gamma + \widetilde{\Gamma}^\gamma{}_{\alpha\beta}\widetilde{Y}^\beta) f_\gamma = X^\lambda(\partial_\lambda Y^\nu + \Gamma^\nu{}_{\lambda\mu} Y^\mu) e_\nu .$$

In the literature, connection coefficients are often defined as objects which transform as (7.29). From our viewpoint, however, they *must* transform according to (7.29) to make $\nabla_X Y$ independent of the coordinate chosen.

Exercise 7.8 Let Γ be an arbitrary connection coefficient. Show that $\Gamma^\lambda{}_{\mu\nu} + t^\lambda{}_{\mu\nu}$ is another connection coefficient provided that $t^\lambda{}_{\mu\nu}$ is a tensor field. Conversely, suppose $\Gamma^\lambda{}_{\mu\nu}$ and $\bar{\Gamma}^\lambda{}_{\mu\nu}$ are connection coefficients. Show that $\Gamma^\lambda{}_{\mu\nu} - \bar{\Gamma}^\lambda{}_{\mu\nu}$ is a component of a tensor of type (1, 2).

7.2.6 The metric connection

So far we have left Γ arbitrary. Now that our manifold is endowed with a metric, we may put reasonable restrictions on the possible form of connections. We demand that the metric $g_{\mu\nu}$ be *covariantly constant*, that is, if two vectors X and Y are parallel transported along any curve, then the inner product between them remains constant under parallel transport. [In example 7.4, we have already assumed this reasonable condition.] Let V be a tangent vector to an arbitrary curve along which the vectors are parallel transported. Then we have

$$0 = \nabla_V[g(X, Y)] = V^\kappa[(\nabla_\kappa g)(X, Y) + g(\nabla_\kappa X, Y) + g(X, \nabla_\kappa Y)]$$
$$= V^\kappa X^\mu Y^\nu (\nabla_\kappa g)_{\mu\nu}$$

where we have noted that $\nabla_\kappa X = \nabla_\kappa Y = 0$. Since this is true for any curves and vectors, we must have

$$(\nabla_\kappa g)_{\mu\nu} = 0 \tag{7.30a}$$

or, from exercise 7.7,

$$\partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\lambda\mu} g_{\kappa\nu} - \Gamma^\kappa{}_{\lambda\nu} g_{\kappa\mu} = 0. \tag{7.30b}$$

If (7.30a) is satisfied, the affine connection ∇ is said to be **metric compatible**, or simply a **metric connection**. We will deal with metric connections only. Cyclic permutations of (λ, μ, ν) yield

$$\partial_\mu g_{\nu\lambda} - \Gamma^\kappa{}_{\mu\nu} g_{\kappa\lambda} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} = 0 \tag{7.30c}$$

$$\partial_\nu g_{\lambda\mu} - \Gamma^\kappa{}_{\nu\lambda} g_{\kappa\mu} - \Gamma^\kappa{}_{\nu\mu} g_{\kappa\lambda} = 0. \tag{7.30d}$$

The combination $-$(7.30b) + (7.30c) + (7.30d) yields

$$-\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} + T^\kappa{}_{\lambda\mu} g_{\kappa\nu} + T^\kappa{}_{\lambda\nu} g_{\kappa\mu} - 2\Gamma^\kappa{}_{(\mu\nu)} g_{\kappa\lambda} = 0 \tag{7.31}$$

where $T^\kappa{}_{\lambda\mu} \equiv 2\Gamma^\kappa{}_{[\lambda\mu]} \equiv \Gamma^\kappa{}_{\lambda\mu} - \Gamma^\kappa{}_{\mu\lambda}$ and $\Gamma^\kappa{}_{(\mu\nu)} \equiv \frac{1}{2}\{\Gamma^\kappa{}_{\mu\nu} + \Gamma^\kappa{}_{\nu\mu}\}$. The tensor $T^\kappa{}_{\lambda\mu}$ is antisymmetric with respect to the lower indices $T^\kappa{}_{\lambda\mu} = -T^\kappa{}_{\mu\lambda}$ and called the *torsion tensor*, see exercise 7.9. The torsion tensor will be studied in detail in the next section. (7.31) is solved for $\Gamma^\kappa{}_{(\mu\nu)}$ to yield

$$\Gamma^\kappa{}_{(\mu\nu)} = \left\{ {\kappa \atop \mu\nu} \right\} + \tfrac{1}{2}(T_\nu{}^\kappa{}_\mu + T_\mu{}^\kappa{}_\nu) \tag{7.32}$$

where $\left\{ {\kappa \atop \mu\nu} \right\}$ are the **Christoffel symbols** defined by

$$\left\{ {\kappa \atop \mu\nu} \right\} = \tfrac{1}{2} g^{\kappa\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{7.33}$$

The connection coefficient Γ is given by

$$\begin{aligned} \Gamma^\kappa{}_{\mu\nu} &= \Gamma^\kappa{}_{(\mu\nu)} + \Gamma^\kappa{}_{[\mu\nu]} \\ &= \left\{ {\kappa \atop \mu\nu} \right\} + \tfrac{1}{2}\,(T_\nu{}^\kappa{}_\mu + T_\mu{}^\kappa{}_\nu + T^\kappa{}_{\mu\nu}). \end{aligned} \tag{7.34}$$

The second term of the last expression of (7.34) is called the **contorsion**, denoted by $K^\kappa{}_{\mu\nu}$:

$$K^\kappa{}_{\mu\nu} \equiv \tfrac{1}{2}(T^\kappa{}_{\mu\nu} + T_\mu{}^\kappa{}_\nu + T_\nu{}^\kappa{}_\mu). \tag{7.35}$$

If the torsion tensor vanishes on a manifold M, the metric connection ∇ is called the **Levi-Civita connection**. Levi-Civita connections are natural generalisations of the connection defined in the classical geometry of surfaces, see §7.4.

Exercise 7.9 Show that $T^\kappa{}_{\mu\nu}$ obeys the tensor transformation rule. [*Hint:* Use (7.29).] Show also that $K^\kappa{}_{[\mu\nu]} = \frac{1}{2} T^\kappa{}_{\mu\nu}$ and $K_{\kappa\mu\nu} = -K_{\nu\mu\kappa}$, where $K_{\kappa\mu\nu} = g_{\kappa\lambda} K^\lambda{}_{\mu\nu}$.

7.3 Curvature and torsion

7.3.1 Definitions

Since Γ is not a tensor, it cannot have an intrinsic geometrical meaning as a measure of how much a manifold is curved. For example, the connection coefficients in example 7.4 vanish if the Cartesian coordinate is employed while they do not in polar coordinates. As intrinsic objects, we defined the **torsion tensor** $T : \mathcal{X}(M) \otimes \mathcal{X}(M) \to \mathcal{X}(M)$ and the **Riemann curvature tensor** (or **Riemann tensor**) $R : \mathcal{X}(M) \otimes \mathcal{X}(M) \otimes \mathcal{X}(M) \to \mathcal{X}(M)$ by

$$T(X, Y) \equiv \nabla_X Y - \nabla_Y X - [X, Y] \tag{7.36}$$

$$R(X, Y, Z) \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{7.37}$$

It is common to write $R(X, Y)Z$ instead of $R(X, Y, Z)$, so that R looks like an operator acting on Z. Clearly they satisfy

$$T(X, Y) = -T(Y, X), \; R(X, Y)Z = -R(Y, X)Z. \tag{7.38}$$

At first sight, T and R seem to be differentiable operators and it is not obvious that they are multilinear objects. We prove the tensorial property of R,

$$\begin{aligned}
R(fX, gY)hZ &= f\nabla_X\{g\nabla_Y(hZ)\} - g\nabla_Y\{f\nabla_X(hZ)\} - fX[g]\nabla_Y(hZ) \\
&\quad + gY[f]\,\nabla_X(hZ) - fg\nabla_{[X,Y]}(hZ) \\
&= fg\nabla_X\{Y[h]Z + h\nabla_Y Z\} - gf\nabla_Y\{X[h]Z + h\nabla_X Z\} \\
&\quad - fg[Z, Y][h]Z - fgh\nabla_{[X,Y]}Z \\
&= fgh\{\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\} = fgh\,R(X, Y)Z.
\end{aligned}$$

Now it is easy to see that R satisfies

$$R(X, Y)Z = X^\lambda Y^\mu Z^\nu R(e_\lambda, e_\mu)e_\nu \tag{7.39}$$

which verifies the tensorial property of R. Since R maps three vector fields to a vector field, it is a tensor field of type $(1, 3)$.

Exercise 7.10 Show that T defined by (7.36) is multilinear,

$$T(X, Y) = X^\mu Y^\nu T(e_\mu, e_\nu) \tag{7.40}$$

and hence a tensor field of type $(1, 2)$.

Since T and R are tensors, their operations on vectors are obtained once their actions on the basis vectors are known. With respect to the coordinate basis $\{e_\mu\}$ and the dual basis $\{dx^\mu\}$, the components of these tensors are given by

$$\begin{aligned} T^{\lambda}{}_{\mu\nu} &= \langle \mathrm{d}x^{\lambda}, T(e_{\mu}, e_{\nu})\rangle = \langle \mathrm{d}x^{\lambda}, \nabla_{\mu}e_{\nu} - \nabla_{\nu}e_{\mu}\rangle \\ &= \langle \mathrm{d}x^{\lambda}, \Gamma^{\eta}{}_{\mu\nu}e_{\eta} - \Gamma^{\eta}{}_{\nu\mu}e_{\eta}\rangle = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} \end{aligned} \tag{7.41}$$

and

$$\begin{aligned} R^{\kappa}{}_{\lambda\mu\nu} &= \langle \mathrm{d}x^{\kappa}, R(e_{\mu}, e_{\nu})e_{\lambda}\rangle = \langle \mathrm{d}x^{\kappa}, \nabla_{\mu}\nabla_{\nu}e_{\lambda} - \nabla_{\nu}\nabla_{\mu}e_{\lambda}\rangle \\ &= \langle \mathrm{d}x^{\kappa}, \nabla_{\mu}(\Gamma^{\eta}{}_{\nu\lambda}e_{\eta}) - \nabla_{\nu}(\Gamma^{\eta}{}_{\mu\lambda}e_{\eta})\rangle \\ &= \langle \mathrm{d}x^{\kappa}, (\partial_{\mu}\Gamma^{\eta}{}_{\nu\lambda})e_{\eta} + \Gamma^{\eta}{}_{\nu\lambda}\Gamma^{\xi}{}_{\mu\eta}e_{\xi} - (\partial_{\nu}\Gamma^{\eta}{}_{\mu\lambda})e_{\eta} - \Gamma^{\eta}{}_{\mu\lambda}\Gamma^{\xi}{}_{\nu\eta}e_{\xi}\rangle \\ &= \partial_{\mu}\Gamma^{\kappa}{}_{\nu\lambda} - \partial_{\nu}\Gamma^{\kappa}{}_{\mu\lambda} + \Gamma^{\eta}{}_{\nu\lambda}\Gamma^{\kappa}{}_{\mu\eta} - \Gamma^{\eta}{}_{\mu\lambda}\Gamma^{\kappa}{}_{\nu\eta}. \end{aligned} \tag{7.42}$$

We readily find (cf (7.38))

$$T^{\lambda}{}_{\mu\nu} = -T^{\lambda}{}_{\nu\mu} \qquad R^{\kappa}{}_{\lambda\mu\nu} = -R^{\kappa}{}_{\lambda\nu\mu}. \tag{7.43}$$

7.3.2 Geometrical meaning of the Riemann tensor and the torsion tensor

Before we proceed further, we examine the geometrical meaning of these tensors. We consider the Riemann tensor first. A crucial observation is that if we parallel transport a vector V at p to q along two different curves C and C', the resulting vectors at q are different in general (figure 7.2). If, however, we parallel transport a vector in a Euclidean space, where the parallel transport is defined in our usual sense, the resulting vector does not depend on the path along which it has been parallel transported. We expect that this non-integrability of parallel transport characterises the intrinsic notion of curvature, which does not depend on the special coordinates chosen. Let us take an infinitesimal parallelogram $pqrs$ whose coordinates are $\{x^{\mu}\}$, $\{x^{\mu} + \varepsilon^{\mu}\}$,

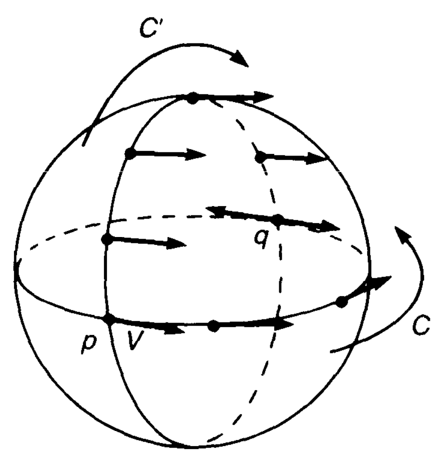


Figure 7.2 It is natural to define V parallel transported along a great circle if the angle V makes with the great circle is kept fixed. If V at p is parallel transported along great circles C and C', the resulting vectors at q point in opposite directions.

$\{x^\mu + \varepsilon^\mu + \delta^\mu\}$, and $\{x^\mu + \delta^\mu\}$ respectively, ε^μ and δ^μ being infinitesimal (figure 7.3). If we parallel transport a vector $V_0 \in T_pM$ along $C = pqr$, we will have a vector $V_C(r) \in T_rM$. The vector V_0 parallel transported to q along C is

$$V^\mu_C(q) = V^\mu_0 - V^\kappa_0 \Gamma^\mu{}_{\nu\kappa}(p)\varepsilon^\nu.$$

Then $V^\mu_C(r)$ is given by

$$\begin{aligned} V^\mu_C(r) &= V^\mu_C(q) - V^\kappa_C(q)\Gamma^\mu{}_{\nu\kappa}(q)\delta^\nu \\ &= V^\mu_0 - V^\kappa_0\Gamma^\mu{}_{\nu\kappa}(p)\varepsilon^\nu - [V^\kappa_0 - V^\rho_0\Gamma^\kappa{}_{\zeta\rho}(p)\varepsilon^\zeta] \\ &\quad \times [\Gamma^\mu{}_{\nu\kappa}(p) + \partial_\lambda\Gamma^\mu{}_{\nu\kappa}(p)\varepsilon^\lambda]\delta^\nu \\ &\simeq V^\mu_0 - V^\kappa_0\Gamma^\mu{}_{\nu\kappa}(p)\varepsilon^\nu - V^\kappa_0\Gamma^\mu{}_{\nu\kappa}(p)\delta^\nu \\ &\quad - V^\kappa_0[\partial_\lambda\Gamma^\mu{}_{\nu\kappa}(p) - \Gamma^\rho{}_{\lambda\kappa}(p)\Gamma^\mu{}_{\nu\rho}(p)]\varepsilon^\lambda\delta^\nu \end{aligned}$$

where we have kept terms of up to order two in ε and δ. Similarly, parallel transport of V_0 along $C' = psr$ yields another vector $V_{C'}(r) \in T_rM$, given by

$$\begin{aligned} V^\mu_{C'}(r) &\simeq V^\mu_0 - V^\kappa_0\Gamma^\mu{}_{\nu\kappa}(p)\delta^\nu - V^\kappa_0\Gamma^\mu{}_{\nu\kappa}(p)\varepsilon^\nu \\ &\quad - V^\kappa_0[\partial_\nu\Gamma^\mu{}_{\lambda\kappa}(p) - \Gamma^\rho{}_{\nu\kappa}(p)\Gamma^\mu{}_{\lambda\rho}(p)]\varepsilon^\lambda\delta^\nu. \end{aligned}$$

The two vectors at r differ by

$$\begin{aligned} V^\mu_{C'}(r) - V^\mu_C(r) &= V^\kappa_0[\partial_\lambda\Gamma^\mu{}_{\nu\kappa}(p) - \partial_\nu\Gamma^\mu{}_{\lambda\kappa}(p) \\ &\quad - \Gamma^\rho{}_{\lambda\kappa}(p)\Gamma^\mu{}_{\nu\rho}(p) + \Gamma^\rho{}_{\nu\kappa}(p)\Gamma^\mu{}_{\lambda\rho}(p)]\varepsilon^\lambda\delta^\nu \\ &= V^\kappa_0 R^\mu{}_{\kappa\lambda\nu}\varepsilon^\lambda\delta^\nu. \end{aligned} \tag{7.44}$$

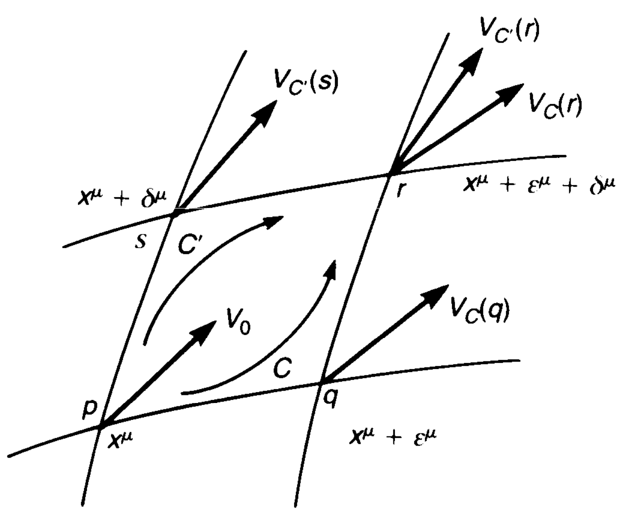


Figure 7.3 A vector V_0 at p is parallel transported along C and C' to yield $V_C(r)$ and $V_{C'}(r)$ at r. The curvature measures the difference between two vectors.

We next look at the geometrical meaning of the torsion tensor. Let $p \in M$ be a point whose coordinates are $\{x^\mu\}$. Let $X = \varepsilon^\mu e_\mu$ and $Y = \delta^\mu e_\mu$ be infinitesimal vectors in T_pM. If these vectors are regarded as small displacements, they define two points q and s near p, whose coordinates are $\{x^\mu + \varepsilon^\mu\}$ and $\{x^\mu + \delta^\mu\}$ respectively (figure 7.4). If we parallel transport X along the line ps, we obtain a vector sr_1 whose component is $\varepsilon^\mu - \varepsilon^\lambda \Gamma^\mu_{\nu\lambda}\delta^\nu$. The displacement vector connecting p and r_1 is

$$pr_1 = ps + sr_1 = \delta^\mu + \varepsilon^\mu - \Gamma^\mu_{\nu\lambda}\varepsilon^\lambda\delta^\nu.$$

Similarly, the parallel transport of δ^μ along pq yields a vector

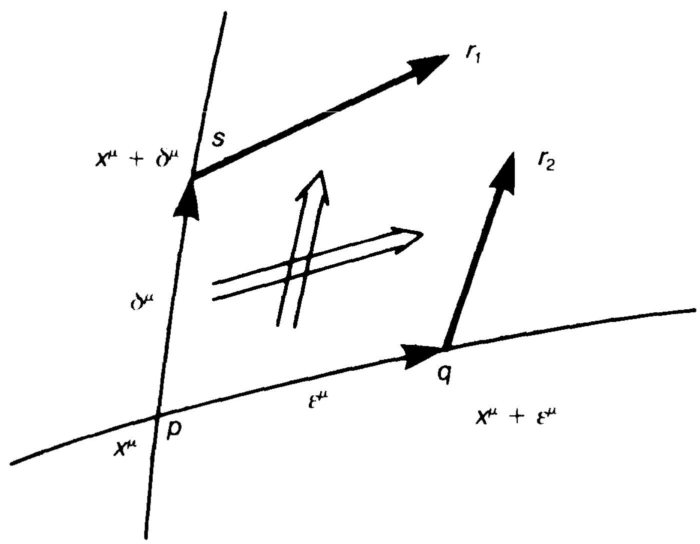


Figure 7.4 The vector qr_2 (sr_1) is the vector ps (pq) parallel transported to q (s). In general, $r_1 \neq r_2$ and the torsion measures the difference r_2r_1.

$$pr_2 = pq + qr_2 = \varepsilon^\mu + \delta^\mu - \Gamma^\mu_{\lambda\nu}\varepsilon^\lambda\delta^\nu.$$

In general r_1 and r_2 do not agree and the difference is

$$r_2r_1 = pr_2 - pr_1 = (\Gamma^\mu_{\nu\lambda} - \Gamma^\mu_{\lambda\nu})\varepsilon^\lambda\delta^\nu = T^\mu{}_{\nu\lambda}\varepsilon^\lambda\delta^\nu. \qquad (7.45)$$

Thus the torsion tensor measures the failure of the closure of the parallelogram made up of the small displacement vectors and their parallel transports.

Example 7.11 Suppose we are navigating on the surface of the Earth. We *define* a vector to be parallel transported if the angle between the vector and the latitude is kept fixed during the navigation. [*Remarks:* This definition of parallel transport is not the *usual* one. For example, the geodesic is not a great circle but a straight line on Mercator's projection. See example 7.17.] Suppose we navigate along a small quadrilateral *pqrs* made up of latitudes and longitudes (figure 7.5(*a*)). We parallel transport a vector at p along *pqr* and *psr*, separately. According to our definition of parallel transport, two vectors at r should agree, hence the curvature tensor vanishes. To find the torsion, we parametrise the points p, q, r and s as in figure 7.5(*b*). We find the torsion by evaluating the difference between pr_1 and pr_2 as in (7.45). If we parallel transport the vector pq along ps, we obtain a vector sr_1, whose length is $R \sin\theta\, d\phi$. On the other hand, a parallel transport of the vector ps along pq yields a vector $qr_2 = qr$. Since sr has a length $R\sin(\theta - d\theta)\, d\phi \simeq R \sin\theta\, d\phi - R\cos\ \theta\, d\theta\, d\phi$, we find that r_1r_2 has a length $R\cos\theta\, d\theta\, d\phi$. Since r_1r_2 is parallel to $-\partial/\partial\phi$, the connection has a torsion $T^{\varphi}{}_{\theta\varphi}$, see (7.45). From $g_{\varphi\varphi} = R^2 \sin^2\theta$, we find that r_1r_2 has components $(0, -\cot\theta\, d\theta\, d\phi)$. Since the ϕ-component of r_1r_2 is equal to $T^{\varphi}{}_{\theta\varphi}\, d\theta\, d\phi$, we obtain $T^{\varphi}{}_{\theta\varphi} = -\cot\theta$.

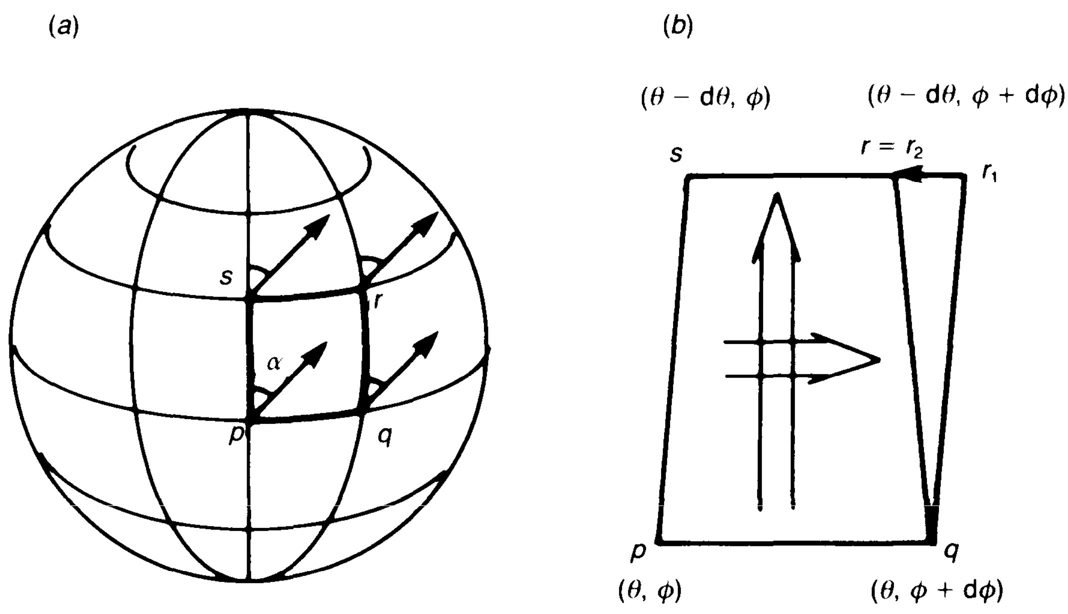


Figure 7.5 (*a*) If a vector makes an angle α with the longitude at p, this angle is kept fixed during parallel transport. (*b*) The vector sr_1 (qr_2) is the vector pq (ps) parallel transported to s (q). The torsion does not vanish.

Note that the basis $\{\partial/\partial\theta, \partial/\partial\phi\}$ is not well defined at the poles. It is known that the sphere S^2 does not admit two vector fields which are linearly independent everywhere on S^2. Any vector field on S^2 must vanish somewhere on S^2 and hence cannot be linearly independent of the other vector field there. If an m-dimensional manifold M admits m vector fields which are linearly independent everywhere, M is said to be **parallelisable**. On a parallelisable manifold, we can use these m vector fields to define a tangent space at each point of M. A vector $V_p \in T_pM$ is defined to be parallel to $V_q \in T_qM$ if all the *components* of V_p at T_pM are equal to those of V_q at T_qM. Since the vector fields are defined throughout M, this parallelism should be independent of the path connecting p and q, hence the Riemann curvature tensor vanishes although the torsion tensor may not in general. For S^m, this is possible only when $m = 1$, 3 and 7, which is closely related to the existence of complex numbers, quaternions and octonions respectively. For definiteness, we consider

$$S^3 = \left\{(x^1, x^2, x^3, x^4) \mid \sum_{i=1}^{4} (x^i)^2 = 1\right\}$$

embedded in $(\mathbb{R}^4, \delta)$. Three orthonormal vectors

$$\begin{aligned} \boldsymbol{e}_1(\boldsymbol{x}) &= (-x^2, x^1, -x^4, x^3) \\ \boldsymbol{e}_2(\boldsymbol{x}) &= (-x^3, x^4, x^1, -x^2) \\ \boldsymbol{e}_3(\boldsymbol{x}) &= (-x^4, -x^3, x^2, x^1) \end{aligned} \tag{7.46}$$

are orthogonal to $\boldsymbol{x} = (x^1, x^2, x^3, x^4)$ and linearly independent everywhere on S^3, hence define the tangent space T_xS^3. Two vectors $\boldsymbol{V}_1(\boldsymbol{x})$ and $\boldsymbol{V}_2(\boldsymbol{y})$ are parallel if $\boldsymbol{V}_1(\boldsymbol{x}) = \Sigma c^i \boldsymbol{e}_i(\boldsymbol{x})$ and $\boldsymbol{V}_2(\boldsymbol{y}) = \Sigma c^i \boldsymbol{e}_i(\boldsymbol{y})$. The connection coefficients are computed from (7.14). Let $\varepsilon \boldsymbol{e}_1(\boldsymbol{x})$ be a small displacement under which $\boldsymbol{x} = (x^1, x^2, x^3, x^4)$ changes to $\boldsymbol{x}' = \boldsymbol{x} + \varepsilon \boldsymbol{e}_1(x) = \{x^1 - \varepsilon x^2, x^2 + \varepsilon x^1, x^3 - \varepsilon x^4, x^4 + \varepsilon x^3\}$. The difference between the basis vectors at $\boldsymbol{x}$ and $\boldsymbol{x}'$ is $\boldsymbol{e}_2(\boldsymbol{x}') - \boldsymbol{e}_2(\boldsymbol{x}) = (-x^3 + \varepsilon x^4, x^4 + \varepsilon x^3, x^1 - \varepsilon x^2, -x^2 - \varepsilon x^1) - (x^1, x^2, x^3, x^4) = -\varepsilon \boldsymbol{e}_3(\boldsymbol{x}) = \varepsilon \Gamma^{\mu}{}_{12} \boldsymbol{e}_\mu(\boldsymbol{x})$, hence $\Gamma^3{}_{12} = -1$, $\Gamma^1{}_{12} = \Gamma^2{}_{12} = 0$. Similarly $\Gamma^3{}_{21} = 1$, hence we find $T^3{}_{12} = -2$. The reader should complete the computation of the connection coefficients and verify that $\Gamma^\lambda{}_{\mu\nu} = -2$ $(+2)$ if $(\lambda\mu\nu)$ is an even (odd) permutation of (123) and vanishes otherwise.

Let us see how this parallelisability of S^3 is related to the existence of quaternions. The multiplication rule of quaternions is

$$\begin{aligned} &(x^1, x^2, x^3, x^4)\cdot(y^1, y^2, y^3, y^4) \\ &\quad = (x^1y^1 - x^2y^2 - x^3y^3 - x^4y^4, x^1y^2 + x^2y^1 + x^3y^4 - x^4y^3, \\ &\qquad x^1y^3 - x^2y^4 + x^3y^1 + x^4y^2, x^1y^4 + x^2y^3 - x^3y^2 + x^4y^1). \end{aligned} \tag{7.47}$$

S^3 may be defined by the set of unit quaternions

$$S^3 = \{(x^1, x^2, x^3, x^4) | x \cdot \bar{x} = (1, 0, 0, 0)\}$$

where the conjugate of x is defined by $\bar{x} = (x^1, -x^2, -x^3, -x^4)$. According to (7.46), the tangent space at $\boldsymbol{x}_0 = (1, 0, 0, 0)$ is spanned by

$$\boldsymbol{e}_1 = (0, 1, 0, 0),\ \boldsymbol{e}_2 = (0, 0, 1, 0),\ \boldsymbol{e}_3 = (0, 0, 0, 1).$$

Then the basis vectors (7.46) of the tangent space at $\boldsymbol{x} = (x^1, x^2, x^3, x^4)$ are expressed as the *quaternion products*

$$\boldsymbol{e}_1(\boldsymbol{x}) = \boldsymbol{e}_1 \cdot \boldsymbol{x},\ \boldsymbol{e}_2(\boldsymbol{x}) = \boldsymbol{e}_2 \cdot \boldsymbol{x},\ \boldsymbol{e}_3(\boldsymbol{x}) = \boldsymbol{e}_3 \cdot \boldsymbol{x}. \tag{7.48}$$

Because of this algebra, it is *always* possible to give a set of basis vectors at an arbitrary point of S^3 once it is given at some point, $\boldsymbol{x}_0 = (1, 0, 0, 0)$, for example.

By the same token, a Lie group is parallelisable. If the set of basis vectors $\{V_1, \ldots, V_m\}$ at the unit element e of a Lie group G is given, we can always find a set of basis vectors of $T_g G$ by the left translation of $\{V_\mu\}$ (see §5.6),

$$\{V_1, \ldots, V_n\} \xrightarrow{L_{g*}} \{X_1|_g, \ldots, X_n|_g\}. \tag{7.49}$$

7.3.3 The Ricci tensor and the scalar curvature

From the Riemann curvature tensor, we construct new tensors by contracting the indices. The **Ricci tensor** Ric is a type $(0, 2)$ tensor defined by

$$Ric(X, Y) \equiv \langle dx^\mu, R(e_\mu, Y)X \rangle \tag{7.50a}$$

whose component is

$$Ric_{\mu\nu} = Ric(e_\mu, e_\nu) = R^\lambda{}_{\mu\lambda\nu}. \tag{7.50b}$$

The **scalar curvature** $\mathcal{R}$ is obtained by further contracting indices,

$$\mathcal{R} \equiv g^{\mu\nu} Ric(e_\mu, e_\nu) = g^{\mu\nu} Ric_{\mu\nu}. \tag{7.51}$$

7.4 Levi-Civita connections

7.4.1 The fundamental theorem

Among affine connections, there is a special connection called the **Levi-Civita connection**, which is a natural generalisation of the connection in the classical differential geometry of surfaces. A connection ∇ is called a **symmetric connection** if the torsion tensor vanishes. In the coordinate basis, connection coefficients of a symmetric connection satisfy

$$\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu}. \tag{7.52}$$

Theorem 7.12 **(The fundamental theorem of (pseudo-) Riemannian geometry)** On a (pseudo-) Riemannian manifold (M, g), there exists a unique *symmetric* connection which is *compatible* with the metric g. This connection is called the **Levi-Civita connection**.

Proof: This follows directly from (7.34). Let $\widetilde{\nabla}$ be an arbitrary connection such that

$$\widetilde{\Gamma}^{\kappa}{}_{\mu\nu} = \left\{ {\kappa \atop \mu\nu} \right\} + K^{\kappa}{}_{\mu\nu}$$

where

$$\left\{ {\kappa \atop \mu\nu} \right\}$$

is the Christoffel symbol and K the contorsion tensor. It was shown in exercise 7.8 that $\Gamma^{\kappa}{}_{\mu\nu} \equiv \widetilde{\Gamma}^{\kappa}{}_{\mu\nu} + t^{\kappa}{}_{\mu\nu}$ is another connection coefficient if t is a tensor field of type $(1, 2)$. Now we choose $t^{\kappa}{}_{\mu\nu} = -K^{\kappa}{}_{\mu\nu}$ so that

$$\Gamma^{\kappa}{}_{\mu\nu} = \left\{ {\kappa \atop \mu\nu} \right\} = \tfrac{1}{2} g^{\kappa\lambda}(\partial_{\mu} g_{\lambda\nu} + \partial_{\nu} g_{\lambda\mu} - \partial_{\lambda} g_{\mu\nu}). \tag{7.53}$$

By construction, this is symmetric and certainly unique given a metric. ∎

Exercise 7.13 Let ∇ be a Levi-Civita connection.

(a) Let $f \in \mathcal{F}(M)$. Show that

$$\nabla_{\mu}\nabla_{\nu} f = \nabla_{\nu}\nabla_{\mu} f. \tag{7.54}$$

(b) Let $\omega \in \Omega^1(M)$. Show that

$$\mathrm{d}\omega = (\nabla_{\mu}\omega)_{\nu}\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}. \tag{7.55}$$

(c) Let $\omega \in \Omega^1(M)$ and let $U \in \mathcal{X}(M)$ be the corresponding vector field; $U^{\mu} = g^{\mu\nu}\omega_{\nu}$. Show that for any $V \in \mathcal{X}(M)$,

$$g(\nabla_X U, V) = \langle \nabla_X \omega, V \rangle. \tag{7.56}$$

Example 7.14

(a) The metric on $\mathbb{R}^2$ in polar coordinates is $g = \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\phi \otimes \mathrm{d}\phi$. The non-vanishing components of the Levi-Civita connection coefficients are $\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\lambda r} = r^{-1}$ and $\Gamma^{r}{}_{\phi\phi} = -r$. This is in agreement with the result obtained in example 7.4.

(*b*) The induced metric on S^2 is $g = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi$. The non-vanishing components of the Levi-Civita connection are

$$\Gamma^{\theta}{}_{\phi\phi} = -\cos\theta\sin\theta \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta. \tag{7.57}$$

7.4.2 The Levi-Civita connection in the classical geometry of surfaces

In the classical differential geometry of surfaces embedded in $\mathbb{R}^3$, Levi-Civita defined the parallelism of vectors at the nearby points p and q in the following sense (figure 7.6). First take the tangent plane at p and a vector V_p at p, which lies in the tangent plane. A vector V_q at q is defined to be parallel to V_p if the projection of V_q to the tangent plane at p is parallel to V_p in our usual sense. Now take two points q and s near p as in figure 7.7 and parallel transport the displacement vectors pq along ps and ps along pq. If the parallelism is defined in the

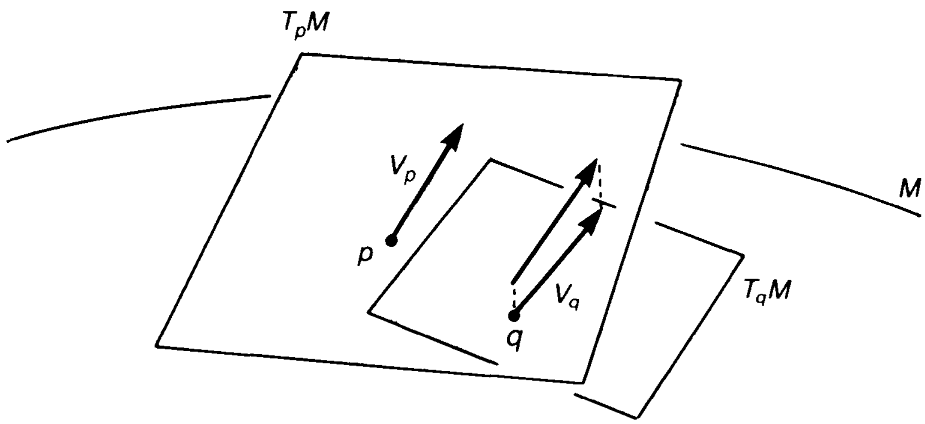


Figure 7.6 On a surface M, a vector $V_p \in T_pM$ is defined to be parallel to $V_{q'} \in T_{q'}M$ if the projection of $V_{q'}$ onto T_pM is parallel to V_p in our ordinary sense of parallelism in $\mathbb{R}^2$.

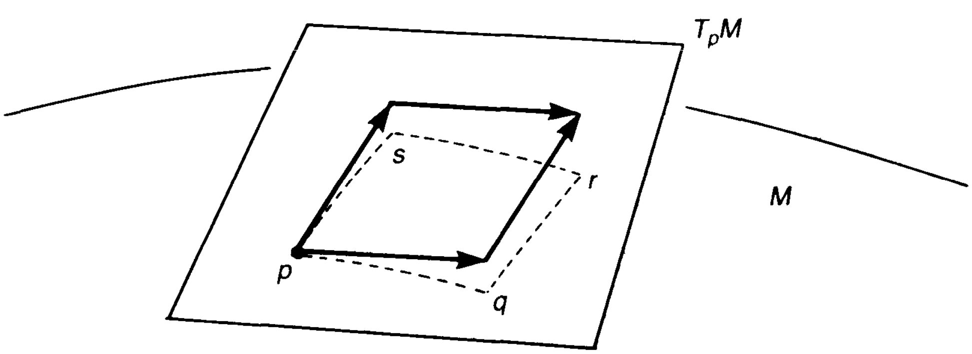


Figure 7.7 If the parallelism is defined in the sense of Levi-Civita, the torsion vanishes identically.

sense of Levi-Civita, the displacement vectors projected to the tangent plane at p form a closed parallelogram, hence this parallelism has vanishing torsion. As has been proved in theorem 7.12, there exists a unique connection which has vanishing torsion, which generalises the parallelism defined here to arbitrary manifolds.

7.4.3 Geodesics

When the Levi-Civita connection is employed, we can compute the connection coefficients, Riemann tensors and many relations involving these by simple routines. Besides this simplicity, the Levi-Civita connection provides a geodesic (defined as the *straightest* possible curve) with another picture, namely the *shortest* possible curve connecting two given points. In Newtonian mechanics, the trajectory of a free particle is the straightest possible as well as the shortest possible curve, that is, a straight line. Einstein proposed that this property should be satisfied in general relativity as well; if gravity is understood as a part of the geometry of spacetime, a freely falling particle should follow the straightest as well as the shortest possible curve. [*Remark:* To be precise, the shortest possible curve is too strong a condition. As we see below, a geodesic defined with respect to the Levi-Civita connection gives the local extremum of the length of a curve connecting two points.]

Example 7.15 In a flat manifold $(\mathbb{R}^m, \delta)$ or $(\mathbb{R}^m, \eta)$, the Levi-Civita connection coefficients Γ vanish identically. Hence the geodesic equation (7.19*b*) is easily solved to yield $x^\mu = A^\mu t + B^\mu$, where A^μ and B^μ are constants.

Exercise 7.16 A metric on a cylinder $S^1 \times \mathbb{R}$ is given by $g = \mathrm{d}\phi \otimes \mathrm{d}\phi + \mathrm{d}z \otimes \mathrm{d}z$, where ϕ is the polar angle of S^1 and z the coordinate of $\mathbb{R}$. Show that the geodesics given by the Levi-Civita connection are helices.

The equivalence of the straightest possible curve and the local extremum of the distance is proved as follows. First we parametrise the curve by the distance s along the curve, $x^\mu = x^\mu(s)$. The length of a path c connecting two points p and q is

$$I(c) = \int_c \mathrm{d}s = \int_c (g_{\mu\nu} x'^\mu x'^\nu)^{1/2}\, \mathrm{d}s \tag{7.58}$$

where $x'^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}s$. Instead of deriving the Euler–Lagrange equation from (7.58), we will solve a slightly easier problem. Let $F \equiv \frac{1}{2}g_{\mu\nu}x'^{\mu}x'^{\nu}$ and write (7.58) as $I(c) = \int_c L(F)\,\mathrm{d}s$. The Euler–Lagrange equation for the original problem takes the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial x'^{\lambda}}\right) - \frac{\partial L}{\partial x^{\lambda}} = 0. \tag{7.59}$$

Then $F = L^2/2$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial F}{\partial x'^{\lambda}}\right) - \frac{\partial F}{\partial x^{\lambda}} = L\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial x'^{\lambda}}\right) - \frac{\partial L}{\partial x^{\lambda}}\right] + \frac{\partial L}{\partial x'^{\lambda}}\frac{\mathrm{d}L}{\mathrm{d}s} = \frac{\partial L}{\partial x'^{\lambda}}\frac{\mathrm{d}L}{\mathrm{d}s}. \tag{7.60}$$

The last expression vanishes since $L \equiv 1$ along the curve; $\mathrm{d}L/\mathrm{d}s = 0$. Now we have proved that F also satisfies the Euler–Lagrange equation provided that L does so. We then have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}(g_{\lambda\mu}x'^{\mu}) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}x'^{\mu}x'^{\nu} \\
&\qquad = \frac{\partial g_{\lambda\mu}}{\partial x^{\nu}}x'^{\mu}x'^{\nu} + g_{\lambda\mu}\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}s^2} - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}x'^{\mu}x'^{\nu} \\
&\qquad = g_{\lambda\mu}\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}s^2} + \frac{1}{2}\left(\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right)\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s} = 0.
\end{aligned} \tag{7.61}$$

If (7.61) is multiplied by $g^{\kappa\lambda}$, we reproduce the geodesic equation (7.19*b*).

Having proved that L and F satisfy the same variational problem, we take advantage of this to compute the Christoffel symbols. Take S^2, for example. F is given by $\frac{1}{2}(\theta'^2 + \sin^2\theta\phi'^2)$ and the Euler–Lagrange equations are

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0 \tag{7.62a}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + 2\cot\theta\frac{\mathrm{d}\phi}{\mathrm{d}s}\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0. \tag{7.62b}$$

It is easy to read off the connection coefficients $\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^{\phi}{}_{\phi\theta} = \Gamma^{\phi}{}_{\theta\phi} = \cot\theta$, see (7.57).

Example 7.17 Let us compute the geodesics of S^2. Rather than solving the geodesic equations (7.62) we find the geodesic by minimising the length of a curve connecting two points on S^2. Without loss of generality, we may assign coordinates (θ_1, ϕ_0) and (θ_2, ϕ_0) to these points. Let $\phi = \phi(\theta)$ be a curve connecting these points. Then the length of the curve is

$$I(c) = \int_{\theta_1}^{\theta_2} \left[1 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right)^2\right]^{1/2} \mathrm{d}\theta \tag{7.63}$$

which is minimised when $\mathrm{d}\phi/\mathrm{d}\theta \equiv 0$, that is $\phi \equiv \phi_0$. Thus the geodesic is a great circle (θ, ϕ_0), $\theta_1 \leqslant \theta \leqslant \theta_2$.

[*Remark:* Solving (7.62) is not very difficult. Let $\theta = \theta(\phi)$ be the equation of the geodesic. Then

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}s} \qquad \frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 + \frac{\mathrm{d}\theta}{\mathrm{d}\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}.$$

Substituting these into (7.62*a*), we obtain

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 + \frac{\mathrm{d}\theta}{\mathrm{d}\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0. \tag{7.64}$$

(7.62*b*) and (7.64) yield

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2} - 2\cot\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right)^2 - \sin\theta\cos\theta = 0. \tag{7.65}$$

If we define $f(\theta) \equiv \cot\theta$, (7.65) becomes

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\phi^2} + f = 0$$

whose general solution is $f(\theta) = \cot\theta = A\cos\phi + B\sin\phi$, or

$$A\sin\theta\cos\phi + B\sin\theta\sin\phi - \cos\theta = 0. \tag{7.66}$$

(7.66) is the equation of a great circle which lies in a plane whose normal vector is $(A, B, -1)$.]

Example 7.18 Let U be the upper-half plane $U \equiv \{(x, y)|y > 0\}$ and introduce the **Poincaré metric**

$$g = y^{-2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y). \tag{7.67}$$

The geodesic equations are

$$x'' - \frac{2}{y}x'y' = 0 \tag{7.68a}$$

$$y'' - \frac{1}{y}[(x')^2 + 3(y')^2] = 0 \tag{7.68b}$$

where $x' \equiv \mathrm{d}x/\mathrm{d}s$ etc. Equation (7.68*a*) is easily integrated, if divided by x', to yield

$$x'/y^2 = 1/R \tag{7.69}$$

where R is a constant. Since the parameter s is taken so that the vector (x', y') has unit length, it satisfies $(x'^2 + y'^2)/y^2 = 1$. From (7.69), this becomes $y^2/R^2 + (y'/y)^2 = 1$ or

$$\mathrm{d}s = \frac{\mathrm{d}y}{y(1 - y^2/R^2)^{1/2}} = \frac{\mathrm{d}t}{\sin t}$$

where we put $y = R\sin t$. Equation (7.69) then becomes

$$x' = \frac{y^2}{R} = R\sin^2 t.$$

Now x is solved for t to yield

$$x = \int x'\,\mathrm{d}s = \int \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t$$
$$= \int R\sin t\,\mathrm{d}t = -R\cos t + x_0.$$

Finally we obtain the solution

$$x = -R\cos t + x_0,\ y = R\sin t \qquad (y > 0) \tag{7.70}$$

which is a circle with radius R centred at $(x_0, 0)$. Maximally extended geodesics are given by $0 < t < \pi$ (figure 7.8) whose length is infinite,

$$l = \int \mathrm{d}s = \int_{0+\varepsilon}^{\pi-\varepsilon} \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = \int_{0+\varepsilon}^{\pi-\varepsilon} \frac{1}{\sin t}\,\mathrm{d}t$$
$$= -\frac{1}{2}\ln\frac{1+\cos t}{1-\cos t}\bigg|_{0+\varepsilon}^{\pi-\varepsilon} \xrightarrow[\varepsilon\to 0]{} \infty.$$

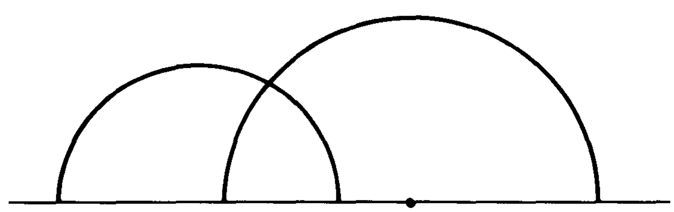


Figure 7.8 Geodesics defined by the Poincaré metric in the upper half plane. The geodesic has an infinite length.

7.4.4 The normal coordinate system

The subject here is not restricted to Levi-Civita connections but it does take an especially simple form when the Levi-Civita connection is employed. Let $c(t)$ be a geodesic in (M, g) defined with respect to a connection ∇, which satisfies

$$c(0) = p \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_p = X = X^\mu e_\mu \in T_pM \tag{7.71}$$

where $\{e_\mu\}$ is the coordinate basis at p. Any geodesic emanating from p

is specified by giving $X \in T_pM$. Take a point q near p. There are many geodesics which connect p and q. However there exists a *unique* geodesic c_q such that $c_q(1) = q$. Let $X_q \in T_pM$ be the tangent vector of this geodesic at p. As long as q is not far from p, q uniquely specifies $X_q = X_q^\mu e_\mu \in T_pM$ and $\varphi : q \to X_q^\mu$ serves as a good coordinate system in the neighbourhood of p. This coordinate system is called the **normal coordinate system** based on p with basis $\{e_\mu\}$. Obviously $\varphi(p) = 0$. We define a map $\mathrm{EXP} : T_pM \to M$ by $\mathrm{EXP}\, X_q = q$. By definition, we have

$$\varphi(\mathrm{EXP}\, X_q^\mu e_\mu) = X_q^\mu. \tag{7.72}$$

With respect to this coordinate system, a geodesic $c(t)$ with $c(0) = p$ and $c(1) = q$ has the coordinate presentation

$$\varphi(c(t)) = X^\mu = X_q^\mu t \tag{7.73}$$

where X_q^μ are the normal coordinates of q.

We now show that Levi-Civita connection coefficients vanish in the normal coordinate system. We write down the geodesic equation in the normal coordinate system,

$$0 = \frac{\mathrm{d}^2 X^\mu}{\mathrm{d}t^2} + \Gamma^\mu{}_{\nu\lambda}(X_q^\kappa t)\frac{\mathrm{d}X^\nu}{\mathrm{d}t}\frac{\mathrm{d}X^\lambda}{\mathrm{d}t} = \Gamma^\mu{}_{\nu\lambda}(X_q^\kappa t)X_q^\nu X_q^\lambda. \tag{7.74}$$

Since $\Gamma^\mu{}_{\nu\lambda}(p)X_q^\nu X_q^\lambda = 0$ for *any* X_q^ν at p for which $t = 0$, we find $\Gamma^\mu{}_{\nu\lambda}(p) + \Gamma^\mu{}_{\lambda\nu}(p) = 0$. Since our connection is symmetric we must have

$$\Gamma^\mu{}_{\nu\lambda}(p) = 0. \tag{7.75}$$

As a consequence, the covariant derivative of any tensor t in this coordinate system takes the extremely simple form at p,

$$\nabla_X t^{\dots}_{\dots} = X[t^{\dots}_{\dots}]. \tag{7.76}$$

Equation (7.74) does not imply that $\Gamma^\lambda{}_{\mu\nu}$ vanishes at q $(\neq p)$. In fact, we find from (7.42) that

$$R^\kappa{}_{\lambda\mu\nu}(p) = \partial_\mu \Gamma^\kappa{}_{\nu\lambda}(p) - \partial_\nu \Gamma^\kappa{}_{\mu\lambda}(p) \tag{7.77}$$

hence $\partial_\mu \Gamma^\kappa{}_{\nu\lambda}(p) \neq 0$ if $R^\kappa{}_{\lambda\mu\nu}(p) \neq 0$.

7.4.5 Riemann curvature tensor with Levi-Civita connection

Let ∇ be the Levi-Civita connection. The components of the Riemann curvature tensor are given by (7.42) with

$$\Gamma^\lambda{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}$$

while the torsion tensor vanishes by definition. Many formulae are simplified if the Levi-Civita connections are employed.

Exercise 7.19

(*a*) Let $g = \mathrm{d}r \otimes \mathrm{d}r + r^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi)$ be the metric of $(\mathbb{R}^3, \delta)$, where $0 \leqslant \theta < \pi$ and $0 \leqslant \phi < 2\pi$. Show, by direct calculation, that all the components of the Riemann curvature tensor with respect to the Levi-Civita connection vanish.

(*b*) The spatially homogeneous and isotropic universe is described by the **Robertson–Walker metric**,

$$g = -\mathrm{d}t \otimes \mathrm{d}t + a^2(t)\left(\frac{\mathrm{d}r \otimes \mathrm{d}r}{1 - kr^2} + r^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi)\right) \tag{7.78}$$

where k is a constant, which may be chosen to be -1, 0 or $+1$ by a suitable rescaling of r and $0 \leqslant \theta < \pi$, $0 \leqslant \phi < 2\pi$. If $k = +1$, r is restricted to $0 \leqslant r < 1$. Compute the Riemann tensor, the Ricci tensor and the scalar curvature.

(*c*) The **Schwarzschild metric** is given by

$$\begin{aligned} g = &-\left(1 - \frac{2M}{r}\right)\mathrm{d}t \otimes \mathrm{d}t + \frac{1}{1 - 2M/r}\mathrm{d}r \otimes \mathrm{d}r \\ &+ r^2(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi) \end{aligned} \tag{7.79}$$

where $0 < 2M < r$, $0 \leqslant \theta < \pi$, $0 \leqslant \phi < 2\pi$. Compute the Riemann tensor, the Ricci tensor and the scalar curvature. [*Remark:* The metric (7.79) describes a black hole of mass M.]

Exercise 7.20 Let R be the Riemann tensor defined with respect to the Levi-Civita connection. Show that

$$\begin{aligned} R_{\kappa\lambda\mu\nu} = &\frac{1}{2}\left(\frac{\partial^2 g_{\kappa\mu}}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 g_{\lambda\mu}}{\partial x^\kappa \partial x^\nu} - \frac{\partial^2 g_{\kappa\nu}}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\mu}\right) \\ &+ g_{\zeta\eta}(\Gamma^\zeta{}_{\kappa\mu}\Gamma^\eta{}_{\lambda\nu} - \Gamma^\zeta{}_{\kappa\nu}\Gamma^\eta{}_{\lambda\mu}) \end{aligned}$$

where $R_{\kappa\lambda\mu\nu} \equiv g_{\kappa\zeta}R^\zeta{}_{\lambda\mu\nu}$. Verify the following symmetries,

$$R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu} \qquad \text{(cf (7.43))} \tag{7.80a}$$

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu} \tag{7.80b}$$

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda} \tag{7.80c}$$

$$Ric_{\mu\nu} = Ric_{\nu\mu}. \tag{7.80d}$$

Theorem 7.21 (**Bianchi identities**) Let R be the Riemann tensor defined with respect to the Levi-Civita connection. Then R satisfies the following identities,

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$$

$$\text{(the **first Bianchi identity**)} \tag{7.81a}$$

$$(\nabla_X R)(Y, Z)V + (\nabla_Z R)(X, Y)V + (\nabla_Y R)(Z, X)V = 0$$

(the **second Bianchi identity**). (7.81b)

Proof: Our proof follows Nomizu (1981). Define the symmetrisor $\mathfrak{S}$ by $\mathfrak{S}\{f(X, Y, Z)\} = f(X, Y, Z) + f(Z, X, Y) + f(Y, Z, X)$. Let us prove the first Bianchi identity $\mathfrak{S}\{R(X, Y)Z\} = 0$. Covariant differentiation of the identity $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$ with respect to Z yields

$$\begin{aligned} 0 &= \nabla_Z\{\nabla_X Y - \nabla_Y X - [X, Y]\} \\ &= \nabla_Z \nabla_X Y - \nabla_Z \nabla_Y X - \{\nabla_{[X,Y]} Z + [Z, [X, Y]]\} \end{aligned}$$

where the torsion-free condition has been used again to derive the second equality. Symmetrising this, we have

$$\begin{aligned} 0 &= \mathfrak{S}\{\nabla_Z \nabla_X Y - \nabla_Z \nabla_Y X - \nabla_{[X,Y]} Z - [Z, [X, Y]]\} \\ &= \mathfrak{S}\{\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\} = \mathfrak{S}\{R(X, Y)Z\} \end{aligned}$$

where the Jacobi identity $\mathfrak{S}\{[X, [Y, Z]]\} = 0$ has been used.

The second Bianchi identity becomes $\mathfrak{S}\{(\nabla_X R)(Y, Z)\}V$, where $\mathfrak{S}$ symmetrises (X, Y, Z) only. If the identity $R(T(X, Y), Z)V = R(\nabla_X Y - \nabla_Y X - [X, Y], Z)V = 0$ is symmetrised, we have

$$\begin{aligned} 0 &= \mathfrak{S}\{R(\nabla_X Y, Z) - R(\nabla_Y X, Z) - R([X, Y], Z)\}V \\ &= \mathfrak{S}\{R(\nabla_Z X, Y) + R(X, \nabla_Z Y) - R([X, Y], Z)\}V. \qquad (7.82) \end{aligned}$$

If we note the Leibnitz rule,

$$\begin{aligned} \nabla_Z\{R(X, Y)V\} &= (\nabla_Z R)(X, Y)V \\ &\quad + R(X, Y)\nabla_Z V + R(\nabla_Z X, Y)V + R(X, \nabla_Z Y)V \end{aligned}$$

(7.82) becomes

$$0 = \mathfrak{S}\{-(\nabla_Z R)(X, Y) + [\nabla_Z, R(X, Y)] - R([X, Y], Z)\}V.$$

The last two terms vanish if $R(X, Y)V = \{[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\}V$ is substituted into them,

$$\begin{aligned} \mathfrak{S}\{[\nabla_Z, R(X, Y)] - R([X, Y], Z)\}V &= \mathfrak{S}\{[\nabla_Z, [\nabla_X, \nabla_Y]] - [\nabla_Z, \nabla_{[X,Y]}] \\ &\quad - [\nabla_{[X,Y]}, \nabla_Z] + \nabla_{[[X,Y],Z]}\}V = 0 \end{aligned}$$

where the Jacobi identities $\mathfrak{S}\{[\nabla_Z, [\nabla_X, \nabla_Y]]\} = \mathfrak{S}\{[[X, Y], Z]\} = 0$ have been used. We finally obtain $\mathfrak{S}\{(\nabla_Z R)(X, Y)\}V = 0$. ■

In components, the Bianchi identities are

$$R^{\kappa}{}_{\lambda\mu\nu} + R^{\kappa}{}_{\mu\nu\lambda} + R^{\kappa}{}_{\nu\lambda\mu} = 0$$

(the first Bianchi identity) (7.83a)

$$(\nabla_\kappa R)^\xi{}_{\lambda\mu\nu} + (\nabla_\mu R)^\xi{}_{\lambda\nu\kappa} + (\nabla_\nu R)^\xi{}_{\lambda\kappa\mu} = 0$$

(the second Bianchi identity). (7.83b)

By contracting the indices ξ and μ of the second Bianchi identity, we obtain an important relation,

$$(\nabla_\kappa Ric)_{\lambda\nu} + (\nabla_\mu R)^\mu{}_{\lambda\nu\kappa} - (\nabla_\nu Ric)_{\lambda\kappa} = 0. \tag{7.84}$$

If the indices λ and ν are further contracted, we have $\nabla_\mu(\mathcal{R}\delta - 2Ric)^\mu{}_\kappa = 0$ or

$$\nabla_\mu G^{\mu\nu} = 0 \tag{7.85}$$

where $G^{\mu\nu}$ is the **Einstein tensor** defined by

$$G^{\mu\nu} = Ric^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}\mathcal{R}. \tag{7.86}$$

Historically, when Einstein formulated general relativity, he first equated the Ricci tensor $Ric^{\mu\nu}$ to the energy momentum tensor $T^{\mu\nu}$. Later he realised that $T^{\mu\nu}$ satisfies the covariant conservation equation $\nabla_\mu T^{\mu\nu} = 0$ while $Ric^{\mu\nu}$ does not. To avoid this difficulty, he proposed that $G^{\mu\nu}$ should be equated to $T^{\mu\nu}$. This new equation is natural in the sense that it can be derived from a scalar action by variation, see §7.10.

Exercise 7.22 Let (M, g) be a two-dimensional manifold with $g = -\mathrm{d}t \otimes \mathrm{d}t + R^2(t)\,\mathrm{d}x \otimes \mathrm{d}x$, where $R(t)$ is an arbitrary function of t. Show that the Einstein tensor vanishes.

The symmetry properties (7.80a–c) restrict the number of independent components of the Riemann tensor. Let m be the dimension of a manifold (M, g). The antisymmetry $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\nu\mu}$ implies there are $N \equiv \binom{m}{2}$ independent choices of the pair, (μ, ν). Similarly, from $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}$, we find there are N independent paris of (κ, λ). Since $R_{\kappa\lambda\mu\nu}$ is symmetric with respect to the interchange of the pairs (κ, λ) and (μ, ν), the number of independent choices of the pairs reduces from N^2 to $\binom{N+1}{2} = \frac{1}{2}N(N+1)$. The first Bianchi identity

$$R_{\kappa\lambda\mu\nu} + R_{\kappa\mu\nu\lambda} + R_{\kappa\nu\lambda\mu} = 0 \tag{7.87}$$

further reduces the number of independent components. The LHS of (7.87) is totally antisymmetric with respect to the interchange of the indices (λ, μ, ν). Furthermore, the antisymmetry (7.80b) ensures that it is totally antisymmetric in all the indices. If $m < 4$, (7.87) is trivially satisfied and it imposes no additional restrictions. If $m \geqslant 4$, (7.87) yields non-trivial constraints only when all the indices are different. The number of constraints is equal to the number of possible ways of choosing four different indices out of m indices, namely $\binom{m}{4}$. Noting that

$$\binom{m}{4} = m(m-1)(m-2)(m-3)/4!$$

vanishes for $m < 4$, the number of independent components of the Riemann tensor is given by

$$F(m) \equiv \tfrac{1}{2}\binom{m}{2}\left[\binom{m}{2}+1\right] - \binom{m}{4} = \tfrac{1}{12}m^2(m^2-1). \qquad (7.88)$$

$F(1) = 0$ implies that one-dimensional manifolds are flat. Since $F(2) = 1$, there is only one independent component R_{1212} on a two-dimensional manifold, other components being either 0 or $\pm R_{1212}$. $F(4) = 20$ is well known in general relativity.

Exercise 7.23 Let (M, g) be a two-dimensional manifold. Show that the Riemann tensor is written as

$$R_{\kappa\lambda\mu\nu} = K(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}) \qquad (7.89)$$

where $K \in \mathcal{F}(M)$. Compute the Ricci tensor to show $Ric_{\mu\nu} \propto g_{\mu\nu}$. Compute the scalar curvature to show $K = \mathcal{R}/2$.

7.5 Holonomy

Let (M, g) be an m-dimensional Riemannian manifold with an affine connection ∇. The connection naturally defines a transformation group at each tangent space T_pM as follows.

Definition 7.24 Let p be a point in (M, g) and consider the set of closed loops at p, $\{c(t)|0 \leqslant t \leqslant 1,\ c(0) = c(1) = p\}$. Take a vector $X \in T_pM$ and parallel transport X along a curve $c(t)$. After a trip along $c(t)$, we end up with a new vector $X_c \in T_pM$. Thus the loop $c(t)$ and the connection ∇ induces a linear transformation

$$P_c : T_pM \to T_pM. \qquad (7.90)$$

The set of these transformations is denoted by $H(p)$ and is called the **holonomy group** at p.

We assume that $H(p)$ acts on T_pM from the right, $P_cX = Xh$, $h \in H(p)$. In components, this becomes $P_cX = X^\mu h_\mu{}^\nu e_\nu$, $\{e_\nu\}$ being the basis of T_pM. It is easy to see that $H(p)$ is a group. The product $P_{c'}P_c$ corresponds to parallel transport along c first and then c'. If we write $P_d = P_{c'}P_c$, the loop d is given by

$$d(t) = \begin{cases} c(2t) & 0 \leqslant t \leqslant \tfrac{1}{2} \\ c'(2t-1) & \tfrac{1}{2} \leqslant t \leqslant 1. \end{cases} \qquad (7.91)$$

The unit element corresponds to the constant map $c_p(t) = p$, $0 \leqslant t \leqslant 1$ and the inverse of P_c is given by $P_{c^{-1}}$, where $c^{-1}(t) \equiv c(1-t)$. Note that $H(p)$ is a subgroup of $\mathrm{GL}(m, \mathbb{R})$, which is the maximal holonomy group possible. $H(p)$ is trivial if and only if the Riemann tensor vanishes. In particular, if (M, g) is parallelisable (see example 7.11), we can make $H(p)$ trivial.

If M is (arcwise-) connected, any two points $p, q \in M$ are connected by a curve a. The curve a defines a map $\tau_a : T_pM \to T_qM$ by parallel transporting a vector in T_pM to T_qM along a. Then the holonomy groups $H(p)$ and $H(q)$ are related by

$$H(q) = \tau_a{}^{-1} H(p) \tau_a \tag{7.92}$$

hence $H(q)$ is isomorphic to $H(p)$.

In general, the holonomy group is a subgroup of $\mathrm{GL}(m, \mathbb{R})$. If ∇ is a metric connection, ∇ preserves the length of a vector, $g_p(P_c(X), P_c(X)) = g_p(X, X)$ for $X \in T_pM$. Then the holonomy group must be a subgroup of $\mathrm{SO}(m)$ if (M, g) is orientable and Riemannian and $\mathrm{SO}(m-1, 1)$ if it is orientable and Lorentzian.

Example 7.25 We work out the holonomy group of the Levi-Civita connection on S^2 with the metric $g = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta \mathrm{d}\phi \otimes \mathrm{d}\phi$. The non-vanishing connection coefficients are $\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$. For simplicity, we take a vector $e_\theta = \partial/\partial\theta$ at a point $(\theta_0, 0)$ and parallel transport it along a circle $\theta = \theta_0$, $0 \leqslant \phi \leqslant 2\pi$. Let X be the vector e_θ parallel transported along the circle. The vector $X = X^\theta e_\theta + X^\phi e_\phi$ satisfies

$$\partial_\phi X^\theta - \sin\theta_0 \cos\theta_0 X^\phi = 0 \tag{7.93a}$$

$$\partial_\phi X^\phi + \cot\theta_0 X^\theta = 0. \tag{7.93b}$$

(7.93) represent the harmonic oscillations. Indeed if we take a ϕ-derivative of (7.93a) and use (7.93b), we have

$$\frac{\mathrm{d}^2 X^\theta}{\mathrm{d}\phi^2} - \sin\theta_0\cos\theta_0 \frac{\mathrm{d}X^\phi}{\mathrm{d}\phi} = \frac{\mathrm{d}^2 X^\theta}{\mathrm{d}\phi^2} - \cos^2\theta_0 X^\theta = 0. \tag{7.94}$$

The general solution is $X^\theta = A\cos(C_0\phi) + B\sin(C_0\phi)$, where $C_0 \equiv \cos\theta_0$. Since $X^\theta = 1$ at $\phi = 0$, we have

$$X^\theta = \cos(C_0\phi) \qquad X^\phi = -\frac{\sin(C_0\phi)}{\sin\theta_0}.$$

After parallel transport along the circle, we end up with

$$X(\phi = 2\pi) = \cos(2\pi C_0)e_\theta - \frac{\sin(2\pi C_0)}{\sin\theta_0} e_\phi. \tag{7.95}$$

Now the vector is rotated by $\Theta = 2\pi\cos\theta_0$ in $T_{(\theta_0,0)}S^2$, with its magnitude kept fixed. If we take a point $p \in S^2$ and a circle in S^2, which passes through p, we can always find a coordinate system such that the circle is given by $\theta = \theta_0$ $(0 \leqslant \theta < \pi)$ and we can apply our calculation above. The rotation angle is $-2\pi \leq \theta < 2\pi$ and we find that the holonomy group at $p \in S^2$ is SO(2).

In general, S^m ($m \geq 2$) admits the holonomy group SO(m). Product manifolds admit more restricted holonomy groups. The following example is taken from Horowitz (1986). Consider six-dimensional manifolds made of the spheres with standard metrics. Examples are S^6, $S^3 \times S^3$, $S^2 \times S^2 \times S^2$, $T^6 = S^1 \times \ldots \times S^1$. Their holonomy groups are

(i) S^6; $H(p) = \mathrm{SO}(6)$
(ii) $S^3 \times S^3$; $H(p) = \mathrm{SO}(3) \times \mathrm{SO}(3)$
(iii) $S^2 \times S^2 \times S^2$; $H(p) = \mathrm{SO}(2) \times \mathrm{SO}(2) \times \mathrm{SO}(2)$
(iv) T^6; $H(p)$ is trivial.

Exercise 7.26 Show that the holonomy group of the Levi-Civita connection of the Poincaré metric given in example 7.18 is SO(2).

7.6 Isometries and conformal transformations

7.6.1 *Isometries*

Definition 7.27 Let (M, g) be a (pseudo-) Riemannian manifold. A diffeomorphism $f : M \to M$ is an **isometry** if it preserves the metric

$$f^* g_{f(p)} = g_p \tag{7.96a}$$

that is, if $g_{f(p)}(f_* X, f_* Y) = g_p(X, Y)$ for $X, Y \in T_pM$.

In components, (7.96a) becomes

$$\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = g_{\mu\nu}(p) \tag{7.96b}$$

where x and y are the coordinates of p and $f(p)$ respectively. The identity map, the composition of the isometries and the inverse of an isometry are isometries; all these isometries form a group. Since an isometry preserves the *length* of a vector, in particular that of an infinitesimal displacement vector, it may be regarded as a *rigid motion*. For example, in $\mathbb{R}^n$, the Euclidean group E^n, that is the set of maps $f : \boldsymbol{x} \mapsto A\boldsymbol{x} + T (A \in \mathrm{SO}(n), T \in \mathbb{R}^n)$, is the isometry group.

7.6.2 *Conformal transformations*

Definition 7.28 Let (M, g) be a (pseudo-) Riemannian manifold. A diffeomorphism $f : M \to M$ is called a **conformal transformation** if it preserves the metric *up to a scale*,

$$f^* g_{f(p)} = e^{2\sigma} g_p, \ \sigma \in \mathcal{F}(M) \tag{7.97a}$$

namely, $g_{f(p)}(f_* X, f_* Y) = e^{2\sigma} g_p(X, Y)$ for $X, Y \in T_pM$.

In components, the condition (7.97a) becomes

$$\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = \mathrm{e}^{2\sigma(p)} g_{\mu\nu}(p). \tag{7.97b}$$

The set of conformal transformations on M is a group, the **conformal group** denoted by Conf(M). Let us define the angle θ between two vectors $X = X^\mu\partial_\mu$, $Y = Y^\mu\partial_\mu \in T_pM$ by

$$\begin{aligned}\cos\theta &\equiv g_p(X, Y)/[g_p(X, X)\cdot g_p(Y, Y)]^{1/2}\\ &= g_{\mu\nu}X^\mu Y^\nu/[g_{\xi\eta}X^\xi X^\eta\cdot g_{\kappa\lambda}Y^\kappa Y^\lambda]^{1/2}.\end{aligned} \tag{7.98}$$

If f is a conformal transformation, the angle θ' between f_*X and f_*Y is given by

$$\cos\theta' = \mathrm{e}^{2\sigma}g_{\mu\nu}X^\mu Y^\nu/[\mathrm{e}^{2\sigma}g_{\xi\eta}X^\xi X^\eta\cdot \mathrm{e}^{2\sigma}g_{\kappa\lambda}Y^\kappa Y^\lambda]^{1/2} = \cos\theta$$

hence f preserves the angle. In other words, f changes the *scale* but not the *shape*.

A concept related to conformal transformations is Weyl rescaling. Let g and $\bar{g}$ be metrics on a manifold M. $\bar{g}$ is said to be **conformally related** to g if

$$\bar{g}_p = \mathrm{e}^{2\sigma(p)} g_p. \tag{7.99}$$

Clearly this is an equivalence relation among the set of metrics on M. The equivalence class is called the **conformal structure**. The transformation $g \rightarrow \mathrm{e}^{2\sigma}g$ is called a **Weyl rescaling**. The set of Weyl rescalings on M is a group denoted by Weyl(M).

Example 7.29 Let $w = f(z)$ be a holomorphic function defined on the complex plane $\mathbb{C}$. [A C^∞-function $f(x, y)$ regarded as a function of $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$ is holomorphic if $\partial_{\bar{z}}f(z, \bar{z}) = 0$.]. We write the real part and the imaginary part of the respective variables as $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$. The map $f : (x, y) \mapsto (u, v)$ is conformal since

$$\begin{aligned}\mathrm{d}u^2 + \mathrm{d}v^2 &= \left(\frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y\right)^2 + \left(\frac{\partial v}{\partial x}\,\mathrm{d}x + \frac{\partial v}{\partial y}\,\mathrm{d}y\right)^2\\ &= \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right](\mathrm{d}x^2 + \mathrm{d}y^2)\end{aligned} \tag{7.100}$$

where use has been made of the Cauchy–Riemann relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(7.100) shows that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a conformal transformation.

Exercise 7.30 Let $f : M \rightarrow M$ be a conformal transformation on a Lorentz manifold (M, g). Show that $f_* : T_pM \rightarrow T_{f(p)}M$ preserves the local light cone structure, namely

$$f_*: \begin{cases} \text{timelike vector} & \mapsto & \text{timelike vector} \\ \text{null vector} & \mapsto & \text{null vector} \\ \text{spacelike vector} & \mapsto & \text{spacelike vector.} \end{cases} \tag{7.101}$$

Let $\bar{g}$ be a metric on M, which is conformally related to g as $\bar{g} = e^{2\sigma(p)}g$. Let us compute the Riemann tensor of $\bar{g}$. We could simply substitute $\bar{g}$ into the defining equation (7.42). However, we follow the elegant coordinate-free derivation of Nomizu (1981). Let K be the difference of the covariant derivatives $\bar{\nabla}$ with respect to $\bar{g}$ and ∇ with respect to g,

$$K(X, Y) \equiv \bar{\nabla}_X Y - \nabla_X Y. \tag{7.102}$$

Proposition 7.31 Let U be a vector field which corresponds to the one-form $d\sigma$; $Z[\sigma] = \langle d\sigma, Z\rangle = g(U, Z)$. Then

$$K(X, Y) = X[\sigma]Y + Y[\sigma]X - g(X, Y)U. \tag{7.103}$$

Proof: It follows from the torsion-free condition that $K(X, Y) = K(Y, X)$. Since $\bar{\nabla}_X \bar{g} = \nabla_X g = 0$, we have

$$X[\bar{g}(Y, Z)] = \bar{\nabla}_X[\bar{g}(Y, Z)] = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}_X \bar{Z})$$

and also

$$\begin{aligned} X[\bar{g}(Y, Z)] &= \nabla_X[e^{2\sigma}g(Y, Z)] \\ &= 2X[\sigma]e^{2\sigma}g(Y, Z) + e^{2\sigma}[g(\nabla_X Y, Z) + g(Y, \nabla_X Z)]. \end{aligned}$$

Taking the difference between these two expressions, we have

$$g(K(X, Y), Z) + g(Y, K(X, Z)) = 2X[\sigma]g(Y, Z). \tag{7.104a}$$

Permutations of (X, Y, Z) yield

$$g(K(Y, X), Z) + g(X, K(Y, Z)) = 2Y[\sigma]g(X, Z) \tag{7.104b}$$

$$g(K(Z, X), Y) + g(X, K(Z, Y)) = 2Z[\sigma]g(X, Y). \tag{7.104c}$$

The combination (7.104a) + (7.104b) − (7.104c) yields

$$g(K(X, Y), Z) = X[\sigma]g(Y, Z) + Y[\sigma]g(X, Z) - Z[\sigma]g(X, Y). \tag{7.105}$$

The last term is modified as

$$Z[\sigma]g(X, Y) = g(U, Z)g(X, Y) = g(g(Y, X)U, Z).$$

Substituting this into (7.105), we find

$$g(K(X, Y) - X[\sigma]Y - Y[\sigma]X + g(X, Y)U, Z) = 0.$$

Since this is true for any Z, we have (7.103). ■

The component expression for K is

$$K(e_\mu, e_\nu) = \bar{\nabla}_\mu e_\nu - \nabla_\mu e_\nu = (\bar{\Gamma}^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\mu\nu})e_\lambda$$
$$= e_\mu[\sigma]e_\nu + e_\nu[\sigma]e_\mu - g(e_\mu, e_\nu)g^{\kappa\lambda}\partial_\kappa\sigma e_\lambda.$$

From this, it is readily seen that

$$\bar{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + \delta^\lambda{}_\nu\partial_\mu\sigma + \delta^\lambda{}_\mu\partial_\nu\sigma - g_{\mu\nu}g^{\kappa\lambda}\partial_\kappa\sigma. \tag{7.106}$$

To find the Riemann curvature tensor, we start from the definition,

$$\begin{aligned}
\bar{R}(X, Y)Z &= \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \\
&= \bar{\nabla}_X[\nabla_Y Z + K(Y, Z)] - \bar{\nabla}_Y[\nabla_X Z + K(X, Z)] \\
&\quad - \{\nabla_{[X,Y]}Z + K([X, Y], Z)\} \\
&= \nabla_X\{\nabla_Y Z + K(Y, Z)\} + K(X, \nabla_Y Z + K(Y, Z)) \\
&\quad - \nabla_Y\{\nabla_X Z + K(X, Z)\} - K(Y, \nabla_X Z + K(X, Z)) \\
&\quad - \{\nabla_{[X,Y]}Z + K([X, Y], Z)\}.
\end{aligned} \tag{7.107}$$

After a straightforward but tedious calculation, we find that

$$\begin{aligned}
\bar{R}(X, Y)Z = R(X, Y)Z &+ \langle\nabla_X \mathrm{d}\sigma, Z\rangle Y - \langle\nabla_Y \mathrm{d}\sigma, Z\rangle X \\
&- g(Y, Z)\nabla_X U + Y[\sigma]Z[\sigma]X \\
&- g(Y, Z)U[\sigma]X + X[\sigma]g(Y, Z)U \\
&+ g(X, Z)\nabla_Y U - X[\sigma]Z[\sigma]Y \\
&+ g(X, Z)U[\sigma]Y - Y[\sigma]g(X, Z)U.
\end{aligned} \tag{7.108}$$

Let us define a type (1, 1) tensor field B by

$$BX \equiv -X[\sigma]U + \nabla_X U + \tfrac{1}{2}U[\sigma]X. \tag{7.109}$$

Since $g(\nabla_Y U, Z) = \langle\nabla_Y \mathrm{d}\sigma, Z\rangle$ (exercise 7.13(*c*)), (7.108) becomes

$$\begin{aligned}
\bar{R}(X, Y)Z = R(X, Y)Z &- [g(Y, Z)BX - g(BX, Z)Y \\
&+ g(BY, Z)X - g(X, Z)BY].
\end{aligned} \tag{7.110}$$

In components, this becomes

$$\bar{R}^\kappa{}_{\lambda\mu\nu} = R^\kappa{}_{\lambda\mu\nu} - g_{\nu\lambda}B_\mu{}^\kappa + g_{\xi\lambda}B_\mu{}^\xi\delta^\kappa{}_\nu - g_{\xi\lambda}B_\nu{}^\xi\delta^\kappa{}_\mu + g_{\mu\lambda}B_\nu{}^\kappa \tag{7.111}$$

where the components of the tensor B are

$$\begin{aligned}
B_\mu{}^\kappa &= -\partial_\mu\sigma U^\kappa + (\nabla_\mu U)^\kappa + \tfrac{1}{2}U[\sigma]\delta_\mu{}^\kappa \\
&= -\partial_\mu\sigma g^{\kappa\lambda}\partial_\lambda\sigma + g^{\kappa\lambda}(\partial_\mu\partial_\lambda\sigma - \Gamma^\xi{}_{\mu\lambda}\partial_\xi\sigma) \\
&\quad + \tfrac{1}{2}g^{\lambda\xi}\partial_\lambda\sigma\partial_\xi\sigma\delta_\mu{}^\kappa.
\end{aligned} \tag{7.112}$$

Note that $B_{\mu\nu} \equiv g_{\nu\lambda}B_\mu{}^\lambda = B_{\nu\mu}$.

By contracting the indices in (7.111), we have

$$\overline{Ric}_{\mu\nu} = Ric_{\mu\nu} - g_{\mu\nu}B_\lambda{}^\lambda - (m-2)B_{\nu\mu} \tag{7.113}$$

$$e^{2\sigma}\bar{\mathcal{R}} = \mathcal{R} - 2(m-1)B_\lambda{}^\lambda \tag{7.114a}$$

where $m = \dim M$. (7.114a) is also written as

$$\bar{g}_{\mu\nu}\bar{\mathcal{R}} = [\mathcal{R} - 2(m-1)B_\lambda{}^\lambda]g_{\mu\nu}. \tag{7.114b}$$

If we eliminate $g_{\mu\nu}B_\lambda{}^\lambda$ and $B_{\mu\nu}$ in $\bar{R}^\kappa{}_{\lambda\mu\nu}$ in favour of $\overline{Ric}$ and $\bar{\mathcal{R}}$ and separate barred and unbarred terms, we find a combination which is independent of σ,

$$\begin{aligned} C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} &+ \frac{1}{m-2}(Ric_{\kappa\mu}g_{\lambda\nu} - Ric_{\lambda\mu}g_{\kappa\nu} + Ric_{\lambda\nu}g_{\kappa\mu} - Ric_{\kappa\nu}g_{\lambda\mu}) \\ &+ \frac{\mathcal{R}}{(m-2)(m-1)}(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}) \end{aligned} \tag{7.115}$$

where $m \geq 4$ (see problem 7.2 for $m = 3$). The tensor C is called the **Weyl tensor**. The reader should verify that $C_{\kappa\lambda\mu\nu} = \bar{C}_{\kappa\lambda\mu\nu}$.

If every point p of a (pseudo-) Riemannian manifold (M, g) has a chart (U, φ) containing p such that $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$, then (M, g) is said to be **conformally flat**. Since the Weyl tensor vanishes for a flat metric, it also vanishes for a conformally flat metric. If $\dim M \geq 4$, then $C = 0$ is the necessary and sufficient condition for conformal flatness (Weyl–Schouten). If $\dim M = 3$, the Weyl tensor vanishes identically; see problem 7.2. If $\dim M = 2$, M is always conformally flat; see next example.

Example 7.32 Any two-dimensional Riemannian manifold (M, g) is conformally flat. Let (x, y) be the original local coordinates with which the metric takes the form

$$ds^2 = g_{xx}\,dx^2 + 2g_{xy}\,dx\,dy + g_{yy}\,dy^2. \tag{7.116}$$

Let $g \equiv g_{xx}g_{yy} - g_{xy}^2$ and write (7.116) as

$$ds^2 = \left(\sqrt{g_{xx}}\,dx + \frac{g_{xy} + i\sqrt{g}}{\sqrt{g_{xx}}}\,dy\right)\left(\sqrt{g_{xx}}\,dx + \frac{g_{xy} - i\sqrt{g}}{\sqrt{g_{xx}}}\,dy\right).$$

According to the theory of differential equations, there exists an integrating factor $\lambda(x, y) = \lambda_1(x, y) + i\lambda_2(x, y)$ such that

$$\lambda\left(\sqrt{g_{xx}}\,dx + \frac{g_{xy} + i\sqrt{g}}{\sqrt{g_{xx}}}\,dy\right) = du + i\,dv \tag{7.117a}$$

$$\bar{\lambda}\left(\sqrt{g_{xx}}\,dx + \frac{g_{xy} - i\sqrt{g}}{\sqrt{g_{xx}}}\,dy\right) = du - i\,dv. \tag{7.117b}$$

Then $ds^2 = (du^2 + dv^2)/|\lambda|^2$ and by setting $|\lambda|^{-2} = e^{2\sigma}$, we have the desired coordinate system. The coordinates (u, v) are called the **isothermal coordinates**. [*Remark:* If the curve u = a constant is regarded as an isothermal curve, v = a constant corresponds to the line of heat flow.]

For example, let $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ be the standard metric of S^2. Noting that

$$\frac{d}{d\theta}\log|\tan(\theta/2)| = \frac{1}{\sin\theta}$$

we find that $f : (\theta, \phi) \mapsto (u, v)$ defined by $u = \log|\tan(\theta/2)|$ and $v = \phi$ yields a conformally flat metric. In fact,

$$ds^2 = \sin^2\theta\left(\frac{d\theta^2}{\sin^2\theta} + d\phi^2\right) = \sin^2\theta(du^2 + dv^2).$$

If (M, g) is a Lorentz manifold, we have integrating factors $\lambda(x, y)$ and $\mu(x, y)$ such that

$$\lambda\left(\sqrt{g_{xx}}\,dx + \frac{g_{xy} + \sqrt{-g}}{\sqrt{g_{xx}}}\,dy\right) = du + dv \tag{7.118a}$$

$$\mu\left(\sqrt{g_{xx}}\,dx + \frac{g_{xy} - \sqrt{-g}}{\sqrt{g_{xx}}}\,dy\right) = du - dv. \tag{7.118b}$$

In terms of the coordinates (u, v) the metric takes the form $ds^2 = \lambda^{-1}\mu^{-1}(du^2 - dv^2)$. $\lambda\mu$ is either positive definite or negative definite and we may set $1/|\lambda\mu| = e^{2\sigma}$ to obtain the form

$$ds^2 = \pm e^{2\sigma}(du^2 - dv^2). \tag{7.119}$$

Exercise 7.33 Let (M, g) be a two-dimensional Lorentz manifold with $g = -dt \otimes dt + t^2 dx \otimes dx$ (the **Milne universe**). Use the transformation $|t| \mapsto e^\eta$ to show that g is conformally flat. In fact it is further simplified by $(\eta, x) \mapsto (u = e^\eta \sinh x, v = e^\eta \cosh x)$. What is the resulting metric?

7.7 Killing vector fields and conformal Killing vector fields

7.7.1 Killing vector fields

Let (M, g) be a Riemannian manifold and $X \in \mathcal{X}(M)$. If an infinitesimal displacement given by εX, ε being infinitesimal, generates an isometry, the vector field X is called a **Killing vector field**. The coordinates x^μ of a point $p \in M$ change to $x^\mu = \varepsilon X^\mu(p)$ under this displacement, see (5.42). If $f : x^\mu \mapsto x^\mu + \varepsilon X^\mu$ is an isometry, it satisfies (7.96b),

$$\frac{\partial(x^\kappa + \varepsilon X^\kappa)}{\partial x^\mu}\frac{\partial(x^\lambda + \varepsilon X^\lambda)}{\partial x^\nu} g_{\kappa\lambda}(x + \varepsilon X) = g_{\mu\nu}(x).$$

After a simple calculation, we find that $g_{\mu\nu}$ and X^μ satisfy the **Killing equation**

$$X^\xi \partial_\xi g_{\mu\nu} + \partial_\mu X^\kappa g_{\kappa\nu} + \partial_\nu X^\lambda g_{\mu\lambda} = 0. \tag{7.120a}$$

From the definition of the Lie derivative, this is written as

$$(\mathcal{L}_X g)_{\mu\nu} = 0. \tag{7.120b}$$

Let $\phi_t : M \to M$ be a one-parameter group of transformations which generate the Killing vector field X. Equation (7.120b) then shows that the local geometry does not change as we move along ϕ_t. In this sense, the Killing vector fields represent the direction of the *symmetry* of a manifold.

A set of Killing vector fields are defined to be dependent if one of them is expressed as a linear combination of others with *constant* coefficients. Thus there may be more Killing vector fields than the dimension of the manifold. [The number of independent symmetries has no direct connection with $\dim M$. The *maximum* number, however, has; see example 7.36.]

Exercise 7.34 Let ∇ be the Levi-Civita connection. Show that the Killing equation is written as

$$(\nabla_\mu X)_\nu + (\nabla_\nu X)_\mu = \partial_\mu X_\nu + \partial_\nu X_\mu - 2\Gamma^\lambda_{\mu\nu} X_\lambda = 0. \tag{7.121}$$

Exercise 7.35 Find three Killing vector fields of $(\mathbb{R}^2, \delta)$. Show that two of them correspond to translations while the third corresponds to a rotation; cf next example.

Example 7.36 Let us work out the Killing vector fields of the Minkowski spacetime $(\mathbb{R}^4, \eta)$, for which all the Levi-Civita connection coefficients vanish. The Killing equation becomes

$$\partial_\mu X_\nu + \partial_\nu X_\mu = 0. \tag{7.122}$$

It is easy to see that X_μ is at most of the first order in x. The constant solutions

$$X_{(i)}{}^\mu = \delta_i{}^\mu \qquad (0 \leqslant i \leqslant 3) \tag{7.123a}$$

correspond to spacetime translations. Next, let $X_\mu = a_{\mu\nu} x^\nu$, $a_{\mu\nu}$ being constant. Equation (7.122) implies that $a_{\mu\nu}$ is antisymmetric with respect to $\mu \leftrightarrow \nu$. Since $\binom{4}{2} = 6$, there are six independent solutions of this form, three of which

$$X_{(j)0} = 0,\ X_{(j)m} = \varepsilon_{jmn} x^n \qquad (1 \leqslant j,\ m,\ n \leqslant 3) \tag{7.123b}$$

correspond to spatial rotations about the x^j axis, while the others

$$X_{(k)0} = x^k,\ X_{(k)m} = -\delta_{km} x^0 \qquad (1 \leqslant k,\ m \leqslant 3) \tag{7.123c}$$

correspond to Lorentz boosts along the x^k axis.

In m-dimensional Minkowski spacetime $(m \geqslant 2)$, there are $m(m+1)/2$ Killing vector fields, m of which generate translations, $(m-1)$, boosts and $(m-1)(m-2)/2$, space rotations. Those spaces (or spacetimes) which admit $m(m+1)/2$ Killing vector fields are called **maximally symmetric spaces**.

Let X and Y be two Killing vector fields. We easily verify

(a) a linear combination $aX + bY$, $(a, b \in \mathbb{R})$ is a Killing vector field,
(b) the Lie bracket $[X, Y]$ is a Killing vector field.

(a) is obvious from the linearity of the covariant derivative. To prove (b), we use (5.58). We have $\mathcal{L}_{[X,Y]}g = \mathcal{L}_X\mathcal{L}_Y g - \mathcal{L}_Y\mathcal{L}_X g = 0$, since $\mathcal{L}_X g = \mathcal{L}_Y g = 0$. Thus all the Killing vector fields form a Lie algebra of the symmetric operations on the manifold M; see next example.

Example 7.37 Let $g = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2\theta\mathrm{d}\phi \otimes \mathrm{d}\phi$ be the standard metric of S^2. The Killing equations (7.121) are

$$\partial_\theta X_\theta + \partial_\theta X_\theta = 0 \tag{7.124a}$$

$$\partial_\phi X_\phi + \partial_\phi X_\phi + 2\sin\theta\cos\theta X_\theta = 0 \tag{7.124b}$$

$$\partial_\theta X_\phi + \partial_\phi X_\theta - 2\cot\theta X_\phi = 0. \tag{7.124c}$$

From (7.124a), it follows that X_θ is independent of $\theta : X_\theta(\theta, \phi) = f(\phi)$. Substituting this into (7.124b), we have

$$X_\phi = -F(\phi)\sin\theta\cos\theta + g(\theta) \tag{7.125}$$

where $F(\phi) = \int^\phi f(\phi)\,\mathrm{d}\phi$. Substitution of (7.125) into (7.124c) yields

$$\begin{aligned} &-F(\phi)(\cos^2\theta - \sin^2\theta) + \frac{\mathrm{d}g}{\mathrm{d}\theta} + \frac{\mathrm{d}f}{\mathrm{d}\phi} \\ &+ 2\cot\theta(F(\phi)\sin\theta\cos\theta - g(\theta)) = 0. \end{aligned}$$

This equation may be separated into

$$\frac{\mathrm{d}g}{\mathrm{d}\theta} - 2\cot\theta g(\theta) = -\frac{\mathrm{d}f}{\mathrm{d}\phi} - F(\phi).$$

Since both sides must be separately constant ($\equiv$ C), we have

$$\frac{\mathrm{d}g}{\mathrm{d}\theta} - 2\cot\theta g(\theta) = C \tag{7.126a}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\phi} + F(\phi) = -C. \tag{7.126b}$$

Equation (7.126a) is solved if we multiply both sides by $\exp(-\int \mathrm{d}\theta 2\cot\theta) = \sin^{-2}\theta$ to make the LHS a total derivative,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin^{-2}\theta g(\theta)) = C\sin^{-2}\theta.$$

We find that

$$g(\theta) = (C_1 - C\cot\theta)\sin^2\theta.$$

Differentiating (7.126b) again, we find that f is harmonic,

$$X_\theta(\phi) = f(\phi) = A\sin\phi + B\cos\phi$$

$$F(\phi) = -A\cos\phi + B\sin\phi - C.$$

Substituting these results into (7.125), we have

$$\begin{aligned} X_\varphi(\theta, \phi) & \\ &= -(-A\cos\phi + B\sin\phi - C)\sin\theta\cos\theta + (C_1 - C\cot\theta)\sin^2\theta \\ &= (A\cos\phi - B\sin\phi)\sin\theta\cos\theta + C_1\sin^2\theta. \end{aligned}$$

A general Killing vector is given by

$$\begin{aligned} X &= X^\theta \frac{\partial}{\partial\theta} + X^\varphi \frac{\partial}{\partial\phi} \\ &= \mathrm{A}\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right) \\ &\quad + B\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right) + C_1\frac{\partial}{\partial\phi}. \end{aligned} \tag{7.127}$$

The basis vectors

$$L_x = -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi} \tag{7.128a}$$

$$L_y = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \tag{7.128b}$$

$$L_z = \frac{\partial}{\partial\phi} \tag{7.128c}$$

generate rotations round the x, y and z axes respectively.

These vectors generate the Lie algebra $\mathfrak{so}(3)$. This reflects the fact that S^2 is the homogeneous space SO(3)/SO(2) and the metric on S^2 retains this SO(3) symmetry (see example 5.65(a)). In general S^n = SO(n + 1)/SO(n) with the usual metric has dim SO(n + 1) = $n(n+1)/2$ Killing vectors and they form the Lie algebra $\mathfrak{so}(n+1)$. The sphere S^n with the usual metric is a maximally symmetric space. We may *squash* S^n so that it has fewer symmetries. For example, if S^2 considered above is squashed along the z axis it has a rotational symmetry around the z axis only and there exists one Killing vector field $L_z = \partial/\partial\phi$.

7.7.2 Conformal Killing vector fields

Let (M, g) be a Riemannian manifold and let $X \in \mathcal{X}(M)$. If an infinitesimal displacement given by εX generates a conformal transformation, the vector field X is called a **conformal Killing vector field** (CKV). Under the displacement $x^\mu \to x^\mu + \varepsilon X^\mu$, we have

$$\frac{\partial(x^\kappa + \varepsilon X^\kappa)}{\partial x^\mu} \frac{\partial(x^\lambda + \varepsilon X^\lambda)}{\partial x^\nu} g_{\kappa\lambda}(x + \varepsilon X) = e^{2\sigma} g_{\mu\nu}(x).$$

Noting that $\sigma \propto \varepsilon$, we set $\sigma = \varepsilon\psi/2$, where $\psi \in \mathcal{F}(M)$. Then we find that $g_{\mu\nu}$ and X^μ satisfy

$$\mathcal{L}_X g_{\mu\nu} = X^\xi \partial_\xi g_{\mu\nu} + \partial_\mu X^\kappa g_{\kappa\nu} + \partial_\nu X^\lambda g_{\mu\lambda} = \psi g_{\mu\nu}. \qquad (7.129a)$$

Equation (7.129a) is easily solved for ψ to yield

$$\psi = m^{-1}(X^\xi g^{\mu\nu} \partial_\xi g_{\mu\nu} + 2\partial_\mu X^\mu) \qquad (7.129b)$$

where $m = \dim M$. We verify

(a) a linear combination of CKV is a CKV; $(\mathcal{L}_{aX+bY} g)_{\mu\nu} = (a\varphi + b\psi) g_{\mu\nu}$ where $a, b \in \mathbb{R}$, $\mathcal{L}_X g_{\mu\nu} = \varphi g_{\mu\nu}$ and $\mathcal{L}_Y g_{\mu\nu} = \psi g_{\mu\nu}$,

(b) the Lie bracket $[X, Y]$ of CKV is again a CKV; $\mathcal{L}_{[X,Y]} g_{\mu\nu} = (X[\psi] - Y[\varphi]) g_{\mu\nu}$.

Example 7.38 Let x^μ be the coordinates of $(\mathbb{R}^m, \delta)$. The vector

$$D \equiv x^\mu \frac{\partial}{\partial x^\mu} \qquad (7.130)$$

(**dilatation vector**) is a CKV. In fact

$$\mathcal{L}_D \delta_{\mu\nu} = \partial_\mu x^\kappa \delta_{\kappa\nu} + \partial_\nu x^\lambda \delta_{\mu\lambda} = 2\delta_{\mu\nu}.$$

7.8 Non-coordinate bases

7.8.1 Definitions

In the coordinate basis, T_pM is spanned by $\{e_\mu\} = \{\partial/\partial x^\mu\}$ and T_p^*M by $\{dx^\mu\}$. If, however, M is endowed with a metric g, there may be an alternative choice. Let us consider the linear combination,

$$\hat{e}_\alpha = e_\alpha{}^\mu (\partial/\partial x^\mu) \qquad \{e_\alpha{}^\mu\} \in \mathrm{GL}(m, \mathbb{R}) \qquad (7.131)$$

where $\det e_\alpha{}^\mu > 0$. In other words, $\{\hat{e}_\alpha\}$ is the frame of basis vectors which is obtained by a $\mathrm{GL}(m, \mathbb{R})$-rotation of the basis $\{e_\mu\}$ preserving the orientation. We require that $\{\hat{e}_\alpha\}$ be orthonormal,

$$g(\hat{e}_\alpha, \hat{e}_\beta) = e_\alpha{}^\mu e_\beta{}^\nu g_{\mu\nu} = \delta_{\alpha\beta}. \qquad (7.132a)$$

If the manifold is Lorentzian, $\delta_{\alpha\beta}$ should be replaced by $\eta_{\alpha\beta}$. We easily reverse (7.132a),

$$g_{\mu\nu} = e^{\alpha}{}_{\mu} e^{\beta}{}_{\nu} \delta_{\alpha\beta} \tag{7.132b}$$

where $e^{\alpha}{}_{\mu}$ is the inverse of $e_{\alpha}{}^{\mu}$; $e^{\alpha}{}_{\mu} e_{\alpha}{}^{\nu} = \delta_{\mu}{}^{\nu}$, $e^{\alpha}{}_{\mu} e_{\beta}{}^{\mu} = \delta^{\alpha}{}_{\beta}$. [We have used the same symbols for a matrix and its inverse. So long as the indices are written explicitly it does not cause confusion.] Since a vector V is independent of the basis chosen, we have $V = V^{\mu} e_{\mu} = V^{\alpha} \hat{e}_{\alpha} = V^{\alpha} e_{\alpha}{}^{\mu} e_{\mu}$. It follows that

$$V^{\mu} = V^{\alpha} e_{\alpha}{}^{\mu} \qquad V^{\alpha} = e^{\alpha}{}_{\mu} V^{\mu}. \tag{7.133}$$

Let us introduce the dual basis $\{\hat{\theta}^{\alpha}\}$ defined by $\langle \hat{\theta}^{\alpha}, \hat{e}_{\beta} \rangle = \delta^{\alpha}{}_{\beta}$. $\hat{\theta}^{\alpha}$ is given by

$$\hat{\theta}^{\alpha} = e^{\alpha}{}_{\mu} \mathrm{d}x^{\mu}. \tag{7.134}$$

In terms of $\{\hat{\theta}^{\alpha}\}$, the metric is

$$g = g_{\mu\nu} \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} = \delta_{\alpha\beta} \hat{\theta}^{\alpha} \otimes \hat{\theta}^{\beta}. \tag{7.135}$$

The bases $\{\hat{e}_{\alpha}\}$ and $\{\hat{\theta}^{\alpha}\}$ are called the **non-coordinate bases**. We use $\kappa, \lambda, \mu, \nu, \ldots (\alpha, \beta, \gamma, \delta, \ldots)$ to denote the coordinate (non-coordinate) basis. The coefficients $e_{\alpha}{}^{\mu}$ are called the **vierbeins** if the space is four dimensional and **vielbeins** if it is *many* dimensional. The non-coordinate basis has a non-vanishing Lie bracket. If the $\{\hat{e}_{\alpha}\}$ are given by (7.131), they satisfy

$$[\hat{e}_{\alpha}, \hat{e}_{\beta}]|_{p} = c_{\alpha\beta}{}^{\gamma}(p) \hat{e}_{\gamma}|_{p} \tag{7.136a}$$

where

$$c_{\alpha\beta}{}^{\gamma}(p) = e^{\gamma}{}_{\nu}[e_{\alpha}{}^{\mu} \partial_{\mu} e_{\beta}{}^{\nu} - e_{\beta}{}^{\mu} \partial_{\mu} e_{\alpha}{}^{\nu}](p). \tag{7.136b}$$

Example 7.39 The standard metric on S^2 is

$$g = \mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2 \theta \mathrm{d}\phi \otimes \mathrm{d}\phi = \hat{\theta}^1 \otimes \hat{\theta}^1 + \hat{\theta}^2 \otimes \hat{\theta}^2 \tag{7.137}$$

where $\hat{\theta}^1 = \mathrm{d}\theta$ and $\hat{\theta}^2 = \sin\theta \mathrm{d}\phi$. The 'zweibeins' are

$$\begin{aligned} e^1{}_{\theta} &= 1 \qquad & e^1{}_{\phi} &= 0 \\ e^2{}_{\theta} &= 0 \qquad & e^2{}_{\phi} &= \sin\theta. \end{aligned} \tag{7.138}$$

The non-vanishing components of $c_{\alpha\beta}{}^{\gamma}$ are $c_{12}{}^{2} = -c_{21}{}^{2} = -\cot\theta$.

Exercise 7.40

(a) Verify the identities,

$$\delta^{\alpha\beta} = g^{\mu\nu} e^{\alpha}{}_{\mu} e^{\beta}{}_{\nu} \qquad g^{\mu\nu} = \delta^{\alpha\beta} e_{\alpha}{}^{\mu} e_{\beta}{}^{\nu}. \tag{7.139}$$

(b) Let γ^{α} be the Dirac matrices in Minkowski spacetime, which satisfy $\{\gamma^{\alpha}, \gamma^{\beta}\} = 2\eta^{\alpha\beta}$. Define the curved spacetime counterparts of the

Dirac matrices by $\gamma^\mu \equiv e_\alpha{}^\mu \gamma^\alpha$. Show that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{7.140}$$

7.8.2 Cartan's structure equations

In §7.3 the curvature tensor R and the torsion tensor T have been defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Let $\{\hat{e}_\alpha\}$ be the non-coordinate basis and $\{\hat{\theta}^\alpha\}$ the dual basis. The vector fields $\{\hat{e}_\alpha\}$ satisfy $[\hat{e}_\alpha, \hat{e}_\beta] = c_{\alpha\beta}{}^\gamma \hat{e}_\gamma$. Define the connection coefficients with respect to the basis $\{\hat{e}_\alpha\}$ by

$$\nabla_\alpha \hat{e}_\beta \equiv \nabla_{\hat{e}_\alpha} \hat{e}_\beta = \Gamma^\gamma_{\alpha\beta} \hat{e}_\gamma. \tag{7.141}$$

Let $\hat{e}_\alpha = e_\alpha{}^\mu e_\mu$. Then (7.141) becomes $e_\alpha{}^\mu(\partial_\mu e_\beta{}^\nu + e_\beta{}^\lambda \Gamma^\nu{}_{\mu\lambda}) e_\nu = \Gamma^\gamma_{\alpha\beta} e_\gamma{}^\nu e_\nu$, from which we find

$$\begin{aligned} \Gamma^\gamma_{\alpha\beta} &= e^\gamma{}_\nu e_\alpha{}^\mu (\partial_\mu e_\beta{}^\nu + e_\beta{}^\lambda \Gamma^\nu{}_{\mu\lambda}) \\ &= e^\gamma{}_\nu e_\alpha{}^\mu \nabla_\mu e_\beta{}^\nu. \end{aligned} \tag{7.142}$$

The components of T and R in this basis are given by

$$\begin{aligned} T^\alpha{}_{\beta\gamma} &= \langle \hat{\theta}^\alpha, T(\hat{e}_\beta, \hat{e}_\gamma) \rangle = \langle \hat{\theta}^\alpha, \nabla_\beta \hat{e}_\gamma - \nabla_\gamma \hat{e}_\beta - [\hat{e}_\beta, \hat{e}_\gamma] \rangle \\ &= \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta} - c_{\beta\gamma}{}^\alpha \end{aligned} \tag{7.143}$$

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} &= \langle \hat{\theta}^\alpha, \nabla_\gamma \nabla_\delta \hat{e}_\beta - \nabla_\delta \nabla_\gamma \hat{e}_\beta - \nabla_{[\hat{e}_\gamma, \hat{e}_\delta]} \hat{e}_\beta \rangle \\ &= \langle \hat{\theta}^\alpha, \nabla_\gamma (\Gamma^\varepsilon{}_{\delta\beta} \hat{e}_\varepsilon) - \nabla_\delta (\Gamma^\varepsilon{}_{\gamma\beta} \hat{e}_\varepsilon) - c_{\gamma\delta}{}^\varepsilon \nabla_\varepsilon \hat{e}_\beta \rangle \\ &= \hat{e}_\gamma[\Gamma^\alpha{}_{\delta\beta}] - \hat{e}_\delta[\Gamma^\alpha{}_{\gamma\beta}] + \Gamma^\varepsilon{}_{\delta\beta} \Gamma^\alpha{}_{\gamma\varepsilon} - \Gamma^\varepsilon{}_{\gamma\beta} \Gamma^\alpha{}_{\delta\varepsilon} - c_{\gamma\delta}{}^\varepsilon \Gamma^\alpha{}_{\varepsilon\beta} \end{aligned} \tag{7.144}$$

where $\nabla_\alpha \equiv \nabla_{\hat{e}_\alpha}$. We define a matrix-valued one-form $\{\omega^\alpha{}_\beta\}$ called the **connection one-form** by

$$\omega^\alpha{}_\beta \equiv \Gamma^\alpha{}_{\gamma\beta} \hat{\theta}^\gamma. \tag{7.145}$$

Theorem 7.41 The connection one-form $\omega^\alpha{}_\beta$ satisfies **Cartan's structure equations**,

$$\mathrm{d}\hat{\theta}^\alpha + \omega^\alpha{}_\beta \wedge \hat{\theta}^\beta = T^\alpha \tag{7.146a}$$

$$\mathrm{d}\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta = R^\alpha{}_\beta \tag{7.146b}$$

where we have introduced the **torsion two-form** $T^\alpha \equiv \frac{1}{2} T^\alpha{}_{\beta\gamma} \hat{\theta}^\beta \wedge \hat{\theta}^\gamma$ and the **curvature two-form** $R^\alpha{}_\beta \equiv \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} \hat{\theta}^\gamma \wedge \hat{\theta}^\delta$.

Proof: Let the LHS of (7.146*a*) act on the basis vectors $\hat{e}_\gamma$ and $\hat{e}_\delta$,

$$\begin{aligned} \mathrm{d}\hat{\theta}^\alpha(\hat{e}_\gamma, \hat{e}_\delta) &+ [\langle \omega^\alpha{}_\beta, \hat{e}_\gamma\rangle\langle \hat{\theta}^\beta, \hat{e}_\delta\rangle - \langle \hat{\theta}^\beta, \hat{e}_\gamma\rangle\langle \omega^\alpha{}_\beta, \hat{e}_\delta\rangle] \\ &= \{\hat{e}_\gamma[\langle \hat{\theta}^\alpha, \hat{e}_\delta\rangle] - \hat{e}_\delta[\langle \hat{\theta}^\alpha, \hat{e}_\gamma\rangle] - \langle \hat{\theta}^\alpha, [\hat{e}_\gamma, \hat{e}_\delta]\rangle\} \\ &\quad + \{\langle \omega^\alpha{}_\delta, \hat{e}_\gamma\rangle - \langle \omega^\alpha{}_\gamma, \hat{e}_\delta\rangle\} \\ &= -c_{\gamma\delta}{}^\alpha + \Gamma^\alpha{}_{\gamma\delta} - \Gamma^\alpha{}_{\delta\gamma} = T^\alpha{}_{\gamma\delta} \end{aligned}$$

where use has been made of (5.70). The RHS acting on $\hat{e}_\gamma$ and $\hat{e}_\delta$ yields

$$\tfrac{1}{2}T^\alpha{}_{\beta\epsilon}[\langle \hat{\theta}^\beta, \hat{e}_\gamma\rangle\langle \hat{\theta}^\epsilon, \hat{e}_\delta\rangle - \langle \hat{\theta}^\epsilon, \hat{e}_\gamma\rangle\langle \theta^\beta, \hat{e}_\delta\rangle] = T^\alpha{}_{\gamma\delta}$$

which verifies (7.146a).

Equation (7.146b) may be proved similarly (exercise). ■

Taking the exterior derivative of (7.146), we have the **Bianchi identities**

$$\mathrm{d}T^\alpha + \omega^\alpha{}_\beta \wedge T^\beta = R^\alpha{}_\beta \wedge \hat{\theta}^\beta \tag{7.147a}$$

$$\mathrm{d}R^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge R^\gamma{}_\beta - R^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta = 0. \tag{7.147b}$$

These are the non-coordinate basis versions of (7.81).

7.8.3 The local frame

In an m-dimensional Riemannian manifold, the metric tensor $g_{\mu\nu}$ has $m(m+1)/2$ degrees of freedom while the vielbein $e_\alpha{}^\mu$ has m^2 degrees of freedom. There are many non-coordinate bases which yield the same metric, g, each of which is related to the other by the *local* orthogonal rotation,

$$\hat{\theta}^\alpha \to \hat{\theta}'^\alpha(p) = \Lambda^\alpha{}_\beta(p)\hat{\theta}^\beta(p) \tag{7.148}$$

at each point p. The vielbein transforms as

$$e^\alpha{}_\mu(p) \to e'^\alpha{}_\mu(p) = \Lambda^\alpha{}_\beta(p)e^\beta{}_\mu(p). \tag{7.149}$$

Unlike the Greek indices, which transform under coordinate changes, the Latin indices transform under the local orthogonal rotation and are inert under coordinate changes. Since the metric tensor is invariant under the rotation, $\Lambda^\alpha{}_\beta$ satisfies

$$\Lambda^\alpha{}_\beta\delta_{\alpha\delta}\Lambda^\delta{}_\gamma = \delta_{\beta\gamma} \qquad \text{if } M \text{ is Riemannian} \tag{7.150a}$$

$$\Lambda^\alpha{}_\beta\eta_{\alpha\delta}\Lambda^\delta{}_\gamma = \eta_{\beta\gamma} \qquad \text{if } M \text{ is Lorentzian.} \tag{7.150b}$$

This implies that $\{\Lambda^\alpha{}_\beta(p)\} \in \mathrm{SO}(m)$ if M is Riemannian with $\dim M = m$ and $\Lambda^\alpha{}_\beta(p) \in \mathrm{SO}(m-1, 1)$ if M is Lorentzian. The dimension of these Lie groups is $m(m-1)/2 = m^2 - m(m+1)/2$, that is the difference between the degrees of freedom of $e_\alpha{}^\mu$ and $g_{\mu\nu}$. Under the local frame rotation $\Lambda^\alpha{}_\beta(p)$, the indices $\alpha, \beta, \gamma, \delta, \ldots$ are rotated while $\kappa, \lambda, \mu, \nu, \ldots$ (world indices) are not affected. Under the rotation (7.148) the basis vector transforms as

$$\hat{e}_\alpha \to \hat{e}'_\alpha = \hat{e}_\beta(\Lambda^{-1})^\beta{}_\alpha. \tag{7.151}$$

Let $t = t^\mu{}_\nu e_\mu \otimes \mathrm{d}x^\nu$ be a tensor field of type (1, 1). In the bases $\{\hat{e}_\alpha\}$ and $\{\hat{\theta}^\alpha\}$, we have $t = t^\alpha{}_\beta \hat{e}_\alpha \otimes \hat{\theta}^\beta$, where $t^\alpha{}_\beta = e_\mu{}^\alpha e_\beta{}^\nu t^\mu{}_\nu$. If the new frames $\{\hat{e}'_\alpha\} = \{\hat{e}_\beta(\Lambda^{-1})^\beta{}_\alpha\}$ and $\{\hat{\theta}'^\alpha\} = \{\Lambda^\alpha{}_\beta\hat{\theta}^\beta\}$ are employed, the tensor t is expressed as

$$t = t'^\alpha{}_\beta \hat{e}'_\alpha \otimes \hat{\theta}'^\beta = t'^\alpha{}_\beta \hat{e}_\gamma(\Lambda^{-1})^\gamma{}_\alpha \otimes \Lambda^\beta{}_\delta\hat{\theta}^\delta$$

from which we find the transformation rule,

$$t^\alpha{}_\beta \to t'^\alpha{}_\beta = \Lambda^\alpha{}_\gamma t^\gamma{}_\delta(\Lambda^{-1})^\delta{}_\beta.$$

To summarise, the upper (lower) non-coordinate indices are rotated by Λ (Λ^{-1}). The change from the coordinate basis to the non-coordinate basis is carried out by multiplications of vielbeins.

From these facts we find the transformation rule of the connection one-form $\omega^\alpha{}_\beta$. The torsion two-form transforms as

$$T^\alpha \to T'^\alpha = \mathrm{d}\hat{\theta}'^\alpha + \omega'^\alpha{}_\beta \wedge \hat{\theta}'^\beta = \Lambda^\alpha{}_\beta[\mathrm{d}\hat{\theta}^\beta + \omega^\beta{}_\gamma \wedge \hat{\theta}^\gamma].$$

Substituting $\hat{\theta}'^\alpha = \Lambda^\alpha{}_\beta\hat{\theta}^\beta$ into this equation, we find that

$$\omega'^\alpha{}_\beta\Lambda^\beta{}_\gamma = \Lambda^\alpha{}_\delta\omega^\delta{}_\gamma - \mathrm{d}\Lambda^\alpha{}_\gamma.$$

Multiplying both sides by Λ^{-1} from the right, we have

$$\omega'^\alpha{}_\beta = \Lambda^\alpha{}_\gamma\omega^\gamma{}_\delta(\Lambda^{-1})^\delta{}_\beta + \Lambda^\alpha{}_\gamma(\mathrm{d}\Lambda^{-1})^\gamma{}_\beta \tag{7.152}$$

where use has been made of the identity $\mathrm{d}\Lambda\,\Lambda^{-1} + \Lambda\,\mathrm{d}\Lambda^{-1} = 0$, which is derived from $\Lambda\Lambda^{-1} = \mathbb{1}$.

The curvature two-form transforms homogeneously as

$$R^\alpha{}_\beta \to R'^\alpha{}_\beta = \Lambda^\alpha{}_\gamma R^\gamma{}_\delta(\Lambda^{-1})^\delta{}_\beta \tag{7.153}$$

under a local frame rotation Λ.

7.8.4 The Levi-Civita connection in a non-coordinate basis

Let ∇ be a Levi-Civita connection on (M, g), which is characterised by the metric compatibility $\nabla_X g = 0$, and the vanishing torsion $\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = 0$. It is interesting to see how these conditions are expressed in the present approach. The components $\Gamma^\lambda{}_{\mu\nu}$ and $\Gamma^\alpha{}_{\beta\gamma}$ are related to each other by (7.142). Let (M, g) be a Riemannian manifold (if (M, g) is Lorentzian, we simply replace $\delta_{\alpha\beta}$ below by $\eta_{\alpha\beta}$). If we define the **Ricci rotation coefficient** $\Gamma_{\alpha\beta\gamma}$ by $\delta_{\alpha\delta}\Gamma^\delta{}_{\beta\gamma}$, the metric compatibility is expressed as

$$\begin{aligned}\Gamma_{\alpha\beta\gamma} &= \delta_{\alpha\delta}e^\delta{}_\lambda e_\beta{}^\mu\nabla_\mu e_\gamma{}^\lambda = -\delta_{\alpha\delta}e_\gamma{}^\lambda e_\beta{}^\mu\nabla_\mu e^\delta{}_\lambda \\ &= -\delta_{\gamma\delta}e^\delta{}_\lambda e_\beta{}^\mu\nabla_\mu e_\alpha{}^\lambda = -\Gamma_{\gamma\beta\alpha}\end{aligned} \tag{7.154}$$

where $\nabla_\mu g = 0$ has been used. In terms of the connection one-form

$\omega_{\alpha\beta} \equiv \delta_{\alpha\gamma}\omega^{\gamma}{}_{\beta}$, this becomes

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}. \tag{7.155}$$

The torsion-free condition is

$$\mathrm{d}\hat{\theta}^{\alpha} + \omega^{\alpha}{}_{\beta} \wedge \hat{\theta}^{\beta} = 0. \tag{7.156}$$

The reader should verify that (7.156) implies the symmetry of the connection coefficient $\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu}$ in the coordinate basis. The condition (7.156) enables us to compute the $c_{\alpha\beta}{}^{\gamma}$ of the basis $\{\hat{e}_{\alpha}\}$. Let us look at the commutation relation

$$c_{\alpha\beta}{}^{\gamma}\hat{e}_{\gamma} = [\hat{e}_{\alpha}, \hat{e}_{\beta}] = \nabla_{\alpha}\hat{e}_{\beta} - \nabla_{\beta}\hat{e}_{\alpha} \tag{7.157}$$

where the final equality follows from the torsion-free condition. From (7.141), we find

$$c_{\alpha\beta}{}^{\gamma} = \Gamma^{\gamma}{}_{\alpha\beta} - \Gamma^{\gamma}{}_{\beta\alpha}. \tag{7.158}$$

Substituting (7.158) into (7.144) we may express the Riemann curvature tensor in terms of Γ only,

$$\begin{aligned} R^{\alpha}{}_{\beta\gamma\delta} = \hat{e}_{\gamma}[\Gamma^{\alpha}{}_{\delta\beta}] - \hat{e}_{\delta}[\Gamma^{\alpha}{}_{\gamma\beta}] + \Gamma^{\epsilon}{}_{\delta\beta}\Gamma^{\alpha}{}_{\gamma\epsilon} - \Gamma^{\epsilon}{}_{\gamma\beta}\Gamma^{\alpha}{}_{\delta\epsilon} \\ - (\Gamma^{\epsilon}{}_{\gamma\delta} - \Gamma^{\epsilon}{}_{\delta\gamma})\Gamma^{\alpha}{}_{\epsilon\beta}. \end{aligned} \tag{7.159}$$

Example 7.42 Let us take the sphere S^2 of example 7.39. The components of $e^{\alpha}{}_{\mu}$ are

$$e^{1}{}_{\theta} = 1,\ e^{1}{}_{\phi} = 0,\ e^{2}{}_{\theta} = 0,\ e^{2}{}_{\phi} = \sin\theta. \tag{7.160}$$

We first note that the metric condition implies $\omega_{11} = \omega_{22} = 0$, hence $\omega^{1}{}_{1} = \omega^{2}{}_{2} = 0$. Other connection one-forms are obtained from the torsion-free conditions,

$$\mathrm{d}(\mathrm{d}\theta) + \omega^{1}{}_{2} \wedge (\sin\theta\,\mathrm{d}\phi) = 0 \tag{7.161a}$$

$$\mathrm{d}(\sin\theta\,\mathrm{d}\phi) + \omega^{2}{}_{1} \wedge \mathrm{d}\theta = 0. \tag{7.161b}$$

From (7.161*b*) we easily see that $\omega^{2}{}_{1} = \cos\theta\,\mathrm{d}\phi$, and the metric condition $\omega_{12} = -\omega_{21}$ implies $\omega^{1}{}_{2} = -\cos\theta\,\mathrm{d}\phi$. The Riemann tensor is also found from Cartan's structure equation,

$$\omega^{1}{}_{2} \wedge \omega^{2}{}_{1} = \tfrac{1}{2}R^{1}{}_{1\alpha\beta}\hat{\theta}^{\alpha} \wedge \hat{\theta}^{\beta} \tag{7.162a}$$

$$\mathrm{d}\omega^{1}{}_{2} = \tfrac{1}{2}R^{1}{}_{2\alpha\beta}\hat{\theta}^{\alpha} \wedge \hat{\theta}^{\beta} \tag{7.162b}$$

$$\mathrm{d}\omega^{2}{}_{1} = \tfrac{1}{2}R^{2}{}_{1\alpha\beta}\hat{\theta}^{\alpha} \wedge \hat{\theta}^{\beta} \tag{7.162c}$$

$$\omega^{2}{}_{1} \wedge \omega^{1}{}_{2} = \tfrac{1}{2}R^{2}{}_{2\alpha\beta}\hat{\theta}^{\alpha} \wedge \hat{\theta}^{\beta}. \tag{7.162d}$$

The non-vanishing components are $R^{1}{}_{212} = -R^{1}{}_{221} = \sin\theta$, $R^{2}{}_{112} = -R^{2}{}_{121} = -\sin\theta$. The transition to the coordinate basis expression is carried out with the help of $e_{\alpha}{}^{\mu}$ and $e_{\mu}{}^{\alpha}$. For example,

$$R^{\theta}{}_{\phi\theta\phi} = e_{\alpha}{}^{\theta} e_{\phi}{}^{\beta} e_{\theta}{}^{\gamma} e_{\phi}{}^{\delta} R^{\alpha}{}_{\beta\gamma\delta} = \frac{1}{\sin^2\theta} R^{1}{}_{212} = \frac{1}{\sin\theta}.$$

Example 7.43 The Schwarzschild metric is given by

$$\begin{aligned} ds^2 &= -(1 - 2M/r)\,dt^2 + \frac{1}{1 - 2M/r}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \\ &= -\hat{\theta}^0 \otimes \hat{\theta}^0 + \hat{\theta}^1 \otimes \hat{\theta}^1 + \hat{\theta}^2 \otimes \hat{\theta}^2 + \hat{\theta}^3 \otimes \hat{\theta}^3 \end{aligned} \tag{7.163}$$

where

$$\begin{aligned} \hat{\theta}^0 &= (1 - 2M/r)^{1/2}\,dt \qquad & \hat{\theta}^1 &= (1 - 2M/r)^{-1/2}\,dr \\ \hat{\theta}^2 &= r\,d\theta \qquad & \hat{\theta}^3 &= r\sin\theta\, d\phi. \end{aligned} \tag{7.164}$$

The parameters run over the range $2M < r$, $0 \leqslant \theta < \pi$ and $0 \leqslant \phi < 2\pi$. The metric condition yields $\omega^0{}_0 = \omega^1{}_1 = \omega^2{}_2 = \omega^3{}_3 = 0$ and the torsion-free conditions are

$$d[(1 - 2M/r)^{1/2}\,dt] + \omega^0{}_\beta \wedge \hat{\theta}^\beta = 0 \tag{7.165a}$$

$$d[(1 - 2M/r)^{-1/2}\,dr] + \omega^1{}_\beta \wedge \hat{\theta}^\beta = 0 \tag{7.165b}$$

$$d(r\,d\theta) + \omega^2{}_\beta \wedge \hat{\theta}^\beta = 0 \tag{7.165c}$$

$$d(r\sin\theta\, d\phi) + \omega^3{}_\beta \wedge \hat{\theta}^\beta = 0. \tag{7.165d}$$

The non-vanishing components of the connection one-forms are

$$\begin{aligned} \omega^0{}_1 &= \omega^1{}_0 = \frac{M}{r^2}\,dt,\ \omega^2{}_1 = -\omega^1{}_2 = \left(1 - \frac{2M}{r}\right)^{1/2} d\theta \\ \omega^3{}_1 &= -\omega^1{}_3 = \left(1 - \frac{2M}{r}\right)^{1/2} \sin\theta\, d\phi,\ \omega^3{}_2 = -\omega^2{}_3 = \cos\theta\, d\phi. \end{aligned} \tag{7.166}$$

The curvature two-forms are found from the structure equations to be

$$\begin{aligned} R^0{}_1 &= R^1{}_0 = \frac{2M}{r^3}\,\hat{\theta}^0 \wedge \hat{\theta}^1,\ R^0{}_2 = R^2{}_0 = -\frac{2M}{r^3}\,\hat{\theta}^0 \wedge \hat{\theta}^2 \\ R^0{}_3 &= R^3{}_0 = -\frac{M}{r^3}\,\hat{\theta}^0 \wedge \hat{\theta}^3,\ R^1{}_2 = -R^2{}_1 = -\frac{M}{r^3}\,\hat{\theta}^1 \wedge \hat{\theta}^2 \\ R^1{}_3 &= -R^3{}_1 = -\frac{M}{r^3}\,\hat{\theta}^1 \wedge \hat{\theta}^3,\ R^2{}_3 = -R^3{}_2 = \frac{2M}{r^3}\,\hat{\theta}^2 \wedge \hat{\theta}^3. \end{aligned} \tag{7.167}$$

7.9 Differential forms and Hodge theory

7.9.1 Invariant volume elements

In §5.5, we have defined the volume element as a non-vanishing m-form on an m-dimensional orientable manifold M. If M is endowed with a metric g, there exists a natural volume element which is invariant under coordinate transformation. Let us define the **invariant volume element** by

$$\Omega_M \equiv \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m \tag{7.168}$$

where $g = \det g_{\mu\nu}$ and x^μ are the coordinates of the chart (U, φ). The m-form Ω_M is indeed invariant under a coordinate change. Let y^λ be the coordinates of another chart (V, ψ) with $U \cap V \neq \emptyset$. In terms of the y-coordinates, the invariant volume element is

$$\left|\det\left(\frac{\partial x^\mu}{\partial y^\kappa}\frac{\partial x^\nu}{\partial y^\lambda} g_{\mu\nu}\right)\right|^{1/2} dy^1 \wedge \ldots \wedge dy^m.$$

Noting that $dy^\lambda = \partial y^\lambda/\partial x^\mu\, dx^\mu$, this becomes

$$\begin{aligned}&|\det(\partial x^\mu/\partial y^\kappa)|\sqrt{|g|}\det(\partial y^\lambda/\partial x^\nu)\, dx^1 \wedge \ldots \wedge dx^m\\ &= \pm\sqrt{|g|}\, dx^1 \wedge \ldots \wedge dx^m.\end{aligned}$$

If x^μ and y^κ define the same orientation, $\det(\partial x^\mu/\partial y^\kappa)$ is strictly positive on $U \cap V$ and Ω_M is invariant under the coordinate change.

Exercise 7.44 Let $\{\hat{\theta}^\alpha\} = \{e^\alpha{}_\mu\, dx^\mu\}$ be the non-coordinate basis. Show that the invariant volume element is given by

$$\begin{aligned}\Omega_M &= |e|\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^m\\ &= \hat{\theta}^1 \wedge \hat{\theta}^2 \wedge \ldots \wedge \hat{\theta}^m\end{aligned} \tag{7.169}$$

where $e = \det e^\alpha{}_\mu$.

Now that we have defined the invariant volume element, it is natural to define an integration of $f \in \mathcal{F}(M)$ over M by

$$\int_M f\Omega_M \equiv \int_M f\sqrt{|g|}\, dx^1 dx^2 \ldots dx^m. \tag{7.170}$$

Obviously (7.170) is invariant under a change of coordinates. In physics, there are many objects which are expressed as volume integrals of this type, see §7.10.

7.9.2 *Duality transformations (Hodge star)*

As noted in §5.4, $\Omega^r(M)$ is isomorphic to $\Omega^{m-r}(M)$ on an m-dimensional manifold M. If M is endowed with a metric g, we can define a natural isomorphism between them called the **Hodge $*$ operation**. Define the totally antisymmetric tensor ε by

$$\varepsilon_{\mu_1\mu_2\ldots\mu_m} = \begin{cases} +1 & \text{if } (\mu_1\mu_2\ldots\mu_m) \text{ is an even permutation of } (12\ldots m)\\ -1 & \text{if } (\mu_1\mu_2\ldots\mu_m) \text{ is an odd permutation of } (12\ldots m)\\ 0 & \text{otherwise.}\end{cases} \tag{7.171a}$$

Note that

$$\varepsilon^{\mu_1\mu_2\ldots\mu_m} = g^{\mu_1\nu_1}g^{\mu_2\nu_2}\ldots g^{\mu_m\nu_m}\varepsilon_{\nu_1\nu_2\ldots\nu_m}$$
$$= g^{-1}\varepsilon_{\mu_1\mu_2\ldots\mu_m}. \tag{7.171b}$$

The Hodge $*$ is a linear map $* : \Omega^r(M) \to \Omega^{m-r}(M)$ whose action on a basis vector of $\Omega^r(M)$ is defined by

$$*(\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_r})$$
$$= \frac{\sqrt{|g|}}{(m-r)!}\,\varepsilon^{\mu_1\ldots\mu_r}{}_{\nu_{r+1}\ldots\nu_m}\,\mathrm{d}x^{\nu_{r+1}} \wedge \ldots \wedge \mathrm{d}x^{\nu_m}. \tag{7.172}$$

It should be noted that $*1$ is the invariant volume element

$$*1 = \frac{\sqrt{|g|}}{m!}\,\varepsilon_{\mu_1\ldots\mu_m}\,\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_m} = \sqrt{|g|}\,\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_m}.$$

For

$$\omega = \frac{1}{r!}\,\omega_{\mu_1\mu_2\ldots\mu_r}\,\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \in \Omega^r(M)$$

we have

$$*\omega = \frac{\sqrt{|g|}}{r!(m-r)!}\omega_{\mu_1\ldots\mu_r}\varepsilon^{\mu_1\ldots\mu_r}{}_{\nu_{r+1}\ldots\nu_m}\,\mathrm{d}x^{\nu_{r+1}} \wedge \ldots \wedge \mathrm{d}x^{\nu_m}. \tag{7.173}$$

If we take the non-coordinate basis $\{\hat{\theta}^\alpha\} = \{e^\alpha{}_\mu \mathrm{d}x^\mu\}$, the $*$ operation becomes

$$*(\hat{\theta}^{\alpha_1} \wedge \ldots \wedge \hat{\theta}^{\alpha_r}) = \frac{1}{(m-r)!}\,\varepsilon^{\alpha_1\ldots\alpha_r}{}_{\beta_{r+1}\ldots\beta_m}\hat{\theta}^{\beta_{r+1}} \wedge \ldots \wedge \hat{\theta}^{\beta_m} \tag{7.174}$$

where

$$\varepsilon_{\alpha_1\ldots\alpha_m} = \begin{cases} +1 \text{ if } (\alpha_1\ldots\alpha_m) \text{ is an even permutation of } (12\ldots m) \\ -1 \text{ if } (\alpha_1\ldots\alpha_m) \text{ is an odd permutation of } (12\ldots m) \\ 0 \quad \text{otherwise} \end{cases} \tag{7.175}$$

and the indices are raised by $\delta^{\alpha\beta}$ or $\eta^{\alpha\beta}$.

Theorem 7.45 Let $\omega \in \Omega^r(M)$. Then

$$**\omega = (-1)^{r(m-r)}\omega \tag{7.176a}$$

if (M, g) is Riemannian and

$$**\omega = (-1)^{1+r(m-r)}\omega \tag{7.176b}$$

if Lorentzian.

Proof: It is simpler to prove (7.176*a*) with a non-coordinate basis. Let

$$\omega = \frac{1}{r!}\,\omega_{\alpha_1\ldots\alpha_r}\,\hat{\theta}^{\alpha_1} \wedge \ldots \wedge \hat{\theta}^{\alpha_r}.$$

Repeated applications of $*$ on ω yield

$$\begin{aligned}
**\omega &= \frac{1}{r!}\,\omega_{\alpha_1\ldots\alpha_r}\,\frac{1}{(m-r)!}\,\varepsilon^{\alpha_1\ldots\alpha_r}{}_{\beta_{r+1}\ldots\beta_m}\\
&\quad\times\frac{1}{r!}\,\varepsilon^{\beta_{r+1}\ldots\beta_m}{}_{\gamma_1\ldots\gamma_r}\,\hat{\theta}^{\gamma_1}\wedge\ldots\wedge\hat{\theta}^{\gamma_r}\\
&= \frac{(-1)^{r(m-r)}}{r!\,r!\,(m-r)!}\sum_{\alpha\beta\gamma}\omega_{\alpha_1\ldots\alpha_r}\varepsilon_{\alpha_1\ldots\alpha_r\beta_{r+1}\ldots\beta_m}\varepsilon_{\gamma_1\ldots\gamma_r\beta_{r+1}\ldots\beta_m}\\
&\quad\times\hat{\theta}^{\gamma_1}\wedge\ldots\wedge\hat{\theta}^{\gamma_r}\\
&= \frac{(-1)^{r(m-r)}}{r!}\,\omega_{\alpha_1\ldots\alpha_r}\hat{\theta}^{\alpha_1}\wedge\ldots\wedge\hat{\theta}^{\alpha_r} = (-1)^{r(m-r)}\omega
\end{aligned}$$

where use has been made of the identity

$$\begin{aligned}
&\sum_{\beta\gamma}\varepsilon_{\alpha_1\ldots\alpha_r\beta_{r+1}\ldots\beta_m}\varepsilon_{\gamma_1\ldots\gamma_r\beta_{r+1}\ldots\beta_m}\hat{\theta}^{\gamma_1}\wedge\ldots\wedge\hat{\theta}^{\gamma_r}\\
&\quad = r!\,(m-r)!\,\hat{\theta}^{\alpha_1}\wedge\ldots\wedge\hat{\theta}^{\alpha_r}.
\end{aligned}$$

The proof of (7.176*b*) is left as an exercise to the reader (use $\det\eta = -1$). ■

Thus we find that $(-1)^{r(m-r)}**$ (or $(-1)^{1+r(m-r)}**$) is an identity map on $\Omega^r(M)$. We define the inverse of $*$ by

$$*^{-1} = (-1)^{r(m-r)}* \qquad (M, g) \text{ is Riemannian} \qquad (7.177a)$$

$$*^{-1} = (-1)^{1+r(m-r)}* \qquad (M, g) \text{ is Lorentzian.} \qquad (7.177b)$$

7.9.3 Inner products of r-forms

Take

$$\omega = \frac{1}{r!}\,\omega_{\mu_1\ldots\mu_r}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}$$

$$\eta = \frac{1}{r!}\,\eta_{\mu_1\ldots\mu_r}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}\in\Omega^r(M).$$

The exterior product $\omega\wedge *\eta$ is an m-form,

$$\begin{aligned}
\omega\wedge *\eta &= \frac{1}{(r!)^2}\,\omega_{\mu_1\ldots\mu_r}\eta_{\nu_1\ldots\nu_r}\,\frac{\sqrt{|g|}}{(m-r)!}\,\varepsilon^{\nu_1\ldots\nu_r}{}_{\mu_{r+1}\ldots\mu_m}\\
&\quad\times\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}\wedge\mathrm{d}x^{\mu_{r+1}}\wedge\ldots\wedge\mathrm{d}x^{\mu_m}\\
&= \frac{1}{r!}\sum_{\mu\nu}\omega_{\mu_1\ldots\mu_r}\eta^{\nu_1\ldots\nu_r}\,\frac{1}{r!(m-r)!}\,\varepsilon_{\nu_1\ldots\nu_r\mu_{r+1}\ldots\mu_m}\\
&\quad\times\varepsilon_{\mu_1\ldots\mu_r\mu_{r+1}\ldots\mu_m}\sqrt{|g|}\,\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^m\\
&= \frac{1}{r!}\,\omega_{\mu_1\ldots\mu_r}\eta^{\mu_1\ldots\mu_r}\sqrt{|g|}\,\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^m. \qquad (7.178)
\end{aligned}$$

This expression shows that the product is symmetric

$$\omega \wedge *\eta = \eta \wedge *\omega. \tag{7.179}$$

Let $\{\hat{\theta}^\alpha\}$ be the non-coordinate basis and

$$\omega = \frac{1}{r!} \omega_{\alpha_1 \ldots \alpha_r} \hat{\theta}^{\alpha_1} \wedge \ldots \wedge \hat{\theta}^{\alpha_r}$$

$$\eta = \frac{1}{r!} \eta_{\alpha_1 \ldots \alpha_r} \hat{\theta}^{\alpha_1} \wedge \ldots \wedge \hat{\theta}^{\alpha_r} \in \Omega^r(M).$$

Equation (7.178) is rewritten as

$$\omega \wedge *\eta = \frac{1}{r!} \omega_{\alpha_1 \ldots \alpha_r} \eta^{\alpha_1 \ldots \alpha_r} \hat{\theta}^1 \wedge \ldots \wedge \hat{\theta}^m. \tag{7.180}$$

Since $\alpha \wedge *\beta$ is an m-form, its integral over M is well defined. Define the **inner product** (ω, η) of two r-forms by

$$\begin{aligned}(\omega, \eta) &\equiv \int \omega \wedge *\eta \\ &= \frac{1}{r!} \int_M \omega_{\mu_1 \ldots \mu_r} \eta^{\mu_1 \ldots \mu_r} \sqrt{|g|}\, \mathrm{d}x^1 \ldots \mathrm{d}x^m.\end{aligned} \tag{7.181}$$

Since $\omega \wedge *\eta = \eta \wedge *\omega$, the inner product is symmetric,

$$(\omega, \eta) = (\eta, \omega). \tag{7.182}$$

If (M, g) is Riemannian, the inner product is positive definite,

$$(\alpha, \alpha) \geqslant 0 \tag{7.183}$$

where the equality is true only when $\alpha = 0$. This is not true if (M, g) is Lorentzian.

7.9.4 *Adjoints of exterior derivatives*

Definition 7.46 Let $\mathrm{d} : \Omega^{r-1}(M) \to \Omega^r(M)$ be the exterior derivative operator. The **adjoint exterior derivative operator** $\mathrm{d}^\dagger : \Omega^r(M) \to \Omega^{r-1}(M)$ is defined by

$$\mathrm{d}^\dagger = (-1)^{mr+m+1} * \mathrm{d} * \tag{7.184a}$$

if (M, g) is Riemannian and

$$\mathrm{d}^\dagger = (-1)^{mr+m} * \mathrm{d} * \tag{7.184b}$$

if Lorentzian, where $m = \dim M$.

In summary, we have the following diagram,

$$\begin{array}{ccc} \Omega^{m-r}(M) & \xrightarrow[\text{[or } (-1)^{mr+m}\mathrm{d}]]{(-1)^{mr+m+1}\mathrm{d}} & \Omega^{m-r+1}(M) \\ \uparrow * & & \downarrow * \\ \Omega^r(M) & \xrightarrow{\quad \mathrm{d}^\dagger \quad} & \Omega^{r-1}(M). \end{array} \tag{7.185}$$

The operator $\mathrm{d}^\dagger$ is nilpotent since d is; $\mathrm{d}^{\dagger 2} = *\mathrm{d}**\mathrm{d}* \propto *\mathrm{d}^2* = 0$.

Theorem 7.47 Let (M, g) be a compact orientable manifold without a boundary and $\alpha \in \Omega^r(M)$, $\beta \in \Omega^{r-1}(M)$. Then

$$(\mathrm{d}\beta, \alpha) = (\beta, \mathrm{d}^\dagger\alpha). \tag{7.186}$$

Proof: Since $\mathrm{d}\beta \wedge *\alpha$ and $\beta \wedge *\mathrm{d}^\dagger\alpha$ are m-forms, their integrals over M are well defined. Let d act on $\beta \wedge *\alpha$,

$$\mathrm{d}(\beta \wedge *\alpha) = \mathrm{d}\beta \wedge *\alpha - (-1)^r\beta \wedge \mathrm{d}*\alpha.$$

Suppose (M, g) is Riemannian. Noting that $\mathrm{d}*\alpha$ is an $(m - r + 1)$-form and inserting the identity map $(-1)^{(m-r+1)[m-(m-r+1)]}** = (-1)^{mr+m+r+1}**$ in front of $\mathrm{d}*\alpha$ in the second term, we have

$$\mathrm{d}(\beta \wedge *\alpha) = \mathrm{d}\beta \wedge *\alpha - (-1)^{mr+m+1}\beta \wedge *(*\mathrm{d}*\alpha).$$

Integrating this equation over M, we have

$$\int_M \mathrm{d}\beta \wedge *\alpha - \int_M \beta \wedge *[(-1)^{mr+m+1}*\mathrm{d}*\alpha] = \int_M \mathrm{d}(\beta \wedge *\alpha)$$
$$= \int_{\partial M} \beta \wedge *\alpha = 0$$

where the last equality follows by assumption. This shows that $(\mathrm{d}\beta, \alpha) = (\beta, \mathrm{d}^\dagger\alpha)$. The reader should check how the proof is modified in the case that (M, g) is Lorentzian. ■

7.9.5 The Laplacian, harmonic forms and the Hodge decompositon theorem

Definition 7.48 The Laplacian $\Delta : \Omega^r(M) \to \Omega^r(M)$ is defined by

$$\Delta = (\mathrm{d} + \mathrm{d}^\dagger)^2 = \mathrm{d}\cdot\mathrm{d}^\dagger + \mathrm{d}^\dagger\cdot\mathrm{d}. \tag{7.187}$$

As an example, we obtain the explicit form of $\Delta : \Omega^0(M) \to \Omega^0(M)$. Let $f \in \mathcal{F}(M)$. Since $\mathrm{d}^\dagger f = 0$, we have

$$\begin{aligned}
\Delta f &= \mathrm{d}^\dagger\mathrm{d}f = -*\mathrm{d}*(\partial_\mu f\,\mathrm{d}x^\mu) \\
&= -*\mathrm{d}\left(\frac{\sqrt{g}}{(m-1)!}\,\partial_\mu f g^{\mu\lambda}\varepsilon_{\lambda\nu_2\ldots\nu_m}\,\mathrm{d}x^{\nu_2} \wedge \ldots \wedge \mathrm{d}x^{\nu_m}\right) \\
&= -*\frac{1}{(m-1)!}\,\partial_\nu[\sqrt{g}\ g^{\lambda\mu}\partial_\mu f]\varepsilon_{\lambda\nu_2\ldots\nu_m}\mathrm{d}x^\nu \wedge \mathrm{d}x^{\nu_2} \wedge \ldots \wedge \mathrm{d}x^{\nu_m} \\
&= -*\partial_\nu[\sqrt{g}\ g^{\nu\mu}\partial_\mu f]g^{-1}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \\
&= -\frac{1}{\sqrt{g}}\,\partial_\nu[\sqrt{g}\ g^{\nu\mu}\partial_\mu f].
\end{aligned} \tag{7.188}$$

Exercise 7.49 Take a one-form $\omega = \omega_\mu\,\mathrm{d}x^\mu$ in the Euclidean space

$(\mathbb{R}^m, \delta)$. Show that

$$\Delta\omega = -\sum_{\mu=1}^{m} (\partial^2\omega_\nu/\partial x^\mu \partial x^\mu)\, \mathrm{d}x^\nu.$$

Example 7.50 In example 5.33, it was shown that half of the Maxwell equations are reduced to the identity, $\mathrm{d}F = \mathrm{d}^2 A = 0$, where $A = A_\mu \,\mathrm{d}x^\mu$ is the vector potential one-form and $F = \mathrm{d}A$ is the electromagnetic two-form. Let ρ be the electric charge density and $\boldsymbol{j}$ the electric current density and form the current one-form $j = \eta_{\mu\nu} j^\nu \,\mathrm{d}x^\mu = -\rho\,\mathrm{d}t + \boldsymbol{j}\cdot\mathrm{d}\boldsymbol{x}$. Then the remaining Maxwell equations become

$$\mathrm{d}^\dagger F = \mathrm{d}^\dagger \mathrm{d}A = j. \tag{7.189a}$$

The component expression is

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho \qquad \boldsymbol{\nabla}\times\boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j}. \tag{7.189b}$$

The vector potential A has a large number of degrees of freedom and we can always choose an A which satisfies the **Lorentz condition** $\mathrm{d}^\dagger A = 0$. Then (7.189a) becomes $(\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d})A = \Delta A = j$.

Let (M, g) be a compact Riemannian manifold. The Laplacian Δ is a *positive* operator on M in the sense that

$$\begin{aligned}(\omega, \Delta\omega) &= (\omega, (\mathrm{d}^\dagger\mathrm{d} + \mathrm{d}\mathrm{d}^\dagger)\omega) \\ &= (\mathrm{d}\omega, \mathrm{d}\omega) + (\mathrm{d}^\dagger\omega, \mathrm{d}^\dagger\omega) \geqslant 0 \end{aligned}\tag{7.190}$$

where (7.183) has been used. An r-form ω is called **harmonic** if $\Delta\omega = 0$ and **closed (co-closed)** if $\mathrm{d}\omega = 0$ ($\mathrm{d}^\dagger\omega = 0$). The following theorem is a direct consequence of (7.190).

Theorem 7.51 An r-form ω is harmonic if and only if ω is closed and co-closed.

An r-form ω_r is called **coexact** if it is written *globally* as

$$\omega_r = \mathrm{d}^\dagger\beta_{r+1} \tag{7.191}$$

where $\beta_{r+1} \in \Omega^{r+1}(M)$. [cf $\omega_r \in \Omega^r(M)$ is exact if $\omega_r = \mathrm{d}\alpha_{r-1}$, $\alpha_{r-1} \in \Omega^{r-1}(M)$.] We denote the set of harmonic r-forms on M by $\mathrm{Harm}^r(M)$ and the set of exact r-forms (coexact r-forms) by $B^r(M) = \mathrm{d}\Omega^{r-1}(M)$ $(\mathrm{d}^\dagger\Omega^{r+1}(M))$.

Theorem 7.52 (**Hodge decomposition theorem**) Let (M, g) be a compact orientable Riemannian manifold without a boundary. Then $\Omega^r(M)$ is *uniquely* decomposed as

$$\Omega^r(M) = \mathrm{d}\Omega^{r-1}(M) \oplus \mathrm{d}^\dagger\Omega^{r+1}(M) \oplus \mathrm{Harm}^r(M). \tag{7.192a}$$

[That is, any r-form ω_r is written globally as

$$\omega_r = \mathrm{d}\alpha_{r-1} + \mathrm{d}^\dagger\beta_{r+1} + \gamma_r \tag{7.192b}$$

where $\alpha_{r-1} \in \Omega^{r-1}(M)$, $\beta_{r+1} \in \Omega^{r+1}(M)$ and $\gamma_r \in \mathrm{Harm}^r(M)$.]

If $r = 0$, we define $\Omega^{-1}(M) = \{0\}$. The proof of this theorem requires the results of the two easy exercises below.

Exercise 7.53 Let (M, g) be as given in theorem 7.52. Show that

$$(\mathrm{d}\alpha_{r-1}, \mathrm{d}^\dagger\beta_{r+1}) = (\mathrm{d}\alpha_{r-1}, \gamma_r) = (\mathrm{d}^\dagger\beta_{r+1}, \gamma_r) = 0. \tag{7.193}$$

Show also that if $\omega_r \in \Omega^r(M)$ satisfies

$$(\mathrm{d}\alpha_{r-1}, \omega_r) = (\mathrm{d}^\dagger\beta_{r+1}, \omega_r) = (\gamma_r, \omega_r) = 0 \tag{7.194}$$

for any $\mathrm{d}\alpha_{r-1} \in \mathrm{d}\Omega^{r-1}(M)$, $\mathrm{d}^\dagger\beta_{r+1} \in \mathrm{d}^\dagger\Omega^{r+1}(M)$ and $\gamma_r \in \mathrm{Harm}^r(M)$, then $\omega_r = 0$.

Exercise 7.54 Suppose $\omega_r \in \Omega^r(M)$ is written as $\omega_r = \Delta\psi_r$ for some $\psi_r \in \Omega^r(M)$. Show that $(\omega_r, \gamma_r) = 0$ for any $\gamma_r \in \mathrm{Harm}^r(M)$. The proof of the converse 'if ω_r is orthogonal to any harmonic r-form, then ω_r is written as $\Delta\psi_r$ for some $\psi_r \in \Omega^r(M)$' is highly technical and we just state that the operator Δ^{-1} (the Green function) is well defined in the present problem and ψ_r is given by $\Delta^{-1}\omega_r$.

Let $P : \Omega^r(M) \to \mathrm{Harm}^r(M)$ be a projection operator to the space of harmonic r-forms. Take an element $\omega_r \in \Omega^r(M)$. Since $\omega_r - P\omega_r$ is orthogonal to $\mathrm{Harm}^r(M)$, it can be written as $\Delta\psi_r$ for some $\psi_r \in \Omega^r(M)$. Then we have

$$\omega_r = \mathrm{d}(\mathrm{d}^\dagger\psi_r) + \mathrm{d}^\dagger(\mathrm{d}\psi_r) + P\omega_r. \tag{7.195}$$

This realises the decomposition of theorem 7.52.

7.9.6 *Harmonic forms and de Rham cohomology groups*

We show that any element of the de Rham cohomology group has a *unique* harmonic representative. Let $[\omega_r] \in H^r(M)$. We first show that $\omega_r \in \mathrm{Harm}^r(M) \oplus \mathrm{d}\Omega^{r-1}(M)$. According to (7.192), ω_r is decomposed as $\omega_r = \gamma_r + \mathrm{d}\alpha_{r-1} + \mathrm{d}^\dagger\beta_{r+1}$. Since $\mathrm{d}\omega_r = 0$, we have

$$0 = (\mathrm{d}\omega_r, \beta_{r+1}) = (\mathrm{d}\mathrm{d}^\dagger\beta_{r+1}, \beta_{r+1}) = (\mathrm{d}^\dagger\beta_{r+1}, \mathrm{d}^\dagger\beta_{r+1}).$$

This is satisfied if and only if $\mathrm{d}^\dagger\beta_{r+1} = 0$. Hence $\omega_r = \gamma_r + \mathrm{d}\alpha_{r-1}$. From (7.195) we have

$$\omega_r = P\omega_r + \mathrm{d}(\mathrm{d}^\dagger\psi) = P\omega_r + \mathrm{d}\cdot\mathrm{d}^\dagger\Delta^{-1}\omega_r. \tag{7.196a}$$

$\gamma_r \equiv P\omega_r$ is the harmonic representative of $[\omega_r]$. Let $\bar{\omega}_r$ be another representative of $[\omega_r]$; $\bar{\omega}_r - \omega_r = \mathrm{d}\eta_{r-1}$, $\eta_{r-1} \in \Omega^{r-1}(M)$. Corresponding to (7.196a), we have

$$\tilde{\omega}_r = P\tilde{\omega}_r + \mathrm{d}(\mathrm{d}^\dagger\Delta^{-1}\tilde{\omega}_r) = P\omega_r + \mathrm{d}(\ldots) \qquad (7.196b)$$

where the last equality follows since $\mathrm{d}\eta_{r-1}$ is orthogonal to $\mathrm{Harm}^r(M)$ and hence its projection to $\mathrm{Harm}^r(M)$ vanishes. (7.196) show that $[\omega]$ has a unique harmonic representative $P\omega$.

The above proof shows that $H^r(M) \subset \mathrm{Harm}^r(M)$. Now we prove that $\mathrm{Harm}^r(M) \subset H^r(M)$. Since $\mathrm{d}\gamma_r = 0$ for any $\gamma_r \in \mathrm{Harm}^r(M)$, we find that $Z^r(M) \supset \mathrm{Harm}^r(M)$. We also have $B^r(M) \cap \mathrm{Harm}^r(M) = \varnothing$ since $B^r(M) = \mathrm{d}\Omega^{r-1}(M)$, see (7.192). Thus every element of $\mathrm{Harm}^r(M)$ is a non-trivial member of $H^r(M)$ and we find that $\mathrm{Harm}^r(M)$ is a vector subspace of $H^r(M)$ and hence $\mathrm{Harm}^r(M) \subset H^r(M)$. We have proved:

Theorem 7.55 (**Hodge's theorem**) On a compact orientable Riemannian manifold (M, g), $H^r(M)$ is isomorphic to $\mathrm{Harm}^r(M)$:

$$H^r(M) \cong \mathrm{Harm}^r(M). \qquad (7.197)$$

The isomorphism is provided by identifying $[\omega] \in H^r(M)$ with $P\omega \in \mathrm{Harm}^r(M)$.

In particular, we have

$$\dim \mathrm{Harm}^r(M) = \dim H^r(M) = b^r \qquad (7.198)$$

b^r being the Betti number. The Euler characteristic is given by

$$\chi(M) = \sum (-1)^r b^r = \sum (-1)^r \dim \mathrm{Harm}^r(M) \qquad (7.199)$$

see theorem 3.33. We note that the LHS is a topological quantity while the RHS is an analytical quantity given by the eigenvalue problem of the Laplacian Δ.

7.10 Aspects of general relativity and the Polyakov string

7.10.1 The Einstein–Hilbert action

This and the next example are taken from Weinberg (1972). The general theory of relativity describes the dynamics of the geometry, that is, the dynamics of $g_{\mu\nu}$. What is the action principle for this theory? As usual, we require that the relevant action should be a scalar. Moreover, it should contain the derivatives of $g_{\mu\nu}$; $\int \sqrt{|g|}\,\mathrm{d}^m x$ cannot describe the dynamics of the metric. The simplest guess will be $S_{\mathrm{EH}} \propto \int \mathcal{R}\sqrt{|g|}\,\mathrm{d}^m x$. Since $\mathcal{R}$ is a scalar and $\sqrt{|g|}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$ is the invariant volume element, S_{EH} is a scalar. In the following, we show that S_{EH} indeed yields the Einstein equation under the variation with respect to the metric. Our connection is restricted to the Levi-Civita connection. We first prove a technical proposition.

Proposition 7.56 Let (g, M) be a (pseudo-) Riemannian manifold. Under the variation $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, $g^{\mu\nu}$, g and $Ric_{\mu\nu}$ change as

$$(a)\ \delta g^{\mu\nu} = -g^{\mu\kappa}g^{\lambda\nu}\delta g_{\kappa\lambda} \tag{7.200}$$

$$(b)\ \delta g = gg^{\mu\nu}\,\delta g_{\mu\nu},\ \delta\sqrt{|g|} = \tfrac{1}{2}\sqrt{|g|}\ g^{\mu\nu}\delta g_{\mu\nu} \tag{7.201}$$

$$(c)\ \delta Ric_{\mu\nu} = \nabla_\kappa \delta\Gamma^\kappa{}_{\nu\mu} - \nabla_\nu \delta\Gamma^\kappa{}_{\kappa\mu}\ \text{(the **Palatini identity**)}. \tag{7.202}$$

Proof: (a) From $g_{\kappa\lambda}g^{\lambda\nu} = \delta_\kappa{}^\nu$, it follows that

$$0 = \delta(g_{\kappa\lambda}g^{\lambda\nu}) = \delta g_{\kappa\lambda}g^{\lambda\nu} + g_{\kappa\lambda}\delta g^{\lambda\nu}.$$

Multiplying by $g^{\mu\kappa}$ we find that $g^{\mu\nu} = -g^{\mu\kappa}g^{\lambda\nu}\delta g_{\kappa\lambda}$.

(b) We first note the matrix identity $\ln(\det g_{\mu\nu}) = \mathrm{tr}(\ln g_{\mu\nu})$. This can be proved by diagonalising $g_{\mu\nu}$. Under the variation δg, the LHS becomes $\delta g \cdot g^{-1}$ while the RHS yields $g^{\mu\nu}\cdot\delta g_{\nu\mu}$, hence $\delta g = gg^{\mu\nu}\delta g_{\mu\nu}$. The rest of (7.201) is easily derived from this.

(c) Let Γ and $\widetilde{\Gamma}$ be two connections. From exercise 7.8, the difference $\delta\Gamma \equiv \widetilde{\Gamma} - \Gamma$ is a tensor of type $(1, 2)$. In the present case, we take $\widetilde{\Gamma}$ to be a connection associated with $g + \delta g$ and Γ with g. We will work in the normal coordinate system in which $\Gamma \equiv 0$ (of course $\partial\Gamma \neq 0$ in general) see §7.4. We find

$$\delta Ric_{\mu\nu} = \partial_\kappa \delta\Gamma^\kappa{}_{\nu\mu} - \partial_\nu \delta\Gamma^\kappa{}_{\kappa\mu} = \nabla_\kappa \delta\Gamma^\kappa{}_{\nu\mu} - \nabla_\nu \delta\Gamma^\kappa{}_{\kappa\mu}.$$

[The reader should verify the second equality.] Since both sides are tensors, this is valid in any coordinate system. ■

We define the **Einstein–Hilbert action** by

$$S_{\mathrm{EH}} \equiv \frac{1}{16\pi G}\int \mathcal{R}\sqrt{-g}\ \mathrm{d}^m x. \tag{7.203}$$

The constant factor is multiplied to reproduce the Newtonian limit when matter is added; see (7.211) and §1.6. We prove that $\delta S_{\mathrm{EH}} = 0$ leads to the Einstein equation. Under the variation $g \to g + \delta g$ such that $\delta g \to 0$ as $|x| \to 0$, the integrand changes as

$$\begin{aligned}
\delta(\mathcal{R}\sqrt{-g}) &= \delta(g^{\mu\nu}Ric_{\mu\nu}\sqrt{-g}) \\
&= \delta g^{\mu\nu}Ric_{\mu\nu}\sqrt{-g} + g^{\mu\nu}\delta Ric_{\mu\nu}\sqrt{-g} + \mathcal{R}\delta(\sqrt{-g}) \\
&= -g^{\mu\kappa}g^{\lambda\nu}\delta g_{\kappa\lambda}Ric_{\mu\nu}\sqrt{-g} \\
&\quad + g^{\mu\nu}(\nabla_\kappa\delta\Gamma^\kappa{}_{\nu\mu} - \nabla_\nu\delta\Gamma^\kappa{}_{\kappa\mu})\sqrt{-g} + \tfrac{1}{2}\mathcal{R}\sqrt{-g}\ g^{\mu\nu}\delta g_{\mu\nu}.
\end{aligned}$$

We note that the second term is a total divergence,

$$\begin{aligned}
&\nabla_\kappa(g^{\mu\nu}\delta\Gamma^\kappa{}_{\nu\mu}\sqrt{-g}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\kappa{}_{\kappa\mu}\sqrt{-g}) \\
&\quad = \partial_\kappa(g^{\mu\nu}\delta\Gamma^\kappa{}_{\mu\nu}\sqrt{-g}) - \partial_\nu(g^{\mu\nu}\delta\Gamma^\kappa{}_{\kappa\mu}\sqrt{-g})
\end{aligned}$$

and hence does not contribute to the variation. From the remaining terms we have

$$\delta S_{\rm EH} = \frac{1}{16\pi G}\int (-Ric^{\mu\nu} + \tfrac{1}{2}\mathcal{R}g^{\mu\nu})\delta g_{\mu\nu}\sqrt{-g}\,\mathrm{d}^m x. \qquad (7.204)$$

If we require that $\delta S_{\rm EH} = 0$ under any variation δg, we obtain the vacuum Einstein equation,

$$G_{\mu\nu} = Ric_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{R} = 0 \qquad (7.205)$$

where the symmetric tensor G is called the **Einstein tensor**.

So far we have considered the gravitational field only. Suppose there exists matter described by an action

$$S_{\rm M} \equiv \int \mathcal{L}(\phi)\sqrt{-g}\,\mathrm{d}^m x \qquad (7.206)$$

where $\mathcal{L}(\phi)$ is the Lagrangian density of the theory. Typical examples are the real scalar field and the Maxwell fields,

$$S_{\rm S} \equiv -\frac{1}{2}\int [g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2]\sqrt{-g}\,\mathrm{d}^m x \qquad (7.207a)$$

$$S_{\rm ED} \equiv -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\sqrt{-g}\,\mathrm{d}^m x \qquad (7.207b)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. If the matter action changes by $\delta S_{\rm M}$ under δg, the **energy–momentum tensor** $T^{\mu\nu}$ is defined by

$$\delta S_{\rm M} = \frac{1}{2}\int T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}\,\mathrm{d}^m x. \qquad (7.208)$$

Since $\delta g_{\mu\nu}$ is symmetric, $T^{\mu\nu}$ is also taken to be so. For example, $T_{\mu\nu}$ of a real scalar field is given by

$$\begin{aligned} T_{\mu\nu}(x) &= 2\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}(x)}S_{\rm S} \\ &= \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}(g^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi + m^2\phi^2). \qquad (7.209)\end{aligned}$$

Suppose we have a gravitational field coupled with a matter field whose action is $S_{\rm M}$. Now our action principle is

$$\delta(S_{\rm EH} + S_{\rm M}) = 0 \qquad (7.210)$$

under $g \to g + \delta g$. From (7.204) and (7.208) we obtain the **Einstein equation**

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \qquad (7.211)$$

Exercise 7.57 We may add an extra scalar to the scalar curvature without spoiling the invariance of the action. For example, we can add a constant called the **cosmological constant** Λ,

$$\widetilde{S}_{\rm EH} = \frac{1}{16\pi G}\int_M (\mathcal{R} + \Lambda)\sqrt{-g}\,\mathrm{d}^m x. \qquad (7.212)$$

Write down the vacuum Einstein equation. Other possible scalars may be such terms as $\mathcal{R}^2$, $\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}$ or $R_{\kappa\lambda\mu\nu}R^{\kappa\lambda\mu\nu}$.

7.10.2 Spinors in curved spacetime

For concreteness, we consider a Dirac spinor ψ in a four-dimensional Lorentz manifold M. The vierbein $e^\alpha{}_\mu$ defined by

$$g_{\mu\nu} = e^\alpha{}_\mu e^\beta{}_\nu \eta_{\alpha\beta} \tag{7.213}$$

defines an orthonormal frame $\{\hat{\theta}^\alpha = e^\alpha{}_\mu \mathrm{d}x^\mu\}$ at each point $p \in M$. As noted before, α, β, γ, ... are the local orthonormal indices while μ, ν, λ, ... are the coordinate indices. With respect to this frame, the Dirac matrices $\gamma^\alpha \equiv e^\alpha{}_\mu \gamma^\mu$ satisfy $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$. Under a local Lorentz transformation $\Lambda^\alpha{}_\beta(p)$, the Dirac spinor transforms as

$$\psi(p) \to \rho(\Lambda)\psi(p) \qquad \bar{\psi}(p) \to \bar{\psi}(p)\rho(\Lambda)^{-1} \tag{7.214}$$

where $\bar{\psi} \equiv \psi^\dagger \gamma^0$ and $\rho(\Lambda)$ is the spinor representation of Λ. To construct an invariant action, we seek a covariant derivative $\nabla_\alpha \psi$ which is a local Lorentz vector and transforms as a spinor,

$$\nabla_\alpha \psi \to \rho(\Lambda)\Lambda_\alpha{}^\beta \nabla_\beta \psi. \tag{7.215}$$

If we find such a $\nabla_\alpha \psi$, an invariant Lagrangian may be given by

$$\mathcal{L} = \bar{\psi} \mathrm{i} \gamma^\alpha \nabla_\alpha \psi. \tag{7.216}$$

We note that $e_\alpha{}^\mu \partial_\mu \psi$ transforms under $\Lambda(p)$ as

$$e_\alpha{}^\mu \partial_\mu \psi \to \Lambda_\alpha{}^\beta e_\beta{}^\mu \partial_\mu \rho(\Lambda)\psi = \Lambda_\alpha{}^\eta e_\beta{}^\mu [\rho(\Lambda)\partial_\mu \psi + \partial_\mu \rho(\Lambda)\psi]. \tag{7.217}$$

Suppose ∇_α is of the form

$$\nabla_\alpha \psi = e_\alpha{}^\mu [\partial_\mu + \Omega_\mu]\psi. \tag{7.218}$$

From (7.215) and (7.217), we find that Ω_μ satisfies

$$\Omega_\mu \to \rho(\Lambda)\Omega_\mu \rho(\Lambda)^{-1} - \partial_\mu \rho(\Lambda)\rho(\Lambda)^{-1}. \tag{7.219}$$

To find the explicit form of Ω_μ, we consider an infinitesimal local Lorentz transformation $\Lambda_\alpha{}^\beta(p) = \delta_\alpha{}^\beta + \varepsilon_\alpha{}^\beta(p)$. The Dirac spinor transforms as

$$\psi \to \exp[\tfrac{1}{2}\mathrm{i}\varepsilon^{\alpha\beta}\Sigma_{\alpha\beta}]\psi \cong [1 + \tfrac{1}{2}\mathrm{i}\varepsilon^{\alpha\beta}\Sigma_{\alpha\beta}]\psi \tag{7.220}$$

where $\Sigma_{\alpha\beta} \equiv \frac{1}{4}\mathrm{i}[\gamma_\alpha, \gamma_\beta]$ is the spinor representation of the generators of the Lorentz transformation. $\Sigma_{\alpha\beta}$ satisfies the $\mathfrak{o}(1, 3)$ Lie algebra

$$\mathrm{i}[\Sigma_{\alpha\beta}, \Sigma_{\gamma\delta}] = \eta_{\gamma\beta}\Sigma_{\alpha\delta} - \eta_{\gamma\alpha}\Sigma_{\beta\delta} + \eta_{\delta\beta}\Sigma_{\gamma\alpha} - \eta_{\delta\alpha}\Sigma_{\gamma\beta}. \tag{7.221}$$

Under the same Lorentz transformation, Ω_μ transforms as

$$\Omega_\mu \to (1 + \tfrac{1}{2}\mathrm{i}\varepsilon^{\alpha\beta}\Sigma_{\alpha\beta})\Omega_\mu(1 - \tfrac{1}{2}\mathrm{i}\varepsilon^{\gamma\delta}\Sigma_{\gamma\delta}) - \tfrac{1}{2}\mathrm{i}\partial_\mu\varepsilon^{\alpha\beta}\Sigma_{\alpha\beta}(1 - \tfrac{1}{2}\mathrm{i}\varepsilon^{\gamma\delta}\Sigma_{\gamma\delta})$$
$$= \Omega_\mu + \tfrac{1}{2}\mathrm{i}\varepsilon^{\alpha\beta}[\Sigma_{\alpha\beta}, \Omega_\mu] - \tfrac{1}{2}\mathrm{i}\partial_\mu\varepsilon^{\alpha\beta}\Sigma_{\alpha\beta}. \tag{7.222}$$

We recall that the connection one-form $\omega^\alpha{}_\beta$ transforms under an infinitesimal Lorentz transformation as (see (7.152))

$$\omega^\alpha{}_\beta \to \omega^\alpha{}_\beta + \varepsilon^\alpha{}_\gamma\omega^\gamma{}_\beta - \omega^\alpha{}_\gamma\varepsilon^\gamma{}_\beta - \mathrm{d}\varepsilon^\alpha{}_\beta \tag{7.223a}$$

or in components,

$$\Gamma^\alpha{}_{\mu\beta} \to \Gamma^\alpha{}_{\mu\beta} + \varepsilon^\alpha{}_\gamma\Gamma^\gamma{}_{\mu\beta} - \Gamma^\alpha{}_{\mu\gamma}\varepsilon^\gamma{}_\beta - \partial_\mu\varepsilon^\alpha{}_\beta. \tag{7.223b}$$

From (7.221), (7.222) and (7.223), we find that the combination

$$\Omega_\mu \equiv \tfrac{1}{2}\mathrm{i}\Gamma^\alpha{}_\mu{}^\beta\Sigma_{\alpha\beta} = \tfrac{1}{2}\mathrm{i}e^\alpha{}_\nu\nabla_\mu e^{\beta\nu}\Sigma_{\alpha\beta} \tag{7.224}$$

satisfies the transformation property (7.219). In fact,

$$\tfrac{1}{2}\mathrm{i}\Gamma^\alpha{}_\mu{}^\beta\Sigma_{\alpha\beta} \to \tfrac{1}{2}\mathrm{i}(\Gamma^\alpha{}_\mu{}^\beta + \varepsilon^\alpha{}_\gamma\Gamma^\gamma{}_\mu{}^\beta - \Gamma^\alpha{}_{\mu\gamma}\varepsilon^{\gamma\beta} - \partial_\mu\varepsilon^{\alpha\beta})\Sigma_{\alpha\beta}$$
$$= \tfrac{1}{2}\mathrm{i}\Gamma^\alpha{}_\mu{}^\beta\Sigma_{\alpha\beta} + \tfrac{1}{2}\mathrm{i}(\varepsilon^\alpha{}_\gamma\Gamma^\gamma{}_\mu{}^\beta\Sigma_{\alpha\beta} - \Gamma^\alpha{}_{\mu\gamma}\varepsilon^{\gamma\beta}\Sigma_{\alpha\beta}) - \tfrac{1}{2}\mathrm{i}\partial_\mu\varepsilon^{\alpha\beta}\Sigma_{\alpha\beta}$$
$$= \tfrac{1}{2}\mathrm{i}\Gamma^\alpha{}_\mu{}^\beta\Sigma_{\alpha\beta} + \tfrac{1}{2}\mathrm{i}\varepsilon^{\alpha\beta}[\Sigma_{\alpha\beta}, \tfrac{1}{2}\mathrm{i}\Gamma^\gamma{}_\mu{}^\delta\Sigma_{\gamma\delta}] - \tfrac{1}{2}\mathrm{i}\partial_\mu\varepsilon^{\alpha\beta}\Sigma_{\alpha\beta}.$$

We finally obtain the Lagrangian which is a scalar both under coordinate changes and local Lorentz rotations,

$$\mathcal{L} \equiv \bar{\psi}\mathrm{i}\gamma^\alpha e_\alpha{}^\mu(\partial_\mu + \tfrac{1}{2}\mathrm{i}\Gamma^\beta{}_\mu{}^\gamma\Sigma_{\beta\gamma})\psi \tag{7.225}$$

and the scalar action

$$S_\psi \equiv \int_M \mathrm{d}^4x\sqrt{-g}\,\bar{\psi}[\mathrm{i}\gamma^\alpha e_\alpha{}^\mu(\partial_\mu + \tfrac{1}{2}\mathrm{i}\Gamma^\beta{}_\mu{}^\gamma\Sigma_{\beta\gamma}) + m]\psi \tag{7.226a}$$

m being the mass of ψ. If ψ is coupled to the gauge field $\mathcal{A}$, the action is given by

$$S_\psi = \int_M \mathrm{d}^4x\sqrt{-g}\,\bar{\psi}[\mathrm{i}\gamma^\alpha e_\alpha{}^\mu(\partial_\mu + \mathcal{A}_\mu + \tfrac{1}{2}\mathrm{i}\Gamma^\alpha{}_\mu{}^\beta\Sigma_{\alpha\beta}) + m]\psi. \tag{7.226b}$$

It is interesting to note that the spin connection term vanishes if $\dim M = 2$. To see this, we rewrite (7.226a) as

$$S_\psi = \frac{1}{2}\int_M \mathrm{d}^2x\sqrt{-g}\,\bar{\psi}[\mathrm{i}\gamma^\mu \overleftrightarrow{\partial}_\mu + \tfrac{1}{2}\mathrm{i}\Gamma^\beta{}_\mu{}^\gamma\{\mathrm{i}\gamma^\mu, \Sigma_{\beta\gamma}\} + m]\psi \tag{7.226a'}$$

where $\gamma^\mu = \gamma^\alpha e_\alpha{}^\mu$ and we have added total derivatives to the Lagrangian to make it Hermitian. The non-vanishing components of Σ are $\Sigma_{01} \propto [\gamma_0, \gamma_1] \propto \gamma_3$, where γ_3 is the two-dimensional analogue of γ_5. Since $\{\gamma^\mu, \gamma_3\} = 0$, the spin connection term drops out from S_ψ.

7.10.3 Symmetries of the Polyakov strings

The bosonic string theory is defined on a two-dimensional Lorentz manifold (M, g). The embedding $f: M \to \mathbb{R}^D$ is defined by $\xi^\alpha \mapsto X^\mu$, where $\{\xi^\alpha\} = (\tau, \sigma)$ are the local coordinates of M, $\xi^0 = \tau$ being timelike and $\xi^1 = \sigma$ spacelike. We assume the physical spacetime is Minkowskian $(\mathbb{R}^D, \eta)$ for simplicity. The **Polyakov action**

$$S = -\frac{1}{2} \int \mathrm{d}^2\xi \sqrt{-g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{7.227}$$

is left invariant under the coordinate reparametrisation Diff(M) since the volume element $\sqrt{-g}\,\mathrm{d}^2\xi$ is invariant and $g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$ is a scalar.

Now we are ready to derive the equation of motion. Our variational parameters are the *embedding* X^μ and the *geometry* $g_{\alpha\beta}$. Under the variation δX^μ, we have the Euler–Lagrange equation

$$\partial_\alpha(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta X_\mu) = 0. \tag{7.228a}$$

Under the variation $\delta g_{\alpha\beta}$, the integrand of S changes as

$$\begin{aligned}\delta(\sqrt{-g}\, g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu) &= \delta\sqrt{-g}\, g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + \sqrt{-g}\,\delta g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \\ &= -\tfrac{1}{2}\sqrt{-g}\, g_{\gamma\delta}\,\delta g^{\gamma\delta} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \\ &\quad + \sqrt{-g}\,\delta g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\end{aligned}$$

where proposition 7.56 has been used. Since this is true for any variation δg, we should have

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \tfrac{1}{2} g_{\alpha\beta}(g^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu) = 0. \tag{7.228b}$$

This is solved for $g_{\alpha\beta}$ to yield

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{7.229}$$

showing that the induced metric (the RHS) agrees with $g_{\alpha\beta}$. Substituting (7.229) into (7.227) to eliminate $g_{\alpha\beta}$, we recover the **Nambu action**,

$$S = -\tfrac{1}{2} \int \mathrm{d}^2\xi [-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)]^{1/2}. \tag{7.230}$$

By construction, the action S is invariant under local reparametrisation of M, $\{\xi^\alpha\} \to \{\xi'^\alpha(\xi)\}$. In addition to this, the action has extra invariances. Under the global **Poincaré transformation** in D-dimensional spacetime,

$$X^\mu \to X'^\mu \equiv \Lambda^\mu{}_\nu X^\nu + a^\mu \tag{7.231}$$

the action S transforms as

$$\begin{aligned}S \to &-\tfrac{1}{2}\int \mathrm{d}^2\xi \sqrt{-g}\, g^{\alpha\beta}\partial_\alpha(\Lambda^\mu{}_\kappa X^\kappa + a^\mu)\partial_\beta(\Lambda^\nu{}_\lambda X^\lambda + a^\nu)\eta_{\mu\nu} \\ &= -\tfrac{1}{2}\int \mathrm{d}^2\xi \sqrt{-g}\, g^{\alpha\beta}\partial_\alpha X^\kappa \partial_\beta X^\lambda (\Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda \eta_{\mu\nu}).\end{aligned}$$

From $\Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda \eta_{\mu\nu} = \eta_{\kappa\lambda}$, we find that S is invariant under global Poincaré transformations. The action S is also invariant under the **Weyl rescaling**, $g_{\alpha\beta} \to e^{2\sigma(\tau,\sigma)} g_{\alpha\beta}(\tau, \sigma)$ keeping (τ, σ) fixed. In fact, S transforms as

$$S \to -\frac{1}{2}\int d^2\xi(-e^{4\sigma}g)^{1/2}e^{-2\sigma}g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

and hence is left invariant. Note that the Weyl rescaling invariance exists only when M is two dimensional, making strings prominent among other extended objects such as membranes.

Since $\dim M = 2$, we can always parametrise the world sheet by the *isothermal coordinate* (example 7.32) so that

$$g_{\alpha\beta} = e^{2\sigma(\tau,\sigma)}\eta_{\alpha\beta}. \tag{7.232}$$

Then the Weyl rescaling invariance allows us to choose the standard metric $\eta_{\alpha\beta}$ on the world sheet. The metric $g_{\alpha\beta}$ has three independent components while the reparametrisation has two degrees of freedom and the Weyl scaling invariance has one. Thus so long as we are dealing with strings, we can choose the standard metric $\eta_{\alpha\beta}$.

We end our analysis of Polyakov strings here. In Chapter 14, we quantise Polyakov strings in the most elegant manner.

Exercise 7.58 Let (M, g) and (N, h) be Riemannian manifolds. Take a chart U of M in which the metric g takes the form

$$g = g_{\mu\nu}(x)\,dx^\mu \otimes dx^\nu.$$

Take a chart V of N on which h takes the form

$$h = G_{\alpha\beta}(\phi)\,d\phi^\alpha \otimes d\phi^\beta.$$

A map $\phi : M \to N$ defined by $x \mapsto \phi(x)$ is called a **harmonic map** if it satisfies

$$\frac{1}{\sqrt{g}}\partial_\mu[\sqrt{g}\,g^{\mu\nu}\partial_\nu\phi^\alpha] + \Gamma^\alpha{}_{\beta\gamma}\partial_\mu\phi^\alpha\partial_\nu\phi^\beta g^{\mu\nu} = 0. \tag{7.233}$$

Show that this is obtained by the variation of the action

$$S \equiv \frac{1}{2}\int d^m x\sqrt{g}\,g^{\mu\nu}\partial_\mu\phi^\alpha\partial_\nu\phi^\beta h_{\alpha\beta}(\phi) \tag{7.234}$$

with respect to ϕ. Applications of harmonic maps to physics are found in Misner (1978) and Sánchez (1988). Mathematical aspects have been reviewed in Eells and Lemaire (1968).

Problems 7

1 Let ∇ be a general connection for which the torsion tensor does not vanish. Show that the first Bianchi identity becomes

$$\mathfrak{S}\{R(X, Y)Z\} = \mathfrak{S}\{T(X, [Y, Z])\} + \mathfrak{S}\{\nabla_X[T(Y, Z)]\}$$

where $\mathfrak{S}$ is the symmetriser defined in theorem 7.23. Show also that the second Bianchi identity is given by

$$\mathfrak{S}\{(\nabla_X R)(Y, Z)\}V = \mathfrak{S}\{R(X, T(Y, Z))\}V$$

where $\mathfrak{S}$ symmetrises X, Y and Z only.

2 Let (M, g) be a conformally flat three-dimensional manifold. Show that the Weyl–Schouten tensor defined by

$$C_{\lambda\mu\nu} \equiv \nabla_\nu Ric_{\lambda\mu} - \nabla_\mu Ric_{\lambda\nu} - \tfrac{1}{4}(g_{\lambda\mu}\partial_\nu \mathcal{R} - g_{\lambda\nu}\partial_\mu \mathcal{R})$$

vanishes. It is known that $C_{\lambda\mu\nu} = 0$ is the necessary and sufficient condition for conformal flatness if $\dim M = 3$.

3 Consider a metric

$$g = -\mathrm{d}t \otimes \mathrm{d}t + \mathrm{d}r \otimes \mathrm{d}r + (1 - 4\mu^2)r^2 \mathrm{d}\phi \otimes \mathrm{d}\phi + \mathrm{d}z \otimes \mathrm{d}z$$

where $0 < \mu < \frac{1}{2}$ and $\mu \neq \frac{1}{4}$. Introduce a new variable

$$\tilde{\phi} \equiv (1 - 4\mu)\phi$$

and show that the metric g reduces to the Minkowski metric. Does this mean that g describes Minkowski spacetime? Compute the Riemann curvature tensor and show that there is a string-like singularity at $r = 0$. This singularity is *conical* (the spacetime is flat except along the line).

8

COMPLEX MANIFOLDS

A differentiable manifold is a topological space which admits differentiable structures. Here we introduce another structure which has relevance in physics. In elementary complex analysis, the partial derivatives are required to satisfy the Cauchy–Riemann relations. We talk not only of the *differentiability* but also of the *analyticity* of a function in this case. A complex manifold admits a complex structure in which each coordinate neighbourhood is homeomorphic to $\mathbb{C}^m$ and the transition from one coordinate system to the other is analytic.

The reader may consult Chern (1979), Goldberg (1962) or Greene (1987) for further details. Griffiths and Harris (1978), Chapter 0 is a concise survey of the present topics. For applications to physics, see Horowitz (1986) and Candelas (1988).

8.1 Complex manifolds

To begin with, we define a holomorphic (or analytic) map on $\mathbb{C}^m$. A complex-valued function $f : \mathbb{C}^m \to \mathbb{C}$ is **holomorphic** if $f = f_1 + \mathrm{i}f_2$ satisfies the **Cauchy–Riemann relations** for each $z^\mu = x^\mu + \mathrm{i}y^\mu$,

$$\frac{\partial f_1}{\partial x^\mu} = \frac{\partial f_2}{\partial y^\mu} \qquad \frac{\partial f_2}{\partial x^\mu} = -\frac{\partial f_1}{\partial y^\mu}. \tag{8.1}$$

A map $(f^1, \ldots, f^n) : \mathbb{C}^m \to \mathbb{C}^n$ is called holomorphic if each function f^λ $(1 \leqslant \lambda \leqslant n)$ is holomorphic.

8.1.1 Definitions

Definition 8.1 M is a **complex manifold** if

(i) M is a topological space.
(ii) M is provided with a family of pairs $\{(U_i, \varphi_i)\}$.
(iii) $\{U_i\}$ is a family of open sets which covers M. φ_i is a homeomorphism from U_i to an open subset U_i' of $\mathbb{C}^m$. [Hence M is even dimensional.]
(iv) Given U_i and U_j such that $U_i \cap U_j \neq \varnothing$, the map $\psi_{ji} = \varphi_j\varphi_i^{-1}$ from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$ is holomorphic.

The number m above is called the complex dimension of M and is denoted as $\dim_{\mathbb{C}} M = m$. The real dimension $2m$ is denoted either by $\dim_{\mathbb{R}} M$ or simply by $\dim M$. Let $z^{\mu} = \varphi_i(p)$ and $w^{\nu} = \varphi_j(p)$ be the (complex) coordinates of a point $p \in U_i \cap U_j$ in the charts (U_i, φ_i) and (U_j, φ_j). Axiom (iv) asserts that the function $w^{\nu} = u^{\nu} + \mathrm{i}v^{\nu}$ $(1 \leqslant \nu \leqslant m)$ is holomorphic in $z^{\mu} = x^{\mu} + \mathrm{i}y^{\mu}$, namely

$$\frac{\partial u^{\nu}}{\partial x^{\mu}} = \frac{\partial v^{\nu}}{\partial y^{\mu}}, \frac{\partial u^{\nu}}{\partial y^{\mu}} = -\frac{\partial v^{\nu}}{\partial x^{\mu}} \qquad 1 \leqslant \mu, \nu \leqslant m.$$

These axioms ensure that calculus on complex manifolds can be carried out independently of the special coordinates chosen. For example, $\mathbb{C}^m$ is the simplest complex manifold. A single chart covers the whole space and φ is the identity map.

Let $\{(U_i, \varphi_i)\}$ and $\{(V_j, \psi_j)\}$ be atlases of M. If the union of two atlases is again an atlas which satisfies the axioms of definition 8.1, they are said to define the same **complex structure**. A complex manifold may carry a number of complex structures (see example 8.3).

8.1.2 Examples

Example 8.2 In exercise 5.5, it was shown that the stereographic coordinates of a point $P(x, y, x) \in S^2 - \{\text{North Pole}\}$ projected from the North Pole are

$$(X, Y) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right)$$

while those of a point $P(x, y, z) \in S^2 - \{\text{South Pole})$ projected from the South Pole are

$$(U, V) = \left(\frac{x}{1 + z}, \frac{-y}{1 + z}\right).$$

[Note the orientation of (U, V) in figure 5.5.] Let us define complex coordinates

$$Z = X + \mathrm{i}Y, \bar{Z} = X - \mathrm{i}Y, W = U + \mathrm{i}V, \bar{W} = U - \mathrm{i}V.$$

W is a holomorphic function of Z,

$$W = \frac{x - \mathrm{i}y}{1 + z} = \frac{1 - z}{1 + z}(X - \mathrm{i}Y) = \frac{X - \mathrm{i}Y}{X^2 + Y^2} = \frac{1}{Z}.$$

Thus S^2 is a complex manifold which is identified with the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Example 8.3 Take a complex plane $\mathbb{C}$ and define a lattice $\mathrm{L}(\omega_1, \omega_2) \equiv \{\omega_1 m + \omega_2 n | m, n \in \mathbb{Z}\}$ where ω_1 and ω_2 are two non-vanishing complex numbers such that $\omega_2/\omega_1 \notin \mathbb{R}$; see figure 8.1. Without loss of generality, we may take $\mathrm{Im}(\omega_2/\omega_1) > 0$. $\mathbb{C}/\mathrm{L}(\omega_1, \omega_2)$ is obtained by identifying the points z_1, $z_2 \in \mathbb{C}$ such that $z_1 - z_2 = \omega_1 m + \omega_2 n$ for some m, $n \in \mathbb{Z}$. Since the opposite sides of the shaded area of figure 8.1 are identified, $\mathbb{C}/\mathrm{L}(\omega_1, \omega_2)$ is homeomorphic to the torus T^2. The complex structure of $\mathbb{C}$ naturally induces that of $\mathbb{C}/\mathrm{L}(\omega_1, \omega_2)$. We say that the pair (ω_1, ω_2) defines a complex structure on T^2. There are many pairs (ω_1, ω_2) which give the same complex structure on T^2.

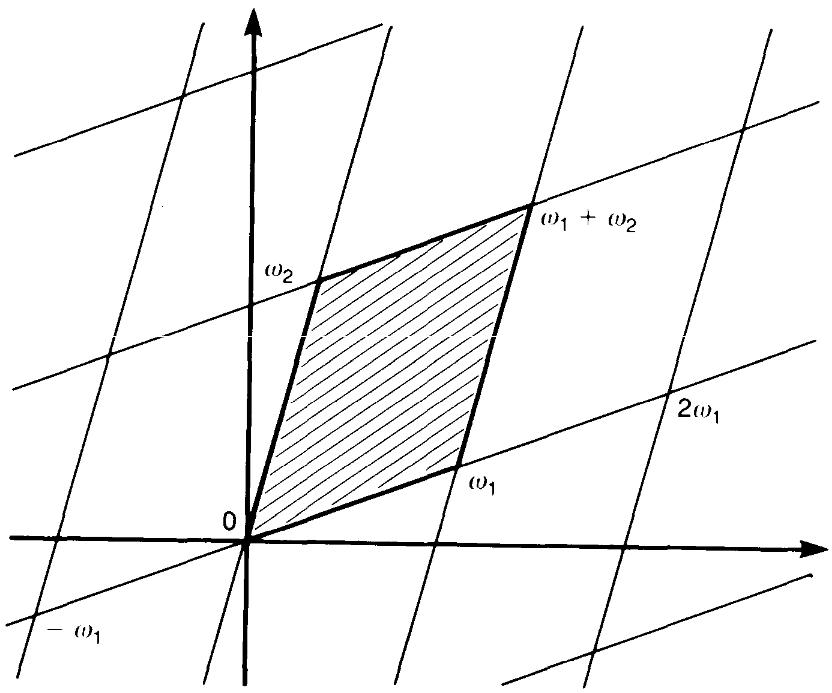


Figure 8.1 Two complex numbers ω_1 and ω_2 define a lattice $\mathrm{L}(\omega_1, \omega_2)$ in the complex plane. $\mathbb{C}/\mathrm{L}(\omega_1, \omega_2)$ is homeomorphic to the torus (the shaded area).

When do pairs (ω_1, ω_2) and (ω_1', ω_2') $(\mathrm{Im}(\omega_2/\omega_1), \mathrm{Im}(\omega_2'/\omega_1') > 0)$ define the same complex structure? We first note that two lattices $\mathrm{L}(\omega_1, \omega_2)$ and $\mathrm{L}(\omega_1', \omega_2')$ coincide if and only if there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) \equiv \mathrm{SL}(2, \mathbb{Z})/\mathbb{Z}_2$$

such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{8.2}$$

[*Remark:* $\mathrm{SL}(2, \mathbb{Z})$ has been defined by (2.3) and two matrices A, $-A \in \mathrm{SL}(2, \mathbb{Z})$ are identified in $\mathrm{PSL}(2, \mathbb{Z})$.]

[*Proof:* Suppose

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Since $\omega_1', \omega_2' \in L(\omega_1, \omega_2)$ we find $L(\omega_1', \omega_2') \subset L(\omega_1, \omega_2)$. From

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$$

we also have $L(\omega_1, \omega_2) \subset L(\omega_1', \omega_2')$. Thus $L(\omega_1, \omega_2) = L(\omega_1', \omega_2')$. Conversely if $L(\omega_1, \omega_2) = L(\omega_1', \omega_2')$, ω_1' and ω_2' are lattice points of $L(\omega_1, \omega_2)$ and can be written as $\omega_1' = d\omega_1 + c\omega_2$ and $\omega_2' = b\omega_1 + a\omega_2$ where $a, b, c, d \in \mathbb{Z}$. Also ω_1 and ω_2 may be expressed as $\omega_1 = d'\omega_1' + c'\omega_2'$ and $\omega_2 = b'\omega_1' + a'\omega_2'$ where $a', b', c', d' \in \mathbb{Z}$. Then we have

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

from which we find

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Equating the determinants of both sides we have $(a'd' - b'c')(ad - bc) = 1$. All the entries being integers, this is possible only when $ad - bc = \pm 1$. Since

$$\mathrm{Im}\left(\frac{\omega_2'}{\omega_1'}\right) = \mathrm{Im}\left(\frac{b\omega_1 + a\omega_2}{d\omega_1 + c\omega_2}\right) = \frac{ad - bc}{|c(\omega_2/\omega_1) + d|^2} \mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$$

we must have $ad - bc > 0$, that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

In fact, it is clear that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

defines the same lattice as

$$-\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and we have to identify those matrices of $SL(2, \mathbb{Z})$ which differ only by their overall signature. Thus two lattices agree if they are related by $PSL(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z})/\mathbb{Z}_2$.]

Assume that there exists a one-to-one holomorphic map h of $\mathbb{C}/L(\omega_1, \omega_2)$ onto $\mathbb{C}/L(\tilde{\omega}_1, \tilde{\omega}_2)$ where $\mathrm{Im}(\omega_2/\omega_1) > 0$, $\mathrm{Im}(\tilde{\omega}_2/\tilde{\omega}_1) > 0$. Let $p : \mathbb{C} \to \mathbb{C}/L(\omega_1, \omega_2)$ and $\tilde{p} : \mathbb{C} \to \mathbb{C}/L(\tilde{\omega}_1, \tilde{\omega}_2)$ be the natural projections. For example, p maps a point in $\mathbb{C}$ to an equivalent point in $\mathbb{C}/L(\omega_1, \omega_2)$. Choose the origin 0 and define $h_*(0)$ to be a point such that $\tilde{p}h_*(0) = hp(0)$ (figure 8.2),

$$\begin{tikzcd} \mathbb{C} \arrow[r, "h_*"] \arrow[d, "p"'] & \mathbb{C} \arrow[d, "\widetilde{p}"] \\ \mathbb{C}/\mathrm{L}(\omega_1, \omega_2) \arrow[r, "h"] & \mathbb{C}/\mathrm{L}(\widetilde{\omega}_1, \widetilde{\omega}_2) \end{tikzcd} \tag{8.3}$$

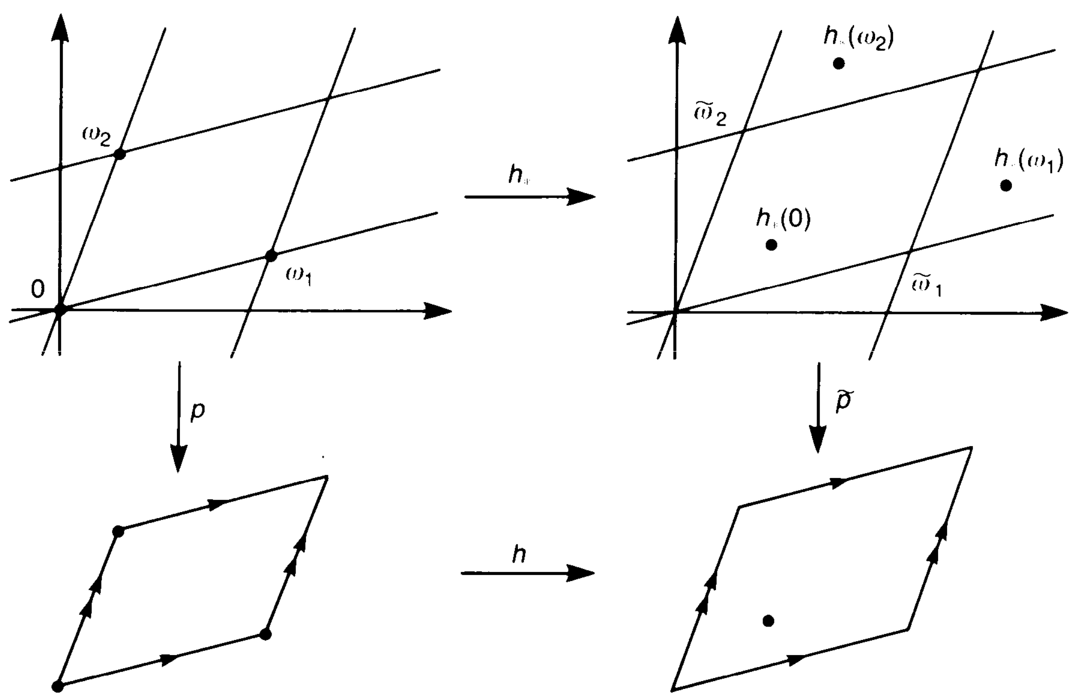


Figure 8.2 A holomorphic bijection $h : \mathbb{C}/\mathrm{L}(\omega_1, \omega_2) \to \mathbb{C}/\mathrm{L}(\widetilde{\omega}_1, \widetilde{\omega}_2)$ and the natural projections $p : \mathbb{C} \to \mathbb{C}/\mathrm{L}(\omega_1, \omega_2)$, $\widetilde{p} : \mathbb{C} \to \mathbb{C}/\mathrm{L}(\widetilde{\omega}_1, \widetilde{\omega}_2)$ define a holomorphic bijection $h_* : \mathbb{C} \to \mathbb{C}$.

Then by analytic continuation from the origin, we obtain a one-to-one holomorphic map h_* of $\mathbb{C}$ onto itself satisfying

$$\widetilde{p}\, h_*(z) = hp(z) \qquad \text{for all } z \in \mathbb{C} \tag{8.4}$$

so that the diagram (8.3) commutes. It is known that a one-to-one holomorphic map of $\mathbb{C}$ onto itself must be of the form $z \to h_*(z) = az + b$, where a, $b \in \mathbb{C}$ and $a \neq 0$. We then have $h_*(\omega_1) - h_*(0) = a\omega_1$ and $h_*(\omega_2) - h_*(0) = a\omega_2$. For h to be well defined as a map of $\mathbb{C}/\mathrm{L}(\omega_1, \omega_2)$ onto $\mathbb{C}/\mathrm{L}(\widetilde{\omega}_1, \widetilde{\omega}_2)$, we must have $a\omega_1$, $a\omega_2 \in \mathrm{L}(\widetilde{\omega}_1, \widetilde{\omega}_2)$, see figure 8.2. By changing the roles of (ω_1, ω_2) and $(\widetilde{\omega}_1, \widetilde{\omega}_2)$, we have $\widetilde{a}\,\widetilde{\omega}_1$, $\widetilde{a}\,\widetilde{\omega}_2 \in \mathrm{L}(\omega_1, \omega_2)$ where $\widetilde{a} \neq 0$ is a complex number. Hence we conclude that if $\mathbb{C}/\mathrm{L}(\omega_1, \omega_2)$ and $\mathbb{C}/\mathrm{L}(\widetilde{\omega}_1, \widetilde{\omega}_2)$ have the same complex structure, there must be a matrix $M \in \mathrm{SL}(2, \mathbb{Z})$ and a complex number λ $(= \widetilde{a}^{-1})$ such that

$$\begin{pmatrix} \widetilde{\omega}_1 \\ \widetilde{\omega}_2 \end{pmatrix} = \lambda M \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{8.5}$$

Conversely we verify that (ω_1, ω_2) and $(\widetilde{\omega}_1, \widetilde{\omega}_2)$ related by (8.5) define the same complex structure. In fact

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ and } M \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

define the same lattice (modulo translation) and we may take $h_* : \mathbb{C} \to \mathbb{C}$ to be $z \mapsto z + b$. $L(\omega_1, \omega_2)$ and $L(\lambda\omega_1, \lambda\omega_2)$ also define the same complex structure. We take in this case $h_* : z \mapsto \lambda z + b$.

We have shown that the complex structure on T^2 is defined by a pair of complex numbers (ω_1, ω_2) modulo a constant factor and $PSL(2, \mathbb{Z})$. To get rid of the constant factor, we introduce the **modular parameter** $\tau \equiv \omega_2/\omega_1 \in H \equiv \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$, to specify the complex structure of T^2. Without loss of generality, we take 1 and τ to be the generators of a lattice. Note, however, that not all of $\tau \in H$ are independent modular parameters. As was shown above, τ and $\tau' = (a\tau + b)/(c\tau + d)$ define the same complex structure if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}).$$

The quotient space $H/PSL(2, \mathbb{Z})$ is shown in figure 8.3, the derivation of which can be found in Koblitz (1984) p100, and Gunning (1962) p4.

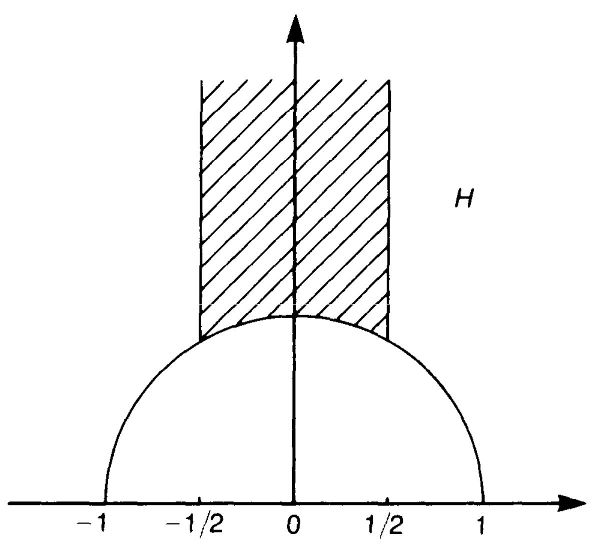


Figure 8.3 The quotient space $H/PSL(2, \mathbb{Z})$.

The change $\tau \to \tau'$ is called the **modular transformation** and is generated by $\tau \to \tau + 1$ and $\tau \to -1/\tau$. The transformation $\tau \to \tau + 1$

generates a **Dehn twist** along the meridian m as follows (figure 8.4(a)). (i) First cut a torus along m. (ii) Then take one of the lips of the cut and rotate it by 2π with the other lip kept fixed. (iii) Then glue the lips together again. The other transformation $\tau \to -1/\tau$ corresponds to changing the roles of the longitude l and the meridian m (figure 8.4(b)).

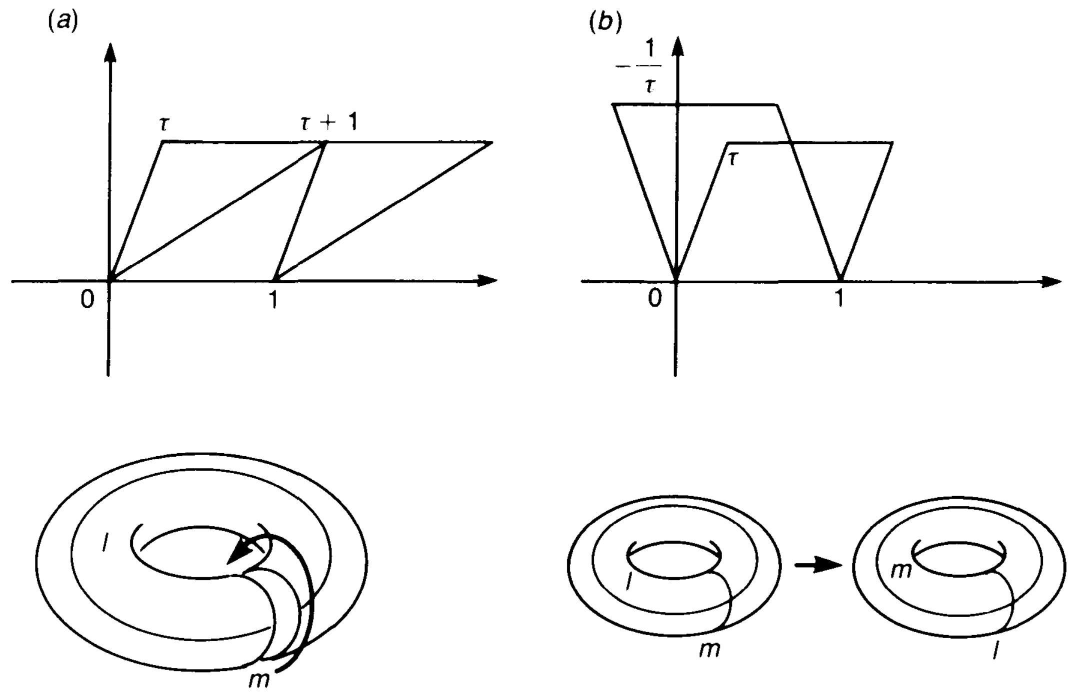


Figure 8.4 (a) Dehn twists generate modular transformations. (b) $\tau \to -1/\tau$ changes the roles of l and m.

Example 8.4 The **complex projective space** $\mathbb{C}P^n$ is defined similarly to $\mathbb{R}P^n$; see example 5.6. $z = (z^0, \ldots, z^n) \in \mathbb{C}^{n+1}$ determines a complex line through the origin if $z \neq 0$. Define an equivalence relation $\sim$ by $z \sim w$ if there exists a complex number $a \neq 0$ such that $w = az$. Then $\mathbb{C}P^n \equiv (\mathbb{C}^{n+1} - \{0\})/\sim$. $n + 1$ numbers $(z^0, \ldots, z^n)$ are called the **homogeneous coordinates**. A chart U_μ is a subset of $\mathbb{C}^{n+1} - \{0\}$ such that $z^\mu \neq 0$. In a chart U_μ, the **inhomogeneous coordinates** are defined by $\xi^\nu_{(\mu)} \equiv z^\nu/z^\mu$ $(\nu \neq \mu)$. In $U_\mu \cap U_\nu \neq \emptyset$, the coordinate transformation $\psi_{\mu\nu} : \mathbb{C}^n \to \mathbb{C}^n$ is

$$\xi^\lambda_{(\nu)} \mapsto \xi^\lambda_{(\mu)} = (z^\nu/z^\mu)\cdot\xi^\lambda_{(\nu)}. \tag{8.6}$$

Accordingly $\psi_{\mu\nu}$ is a multiplication by z^ν/z^μ, which is, of course, holomorphic.

Example 8.5 The **complex Grassmann manifolds** $G_{k,n}(\mathbb{C})$ are defined similarly to the real Grassmann manifolds; see example 5.7. $G_{k,n}(\mathbb{C})$ is the set of complex k-dimensional subspaces of $\mathbb{C}^n$. Note that $\mathbb{C}P^n = G_{1,n+1}(\mathbb{C})$.

Let $M_{k,n}(\mathbb{C})$ be the set of $k \times n$ matrices of rank k ($k \leq n$). Take A, $B \in M_{k,n}(\mathbb{C})$ and define an equivalence relation $\sim$ by $A \sim B$ if there exists $g \in \mathrm{GL}(k, \mathbb{C})$ such that $B = gA$. We identify $G_{k,n}(\mathbb{C})$ with $M_{k,n}(\mathbb{C})/\mathrm{GL}(k, \mathbb{C})$. Let $\{A_1, \ldots, A_l\}$ be the collection of all the $k \times k$ minors of $A \in M_{k,n}(\mathbb{C})$. We define the chart U_α to be a subset of $G_{k,n}(\mathbb{C})$ such that $\det A_\alpha \neq 0$. The $k(n-k)$ coordinates on U_α are given by the non-trivial entries of the matrix $A_\alpha^{-1}A$. See example 5.7 for details.

Example 8.6 The common zeros of a set of homogeneous polynomials are a compact submanifold of $\mathbb{C}P^n$ called an **algebraic variety**. For example, let $P(z^0, \ldots, z^n)$ be a homogeneous polynomial of degree d. If $a \neq 0$ is a complex number, P satisfies

$$P(az^0, \ldots, az^n) = a^d P(z^0, \ldots, z^n).$$

This shows that the zeros of P are defined on $\mathbb{C}P^n$; if $P(z^0, \ldots, z^n) = 0$ then $P([z^0, \ldots, z^n]) = 0$. For definiteness, consider

$$P(z^0, z^1, z^2) = (z^0)^2 + (z^1)^2 + (z^2)^2$$

and define N by

$$N = \{[z^0, z^1, z^2] \in \mathbb{C}P^2 | P(z^0, z^1, z^2) = 0\}. \tag{8.7}$$

We define U_μ as in example 8.4. In $N \cap U_0$, we have

$$[\xi^1_{(0)}]^2 + [\xi^2_{(0)}]^2 + 1 = 0$$

where $\xi^\mu_{(0)} = z^\mu/z^0$ (note that $z^0 \neq 0$). Consider a holomorphic change of coordinates $(\xi^1_{(0)}, \xi^2_{(0)}) \mapsto (\eta^1 = \xi^1_{(0)}, \eta^2 = [\xi^1_{(0)}]^2 + [\xi^2_{(0)}]^2 + 1)$. $\partial(\eta^1, \eta^2)/\partial(\xi^1_{(0)}, \xi^2_{(0)}) \neq 0$ unless $\xi^2_{(0)} = z^2 = 0$. Then $N \cap U_0 \cap U_2 = \{(\eta^1, \eta^2) \in \mathbb{C}^2 | \eta^2 = 0\}$ is clearly a one-dimensional submanifold of $\mathbb{C}^2$. If $\xi^2_{(0)} = z^2 = 0$, we have $(\xi^1_{(0)}, \xi^2_{(0)}) \mapsto (\zeta^1 = [\xi^1_{(0)}]^2 + [\xi^2_{(0)}]^2 + 1, \zeta^2 = \xi^2_{(0)})$ for which the Jacobian does not vanish unless $\xi^1_{(0)} = z^1 = 0$. Then $N \cap U_0 \cap U_1 = \{(\zeta^1, \zeta^2) \in \mathbb{C}^2 | \zeta^1 = 0\}$ is a one-dimensional submanifold of $\mathbb{C}^2$. On $N \cap U_0 \cap U_1 \cap U_2$, the coordinate change $\eta^1 \mapsto \zeta^2$ is a multiplication by z^2/z^1 and hence is holomorphic. In this way, we may define a one-dimensional compact submanifold N of $\mathbb{C}P^2$.

A complex manifold is a differentiable manifold. For example, $\mathbb{C}^m$ is regarded as $\mathbb{R}^{2m}$ by the identification $z^\mu = x^\mu + \mathrm{i}y^\mu$, $x^\mu, y^\mu \in \mathbb{R}$. Similarly, any chart U of a complex manifold has coordinates $(z^1, \ldots, z^m)$ which may be understood as real coordinates $(x^1, y^1, \ldots, x^m, y^m)$. The analytic property of the coordinate transformation functions ensures that they are differentiable when the manifold is regarded as a $2m$-dimensional differentiable manifold.

8.2 Calculus on complex manifolds

8.2.1 Holomorphic maps

Let $f: M \to N$, M and N being complex manifolds with $\dim_{\mathbb{C}} M = m$ and $\dim_{\mathbb{C}} N = n$. Take a point p in a chart (U, φ) of M. Let (V, ψ) be a chart of N such that $f(p) \in V$. If we write $\{z^\mu\} = \varphi(p)$ and $\{w^\nu\} = \psi(f(p))$, we have a map $\psi f \varphi^{-1}: \mathbb{C}^m \to \mathbb{C}^n$. If each function w^ν $(1 \leqslant \nu \leqslant n)$ is a holomorphic function of z^μ, f is called a **holomorphic map**. This definition is independent of the special coordinates chosen. In fact, let (U', φ') be another chart such that $U \cap U' \neq \emptyset$ and $z'^\lambda = x'^\lambda + \mathrm{i}y'^\lambda$ be the coordinates. Take a point $p \in U \cap U'$. If $w^\nu = u^\nu + \mathrm{i}v^\nu$ is a holomorphic function with respect to z, then

$$\frac{\partial u^\nu}{\partial x'^\lambda} = \frac{\partial u^\nu}{\partial x^\mu}\frac{\partial x^\mu}{\partial x'^\lambda} + \frac{\partial u^\nu}{\partial y^\mu}\frac{\partial y^\mu}{\partial x'^\lambda} = \frac{\partial v^\nu}{\partial y^\mu}\frac{\partial y^\mu}{\partial y'^\lambda} + \frac{\partial v^\nu}{\partial x^\mu}\frac{\partial x^\mu}{\partial y'^\lambda} = \frac{\partial v^\nu}{\partial y'^\lambda}.$$

We also find $\partial u^\nu/\partial y'^\lambda = -\partial v^\nu/\partial x'^\lambda$. Thus w^ν is holomorphic with respect to z' also. It can be shown that the holomorphic property is also independent of the choice of chart in N.

Let M and N be complex manifolds. M is **biholomorphic** to N if there exists a diffeomorphism $f: M \to N$ which is also holomorphic (then $f^{-1}: N \to M$ is automatically holomorphic). The map f is called a **biholomorphism**.

A **holomorphic function** is a holomorphic map $f: M \to \mathbb{C}$. There is a striking theorem; any holomorphic function on a *compact* complex manifold is *constant*. This is a generalisation of the maximum principle of elementary complex analysis, see Wells (1980). The set of holomorphic functions on M is denoted by $\mathcal{O}(M)$. $\mathcal{O}(U)$ is the set of holomorphic functions on $U \subset M$.

8.2.2 Complexifications

Let M be a differentiable manifold with $\dim_{\mathbb{R}} M = m$. If $f: M \to \mathbb{C}$ is decomposed as $f = g + \mathrm{i}h$ where $g, h \in \mathcal{F}(M)$, then f is a complex-valued smooth function. The set of complex-valued smooth functions on M is called the **complexification** of $\mathcal{F}(M)$, denoted by $\mathcal{F}(M)^{\mathbb{C}}$. A complexified function does not satisfy the Cauchy–Riemann relation in general. For $f = g + \mathrm{i}h \in \mathcal{F}(M)^{\mathbb{C}}$, the **complex conjugate** of f is $\bar{f} \equiv g - \mathrm{i}h$. f is real if and only if $f = \bar{f}$.

Before we consider the complexification of T_pM, we define the complexification $V^{\mathbb{C}}$ of a general vector space V with $\dim_{\mathbb{R}} V = m$. An element of $V^{\mathbb{C}}$ is given by $X + \mathrm{i}Y$ where $X, Y \in V$. The space $V^{\mathbb{C}}$ becomes a complex vector space of complex dimension m if the addition and the scalar multiplication by a complex number $a + \mathrm{i}b$ are defined by

$$(X_1 + iY_1) + (X_2 + iY_2) = (X_1 + X_2) + i(Y_1 + Y_2)$$
$$(a + ib)(X + iY) = (aX - bY) + i(bX + aY).$$

V is a vector subspace of $V^{\mathbb{C}}$ since $X \in V$ and $X + i0 \in V^{\mathbb{C}}$ may be identified. Vectors in V are said to be **real**. The complex conjugate of $Z = X + iY$ is $\bar{Z} \equiv X - iY$. Z is real if $Z = \bar{Z}$.

A linear operator A on V is *extended* to act on $V^{\mathbb{C}}$ as

$$A(X + iY) \equiv A(X) + iA(Y). \tag{8.8}$$

If $A : V \to \mathbb{R}$ is a linear function ($A \in V^*$), its extension is a complex-valued linear function on $V^{\mathbb{C}}$, $A : V^{\mathbb{C}} \to \mathbb{C}$. In general, any tensor defined on V and V^* is extended so that it is defined on $V^{\mathbb{C}}$ and $(V^*)^{\mathbb{C}}$. An extended tensor is complexified as $t = t_1 + it_2$, where t_1 and t_2 are tensors of the same type. The conjugate of t is $\bar{t} \equiv t_1 - it_2$. If $t = \bar{t}$, the tensor is said to be real. For example $A : V^{\mathbb{C}} \to \mathbb{C}$ is real if $\overline{A(X + iY)} = A(X - iY)$.

Let $\{e_k\}$ be a basis of V. If the basis vectors are regarded as complex vectors, the *same* basis $\{e_k\}$ becomes a basis of $V^{\mathbb{C}}$. To see this, let $X = X^k e_k$, $Y = Y^k e_k \in V$. Then $Z = X + iY$ is *uniquely* expressed as $(X^k + iY^k)e_k$. We find $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{\mathbb{C}}$.

Now we are ready to complexify the tangent space T_pM. If V above is replaced by T_pM, we have the complexification $T_pM^{\mathbb{C}}$ of T_pM, whose element is expressed as $Z = X + iY$, $X, Y \in T_pM$. Z acts on a function $f = f_1 + if_2 \in \mathcal{F}(M)^{\mathbb{C}}$ as

$$\begin{aligned} Z[f] &= X[f_1 + if_2] + iY[f_1 + if_2] \\ &= X[f_1] - Y[f_2] + i\{X[f_2] + Y[f_1]\}. \end{aligned} \tag{8.9}$$

The dual vector space T_p^*M is complexified if $\omega, \eta \in T_p^*M$ are combined as $\zeta = \omega + i\eta$. The set of complexified dual vectors is denoted by $(T_p^*M)^{\mathbb{C}}$. Any tensor t is extended so that it is defined on $T_pM^{\mathbb{C}}$ and $(T_p^*M)^{\mathbb{C}}$ and then complexified.

Exercise 8.7 Show that $(T_p^*M)^{\mathbb{C}} = (T_pM^{\mathbb{C}})^*$. From now on, we denote the complexified dual vector space simply by $T_p^*M^{\mathbb{C}}$.

Given smooth vector fields $X, Y \in \mathcal{X}(M)$, we define a **complex vector field** $Z = X + iY$. Clearly $Z|_p \in T_pM^{\mathbb{C}}$. The set of complex vector fields is the complexification of $\mathcal{X}(M)$ and is denoted by $\mathcal{X}(M)^{\mathbb{C}}$. The **conjugate vector field** of $Z = X + iY$ is $\bar{Z} = X - iY$. $Z = \bar{Z}$ if $Z \in \mathcal{X}(M)$, hence $\mathcal{X}(M)^{\mathbb{C}} \supset \mathcal{X}(M)$. The Lie bracket of $Z = X + iY$, $W = U + iV \in \mathcal{X}(M)^{\mathbb{C}}$ is

$$[X + iY, U + iV] = \{[X, U] - [Y, V]\} + i\{[X, V] + [Y, U]\}. \tag{8.10}$$

The complexification of a tensor field of type (p, q) is defined in an obvious manner. If $\omega, \eta \in \Omega^1(M)$, $\xi \equiv \omega + \mathrm{i}\eta \in \Omega^1(M)^{\mathbb{C}}$ is a complexified one-form.

8.2.3 *Almost complex structure*

Since a complex manifold is also a differentiable manifold, we may use the framework developed in Chapter 5. We then put appropriate *constraints* on the results. Let us look at the tangent space of a complex manifold M with $\dim_{\mathbb{C}} M = m$. The tangent space T_pM is spanned by $2m$ vectors

$$\{\partial/\partial x^1, \ldots, \partial/\partial x^m; \partial/\partial y^1, \ldots, \partial/\partial y^m\} \tag{8.11}$$

where $z^\mu = x^\mu + \mathrm{i}y^\mu$ are the coordinates of p in a chart (U, φ). With the same coordinates, T_p^*M is spanned by

$$\{\mathrm{d}x^1, \ldots, \mathrm{d}x^m; \mathrm{d}y^1, \ldots, \mathrm{d}y^m\}. \tag{8.12}$$

Let us define $2m$ vectors,

$$\partial/\partial z^\mu \equiv \tfrac{1}{2}\{\partial/\partial x^\mu - \mathrm{i}\partial/\partial y^\mu\} \tag{8.13a}$$

$$\partial/\partial \bar{z}^\mu \equiv \tfrac{1}{2}\{\partial/\partial x^\mu + \mathrm{i}\partial/\partial y^\mu\} \tag{8.13b}$$

where $1 \leqslant \mu \leqslant m$. Clearly they form a basis of the $2m$-dimensional (complex) vector space $T_pM^{\mathbb{C}}$. Note that $\overline{\partial/\partial z^\mu} = \partial/\partial \bar{z}^\mu$. Correspondingly $2m$ one-forms

$$\mathrm{d}z^\mu \equiv \mathrm{d}x^\mu + \mathrm{i}\mathrm{d}y^\mu \qquad \mathrm{d}\bar{z}^\mu \equiv \mathrm{d}x^\mu - \mathrm{i}\mathrm{d}y^\mu \tag{8.14}$$

form the basis of $T_p^*M^{\mathbb{C}}$. They are dual to (8.13),

$$\langle \mathrm{d}z^\mu, \partial/\partial \bar{z}^\nu\rangle = \langle \mathrm{d}\bar{z}^\mu, \partial/\partial z^\nu\rangle = 0 \tag{8.15a}$$

$$\langle \mathrm{d}z^\mu, \partial/\partial z^\nu\rangle = \langle \mathrm{d}\bar{z}^\mu, \partial/\partial \bar{z}^\nu\rangle = \delta^\mu{}_\nu. \tag{8.15b}$$

Let M be a complex manifold and define a linear map $J_p : T_pM \to T_pM$ by

$$J_p \cdot \partial/\partial x^\mu = \partial/\partial y^\mu \qquad J_p \cdot \partial/\partial y^\mu = -\partial/\partial x^\mu. \tag{8.16}$$

J_p is a *real* tensor of type $(1, 1)$. Note that

$$J_p^2 = -1_p \tag{8.17}$$

where 1_p is the identity map on T_pM. Roughly speaking J_p corresponds to the multiplication by $\pm\mathrm{i}$. The action of J_p is independent of the chart. In fact, let (U, φ) and (V, ψ) be overlapping charts with $\varphi(p) = z^\mu = x^\mu + \mathrm{i}y^\mu$ and $\psi(p) = w^\mu = u^\mu + \mathrm{i}v^\mu$. On $U \cap V$, the functions $z^\mu = z^\mu(w)$ satisfy the Cauchy–Riemann relations. Then we find

$$J_p \frac{\partial}{\partial u^\mu} = J_p\left(\frac{\partial x^\nu}{\partial u^\mu}\frac{\partial}{\partial x^\nu} + \frac{\partial y^\nu}{\partial u^\mu}\frac{\partial}{\partial y^\nu}\right) = \frac{\partial y^\nu}{\partial v^\mu}\frac{\partial}{\partial y^\nu} + \frac{\partial x^\nu}{\partial v^\mu}\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial v^\mu}.$$

We also find that $J_p\partial/\partial v^\mu = -\partial/\partial u^\mu$. Accordingly J_p takes the form

$$J_p = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{8.18}$$

with respect to the basis (8.11). Since all the components of J_p are constant at any point, we may define a smooth tensor field J whose components at p are (8.18). The tensor field J is called the **almost complex structure** of a complex manifold M. Note that any $2m$-dimensional manifold *locally* admits a tensor J which squares to $-\mathbb{1}$. However, J may be patched across charts and defined *globally* only on a complex manifold. The tensor J completely specifies the complex structure.

J_p is extended so that it may be defined on $T_pM^{\mathbb{C}}$,

$$J_p(X + \mathrm{i}Y) \equiv J_pX + \mathrm{i}J_pY. \tag{8.19}$$

It follows from (8.16) that

$$J_p\partial/\partial z^\mu = \mathrm{i}\partial/\partial z^\mu \qquad J_p\partial/\partial \bar{z}^\mu = -\mathrm{i}\partial/\partial \bar{z}^\mu. \tag{8.20}$$

Thus we have an expression for J_p in (anti-) holomorphic bases,

$$J_p = \mathrm{i}\,\mathrm{d}z^\mu \otimes \partial/\partial z^\mu - \mathrm{i}\,\mathrm{d}\bar{z}^\mu \otimes \partial/\partial \bar{z}^\mu \tag{8.21}$$

whose components are given by

$$J_p = \begin{pmatrix} \mathrm{i}\mathbb{1} & 0 \\ 0 & -\mathrm{i}\mathbb{1} \end{pmatrix}. \tag{8.22}$$

Let $Z \in T_pM^{\mathbb{C}}$ be a vector of the form $Z = Z^\mu\partial/\partial z^\mu$. Then Z is an eigenvector of J_p; $J_pZ = \mathrm{i}Z$. Similarly if $Z = Z^\mu\partial/\partial\bar{z}^\mu$, it satisfies $J_pZ = -\mathrm{i}Z$. In this way $T_pM^{\mathbb{C}}$ of a complex manifold is separated into two *disjoint* vector spaces,

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^- \tag{8.23}$$

where

$$T_pM^\pm = \{Z \in T_pM^{\mathbb{C}} | J_pZ = \pm\mathrm{i}Z\}. \tag{8.24}$$

We define the projection operators $\mathcal{P}^\pm : T_pM^{\mathbb{C}} \to T_pM^\pm$ by

$$\mathcal{P}^\pm \equiv \tfrac{1}{2}(\mathbb{1} \mp \mathrm{i}J_p). \tag{8.25}$$

We have $J_p\mathcal{P}^\pm Z = \frac{1}{2}(J_p \mp \mathrm{i}J_p^2)Z = \pm\mathrm{i}\mathcal{P}^\pm Z$ for any $Z \in T_pM^{\mathbb{C}}$. Hence

$$Z^\pm \equiv \mathcal{P}^\pm Z \in T_pM^\pm. \tag{8.26}$$

Now $Z \in T_pM^{\mathbb{C}}$ is uniquely decomposed as $Z = Z^+ + Z^-$. T_pM^+ is spanned by $\{\partial/\partial z^\mu\}$ and T_pM^- by $\{\partial/\partial \bar{z}^\mu\}$. $Z \in T_pM^+$ is called a **holomorphic vector** while $Z \in T_pM^-$ is called an **antiholomorphic vector**. We readily verify that

$$T_pM^- = \overline{T_pM^+} = \{\bar{Z} | Z \in T_pM^+\}. \tag{8.27}$$

Note that

$$\dim_{\mathbb{C}} T_pM^+ = \dim_{\mathbb{C}} T_pM^- = \tfrac{1}{2}\dim_{\mathbb{C}} T_pM^{\mathbb{C}} = \dim_{\mathbb{C}} M.$$

Exercise 8.8 Let (U, φ) and (V, ψ) be overlapping charts on a complex manifold M and let $z^\mu = \varphi(p)$ and $w^\mu = \psi(p)$. Verify that $X = X^\mu \partial/\partial z^\mu$ expressed in the coordinates w^μ contains a holomorphic basis $\{\partial/\partial w^\mu\}$ only. Thus the separation of $T_pM^{\mathbb{C}}$ into $T_pM^\pm$ is independent of charts (note that J is defined independently of charts).

Given a complexified vector field $Z \in \mathcal{X}(M)^{\mathbb{C}}$, we obtain a new vector field $JZ \in \mathcal{X}(M)^{\mathbb{C}}$ defined at each point of M by $JZ|_p = J_p \cdot Z|_p$. The vector field Z is naturally separated as

$$Z = Z^+ + Z^- \tag{8.28}$$

where $Z^\pm = \mathcal{P}^\pm Z$. Z^+ (Z^-) is called a **holomorphic (antiholomorphic) vector field**. Accordingly once J is given, $\mathcal{X}(M)^{\mathbb{C}}$ is decomposed uniquely as

$$\mathcal{X}(M)^{\mathbb{C}} = \mathcal{X}(M)^+ \oplus \mathcal{X}(M)^-. \tag{8.29}$$

$Z = Z^+ + Z^- \in \mathcal{X}(M)^{\mathbb{C}}$ is real if and only if $Z^+ = \overline{Z^-}$.

Exercise 8.9 Let $X, Y \in \mathcal{X}(M)^+$. Show that $[X, Y] \in \mathcal{X}(M)^+$. [If $X, Y \in \mathcal{X}(M)^-$, then $[X, Y] \in \mathcal{X}(M)^-$.]

8.3 Complex differential forms

On a complex manifold, we define complex differential forms by which we will discuss such topological properties as cohomology groups.

8.3.1 Complexification of real differential forms

Let M be a *differentiable* manifold with $\dim_{\mathbb{R}} M = m$. Take two q-forms $\omega, \eta \in \Omega_p^q(M)$ at p and define a **complex q-form** $\zeta = \omega + i\eta$. We denote the vector space of complex q-forms at p by $\Omega_p^q(M)^{\mathbb{C}}$. Clearly $\Omega_p^q(M) \subset \Omega_p^q(M)^{\mathbb{C}}$. The conjugate of ζ is $\bar{\zeta} = \omega - i\eta$. A complex q-form ζ is real if $\zeta = \bar{\zeta}$.

Exercise 8.10 Let $\omega \in \Omega_p^q(M)^{\mathbb{C}}$. Show that

$$\bar{\omega}(V_1, \ldots, V_q) = \overline{\omega(\bar{V}_1, \ldots, \bar{V}_q)} \qquad V_i \in T_pM^{\mathbb{C}}. \tag{8.30}$$

Show also that $\overline{\omega + \eta} = \bar{\omega} + \bar{\eta}$, $\overline{\lambda\omega} = \bar{\lambda}\bar{\omega}$ and $\bar{\bar{\omega}} = \omega$, where ω, $\eta \in \Omega_p^q(M)^{\mathbb{C}}$ and $\lambda \in \mathbb{C}$.

A complex q-form ω defined on a differentiable manifold M is a smooth assignment of an element of $\Omega_p^q(M)^{\mathbb{C}}$. The set of complex q-forms is denoted by $\Omega^q(M)^{\mathbb{C}}$. A complex q-form ζ is uniquely decomposed as $\zeta = \omega + \mathrm{i}\eta$, where ω, $\eta \in \Omega^q(M)$.

The exterior product of $\zeta = \omega + \mathrm{i}\eta$ and $\xi = \varphi + \mathrm{i}\psi$ is defined by

$$\begin{aligned}\zeta \wedge \xi &= (\omega + \mathrm{i}\eta) \wedge (\varphi + \mathrm{i}\psi) \\ &= (\omega \wedge \varphi - \eta \wedge \psi) + \mathrm{i}(\omega \wedge \psi + \eta \wedge \varphi). \end{aligned} \tag{8.31}$$

The exterior derivative d acts on $\zeta = \omega + \mathrm{i}\eta$ as

$$\mathrm{d}\zeta = \mathrm{d}\omega + \mathrm{i}\,\mathrm{d}\eta. \tag{8.32}$$

d is a real operator: $\overline{\mathrm{d}\zeta} = \mathrm{d}\omega - \mathrm{i}\,\mathrm{d}\eta = \mathrm{d}\bar{\zeta}$.

Exercise 8.11 Let $\omega \in \Omega^q(M)^{\mathbb{C}}$ and $\xi \in \Omega^r(M)^{\mathbb{C}}$. Show that

$$\omega \wedge \xi = (-1)^{qr}\xi \wedge \omega \tag{8.33}$$

$$\mathrm{d}(\omega \wedge \xi) = \mathrm{d}\omega \wedge \xi + (-1)^q\omega \wedge \mathrm{d}\xi. \tag{8.34}$$

8.3.2 Differential forms on complex manifolds

Now we restrict ourselves to complex manifolds in which we have the decompositions $T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-$ and $\mathscr{X}(M)^{\mathbb{C}} = \mathscr{X}(M)^+ \oplus \mathscr{X}(M)^-$.

Definition 8.12 Let M be a complex manifold with $\dim_{\mathbb{C}} M = m$. Let $\omega \in \Omega_p^q(M)^{\mathbb{C}}$ $(q \leqslant 2m)$ and r, s be positive integers such that $r + s = q$. Let $V_i \in T_pM^{\mathbb{C}}$ $(1 \leqslant i \leqslant q)$ be vectors in either T_pM^+ or T_pM^-. If $\omega(V_1, \ldots, V_q) = 0$ unless r of the V_i are in T_pM^+ and s of the V_i are in T_pM^-, ω is said to be of **bidegree** (r, s) or simply an (r, s)-form. The set of (r, s)-forms at p is denoted by $\Omega_p^{r,s}(M)$. If an (r, s)-form is assigned smoothly at each point of M, we have an (r, s)-form defined over M. The set of (r, s)-forms over M is denoted by $\Omega^{r,s}(M)$.

Take a chart (U, φ) with the complex coordinates $\varphi(p) = z^\mu$. We take the bases (8.13) for the tangent spaces $T_pM^\pm$. The dual bases are given by (8.14). Note that $\mathrm{d}z^\mu$ is of bidegree $(1, 0)$ since $\langle \mathrm{d}z^\mu, \partial/\partial\bar{z}^\nu \rangle = 0$ and $\mathrm{d}\bar{z}^\mu$ is of bidegree $(0, 1)$. With these bases, a form ω of bidegree (r, s) is written as

$$\omega = \frac{1}{r!s!}\,\omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s}\,\mathrm{d}z^{\mu_1} \wedge \ldots \wedge \mathrm{d}z^{\mu_r} \wedge \mathrm{d}\bar{z}^{\nu_1} \wedge \ldots \wedge \mathrm{d}\bar{z}^{\nu_s}. \tag{8.35}$$

The set $\{\mathrm{d}z^{\mu_1} \wedge \ldots \wedge \mathrm{d}z^{\mu_r} \wedge \mathrm{d}\bar{z}^{\nu_1} \wedge \ldots \wedge \mathrm{d}\bar{z}^{\mu_s}\}$ is the basis of $\Omega_p^{r,s}(M)$.

The components are totally antisymmetric in the μ and ν separately. Let z^μ and w^μ be two overlapping coordinates. The reader should verify that an (r, s)-form in the z^μ coordinate system is also an (r, s)-form in the w^μ system.

Proposition 8.13 Let M be a complex manifold of $\dim_{\mathbb{C}} M = m$ and ω and ξ be complex differential forms on M.

(*a*) If $\omega \in \Omega^{q,r}(M)$, then $\bar{\omega} \in \Omega^{r,q}(M)$.
(*b*) If $\omega \in \Omega^{q,r}(M)$ and $\xi \in \Omega^{q',r'}(M)$, then $\omega \wedge \xi \in \Omega^{q+q',r+r'}(M)$.
(*c*) A complex q-form ω is *uniquely* written as

$$\omega = \sum_{r+s=q} \omega^{(r,s)} \tag{8.36a}$$

where $\omega^{(r,s)} \in \Omega^{r,s}(M)$. Thus we have the decomposition

$$\Omega^q(M)^{\mathbb{C}} = \bigoplus_{r+s=q} \Omega^{r,s}(M). \tag{8.36b}$$

The proof is easy and is left to the reader. Now any q-form ω is decomposed as

$$\begin{aligned}\omega &= \sum_{r+s=q} \omega^{(r,s)} \\ &= \sum_{r+s=q} \frac{1}{r!s!}\, \omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s}\, \mathrm{d}z^{\mu_1} \wedge \ldots \wedge \mathrm{d}z^{\mu_r} \wedge \mathrm{d}\bar{z}^{\nu_1} \wedge \ldots \wedge \mathrm{d}\bar{z}^{\nu_s}\end{aligned} \tag{8.37}$$

where

$$\omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s} = \omega(\partial/\partial z^{\mu_1}, \ldots, \partial/\partial z^{\mu_r}, \partial/\partial \bar{z}^{\nu_1}, \ldots, \partial/\partial \bar{z}^{\nu_s}). \tag{8.38}$$

Exercise 8.14 Verify that

$$\dim_{\mathbb{R}} \Omega_p^{r,s}(M) = \begin{cases} \begin{pmatrix} m \\ r \end{pmatrix}\begin{pmatrix} m \\ s \end{pmatrix} & \text{if } 0 \leqslant r, s \leqslant m \\ 0 & \text{otherwise.} \end{cases}$$

Show also that $\dim_{\mathbb{R}} \Omega_p^q(M)^{\mathbb{C}} = \Sigma_{r+s=q} \dim_{\mathbb{R}} \Omega_p^{r,s}(M) = \binom{2m}{q}$.

8.3.3 *Dolbeault operators*

Let us compute the exterior derivative of an (r, s)-form ω. From (8.35), we find

$$\begin{aligned}\mathrm{d}\omega = \frac{1}{r!s!}&\left(\frac{\partial}{\partial z^\lambda}\, \omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s}\, \mathrm{d}z^\lambda + \frac{\partial}{\partial \bar{z}^\lambda}\, \omega_{\mu_1\ldots\mu_r\bar{\nu}_1\ldots\bar{\nu}_s} \mathrm{d}\bar{z}^\lambda\right) \\ &\times \mathrm{d}z^{\mu_1} \wedge \ldots \wedge \mathrm{d}z^{\mu_r} \wedge \mathrm{d}\bar{z}^{\nu_1} \wedge \ldots \wedge \mathrm{d}\bar{z}^{\nu_s}.\end{aligned} \tag{8.39}$$

$\mathrm{d}\omega$ is a mixture of an $(r+1, s)$-form and an $(r, s+1)$-form. We separate the action of d according to its destinations,

$$\mathrm{d} = \partial + \bar{\partial} \tag{8.40}$$

where $\partial : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M)$ and $\bar{\partial} : \Omega^{r,s}(M) \to \Omega^{r,s+1}(M)$. For example, if $\omega = \omega_{\mu\bar{\nu}} \mathrm{d}z^{\mu} \wedge \mathrm{d}\bar{z}^{\nu}$, $\partial\omega = \partial\omega_{\mu\bar{\nu}}/\partial z^{\lambda} \mathrm{d}z^{\lambda} \wedge \mathrm{d}z^{\mu} \wedge \mathrm{d}\bar{z}^{\nu}$ and $\bar{\partial}\omega = \partial\omega_{\mu\bar{\nu}}/\partial\bar{z}^{\lambda} \mathrm{d}\bar{z}^{\lambda} \wedge \mathrm{d}z^{\mu} \wedge \mathrm{d}\bar{z}^{\nu} = -\partial\omega_{\mu\bar{\nu}}/\partial\bar{z}^{\lambda} \mathrm{d}z^{\mu} \wedge \mathrm{d}\bar{z}^{\lambda} \wedge \mathrm{d}\bar{z}^{\nu}$. The operators ∂ and $\bar{\partial}$ are called the **Dolbeault operators**.

If ω is a general q-form given by (8.37), the actions of ∂ and $\bar{\partial}$ on ω are defined by

$$\partial\omega = \sum_{r+s=q} \partial\omega^{(r,s)} \qquad \bar{\partial}\omega = \sum_{r+s=q} \bar{\partial}\omega^{(r,s)}. \tag{8.41}$$

Theorem 8.15 Let M be a complex manifold and let $\omega \in \Omega^{q}(M)^{\mathbb{C}}$ and $\xi \in \Omega^{p}(M)^{\mathbb{C}}$. Then

(*a*) $\partial\partial\omega = (\partial\bar{\partial} + \bar{\partial}\partial)\omega = \bar{\partial}\bar{\partial}\omega = 0$ (8.42*a*)

(*b*) $\partial\bar{\omega} = \overline{\bar{\partial}\omega}$, $\bar{\partial}\bar{\omega} = \overline{\partial\omega}$ (8.42*b*)

(*c*) $\partial(\omega \wedge \xi) = \partial\omega \wedge \xi + (-1)^{q}\omega \wedge \partial\xi$

$\bar{\partial}(\omega \wedge \xi) = \bar{\partial}\omega \wedge \xi + (-1)^{q}\omega \wedge \bar{\partial}\xi.$ (8.42*c*)

Proof: It is sufficient to prove them when ω is of bidegree (r, s). (*a*) Since $\mathrm{d} = \partial + \bar{\partial}$, we have

$$0 = \mathrm{d}^2\omega = (\partial + \bar{\partial})(\partial + \bar{\partial})\omega = \partial\partial\omega + (\partial\bar{\partial} + \bar{\partial}\partial)\omega + \bar{\partial}\bar{\partial}\omega.$$

The three terms of the RHS are of bidegrees $(r+2, s)$, $(r+1, s+1)$ and $(r, s+2)$ respectively. From theorem 8.13(*c*), each term must vanish separately. (*b*) Since $\mathrm{d}\bar{\omega} = \overline{\mathrm{d}\omega}$, we have

$$\partial\bar{\omega} + \bar{\partial}\bar{\omega} = \mathrm{d}\bar{\omega} = \overline{(\partial + \bar{\partial})\omega} = \overline{\partial\omega} + \overline{\bar{\partial}\omega}.$$

Noting that $\partial\bar{\omega}$ and $\overline{\bar{\partial}\omega}$ are of bidegree $(s+1, r)$ and $\bar{\partial}\bar{\omega}$ and $\overline{\partial\omega}$ are of $(s, r+1)$, we conclude that $\partial\bar{\omega} = \overline{\bar{\partial}\omega}$ and $\bar{\partial}\bar{\omega} = \overline{\partial\omega}$. (*c*) We assume ω is of bidegree (r, s) and ξ of (r', s'). (8.42*c*) is proved by separating $\mathrm{d}(\omega \wedge \xi) = \mathrm{d}\omega \wedge \xi + (-1)^{q}\omega \wedge \mathrm{d}\xi$, into forms of bidegrees $(r + r' + 1, s + s')$ and $(r + r', s + s' + 1)$. ■

Definition 8.16 Let M be a complex manifold. If $\omega \in \Omega^{r,0}(M)$ satisfies $\bar{\partial}\omega = 0$, the r-form ω is called a **holomorphic r-form**.

Let us look at a holomorphic 0-form $f \in \mathcal{F}(U)^{\mathbb{C}}$ on a chart (U, φ). The condition $\bar{\partial}f = 0$ becomes

$$\frac{\partial f}{\partial \bar{z}^{\lambda}} = 0 \qquad \text{for } 1 \leqslant \lambda \leqslant m = \dim_{\mathbb{C}} M. \tag{8.43}$$

A holomorphic 0-form is just a holomorphic function, $f \in \mathcal{O}(U)$. Let

$\omega \in \Omega^{r,0}(M)$, where $1 \leqslant r \leqslant m = \dim_{\mathbb{C}} M$. On a chart (U, φ), we have

$$\omega = \frac{1}{r!}\,\omega_{\mu_1\ldots\mu_r}\,\mathrm{d}z^{\mu_1} \wedge \ldots \wedge \mathrm{d}z^{\mu_r}. \tag{8.44}$$

Then $\bar{\partial}\omega = 0$ if and only if

$$\frac{\partial}{\partial \bar{z}^{\lambda}}\,\omega_{\mu_1\ldots\mu_r} = 0$$

namely if $\omega_{\mu_1\ldots\mu_r}$ are holomorphic functions on U.

Let $\dim_{\mathbb{C}} M = m$. The sequence of $\mathbb{C}$-linear maps

$$\begin{aligned}\Omega^{r,0}(M) \xrightarrow{\bar{\partial}} \Omega^{r,1}(M) \xrightarrow{\bar{\partial}} \ldots \\ \ldots \xrightarrow{\bar{\partial}} \Omega^{r,m-1}(M) \xrightarrow{\bar{\partial}} \Omega^{r,m}(M)\end{aligned} \tag{8.45}$$

is called the **Dolbeault complex**. Note that $\bar{\partial}^2 = 0$. The set of $\bar{\partial}$-closed (r, s)-forms (those $\omega \in \Omega^{r,s}(M)$ such that $\bar{\partial}\omega = 0$) is called the **(r, s)-cocycle** and is denoted by $Z^{r,s}_{\bar{\partial}}(M)$. The set of $\bar{\partial}$-exact (r, s)-forms (those $\omega \in \Omega^{r,s}(M)$ such that $\omega = \bar{\partial}\eta$ for some $\eta \in \Omega^{r,s-1}(M)$) is called the **$(r, s)$-coboundary** and is denoted by $B^{r,s}_{\bar{\partial}}(M)$. The complex vector space

$$H^{r,s}_{\bar{\partial}}(M) \equiv Z^{r,s}_{\bar{\partial}}(M)/B^{r,s}_{\bar{\partial}}(M) \tag{8.46}$$

is called the **(r, s)th $\bar{\partial}$-cohomology group**, see §8.6.

8.4 Hermitian manifolds and Hermitian differential geometry

Let M be a complex manifold with $\dim_{\mathbb{C}} M = m$ and let g be a Riemannian metric of M as a differentiable manifold. Take $Z = X + \mathrm{i}Y$, $W = U + \mathrm{i}V \in T_pM^{\mathbb{C}}$ and extend g so that

$$g_p(Z, W) = g_p(X, U) - g_p(Y, V) + \mathrm{i}[g_p(X, V) + g_p(Y, U)]. \tag{8.47}$$

The components of g with respect to the bases (8.13) are

$$g_{\mu\nu}(p) = g_p(\partial/\partial z^{\mu}, \partial/\partial z^{\nu}) \tag{8.48a}$$

$$g_{\mu\bar{\nu}}(p) = g_p(\partial/\partial z^{\mu}, \partial/\partial \bar{z}^{\nu}) \tag{8.48b}$$

$$g_{\bar{\mu}\nu}(p) = g_p(\partial/\partial \bar{z}^{\mu}, \partial/\partial z^{\nu}) \tag{8.48c}$$

$$g_{\bar{\mu}\bar{\nu}}(p) = g_p(\partial/\partial \bar{z}^{\mu}, \partial/\partial \bar{z}^{\nu}). \tag{8.48d}$$

We easily verify that

$$g_{\mu\nu} = g_{\nu\mu},\ g_{\bar{\mu}\bar{\nu}} = g_{\bar{\nu}\bar{\mu}},\ g_{\bar{\mu}\nu} = g_{\nu\bar{\mu}},\ \overline{g_{\mu\bar{\nu}}} = g_{\bar{\mu}\nu},\ \overline{g_{\mu\nu}} = g_{\bar{\mu}\bar{\nu}}. \tag{8.49}$$

8.4.1 The Hermitian metric

If a Riemannian metric g of a complex manifold M satisfies

$$g_p(J_pX, J_pY) = g_p(X, Y) \tag{8.50}$$

at each point $p \in M$ and for any $X, Y \in T_pM$, g is said to be a **Hermitian metric**. The pair (M, g) is called a **Hermitian manifold**. J_pX is orthogonal to X with respect to a Hermitian metric,

$$g_p(J_pX, X) = g_p(J_p^2X, J_pX) = -g_p(J_pX, X) = 0. \tag{8.51}$$

Theorem 8.17 A complex manifold always admits a Hermitian metric.

Proof: Let g be any Riemannian metric of a complex manifold M. Define a new metric $\hat{g}$ by

$$\hat{g}_p(X, Y) \equiv \tfrac{1}{2}[g_p(X, Y) + g_p(JX, JY)]. \tag{8.52}$$

Clearly $\hat{g}_p(JX, JY) = \hat{g}_p(X, Y)$. Moreover, $\hat{g}$ is positive definite provided that g is. Hence $\hat{g}$ is a Hermitian metric on M. ■

Let g be a Hermitian metric on a complex manifold M. From (8.50), we find that $g_{\mu\nu} = g(\partial/\partial z^\mu, \partial/\partial z^\nu) = g(J\partial/\partial z^\mu, J\partial/\partial z^\nu) = -g(\partial/\partial z^\mu, \partial/\partial z^\nu) = -g_{\mu\nu}$, hence $g_{\mu\nu} = 0$. We also find that $g_{\bar{\mu}\bar{\nu}} = 0$. Thus the Hermitian metric g takes the form

$$g = g_{\mu\bar{\nu}} \mathrm{d}z^\mu \otimes \mathrm{d}\bar{z}^\nu + g_{\bar{\mu}\nu} \mathrm{d}\bar{z}^\mu \otimes \mathrm{d}z^\nu. \tag{8.53}$$

[*Remark:* Take $X, Y \in T_pM^+$. Define an inner product h_p in T_pM^+ by

$$h_p(X, Y) \equiv g_p(X, \bar{Y}). \tag{8.54}$$

It is easy to see that h_p is a positive-definite Hermitian form in T_pM^+. In fact,

$$\overline{h(X, Y)} = \overline{g(X, \bar{Y})} = g(\bar{X}, Y) = h(Y, \bar{X})$$

and $h(X, X) = g(X, \bar{X}) = g(X_1, X_1) + g(X_2, X_2) \geqslant 0$ for $X = X_1 + \mathrm{i}X_2$. This is why a metric g satisfying (8.50) is called Hermitian.]

8.4.2 *Kähler form*

Let (M, g) be a Hermitian manifold. Define a tensor field Ω whose action on $X, Y \in T_pM$ is

$$\Omega_p(X, Y) = g_p(J_pX, Y). \tag{8.55}$$

Note that Ω is antisymmetric, $\Omega(X, Y) = g(JX, Y) = g(J^2X, JY) = -g(JY, X) = -\Omega(Y, X)$. Hence Ω defines a two-form called the **Kähler form** of a Hermitian metric g. Observe that Ω is invariant under the action of J,

$$\Omega(JX, JY) = g(J^2X, JY) = g(J^3X, J^2Y) = \Omega(X, Y). \tag{8.56}$$

If the domain is extended from T_pM to $T_pM^{\mathbb{C}}$, Ω is a two-form of bidegree (1, 1). Indeed, for the metric (8.53), we have $\Omega(\partial/\partial z^\mu, \partial/\partial z^\nu) = g(J\cdot\partial/\partial z^\mu, \partial/\partial z^\nu) = \mathrm{i}g_{\mu\nu} = 0$. We also have $\Omega(\partial/\partial \bar{z}^\mu, \partial/\partial \bar{z}^\nu) = 0$ and $\Omega(\partial/\partial z^\mu, \partial/\partial \bar{z}^\nu) = \mathrm{i}g_{\mu\bar{\nu}} = -\Omega(\partial/\partial \bar{z}^\nu, \partial/\partial z^\mu)$. Thus, the components of Ω are

$$\Omega_{\mu\nu} = \Omega_{\bar{\mu}\bar{\nu}} = 0 \qquad \Omega_{\mu\bar{\nu}} = -\Omega_{\bar{\nu}\mu} = \mathrm{i}g_{\mu\bar{\nu}}. \tag{8.57}$$

We may write

$$\Omega = \mathrm{i}g_{\mu\bar{\nu}}\mathrm{d}z^\mu \otimes \mathrm{d}\bar{z}^\nu - \mathrm{i}g_{\bar{\nu}\mu}\mathrm{d}\bar{z}^\nu \otimes \mathrm{d}z^\mu = \mathrm{i}g_{\mu\bar{\nu}}\mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\nu. \tag{8.58}$$

Ω is also written as

$$\Omega = -J_{\mu\bar{\nu}}\mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\nu \tag{8.59}$$

where $J_{\mu\bar{\nu}} = g_{\mu\bar{\lambda}}J^{\bar{\lambda}}{}_{\bar{\nu}} = -\mathrm{i}g_{\mu\bar{\nu}}$. Ω is a real form

$$\bar{\Omega} = \overline{-\mathrm{i}g_{\mu\bar{\nu}}}\mathrm{d}\bar{z}^\mu \wedge \mathrm{d}z^\nu = \mathrm{i}g_{\nu\bar{\mu}} - \mathrm{d}z^\nu \wedge \mathrm{d}\bar{z}^\mu = \Omega. \tag{8.60}$$

Making use of the Kähler form, we show that any Hermitian manifold, and hence any complex manifold, is orientable. We first note that we may choose an orthonormal basis $\{\hat{e}_1, J\hat{e}_1, \ldots, \hat{e}_m, J\hat{e}_m\}$. In fact, if $g(\hat{e}_1, \hat{e}_1) = 1$, it follows that $g(J\hat{e}_1, J\hat{e}_1) = g(\hat{e}_1, \hat{e}_1) = 1$ and $g(\hat{e}_1, J\hat{e}_1) = -g(J\hat{e}_1, \hat{e}_1) = 0$. Thus $\hat{e}_1$ and $J\hat{e}_1$ form an orthonormal basis of a two-dimensional subspace. Now take $\hat{e}_2$ which is orthonormal to $\hat{e}_1$ and $J\hat{e}_1$ and form the subspace $\{\hat{e}_2, J\hat{e}_2\}$ .... Repeating this procedure we obtain an orthonormal basis $\{\hat{e}_1, J\hat{e}_1, \ldots, \hat{e}_m, J\hat{e}_m\}$.

Lemma 8.18 Let Ω be the Kähler form of a Hermitian manifold with $\dim_{\mathbb{C}} M = m$. Then

$$\underbrace{\Omega \wedge \ldots \wedge \Omega}_{m}$$

is a nowhere vanishing $2m$-form.

Proof: For the orthonormal basis above, we have

$$\Omega(\hat{e}_i, J\hat{e}_j) = g(J\hat{e}_i, J\hat{e}_j) = \delta_{ij} \qquad \Omega(\hat{e}_i, \hat{e}_j) = \Omega(J\hat{e}_i, J\hat{e}_j) = 0.$$

Then it follows that

$$\begin{aligned}&\underbrace{(\Omega \wedge \ldots \wedge \Omega)}_{m}(\hat{e}_1, J\hat{e}_1, \ldots, \hat{e}_m, J\hat{e}_m)\\ &\quad = \sum_P \Omega(\hat{e}_{P(1)}, J\hat{e}_{P(1)}) \ldots \Omega(\hat{e}_{P(m)}, J\hat{e}_{P(m)})\\ &\quad = m!\Omega(\hat{e}_1, J\hat{e}_1) \ldots \Omega(\hat{e}_m, J\hat{e}_m) = m!.\end{aligned}$$

This shows that $\Omega \wedge \ldots \wedge \Omega$ cannot vanish at any point. ■

Since the *real* $2m$-form $\Omega \wedge \ldots \wedge \Omega$ vanishes nowhere, it serves as a volume element. Thus we obtain the following theorem.

Theorem 8.19 A complex manifold is orientable.

8.4.3 Covariant derivatives

Let (M, g) be a Hermitian manifold. We define a connection which is compatible with the complex structure. It is natural to assume that a holomorphic vector $V \in T_pM^+$ parallel transported to another point q is again a holomorphic vector $\widetilde{V}(q) \in T_qM^+$. We show below that the almost complex structure is covariantly conserved under this requirement. Let $\{z^\mu\}$ and $\{z^\mu + \Delta z^\mu\}$ be the coordinates of p and q respectively and let $V = V^\mu \partial/\partial z^\mu|_p$ and $\widetilde{V}(q) = \widetilde{V}^\mu(z + \Delta z)\partial/\partial z^\mu|_q$. We assume that (cf (7.9))

$$\widetilde{V}^\mu(z + \Delta z) = V^\mu - V^\lambda \Gamma^\mu{}_{\nu\lambda}(z)\Delta z^\nu. \tag{8.61}$$

Then the basis vectors satisfy (cf (7.14))

$$\nabla_\mu \partial/\partial z^\nu = \Gamma^\lambda{}_{\mu\nu}\partial/\partial z^\lambda. \tag{8.62a}$$

Since $\partial/\partial \bar{z}^\mu$ is a conjugate vector field of $\partial/\partial z^\mu$, we have

$$\nabla_{\bar{\mu}} \partial/\partial \bar{z}^\nu = \Gamma^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}}\partial/\partial \bar{z}^\lambda \tag{8.62b}$$

where $\Gamma^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}} = \overline{\Gamma^\lambda{}_{\mu\nu}}$. $\Gamma^\lambda{}_{\mu\nu}$ and $\Gamma^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}}$ are the only non-vanishing components of the connection coefficients. Note that $\nabla_\mu \partial/\partial \bar{z}^\nu = \nabla_{\bar{\mu}}\partial/\partial z^\nu = 0$. For the dual basis, non-vanishing covariant derivatives are

$$\nabla_\mu \mathrm{d}z^\nu = -\Gamma^\nu{}_{\mu\lambda}\mathrm{d}z^\lambda \qquad \nabla_{\bar{\mu}}\mathrm{d}\bar{z}^\nu = -\Gamma^{\bar{\nu}}{}_{\bar{\mu}\bar{\lambda}}\mathrm{d}\bar{z}^\lambda. \tag{8.63}$$

The covariant derivative of $X^+ = X^\mu \partial/\partial z^\mu \in \mathcal{X}(M)^+$ is

$$\nabla_\mu X^+ = (\partial_\mu X^\lambda + X^\nu \Gamma^\lambda{}_{\mu\nu})\partial/\partial z^\lambda \tag{8.64}$$

where $\partial_\mu \equiv \partial/\partial z^\mu$. For $X^- = X^{\bar{\mu}}\partial/\partial \bar{z}^\mu \in \mathcal{X}(M)^-$, we have

$$\nabla_\mu X^- = (\partial_\mu X^{\bar{\lambda}})\partial/\partial \bar{z}^\lambda \tag{8.65}$$

since $\Gamma^{\bar{\lambda}}{}_{\mu\nu} = \Gamma^{\bar{\lambda}}{}_{\mu\bar{\nu}} = 0$. As far as antiholomorphic vectors are concerned, ∇_μ works as the ordinary derivative ∂_μ. Similarly we have

$$\nabla_{\bar{\mu}} X^+ = (\partial_{\bar{\mu}} X^\lambda)\partial/\partial z^\lambda \tag{8.66}$$

$$\nabla_{\bar{\mu}} X^- = (\partial_{\bar{\mu}} X^{\bar{\lambda}} + X^{\bar{\nu}}\Gamma^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}})\partial/\partial \bar{z}^\lambda. \tag{8.67}$$

It is easy to generalise this to an arbitrary tensor field. For example, if $t = t_{\mu\nu}{}^{\bar{\lambda}}\mathrm{d}z^\mu \otimes \mathrm{d}z^\nu \otimes \partial/\partial \bar{z}^\lambda$, we have

$$(\nabla_\kappa t)_{\mu\nu}{}^{\bar{\lambda}} = \partial_\kappa t_{\mu\nu}{}^{\bar{\lambda}} - t_{\xi\nu}{}^{\bar{\lambda}}\Gamma^\xi{}_{\kappa\mu} - t_{\mu\xi}{}^{\bar{\lambda}}\Gamma^\xi{}_{\kappa\nu}$$

$$(\nabla_{\bar{\kappa}} t)_{\mu\nu}{}^{\bar{\lambda}} = \partial_{\bar{\kappa}} t_{\mu\nu}{}^{\bar{\lambda}} + t_{\mu\nu}{}^{\bar{\xi}}\Gamma^{\bar{\lambda}}{}_{\bar{\kappa}\bar{\xi}}.$$

We require the **metric compatibility** as in §7.2. We demand that $\nabla_\kappa g_{\mu\bar\nu} = \nabla_{\bar\kappa} g_{\mu\bar\nu} = 0$. In components, we have

$$\partial_\kappa g_{\mu\bar\nu} - g_{\lambda\bar\nu}\Gamma^\lambda{}_{\kappa\mu} = 0 \qquad \partial_{\bar\kappa} g_{\mu\bar\nu} - g_{\mu\bar\lambda}\Gamma^{\bar\lambda}{}_{\bar\kappa\bar\nu} = 0. \tag{8.68}$$

The connection coefficients are easily read off:

$$\Gamma^\lambda{}_{\kappa\mu} = g^{\bar\nu\lambda}\partial_\kappa g_{\mu\bar\nu} \qquad \Gamma^{\bar\lambda}{}_{\bar\kappa\bar\nu} = g^{\bar\lambda\mu}\partial_{\bar\kappa} g_{\mu\bar\nu} \tag{8.69}$$

where $\{g^{\bar\nu\lambda}\}$ is the inverse matrix of $g_{\mu\bar\nu}$; $g_{\mu\bar\lambda}g^{\bar\lambda\nu} = \delta_\mu{}^\nu$, $g^{\bar\nu\lambda}g_{\lambda\bar\mu} = \delta^{\bar\nu}{}_{\bar\mu}$. A metric-compatible connection for which Γ(mixed indices) $= 0$ is called the **Hermitian connection**. By construction, this is unique and given by (8.69).

Theorem 8.20 The almost complex structure J is covariantly constant with respect to the *Hermitian* connection,

$$(\nabla_\kappa J)_\nu{}^\mu = (\nabla_{\bar\kappa} J)_\nu{}^\mu = (\nabla_\kappa J)_{\bar\nu}{}^{\bar\mu} = (\nabla_{\bar\kappa} J)_{\bar\nu}{}^{\bar\mu} = 0. \tag{8.70}$$

Proof: We prove the first equality. From (8.22), we find

$$(\nabla_\kappa J)_\nu{}^\mu = \partial_\kappa \mathrm{i} - \mathrm{i}\delta_\xi{}^\mu\Gamma^\xi{}_{\kappa\nu} + \mathrm{i}\delta_\nu{}^\xi\Gamma^\mu{}_{\kappa\xi} = 0.$$

Other equalities follow from similar calculations. ∎

8.4.4 Torsion and curvature

The torsion tensor T and the Riemann curvature tensor R are defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{8.71}$$

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X, Y]}Z. \tag{8.72}$$

We find that

$$T(\partial/\partial z^\mu, \partial/\partial z^\nu) = (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu})\partial/\partial z^\lambda$$

$$T(\partial/\partial z^\mu, \partial/\partial \bar z^\nu) = T(\partial/\partial \bar z^\mu, \partial/\partial z^\nu) = 0$$

$$T(\partial/\partial \bar z^\mu, \partial/\partial \bar z^\nu) = (\Gamma^{\bar\lambda}{}_{\bar\mu\bar\nu} - \Gamma^{\bar\lambda}{}_{\bar\nu\bar\mu})\partial/\partial \bar z^\lambda.$$

The non-vanishing components are

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = g^{\bar\xi\lambda}(\partial_\mu g_{\nu\bar\xi} - \partial_\nu g_{\mu\bar\xi}) \tag{8.73a}$$

$$T^{\bar\lambda}{}_{\bar\mu\bar\nu} = \Gamma^{\bar\lambda}{}_{\bar\mu\bar\nu} - \Gamma^{\bar\lambda}{}_{\bar\nu\bar\mu} = g^{\bar\lambda\xi}(\partial_{\bar\mu} g_{\bar\nu\xi} - \partial_{\bar\nu} g_{\bar\mu\xi}). \tag{8.73b}$$

As for the Riemann tensor, we find, for example, that

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\kappa{}_{\nu\lambda} - \partial_\nu\Gamma^\kappa{}_{\mu\lambda} + \Gamma^\eta{}_{\nu\lambda}\Gamma^\kappa{}_{\mu\eta} - \Gamma^\eta{}_{\mu\lambda}\Gamma^\kappa{}_{\nu\eta}.$$

If (8.69) is substituted, we find that

$$
\begin{aligned}
R^{\kappa}{}_{\lambda\mu\nu} &= \partial_\mu g^{\bar{\xi}\kappa}\partial_\nu g_{\lambda\bar{\xi}} + g^{\bar{\xi}\kappa}\partial_\mu\partial_\nu g_{\lambda\bar{\xi}} - \partial_\nu g^{\bar{\xi}\kappa}\partial_\mu g_{\lambda\bar{\xi}} - g^{\bar{\xi}\kappa}\partial_\mu\partial_\nu g_{\lambda\bar{\xi}} \\
&\quad + g^{\bar{\xi}\eta}\partial_\nu g_{\lambda\bar{\xi}} g^{\bar{\zeta}\kappa}\partial_\mu g_{\eta\bar{\zeta}} - g^{\bar{\xi}\eta}\partial_\mu g_{\lambda\bar{\xi}} g^{\bar{\zeta}\kappa}\partial_\nu g_{\eta\bar{\xi}} = 0
\end{aligned}
$$

where use has been made of the identity $g^{\bar{\zeta}\kappa}\partial_\mu g_{\eta\bar{\zeta}} = -g_{\eta\bar{\zeta}}\partial_\mu g^{\bar{\zeta}\kappa}$ etc. In general we find

$$R^{\kappa}{}_{\bar{\lambda}AB} = R^{\bar{\kappa}}{}_{\lambda AB} = R^{A}{}_{B\kappa\lambda} = R^{A}{}_{B\bar{\kappa}\bar{\lambda}} = 0 \tag{8.74}$$

where A and B are any (holomorphic or antiholomorphic) indices. As a result, we are left only with the components $R^{\kappa}{}_{\lambda\bar{\mu}\nu}$, $R^{\kappa}{}_{\lambda\mu\bar{\nu}}$, $R^{\bar{\kappa}}{}_{\bar{\lambda}\bar{\mu}\nu}$ and $R^{\bar{\kappa}}{}_{\bar{\lambda}\mu\bar{\nu}}$. Note that we have a trivial symmetry $R^{\kappa}{}_{\lambda\bar{\mu}\nu} = -R^{\kappa}{}_{\lambda\nu\bar{\mu}}$. So the independent components are reduced to $R^{\kappa}{}_{\lambda\bar{\mu}\nu}$ and $R^{\bar{\kappa}}{}_{\bar{\lambda}\mu\bar{\nu}} = \overline{R^{\kappa}{}_{\lambda\bar{\mu}\nu}}$. We find that

$$R^{\kappa}{}_{\lambda\bar{\mu}\nu} = \partial_{\bar{\mu}}\Gamma^{\kappa}{}_{\nu\lambda} = \partial_{\bar{\mu}}(g^{\bar{\xi}\kappa}\partial_\nu g_{\lambda\bar{\xi}}) \tag{8.75a}$$

$$R^{\bar{\kappa}}{}_{\bar{\lambda}\mu\bar{\nu}} = \partial_\mu\Gamma^{\bar{\kappa}}{}_{\bar{\nu}\bar{\lambda}} = \partial_\mu(g^{\bar{\kappa}\xi}\partial_{\bar{\nu}} g_{\xi\bar{\lambda}}). \tag{8.75b}$$

Exercise 8.21 Show that

$$R_{\bar{\kappa}\lambda\bar{\mu}\nu} \equiv g_{\bar{\kappa}\xi}R^{\xi}{}_{\lambda\bar{\mu}\nu} = \partial_{\bar{\mu}}\partial_\nu g_{\lambda\bar{\kappa}} - g^{\bar{\eta}\xi}\partial_{\bar{\mu}} g_{\bar{\kappa}\xi}\partial_\nu g_{\lambda\bar{\eta}} \tag{8.76a}$$

$$R_{\kappa\bar{\lambda}\mu\bar{\nu}} \equiv g_{\kappa\bar{\xi}}R^{\bar{\xi}}{}_{\bar{\lambda}\mu\bar{\nu}} = \partial_\mu\partial_{\bar{\nu}} g_{\bar{\lambda}\kappa} - g^{\eta\bar{\xi}}\partial_\mu g_{\kappa\bar{\xi}}\partial_{\bar{\nu}} g_{\bar{\lambda}\eta} \tag{8.76b}$$

$$R_{\bar{\kappa}\lambda\mu\bar{\nu}} \equiv g_{\bar{\kappa}\xi}R^{\xi}{}_{\lambda\mu\bar{\nu}} = -R_{\bar{\kappa}\lambda\bar{\nu}\mu} \tag{8.76c}$$

$$R_{\kappa\bar{\lambda}\bar{\mu}\nu} \equiv g_{\kappa\bar{\xi}}R^{\bar{\xi}}{}_{\bar{\lambda}\bar{\mu}\nu} = -R_{\kappa\bar{\lambda}\nu\bar{\mu}}. \tag{8.76d}$$

Verify the symmetries

$$R_{\bar{\kappa}\lambda\bar{\mu}\nu} = -R_{\lambda\bar{\kappa}\bar{\mu}\nu} \qquad R_{\kappa\bar{\lambda}\mu\bar{\nu}} = R_{\bar{\lambda}\kappa\mu\bar{\nu}}. \tag{8.77}$$

Let us contract the indices of the Riemann tensor as

$$\mathfrak{R}_{\mu\bar{\nu}} \equiv R^{\kappa}{}_{\kappa\mu\bar{\nu}} = -\partial_{\bar{\nu}}(g^{\kappa\bar{\xi}}\partial_\mu g_{\kappa\bar{\xi}}) = -\partial_{\bar{\nu}}\partial_\mu \ln G \tag{8.78}$$

where $G \equiv \det(g_{\mu\bar{\nu}}) = \sqrt{g}$. To obtain the last equality, we used an identity $\delta G = Gg^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$; see (7.201). We define the **Ricci form** by

$$\mathfrak{R} \equiv \mathrm{i}\mathfrak{R}_{\mu\bar{\nu}}\,\mathrm{d}z^{\mu} \wedge \mathrm{d}\bar{z}^{\nu} = \mathrm{i}\partial\bar{\partial}\ln G. \tag{8.79}$$

$\mathfrak{R}$ is a *real* form; $\bar{\mathfrak{R}} = -\mathrm{i}\overline{\partial\bar{\partial}}\ln G = -\mathrm{i}\bar{\partial}\partial\ln G = \mathfrak{R}$. From the identity $\partial\bar{\partial} = -\frac{1}{2}\mathrm{d}(\partial - \bar{\partial})$, we find $\mathfrak{R}$ is closed; $\mathrm{d}\mathfrak{R} \propto \mathrm{d}^2(\partial - \bar{\partial})\ln G = 0$. However this does not imply that $\mathfrak{R}$ is exact. In fact, G is not a scalar and $(\partial - \bar{\partial})\ln G$ is not defined globally. $\mathfrak{R}$ defines a non-trivial element $c_1(M) \equiv [\mathfrak{R}/2\pi] \in H^2(M; \mathbb{R})$ called the **first Chern class**. We discuss this further in §11.2.

Proposition 8.22 The first Chern class $c_1(M)$ is invariant under a smooth change of the metric $g \to g + \delta g$.

Proof: It follows from (7.201) that $\delta\ln G = g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$. Then

$$\delta\mathfrak{R} = \delta\mathrm{i}\partial\bar{\partial}\ln G = \mathrm{i}\partial\bar{\partial}g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}} = -\tfrac{1}{2}\mathrm{d}(\partial - \bar{\partial})\mathrm{i}g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}.$$

Since $g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ is a *scalar*, $\omega \equiv -\frac{1}{2}\mathrm{i}(\partial - \bar{\partial})g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ is a well defined one-form on M. Thus $\delta\mathfrak{R} = \mathrm{d}\omega$ is an exact two-form and $[\mathfrak{R}] = [\mathfrak{R} + \delta\mathfrak{R}]$, namely $c_1(M)$ is left invariant under $g \to g + \delta g$. ■

8.5 Kähler manifolds and Kähler differential geometry

8.5.1 Definitions

Definition 8.23 A **Kähler manifold** is a Hermitian manifold (M, g) whose Kähler form Ω is closed: $\mathrm{d}\Omega = 0$. The metric g is called the **Kähler metric** of M. [*Warning:* Not all complex manifolds admit Kähler metrics.]

Theorem 8.24 A Hermitian manifold (M, g) is a Kähler manifold if and only if the almost complex structure J satisfies

$$\nabla_\mu J = 0 \tag{8.80}$$

where ∇_μ is the *Levi-Civita* connection associated with g.

Proof: We first note that for any r-form ω, $\mathrm{d}\omega$ is written as

$$\mathrm{d}\omega = \nabla\omega \equiv \frac{1}{r!}\nabla_\mu \omega_{\nu_1\ldots\nu_r}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_r} \tag{8.81}$$

[For example,

$$\begin{aligned}
\nabla\Omega &= \tfrac{1}{2}\nabla_\lambda\Omega_{\mu\nu}\,\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \tfrac{1}{2}(\partial_\lambda\Omega_{\mu\nu} - \Gamma^\kappa{}_{\lambda\mu}\Omega_{\kappa\nu} - \Gamma^\kappa{}_{\lambda\nu}\Omega_{\mu\kappa})\,\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \tfrac{1}{2}\partial_\lambda\Omega_{\mu\nu}\,\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}\Omega
\end{aligned}$$

since Γ is symmetric.] Now we prove that $\nabla_\mu J = 0$ if and only if $\nabla_\mu\Omega = 0$. We verify the following equalities,

$$\begin{aligned}
(\nabla_Z\Omega)(X, Y) &= \nabla_Z[\Omega(X, Y)] - \Omega(\nabla_Z X, Y) - \Omega(X, \nabla_Z Y) \\
&= \nabla_Z[g(JX, Y)] - g(J\nabla_Z X, Y) - g(JX, \nabla_Z Y) \\
&= (\nabla_Z g)(JX, Y) + g(\nabla_Z JX, Y) - g(J\nabla_Z X, Y) \\
&= g(\nabla_Z JX - J\nabla_Z X, Y) = g((\nabla_Z J)X, Y)
\end{aligned}$$

where $\nabla_Z g = 0$ has been used. Since this is true for any X, Y, Z, it follows that $\nabla_Z\Omega = 0$ if and only if $\nabla_Z J = 0$. ■

Theorems 8.20 and 8.24 show that in the Kähler manifold, the Riemann structure is compatible with the Hermitian structure.

Let g be a Kähler metric. Since $\mathrm{d}\Omega = 0$, we have

$$
\begin{aligned}
(\partial + \bar\partial)\mathrm{i}g_{\mu\bar\nu}\mathrm{d}z^\mu \wedge \mathrm{d}\bar z^\nu
&= \mathrm{i}\partial_\lambda g_{\mu\bar\nu}\mathrm{d}z^\lambda \wedge \mathrm{d}z^\mu \wedge \mathrm{d}\bar z^\nu + \mathrm{i}\partial_{\bar\lambda} g_{\mu\bar\nu}\mathrm{d}\bar z^\lambda \wedge \mathrm{d}z^\mu \wedge \mathrm{d}\bar z^\nu \\
&= \tfrac{1}{2}\mathrm{i}(\partial_\lambda g_{\mu\bar\nu} - \partial_\mu g_{\lambda\bar\nu})\mathrm{d}z^\lambda \wedge \mathrm{d}z^\mu \wedge \mathrm{d}\bar z^\nu \\
&\quad + \tfrac{1}{2}\mathrm{i}(\partial_{\bar\lambda} g_{\mu\bar\nu} - \partial_{\bar\nu} g_{\mu\bar\lambda})\mathrm{d}\bar z^\lambda \wedge \mathrm{d}z^\mu \wedge \mathrm{d}\bar z^\nu = 0
\end{aligned}
$$

from which we find

$$
\frac{\partial g_{\mu\bar\nu}}{\partial z^\lambda} = \frac{\partial g_{\lambda\bar\nu}}{\partial z^\mu} \qquad \frac{\partial g_{\mu\bar\nu}}{\partial \bar z^\lambda} = \frac{\partial g_{\mu\bar\lambda}}{\partial \bar z^\nu}. \tag{8.82}
$$

Suppose that a Hermitian metric g is given on a chart U_i by

$$
g_{\mu\bar\nu} = \partial_\mu\partial_{\bar\nu}\mathcal{K}_i \tag{8.83}
$$

where $\mathcal{K}_i \in \mathcal{F}(U_i)$. Clearly this metric satisfies the condition (8.82), hence it is Kähler. Conversely it can be shown that any Kähler metric is *locally* expressed as (8.83). The function $\mathcal{K}_i$ is called the **Kähler potential** of a Kähler metric. If follows that $\Omega = \mathrm{i}\partial\bar\partial\mathcal{K}_i$ on U_i.

Let (U_i, φ_i) and (U_j, φ_j) be overlapping charts. On $U_i \cap U_j$, we have

$$
\frac{\partial}{\partial z^\mu}\frac{\partial}{\partial \bar z^\nu}\mathcal{K}_i\,\mathrm{d}z^\mu \otimes \mathrm{d}\bar z^\nu = \frac{\partial}{\partial w^\alpha}\frac{\partial}{\partial \bar w^\beta}\mathcal{K}_j\,\mathrm{d}w^\alpha \otimes \mathrm{d}\bar w^\beta
$$

where $z = \varphi_i(p)$ and $w = \varphi_j(p)$. It follows that

$$
\frac{\partial w^\alpha}{\partial z^\mu}\frac{\partial \bar w^\beta}{\partial \bar z^\nu}\frac{\partial}{\partial w^\alpha}\frac{\partial}{\partial \bar w^\beta}\mathcal{K}_j = \frac{\partial}{\partial z^\mu}\frac{\partial}{\partial \bar z^\nu}\mathcal{K}_i. \tag{8.84}
$$

This is satisfied if and only if $\mathcal{K}_j(w, \bar w) = \mathcal{K}_i(z, \bar z) + \phi_{ij}(z) + \psi_{ij}(\bar z)$ where $\phi_{ij}(\psi_{ij})$ is holomorphic (antiholomorphic) in z.

Exercise 8.25 Let M be a compact Kähler manifold without a boundary. Show that

$$
\Omega^m \equiv \underbrace{\Omega \wedge \ldots \wedge \Omega}_{m}
$$

is closed but not exact where $m = \dim_{\mathbb{C}} M$ [*Hint:* Use Stokes' theorem.] Thus the $2m$th Betti number cannot vanish, $b^{2m} \geqslant 1$. We will see later that $b^{2p} \geqslant 1$ for $1 \leqslant p \leqslant m$.

Example 8.26 Let $M = \mathbb{C}^m = \{(z^1, \ldots, z^m)\}$. $\mathbb{C}^m$ is identified with $\mathbb{R}^{2m}$ by the identification $z^\mu \to x^\mu + \mathrm{i}y^\mu$. Let δ be the Euclidean metric of $\mathbb{R}^{2m}$,

$$
\begin{aligned}
&\delta(\partial/\partial x^\mu, \partial/\partial x^\nu) = \delta(\partial/\partial y^\mu, \partial/\partial y^\nu) = \delta_{\mu\nu} \\
&\delta(\partial/\partial x^\mu, \partial/\partial y^\nu) = 0.
\end{aligned} \tag{8.85a}
$$

Noting that $J\partial/\partial x^\mu = \partial/\partial y^\mu$ and $J\partial/\partial y^\mu = -\partial/\partial x^\mu$, we find that δ is a Hermitian metric. In complex coordinates, we have

$$\begin{aligned}\delta(\partial/\partial z^\mu, \partial/\partial z^\nu) &= \delta(\partial/\partial \bar{z}^\mu, \partial/\partial \bar{z}^\nu) = 0\\ \delta(\partial/\partial z^\mu, \partial/\partial \bar{z}^\nu) &= \delta(\partial/\partial \bar{z}^\mu, \partial/\partial z^\nu) = \tfrac{1}{2}\delta_{\mu\nu}.\end{aligned} \tag{8.85b}$$

The Kähler form is given by

$$\Omega = \frac{\mathrm{i}}{2}\sum_{\mu=1}^{m} \mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\mu = \sum_{\mu=1}^{m} \mathrm{d}x^\mu \wedge \mathrm{d}y^\mu. \tag{8.86}$$

Clearly, $\mathrm{d}\Omega = 0$ and we find that the Euclidean metric δ of $\mathbb{R}^{2m}$ is a Kähler metric of $\mathbb{C}^m$. The Kähler potential is

$$\mathcal{K} = \frac{1}{2}\sum z^\mu \bar{z}^\mu. \tag{8.87}$$

The Kähler manifold $\mathbb{C}^m$ is called the **complex Euclidean space**.

Example 8.27 Any orientable complex manifold M with $\dim_{\mathbb{C}} M = 1$ is Kähler. Take a Hermitian metric g whose Kähler form is Ω. Since Ω is a real two-form, a three-form $\mathrm{d}\Omega$ has to vanish on M. One-dimensional compact orientable complex manifolds are known as **Riemann surfaces**.

Example 8.28 A complex projective space $\mathbb{C}P^n$ is a Kähler manifold. Let $(U_\alpha, \varphi_\alpha)$ be a chart whose inhomogeneous coordinates are $\varphi_\alpha(p) = \xi^\nu{}_{(\alpha)}$, $\nu \neq \alpha$ (see example 8.4). It is convenient to introduce a tidier notation $\{\zeta^\nu{}_{(\alpha)} | 1 \leqslant \nu \leqslant n\}$ by

$$\xi^\nu{}_{(\alpha)} = \zeta^\nu{}_{(\alpha)} \quad (\nu \leqslant \alpha - 1) \qquad \xi^{\nu+1}{}_{(\alpha)} = \zeta^\nu{}_{(\alpha)} \quad (\nu \geqslant \alpha). \tag{8.88}$$

$\{\zeta^\nu{}_{(\alpha)}\}$ is just a renaming of $\{\xi^\nu{}_{(\alpha)}\}$. Define a positive-definite function $\mathcal{K}_\alpha : U_\alpha \to \mathbb{R}$ by

$$\mathcal{K}_\alpha(p) \equiv \sum_{\nu=1}^{m} |\zeta^\nu{}_{(\alpha)}(p)|^2 + 1 = \sum_{\nu=1}^{m+1} \left|\frac{z^\nu}{z^\alpha}\right|^2. \tag{8.89}$$

On a point $p \in U_\alpha \cap U_\beta$, $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ are related by

$$\mathcal{K}_\alpha(p) = \left|\frac{z^\beta}{z^\alpha}\right|^2 \mathcal{K}_\beta(p). \tag{8.90}$$

Then it follows that

$$\ln \mathcal{K}_\alpha = \ln \mathcal{K}_\beta + \ln(z^\beta/z^\alpha) + \overline{\ln(z^\beta/z^\alpha)}. \tag{8.91}$$

Since z^β/z^α is a holomorphic function, we have $\bar{\partial} \ln z^\beta/z^\alpha = 0$. Also

$$\partial\, \overline{\ln z^\beta/z^\alpha} = \overline{\bar{\partial} \ln z^\beta/z^\alpha} = 0.$$

Then it follows that

$$\partial\bar{\partial} \ln \mathcal{K}_\alpha = \partial\bar{\partial} \ln \mathcal{K}_\beta. \tag{8.92}$$

A closed two-form Ω is locally defined by

$$\Omega \equiv \mathrm{i}\partial\bar{\partial} \ln \mathcal{K}_\alpha. \tag{8.93}$$

There exists a Hermitian metric whose Kähler form is Ω. Take X, $Y \in T_p\mathbb{C}P^n$ and define $g : T_p\mathbb{C}P^n \otimes T_p\mathbb{C}P^n \to \mathbb{R}$ by $g(X, Y) = \Omega(X, JY)$. To prove that g is a Hermitian metric, we have to show that g satisfies (8.50) and is positive definite. The Hermiticity is obvious since $g(JX, JY) = -\Omega(JX, Y) = \Omega(Y, JX) = g(X, Y)$. Next we show that g is positive definite. On a chart $(U_\alpha, \varphi_\alpha)$, we have

$$\Omega = \mathrm{i}\,\frac{\partial^2 \ln \mathcal{K}}{\partial \zeta^\mu \partial \bar{\zeta}^\nu}\,\mathrm{d}\zeta^\mu \wedge \mathrm{d}\bar{\zeta}^\nu \tag{8.94}$$

where we have dropped the subscript (α) to simplify the notation. If we substitute the expression (8.89) for $\mathcal{K}$ on U_α, we have

$$\Omega = \mathrm{i}\sum_{\mu,\nu} \frac{\delta_{\mu\nu}(\Sigma|\zeta^\lambda|^2 + 1) - \zeta^\mu\bar{\zeta}^\nu}{(\Sigma|\zeta^\lambda|^2 + 1)^2}\,\mathrm{d}\zeta^\mu \wedge \mathrm{d}\bar{\zeta}^\nu. \tag{8.95}$$

Let X be a real vector, $X = X^\mu\partial/\partial\zeta^\mu + \bar{X}^\mu\partial/\partial\bar{\zeta}^\mu$ and $JX = \mathrm{i}X^\mu\partial/\partial\zeta^\mu - \mathrm{i}\bar{X}^\mu\partial/\partial\bar{\zeta}^\mu$. Then

$$\begin{aligned}
g(X, X) = \Omega(X, JX) &= 2\sum_{\mu,\nu} \frac{\delta_{\mu\nu}(\Sigma|\zeta^\lambda|^2 + 1) - \zeta^\mu\bar{\zeta}^\nu}{(\Sigma|\zeta^\lambda|^2 + 1)^2}\,X^\mu\bar{X}^\nu \\
&= 2\left[\sum_\mu |X^\mu|^2\left(\sum_\lambda |\zeta^\lambda|^2 + 1\right) - \left|\sum_\mu X^\mu\zeta^\mu\right|^2\right]\left(\sum_\lambda |\zeta^\lambda|^2 + 1\right)^{-2}.
\end{aligned}$$

From the Schwarz inequality $\Sigma_\mu|X^\mu|^2\cdot\Sigma_\lambda|\zeta^\lambda|^2 \geq \Sigma|X^\mu\zeta^\mu|^2$, we find the metric g is positive definite. This metric is called the **Fubini–Study metric** of $\mathbb{C}P^n$.

A few useful facts are,

(*a*) S^2 is the only sphere which admits a complex structure. Since $S^2 \simeq \mathbb{C}P^1$, it is a Kähler manifold.

(*b*) A product of two odd-dimensional spheres $S^{2m+1} \times S^{2n+1}$ always admits a complex structure. This complex structure does not admit a Kähler metric.

(*c*) Any complex submanifold of a Kähler manifold is Kähler.

8.5.2 Kähler geometry

A Kähler metric g is characterised by (8.82),

$$\frac{\partial g_{\mu\bar{\nu}}}{\partial z^\lambda} = \frac{\partial g_{\lambda\bar{\nu}}}{\partial z^\mu} \qquad \frac{\partial g_{\mu\bar{\nu}}}{\partial \bar{z}^\lambda} = \frac{\partial g_{\lambda\bar{\nu}}}{\partial \bar{z}^\mu}.$$

This ensures that the Kähler metric is *torsion free*,

$$T^\lambda{}_{\mu\nu} = g^{\bar{\xi}\lambda}\,(\partial_\mu g_{\nu\bar{\xi}} - \partial_\nu g_{\mu\bar{\xi}}) = 0 \tag{8.96a}$$

$$T^{\bar{\lambda}}{}_{\bar{\mu}\bar{\nu}} = g^{\bar{\lambda}\xi}(\partial_{\bar{\mu}} g_{\bar{\nu}\xi} - \partial_{\bar{\nu}} g_{\bar{\mu}\xi}) = 0. \tag{8.96b}$$

In this sense, the Kähler metric defines a connection which is very similar to the Levi-Civita connection. Now the Riemann tensor has an extra symmetry

$$R^{\kappa}{}_{\lambda\mu\bar{\nu}} = -\partial_{\bar{\nu}}(g^{\bar{\xi}\kappa}\partial_{\mu}g_{\lambda\bar{\xi}}) = -\partial_{\bar{\nu}}(g^{\bar{\xi}\kappa}\partial_{\lambda}g_{\mu\bar{\xi}}) = R^{\kappa}{}_{\mu\lambda\bar{\nu}} \tag{8.97}$$

as well as those obtained from (8.97) by known symmetry operations,

$$R^{\bar{\kappa}}{}_{\bar{\lambda}\bar{\mu}\nu} = R^{\bar{\kappa}}{}_{\bar{\mu}\bar{\lambda}\nu}, \; R^{\kappa}{}_{\lambda\bar{\mu}\nu} = R^{\kappa}{}_{\nu\bar{\mu}\lambda}, \; R^{\bar{\kappa}}{}_{\bar{\lambda}\mu\bar{\nu}} = R^{\bar{\kappa}}{}_{\bar{\nu}\mu\bar{\lambda}}. \tag{8.98}$$

The Ricci form $\mathfrak{R}$ is defined as before,

$$\mathfrak{R} = -\mathrm{i}\partial_{\bar{\nu}}\partial_{\mu}\ln G\,\mathrm{d}z^{\mu} \wedge \mathrm{d}\bar{z}^{\nu}.$$

Because of (8.97), the components of the Ricci form agree with $Ric_{\mu\bar{\nu}}$: $\mathfrak{R}_{\mu\bar{\nu}} \equiv R^{\kappa}{}_{\kappa\mu\bar{\nu}} = R^{\kappa}{}_{\mu\kappa\bar{\nu}} = Ric_{\mu\bar{\nu}}$. If $Ric = \mathfrak{R} = 0$, the Kähler metric is said to be **Ricci-flat**.

Theorem 8.29 Let (M, g) be a Kähler manifold. If M admits a Ricci-flat metric h, then its first Chern class must vanish.

Proof: By assumption, $\mathfrak{R} = 0$ for the metric h. As was shown in the previous section, $\mathfrak{R}(g) - \mathfrak{R}(h) = \mathfrak{R}(g) = \mathrm{d}\omega$. Hence $c_1(M)$ computed from g agrees with that computed from h and hence vanishes. ∎

A compact Kähler manifold with vanishing first Chern class is called a **Calabi–Yau manifold**. Calabi (1957) conjectured that if $c_1(M) = 0$, the Kähler manifold M admits a Ricci-flat metric. This is proved by Yau (1977). Calabi–Yau manifolds with $\dim_{\mathbb{C}} M = 3$ have been proposed as candidates for superstring compactification (see Horowitz (1986) and Candelas (1988)).

8.5.3 The holonomy group of Kähler manifolds

Before we close this section, we briefly look at the holonomy groups of Kähler manifolds. Let (M, g) be a Hermitian manifold with $\dim_{\mathbb{C}} M = m$. Take a vector $X \in T_pM^+$ and parallel transport it along a loop c at p. Then we end up with a vector $X' \in T_pM^+$ given by $X'^{\mu} = X^{\nu}h_{\nu}{}^{\mu}(c)$. Note that ∇ does not mix the holomorphic indices with antiholomorphic indices, hence X' has no components in T_pM^-. Moreover, ∇ preserves the length of a vector. These facts tell us that $(h_{\mu}{}^{\nu}(c))$ is contained in $\mathrm{U}(m) \subset \mathrm{O}(2m)$.

Theorem 8.30 If g is the Ricci-flat metric of an m-dimensional Calabi–Yau maifold M, the holonomy group is contained in $\mathrm{SU}(m)$.

Proof: Our proof is sketchy. If $X = X^{\mu}\partial/\partial z^{\mu} \in T_pM^+$ is parallel transported along the small parallelogram in figure 8.5 back to p, we have $X' \in T_pM^+$ whose components are (cf (7.44))

$$X'^{\mu} = X^{\mu} + X^{\nu}R^{\mu}{}_{\nu\kappa\bar{\lambda}}\varepsilon^{\kappa}\bar{\delta}^{\lambda} \tag{8.99}$$

from which we find

$$h_{\mu}{}^{\nu} = \delta_{\mu}{}^{\nu} + R^{\nu}{}_{\mu\kappa\bar{\lambda}}\varepsilon^{\kappa}\bar{\delta}^{\lambda}. \tag{8.100}$$

$\mathrm{U}(m)$ is decomposed as $\mathrm{U}(m) = \mathrm{SU}(m) \times \mathrm{U}(1)$ in the vicinity of the unit element. In particular, the Lie algebra $\mathfrak{u}(m) \simeq T_e\,(\mathrm{U}(m))$ is separated into

$$\mathfrak{u}(m) = \mathfrak{su}(m) \oplus \mathfrak{u}(1). \tag{8.101}$$

$\mathfrak{su}(m)$ is the traceless part of $\mathfrak{u}(m)$ while $\mathfrak{u}(1)$ contains the trace. Since the present metric is Ricci-flat, the $\mathfrak{u}(1)$ part vanishes,

$$R^{\kappa}{}_{\kappa\mu\bar{\nu}}\varepsilon^{\mu}\bar{\delta}^{\nu} = \mathfrak{R}_{\mu\bar{\nu}}\varepsilon^{\mu}\bar{\delta}^{\nu} = 0.$$

This shows that the holonomy group is contained in $\mathrm{SU}(m)$. ∎

[*Remark:* Strictly speaking, we have only shown that the restricted holonomy group is contained in $\mathrm{SU}(m)$. This statement remains true even when M is multiply connected.]

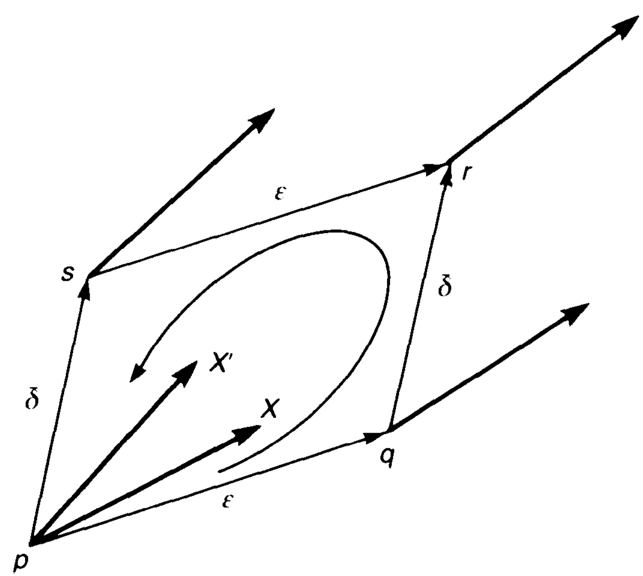


Figure 8.5 $X \in T_pM^+$ is parallel transported along $pqrs$ and comes back as a vector $X' \in T_pM^+$.

8.6 Harmonic forms and $\bar{\partial}$-cohomology groups

The (r, s)**th $\bar{\partial}$-cohomology group** has been defined by

$$H^{r,s}_{\bar{\partial}}(M) \equiv Z^{r,s}_{\bar{\partial}}(M)/B^{r,s}_{\bar{\partial}}(M). \tag{8.102}$$

An element $[\omega] \in H^{r,s}_{\bar{\partial}}(M)$ is an equivalence class of $\bar{\partial}$-closed forms of bidegree (r, s) which differ from ω by a $\bar{\partial}$-exact form,

$$[\omega] = \{\eta \in \Omega^{r,s}(M) | \omega - \eta = \bar{\partial}\psi, \ \psi \in \Omega^{r,s-1}(M)\}. \tag{8.103}$$

Clearly $H^{r,s}_{\bar{\partial}}(M)$ is a complex vector space. Similarly to the de Rham cohomology groups, the $\bar{\partial}$-cohomology groups of $\mathbb{C}^m$ are trivial, that is, all the closed (r, s)-forms are exact. The $\bar{\partial}$-cohomology groups measure the topological non-triviality of a complex manifold M.

8.6.1 The adjoint operators $\partial^\dagger$ and $\bar{\partial}^\dagger$

Let M be a Hermitian manifold with $\dim_{\mathbb{C}} M = m$. Define the inner product between $\alpha, \beta \in \Omega^{r,s}(M)$ $(0 \leqslant r, s \leqslant m)$ by

$$(\alpha, \beta) \equiv \int_M \alpha \wedge \bar{*}\beta \tag{8.104}$$

where $\bar{*} : \Omega^{r,s}(M) \to \Omega^{m-r,m-s}(M)$ is the **Hodge star** defined by

$$\bar{*}\beta \equiv \overline{*\beta} = *\bar{\beta} \tag{8.105}$$

where $*\beta$ is computed according to (7.173) extended to $\Omega^{r+s}(M)^{\mathbb{C}}$. [*Remark:* $*$ maps an (r, s)-form to an $(m - s, m - r)$-form since it acts on a basis of $\Omega^{r,s}(M)$, up to an irrelevant factor, as

$$*\mathrm{d}z^{\mu_1} \wedge \ldots \wedge \mathrm{d}z^{\mu_r} \wedge \mathrm{d}\bar{z}^{\nu_1} \wedge \ldots \wedge \mathrm{d}\bar{z}^{\nu_s} \sim \varepsilon^{\mu_1 \ldots \mu_r}{}_{\bar{\mu}_{r+1} \ldots \bar{\mu}_m} \varepsilon^{\bar{\nu}_1 \ldots \bar{\nu}_s}{}_{\nu_{s+1} \ldots \nu_m}$$
$$\times \, \mathrm{d}\bar{z}^{\mu_{r+1}} \wedge \ldots \wedge \mathrm{d}\bar{z}^{\mu_m} \wedge \mathrm{d}z^{\nu_{s+1}} \wedge \ldots \wedge \mathrm{d}z^{\nu_m}.$$

Note that the above ε-symbols are only non-vanishing components in a Hermitian manifold. Now it follows that $\bar{*} : \Omega^{r,s}(M) \to \Omega^{m-r,m-s}(M)$.]

We define the adjoint operators $\partial^\dagger$ and $\bar{\partial}^\dagger$ of ∂ and $\bar{\partial}$ by

$$(\alpha, \partial\beta) = (\partial^\dagger\alpha, \beta), \ (\alpha, \bar{\partial}\beta) = (\bar{\partial}^\dagger\alpha, \beta). \tag{8.106}$$

The operators $\partial^\dagger$ and $\bar{\partial}^\dagger$ change the bidegrees as $\partial^\dagger : \Omega^{r,s}(M) \to \Omega^{r-1,s}(M)$ and $\bar{\partial}^\dagger : \Omega^{r,s}(M) \to \Omega^{r,s-1}(M)$. Clearly $\mathrm{d}^\dagger = \partial^\dagger + \bar{\partial}^\dagger$. Noting that a complex manifold M is even dimensional as a differentiable manfiold, we have (see (7.184*a*))

$$\mathrm{d}^\dagger = -*\mathrm{d}*. \tag{8.107}$$

Proposition 8.31 $\partial^\dagger = -*\bar{\partial}*$, $\bar{\partial}^\dagger = -*\partial*$. $\quad$ (8.108)

Proof: Let $\omega \in \Omega^{r-1,s}(M)$ and $\psi \in \Omega^{r,s}(M)$. If we note that $\omega \wedge \bar{*}\psi \in \Omega^{m-1,m}(M)$ and hence $\bar{\partial}(\omega \wedge \bar{*}\psi) = 0$, we find that

$$\begin{aligned}
\mathrm{d}(\omega \wedge \bar{*}\psi) = \partial(\omega \wedge \bar{*}\psi) &= \mathrm{d}\omega \wedge \bar{*}\psi + (-1)^{r+s-1}\omega \wedge \partial(\bar{*}\psi) \\
&= \partial\omega \wedge \bar{*}\psi + (-1)^{r+s-1} \, \omega \wedge (-1)^{r+s+1}\bar{*}\bar{*}\partial(\bar{*}\psi) \\
&= \partial\omega \wedge \bar{*}\psi + \omega \wedge \overline{**\bar{\partial}*\psi}
\end{aligned} \tag{8.109}$$

where use has been made of the facts $\partial \overline{*\psi} \in \Omega^{2m-r-s-1}(M)$, $\overline{**\beta} = **\beta$ and (7.176a). If (8.109) is integrated over a compact complex manifold M with no boundary, we have

$$0 = (\partial\omega, \psi) + (\omega, \overline{*\partial\overline{*\psi}}).$$

The second term is

$$(\omega, \overline{*\partial*\bar{\psi}}) = (\omega, \overline{*\bar{\partial}*\bar{\bar{\psi}}}) = (\omega, *\bar{\partial}*\psi).$$

We finally find $0 = (\partial\omega, \psi) + (\omega, *\bar{\partial}*\psi)$, namely $\partial^{\dagger} = -*\bar{\partial}*$. The other formula $\bar{\partial}^{\dagger} = -*\partial*$ follows similarly. ■

As a corollary of proposition 8.31, we have

$$(\partial^{\dagger})^2 = (\bar{\partial}^{\dagger})^2 = 0. \tag{8.110}$$

8.6.2 *Laplacians and the Hodge theorem*

Besides the usual Laplacian $\Delta = (\mathrm{dd}^{\dagger} + \mathrm{d}^{\dagger}\mathrm{d})$, we define other Laplacians Δ_{∂} and $\Delta_{\bar{\partial}}$ on a Hermitian manifold,

$$\Delta_{\partial} \equiv (\partial + \partial^{\dagger})^2 = \partial\partial^{\dagger} + \partial^{\dagger}\partial \tag{8.111a}$$

$$\Delta_{\bar{\partial}} \equiv (\bar{\partial} + \bar{\partial}^{\dagger})^2 = \bar{\partial}\bar{\partial}^{\dagger} + \bar{\partial}^{\dagger}\bar{\partial}. \tag{8.111b}$$

An (r, s)-form ω which satisfies $\Delta_{\partial}\omega = 0$ $(\Delta_{\bar{\partial}}\omega = 0)$ is said to be **∂-harmonic** (**$\bar{\partial}$-harmonic**). If $\Delta_{\partial}\omega = 0$ $(\Delta_{\bar{\partial}}\omega = 0)$, ω satisfies $\partial\omega = \partial^{\dagger}\omega = 0$ $(\bar{\partial}\omega = \bar{\partial}^{\dagger}\omega = 0)$.

We have the complex version of the Hodge decomposition. Let $\mathrm{Harm}^{r,s}_{\bar{\partial}}(M)$ be the set of $\bar{\partial}$-harmonic (r, s)-forms,

$$\mathrm{Harm}^{r,s}_{\bar{\partial}}(M) \equiv \{\omega \in \Omega^{r,s}(M) | \Delta_{\bar{\partial}}\omega = 0\}. \tag{8.112}$$

Theorem 8.32 (**Hodge's theorem**) $\Omega^{r,s}(M)$ has a *unique* orthogonal decomposition

$$\Omega^{r,s}(M) = \bar{\partial}\Omega^{r,s-1}(M) \oplus \bar{\partial}^{\dagger}\Omega^{r,s+1}(M) \oplus \mathrm{Harm}^{r,s}_{\bar{\partial}}(M) \tag{8.113a}$$

namely an (r, s)-form ω is uniquely expressed as

$$\omega = \bar{\partial}\alpha + \bar{\partial}^{\dagger}\beta + \gamma \tag{8.113b}$$

where $\alpha \in \Omega^{r,s-1}(M)$, $\beta \in \Omega^{r,s+1}(M)$ and $\gamma \in \mathrm{Harm}^{r,s}_{\bar{\partial}}(M)$.

The proof is found in lecture 22, Schwartz (1986). If ω is $\bar{\partial}$-closed, we have $\bar{\partial}\omega = \bar{\partial}\bar{\partial}^{\dagger}\beta = 0$. Then $0 = \langle\beta, \bar{\partial}\bar{\partial}^{\dagger}\beta\rangle = \langle\bar{\partial}^{\dagger}\beta, \bar{\partial}^{\dagger}\beta\rangle \geqslant 0$ implies $\bar{\partial}^{\dagger}\beta = 0$. Thus any closed (r, s)-form ω is written as $\omega = \gamma + \bar{\partial}\alpha$, $\alpha \in \Omega^{r,s-1}(M)$. This shows that $H^{r,s}_{\bar{\partial}}(M) \subset \mathrm{Harm}^{r,s}_{\bar{\partial}}(M)$. Note also that $\mathrm{Harm}^{r,s}_{\bar{\partial}}(M) \subset Z^{r,s}_{\bar{\partial}}(M)$ since $\bar{\partial}\gamma = 0$ for $\gamma \in \mathrm{Harm}^{r,s}_{\bar{\partial}}(M)$. Moreover, $\mathrm{Harm}^{r,s}_{\bar{\partial}}(M) \cap B^{r,s}_{\bar{\partial}}(M) = \varnothing$ since $B^{r,s}_{\bar{\partial}}(M) = \bar{\partial}\Omega^{r,s-1}(M)$ is orthogonal to $\mathrm{Harm}^{r,s}_{\bar{\partial}}(M)$. Then it follows that $\mathrm{Harm}^{r,s}_{\bar{\partial}}(M) \cong H^{r,s}_{\bar{\partial}}(M)$. If

$P : \Omega^{r,s}(M) \to \mathrm{Harm}^{r,s}_{\bar\partial}(M)$ denotes the projection operator to a harmonic (r, s)-form, $[\omega] \in H^{r,s}_{\bar\partial}(M)$ has a unique harmonic representative $P\omega \in \mathrm{Harm}^{r,s}_{\bar\partial}(M)$.

8.6.3 Laplacians on a Kähler manifold

In a general Hermitian manifold, there exist no particular relationships among the Laplacians Δ, Δ_∂ and $\Delta_{\bar\partial}$. However, if M is a Kähler manifold, they are essentially the *same*. [Note that the Levi-Civita connection is compatible with the Hermitian connection in a Kähler manifold.]

Theorem 8.33 Let M be a Kähler manifold. Then

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}. \tag{8.114}$$

The proof requires some technicalities and we simply refer to Schwartz (1986) and Goldberg (1962). This theorem puts constraints on the cohomology groups of a Kähler manifold M. A form ω which satisfies $\bar\partial\omega = \bar\partial^\dagger\omega = 0$ also satisfies $\partial\omega = \partial^\dagger\omega = 0$. Let ω be a holomorphic p-form; $\bar\partial\omega = 0$. Since ω contains no $\mathrm{d}\bar z^\mu$ in its expansion, we have $\bar\partial^\dagger\omega = 0$, hence $\Delta_{\bar\partial}\omega = (\bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial)\omega = 0$. According to theorem 8.33, we then have $\Delta\omega = 0$, that is *any holomorphic form is automatically harmonic* with respect to the Kähler metric. Conversely $\Delta\omega = 0$ implies $\bar\partial\omega = 0$, hence every harmonic form of bidegree $(p, 0)$ is holomorphic.

8.6.4 The Hodge numbers of Kähler manifolds

The complex dimension of $H^{r,s}_{\bar\partial}(M)$ is called the **Hodge number** $b^{r,s}$. The cohomolgy groups of a complex manifold are summarised by the **Hodge diamond**,

$$\left(\begin{array}{ccccccc}
 & & & b^{m,m} & & & \\
 & & b^{m,m-1} & & b^{m-1,m} & & \\
 & & & \cdots & & & \\
b^{m,0} & b^{m-1,1} & & \cdots & & b^{1,m-1} & b^{0,m} \\
 & & & \cdots & & & \\
 & & b^{1,0} & & b^{0,1} & & \\
 & & & b^{0,0} & & &
\end{array}\right). \tag{8.115}$$

These $(m + 1)^2$ Hodge numbers are far from independent as we shall see below.

Theorem 8.34 Let M be a Kähler manifold with $\dim_{\mathbb{C}} M = m$. Then the Hodge numbers satisfy

$$(a)\ b^{r,s} = b^{s,r} \tag{8.116}$$

$$(b)\ b^{r,s} = b^{m-r,m-s}. \tag{8.117}$$

Proof: (*a*) If $\omega \in \Omega^{r,s}(M)$ is harmonic, it satisfies $\Delta_{\bar\partial}\omega = \Delta_{\partial}\omega = 0$. Then the (s, r)-form $\bar\omega$ is also harmonic, $\Delta_{\bar\partial}\bar\omega = 0$ since $\Delta_{\bar\partial}\bar\omega = \overline{\Delta_{\partial}\omega} = \overline{\Delta_{\bar\partial}\omega} = 0$ (note that $\Delta_{\partial} = \Delta_{\bar\partial}$). Thus for any harmonic form of bidegree (r, s), there exists a harmonic form of bidegree (s, r) and vice versa. Thus it follows that $b^{s,r} = b^{r,s}$.

(*b*) Let $\omega \in H^{r,s}_{\bar\partial}(M)$ and $\psi \in H^{m-r,m-s}_{\bar\partial}(M)$. Then $\omega \wedge \psi$ is a volume element and it can be shown (Schwartz 1986) that $\int_M \omega \wedge \psi$ defines a *non-singular* map $H^{r,s}_{\bar\partial}(M) \times H^{m-r,m-s}_{\bar\partial}(M) \to \mathbb{C}$, hence the duality between $H^{r,s}_{\bar\partial}(M)$ and $H^{m-r,m-s}_{\bar\partial}(M)$. This shows that $H^{r,s}_{\bar\partial}(M)$ is isomorphic to $H^{m-r,m-s}_{\bar\partial}(M)$ as a vector space and we have $\dim_{\mathbb{C}} H^{r,s}_{\bar\partial}(M) = \dim_{\mathbb{C}} H^{m-r,m-s}_{\bar\partial}(M)$, hence $b^{r,s} = b^{m-r,m-s}$. ■

Accordingly the Hodge diamond ot a Kähler manifold is symmetric about the vertical and horizontal lines. These symmetries reduce the number of independent Hodge numbers to $(\frac{1}{2}m + 1)^2$ if m is even and $\frac{1}{4}(m + 1)(m + 3)$ if m is odd.

In a general Hermitian manifold, there are no direct relations between the Betti numbers and the Hodge numbers. If M is a Kähler manifold, however, theorem 8.35 establishes close relationships between them.

Theorem 8.35 Let M be a Kähler manifold with $\dim_{\mathbb{C}} M = m$ and $\partial M = \varnothing$. Then the Betti numbers b^p $(1 \leqslant p \leqslant 2m)$ satisfy the following conditions;

(*a*) $b^p = \sum_{r+s=p} b^{r,s}$ (8.118)

(*b*) b^{2p-1} is even $\qquad (1 \leqslant p \leqslant m)$ (8.119)

(*c*) $b^{2p} \geqslant 1 \qquad (1 \leqslant p \leqslant m).$ (8.120)

Proof: (*a*) $H^{r,s}_{\bar\partial}(M)$ is a complex vector space spanned by $\Delta_{\bar\partial}$-harmonic (r, s)-forms, $H^{r,s}_{\bar\partial}(M) = \{[\omega] | \omega \in \Omega^{r,s}(M),\ \Delta_{\bar\partial}\omega = 0\}$. On the other hand, $H^p(M)$ is a real vector space spanned by Δ-harmonic p-forms, $H^p(M) = \{[\omega] | \omega \in \Omega^p(M), \Delta\omega = 0\}$. Then the complexification of $H^p(M)$ is $H^p(M)^{\mathbb{C}} = \{[\omega] | \omega \in \Omega^p(M)^{\mathbb{C}}, \Delta\omega = 0\}$. Since M is Kähler, any form ω which satisfies $\Delta_{\bar\partial}\omega = 0$ also satsifies $\Delta\omega = 0$ and vice versa. Since

$$\Omega^p(M)^{\mathbb{C}} = \bigoplus_{r+s=p} \Omega^{r,s}(M)$$

we find that

$$H^p(M)^{\mathbb{C}} = \bigoplus_{r+s=p} H^{r,s}(M). \tag{8.121}$$

Noting that $\dim_{\mathbb{R}} H^p(M) = \dim_{\mathbb{C}} H^p(M)^{\mathbb{C}}$, we obtain $b^p = \sum_{r+s=p} b^{r,s}$.

(*b*) From (*a*) and (8.116), it follows that

$$b^{2p-1} = \sum_{r+s=2p-1} b^{r,s} = 2 \sum_{\substack{r+s=2p-1 \\ r>s}} b^{r,s}.$$

Thus b^{2p-1} must be even.

(c) The crucial observation is that the Kähler form Ω is a closed *real* two-form, $\mathrm{d}\Omega = 0$, and the real $2p$-form

$$\Omega^p \equiv \underbrace{\Omega \wedge \ldots \wedge \Omega}_{p}$$

is also closed, $\mathrm{d}\Omega^p = 0$. We show that Ω^p is not exact. Suppose $\Omega^p = \mathrm{d}\eta$ for some $\eta \in \Omega^{2p-1}(M)$. Then $\Omega^m = \Omega^{m-p} \wedge \Omega^p = \mathrm{d}(\Omega^{m-p} \wedge \eta)$. By Stokes' theorem, we have

$$\int_M \Omega^m = \int_M \mathrm{d}(\Omega^{m-p} \wedge \eta) = \int_{\partial M} \Omega^{m-p} \wedge \eta = 0.$$

Since the LHS is the volume of M, this is in contradiction. Thus there is at least one non-trivial element of $H^{2p}(M)$ and we have proved that $b^{2p} \geqslant 1$. ■

If a Kähler manifold is Ricci-flat, there exists an extra relationship among the Hodge numbers, which further reduces the independent Hodge numbers (Horowitz (1986) and Candelas (1988)).

8.7 Almost complex manifolds

This and the next section deal with spaces which are closely related to complex manifolds. These are somewhat specialised topics and may be omitted on a first reading.

8.7.1 *Definitions*

There are some differentiable manifolds which carry a similar structure to complex manifolds. To study these manifolds, we somewhat relax the condition (8.16) and require a weaker condition below.

Definition 8.36 Let M be a differentiable manifold. The pair (M, J), or simply M, is called an **almost complex manifold** if there exists a tensor field J of type $(1, 1)$ such that at each point p of M, $J_p^2 = -\mathbb{1}_p$. The tensor field J is also called the **almost complex structure**.

Since $J_p^2 = -\mathbb{1}_p$, J_p has eigenvalues $\pm\mathrm{i}$. If there are m i, then there must be an equal number of $-\mathrm{i}$, hence J_p is a $2m \times 2m$ matrix. Thus M is an even-dimensional manifold. Note that not all even-dimensional manifolds are almost complex manifolds. For example, S^4 is not an almost complex manifold (Steenrod 1951). Note also that we now require a weaker condition $J_p^2 = -1_p$. Of course the tensor J_p defined by (8.16) satisfies $J_p^2 = -1_p$ hence a complex manifold is an almost complex manifold. There are almost complex manifolds which are *not* complex manifolds. For example, it is known that S^6 admits an almost complex structure, although it is *not* a complex manifold (Fröhlicher 1955).

Let us complexify a tangent space of an almost complex manifold (M, J). Given a linear transformation J_p at T_pM such that $J_p^2 = -1_p$, we extend J_p to a $\mathbb{C}$-linear map defined on $T_pM^{\mathbb{C}}$. J_p defined on $T_pM^{\mathbb{C}}$ also satisfies $J_p^2 = -1_p$,

$$J_p^2(X + \mathrm{i}Y) = J_p^2 X + \mathrm{i}J_p^2 Y = -X + \mathrm{i}(-Y) = -(X + \mathrm{i}Y)$$

where $X, Y \in T_pM$. Let us divide $T_pM^{\mathbb{C}}$ into two disjoint vector subspaces, according to the eigenvalue of J_p,

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^- \tag{8.122}$$

where

$$T_pM^{\pm} = \{Z \in T_pM^{\mathbb{C}} | J_pZ = \pm \mathrm{i}Z\}. \tag{8.123}$$

Any vector $V \in T_pM^{\mathbb{C}}$ is written as $V = W_1 + \overline{W_2}$, where $W_1, W_2 \in T_pM^+$. Note that $J_pV = \mathrm{i}W_1 - \mathrm{i}\overline{W_2}$. At this stage the reader might have noticed that we can follow the classification scheme of vectors and vector fields developed for the complex manifolds in §8.2. In fact, the only difference is that on a complex manifold the almost complex structure is explicitly given by (8.18), while on an almost complex manifold, it is required to satisfy the less strict condition $J_p^2 = -1_p$. To classify the complexified tangent spaces and complexified vector spaces, we only need the latter condition. Accordingly we separate $T_pM^{\mathbb{C}}$ into $T_pM^{\pm}$ and $\mathcal{X}(M)^{\mathbb{C}}$ into $\mathcal{X}(M)^{\pm}$, although there does not necessarily exist a basis of T_pM^+ of the form $\{\partial/\partial z^{\mu}\}$. For example, we may still define the projection operators

$$\mathcal{P}^{\pm} \equiv \tfrac{1}{2}(1_p \mp \mathrm{i}J_p) : T_pM^{\mathbb{C}} \rightarrow T_pM^{\pm}. \tag{8.124}$$

We call a vector in T_pM^+ (T_pM^-) a holomorphic (antiholomorphic) vector and a vector field in $\mathcal{X}(M)^+(\mathcal{X}(M)^-)$ a holomorphic (antiholomorphic) vector field.

Definition 8.37 Let (M, J) be an almost complex manifold. If the Lie bracket of any holomorphic vector fields $X, Y \in \mathcal{X}^+(M)$ is again a holomorphic vector field, $[X, Y] \in \mathcal{X}^+(M)$, the almost complex structure J is said to be **integrable**.

Let (M, J) be an almost complex manifold. Define the **Nijenhuis tensor field** $N : \mathcal{X}(M) \otimes \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ by

$$N(X, Y) \equiv [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]. \quad (8.125)$$

Given a basis $\{\partial/\partial x^\mu\}$ and the dual basis $\{\mathrm{d}x^\mu\}$, the almost complex structure is expressed as $J = J_\mu{}^\nu \mathrm{d}x^\mu \otimes \partial/\partial x^\nu$. The component expression of N is

$$\begin{aligned} N(X, Y) &= (X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu) e_\mu \\ &\quad + J_\lambda{}^\mu \{J_\kappa{}^\nu X^\kappa \partial_\nu Y^\lambda - Y^\nu \partial_\nu (J_\kappa{}^\lambda X^\kappa)\} e_\mu \\ &\quad + J_\lambda{}^\mu \{X^\nu \partial_\nu (J_\kappa{}^\lambda Y^\kappa) - J_\kappa{}^\nu Y^\kappa \partial_\nu X^\lambda\} e_\mu \\ &\quad - \{J_\kappa{}^\nu X^\kappa \partial_\nu (J_\lambda{}^\mu Y^\lambda) - J_\kappa{}^\nu Y^\kappa \partial_\nu (J_\lambda{}^\mu X^\lambda)\} e_\mu \\ &= X^\kappa Y^\nu [-J_\lambda{}^\mu (\partial_\nu J_\kappa{}^\lambda) + J_\lambda{}^\mu (\partial_\kappa J_\nu{}^\lambda) \\ &\quad - J_\kappa{}^\lambda (\partial_\lambda J_\nu{}^\mu) + J_\nu{}^\lambda (\partial_\lambda J_\kappa{}^\mu)] e_\mu. \end{aligned} \quad (8.126)$$

Thus N is indeed linear in X and Y and hence a tensor. If J is a complex structure, J is given by (8.18) and the Nijenhuis tensor field trivially vanishes.

Theorem 8.38 An almost complex structure J on a manifold M is integrable if and only if $N(A, B) = 0$ for any $A, B \in \mathcal{X}(M)$.

Proof: Let $Z = X + \mathrm{i}Y$, $W = U + \mathrm{i}V \in \mathcal{X}^{\mathbb{C}}(M)$. We extend the Nijenhuis tensor field so that its action on vector fields in $\mathcal{X}^{\mathbb{C}}(M)$ is given by

$$\begin{aligned} N(Z, W) &= [Z, W] + J[JZ, W] + J[Z, JW] - [JZ, JW] \\ &= \{N(X, U) - N(Y, V)\} + \mathrm{i}\{N(X, V) + N(Y, U)\}. \end{aligned} \quad (8.127)$$

Suppose that $N(A, B) = 0$ for any $A, B \in \mathcal{X}(M)$. From (8.127), it turns out that $N(Z, W) = 0$ for $Z, W \in \mathcal{X}^{\mathbb{C}}(M)$. Let $Z, W \in \mathcal{X}^+(M) \subset \mathcal{X}^{\mathbb{C}}(M)$. Since $JZ = \mathrm{i}Z$ and $JW = \mathrm{i}W$, we have $N(Z, W) = 2\{[Z, W] + \mathrm{i}J[Z, W]\}$. By assumption, $N(Z, W) = 0$ and we find $[Z, W] = -\mathrm{i}J[Z, W]$ or $J[Z, W] = \mathrm{i}[Z, W]$, that is, $[Z, W] \in \mathcal{X}^+(M)$. Thus the almost complex structure is integrable.

Conversely suppose that J is integrable. Since $\mathcal{X}^{\mathbb{C}}(M)$ is a direct sum of $\mathcal{X}^+(M)$ and $\mathcal{X}^-(M)$, we can separate $Z, W \in \mathcal{X}^{\mathbb{C}}(M)$ as $Z = Z^+ + Z^-$ and $W = W^+ + W^-$. Then

$$N(Z, W) = N(Z^+, W^+) + N(Z^+, W^-) + N(Z^-, W^+) + N(Z^-, W^-).$$

Since $JZ^\pm = \pm\mathrm{i}Z^\pm$ and $JW^\pm = \pm\mathrm{i}W^\pm$, it is easy to see that $N(Z^+, W^-) = N(Z^-, W^+) = 0$. We also have

$$\begin{aligned} N(Z^+, W^+) &= [Z^+, W^+] + J[\mathrm{i}Z^+, W^+] + J[Z^+, \mathrm{i}W^+] - [\mathrm{i}Z^+, \mathrm{i}W^+] \\ &= 2[Z^+, W^+] - 2[Z^+, W^+] = 0 \end{aligned}$$

since $J[Z^+, W^+] = \mathrm{i}[Z^+, W^+]$. Similarly $N(Z^-, W^-)$ vanishes and we have shown that $N(Z, W) = 0$ for any $Z, W \in \mathcal{X}^{\mathbb{C}}(M)$. In particular, it should vanish for $Z, W \in \mathcal{X}(M)$. ■

If M is a complex manifold, the complex structure J is a constant tensor field and the Nijenhuis tensor field vanishes. What about the converse? We now state an important (and difficult to prove) theorem.

Theorem 8.39 (Newlander and Nirenberg 1957) Let (M, J) be a $2m$-dimensional almost complex manifold. If J is integrable, the manifold M is a complex manifold with the almost complex structure J.

In summary we have,

$$\begin{array}{c}\text{Integrable almost}\\ \text{complex structure}\end{array} = \begin{array}{c}\text{Vanishing Nijenhuis}\\ \text{tensor field}\end{array} = \text{Complex manifold.}$$

8.8 Orbifolds

Let M be a manifold and let G be a *discrete* group which acts on M. Then the quotient space $\Gamma \equiv M/G$ is called an **orbifold**. As we will see below there are fixed points in M, which do not transform under the action of G. These points are singular and the orbifold is not a manifold in general. Thus, even though we start with a simple manifold M, the orbifold M/G may have quite a complicated topology.

8.8.1 One-dimensional examples

To obtain a concrete idea, let us consider a simple example. Take $M = \mathbb{R}^2$ which is to be identified with the complex plane $\mathbb{C}$. Let us take $G = \mathbb{Z}_3$ and identify the points z, $\mathrm{e}^{\mathrm{i}2\pi/3}z$ and $\mathrm{e}^{\mathrm{i}4\pi/3}z$. The orbifold M/G consists of a third of the complex plane and after the identification of the edges we end up with a cone, see figure 8.6. It is interesting to see what the holonomy group of this orbifold is. We use the flat connection induced by the Euclidean metric of $\mathbb{C}$. Then after the parallel transport of a vector V along the loop C (this is indeed a loop!) we obtain a vector $\widetilde{V}$ which is different from V after the identification. Observe that the angle between V and $\widetilde{V}$ is $2\pi/3$. It is easy to verify that the holomony group is $\mathbb{Z}_3$. Since the holonomy is trivial for the loop C_0 which does not enircle the origin, we find that the curvature is singular at the origin (recall that the curvature measures the non-triviality of the holonomy, see §7.3). In general the fixed points (the origin in the present case) are singular points of the curvature. Note, however, that $\mathbb{C}/\mathbb{Z}_3$ *is* a manifold since it has an open covering homeomorphic to $\mathbb{R}^2$.

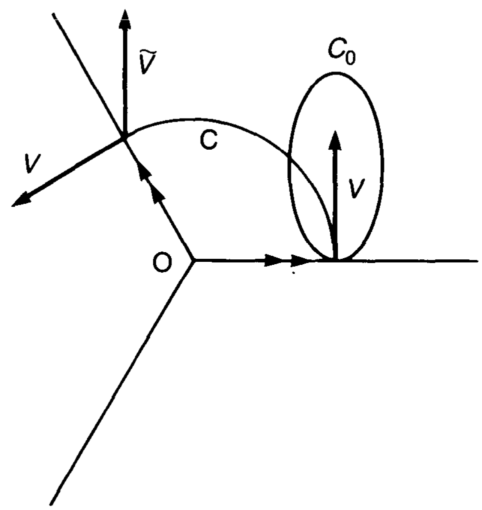


Figure 8.6 The orbifold $\mathbb{C}/\mathbb{Z}_3$ is a third of the complex plane. The edges of the orbifold are identified as shown in the figure. V becomes a vector $\widetilde{V}$ after parallel transportation along C. The angle between V and $\widetilde{V}$ is $2\pi/3$.

A less trivial example is obtained by taking the torus as the manifold. We identify the points z and $z + m + ne^{\pi i/3}$ in the complex plane; see figure 8.7(a). If we identify the edges of the parallelogram OPQR, we have the torus T^2. Let $\mathbb{Z}_3$ act on T^2 as $\alpha : z \mapsto e^{2\pi i/3}z$. We find that there are three inequivalent fixed points $z = (n/\sqrt{3})e^{i\pi/6}$ where $n = 0$, 1 and 2. This orbifold $\Gamma = \mathbb{C}/\mathbb{Z}_3$ consists of two triangles surrounding a hollow; see figure 8.7(b). If the flat connection induced by the flat metric of the torus is employed to define the parallel transport of vectors, we find that the holonomy around each fixed point is $\mathbb{Z}_3$.

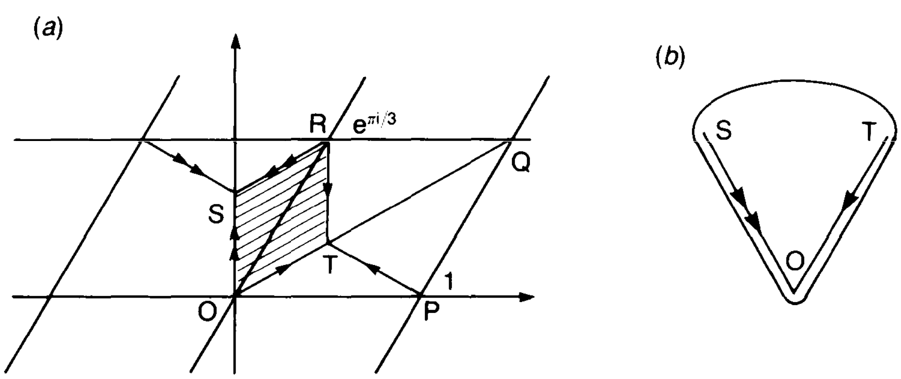


Figure 8.7 Under the action of $\mathbb{Z}_3$, points of the torus T are identified. The shaded area is the orbifold $T/\mathbb{Z}_3$. If the edges of the orbifold are identified, we end up with the object in figure 8.7(b), which is homeomorphic to the sphere S^2.

8.8.2 Three-dimensional examples

Orbifolds with three complex dimensions have been proposed as candidates for superstring compactification. The detailed treatment of this subject is outside the scope of this book and the reader should consult Dixson *et al* (1985, 1986) and Green *et al* (1987).

Let $T = \mathbb{C}^3/L$ be a three-dimensional complex torus, where L is a lattice in $\mathbb{C}^3$. For definiteness, let (z_1, z_2, z_3) be the coordinates of $\mathbb{C}^3$ and identify z_i and $z_i + m + ne^{\pi i/3}$. Under this identification, T is identified with a product of three tori, $T = T_1 \times T_2 \times T_3$. T admits, as before, the action of $\mathbb{Z}_3$ defined by $\alpha : z_i \mapsto e^{2\pi i/3} z_i$. If each z_i takes one of the values 0, $(1/\sqrt{3})e^{i\pi/6}$, $(2/\sqrt{3})e^{i\pi/6}$, the action of α leaves the point (z_i) invariant. Thus there are $3^3 = 27$ fixed points in the orbifold. In the present case, the fixed point is a conical singularity (figure 8.8) and the orbifold cannot be a manifold. [*Remarks:* The appearance of the conical singularity can be understood more easily from a simpler example. Let $(x, y) \in \mathbb{C}^2$ and let $\mathbb{Z}_2$ act on $\mathbb{C}^2$ as $(x, y) \mapsto \pm(x, y)$. Then the orbifold $\Gamma = \mathbb{C}^2/\mathbb{Z}_2$ has a conical singularity at the origin. In fact let $[(x, y)] \rightarrow (x^2, xy, y^2) \equiv (X, Y, Z)$ be an embedding of Γ in $\mathbb{C}^3$. Note that X, Y and Z satisfy a relation $Y^2 = XZ$. If X, Y and Z are thought of as real variables, this is simply the equation of a cone.]

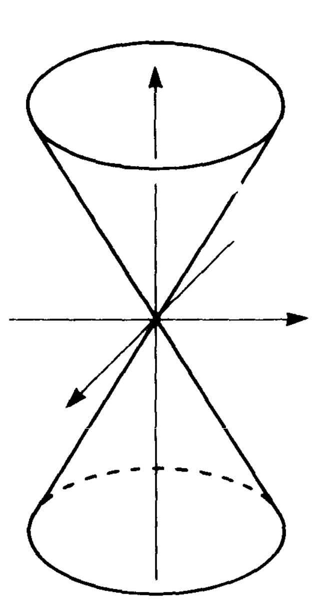


Figure 8.8 The conical singularity. The origin does not look like $\mathbb{R}^n$ or $\mathbb{C}^n$.

9

FIBRE BUNDLES

A manifold is a topological space which looks locally like $\mathbb{R}^m$, but not necessarily so globally. By introducing a chart, we give a local Euclidean structure to a manifold, which enables us to use the conventional calculus of several variables. A fibre bundle is, so to speak, a topological space which looks locally like a direct product of two topological spaces. Many theories in physics, such as general relativity and gauge theories, are described naturally in terms of fibre bundles.

Relevant references are Choquet-Bruhat and DeWitt-Morette (1982), Eguchi *et al* (1980) and Nash and Sen (1983). A complete analysis is found in Kobayashi and Nomizu (1963, 1969).

9.1 Tangent bundles

For clarification, we begin our exposition with a motivating example. A **tangent bundle** TM over an m-dimensional manifold M is a collection of all the tangent spaces of M:

$$TM \equiv \bigcup_{p \in M} T_pM. \tag{9.1}$$

The manifold M over which TM is defined is called the **base space**. Let $\{U_i\}$ be an open covering of M. If $x^\mu = \varphi_i(p)$ is the coordinate on U_i, an element of

$$TU_i \equiv \bigcup_{p \in U_i} T_pM$$

is specified by a point $p \in M$ and a vector $V = V^\mu(p)\partial/\partial x^\mu|_p \in T_pM$. Noting that U_i is homeomorphic to an open subset $\varphi_i(U_i)$ of $\mathbb{R}^m$ and each T_pM is homeomorphic to $\mathbb{R}^m$, we find that TU_i is identified with a direct product $\mathbb{R}^m \times \mathbb{R}^m$ (figure 9.1). If $(p, V) \in TU_i$, the identification is given by $(p, V) \mapsto (x^\mu(p), V^\mu(p))$. TU_i is a $2m$-dimensional differentiable manifold. What is more, TU_i is decomposed into a direct product $U_i \times \mathbb{R}^m$. If we pick up a point u of TU_i, we can systematically decompose the information u contains into a point $p \in M$ and a vector $V \in T_pM$. Thus we are naturally led to the concept of **projection** $\pi : TU_i \to U_i$ (figure 9.1). For any point $u \in TU_i$, $\pi(u)$ is a point $p \in U_i$ at which the vector is defined. The information about the vector is completely lost under the projection. Observe that $\pi^{-1}(p) = T_pM$. In the context of the theory of fibre bundles, T_pM is called the **fibre** at p.

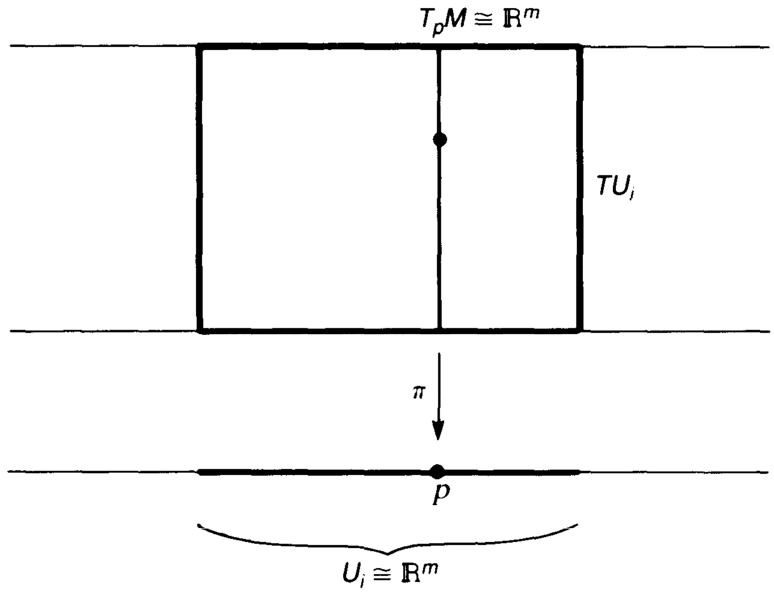


Figure 9.1 A local piece $TU_i \cong \mathbb{R}^m \times \mathbb{R}^m$ of a tangent bundle TM. π projects a vector $V \in T_pM$ to p.

It is obvious by construction that if $M = \mathbb{R}^m$, the tangent bundle itself is expressed as a direct product $\mathbb{R}^m \times \mathbb{R}^m$. However, this is not always the case and the non-trivial structure of the tangent bundle measures the topological non-triviality of M. To see this we have to look not only at a single chart U_i but also at other charts. Let U_j be a chart such that $U_i \cap U_j \neq \varnothing$ and let $y^\mu = \psi(p)$ be the coordinates on U_j. Take a vector $V \in T_pM$ where $p \in U_i \cap U_j$. V has two coordinate presentations,

$$V = V^\mu \partial/\partial x^\mu|_p = \widetilde{V}^\mu \partial/\partial y^\mu|_p. \tag{9.2}$$

It is easy to see that

$$\widetilde{V}^\nu = (\partial y^\nu/\partial x^\mu)_p V^\mu. \tag{9.3}$$

For $\{x^\mu\}$ and $\{y^\nu\}$ to be good coordinate systems, the matrix $(G^\nu{}_\mu) \equiv (\partial y^\nu/\partial x^\mu)$ must be non-singular: $(G^\nu{}_\mu) \in \mathrm{GL}(m, \mathbb{R})$. Thus fibre coordinates are rotated by an element of $\mathrm{GL}(m, \mathbb{R})$ whenever we change the coordinates. The *group* $\mathrm{GL}(m, \mathbb{R})$ is called the **structure group** of TM. In this way fibres are interwoven together to form a tangent bundle, which consequently may have quite a complicated topological structure.

We note *en passant* that the projection π can be defined *globally* on M. It is obvious that $\pi(u) = p$ does not depend on a special coordinate chosen. Thus $\pi : TM \to M$ is defined globally with no reference to local charts.

Let X be a vector field on M. X assigns a vector $X|_p \in T_pM$ at each point $p \in M$. From our viewpoint, X is looked upon as a smooth map $M \to TM$. This map is not utterly arbitrary since a point p must be mapped to a point $u \in TM$ such that $\pi(u) = p$. We define a **section** (or a **cross section**) of TM as a smooth map $s : M \to TM$ such that $\pi s = \mathrm{id}_M$. If a section $s_i : U_i \to TU_i$ is defined only on a chart U_i, it is called a **local section**.

9.2 Fibre bundles

The tangent bundle in the previous section is an example of a more general framework called a fibre bundle. Definitions are now in order.

9.2.1 Definitions

Definition 9.1 A (differentiable) **fibre bundle** (E, π, M, F, G) consists of the following elements:

(i) A differentiable manifold E called the **total space**.
(ii) A differentiable manifold M called the **base space**.
(iii) A differentiable manifold F called the **fibre** (or **typical fibre**).
(iv) A surjection $\pi : E \to M$ called the **projection**. The inverse image $\pi^{-1}(p) \equiv F_p \cong F$ is called the fibre at p.
(v) A Lie group G called the **structure group**, which acts on F on the left.
(vi) A set of open covering $\{U_i\}$ of M with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi\phi_i(p, f) = p$. The map ϕ_i is called the **local trivialisation** since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ *onto* the direct product $U_i \times F$.
(vii) If we write $\phi_i(p, f) = \phi_{i,p}(f)$, the map $\phi_{i,p} : F \to F_p$ is a diffeomorphism. On $U_i \cap U_j \neq \emptyset$, we require that $t_{ij}(p) \equiv \phi_{i,p}^{-1}\phi_{j,p} : F \to F$ be an element of G. Then ϕ_i and ϕ_j are related by a smooth map $t_{ij} : U_i \cap U_j \to G$ as (figure 9.2)

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f). \tag{9.4}$$

$\{t_{ij}\}$ are called the **transition functions**.

[*Remarks:* We often use a shorthand notation $E \xrightarrow{\pi} M$ or simply E to denote a fibre bundle (E, π, M, F, G).

Strictly speaking, the definition of a fibre bundle should be independent of the special covering $\{U_i\}$ of M. In the mathematical literature,

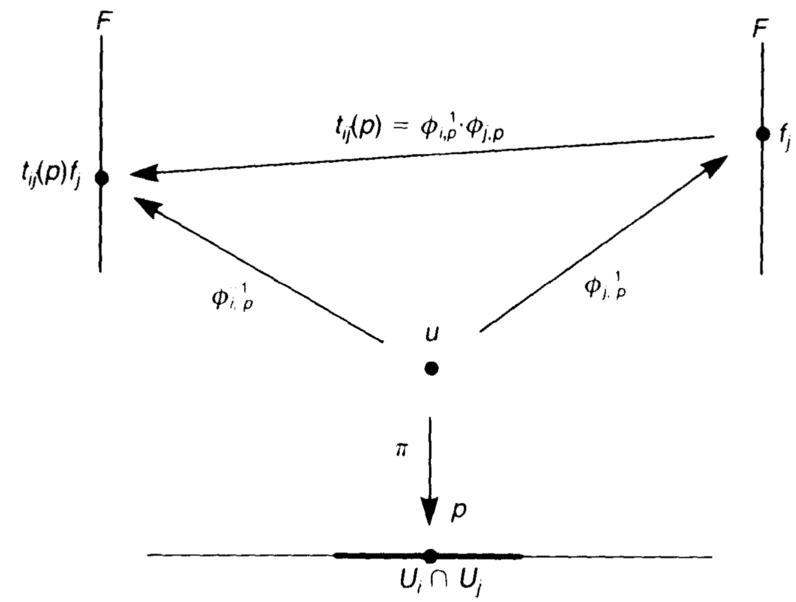


Figure 9.2 On the overlap $U_i \cap U_j$, two elements $f_i, f_j \in F$ are assigned to $u \in \pi^{-1}(p)$, $p \in U_i \cap U_j$. They are related by $t_{ij}(p)$ as $f_i = t_{ij}(p) f_j$.

the above definition is employed to define a **coordinate bundle** $(E, \pi, M, F, G, \{U_i\}, \{\phi_i\})$. Two coordinate bundles $(E, \pi, M, F, G, \{U_i\}, \{\phi_i\})$ and $(E, \pi, M, F, G, \{V_i\}, \{\psi_i\})$ are said to be **equivalent** if $(E, \pi, M, F, G, \{U_i\} \cup \{V_i\}, \{\phi_i\} \cup \{\psi_i\})$ is again a coordinate bundle. A fibre bundle is defined as an equivalence class of coordinate bundles. In practical applications in physics, however, we always employ a certain definite covering and make no distinction between a coordinate bundle and a fibre bundle.]

We need to clarify several points. Let us take a chart U_i of the base space M. $\pi^{-1}(U_i)$ is a direct product diffeomorphic to $U_i \times F$, $\phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F$ being the diffeomorphism. If $U_i \cap U_j \neq \emptyset$, we have two maps ϕ_i and ϕ_j on $U_i \cap U_j$. Let us take a point u such that $\pi(u) = p \in U_i \cap U_j$. We then assign two elements of F, one by ϕ_i^{-1}, the other by ϕ_j^{-1},

$$\phi_i^{-1}(u) = (p, f_i), \ \phi_j^{-1}(u) = (p, f_j) \tag{9.5}$$

see figure 9.2. There exists a map $t_{ij} : U_i \cap U_j \to G$ which relates f_i and f_j as $f_i = t_{ij}(p) f_j$. This is also written as (9.4).

We require that the transition functions satisfy the consistency conditions

$$t_{ii}(p) = \text{identity map} \qquad (p \in U_i) \tag{9.6a}$$

$$t_{ij}(p) = t_{ji}(p)^{-1} \qquad (p \in U_i \cap U_j) \tag{9.6b}$$

$$t_{ij}(p) t_{jk}(p) = t_{ik}(p) \qquad (p \in U_i \cap U_j \cap U_k). \tag{9.6c}$$

Unless these conditions are satisfied, local pieces of a fibre bundle cannot be glued together consistently. If all the transition functions can be taken to be identity maps, the fibre bundle is called a **trivial bundle**. A trivial bundle is a direct product $M \times F$.

Given a fibre bundle $E \xrightarrow{\pi} M$, the possible set of transition functions is obviously far from unique. Let $\{U_i\}$ be a covering of M and $\{\phi_i\}$ and $\{\widetilde{\phi}_i\}$ be two sets of local trivialisations giving rise to the same fibre bundle. The transition functions of respective local trivialisations are

$$t_{ij}(p) = \phi_{i,p}^{-1}\phi_{j,p} \tag{9.7a}$$

$$\widetilde{t}_{ij}(p) = \widetilde{\phi}_{i,p}^{-1}\widetilde{\phi}_{j,p}. \tag{9.7b}$$

Define a map $g_i(p) : F \to F$ at each point $p \in M$ by

$$g_i(p) \equiv \phi_{i,p}^{-1}\widetilde{\phi}_{i,p}. \tag{9.8}$$

We require that $g_i(p)$ be a homeomorphism which belongs to G. This requirement must certainly be fulfilled if $\{\phi_i\}$ and $\{\widetilde{\phi}_i\}$ describe the same fibre bundle. It is easily seen from (9.7) and (9.8) that

$$\widetilde{t}_{ij}(p) = g_i(p)^{-1}t_{ij}(p)g_j(p). \tag{9.9}$$

In the practical situations which we shall encounter later, t_{ij} are the gauge transformations required for pasting local charts together, while g_i corresponds to the gauge degrees of freedom within a chart U_i. If the bundle is trivial, we may put all the transition functions to be identity maps. Then the most general form of the transition functions is

$$t_{ij}(p) = g_i(p)^{-1}g_j(p). \tag{9.10}$$

Let $E \xrightarrow{\pi} M$ be a fibre bundle. A **section** (or a **cross section**) $s : M \to E$ is a smooth map which satisfies $\pi s = \mathrm{id}_M$. Clearly $s(p) = s|_p$ is an element of $F_p = \pi^{-1}(p)$. The set of sections on M is denoted by $\Gamma(M, E)$. If $U \subset M$, we may talk of a **local section** which is defined only on U. $\Gamma(U, E)$ denotes the set of local sections on U. For example, $\Gamma(M, TM)$ is identified with the set of vector fields $\mathcal{X}(M)$. It should be noted that not all fibre bundles admit global sections.

Example 9.2 Let E be a fibre bundle $E \xrightarrow{\pi} S^1$ with a typical fibre $F = [-1, 1]$. Let $U_1 = (0, 2\pi)$ and $U_2 = (-\pi, \pi)$ be an open covering of S^1 and let $A = (0, \pi)$ and $B = (\pi, 2\pi)$ be the intersection $U_1 \cap U_2$, see figure 9.3. The local trivialisations ϕ_1 and ϕ_2 are given by

$$\phi_1^{-1}(u) = (\theta, t) \qquad \phi_2^{-1}(u) = (\theta, t)$$

for $\theta \in A$ and $t \in F$. The transition function $t_{12}(\theta)$, $\theta \in A$, is the identity map $t_{12}(\theta) : t \mapsto t$. We have two choices on B;

(I) $\phi_1^{-1}(u) = (\theta, t) \qquad \phi_2^{-1}(u) = (\theta, t)$

(II) $\phi_1^{-1}(u) = (\theta, t) \qquad \phi_2^{-1}(u) = (\theta, -t)$.

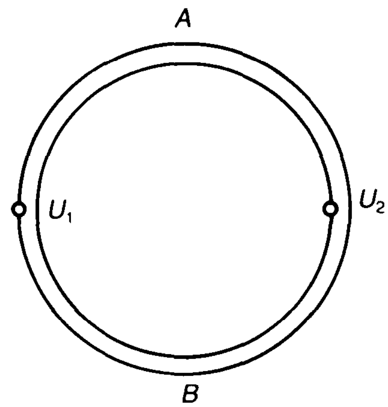


Figure 9.3 The base space S^1 and two charts U_1 and U_2 over which the fibre bundle is trivial.

For case (I), we find that $t_{12}(\theta)$ is the identity map and two pieces of the local bundles are glued together to form a cylinder (figure 9.4(a)). For case (II), we have $t_{12}(\theta) : t \mapsto -t$, $\theta \in B$, and obtain the **Möbius strip** (figure 9.4(b)). Thus a cylinder has the trivial structure group $G = \{e\}$ where e is the identity map of F onto F. The Möbius strip has $G = \{e, g\}$ where $g : t \mapsto -t$. Since $g^2 = e$, we find $G \cong \mathbb{Z}_2$. A cylinder is a trivial bundle $S^1 \times F$, while the Möbius strip is not. [*Remark:* The group $\mathbb{Z}_2$ is not a Lie group. This is the only occasion we use a discrete group for the structure group.]

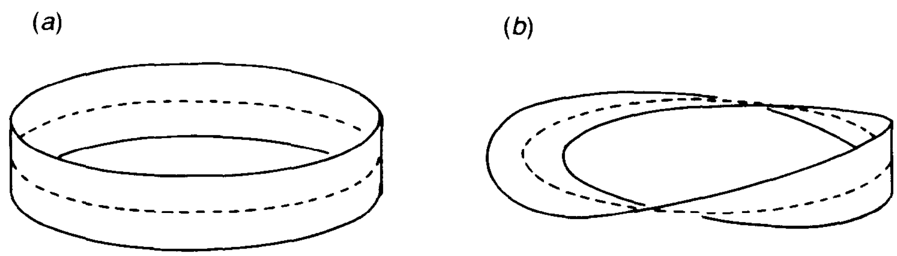


Figure 9.4 Two fibre bundles over S^1: (a) is the cylinder which is a trivial bundle $S^1 \times I$; (b) is the Möbius strip.

9.2.2 Reconstruction of fibre bundle

What is the minimal information required to construct a fibre bundle? We now show that for given M, $\{U_i\}$, $t_{ij}(p)$, F and G, we can reconstruct the fibre bundle (E, π, M, F, G). This amounts to finding a unique π, E, and ϕ_i from given data. Let us define

$$X \equiv \bigcup_i U_i \times F. \tag{9.11}$$

Introduce an equivalent relation $\sim$ between $(p, f) \in U_i \times F$ and $(q, f') \in U_j \times F$ by $(p, f) \sim (q, f')$ if and only if $p = q$ and $f' = t_{ij}(p)f$.

A fibre bundle E is then defined by

$$E = X/\sim. \tag{9.12}$$

Denote an element of E by $[(p, f)]$. The projection is given by

$$\pi : [(p, f)] \mapsto p. \tag{9.13}$$

The local trivialisation $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ is given by

$$\phi_i : (p, f) \mapsto [(p, f)]. \tag{9.14}$$

The reader should verify that E, π and $\{\phi_i\}$ thus defined satisfy all the axioms of fibre bundles. Thus the given data reconstruct a fibre bundle E uniquely.

This procedure may be employed to construct a new fibre bundle from an old one. Let (E, π, M, F, G) be a fibre bundle. Associated with this bundle is a new bundle whose base space is M, transition function $t_{ij}(p)$, structure group G and fibre F' on which G acts. Examples of associated bundles will be given later.

9.2.3 *Bundle maps*

Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be fibre bundles. A *smooth* map $\bar{f} : E' \to E$ is called a **bundle map** if it maps each fibre F'_p of E' onto F_q of E. Then $\bar{f}$ naturally induces a smooth map $f : M' \to M$ such that $f(p) = q$ (figure 9.5). Observe that the diagram

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f} & M \end{array} \qquad \left(\begin{array}{ccc} u & \xrightarrow{\bar{f}} & \bar{f}(u) \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ p & \xrightarrow{f} & q \end{array} \right) \tag{9.15}$$

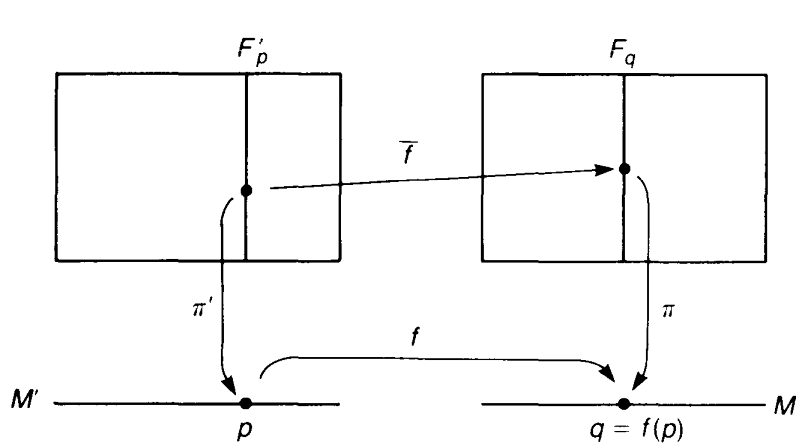


Figure 9.5 A bundle map $\bar{f} : E' \to E$ induces a map $f : M' \to M$.

commutes. [*Caution:* A smooth map $\bar{f} : E' \to E$ is not necessarily a bundle map. It may map $u, v \in F'_p$ of E' to $\bar{f}(u)$ and $\bar{f}(v)$ on *different* fibres of E so that $\pi(\bar{f}(u)) \neq \pi(\bar{f}(v))$.]

9.2.4 *Equivalent bundles*

Two bundles $E' \xrightarrow{\pi'} M$ and $E \xrightarrow{\pi} M$ are **equivalent** if there exists a bundle map $\bar{f} : E' \to E$ such that $f : M \to M$ is the identity map and $\bar{f}$ is a diffeomorphism:

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\mathrm{id}_M} & M \end{array} . \tag{9.16}$$

This definition of equivalent bundles is in harmony with that given in the remarks following definition 9.1.

9.2.5 *Pullback bundles*

Let $E \xrightarrow{\pi} M$ be a fibre bundle with typical fibre F. If a map $f : N \to M$ is given, the pair (E, f) defines a new fibre bundle over N with the same fibre F (figure 9.6). Let f^*E be a subspace of $N \times E$, which consists of points (p, u) such that $f(p) = \pi(u)$. $f^*E \equiv \{(p, u) \in N \times E | f(p) = \pi(u)\}$ is called the **pullback** of E by f. The fibre F_p of f^*E is just a copy of the fibre $F_{f(p)}$ of E. If we define $f^*E \xrightarrow{\pi_1} N$ by $\pi_1 : (p, u) \mapsto p$ and $\pi_2 : f^*E \to E$ by $(p, u) \mapsto u$, the pullback f^*E may be endowed with the structure of a fibre bundle and we obtain the following bundle map,

$$\begin{array}{ccc} f^*E & \xrightarrow{\pi_2} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M \end{array} \qquad \left(\begin{array}{ccc} (p, u) & \xrightarrow{\pi_2} & u \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ p & \xrightarrow{f} & f(p) \end{array} \right). \tag{9.17}$$

The commutativity of the diagram follows since $\pi(\pi_2(p, u)) = \pi(u) = f(p) = f(\pi_1(p, u))$ for $(p, u) \in f^*E$. In particular if $N = M$ and $f = \mathrm{id}_M$, then two fibre bundles f^*E and E are equivalent.

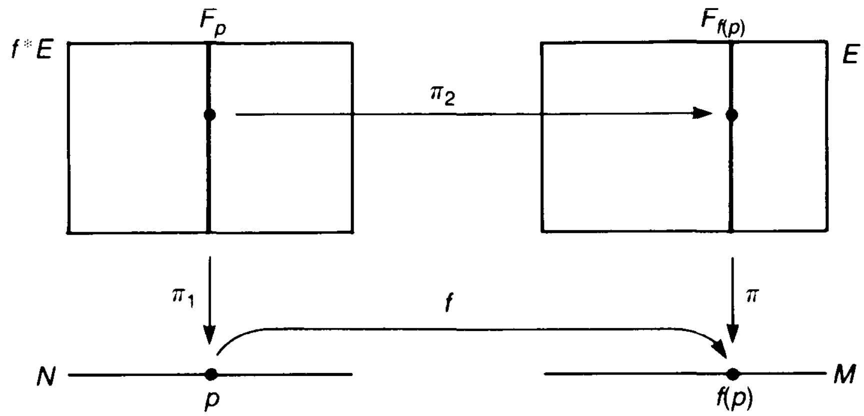


Figure 9.6 Given a fibre bundle $E \xrightarrow{\pi} M$, a map $f : N \to M$ defines a pullback bundle f^*E over N.

Let $\{U_i\}$ be a covering of M and $\{\phi_i\}$ be local trivialisations. $\{f^{-1}(U_i)\}$ defines a covering of N such that f^*E is locally trivial. Take $u \in E$ such that $\pi(u) = f(p) \in U_i$ for some $p \in N$. If $\phi_i^{-1}(u) = (f(p), f_i)$ we find $\psi_i^{-1}(p, u) = (p, f_i)$ where ψ_i is the local trivialisation of f^*E. The transition function t_{ij} at $f(p) \in U_i \cap U_j$ maps f_j to $f_i = t_{ij}(f(p))f_j$. The corresponding transition function t_{ij}^* of f^*E at $p \in f^{-1}(U_i) \cap f^{-1}(U_j)$ also maps f_j to f_i; see figure 9.7. This shows that

$$t_{ij}^*(p) = t_{ij}(f(p)). \tag{9.18}$$

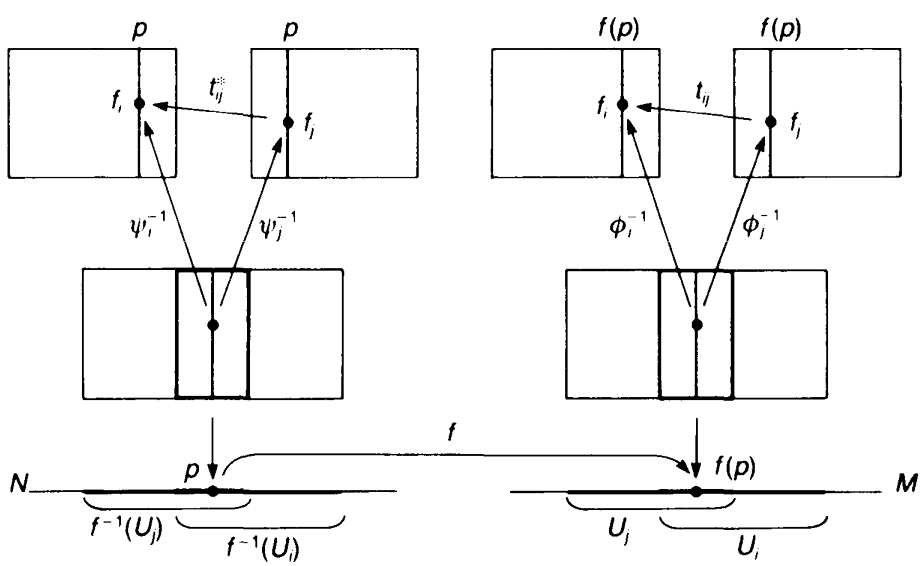


Figure 9.7 The transition function t_{ij}^* of the pullback bundle f^*E is a pullback of the transition function t_{ij} of E.

Example 9.3 Let M and N be differentiable manifolds with $\dim M = \dim N = m$. Let $f : N \to M$ be a smooth map. The map f induces a map $\pi_2 : TN \to TM$ such that the diagram below commutes,

$$\begin{array}{ccc} TN & \xrightarrow{\pi_2} & TM \\ \pi_1 \downarrow & & \downarrow \pi. \\ N & \xrightarrow{f} & M \end{array} \tag{9.19}$$

Let $W = W^\nu \partial/\partial y^\nu$ be a vector of T_pN and $V = V^\mu \partial/\partial x^\mu$ be the corresponding vector of $T_{f(p)}M$. If TN is a pullback bundle $f^*(TM)$, π_2 maps T_pN to $T_{f(p)}M$ *diffeomorphically*. This is possible if and only if π_2 has the maximal rank m at each point of N. Let $\varphi(f(p)) = (f^1(y), \ldots, f^m(y))$ be the coordinates of $f(p)$ in a chart (U, φ) of M, where $y = \psi(p)$ are the coordinates of p in a chart (V, ψ) of N. The maximal rank condition is given by $\det(\partial f^\mu(y)/\partial y^\nu)_p \neq 0$ for $p \in N$.

9.2.6 *Homotopy axiom*

Let f and g be maps from M' to M. They are said to be **homotopic** if there exists a smooth map $F : M' \times [0, 1] \to M$ such that $F(p, 0) = f(p)$ and $F(p, 1) = g(p)$ for any $p \in M'$, see §4.2.

Theorem 9.4 Let $E \xrightarrow{\pi} M$ be a fibre bundle with fibre F and let f and g be homotopic maps from N to M. Then f^*E and g^*E are equivalent bundles over N.

The proof is found in Steenrod (1951). Let M be a manifold which is contractible to a point. Then there exists a homotopy $F : M \times I \to M$ such that

$$F(p, 0) = p, \; F(p, 1) = p_0$$

where $p_0 \in M$ is a fixed point. Let $E \xrightarrow{\pi} M$ be a fibre bundle over M and consider pullback bundles h_0^*E and h_1^*E, where $h_t(p) \equiv F(p, t)$. The fibre bundle h_1^*E is defined over a single point p_0, and hence is a trivial bundle; $h_1^*E \simeq \{p_0\} \times F$. On the other hand $h_0^*E = E$ since h_0 is the identity map. According to theorem 9.4, $h_1^*E = E$ is equivalent to h_0^*E, hence E is a trivial bundle. For example, the tangent bundle $T\mathbb{R}^m$ is trivial. We have obtained the following corollary.

Corollary 9.5 Let $E \xrightarrow{\pi} M$ be a fibre bundle. E is trivial if M is contractible to a point.

9.3 Vector bundles

9.3.1 Definitions and examples

A **vector bundle** $E \xrightarrow{\pi} M$ is a fibre bundle whose fibre is a vector space. Let F be $\mathbb{R}^k$ and M be an m-dimensional manifold. It is common to call k the **fibre dimension** and denote it by $\dim E$, although the total space E is $m + k$ dimensional. The transition functions belong to $\mathrm{GL}(k, \mathbb{R})$, since it maps a vector space onto another vector space of the same dimension isomorphically. If F is a complex vector space $\mathbb{C}^k$, the structure group is $\mathrm{GL}(k, \mathbb{C})$.

Example 9.6 A tangent bundle TM over an m-dimensional manifold M is a vector bundle whose typical fibre is $\mathbb{R}^m$, see §9.1. Let u be a point in TM such that $\pi(u) = p \in U_i \cap U_j$, where $\{U_i\}$ covers M. Let $x^\mu = \varphi_i(p)$ ($y^\mu = \varphi_j(p)$) be the coordinate system of $U_i(U_j)$. The vector V corresponding to u is expressed as $V = V^\mu \partial/\partial x^\mu|_p = \widetilde{V}^\mu \partial/\partial y^\mu|_p$. The local trivialisations are

$$\phi_i^{-1}(u) = (p, \{V^\mu\}), \ \phi_j^{-1}(u) = (p, \{\widetilde{V}^\mu\}). \tag{9.20}$$

The fibre coordinates $\{V^\mu\}$ and $\{\widetilde{V}^\mu\}$ are related as

$$V^\mu = G^\mu{}_\nu(p)\widetilde{V}^\nu \tag{9.21}$$

where $\{G^\mu{}_\nu(p)\} = \{(\partial x^\mu/\partial y^\nu)_p\} \in \mathrm{GL}(m, \mathbb{R})$. Hence a tangent bundle is $(TM, \pi, M, \mathbb{R}^m, \mathrm{GL}(m, \mathbb{R}))$. Sections of TM are the vector fields on M; $\mathcal{X}(M) = \Gamma(M, TM)$.

For concreteness let us work out TS^2. Let the pair $U_\mathrm{N} \equiv S^2 - \{\text{South Pole}\}$ and $U_\mathrm{S} \equiv S^2 - \{\text{North Pole}\}$ be an open covering of S^2. Let (X, Y) and (U, V) be the respective stereographic coordinates (example 8.2). They are related as

$$U = X/(X^2 + Y^2) \qquad V = -Y/(X^2 + Y^2). \tag{9.22}$$

Take $u \in TM$ such that $\pi(u) = p \in U_\mathrm{N} \cap U_\mathrm{S}$. Let ϕ_N and ϕ_S be the respective local trivialisations such that $\phi_\mathrm{N}^{-1}(u) = (p, V_\mathrm{N}^\mu)$ and $\phi_\mathrm{S}^{-1}(u) = (p, V_\mathrm{S}^\nu)$. The transition function is

$$t_\mathrm{SN}(p) = \frac{\partial(U, V)}{\partial(X, Y)} = r^{-2}\begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \tag{9.23}$$

where we have put $X = r\cos\theta$ and $Y = r\sin\theta$. The transition of the components of the tangent vectors consists of a rotation of $\{V_i^\mu\}$ by an angle 2θ followed by a rescaling. The reader should verify that $t_\mathrm{NS}(p) = t_\mathrm{SN}(p)^{-1}$.

Example 9.7 Let M be an m-dimensional manifold embedded in $\mathbb{R}^{m+k}$. Let N_pM be the vector space which is normal to T_pM in $\mathbb{R}^{m+k}$, that is,

$U \cdot V = 0$ for any $U \in N_pM$ and $V \in T_pM$ with respect to the Euclidean metric in $\mathbb{R}^{m+k}$. The vector space N_pM is isomorphic to $\mathbb{R}^k$. The **normal bundle**

$$NM \equiv \bigcup_{p \in M} N_pM$$

is a vector bundle with the typical fibre $\mathbb{R}^k$.

Consider the sphere S^2 embedded in $\mathbb{R}^3$. The normal bundle NS^2 is imagined as S^2 whose surface is pierced perpendicularly by straight lines. NS^2 is a trivial bundle $S^2 \times \mathbb{R}$.

A vector bundle whose fibre is one dimensional ($F = \mathbb{R}$ or $\mathbb{C}$) is called a **line bundle**. A cylinder $S^1 \times \mathbb{R}$ is a trivial $\mathbb{R}$-line bundle. A Möbius strip is also a real line bundle. The structure group $\mathrm{GL}(1, \mathbb{R}) = \mathbb{R} - \{0\}$ or $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C} - \{0\}$ is Abelian.

In the following, we often consider the **canonical line bundle** L. Recall that an element p of $\mathbb{C}P^n$ is a complex line in $\mathbb{C}^{n+1}$ through the origin (example 8.4). The fibre $\pi^{-1}(p)$ of L is defined to be the line in $\mathbb{C}^{n+1}$, which belongs to p. More formally, let $I^{n+1} \equiv \mathbb{C}P^n \times \mathbb{C}^{n+1}$ be a trivial bundle over $\mathbb{C}P^n$. If we write an element of I^{n+1} as (p, v), $p \in \mathbb{C}P^n$, $v \in \mathbb{C}^{n+1}$, L is defined by $L \equiv \{(p, v) \in I^{n+1} | v = ap,\ a \in \mathbb{C}\}$. The projection is $(p, v) \xrightarrow{\pi} p$.

Example 9.8 The (trivial) complex line bundle $L \equiv \mathbb{R}^3 \times \mathbb{C}$ is associated with the non-relativistic quantum mechanics defined on $\mathbb{R}^3$. The wavefunction $\psi(\boldsymbol{x})$ is simply a section of L.

Let us consider a wavefunction $\psi(\boldsymbol{x})$ in the field of a magnetic monopole studied in §1.3. When a monopole is at the origin, $\psi(\boldsymbol{x})$ is defined on $\mathbb{R}^3 - \{0\}$ and we have a complex line bundle over $\mathbb{R}^3 - \{0\}$. If we are interested only in the wavefunction on S^2 surrounding the monopole, we have a complex line bundle over S^2. Note that S^2 is a deformation retract of $\mathbb{R}^3 - \{0\}$.

9.3.2 *Frames*

On a tangent bundle TM, each fibre has a natural basis $\{\partial/\partial x^\mu\}$ given by the coordinate system x^μ on a chart U_i. We may also employ the orthonormal basis $\{\hat{e}_\alpha\}$ if M is endowed with a metric. $\partial/\partial x^\mu$ or $\hat{e}_\alpha$ is a vector field on U_i and the set $\{\partial/\partial x^\mu\}$ or $\{\hat{e}_\alpha\}$ forms linearly independent vector fields over U_i. It is always possible to choose m linearly independent tangent vectors over U_i but it is not necessarily the case throughout M. By definition, the components of the basis vectors are

$$\partial/\partial x^\mu = (0, \ldots 0, \underset{\mu}{1}, 0, \ldots, 0)$$

or

$$\hat{e}_\alpha = (0, \ldots, 0, \underset{\alpha}{1}, 0, \ldots, 0).$$

These vectors define a (local) **frame** over U_i, see below.

Let $E \xrightarrow{\pi} M$ be a vector bundle whose fibre is $\mathbb{R}^k$ (or $\mathbb{C}^k$). On a chart U_i, the piece $\pi^{-1}(U_i)$ is trivial, $\pi^{-1}(U_i) \cong U_i \times \mathbb{R}^k$, and we may choose k linearly independent sections $\{e_1(p), \ldots, e_k(p)\}$ over U_i. These sections are said to define a **frame** over U_i. Given a frame over U_i, we have a natural map $F_p \to F$ ($= \mathbb{R}^k$ or $\mathbb{C}^k$) given by

$$V = V^\alpha e_\alpha(p) \mapsto \{V^\alpha\} \in F. \tag{9.24}$$

The local trivialisation is

$$\phi_i^{-1}(V) = (p, \{V^\alpha(p)\}). \tag{9.25}$$

By definition, we have

$$\phi_i(p, \{0, \ldots, 0, \underset{\alpha}{1}, 0, \ldots, 0\}) = e_\alpha(p). \tag{9.26}$$

Let $U_i \cap U_j \neq \emptyset$ and consider the change of frames. On U_i we have a frame $\{e_1(p), \ldots, e_k(p)\}$ and on U_j, $\{\widetilde{e}_1(p), \ldots, \widetilde{e}_k(p)\}$, where $p \in U_i \cap U_j$. A vector $\widetilde{e}_\beta(p)$ is expressed as

$$\widetilde{e}_\beta(p) = e_\alpha(p) G(p)^\alpha{}_\beta \tag{9.27}$$

where $(G(p)^\alpha{}_\beta) \in \mathrm{GL}(k, \mathbb{R})$ or $\mathrm{GL}(k, \mathbb{C})$. Any vector $V \in \pi^{-1}(p)$ is expressed as

$$V = V^\alpha e_\alpha(p) = \widetilde{V}^\alpha \widetilde{e}_\alpha(p). \tag{9.28}$$

From (9.27) and (9.28) we find that

$$\widetilde{V}^\beta = G^{-1}(p)^\beta{}_\alpha V^\alpha \tag{9.29}$$

where $G^{-1}(p)^\beta{}_\alpha G(p)^\alpha{}_\gamma = G(p)^\beta{}_\alpha G^{-1}(p)^\alpha{}_\gamma = \delta^\beta{}_\gamma$. Thus we find that the transition function $t_{ji}(p)$ is given by a matrix $G^{-1}(p)$.

9.3.3 Cotangent bundles and dual bundles

The **cotangent bundle** $T^*M \equiv \cup_{p \in M} T_p^*M$ is defined similarly to the tangent bundle. On a chart U_i whose coordinates are x^μ, the basis of T_p^*M is taken to be $\{\mathrm{d}x^1, \ldots, \mathrm{d}x^m\}$, which is dual to $\{\partial/\partial x^\mu\}$. Let y^μ be the coordinates of U_j such that $U_i \cap U_j \neq \emptyset$. For $p \in U_i \cap U_j$, we have the transformation,

$$\mathrm{d}y^\mu = \mathrm{d}x^\nu \left(\frac{\partial y^\mu}{\partial x^\nu}\right)_p. \tag{9.30}$$

A one-form ω is expressed, in both coordinate systems, as

$$\omega = \omega_\mu \mathrm{d}x^\mu = \widetilde{\omega}_\mu \mathrm{d}y^\mu$$

from which we find that

$$\widetilde{\omega}_{\mu} = G_{\mu}{}^{\nu}(p)\omega_{\nu} \tag{9.31}$$

where $G(p)_{\mu}{}^{\nu} \equiv (\partial x^{\nu}/\partial y^{\mu})_p$ corresponds to the transition function $t_{ji}(p)$. Note that $\Gamma(M, T^*M) = \Omega^1(M)$.

The cotangent bundle above is easily extended to more general cases. Given a vector bundle $E \xrightarrow{\pi} M$ with the fibre F, we may define its **dual bundle** $E^* \xrightarrow{\pi'} M$. The fibre F^* of E^* is the set of linear maps of F to $\mathbb{R}$ (or $\mathbb{C}$). Given a general basis $\{e_{\alpha}(p)\}$ of F_p, we define the dual basis $\{\theta^{\alpha}(p)\}$ of F_p^* by $\langle\theta^{\alpha}(p), e_{\beta}(p)\rangle = \delta^{\alpha}{}_{\beta}$.

9.3.4 *Sections of vector bundles*

Let s and s' be sections of a vector bundle $E \xrightarrow{\pi} M$. The vector addition and the scalar multiplication are pointwisely defined as

$$(s + s')(p) = s(p) + s'(p) \tag{9.32a}$$

$$(fs)(p) = f(p)s(p) \tag{9.32b}$$

where $p \in M$ and $f \in \mathcal{F}(M)$. The null vector 0 of each fibre is left invariant under $\mathrm{GL}(k, \mathbb{R})$ (or $\mathrm{GL}(k, \mathbb{C})$) and plays a distinguished role. Any vector bundle E admits a global section called the **null section** $s_0 \in \Gamma(M, E)$ such that $\phi_i^{-1}(s_0(p)) = (p, 0)$ in any local trivialisation.

For example, let us consider sections of the canonical line bundle L over $\mathbb{C}P^n$. Let $\xi^{\nu}{}_{(\mu)}$ be the inhomogeneous coordinates on U_{μ}. The local section s_{μ} over U_{μ} is of the form

$$s_{\mu} = \{\xi^0{}_{(\mu)}, \ldots, 1, \ldots, \xi^n{}_{(\mu)}\} \in \mathbb{C}^{n+1}.$$

The transition from one coordinate system to the other is carried out by a scalar multiplication: $s_{\nu} = (z^{\mu}/z^{\nu})s_{\mu}$. Let L^* be the dual bundle of L. Corresponding to s_{μ}, we may choose a dual section s_{μ}^* such that $s_{\mu}^*(s_{\mu}) = 1$. From this, we find that the transition of s_{μ}^* is given by $s_{\nu}^* = (z^{\nu}/z^{\mu})s_{\mu}^*$.

A **fibre metric** $h_{ij}(p)$ is also defined pointwisely. Let s and s' be sections over U_i. The inner product between s and s' at p is defined by

$$(s, s')_p = h_{ij}(p)s^i(p)s'^j(p) \tag{9.33a}$$

if the fibre is $\mathbb{R}^k$. If the fibre is $\mathbb{C}^k$ we define

$$(s, s')_p = h_{ij}(p)\overline{s^i(p)}s'^j(p). \tag{9.33b}$$

We have more about this subject in §10.4.

9.3.5 *The product bundle and Whitney sum bundle*

Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M'$ be vector bundles with fibres F and F'

respectively. The **product bundle**

$$E \times E' \xrightarrow{\pi \times \pi'} M \times M' \tag{9.34}$$

is a fibre bundle whose typical fibre is $F \oplus F'$. [A vector in $F \oplus F'$ is written as

$$\begin{pmatrix} V \\ W \end{pmatrix} \qquad \text{where } V \in F \text{ and } W \in F'.$$

Vector addition and scalar multiplication are defined by

$$\begin{pmatrix} V \\ W \end{pmatrix} + \begin{pmatrix} V' \\ W' \end{pmatrix} = \begin{pmatrix} V + V' \\ W + W' \end{pmatrix}$$

and

$$\lambda \begin{pmatrix} V \\ W \end{pmatrix} = \begin{pmatrix} \lambda V \\ \lambda W \end{pmatrix}.$$

Let $\{e_\alpha\}$ and $\{f_\beta\}$ be bases of F and F' respectively. Then $\{e_\alpha\} \cup \{f_\beta\}$ is a basis of $F \oplus F'$ and we find that $\dim(F \oplus F') = \dim F + \dim F'$.] If $\pi(u) = p$ and $\pi'(u') = p'$ the projection $\pi \times \pi'$ acts on $(u, u') \in E \times E'$ as

$$\pi \times \pi'(u, u') = (p, p'). \tag{9.35}$$

The fibre at (p, p') is $F_p \oplus F'_{p'}$. For example, if $M = M_1 \times M_2$, we have $TM = TM_1 \times TM_2$.

Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M$ be vector bundles with fibres F and F' respectively. The **Whitney sum bundle** $E \oplus E'$ is a pullback bundle of $E \times E'$ by $f : M \to M \times M$ defined by $f(p) = (p, p)$,

$$\begin{tikzcd} E \oplus E' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E \times E' \arrow[d, "\pi \times \pi'"] \\ M \arrow[r, "f"] & M \times M \end{tikzcd} \tag{9.36}$$

Thus $E \oplus E' = \{(u, u') \in E \times E' | \pi \times \pi'(u, u') = (p, p)\}$. The fibre of a Whitney sum bundle is $F \oplus F'$. $(\pi \times \pi')^{-1}(p)$ is isomorphic to $\pi^{-1}(p) \oplus \pi'^{-1}(p) = F_p \oplus F'_p$. In a word, $E \oplus E'$ is a bundle over M whose fibre at p is $F_p \oplus F'_p$. Let $\{U_i\}$ be an open covering of M and $\{t^E_{ij}\}$ and $\{t^{E'}_{ij}\}$ be the transition functions of E and E' respectively. Then the transition function T_{ij} of $E \oplus E'$ is a $(\dim F + \dim F') \times (\dim F + \dim F')$ matrix

$$T_{ij}(p) = \begin{pmatrix} t^E_{ij}(p) & 0 \\ 0 & t^{E'}_{ij}(p) \end{pmatrix} \tag{9.37}$$

which acts on $F \oplus F'$ on the left.

Example 9.9 Let $E = TS^2$ and $E' = NS^2$ defined in $\mathbb{R}^3$. Take $u \in TS^2$ and $v \in NS^2$ whose local trivialisations are $\phi_i^{-1}(u) = (p, V)$ and $\psi_i^{-1}(v) = (q, W)$ respectively where $p, q \in S^2$, $V \in \mathbb{R}^2$ and $W \in \mathbb{R}$. If (u, v) is a point of the *product bundle* $E \times E'$, we have a local trivialisation $\Phi_{i,j} = \phi_i \times \psi_j$ such that

$$\Phi_{i,j}^{-1}(u, v) = (p, q; V, W). \tag{9.38a}$$

If, on the other hand, $(u, v) \in E \oplus E'$, u and v satisfy the stronger condition $\pi(u) = \pi'(v)$ $(= p$, say). Thus we have

$$\Phi_i^{-1}(u, v) = (p; V, W). \tag{9.38b}$$

The Whitney sum $TS^2 \oplus NS^2$, S^2 being embedded in $\mathbb{R}^3$, is a trivial bundle over S^2, whose fibre is isomorphic to $\mathbb{R}^3$.

9.3.6 Tensor product bundles

Let $E \xrightarrow{\pi} M$ and $E' \xrightarrow{\pi'} M$ be vector bundles over M. The **tensor product bundle** $E \otimes E'$ is obtained by assigning the tensor product of fibres $F_p \otimes F'_p$ to each point $p \in M$. If $\{e_\alpha\}$ and $\{f_\beta\}$ are bases of F and F', $F \otimes F'$ is spanned by $\{e_\alpha \otimes f_\beta\}$ and hence $\dim(E \otimes E') = \dim(E) \times \dim(E')$.

Let $\otimes^r E \equiv E \otimes \ldots \otimes E$ be the tensor product bundle of $r\,E$. If $\{e_\alpha\}$ is the basis of the fibre F of E, the fibre of $\otimes^r E$ is spanned by $\{e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_r}\}$. If we define $\wedge$ by

$$e_\alpha \wedge e_\beta \equiv e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha \tag{9.39}$$

we have a bundle $\Lambda^r(E)$ of totally antisymmetric tensors spanned by $\{e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_r}\}$. In particular $\Omega^r(M)$, the space of r-forms on M, is identified with $\Gamma(M, \Lambda^r(T^*M))$.

Exercise 9.10 Let E_1, E_2 and E_3 be vector bundles over M. Show that $\otimes$ is distributive:

$$E_1 \otimes (E_2 \oplus E_3) = (E_1 \otimes E_2) \oplus (E_1 \otimes E_3). \tag{9.40}$$

Express the transition functions of $E_1 \otimes (E_2 \oplus E_3)$ in terms of those of E_1, E_2 and E_3.

9.4 Principal bundles

9.4.1 Definitions

A **principal bundle** has a fibre F which is identical with the structure group G. A principal bundle $P \xrightarrow{\pi} M$ is also denoted by $P(M, G)$ and is often called a G **bundle** over M.

The transition function acts on the fibre on the left as before. In addition, we may also define the action of G on F *on the right*. Let $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ be the local trivialisation given by $\phi_i^{-1}(u) = (p, g_i)$, where $u \in \pi^{-1}(U_i)$ and $p = \pi(u)$. The right action of G on $\pi^{-1}(U_i)$ is defined by $\phi_i^{-1}(ua) = (p, g_i a)$, that is, (figure 9.8)

$$ua = \phi_i(p, g_i a) \tag{9.41}$$

for any $a \in G$ and $u \in \pi^{-1}(p)$. Since the right action commutes with the left action, this definition is independent of the local trivialisations. In fact, if $p \in U_i \cap U_j$,

$$ua = \phi_j(p, g_j a) = \phi_j(p, t_{ji}(p) g_i a) = \phi_i(p, g_i a).$$

Thus the right multiplication is defined without reference to the local trivialisations. This is denoted by $P \times G \to P$ or $(u, a) \mapsto ua$. Note that $\pi(ua) = \pi(u)$. The right action of G on $\pi^{-1}(p)$ is *transitive* since G acts on G transitively on the right and $F_p = \pi^{-1}(p)$ is diffeomorphic to G. Thus for any $u_1, u_2 \in \pi^{-1}(p)$ there exists an element a of G such that $u_1 = u_2 a$. Then, if $\pi(u) = p$, we can construct the whole fibre as $\pi^{-1}(p) = \{ua | a \in G\}$. The action is also *free*; if $ua = u$ for some $u \in P$, a must be the unit element e of G. In fact, if $u = \phi_i(p, g_i)$, we have $\phi_i(p, g_i a) = \phi_i(p, g_i)a = ua = u = \phi_i(p, g_i)$. Since ϕ_i is bijective, we must have $g_i a = g_i$, that is, $a = e$.

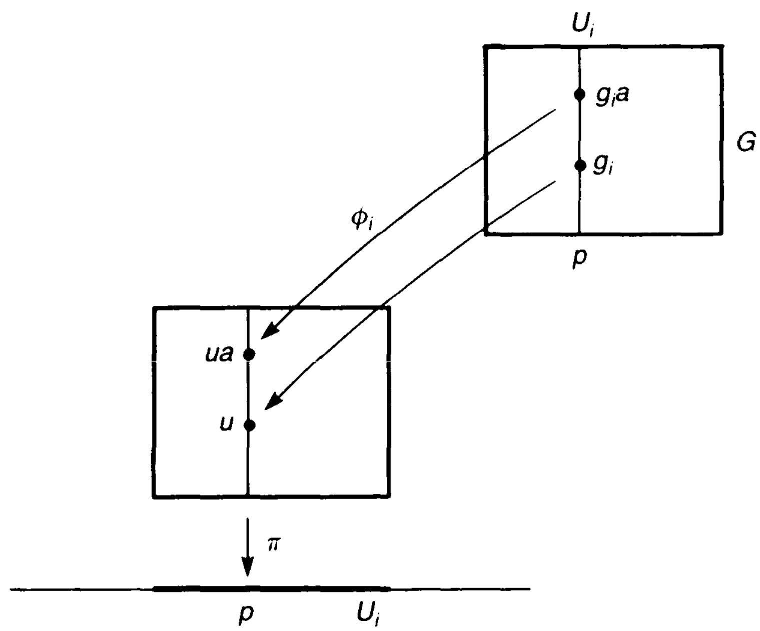


Figure 9.8 The right action of G on P.

Given a section $s_i(p)$ over U_i, we define a preferred local trivialisation $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ as follows. For $u \in \pi^{-1}(p)$, $p \in U_i$, there is a *unique* element $g_u \in G$ such that $u = s_i(p) g_u$. Then we define ϕ_i by

$\phi_i^{-1}(u) = (p, g_u)$. In this local trivialisation, the section $s_i(p)$ is expressed as

$$s_i(p) = \phi_i(p, e). \tag{9.42}$$

This local trivialisation is called the **canonical local trivialisation**. By definition $\phi_i(p, g) = \phi_i(p, e)g = s_i(p)g$. If $p \in U_i \cap U_j$, two sections $s_i(p)$ and $s_j(p)$ are related by the transition function $t_{ij}(p)$ as follows

$$\begin{aligned} s_i(p) &= \phi_i(p, e) = \phi_j(p, t_{ji}(p)e) = \phi_j(p, t_{ji}(p)) \\ &= \phi_j(p, e)t_{ji}(p) = s_j(p)t_{ji}(p). \end{aligned} \tag{9.43}$$

Example 9.11 Let P be a principal bundle with fibre $\mathrm{U}(1) = S^1$ and the base space S^2. This principal bundle represents the topological setting of the magnetic monopole (§1.3). Let $\{U_{\mathrm{N}}, U_{\mathrm{S}}\}$ be an open covering of S^2, U_{N} (U_{S}) being the northern (southern) hemisphere. If we parametrise S^2 by the usual polar angles, we have

$$U_{\mathrm{N}} = \{(\theta, \phi) | 0 \leqslant \theta \leqslant \pi/2 + \varepsilon, 0 \leqslant \phi < 2\pi\}$$

$$U_{\mathrm{S}} = \{(\theta, \phi) | \pi/2 - \varepsilon \leqslant \theta \leqslant \pi, 0 \leqslant \phi < 2\pi\}.$$

The intersection $U_{\mathrm{N}} \cap U_{\mathrm{S}}$ is a strip which is essentially the equator. Let ϕ_{N} and ϕ_{S} be the local trivialisations such that

$$\phi_{\mathrm{N}}^{-1}(u) = (p, \exp(\mathrm{i}\alpha_{\mathrm{N}})), \ \phi_{\mathrm{S}}^{-1}(u) = (p, \exp(\mathrm{i}\alpha_{\mathrm{S}})) \tag{9.44}$$

where $p = \pi(u)$. Take a transition function t_{NS} of the form $\mathrm{e}^{\mathrm{i}n\phi}$, where n must be an integer so that $t_{\mathrm{NS}}(p)$ may be uniquely defined on the equator. Since t_{NS} maps the equator S^1 to $\mathrm{U}(1)$, this integer characterises the homotopy group $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$. The fibre coordinates α_{N} and α_{S} are related on the equator as

$$\mathrm{e}^{\mathrm{i}\alpha_{\mathrm{N}}} = \mathrm{e}^{\mathrm{i}n\phi}\mathrm{e}^{\mathrm{i}\alpha_{\mathrm{S}}}. \tag{9.45}$$

If $n = 0$, the transition function is the unit element of $\mathrm{U}(1)$ and we have a trivial bundle $P = S^2 \times S^1$. If $n \neq 0$, the $\mathrm{U}(1)$-bundle P_n is twisted. It is remarkable that the topological structure of a fibre bundle is characterised by an integer.

Since $\mathrm{U}(1)$ is Abelian, the right action and the left action are equivalent. Under the right action $g = \mathrm{e}^{\mathrm{i}\lambda}$, we have

$$\phi_{\mathrm{N}}^{-1}(ug) = (p, \mathrm{e}^{\mathrm{i}(\alpha_{\mathrm{N}}+\lambda)}) \tag{9.46a}$$

$$\phi_{\mathrm{S}}^{-1}(ug) = (p, \mathrm{e}^{\mathrm{i}(\alpha_{\mathrm{S}}+\lambda)}). \tag{9.46b}$$

The right action corresponds to the $\mathrm{U}(1)$-gauge transformation.

Example 9.12 If we identify all the infinite points of the Euclidean space $\mathbb{R}^m$, we have the one-point compactification $S^m = \mathbb{R}^m \cup \{\infty\}$. If a trivial G bundle is defined over $\mathbb{R}^m$, we shall have a new G bundle over S^m after compactification, which is not necessarily trivial. Let P be an

SU(2) bundle over S^4 obtained from $\mathbb{R}^4$ by one-point compactification. This principal bundle represents an SU(2) instanton (§1.4). Introduce an open covering $\{U_N, U_S\}$ of S^4,

$$U_N = \{(x, y, z, t)|x^2 + y^2 + z^2 + t^2 \leqslant R^2 + \varepsilon\}$$
$$U_S = \{(x, y, z, t)|R^2 - \varepsilon \leqslant x^2 + y^2 + z^2 + t^2\}$$

where R is a positive constant and ε is an infinitesimal positive number. The thin intersection $U_N \cap U_S$ is essentially S^3. Let $t_{NS}(p)$ be the transition function defined at $p \in U_N \cap U_S$. Since t_{NS} maps S^3 to SU(2), it is classified by $\pi_3(\mathrm{SU}(2)) = \mathbb{Z}$. The integer characterising the bundle is called the **instanton number**. If $t_{NS}(p)$ is taken to be unity, we have a trivial bundle $P_0 = S^3 \times \mathrm{SU}(2)$, which corresponds to the homotopy class 0. Non-trivial bundles are obtained as follows. We first note that $\mathrm{SU}(2) \cong S^3$ (example 4.50). An element $A \in \mathrm{SU}(2)$ is written as

$$A = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

where $|u|^2 + |v|^2 = 1$. Separating u and v into real and imaginary parts as $u = t + \mathrm{i}z$ and $v = y + \mathrm{i}x$, we find $t^2 + x^2 + y^2 + z^2 = 1$. Thus SU(2) is regarded as the unit sphere S^3 and $\pi_3(\mathrm{SU}(2)) \cong \pi_3(S^3) \cong \mathbb{Z}$ classifies maps from S^3 to $\mathrm{SU}(2) \cong S^3$. The *identity map* $f : S^3 \to S^3 \cong \mathrm{SU}(2)$ is

$$\begin{aligned} f : (x, y, z, t) \mapsto & \begin{pmatrix} t + \mathrm{i}z & y + \mathrm{i}x \\ -y + \mathrm{i}x & t - \mathrm{i}z \end{pmatrix} \\ & = t\mathbb{1} + \mathrm{i}(x\sigma_x + y\sigma_y + z\sigma_z) \end{aligned} \tag{9.47}$$

where

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the σ_μ are the Pauli matrices. Let us take a point $p = (x, y, z, t) \in U_N \cap U_S$. If $R = (x^2 + y^2 + z^2 + t^2)^{1/2}$ denotes the length of p, the vector $(x/R, y/R, z/R, t/R)$ has unit length. We assign an element of SU(2) to the point p as

$$t_{NS}(p) = \frac{1}{R}\left(t\mathbb{1} + \mathrm{i}\sum_i x^i\sigma_i\right). \tag{9.48}$$

Let ϕ_N and ϕ_S be the local trivialisations,

$$\phi_N^{-1}(u) = (p, g_N) \qquad \phi_S^{-1}(u) = (p, g_S) \tag{9.49}$$

where $p = \pi(u)$ and $g_N, g_S \in \mathrm{SU}(2)$. On $U_N \cap U_S$, we have

$$g_N = \frac{1}{R}\left(t\mathbb{1} + \mathrm{i}\sum_i x^i\sigma_i\right) g_S. \tag{9.50}$$

While $(t, \boldsymbol{x})$ scans S^3 once, $t_{\mathrm{NS}}(p)$ sweeps SU(2) once, hence this bundle corresponds to the homotopy class 1 of $\pi_3(\mathrm{SU}(2))$. It is not difficult to see that the transition function corresponding to the homotopy class n is given by

$$t_{\mathrm{NS}}(p) = \frac{1}{R}\left(t\mathbb{1} + \mathrm{i}\sum_i x^i\sigma_i\right)^n. \tag{9.51}$$

To continue our study of monopoles and instantons, we have to introduce connections (the *gauge potentials*) on the fibre bundle. We will come back to these topics in the next chapter.

Example 9.13 Hopf has shown that S^3 is a U(1) bundle over S^2. The unit three-sphere embedded in $\mathbb{R}^4$ may be given by

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1.$$

If we introduce $z^0 = x^1 + \mathrm{i}x^2$ and $z^1 = x^3 + \mathrm{i}x^4$, this becomes

$$|z^0|^2 + |z^1|^2 = 1. \tag{9.52}$$

Let us parametrise S^2 as

$$(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 1.$$

The **Hopf map** $\pi : S^3 \to S^2$ is defined by

$$\begin{aligned}
\xi^1 &= 2(x^1x^3 + x^2x^4)\\
\xi^2 &= 2(x^2x^3 - x^1x^4)\\
\xi^3 &= (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2.
\end{aligned} \tag{9.53}$$

It is easily verified that π maps S^3 to S^2 since

$$(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = [(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2]^2 = 1.$$

Let (X, Y) be the stereographic projection coordinates of a point in the southern hemisphere U_{S} of S^2 from the North Pole. If we take a complex plane which contains the equator of S^2, $Z = X + \mathrm{i}Y$ is within the circle of unit radius. We found in example 8.2 that (figure 9.9)

$$Z = \frac{\xi^1 + \mathrm{i}\xi^2}{1 - \xi^3} = \frac{x^1 + \mathrm{i}x^2}{x^3 + \mathrm{i}x^4} = \frac{z^0}{z^1} \qquad (\xi \in U_{\mathrm{S}}). \tag{9.54a}$$

Observe that Z is invariant under

$$(z^0, z^1) \mapsto (\lambda z^0, \lambda z^1)$$

where $\lambda \in \mathrm{U}(1)$. Since $|\lambda| = 1$, the point $(\lambda z^0, \lambda z^1)$ is also in S^3. The stereographic coordinates (U, V) of the northern hemisphere U_{N} projected from the South Pole are given by

$$W = U + \mathrm{i}V = \frac{\xi^1 - \mathrm{i}\xi^2}{1 + \xi^3} = \frac{x^3 + \mathrm{i}x^4}{x^1 + \mathrm{i}x^2} = \frac{z^1}{z^0} \qquad (\xi \in U_{\mathrm{N}}). \tag{9.54b}$$

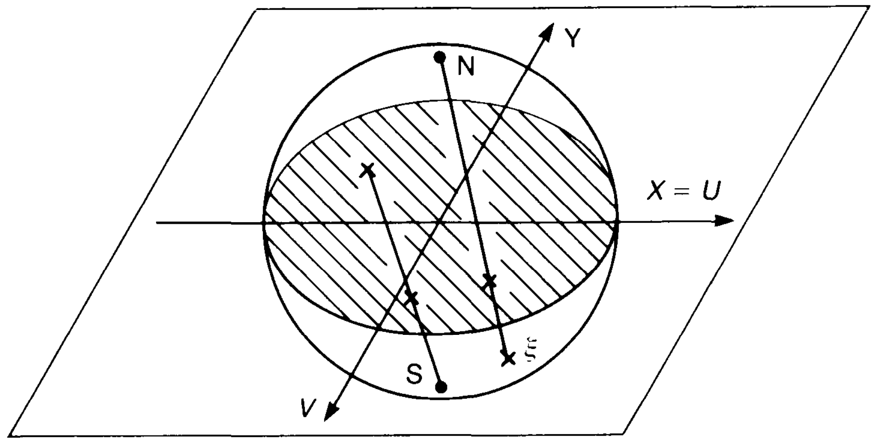


Figure 9.9 Stereographic coordinates of the sphere S^2. (X, Y) is the projection from the North Pole while (U, V) is the projection from the South Pole.

Note that $Z = 1/W$ on the equator $U_N \cap U_S$.

The fibre bundle structure is given as follows. We first define the local trivialisations,

$$\phi_S^{-1} : \pi^{-1}(U_S) \to U_S \times \mathrm{U}(1)$$

by $(z^0, z^1) \mapsto (z^0/z^1, z^1/|z^1|)$ (9.55*a*)

$$\phi_N^{-1} : \pi^{-1}(U_N) \to U_N \times \mathrm{U}(1)$$

by $(z^0, z^1) \mapsto (z^1/z^0, z^0/|z^0|)$. (9.55*b*)

Observe that these local trivialisations are well defined on each chart. For example, $z^0 \neq 0$ on U_N, hence both $z^1/z^0 = U + \mathrm{i}V$ and $z^0/|z^0|$ are non-singular. On the equator, $\xi^3 = 0$, we have $|z^0| = |z^1| = 1/\sqrt{2}$. Accordingly, the local trivialisations on the equator are

$$\phi_S^{-1} : (z^0, z^1) \mapsto (z^0/z^1, \sqrt{2}z^1) \tag{9.56a}$$

and

$$\phi_N^{-1} : (z^0, z^1) \mapsto (z^1/z^0, \sqrt{2}z^0). \tag{9.56b}$$

The transition function on the equator is

$$t_{NS}(\xi) = \frac{\sqrt{2}z^0}{\sqrt{2}z^1} = \xi^1 + \mathrm{i}\xi^2 \in \mathrm{U}(1). \tag{9.57}$$

If we circumnavigate the equator, $t_{NS}(\xi)$ traverses the unit circle in the complex plane once, hence the U(1) bundle $S^3 \xrightarrow{\pi} S^2$ is characterised by the homotopy class 1 of $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$. Trautman (1977), Minami (1979) and Ryder (1980) have pointed out that a magnetic monopole of unit strength is described by the Hopf map $S^3 \xrightarrow{\pi} S^2$.

The Hopf map can be understood from a slightly different point of view. We regard S^3 as a complex one-sphere

$$S^1_{\mathbb{C}} = \{(z^0, z^1) \in \mathbb{C}^2 \| z^0|^2 + |z^1|^2 = 1\}.$$

Define a map $\pi : S^1_{\mathbb{C}} \to \mathbb{C}P^1$ by

$$(z^0, z^1) \mapsto [(z^0, z^1)] = \{\lambda(z^0, z^1) | \lambda \in \mathbb{C}\}. \tag{9.58}$$

Under this map, points of S^3 of the form $\lambda(z^0, z^1)$, $|\lambda| = 1$ are mapped to a single point of $\mathbb{C}P^1 = S^2$. This is the Hopf map $\pi : S^3 \to S^2$ obtained above. This is easily generalised to the case of the quaternion $\mathbb{H}$. The quaternion algebra is defined by the product table,

$$i^2 = j^2 = k^2 = -1 \qquad ij = -ji = k$$
$$jk = -kj = i \qquad ki = -ik = j.$$

An arbitrary element of $\mathbb{H}$ is written as

$$q = t + ix + jy + kz. \tag{9.59}$$

Clearly the unit quaternion $|q| = (t^2 + x^2 + y^2 + z^2)^{1/2} = 1$ represents $S^3 \cong \mathrm{SU}(2)$. The quaternion one-sphere is given by

$$S^1_{\mathbb{H}} = \{q^0, q^1) \in \mathbb{H}^2 \| q^0|^2 + |q^1|^2 = 1\} \tag{9.60}$$

which represents S^7. The Hopf map in this case takes the form

$$\pi : S^1_{\mathbb{H}} \to \mathbb{H}P^1 \tag{9.61}$$

where $\mathbb{H}P^1$ is the quaternion projective space whose element is

$$[(q^0, q^1)] = \{\eta(q^0, q^1) \in \mathbb{H}^2 | \eta \in \mathbb{H}\}. \tag{9.62}$$

Under this map, points of S^7 with $|\eta| = 1$ are mapped to a single point of $\mathbb{H}P^1 = S^4$ and we have the Hopf map

$$\pi : S^7 \to S^4. \tag{9.63}$$

The fibre is the unit quaternion $S^3 = \mathrm{SU}(2)$. The transition function defined by the Hopf map belongs to the class 1 of $\pi_3(\mathrm{SU}(2)) \simeq \mathbb{Z}$. An instanton of unit strength is described in terms of this Hopf map.

Octonions define a Hopf map $\pi : S^{15} \to S^8$. This is different from other Hopf maps in that the fibre S^7 is not really a group. So far we have not found an application of this map in physics.

Example 9.14 Let H be a closed Lie subgroup of a Lie group G. We show that G is a principal bundle with fibre H and base space $M \equiv G/H$. Define the right action of H on G by $g \mapsto ga$, $g \in G$, $a \in H$. The right action is differentiable since G is a Lie group. Define the projection $\pi : G \to M = G/H$ by the map $\pi : g \mapsto [g] = \{gh | h \in H\}$. Clearly, g, $ga \in G$ are mapped to the same point $[g]$ hence $\pi(g) = \pi(ga)$ $(= [g])$. To define local trivialisations, we need to define a map $f_i : G \to H$ on each chart U_i. Let s be a local section over U_i and let $g \in \pi^{-1}([g])$. Define f_i by $f_i(g) \equiv s([g])^{-1}g$. Since $s([g])$ is a section at $[g]$, it is expressed as ga for some $a \in H$ and accordingly $s([g])^{-1}g = a^{-1}g^{-1}g = a^{-1} \in H$. Then we define the local trivialisation $\phi_i : U_i \times H \to G$ by

$$\phi_i^{-1}(g) = ([g], f_i(g)). \tag{9.64}$$

It is easy to see that $f_i(ga) = f_i(g)a$ $(a \in H)$ hence $\phi_i^{-1}(ua) = (p, f_i(u)a)$ is satisfied. Useful examples are (see example 5.65)

$$\mathrm{O}(n)/\mathrm{O}(n-1) = \mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1} \tag{9.65}$$

$$\mathrm{U}(n)/\mathrm{U}(n-1) = \mathrm{SU}(n)/\mathrm{SU}(n-1) = S^{2n-1}. \tag{9.66}$$

9.4.2 Associated bundles

Given a principal fibre bundle $P(M, G)$, we may construct an **associated fibre bundle** as follows. Let G act on a manifold F on the left. Define an action of $g \in G$ on $P \times F$ by

$$(u, f) \to (ug, g^{-1}f) \tag{9.67}$$

where $u \in P$ and $f \in F$. Then the associated fibre bundle (E, π, M, G, F, P) is an equivalence class $P \times F/G$ in which two points (u, f) and $(ug, g^{-1}f)$ are identified.

Let us consider the case in which F is a k-dimensional vector space V. Let ρ be the k-dimensional representation of G. The **associated vector bundle** $P \times_\rho V$ is defined by identifying the points (u, v) and $(ug, \rho(g)^{-1}v)$ of $P \times V$, where $u \in P$, $g \in G$ and $v \in V$. For example, associated with $P(M, \mathrm{GL}(k, \mathbb{R}))$ is a vector bundle over M with fibre $\mathbb{R}^k$. The fibre bundle structure of an associated vector bundle $E = P \times_\rho V$ is given as follows. The projection $\pi_E : E \to M$ is defined by $\pi_E(u, v) = \pi(u)$. This projection is well defined since $\pi(u) = \pi(ug)$ implies $\pi_E(ug, \rho(g)^{-1}v) = \pi(ug) = \pi_E(u, v)$. The local trivialisation is given by $\psi_i : U_i \times V \to \pi_E^{-1}(U_i)$. The transition function of E is given by $\rho(t_{ij}(p))$ where $t_{ij}(p)$ is that of P.

Conversely a vector bundle naturally induces a principal bundle associated with it. Let $E \xrightarrow{\pi} M$ be a vector bundle with $\dim E = k$ (the fibre is $\mathbb{R}^k$ or $\mathbb{C}^k$). Then E induces a principal bundle $P(E) \equiv P(M, G)$ over M by employing the same transition functions. The structure group G is either $\mathrm{GL}(k, \mathbb{R})$ or $\mathrm{GL}(k, \mathbb{C})$. Explicit construction of $P(E)$ is carried out following the reconstruction process described in §9.1.

Example 9.15 Associated with a tangent bundle TM over an m-dimensional manifold M is a principal bundle called the **frame bundle** $LM \equiv \cup_{p \in M} L_pM$, where L_pM is the set of frames at p. We introduce coordinates x^μ on a chart U_i. T_pM has a natural basis $\{\partial/\partial x^\mu\}$ on U_i. A frame $u = \{X_1, \ldots, X_m\}$ at p is expressed as

$$X_\alpha = X^\mu{}_\alpha \partial/\partial x^\mu|_p \qquad 1 \leq \alpha \leq m \tag{9.68}$$

where $(X^\mu{}_\alpha)$ is an element of $\mathrm{GL}(m, \mathbb{R})$ so that $\{X_\alpha\}$ are linearly independent. We define the local trivialisation $\phi_i : U_i \times \mathrm{GL}(m, \mathbb{R}) \to \pi^{-1}(U_i)$ by $\phi_i^{-1}(u) = (p, (X^\mu{}_\alpha))$. The bundle structure of LM is defined as follows. (a) If $u = (X_1, \ldots, X_m)$ is a frame at p, we define $\pi_L : LM \to M$ by $\pi_L(u) = p$. (b) The action of $a = (a^i{}_j) \in \mathrm{GL}(m, \mathbb{R})$ on the frame $u = (X_1, \ldots, X_m)$ is given by $(u, a) \mapsto ua$, where ua is a new frame at p, defined by

$$Y_\beta = X_\alpha a^\alpha{}_\beta. \tag{9.69}$$

Conversely given any frames $\{X_\alpha\}$ and $\{Y_\beta\}$ there exists an element of $\mathrm{GL}(m, \mathbb{R})$ such that (9.69) is satisfied. Thus $\mathrm{GL}(m, \mathbb{R})$ acts on LM transitively. (c) Let U_i and U_j be overlapping charts with the coordinates x^μ and y^μ respectively. If $p \in U_i \cap U_j$, we have

$$X_\alpha = X^\mu{}_\alpha \partial/\partial x^\mu|_p = \widetilde{X}^\mu{}_\alpha \partial/\partial y^\mu|_p \tag{9.70}$$

where $(X^\mu{}_\alpha)$, $(\widetilde{X}^\mu{}_\alpha) \in \mathrm{GL}(m, \mathbb{R})$. Since $X^\mu{}_\alpha = (\partial x^\mu/\partial y^\nu)_p \widetilde{X}^\nu{}_\alpha$, we find the transition function $t^L_{ij}(p)$ to be

$$t^L_{ij}(p) = ((\partial x^\mu/\partial y^\nu)_p) \in \mathrm{GL}(m, \mathbb{R}). \tag{9.71}$$

Accordingly, given TM, we have constructed a frame bundle LM with the same transition functions.

In general relativity, the right action corresponds to the local Lorentz transformation while the left action corresponds to the general coordinate transformation. It turns out that the frame bundle is the most natural framework in which to incorporate these transformations. If $\{X_\alpha\}$ is normalised by introducing a metric, the matrix $(X^\mu{}_\alpha)$ becomes the vierbein and the structure group reduces to $\mathrm{O}(m)$; see §7.8.

Example 9.16 A spinor field on M is a section of a **spin bundle** which we now define. Since $\mathrm{GL}(k, \mathbb{R})$ has no spinor representation, we need to introduce an orthonormal frame bundle whose structure group is $\mathrm{SO}(k)$. As we mentioned in example 4.50, $\mathrm{SPIN}(k)$ is the universal covering group of $\mathrm{SO}(k)$. [To define a spin bundle, we have to check whether the $\mathrm{SO}(k)$ bundle *lifts* to a $\mathrm{SPIN}(k)$ bundle over M. The obstruction to this lifting is discussed in §11.6.]

To be specific, let us consider a spin bundle associated with the four-dimensional Lorentz frame bundle LM, where M is a four-dimensional Lorentz manifold. We are interested in a frame with a definite spacetime orientation as well as a time orientation. The structure group is then reduced to

$$O_{\uparrow}^{+}(3, 1) \equiv \{\Lambda \in O(3, 1) | \det\Lambda = +1, \Lambda_0{}^0 > 0\}. \tag{9.72}$$

The universal covering group of $O_{\uparrow}^{+}(3, 1)$ is $SL(2, \mathbb{C})$, see example 5.57(c). The homomorphism $\varphi : SL(2, \mathbb{C}) \to O_{\uparrow}^{+}(3, 1)$ is a $2:1$ map with $\ker\varphi = \{\mathbb{1}, -\mathbb{1}\}$. The Weyl spinor is a section of the fibre bundle $(W, \pi, M, \mathbb{C}^2, SL(2, \mathbb{C}))$. The Dirac spinor is a section of

$$(D, \pi, M, \mathbb{C}^4, SL(2, \mathbb{C}) \oplus \overline{SL(2, \mathbb{C})}).$$

A section of W is a $(1/2, 0)$ representation of $O_{\uparrow}^{+}(3, 1)$ and a section of $(\overline{W}, \pi, M, \mathbb{C}^2, \overline{SL(2, \mathbb{C})})$ is a $(0, 1/2)$ representation, see Ramond (1981) for example. A Dirac spinor belongs to $(1/2, 0) \oplus (0, 1/2)$.

The general structure of the spin bundle will be worked out in §11.6.

9.4.3 *Triviality of bundles*

A fibre bundle is trivial if it is expressed as a direct product of the base space and the fibre. The following theorem gives the condition under which a fibre bundle is trivial.

Theorem 9.17 A principal bundle is trivial if and only if it admits a *global* section.

Proof: Let (P, π, M, G) be a principal bundle over M and let $s \in \Gamma(M, P)$ be a global section. This section may be used to show that there exists a homeomorphism between P and $M \times G$. If a is an element of G, the product $s(p)a$ belongs to the fibre at p. Since the right action is transitive and free, any element $u \in P$ is uniquely written as $s(p)a$ for some $p \in M$ and $a \in G$. Define a map $\Phi : P \to M \times G$ by

$$\Phi : s(p)a \mapsto (p, a). \tag{9.73}$$

It is easily verified that Φ is indeed a homeomorphism and we have shown that P is a trivial bundle $M \times G$.

Conversely suppose $P \cong M \times G$. Let $\phi : M \times G \to P$ be a trivialisation. Take a fixed element $g \in G$. Then $s_g : M \to P$ defined by $s_g(p) = \phi(p, g)$ is a global section. ■

Is there a corresponding theorem for vector bundles? We know that any vector bundle admits a global null section. Thus we cannot simply replace P by E in theorem 9.19. Let us consider the associated principal bundle $P(E)$ of E. By definition, E and $P(E)$ share the same set of transition functions. Since the twisting of a bundle is described purely by the transition functions, we obtain the following corollary.

Corollary 9.18 A vector bundle E is trivial if and only if its associated principal bundle $P(E)$ admits a global section.

Problems 9

1 Let L be the real line bundle over S^1 (that is L is either the cylinder $S^1 \times \mathbb{R}$ or the Möbius strip). Show that the Whitney sum $L \oplus L$ is a trivial bundle. Sketch $L \oplus L$ to confirm the result.
2 Let Ω_n be the volume element of S^n normalised as $\int_{S^n} \Omega_n = 1$. Let $f: S^{2n-1} \to S^n$ be a smooth map and consider the pullback $f^*\Omega_n$.

(*a*) Show that $f^*\Omega_n$ is closed and written as $\mathrm{d}\omega_{n-1}$, where ω_{n-1} is an $(n-1)$-form on S^{2n-1}.

(*b*) Show that the **Hopf invariant**

$$H(f) \equiv \int_{S^{2n-1}} \omega_{n-1} \wedge \mathrm{d}\omega_{n-1}$$

is independent of the choice of ω_{n-1}.

(*c*) Show that if f is homotopic to g, then $H(f) = H(g)$.

(*d*) Show that $H(f) = 0$ if n is odd. [*Hint:* Use $\omega_{n-1} \wedge \mathrm{d}\omega_{n-1} = \frac{1}{2}\mathrm{d}(\omega_{n-1} \wedge \omega_{n-1})$.]

(*e*) Compute the Hopf invariant of the map $\pi: S^3 \to S^2$ defined in example 9.13.

10

CONNECTIONS ON FIBRE BUNDLES

In Chapter 7 we have introduced connections in Riemannian manifolds which enable us to compare vectors in different tangent spaces. In the present chapter connections on fibre bundles are defined in an abstract though geometrical way.

We first define a connection on a principal bundle. Our abstract definition is realised concretely by introducing the connection one-form whose local form is well known to physicists as a gauge potential. The Yang–Mills field strength is defined as the curvature associated with the connection. A connection on a principal bundle naturally defines a covariant derivative in the associated vector bundle. We reproduce the results obtained in Chapter 7, applying our approach to tangent bundles. We conclude this chapter with a few applications of connections to physics: to gauge field theories and Berry's phase.

We follow the line of Choquet-Bruhat and DeWitt-Morette (1982), Kobayashi (1984) and Nomizu (1981). Details will be found in the classic books by Kobayashi and Nomizu (1963, 1968). See also Daniel and Viallet (1980) for a quick review.

10.1 Connections on principal bundles

There are several equivalent definitions of a connection on a principal bundle. Our approach is based on the *separation* of tangent space T_uP into 'vertical' and 'horizontal' subspaces. Although this approach seems to be abstract, it is advantageous compared with other approaches in that it clarifies the geometrical pictures involved and is defined independently of special local trivialisations. Connections are also defined as $\mathfrak{g}$-valued one-forms which satisfy certain axioms. These definitions are shown to be equivalent.

We briefly summarise the basic facts on Lie groups and Lie algebras, since we shall make extensive use of these (see §5.6 for details). Let G be a Lie group. The left action L_g and the right action R_g are defined by $L_gh = gh$ and $R_gh = hg$ for g, $h \in G$. L_g induces a map $L_{g*} : T_h(G) \to T_{gh}(G)$. A left-invariant vector field X satisfies $L_{g*}X|_h = X|_{gh}$. Left-invariant vector fields form a Lie algebra of G, denoted by $\mathfrak{g}$. Since $X \in \mathfrak{g}$ is specified by its value at the unit element e, and vice versa, there exists a vector space isomorphism $\mathfrak{g} \simeq T_e(G)$. The Lie algebra $\mathfrak{g}$ is closed under the Lie bracket, $[T_\alpha, T_\beta] = f_{\alpha\beta}{}^{\gamma}T_\gamma$ where

$\{T_\alpha\}$ is a set of generators of $\mathfrak{g}$. $f_{\alpha\beta}{}^{\gamma}$ are called the structure constants. The adjoint action $\mathrm{ad}: G \to G$ is defined by $\mathrm{ad}_g h = ghg^{-1}$. The tangent map of ad_g is called the adjoint map and is denoted by $\mathrm{Ad}_g: T_h(G) \to T_{ghg^{-1}}(G)$. If restricted to $T_e(G) \simeq \mathfrak{g}$, Ad_g maps $\mathfrak{g}$ onto itself; $\mathrm{Ad}_g: \mathfrak{g} \to \mathfrak{g}$ as $A \mapsto gAg^{-1}$, $A \in \mathfrak{g}$.

10.1.1 Definitions

Let u be an element of a principal bundle $P(M, G)$ and let G_p be the fibre at $p = \pi(u)$. The **vertical subspace** V_uP is a subspace of T_uP which is tangent to G_p at u. [*Warning:* T_uP is the tangent space of P and should not be confused with the tangent space T_pM of M.] Let us see how V_uP is constructed. Take an element A of $\mathfrak{g}$. By the right action

$$R_{\exp(tA)}u = u\exp(tA)$$

a curve through u is defined in P. Since $\pi(u) = \pi(u\exp(tA)) = p$, this curve lies within G_p. Define a vector $A^\# \in T_uP$ by

$$A^\# f(u) = \frac{\mathrm{d}}{\mathrm{d}t} f(u\exp(tA))\Big|_{t=0} \tag{10.1}$$

where $f: P \to \mathbb{R}$ is an arbitrary smooth function. The vector $A^\#$ is tangent to G_p at u, hence $A^\# \in V_uP$. In this way we define a vector $A^\#$ at each point of P and construct a vector field $A^\#$, called the **fundamental vector field** generated by A. There is a vector space isomorphism $\#: \mathfrak{g} \to V_uP$ given by $A \mapsto A^\#$. The **horizontal subspace** H_uP is a complement of V_uP in T_uP and is uniquely specified if a connection is defined in P.

Exercise 10.1

(a) Show that $\pi_* X = 0$ for $X \in V_uP$.

(b) Show that $\#$ preserves the Lie algebra structure:

$$[A^\#, B^\#] = [A, B]^\#. \tag{10.2}$$

Definition 10.2 Let $P(M, G)$ be a principal bundle. A **connection** on P is a unique separation of the tangent space T_uP into the vertical subspace V_uP and the horizontal subspace H_uP such that

(i) $T_uP = H_uP \oplus V_uP$.

(ii) A smooth vector field X on P is separated into smooth vector fields $X^H \in H_uP$ and $X^V \in V_uP$ as $X = X^H + X^V$.

(iii) $H_{ug}P = R_{g*}H_uP$ for arbitrary $u \in P$ and $g \in G$; see figure 10.1.

The condition (iii) states that horizontal subspaces H_uP and $H_{ug}P$ on the *same* fibre are related by a linear map R_{g*} induced by the right action. Accordingly a subspace H_uP at u generates all the horizontal subspaces on the *same* fibre. This condition ensures that if a point u is parallel transported, so is its constant multiple ug, $g \in G$; see below.

At this point, the reader might feel rather uneasy about our definition of a connection. At first sight, this definition seems to have nothing to do with the gauge potential or the field strength. We clarify these points after we introduce the connection one-form on P. We again stress that our definition, which is based on the separation $T_uP = V_uP \oplus H_uP$ is purely geometrical and is defined independently of any extra information. Although the connection becomes more tractable in the following, the geometrical picture and its intrinsic nature are generally obscured.

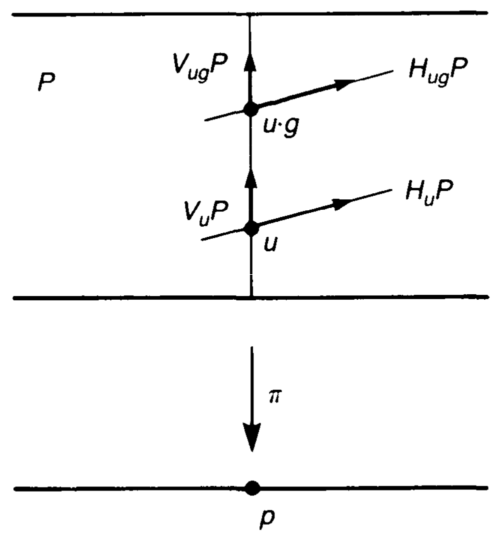


Figure 10.1 The horizontal subspace $H_{ug}P$ is obtained from H_uP by the right action.

10.1.2 The connection one-form

In practical computations, we need to separate T_uP into V_uP and H_uP in a systematic way. This can be achieved by introducing a Lie-algebra-valued one-form $\omega \in \mathfrak{g} \otimes T^*P$ called the **connection one-form**.

Definition 10.3 A connection one-form $\omega \in \mathfrak{g} \otimes T^*P$ is a *projection* of T_uP onto the vertical component $V_uP \simeq \mathfrak{g}$. The projection property is summarised by the following requirements,

$$\text{(i) } \omega(A^{\#}) = A \qquad A \in \mathfrak{g} \tag{10.3a}$$

$$\text{(ii) } R_g^*\omega = \mathrm{Ad}_{g^{-1}}\omega \tag{10.3b}$$

that is, for $X \in T_uP$,

$$R_g^*\omega_{ug}(X) = \omega_{ug}(R_{g*}X) = g^{-1}\omega_u(X)g. \tag{10.3b$'$}$$

Define the horizontal subspace H_uP by the kernel of ω,

$$H_uP \equiv \{X \in T_uP | \omega(X) = 0\}. \tag{10.4}$$

To show that this definition is consistent with definition 10.2, we prove the following proposition.

Proposition 10.4 The horizontal subspaces (10.4) satisfy

$$R_{g*}H_uP = H_{ug}P. \tag{10.5}$$

Proof: Fix a point $u \in P$ and define H_uP by (10.4). Take $X \in H_uP$ and construct $R_{g*}X \in T_{ug}P$. We find

$$\omega(R_{g*}X) = R_g^*\omega(X) = g^{-1}\omega(X)g = 0$$

since $\omega(X) = 0$. Accordingly, $R_{g*}X \in H_{ug}P$. We note that R_{g*} is an invertible linear map. Hence any vector $Y \in H_{ug}P$ is expressed as $Y = R_{g*}X$ for some $X \in H_uP$. This proves (10.5). ■

We have shown that the definition of the connection one-form ω is equivalent to that of the connection, since ω separates T_uP into $H_uP \oplus V_uP$ in harmony with the axioms of definition 10.2. The connection one-form ω defined here is known as the **Ehresmann connection** in the literature.

10.1.3 The local connection form and gauge potential

Let $\{U_i\}$ be an open covering of M and let σ_i be a local section defined on each U_i. It is convenient to introduce a Lie-algebra-valued one-form $\mathcal{A}_i$ on U_i by

$$\mathcal{A}_i \equiv \sigma_i^*\omega \in \mathfrak{g} \otimes \Omega^1(U_i). \tag{10.6}$$

Conversely, given a Lie-algebra-valued one-form $\mathcal{A}_i$ on U_i, we can reconstruct a connection one-form ω whose pullback by σ_i^* is $\mathcal{A}_i$.

Theorem 10.5 Given a $\mathfrak{g}$-valued one-form $\mathcal{A}_i$ on U_i and a local section $\sigma_i : U_i \to \pi^{-1}(U_i)$, there exists a connection one-form ω such that $\mathcal{A}_i = \sigma_i^*\omega$ on U_i.

Proof: Let us define a $\mathfrak{g}$-valued one-form ω_i on P by

$$\omega_i \equiv g_i^{-1}\pi^*\mathcal{A}_i g_i + g_i^{-1}\mathrm{d}_P g_i \tag{10.7}$$

where d_P is the exterior derivative on P and g_i is the **canonical local trivialisation** defined by $\phi_i^{-1}(u) = (p, g_i)$ for $u = \sigma_i(p)g_i$. We first show that $\sigma_i^*\omega_i = \mathcal{A}_i$. For $X \in T_pM$, we have

$$\begin{aligned}\sigma_i^*\omega_i(X) &= \omega_i(\sigma_{i*}X) = \pi^*\mathcal{A}_i(\sigma_{i*}X) + \mathrm{d}_P g_i(\sigma_{i*}X)\\ &= \mathcal{A}_i(\pi_*\sigma_{i*}X) + \mathrm{d}_P g_i(\sigma_{i*}X)\end{aligned}$$

where we have noted that $\sigma_{i*}X \in T_{\sigma_i}P$ and $g_i = e$ at σ_i, see figure 10.2. We further note that $\pi_*\sigma_{i*} = \mathrm{id}_{T_p(M)}$ and $\mathrm{d}_P g_i(\sigma_{i*}X) = 0$ since $g \equiv e$

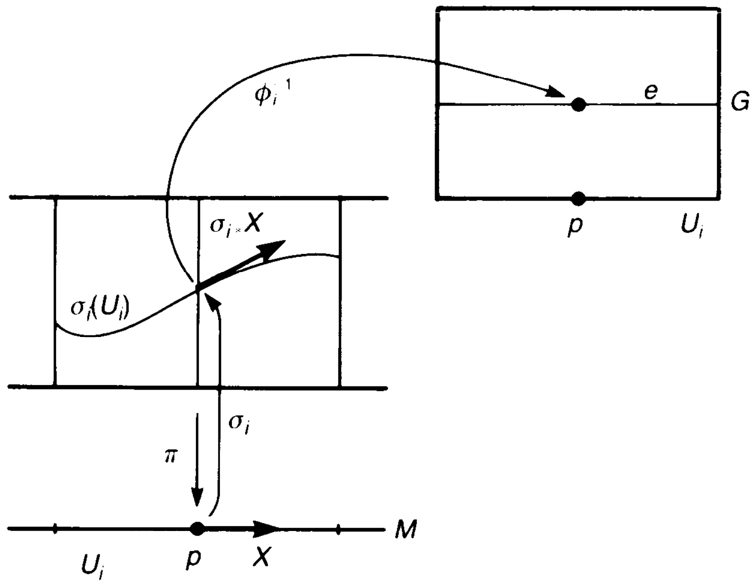


Figure 10.2 The canonical local trivialisation defined by the local section σ_i over U_i.

along $\sigma_{i*}X$. Thus we have obtained $\sigma_i^*\omega_i(X) = \mathcal{A}_i(X)$.

Next we show that ω_i satisfies the axioms of a connection one-form given in definition 10.3.

(i) Let $X = A^{\#} \in V_uP$, $A \in \mathfrak{g}$. It follows from exercise 10.1(a) that $\pi_*X = 0$. Now we have

$$\omega_i(A^{\#}) = g_i^{-1}\mathrm{d}_P g_i(A^{\#}) = g_i(u)^{-1} \left.\frac{\mathrm{d}g(u\exp(tA))}{\mathrm{d}t}\right|_{t=0}$$

$$= g_i(u)^{-1}g_i(u) \left.\frac{\mathrm{d}\exp(tA)}{\mathrm{d}t}\right|_{t=0} = A.$$

(ii) Take $X \in T_uP$ and $h \in G$. We have

$$R_h^*\omega_i(X) = \omega_i(R_{h*}X) = g_{iuh}^{-1}\mathcal{A}_i(\pi_*R_{h*}X)g_{iuh} + g_{iuh}^{-1}\mathrm{d}_P g_{iuh}(R_{h*}X).$$

Since $g_{iuh} = g_{iu}h$ and $\pi_*R_{h*}X = \pi_*X$ (note that $\pi R_h = \pi$), we have

$$R_h^*\omega_i(X) = h^{-1}g_{iu}^{-1}\mathcal{A}_i(\pi_*X)g_{iu}h + h^{-1}g_{iu}^{-1}\mathrm{d}_P g_{iu}(X)h$$

$$= h^{-1}\omega_i(X)h$$

where we have noted that

$$g_{iuh}^{-1}\mathrm{d}_P g_{iuh}(R_{h*}X) = g_{iuh}^{-1} \left.\frac{\mathrm{d}}{\mathrm{d}t}\, g_{i\gamma(t)h}\right|_{t=0}$$

$$= h^{-1}g_{iu}^{-1} \left.\frac{\mathrm{d}}{\mathrm{d}t}\, g_{i\gamma(t)}\right|_{t=0} h = h^{-1}g_{iu}^{-1}\mathrm{d}_P g_{iu}(X)h.$$

Here $\gamma(t)$ is a curve through $u = \gamma(0)$, whose tangent vector at u is X.

Hence the $\mathfrak{g}$-valued one-form ω_i defined by (10.7) indeed satisfies $\mathcal{A}_i = \sigma_i^*\omega_i$ and the axioms of a connection one-form. ∎

For ω to be defined *uniquely* on P, that is, for the separation $T_uP = H_uP \oplus V_uP$ to be unique, we must have $\omega_i = \omega_j$ on $U_i \cap U_j$. A unique one-form ω is then defined throughout P by $\omega|_{U_i} = \omega_i$. To fulfil this condition, the local forms $\mathcal{A}_i$ have to satisfy a peculiar transformation property similar to that of the Christoffel symbols. We first prove a technical lemma.

Lemma 10.6 Let $P(M, G)$ be a principal bundle and σ_i (σ_j) be a local section over U_i (U_j) such that $U_i \cap U_j \neq \emptyset$. For $X \in T_pM$ ($p \in U_i \cap U_j$), $\sigma_{i*}X$ and $\sigma_{j*}X$ satisfy

$$\sigma_{j*}X = R_{t_{ij}*}(\sigma_{i*}X) + (t_{ij}^{-1}\,\mathrm{d}t_{ij}(X))^{\#} \tag{10.8}$$

where $t_{ij} : U_i \cap U_j \to G$ is the transition function.

Proof: Take a curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X$. Since $\sigma_i(p)$ and $\sigma_j(p)$ are related by the transition function as $\sigma_j(p) = \sigma_i(p)t_{ij}(p)$ (see (9.43)), we have

$$\begin{aligned}
\sigma_{j*}X &= \frac{\mathrm{d}}{\mathrm{d}t}\,\sigma_j(\gamma(t))\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\,\{\sigma_i(t)t_{ij}(t)\}\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\,\sigma_i(t)\cdot t_{ij}(p) + \sigma_i(p)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\,t_{ij}(t)\bigg|_{t=0} \\
&= R_{t_{ij}*}(\sigma_{i*}X) + \sigma_j(p)t_{ij}(p)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\,t_{ij}(t)\bigg|_{t=0}
\end{aligned}$$

where $\sigma_i(t)$ stands for $\sigma_i(\gamma(t))$ and we assumed that G is a matrix group for which $R_{g*}X = Xg$. We note that

$$\begin{aligned}
t_{ij}(p)^{-1}\,\mathrm{d}t_{ij}(X) &= t_{ij}(p)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\,t_{ij}(t)\bigg|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\,[t_{ij}(p)^{-1}t_{ij}(t)]\bigg|_{t=0} \in T_e(G) \simeq \mathfrak{g}.
\end{aligned}$$

[Note that $t_{ij}(p)^{-1}t_{ij}(\gamma(t)) = e$ at $t = 0$.] This shows that the second term of $\sigma_{j*}X$ represents the vector field $(t_{ij}^{-1}\,\mathrm{d}t_{ij}(X))^{\#}$ at $\sigma_j(p)$. ∎

The compatibility condition is easily obtained by applying the connection one-form ω on (10.8). We find that

$$\begin{aligned}
\sigma_j^*\omega(X) &= R_{t_{ij}}^*\omega(\sigma_{i*}X) + t_{ij}^{-1}\,\mathrm{d}t_{ij}(X) \\
&= t_{ij}^{-1}\omega(\sigma_{i*}X)t_{ij} + t_{ij}^{-1}\,\mathrm{d}t_{ij}(X)
\end{aligned}$$

where the axioms of definition 10.3 have been used. Since this is true for any $X \in T_pM$, the above equation reduces to

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}. \tag{10.9}$$

This is the **compatibility condition** we have been seeking.

Conversely, given an open covering $\{U_i\}$, the local sections $\{\sigma_i\}$ and the local forms $\{\mathcal{A}_i\}$ which satisfy (10.9), we may construct the $\mathfrak{g}$-valued one-form ω over P. Since a non-trivial principal bundle does not admit a global section, the pullback $\mathcal{A}_i = \sigma_i^*\omega$ exists locally but not necessarily globally. In gauge theories, $\mathcal{A}_i$ is identified with the **gauge potential (Yang–Mills potential)**. As we have seen in the monopole case, the monopole field $\boldsymbol{B} = g\boldsymbol{r}/r^3$ does not admit a single gauge potential and we require at least two $\mathcal{A}_i$ to describe this U(1) bundle over S^2.

Exercise 10.7 Let $P(M, G)$ be a principal bundle over M and let U be a chart of M. Take local sections σ_1 and σ_2 over U such that $\sigma_2(p) = \sigma_1(p)g(p)$. Show that the corresponding local forms $\mathcal{A}_1$ and $\mathcal{A}_2$ are related as

$$\mathcal{A}_2 = g^{-1}\mathcal{A}_1 g + g^{-1}\mathrm{d}g. \tag{10.10a}$$

In components, this becomes

$$\mathcal{A}_{2\mu} = g^{-1}(p)\mathcal{A}_{1\mu}(p)g(p) + g^{-1}(p)\partial_\mu g(p) \tag{10.10b}$$

which is simply the **gauge transformation** defined in §1.2.

Example 10.8 Let P be a U(1) bundle over M. Take overlapping charts U_i and U_j. Let $\mathcal{A}_i$ ($\mathcal{A}_j$) be a local connection form on U_i (U_j). The transition function $t_{ij} : U_i \cap U_j \to \mathrm{U}(1)$ is given by

$$t_{ij}(p) = \exp[\mathrm{i}\chi(p)] \qquad \chi(p) \in \mathbb{R}. \tag{10.11}$$

$\mathcal{A}_i$ and $\mathcal{A}_j$ are related as

$$\begin{aligned}\mathcal{A}_j(p) &= t_{ij}(p)^{-1}\mathcal{A}_i(p)t_{ij}(p) + t_{ij}(p)^{-1}\,\mathrm{d}t_{ij}(p)\\ &= \mathcal{A}_i(p) + \mathrm{i}\,\mathrm{d}\chi(p).\end{aligned} \tag{10.12a}$$

In components, we have the familiar expression

$$\mathcal{A}_{j\mu} = \mathcal{A}_{i\mu} + \mathrm{i}\partial_\mu\chi. \tag{10.12b}$$

Our connection $\mathcal{A}_\mu$ differs from the standard vector potential A_μ by the Lie algebra factor, $\mathcal{A}_\mu = \mathrm{i}A_\mu$.

Here we note again that ω is defined globally over the bundle $P(M, G)$. Although there are many connection one-forms on $P(M, G)$, they share the same global information about the bundle. On the contrary, an individual local piece (gauge potential) $\mathcal{A}_i$ is associated with the *trivial* bundle $\pi^{-1}(U_i)$ and cannot have any global information on P. It is ω, or equivalently the *total* of $\{\mathcal{A}_i\}$ satisfying the compatibility condition (10.9), which carries the global information about the bundle.

10.1.4 Horizontal lift and parallel transport

In Chapter 7, parallel transport of a vector has been defined as transport *without change*. Parallel transport of an element of a principal bundle along a curve in M is provided by the 'horizontal lift' of the curve.

Definition 10.9 Let $P(M, G)$ be a G bundle and let $\gamma : [0, 1] \rightarrow M$ be a curve in M. A curve $\widetilde{\gamma} : [0, 1] \rightarrow P$ is said to be a **horizontal lift** of γ if $\pi\widetilde{\gamma} = \gamma$ and the tangent vector to $\widetilde{\gamma}(t)$ always belongs to $H_{\widetilde{\gamma}(t)}P$.

Let $\widetilde{X}$ be a tangent vector to $\widetilde{\gamma}$. Then it satisfies $\omega(\widetilde{X}) = 0$ by definition. This condition is an ordinary differential equation (ODE) and the fundamental theorem of ODE guarantees the local existence and uniqueness of the horizontal lift.

Theorem 10.10 Let $\gamma : [0, 1] \rightarrow M$ be a curve in M and let $u_0 \in \pi^{-1}(\gamma(0))$. Then there exists a unique horizontal lift $\widetilde{\gamma}(t)$ in P such that $\widetilde{\gamma}(0) = u_0$.

Let us construct such a curve $\widetilde{\gamma}$. Let U_i be a chart which contains γ and take a section σ_i over U_i. If there exists a horizontal lift $\widetilde{\gamma}$, it may be expressed as $\widetilde{\gamma}(t) = \sigma_i(\gamma(t))g_i(t)$, where $g_i(t)$ stands for $g_i(\gamma(t)) \in G$. Without loss of generality, we may take a section such that $\sigma_i(\gamma(0)) = \widetilde{\gamma}(0)$, that is $g_i(0) = e$. Let X be a tangent vector to $\gamma(t)$ at $\gamma(0)$. Then $\widetilde{X} = \widetilde{\gamma}_* X$ is tangent to $\widetilde{\gamma}$ at $u_0 = \widetilde{\gamma}(0)$. Since the tangent vector $\widetilde{X}$ is horizontal, it satisfies $\omega(\widetilde{X}) = 0$. A slight modification of lemma 10.6 yields

$$\widetilde{X} = g_i(t)^{-1}\sigma_{i*}Xg_i(t) + [g_i(t)^{-1}\,\mathrm{d}g_i(X)]^{\#}.$$

By applying ω on the above equation, we find

$$0 = \omega(\widetilde{X}) = g_i(t)^{-1}\omega(\sigma_{i*}X)g_i(t) + g_i(t)^{-1}\frac{\mathrm{d}g_i(t)}{\mathrm{d}t}.$$

Multiplying on the left by $g_i(t)$, we have

$$\frac{\mathrm{d}g_i(t)}{\mathrm{d}t} = -\omega(\sigma_{i*}X)g_i(t). \tag{10.13a}$$

The fundamental theorem of ODE guarantees the existence and uniqueness of the solution of (10.13a).

Since $\omega(\sigma_{i*}X) = \sigma_i^*\omega(X) = \mathcal{A}_i(X)$, (10.13$a$) is expressed in a local form,

$$\frac{\mathrm{d}g_i(t)}{\mathrm{d}t} = -\mathcal{A}_i(X)g_i(t) \tag{10.13b}$$

whose formal solution with $g_i(0) = e$ is

$$\begin{aligned} g_i(\gamma(t)) &= \mathrm{P}\exp\left(-\int_0^t \mathcal{A}_{i\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\,\mathrm{d}t\right) \\ &= \mathrm{P}\exp\left(-\int_{\gamma(0)}^{\gamma(t)} \mathcal{A}_{i\mu}(\gamma(t))\,\mathrm{d}x^\mu\right). \end{aligned} \tag{10.14}$$

where P is a path-ordering operator along $\gamma(t)$. The horizontal lift is expressed as $\tilde{\gamma}(t) = \sigma_i(\gamma(t))g_i(\gamma(t))$.

Corollary 10.11 Let $\tilde{\gamma}'$ be another horizontal lift of γ, such that $\tilde{\gamma}'(0) = \gamma(0)g$. Then $\tilde{\gamma}'(t) = \tilde{\gamma}(t)g$ for all $t \in [0, 1]$.

Proof: We first note that the horizontal subspace is right invariant, $R_{g*}H_uP = H_{ug}P$. Let $\tilde{\gamma}$ be a horizontal lift of γ. Then $\tilde{\gamma}_g : t \mapsto \tilde{\gamma}(t)g$ is also a horizontal lift of $\gamma(t)$ since its tangent vector belongs to $H_{\tilde{\gamma}g}P$. From theorem 10.10 we find $\tilde{\gamma}'$ is the unique horizontal lift which starts at $\tilde{\gamma}(0)g$. ■

Example 10.12 Let us consider the bundle $P(M, \mathbb{R}) \cong M \times \mathbb{R}$ where $M = \mathbb{R}^2 - \{0\}$. Let $\phi : ((x, y), f) \mapsto u \in P$ be a local trivialisation, where (x, y) are the coordinates of M while f is that of the additive group $\mathbb{R}$. Let

$$\omega = \frac{y\,\mathrm{d}x - x\,\mathrm{d}y}{x^2 + y^2} + \mathrm{d}f$$

be a connection one-form. It is easily verified that ω satisfies the axioms of the connection one-form. In fact, for $A^\# = A\partial/\partial f$, $A \in \mathbb{R}$ being an element of the Lie algebra of additive group, we have $\omega(A^\#) = A$. Furthermore $R_g{}^*\omega = \omega = g^{-1}\omega g$, since $\mathbb{R}$ is Abelian. Let $\gamma : [0, 1] \to M$ be a curve $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. Let us work out a horizontal lift which starts at $((1, 0), 0)$. Let

$$X = \frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial}{\partial y} + \frac{\mathrm{d}f}{\mathrm{d}t}\frac{\partial}{\partial f}$$

be tangent to $\tilde{\gamma}(t)$. For X to be horizontal, it must satisfy

$$0 = \omega(X) = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{y}{r^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{x}{r^2} + \frac{\mathrm{d}f}{\mathrm{d}t} = -2\pi + \frac{\mathrm{d}f}{\mathrm{d}t}.$$

The solution is easily found to be $f = 2\pi t + \text{const}$. We finally find the horizontal lift $\tilde{\gamma}$ passing through $((1, 0), 0)$,

$$\tilde{\gamma}(t) = ((\cos 2\pi t, \sin 2\pi t), 2\pi t) \tag{10.15}$$

which is a helix over the unit circle.

Under the group action (right or left does not matter), f translates to $f + g$, $g \in \mathbb{R}$. The shifted horizontal lift is

$$\tilde{\gamma}_g(t) = ((\cos 2\pi t, \sin 2\pi t), 2\pi t + g). \tag{10.16}$$

Let $\gamma : [0, 1] \to M$ be a curve. Take a point $u_0 \in \pi^{-1}(\gamma(0))$. There is a unique horizontal lift $\tilde{\gamma}(t)$ of $\gamma(t)$ through u_0, and hence a unique point $u_1 = \tilde{\gamma}(1) \in \pi^{-1}(\gamma(1))$, see figure 10.3. The point u_1 is called the **parallel transport** of u_0 along the curve $\tilde{\gamma}$. This defines a map $\Gamma(\tilde{\gamma}) : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$ such that $u_0 \mapsto u_1$. If the local form (10.14) is employed, we have

$$u_1 = \sigma_i(1) \mathrm{P} \exp\left(- \int_0^1 \mathcal{A}_{i\mu} \frac{\mathrm{d}x^\mu(\gamma(t))}{\mathrm{d}t} \, \mathrm{d}t \right) \tag{10.17}$$

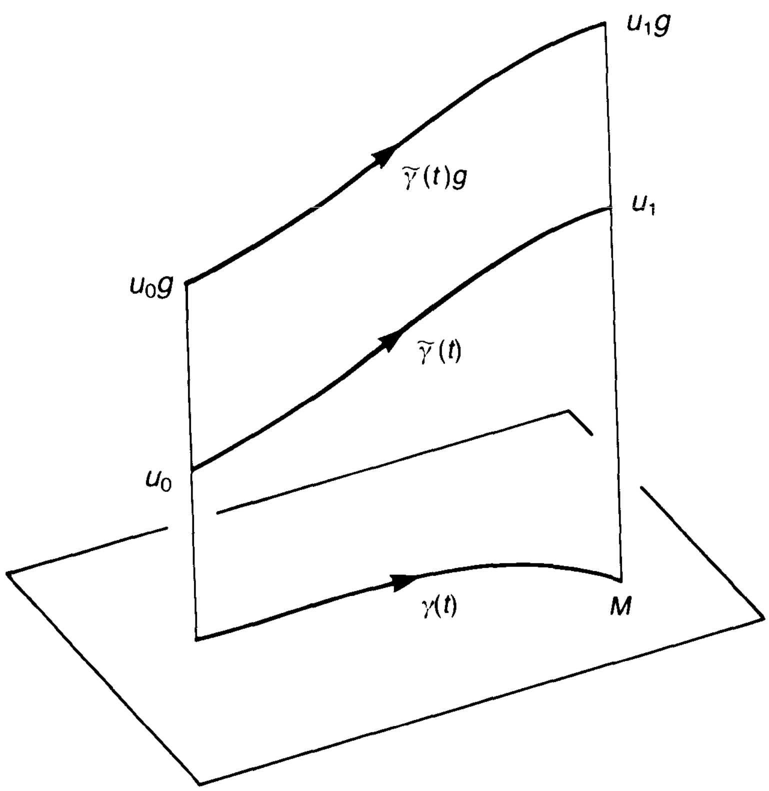


Figure 10.3 A curve $\gamma(t)$ in M and its horizontal lifts $\tilde{\gamma}(t)$ and $\tilde{\gamma}(t)g$.

Corollary 10.11 ensures that $\Gamma(\tilde{\gamma})$ commutes with the right action R_g. First note that $R_g\Gamma(\tilde{\gamma})(u_0) = u_1 g$ and $\Gamma(\tilde{\gamma})R_g(u_0) = \Gamma(\tilde{\gamma})(u_0 g)$. Observe that $\tilde{\gamma}(t)g$ is a horizontal lift through $u_0 g$ *and* $u_1 g$. From the uniqueness of the horizontal lift through $u_0 g$, we have $u_1 g = \Gamma(\tilde{\gamma})(u_0 g)$, that is, $R_g\Gamma(\tilde{\gamma})(u_0) = \Gamma(\tilde{\gamma})R_g(u_0)$. Since this is true for any $u_0 \in \pi^{-1}(\gamma(0))$, we have

$$R_g\Gamma(\tilde{\gamma}) = \Gamma(\tilde{\gamma})R_g. \tag{10.18}$$

Exercise 10.13 Let $\tilde{\gamma}$ be a horizontal lift of $\gamma : [0, 1] \to M$. Consider a map $\Gamma(\tilde{\gamma}^{-1}) : \pi^{-1}(\gamma(1)) \to \pi^{-1}(\gamma(0))$ where $\tilde{\gamma}^{-1}(t) = \tilde{\gamma}(1 - t)$. Show that

$$\Gamma(\widetilde{\gamma}^{-1}) = \Gamma(\widetilde{\gamma})^{-1}. \tag{10.19}$$

Consider two curves $\alpha : [0, 1] \to M$ and $\beta : [0, 1] \to M$ such that $\alpha(1) = \beta(0)$. Define the product $\alpha * \beta : [0, 1] \to M$ by

$$\alpha * \beta = \begin{cases} \alpha(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ \beta(2t - 1) & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Let $\Gamma(\widetilde{\alpha}) : \pi^{-1}(\alpha(0)) \to \pi^{-1}(\alpha(1))$ and $\Gamma(\widetilde{\beta}) : \pi^{-1}(\beta(0)) \to \pi^{-1}(\beta(1))$. Show that

$$\Gamma(\widetilde{\alpha * \beta}) = \Gamma(\widetilde{\beta})\Gamma(\widetilde{\alpha}). \tag{10.20}$$

Exercise 10.14 Let us write $u \sim v$, if $u, v \in P$ are on the same horizontal lift. Show that $\sim$ is an equivalence relation.

10.2 Holonomy

10.2.1 Definitions

Let $P(M, G)$ be a principal bundle and let $\gamma : [0, 1] \to M$ be a curve whose horizontal lift through $u_0 \in \pi^{-1}(\gamma(0))$ is $\widetilde{\gamma}$. In the last section, we defined a map $\Gamma(\widetilde{\gamma}) : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$ which maps a point $u_0 = \widetilde{\gamma}(0)$ to $u_1 = \widetilde{\gamma}(1)$. Let us consider two curves $\alpha, \beta : [0, 1] \to M$ with $\alpha(0) = \beta(0) = p_0$ and $\alpha(1) = \beta(1) = p_1$. Take horizontal lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ of α and β such that $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = u_0$. Then $\widetilde{\alpha}(1)$ is not necessarily equal to $\widetilde{\beta}(1)$. This shows that if we consider a *loop* $\gamma : [0, 1] \to M$ at $p = \gamma(0) = \gamma(1)$, we have $\widetilde{\gamma}(0) \neq \widetilde{\gamma}(1)$ in general. A loop γ defines a *transformation* $\tau_\gamma : \pi^{-1}(p) \to \pi^{-1}(p)$ on the fibre. This transformation is compatible with the right action of the group,

$$\tau_\gamma(ug) = \tau_\gamma(u)g \tag{10.21}$$

which follows immediately from (10.18). We note that τ_γ depends not only on the loop γ but also on the connection.

Example 10.15 Consider an $\mathbb{R}$-bundle over $M = \mathbb{R}^2 - \{0\}$. The connection one-form ω and the loop γ in example 10.12 define a map $\tau_\gamma : \pi^{-1}((1, 0)) \to \pi^{-1}((1, 0))$ given by $g \mapsto g + 2\pi$, $g \in \mathbb{R}$.

Take a point $u \in P$ with $\pi(u) = p$ and consider the set of loops $C_p(M)$ at p; $C_p(M) \equiv \{\gamma : [0, 1] \to M | \gamma(0) = \gamma(1) = p\}$. The set of elements

$$\Phi_u \equiv \{g \in G | \tau_\gamma(u) = ug, \gamma \in C_p(M)\} \tag{10.22}$$

is a subgroup of the structure group G and is called the **holonomy group** at u. The group property of Φ_u is easily derived from exercise 10.13. If α, β and $\gamma = \alpha * \beta$ are loops at p, we have $\tau_\gamma = \tau_\beta \tau_\alpha$ hence

$$\tau_\gamma(u) = \tau_\beta \tau_\alpha(u) = \tau_\beta(ug_\alpha) = \tau_\beta(u)g_\alpha = ug_\beta g_\alpha$$

where $\tau_\alpha(u) = ug_\alpha$ etc. This shows that

$$g_\gamma = g_\beta g_\alpha. \tag{10.23}$$

The constant loop $c : [0, 1] \mapsto p$ defines the identity transformation $\tau_c : u \mapsto u$. The inverse loop γ^{-1} of γ induces the inverse transformation $\tau_{\gamma^{-1}} = \tau_\gamma^{-1}$, hence $g_{\gamma^{-1}} = g_\gamma^{-1}$.

Exercise 10.16

(*a*) Let $\tau_\alpha(u) = ug_\alpha$. Show that

$$\tau_\alpha(ug) = ug(\mathrm{ad}_g g_\alpha) = ug(g^{-1}g_\alpha g). \tag{10.24}$$

Verify that

$$\Phi_{ua} \cong a^{-1}\Phi_u a. \tag{10.25}$$

(*b*) Let u, $u' \in P$ be points on the same horizontal lift $\tilde{\gamma}$. Show that $\Phi_u \cong \Phi_{u'}$.

(*c*) Suppose that M is connected. Show that all Φ_u are isomorphic to each other.

Exercise 10.17 Let $\mathcal{A}_i = \mathcal{A}_{i\mu} \mathrm{d}x^\mu$ be a gauge potential over U_i and γ a curve in U_i. Let $\tau_\gamma(u) = ug_\gamma$. Use (10.14) to show that

$$g_\gamma = \exp\left(- \oint_\gamma \mathcal{A}_{i\mu} \mathrm{d}x^\mu \right). \tag{10.26}$$

Let $C_p^0(M)$ denote the set of loops at p, which are homotopic to the constant loop at p. The group

$$\Phi_u^0 \equiv \{g \in G | \tau_\gamma(u) = ug, \gamma \in C_p^0(M)\} \tag{10.27}$$

is called the **restricted holonomy group**.

10.3 Curvature

10.3.1 Covariant derivatives in principal bundles

We defined the exterior derivative $\mathrm{d} : \Omega^r(M) \to \Omega^{r+1}(M)$ in Chapter 5. An r-form is a real-valued form acting on vectors,

$$\eta : TM \otimes \ldots \otimes TM \to \mathbb{R}.$$

We will generalise this operation so that we can differentiate a vector-valued r-form $\phi \in \Omega^r(P) \otimes V$,

$$\phi : TP \otimes \ldots \otimes TP \to V$$

where V is a vector space of dimension k. The most general form of ϕ is $\phi = \Sigma_{\alpha=1}^{k} \phi^{\alpha} \otimes e_{\alpha}$, $\{e_{\alpha}\}$ being a basis of V and $\phi^{\alpha} \in \Omega^{r}(P)$.

A connection ω on a principal bundle $P(M, G)$ separates T_uP into $H_uP \oplus V_uP$. Accordingly a vector $X \in T_uP$ is decomposed as $X = X^H + X^V$ where $X^H \in H_uP$ and $X^V \in V_uP$.

Definition 10.18 Let $\phi \in \Omega^{r}(P) \otimes V$ and $X_1, \ldots, X_{r+1} \in T_uP$. The **covariant derivative** of ϕ is defined by

$$\mathrm{D}\phi(X_1, \ldots, X_{r+1}) \equiv \mathrm{d}_P\phi(X_1^H, \ldots, X_{r+1}^H) \tag{10.28}$$

where $\mathrm{d}_P\phi \equiv \mathrm{d}_P\phi^{\alpha} \otimes e_{\alpha}$.

10.3.2 Curvature

Definition 10.19 The **curvature two-form** Ω is the covariant derivative of the connection one-form ω,

$$\Omega \equiv \mathrm{D}\omega \in \Omega^2(P) \otimes \mathfrak{g}. \tag{10.29}$$

Proposition 10.20 The curvature two-form satisfies (cf (10.3b))

$$R_a^*\Omega = a^{-1}\Omega a \qquad a \in G. \tag{10.30}$$

Proof: We first note that $(R_{a*}X)^H = R_{a*}(X^H)$ (R_{a*} preserves the horizontal subspaces) and $\mathrm{d}_P R_a^* = R_a^* \mathrm{d}_P$. By definition we have

$$\begin{aligned} R_a^*\Omega(X, Y) &= \Omega(R_{a*}X, R_{a*}Y) = \mathrm{d}_P\omega((R_{a*}X)^H, (R_{a*}Y)^H) \\ &= \mathrm{d}_P\omega(R_{a*}X^H, R_{a*}Y^H) = R_a^*\mathrm{d}_P\omega(X^H, Y^H) \\ &= \mathrm{d}_P R_a^*\omega(X^H, Y^H) \\ &= \mathrm{d}_P(a^{-1}\omega a)(X^H, Y^H) = a^{-1}\mathrm{d}_P\omega(X^H, Y^H)a \\ &= a^{-1}\Omega(X, Y)a \end{aligned}$$

where we noted that a is a constant element and hence $\mathrm{d}_P a = 0$. ∎

Take a $\mathfrak{g}$-valued p-form $\zeta = \zeta^{\alpha} \otimes T_{\alpha}$ and a $\mathfrak{g}$-valued q-form $\eta = \eta^{\alpha} \otimes T_{\alpha}$ where $\zeta^{\alpha} \in \Omega^{p}(M)$, $\eta^{\alpha} \in \Omega^{q}(M)$ and $\{T_{\alpha}\}$ is a basis of $\mathfrak{g}$. Define the commutator of ζ and η by

$$\begin{aligned} [\zeta, \eta] &\equiv \zeta \wedge \eta - (-1)^{pq}\eta \wedge \zeta \\ &= T_{\alpha}T_{\beta}\zeta^{\alpha} \wedge \eta^{\beta} - (-1)^{pq}T_{\beta}T_{\alpha}\eta^{\beta} \wedge \zeta^{\alpha} \\ &= [T_{\alpha}, T_{\beta}] \otimes \zeta^{\alpha} \wedge \eta^{\beta} = f_{\alpha\beta}{}^{\gamma}T_{\gamma} \otimes \zeta^{\alpha} \wedge \eta^{\beta}. \end{aligned} \tag{10.31}$$

If we put $\zeta = \eta$ in (10.31), we have

$$[\zeta, \zeta] = 2\zeta \wedge \zeta = f_{\alpha\beta}{}^{\gamma}T_{\gamma} \otimes \zeta^{\alpha} \wedge \zeta^{\beta}.$$

Lemma 10.21 Let $X \in H_uP$ and $Y \in V_uP$. Then $[X, Y] \in H_uP$.

Proof: Let Y be a vector field generated by $g(t)$, then

$$[Y, X] = \lim_{t \to 0} t^{-1} (R_{g(t)*}X - X).$$

Since a connection satisfies $R_{g*}H_uP = H_{ug}P$, the vector $R_{g(t)*}Y$ is horizontal and so is $[Y, X]$. ■

Theorem 10.22 Let $X, Y \in T_uP$. Then Ω and ω satisfy **Cartan's structure equation**

$$\Omega(X, Y) = \mathrm{d}_P\omega(X, Y) + [\omega(X), \omega(Y)] \qquad (10.32a)$$

which is also written as

$$\Omega = \mathrm{d}_P\omega + \omega \wedge \omega. \qquad (10.32b)$$

[To derive (10.32b) from (10.32a), we note that

$$\begin{aligned}[\omega, \omega](X, Y) &= [T_\alpha, T_\beta]\omega^\alpha \wedge \omega^\beta(X, Y)\\ &= [T_\alpha, T_\beta][\omega^\alpha(X)\omega^\beta(Y) - \omega^\beta(X)\omega^\alpha(Y)]\\ &= [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] = 2[\omega(X), \omega(Y)].\end{aligned}$$

Hence $\Omega(X, Y) = (\mathrm{d}_P\omega + \frac{1}{2}[\omega, \omega])(X, Y) = (\mathrm{d}_P\omega + \omega \wedge \omega)(X, Y)$.]

Proof: We consider the following three cases separately:

(1) Let $X, Y \in H_uP$. Then $\omega(X) = \omega(Y) = 0$ by definition. From definition 10.19, we have $\Omega(X, Y) = \mathrm{d}_P\omega(X^H, Y^H) = \mathrm{d}_P\omega(X, Y)$, since $X = X^H$ and $Y = Y^H$.

(2) Let $X \in H_uP$, $Y \in V_uP$. Since $Y^H = 0$, we have $\Omega(X, Y) = 0$. We also have $\omega(X) = 0$. Thus we need to prove $\mathrm{d}_P\omega(X, Y) = 0$. From (5.70), we have

$$\mathrm{d}_P\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) = X\omega(Y) - \omega([X, Y]).$$

Since $Y \in V_uP$, there is an element $V \in \mathfrak{g}$ such that $Y = V^\#$. Then $\omega(Y) = V$ is constant, hence $X\omega(Y) = X \cdot V = 0$. From lemma 10.21, we have $[X, Y] \in H_uP$ so that $\omega([X, Y]) = 0$ and we find $\mathrm{d}_P\omega(X, Y) = 0$.

(3) For $X, Y \in V_uP$, we have $\Omega(X, Y) = 0$. We find that in this case,

$$\mathrm{d}_P\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) = -\omega([X, Y]).$$

We note that X and Y are closed under the Lie bracket, $[X, Y] \in V_uP$, see exercise 10.1(b). Then there exists $A \in \mathfrak{g}$ such that

$$\omega([X, Y]) = A$$

where $A^\# = [X, Y]$. Let $B^\# = X$ and $C^\# = Y$. Then $[\omega(X), \omega(Y)] = [B, C] = A$ since $[B, C]^\# = [B^\#, C^\#]$. Thus we have shown that

$$0 = \mathrm{d}_P\omega(X, Y) + \omega([X, Y]) = \mathrm{d}_P\omega(X, Y) + [\omega(X), \omega(Y)].$$

Since Ω is linear and skew-symmetric, the above three cases are sufficient to show that (10.32) is true for any vectors. ■

10.3.3 Geometrical meaning of the curvature and the Ambrose–Singer theorem

In Chapter 7 we have shown that the Riemann curvature tensor expresses the non-commutativity of the parallel transport of vectors. There is a similar interpretation of curvature on principal bundles. We first show that $\Omega(X, Y)$ yields the vertical component of the Lie bracket $[X, Y]$ of horizontal vectors $X, Y \in H_uP$. It follows from $\omega(X) = \omega(Y) = 0$ that

$$\mathrm{d}_P\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) = -\omega([X, Y]).$$

Since $X^H = X$, $Y^H = Y$, we have

$$\Omega(X, Y) = \mathrm{d}_P\omega(X, Y) = -\omega([X, Y]). \tag{10.33}$$

Let us consider a coordinate system $\{x^\mu\}$ on a chart U. Let $V = \partial/\partial x^1$ and $W = \partial/\partial x^2$. Take an infinitesimal parallelogram γ whose corners are $\mathrm{O} = \{0\}$, $\mathrm{P} = \{\varepsilon, 0, \ldots, 0\}$, $\mathrm{Q} = \{\varepsilon, \delta, 0, \ldots, 0\}$ and $\mathrm{R} = \{0, \delta, 0, \ldots, 0\}$. Consider the horizontal lift $\tilde{\gamma}$ of γ. Let X, $Y \in H_uP$ such that $\pi_*X = \varepsilon V$ and $\pi_*Y = \delta W$. Then

$$\pi_*([X, Y]^H) = \varepsilon\delta[V, W] = \varepsilon\delta\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right] = 0 \tag{10.34}$$

that is, $[X, Y]$ is *vertical*. The above consideration shows that the horizontal lift $\tilde{\gamma}$ of a loop γ fails to close. This failure is proportional to the vertical vector $[X, Y]$ connecting the initial point and the final point on the *same* fibre. The curvature measures this distance,

$$\Omega(X, Y) = -\omega([X, Y]) = A \tag{10.35}$$

where A is an element of $\mathfrak{g}$ such that $[X, Y] = A^\#$.

Since the discrepancy between the initial and final points of the horizontal lift of a closed curve is simply the holonomy, we expect that the holonomy group is expressed in terms of the curvature.

Theorem 10.23 **(Ambrose–Singer)** Let $P(M, G)$ be a G bundle over a connected manifold M. The Lie algebra $\mathfrak{h}$ of the holonomy group Φ_{u_0} of a point $u_0 \in P$ agrees with the subalgebra of $\mathfrak{g}$ spanned by the elements of the form

$$\Omega_u(X, Y) \qquad X, Y \in H_uP \tag{10.36}$$

where $u \in P$ is a point on the same horizontal lift as u_0. [See Choquet-Bruhat *et al* (1982) for the proof.]

10.3.4 Local form of the curvature

The local form $\mathcal{F}$ of the curvature Ω is defined by

$$\mathcal{F} \equiv \sigma^*\Omega \tag{10.37}$$

where σ is a local section defined on a chart U of M (cf $\mathcal{A} = \sigma^*\omega$). $\mathcal{F}$ is expressed in terms of the gauge potential $\mathcal{A}$ as

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \tag{10.38a}$$

where d is the exterior derivative on M. The action of $\mathcal{F}$ on the vectors of TM is given by

$$\mathcal{F}(X, Y) = \mathrm{d}\mathcal{A}(X, Y) + [\mathcal{A}(X), \mathcal{A}(Y)]. \tag{10.38b}$$

To prove (10.38*a*) we note that $\mathcal{A} = \sigma^*\omega$, $\sigma^*\mathrm{d}_P\omega = \mathrm{d}\sigma^*\omega$ and $\sigma^*(\zeta \wedge \eta) = \sigma^*\zeta \wedge \sigma^*\eta$. From Cartan's structure equation, we find

$$\mathcal{F} = \sigma^*(\mathrm{d}_P\omega + \omega \wedge \omega) = \mathrm{d}\sigma^*\omega + \sigma^*\omega \wedge \sigma^*\omega = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}.$$

Next we find the component expression of $\mathcal{F}$ on a chart U whose coordinates are $x^\mu = \varphi(p)$. Let $\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu (\mathcal{A}_\mu \in \mathfrak{g})$ be the gauge potential. If we write $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, a direct computation yields

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{10.39}$$

$\mathcal{F}$ is also called the curvature two-form and is identified with the **(Yang–Mills) field strength**. To avoid confusion, we call Ω the curvature and $\mathcal{F}$ the (Yang–Mills) field strength. Since $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$ are $\mathfrak{g}$-valued functions, they can be expanded in terms of the basis $\{T_\alpha\}$ of $\mathfrak{g}$ as

$$\mathcal{A}_\mu = A_\mu{}^\alpha T_\alpha \qquad \mathcal{F}_{\mu\nu} = F_{\mu\nu}{}^\alpha T_\alpha. \tag{10.40}$$

The basis vectors satisfy the usual commutation relations $[T_\alpha, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma$. We then obtain the well known expression

$$F_{\mu\nu}{}^\alpha = \partial_\mu A_\nu{}^\alpha - \partial_\nu A_\mu{}^\alpha + f_{\beta\gamma}{}^\alpha A_\mu{}^\beta A_\nu{}^\gamma. \tag{10.41}$$

Theorem 10.24 Let U_i and U_j be overlapping charts of M and let $\mathcal{F}_i$ and $\mathcal{F}_j$ be field strengths on the respective charts. On $U_i \cap U_j$, they satisfy the compatibility condition,

$$\mathcal{F}_j = \mathrm{Ad}_{t_{ij}^{-1}}\mathcal{F}_i = t_{ij}^{-1}\mathcal{F}_i t_{ij} \tag{10.42}$$

where t_{ij} is the transition function on $U_i \cap U_j$.

Proof: Introduce the corresponding gauge potentials $\mathcal{A}_i$ and $\mathcal{A}_j$,

$$\mathcal{F}_i = \mathrm{d}\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i \qquad \mathcal{F}_j = \mathrm{d}\mathcal{A}_j + \mathcal{A}_j \wedge \mathcal{A}_j.$$

Substituting $\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}$ into $\mathcal{F}_j$, we verify that

$$\begin{aligned}
\mathcal{F}_j &= \mathrm{d}(t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}) \\
&\quad + (t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}) \wedge (t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}) \\
&= [-t_{ij}^{-1}\mathrm{d}t_{ij} \wedge t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}\mathcal{A}_i t_{ij} \\
&\quad - t_{ij}^{-1}\mathcal{A}_i \wedge \mathrm{d}t_{ij} - t_{ij}^{-1}\mathrm{d}t_{ij}t_{ij}^{-1} \wedge \mathrm{d}t_{ij}] \\
&\quad + [t_{ij}^{-1}\mathcal{A}_i \wedge \mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathcal{A}_i \wedge \mathrm{d}t_{ij} \\
&\quad + t_{ij}^{-1}\mathrm{d}t_{ij}t_{ij}^{-1} \wedge \mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij} \wedge t_{ij}^{-1}\mathrm{d}t_{ij}] \\
&= t_{ij}^{-1}(\mathrm{d}\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i)t_{ij} = t_{ij}^{-1}\mathcal{F}_i t_{ij}
\end{aligned}$$

where use has been made of the identity $\mathrm{d}t^{-1} = -t^{-1}\mathrm{d}t\, t^{-1}$. ■

Exercise 10.25 If the gauge potential $\mathcal{A}$ is written locally as $\mathcal{A} = g^{-1}\mathrm{d}g$, $\mathcal{A}$ is called a **pure gauge**. Show that the field strength $\mathcal{F}$ vanishes for a pure gauge $\mathcal{A}$. [It can be shown that the converse is also true. If $\mathcal{F} = 0$ on a chart U, the gauge potential may be expressed locally as $\mathcal{A} = g^{-1}\mathrm{d}g$.]

10.3.5 The Bianchi identity

Since ω and Ω are $\mathfrak{g}$-valued, we expand them in terms of the basis $\{T_\alpha\}$ of $\mathfrak{g}$ as $\omega = \omega^\alpha T_\alpha$, $\Omega = \Omega^\alpha T_\alpha$. Then (10.32*b*) becomes

$$\Omega^\alpha = \mathrm{d}_P\omega^\alpha + f_{\beta\gamma}{}^\alpha \omega^\beta \wedge \omega^\gamma. \tag{10.43}$$

Exterior differentiation of (10.43) yields

$$\mathrm{d}_P\Omega^\alpha = f_{\beta\gamma}{}^\alpha \mathrm{d}_P\omega^\beta \wedge \omega^\gamma + f_{\beta\gamma}{}^\alpha \omega^\beta \wedge \mathrm{d}_P\omega^\gamma. \tag{10.44}$$

If we note that $\omega(X) = 0$ for a horizontal vector X, we have

$$\mathrm{D}\Omega(X, Y, Z) = \mathrm{d}_P\Omega(X^H, Y^H, Z^H) = 0$$

where $X, Y, Z \in T_uP$. Thus we have proved the **Bianchi identity**

$$\mathrm{D}\Omega = 0. \tag{10.45}$$

Let us find the local form of the Bianchi identity. Operating with σ^* on (10.44), we find that $\sigma^*\mathrm{d}_P\Omega = \mathrm{d}\cdot\sigma^*\Omega = \mathrm{d}\mathcal{F}$ for the LHS and

$$\begin{aligned}
\sigma^*(\mathrm{d}_P\omega \wedge \omega - \omega \wedge \mathrm{d}_P\omega) &= \mathrm{d}\sigma^*\omega \wedge \sigma^*\omega - \sigma^*\omega \wedge \mathrm{d}\sigma^*\omega \\
&= \mathrm{d}\mathcal{A} \wedge \mathcal{A} - \mathcal{A} \wedge \mathrm{d}\mathcal{A} = \mathcal{F} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{F}
\end{aligned}$$

for the RHS. Thus we have obtained that

$$\mathcal{D}\mathcal{F} = \mathrm{d}\mathcal{F} + \mathcal{A} \wedge \mathcal{F} - \mathcal{F} \wedge \mathcal{A} = \mathrm{d}\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0 \tag{10.46}$$

where the action of $\mathcal{D}$ on a $\mathfrak{g}$-valued p-form η on M is defined by

$$\mathcal{D}\eta \equiv \mathrm{d}\eta + [\mathcal{A}, \eta]. \tag{10.47}$$

Note that $\mathcal{D}\mathcal{F} = \mathrm{d}\mathcal{F}$ for $G = \mathrm{U}(1)$.

10.4 The covariant derivative on associated vector bundles

A connection one-form ω on a principal bundle $P(M, G)$ enables us to define the covariant derivative in associated bundles of P in a natural way.

10.4.1 The covariant derivative on associated bundles

In physics, we often need to differentiate sections of a vector bundle which is associated with a certain principal bundle. For example, a charged scalar field in QED is regarded as a section of a complex line bundle associated with a U(1) bundle $P(M, \mathrm{U}(1))$. Differentiating sections covariantly is very important in constructing gauge-invariant actions.

Let $P(M, G)$ be a G bundle with the projection π_P. Let us take a chart U_i of M and a section σ_i over U_i. We take the canonical trivialisation $\phi_i(p, e) = \sigma_i(p)$. Let $\tilde{\gamma}$ be a horizontal lift of a curve $\gamma : [0, 1] \to U_i$. We denote $\gamma(0) = p_0$ and $\tilde{\gamma}(0) = u_0$. Associated with P is a vector bundle $E = P \times_\rho V$ with the projection π_E; see §9.4. Let $X \in T_{p_0}M$ be a tangent vector to $\gamma(t)$ at p_0. Let $s \in \Gamma(M, E)$ be a section, or a vector field, on M. Write an element of E as $[(u, v)] = \{(ug, \rho(g)^{-1}v) | u \in P, v \in V, g \in G\}$. Taking a representative of the equivalence class amounts to fixing the gauge. We choose the following form,

$$s(p) = [(\sigma_i(p), \xi(p))] \tag{10.48}$$

as a representative.

Now we define the parallel transport of a vector in E along a curve γ in M. Of course, a naive guess 'ξ is parallel transported if $\xi(\gamma(t))$ is constant along $\gamma(t)$' does not make sense since this statement depends on the choice of the section $\sigma_i(p)$. We define a vector to be parallel transported if it is constant with respect to a *horizontal lift* $\tilde{\gamma}$ of γ in P. In other words, a section $s(\gamma(t)) = [(\tilde{\gamma}(t), \eta(\gamma(t)))]$ is parallel transported if η is constant along $\gamma(t)$. This definition is intrinsic since if $\tilde{\gamma}'$ is another horizontal lift of γ, then it can be written as $\tilde{\gamma}'(t) = \tilde{\gamma}(t)a$, $a \in G$ and we have (we omit ρ to simplify the notation)

$$[(\tilde{\gamma}(t), \eta(t))] = [(\tilde{\gamma}'(t)a^{-1}, \eta(t))] = [(\tilde{\gamma}'(t), a^{-1}\eta(t))]$$

where $\eta(t)$ stands for $\eta(\gamma(t))$. Hence if $\eta(t)$ is constant along $\gamma(t)$, so is its constant multiple $a^{-1}\eta(t)$.

Now the definition of covariant derivative is in order. Let $s(p)$ be a section of E. Along a curve $\gamma : [0, 1] \to M$ we have $s(t) = [(\tilde{\gamma}(t), \eta(t))]$, where $\tilde{\gamma}(t)$ is an arbitrary horizontal lift of $\gamma(t)$. The **covariant derivative** of $s(t)$ along $\gamma(t)$ at $p_0 = \gamma(0)$ is defined by

$$\nabla_X s \equiv \left[\left(\tilde{\gamma}(0), \frac{\mathrm{d}}{\mathrm{d}t}\eta(\gamma(t))\bigg|_{t=0}\right)\right] \tag{10.49}$$

where X is the tangent vector to $\gamma(t)$ at p_0. For the covariant derivative to be really intrinsic, it should not depend on the *extra* information, that is the special horizontal lift. Let $\tilde{\gamma}'(t) = \tilde{\gamma}(t)a$ $(a \in G)$ be another horizontal lift of γ. If $\tilde{\gamma}'(t)$ is chosen to be *the* horizontal lift, we have a representative $[(\tilde{\gamma}'(t), a^{-1}\eta(t))]$. The covariant derivative is now given by

$$\left[\left(\tilde{\gamma}'(0), \frac{\mathrm{d}}{\mathrm{d}t}\{a^{-1}\eta(t)\}\bigg|_{t=0}\right)\right] = \left[\left(\tilde{\gamma}'(0)a^{-1}, \frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\bigg|_{t=0}\right)\right]$$

which agrees with (10.49). Hence $\nabla_X s$ depends only on the tangent vector X and the sections $s \in \Gamma(M, E)$ and not on the horizontal lift $\tilde{\gamma}(t)$. Our definition depends only on a curve γ and a connection and not on local trivialisations. The local form of the covariant derivative is useful in practical computations and will be given below.

So far we have defined the covariant derivative at a point $p_0 = \gamma(0)$. It is clear that if X is a vector field, ∇_X maps a section s to a new section $\nabla_X s$, hence ∇_X is regarded as a map $\Gamma(M, E) \to \Gamma(M, E)$. To be more precise, take $X \in \mathcal{X}(M)$ whose value at p is $X_p \in T_pM$. There is a curve $\gamma(t)$ such that $\gamma(0) = p$ and its tangent at p is X_p. Then any horizontal lift $\tilde{\gamma}(t)$ of γ enables us to compute the covariant derivative $\nabla_X s|_p \equiv \nabla_{X_p} s$. We also define a map $\nabla : \Gamma(M, E) \to \Gamma(M, E) \otimes \Omega^1(M)$ by

$$\nabla s(X) \equiv \nabla_X s \qquad X \in \mathcal{X}(M) \qquad s \in \Gamma(M, E). \tag{10.50}$$

Exercise 10.26 Show that

$$(a)\ \nabla_X(a_1 s_1 + a_2 s_2) = a_1 \nabla_X s_1 + a_2 \nabla_X s_2 \tag{10.51a}$$

$$(a')\ \nabla(a_1 s_1 + a_2 s_2) = a_1 \nabla s_1 + a_2 \nabla s_2 \tag{10.51a'}$$

$$(b)\ \nabla_{(a_1 X_1 + a_2 X_2)} s = a_1 \nabla_{X_1} s + a_2 \nabla_{X_2} s \tag{10.51b}$$

$$(c)\ \nabla_X(fs) = X[f]s + f\nabla_X s \tag{10.51c}$$

$$(c')\ \nabla(fs) = (\mathrm{d}f)s + f\nabla s \tag{10.51c'}$$

$$(d)\ \nabla_{fX} s = f\nabla_X s \tag{10.51d}$$

where $a_i \in \mathbb{R}$, s, $s_i \in \Gamma(M, E)$ and $f \in \mathcal{F}(M)$.

10.4.2 A local expression for the covariant derivative

In practical computations it is convenient to have a local coordinate representation of the covariant derivative. Let $P(M, G)$ be a G bundle and $E = P \times_\rho G$ be an associate vector bundle. Take a local section $\sigma_i \in \Gamma(U_i, P)$ and employ the canonical trivialisation $\sigma_i(p) = \phi_i(p, e)$. Let $\gamma : [0, 1] \to M$ be a curve in U_i and $\tilde{\gamma}$ its horizontal lift. $\tilde{\gamma}$ is written as

$$\tilde{\gamma}(t) = \sigma_i(t) g_i(t) \tag{10.52}$$

where $g_i(t) \equiv g_i(\gamma(t)) \in G$. Take a section $e_\alpha(p) \equiv [(\sigma_i(p), e_\alpha{}^0)]$ of E, where $e_\alpha{}^0$ is the αth basis vector of V; $(e_\alpha{}^0)^\beta = \delta_\alpha{}^\beta$. We have

$$\begin{aligned} e_\alpha(t) &= [(\tilde{\gamma}(t) g_i(t)^{-1}, e_\alpha{}^0)] \\ &= [(\tilde{\gamma}(t), g_i(t)^{-1} e_\alpha{}^0)]. \end{aligned} \tag{10.53}$$

$g_i(t)^{-1}$ acts on $e_\alpha{}^0$ to compensate for the change of basis along γ. The covariant derivative of e_α is then given by

$$\begin{aligned} \nabla_X e_\alpha &= \left[\left(\tilde{\gamma}(0), \frac{\mathrm{d}}{\mathrm{d}t}\{g_i(t)^{-1} e_\alpha{}^0\}\bigg|_{t=0}\right)\right] \\ &= \left[\left(\tilde{\gamma}(0), -g_i(t)^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}t} g_i(t)\right\} g_i(t)^{-1} e_\alpha{}^0\bigg|_{t=0}\right)\right] \\ &= [(\tilde{\gamma}(0) g_i(0)^{-1}, \mathcal{A}_i(X) e_\alpha{}^0)] \end{aligned} \tag{10.54}$$

where (10.13b) has been used. From (10.54) we find the local expression,

$$\nabla_X e_\alpha = [(\sigma_i(0), \mathcal{A}_i(X) e_\alpha{}^0)]. \tag{10.55}$$

Let $\mathcal{A}_i = \mathcal{A}_{i\mu}\,\mathrm{d}x^\mu = \mathcal{A}_{i\mu}{}^\alpha{}_\beta\,\mathrm{d}x^\mu$, where $\mathcal{A}_{i\mu}{}^\alpha{}_\beta \equiv \mathcal{A}_{i\mu}{}^\gamma (T_\gamma)^\alpha{}_\beta$. The second entry of (10.55) is

$$\mathcal{A}_i(X) e_\alpha{}^0 = \frac{\mathrm{d}x^\mu}{\mathrm{d}t} e_\beta{}^0 \mathcal{A}_{i\mu}{}^\beta{}_\gamma \delta_\alpha{}^\gamma = \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \mathcal{A}_{i\mu}{}^\beta{}_\alpha e_\beta{}^0.$$

Substituting this into (10.55), we finally have

$$\nabla_X e_\alpha = \left[\left(\sigma_i(0), \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \mathcal{A}_{i\mu}{}^\beta{}_\alpha e_\beta{}^0\right)\right] = \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \mathcal{A}_{i\mu}{}^\beta{}_\alpha e_\beta \tag{10.56a}$$

or

$$\nabla e_\alpha = \mathcal{A}_i{}^\beta{}_\alpha e_\beta. \tag{10.56b}$$

In particular, for a coordinate curve x^μ, we have

$$\nabla_{\partial/\partial x^\mu} e_\alpha = \mathcal{A}_{i\mu}{}^\beta{}_\alpha e_\beta. \tag{10.57}$$

It is remarkable that a connection $\mathcal{A}$ on a principal bundle P completely specifies the covariant derivative on an associated bundle E (modulo representations).

Exercise 10.27 Let $s(p) = [(\sigma_i(p), \xi_i(p))] = \xi_i{}^\alpha(p) e_\alpha$ be a general section of E, where $\xi_i(p) = \xi_i{}^\alpha(p) e_\alpha{}^0$. Use the results of exercise 10.26 to verify that

$$\nabla_X s = \left[\left(\sigma_i(0), \frac{\mathrm{d}\xi_i}{\mathrm{d}t} + \mathcal{A}_i(X)\xi_i\Big|_{t=0}\right)\right] = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\left\{\frac{\partial \xi_i{}^\alpha}{\partial x^\mu} + \mathcal{A}_{i\mu}{}^\alpha{}_\beta \xi_i{}^\beta\right\} e_\alpha . \tag{10.58}$$

By construction, the covariant derivative is independent of the local trivialisation. This is also observed from the local form of $\nabla_X s$. Let $\sigma_i(p)$ and $\sigma_j(p)$ be local sections on overlapping charts U_i and U_j. On $U_i \cap U_j$ we have $\sigma_j(p) = \sigma_i(p)t_{ij}(p)$. In the i-trivialisation, we have

$$\begin{aligned}
\nabla_X s &= \left[\left(\sigma_i(0), \frac{\mathrm{d}\xi_i}{\mathrm{d}t} + \mathcal{A}_i(X)\xi_i\Big|_{t=0}\right)\right] \\
&= \left[\left(\sigma_j(0)\cdot t_{ij}^{-1}, \frac{\mathrm{d}}{\mathrm{d}t}(t_{ij}\xi_j) + \mathcal{A}_i(X)t_{ij}\xi_j\Big|_{t=0}\right)\right] \\
&= \left[\left(\sigma_j(0), \frac{\mathrm{d}\xi_j}{\mathrm{d}t} + \mathcal{A}_j(X)\xi_j\Big|_{t=0}\right)\right]
\end{aligned} \tag{10.59}$$

where use has been made of the condition (10.9). The last line of (10.59) is $\nabla_X s$ expressed in the j-trivialisation.

We have found that the covariant derivative defined by (10.49) is independent of the horizontal lift as well as the local section. The gauge potential $\mathcal{A}_i$ transforms under the change of local trivialisation so that $\nabla_X s$ is a well defined section of E. In this sense, ∇_X is the most natural derivative on an associated vector bundle, which is compatible with the connection on the principle bundle P.

Example 10.28 Let us recover the results obtained in §7.2. Let FM be a frame bundle over M and let TM be its associated bundle. We note $FM = P(M, \mathrm{GL}(m, \mathbb{R}))$ and $TM = FM \times_\rho \mathbb{R}^m$, where $m = \dim M$ and ρ is the $m \times m$ matrix representation of $\mathrm{GL}(m, \mathbb{R})$. Elements of $\mathfrak{gl}(m, \mathbb{R})$ are $m \times m$ matrices. Let us rewrite the local connection form $\mathcal{A}_i$ as $\mathcal{A}_i = \Gamma^\alpha{}_{\mu\beta}\mathrm{d}x^\mu$. We then find that

$$\nabla_{\partial/\partial x^\mu} e_\alpha = [(\sigma_i(0), \Gamma_\mu e_\alpha{}^0)] = \Gamma^\beta{}_{\mu\alpha} e_\beta \tag{10.60}$$

which should be compared with (7.14). For a general section (vector field) $s(p) = [(\sigma_i(p), X_i(p))] = X_i{}^\alpha(p) e_\alpha$, we have

$$\nabla_{\partial/\partial x^\mu} s = \left(\frac{\partial}{\partial x^\mu} X_i{}^\alpha + \Gamma^\alpha{}_{\mu\beta} X^\beta\right) e_\alpha \tag{10.61}$$

which reproduces the result of §7.2. It is evident that the roles played by the indices α, β and μ in $\Gamma^\alpha{}_{\mu\beta}$ are very different in their characters; μ is the $\Omega^1(M)$ index while α and β are the $\mathfrak{gl}(m, \mathbb{R})$ indices.

Example 10.29 Let us consider the U(1) gauge field coupled to a complex scalar field ϕ. The relevant fibre bundles are the U(1) bundle $P(M, \mathrm{U}(1))$ and the associated bundle $E = P \times_\rho \mathbb{C}$ where ρ is the natural identification of an element of U(1) with a complex number. The local expression for ω is $\mathcal{A}_i = \mathcal{A}_{i\mu}\,\mathrm{d}x^\mu$, where $\mathcal{A}_{i\mu} = \mathcal{A}_i(\partial/\partial x^\mu)$ is the vector potential of Maxwell's theory. Let γ be a curve in M with tangent vector X at $\gamma(0)$. Take a local section σ_i and express a horizontal lift $\tilde{\gamma}$ of γ as $\tilde{\gamma}(t) = \sigma_i(t)\mathrm{e}^{\mathrm{i}\varphi(t)}$. If $1 \in \mathbb{C}$ is taken to be the basis vector, the basis section is

$$e = [(\sigma_i(p), 1)].$$

Let $\phi(p) = [(\sigma_i(p), \Phi(p))] = \Phi(p)e$ ($\Phi : M \to \mathbb{C}$) be a section of E, which is identified with a complex scalar field. With respect to $\tilde{\gamma}(t)$, the section is given by

$$\phi(t) = \Phi(t)[(\tilde{\gamma}(t), g(t)^{-1})] \tag{10.62}$$

where $g(t) = \mathrm{e}^{\mathrm{i}\varphi(t)}$. The covariant derivative of ϕ along γ is

$$\begin{aligned} \nabla_X\phi &= \frac{\mathrm{d}\Phi}{\mathrm{d}t}[(\tilde{\gamma}(0), g(0)^{-1})] + \Phi(0)[(\tilde{\gamma}(0), g(0)^{-1}\mathcal{A}_i(X)\cdot 1)] \\ &= \left(\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \mathcal{A}_{i\mu}\Phi\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right)e = X^\mu\left(\frac{\partial\Phi}{\partial x^\mu} + \mathcal{A}_{i\mu}\Phi\right)e. \end{aligned} \tag{10.63}$$

Example 10.30 Let us consider the SU(2) Yang–Mills theory on M. The relevant bundles are the SU(2) bundle $P(M, \mathrm{SU}(2))$ and its associated bundle $E = P \times_\rho \mathbb{C}^2$, where we have taken the two-dimensional representation. The gauge potential on a chart U_i is

$$\mathcal{A}_i = \mathcal{A}_{i\mu}\,\mathrm{d}x^\mu = A_{i\mu}{}^{\alpha}\left(\frac{\sigma_\alpha}{2\mathrm{i}}\right)\mathrm{d}x^\mu \tag{10.64}$$

where $\sigma_\alpha/2\mathrm{i}$ are generators of SU(2), σ_α being the Pauli matrices. Let $e_\alpha{}^0$ ($\alpha = 1, 2$) be basis vectors of $\mathbb{C}^2$ and consider sections

$$e_\alpha(p) \equiv [(\sigma_i(p), e_\alpha{}^0)] \tag{10.65}$$

where $\sigma_i(p)$ defines a canonical trivialisation of P over U_i. Let $\phi(p) = [(\sigma_i(p), \Phi^\alpha(p)e_\alpha{}^0)]$ be a section of E over M. Along a horizontal lift $\tilde{\gamma}(t) = \sigma_i(t)g(t)$, we have

$$\phi(t) = [(\tilde{\gamma}(t), g(t)^{-1}\Phi^\alpha(t)e_\alpha{}^0)]. \tag{10.66}$$

The covariant derivative of ϕ along $X = \mathrm{d}/\mathrm{d}t$ is

$$\begin{aligned} \nabla_X\phi &= \left[\left(\tilde{\gamma}(0), g(0)^{-1}\frac{\mathrm{d}\Phi^\alpha(0)}{\mathrm{d}t}e_\alpha{}^0\right)\right] \\ &\quad + [(\tilde{\gamma}(0), g(0)^{-1}\mathcal{A}_i(X)^\alpha{}_\beta\Phi^\beta(0)e_\alpha{}^0)] \\ &= X^\mu\left(\frac{\partial\Phi^\alpha}{\partial x^\mu} + \mathcal{A}_{i\mu}{}^\alpha{}_\beta\Phi^\beta\right)e_\alpha \end{aligned} \tag{10.67}$$

where (10.13b) has been used to obtain the last equality.

Exercise 10.31 Let us consider an associated adjoint bundle $E_{\mathfrak{g}} \equiv P \times_{\mathrm{Ad}} \mathfrak{g}$, where the action of G on $\mathfrak{g}$ is the adjoint action $V \to \mathrm{Ad}_g V = g^{-1}Vg$, $V \in \mathfrak{g}$ and $g \in G$. Take a local section $\sigma_i \in \Gamma(U_i, P)$ such that $\tilde{\gamma}(t) = \sigma_i(t)g(t)$. Take a section $s(p) = [(\sigma_i(p), V(p))]$ of $E_{\mathfrak{g}}$, where $V(p) = V^\alpha(p)T_\alpha$, $\{T_\alpha\}$ being the basis of $\mathfrak{g}$. Define the covariant derivative $\mathcal{D}_X s$ by

$$\mathcal{D}_X s \equiv \left[\left(\tilde{\gamma}(0), \frac{\mathrm{d}}{\mathrm{d}t}\{\mathrm{Ad}_{g(t)^{-1}}V(t)\}|_{t=0}\right)\right]. \tag{10.68a}$$

Show that

$$\begin{aligned}\mathcal{D}_X s &= \left[\left(\sigma_i(0), \frac{\mathrm{d}V(t)}{\mathrm{d}t} + [\mathcal{A}_i(X), V(t)]|_{t=0}\right)\right] \\ &= X^\mu\left[\frac{\partial V^\alpha}{\partial x^\mu} + f_{\beta\gamma}{}^\alpha \mathcal{A}_{i\mu}{}^\beta V^\gamma\right][(\sigma_i(0), T_\alpha)].\end{aligned} \tag{10.68b}$$

10.4.3 *Curvature rederived*

The covariant derivative $\nabla_X s$ defines an operator $\nabla : \Gamma(M, E) \to \Gamma(M, E \otimes \Omega^1(M))$ by (10.50). More generally the action of ∇ on a vector-valued p-form $s \otimes \eta$, $\eta \in \Omega^p(M)$, is defined by

$$\nabla(s \otimes \eta) \equiv (\nabla s) \wedge \eta + s \otimes \mathrm{d}\eta. \tag{10.69}$$

Let U_i be a chart of M and σ_i a section of P over U_i. We take the canonical local trivialisation over U_i. We now prove

$$\nabla\nabla e_\alpha = e_\beta \otimes \mathcal{F}_i{}^\beta{}_\alpha \tag{10.70}$$

where $e_\alpha = [(\sigma_i, e_\alpha{}^0)] \in \Gamma(U_i, E)$. In fact by straightforward computation, we find

$$\begin{aligned}\nabla\nabla e_\alpha &= \nabla(e_\beta \otimes \mathcal{A}_i{}^\beta{}_\alpha) = \nabla e_\beta \wedge \mathcal{A}_i{}^\beta{}_\alpha + e_\beta \otimes \mathrm{d}\mathcal{A}_i{}^\beta{}_\alpha \\ &= e_\beta \otimes (\mathrm{d}\mathcal{A}_i{}^\beta{}_\alpha + \mathcal{A}_i{}^\beta{}_\gamma \wedge \mathcal{A}_i{}^\gamma{}_\alpha) = e_\beta \otimes \mathcal{F}_i{}^\beta{}_\alpha.\end{aligned}$$

Exercise 10.32 Let $s(p) = \xi^\alpha(p)e_\alpha(p)$ be a section of E. Show that

$$\nabla\nabla s = e_\alpha \otimes \mathcal{F}_i{}^\alpha{}_\beta \xi^\beta. \tag{10.71}$$

10.4.4 *A connection which preserves the inner product*

Let $E \xrightarrow{\pi} M$ be a vector bundle with a positive-definite symmetric inner product g whose action is defined at each point $p \in M$ by

$$g_p : \pi^{-1}(p) \otimes \pi^{-1}(p) \to \mathbb{R}. \tag{10.72}$$

g is said to define a **Riemannian structure** on E. A connection ∇ is called a **metric connection** if it preserves the inner product,

$$\mathrm{d}[g(s, s')] = g(\nabla s, s') + g(s, \nabla s'). \tag{10.73}$$

In particular, if we take $s = e_\alpha$, $s' = e_\beta$ and set $g(e_\alpha, e_\beta) = g_{\alpha\beta}$, we find

$$\mathrm{d}g_{\alpha\beta} = \mathfrak{A}^{\gamma}{}_{\alpha} g_{\gamma\beta} + \mathfrak{A}^{\gamma}{}_{\beta} g_{\alpha\gamma}. \tag{10.74}$$

This should be compared with (7.30*b*). If $E = TM$ and, moreover, the torsion-free condition is imposed, our connection reduces to the Levi-Civita connection of the Riemannian geometry.

Given an inner product, we may take an **orthonormal frame** $\{\hat{e}_\alpha\}$ such that $g(\hat{e}_\alpha, \hat{e}_\beta) = \delta_{\alpha\beta}$. The structure group G is taken to be $\mathrm{O}(k)$, k being the dimension of the fibre. The Lie algebra $\mathfrak{o}(k)$ is a vector space of skew-symmetric matrices and the connection one-form ω satisfies

$$\mathfrak{A}^{\alpha}{}_{\beta} = -\mathfrak{A}^{\beta}{}_{\alpha}. \tag{10.75}$$

Theorem 10.33 Let E be a vector bundle with inner product g and let ∇ be the covariant derivative associated with the *orthonormal* frame. Then ∇ is a metric connection.

Proof: Since g is bilinear, it suffices to show that

$$\mathrm{d}[g(s, s')] = g(\nabla s, s') + g(s, \nabla s')$$

for $s = f\hat{e}_\alpha$ and $s' = f'\hat{e}_\beta$ where $f, f' \in \mathcal{F}(M)$. In fact, the LHS is $\mathrm{d}[g(f\hat{e}_\alpha, f'\hat{e}_\beta)] = \mathrm{d}[ff'\delta_{\alpha\beta}] = \mathrm{d}(ff')\delta_{\alpha\beta}$, while the RHS is

$$\begin{aligned} g(\nabla f\hat{e}_\alpha, f'\hat{e}_\beta) + g(f\hat{e}_\alpha, \nabla f'\hat{e}_\beta) &= g(\mathrm{d}f\hat{e}_\alpha + f\hat{e}_\gamma \mathfrak{A}^{\gamma}{}_{\alpha}, f'\hat{e}_\beta) \\ &\quad + g(f\hat{e}_\alpha, \mathrm{d}f'\hat{e}_\beta + f'\hat{e}_\gamma \mathfrak{A}^{\gamma}{}_{\beta}) \\ &= \mathrm{d}ff'\delta_{\alpha\beta} + ff'\mathfrak{A}^{\gamma}{}_{\alpha}\delta_{\gamma\beta} + f\mathrm{d}f'\delta_{\alpha\beta} + ff'\mathfrak{A}^{\gamma}{}_{\beta}\delta_{\alpha\gamma} \\ &= \mathrm{d}(ff')\delta_{\alpha\beta} \end{aligned}$$

where (10.75) has been used to obtain the final equality. ■

10.4.5 Holomorphic vector bundles and Hermitian inner products

Definition 10.34 Let E and M be complex manifolds and $\pi : E \to M$ be a holomorphic surjection. E is a **holomorphic vector bundle** if:

(i) the typical fibre is $\mathbb{C}^k$ and the structure group is $\mathrm{GL}(k, \mathbb{C})$,

(ii) the local trivialisation $\phi_i : U_i \times \mathbb{C}^k \to \pi^{-1}(U_i)$ is a biholomorphism,

(iii) the transition function $t_{ij} : U_i \cap U_j \to G = \mathrm{GL}(k, \mathbb{C})$ is a holomorphic map.

For example, let M be a complex manifold with $\dim_{\mathbb{C}} M = m$. The **holomorphic tangent bundle** $TM^+ \equiv \bigcup_{p\in M} T_pM^+$ is a holomorphic vector bundle. The typical fibre is $\mathbb{C}^m$ and the local basis is $\{\partial/\partial z^\mu\}$.

Let h be an inner product on a holomorphic vector bundle whose action at $p \in M$ is $h_p : \pi^{-1}(p) \times \pi^{-1}(p) \to \mathbb{C}$. The most natural inner product is a **Hermitian structure** which satisfies

(i) $h_p(u, av + bw) = ah_p(u, v) + bh_p(u, w)$
for $u, v, w \in \pi^{-1}(p)$, $a, b \in \mathbb{C}$,
(ii) $h_p(u, v) = \overline{h_p(v, u)} \qquad u, v \in \pi^{-1}(p)$,
(iii) $h_p(u, u) \geqslant 0$, $h_p(u, u) = 0$ if and only if $u = \phi_i(p, 0)$,
(iv) $h(s_1, s_2) \in \mathcal{F}(M)^{\mathbb{C}}$ for $s_1, s_2 \in \Gamma(M, E)$.

A set of sections $\{\hat{e}_1, \ldots, \hat{e}_k\}$ is a **unitary frame** if

$$h(\hat{e}_i, \hat{e}_j) = \delta_{ij}. \tag{10.76}$$

The unitary frame bundle LM is not a holomorphic vector bundle since the structure group $\mathrm{U}(m)$ is not a complex manifold.

Given a Hermitian structure h, we define a connection which is compatible with h. The **Hermitian connection** ∇ is a linear map $\Gamma(M, E) \to \Gamma(M, E \otimes T^*M^{\mathbb{C}})$ which satisfies

(i) $\nabla(fs) = (\mathrm{d}f)s + f\nabla s \qquad f \in \mathcal{F}(M)^{\mathbb{C}}, s \in \Gamma(M, E)$
(ii) $\mathrm{d}[h(s_1, s_2)] = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$,
(iii) according to the destination, we separate the action of ∇ as $\nabla s = \mathrm{D}s + \overline{\mathrm{D}}s$, $\mathrm{D}s$ ($\overline{\mathrm{D}}s$) being a $(1, 0)$-form ($(0, 1)$-form) valued section. We demand that $\overline{\mathrm{D}} = \bar{\partial}$.

It can be shown that given E and a Hermitian metric h, there exists a *unique* Hermitian connection ∇. The curvature is defined from the Hermitian connection. Let $\{\hat{e}_1, \ldots, \hat{e}_k\}$ be a unitary frame and define the local connection form $\mathcal{A}^\beta{}_\alpha$ by

$$\nabla \hat{e}_\alpha = \hat{e}_\beta \mathcal{A}^\beta{}_\alpha. \tag{10.77}$$

The field strength is defined by

$$\mathcal{F} \equiv \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}. \tag{10.78}$$

We verify that

$$\nabla\nabla \hat{e}_\alpha = \nabla(\hat{e}_\beta \mathcal{A}^\beta{}_\alpha) = \hat{e}_\beta \mathcal{F}^\beta{}_\alpha. \tag{10.79}$$

We prove that both $\mathcal{A}$ and $\mathcal{F}$ are skew-Hermitian:

$$\begin{aligned}
0 = \mathrm{d}\delta_{\alpha\beta} &= \mathrm{d}h(\hat{e}_\alpha, \hat{e}_\beta) \\
&= h(\nabla \hat{e}_\alpha, \hat{e}_\beta) + h(\hat{e}_\alpha, \nabla \hat{e}_\beta) = \overline{\mathcal{A}^\beta{}_\alpha} + \mathcal{A}^\alpha{}_\beta \\
\mathcal{F}^\beta{}_\alpha + \overline{\mathcal{F}^\alpha{}_\beta} &= \mathrm{d}\mathcal{A}^\beta{}_\alpha + \mathcal{A}^\beta{}_\gamma \wedge \mathcal{A}^\gamma{}_\alpha + \mathrm{d}\overline{\mathcal{A}^\alpha{}_\beta} + \overline{\mathcal{A}^\alpha{}_\gamma} \wedge \overline{\mathcal{A}^\gamma{}_\alpha} \\
&= \mathrm{d}(\mathcal{A}^\beta{}_\alpha - \mathcal{A}^\beta{}_\alpha) + \mathcal{A}^\beta{}_\gamma \wedge \mathcal{A}^\gamma{}_\alpha + \mathcal{A}^\gamma{}_\alpha \wedge \mathcal{A}^\alpha{}_\gamma = 0.
\end{aligned}$$

Thus we have shown that

$$\mathcal{A}^\alpha{}_\beta = -\overline{\mathcal{A}^\beta{}_\alpha} \qquad \mathcal{F}^\beta{}_\alpha = -\overline{\mathcal{F}^\alpha{}_\beta}. \tag{10.80}$$

Next we show that $\mathcal{F}$ is a $(1, 1)$-form. Let $\{\hat{e}_\alpha\}$ be a unitary frame. $\mathcal{F}$ cannot have a component of bidegree-$(0, 2)$ since

$$\hat{e}_\beta \mathcal{F}^\beta{}_\alpha = \nabla\nabla\hat{e}_\alpha = (\mathrm{D} + \bar{\partial})(\mathrm{D} + \bar{\partial})\hat{e}_\alpha = \mathrm{DD}\hat{e}_\alpha + (\mathrm{D}\bar{\partial} + \bar{\partial}\mathrm{D})\hat{e}_\alpha.$$

From $\mathcal{F}^\beta{}_\alpha = -\overline{\mathcal{F}^\alpha{}_\beta}$ it follows that $\overline{\mathcal{F}}$ has no component of bidegree-$(0, 2)$, and hence $\mathcal{F}$ has no component of bidegree-$(2, 0)$ either. Thus $\mathcal{F}^\beta{}_\alpha$ is a two-form of bidegree-$(1, 1)$.

10.5 Gauge theories

As we have remarked several times, a gauge potential can be regarded as a local expression for a connection in a principal bundle. The Yang–Mills field strength is then identified with the local form of the curvature associated with the connection. We will summarise here the relevant aspects of gauge theories from the geometrical viewpoint.

10.5.1 U(1) gauge theory

Maxwell's theory of electromagnetism is described by the U(1) gauge group. U(1) is Abelian and one dimensional, hence we omit all the group indices α, β, . . . and put the structure constants $f_{\alpha\beta}{}^\gamma = 0$. Suppose the base space M is a four-dimensional Minkowski spacetime. From corollary 9.5, we find that the U(1) bundle P is trivial, namely $P = \mathbb{R}^4 \times \mathrm{U}(1)$ and a single local trivialisation over M is required. The gauge potential is simply

$$\mathcal{A} = \mathcal{A}_\mu \, \mathrm{d}x^\mu. \tag{10.81}$$

Our gauge potential $\mathcal{A}$ differs from the usual vector potential A by the Lie algebra factor i : $\mathcal{A}_\mu = \mathrm{i}A_\mu$. The field strength is

$$\mathcal{F} = \mathrm{d}\mathcal{A}. \tag{10.82a}$$

In components, we have

$$\mathcal{F}_{\mu\nu} = \partial\mathcal{A}_\nu/\partial x^\mu - \partial\mathcal{A}_\mu/\partial x^\nu. \tag{10.82b}$$

$\mathcal{F}$ satisfies the Bianchi identity,

$$\mathrm{d}\mathcal{F} = \mathcal{F} \wedge \mathcal{A} - \mathcal{A} \wedge \mathcal{F} = 0. \tag{10.83a}$$

This should be expected from the outset since $\mathcal{F}$ is exact, $\mathcal{F} = \mathrm{d}\mathcal{A}$; and hence closed, $\mathrm{d}\mathcal{F} = \mathrm{d}^2\mathcal{A} = 0$. In components, we have

$$\partial_\lambda\mathcal{F}_{\mu\nu} + \partial_\nu\mathcal{F}_{\lambda\mu} + \partial_\mu\mathcal{F}_{\nu\lambda} = 0. \tag{10.83b}$$

If we identify the components $\mathcal{F}_{\mu\nu} \equiv \mathrm{i}F_{\mu\nu}$ with the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ as

$$E_i = F_{i0},\ B_i = \tfrac{1}{2}\varepsilon_{ijk}F_{jk} \qquad (i, j, k = 1, 2, 3) \tag{10.84}$$

(10.83*b*) reduces to two of Maxwell's equations,

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0. \tag{10.83c}$$

These equations are *geometrical* rather than *dynamical*. To find the dynamics, we have to specify the action. The **Maxwell action** $\mathcal{S}_M[\mathcal{A}]$ is a functional of $\mathcal{A}$ and is given by

$$\mathcal{S}_M[\mathcal{A}] \equiv \frac{1}{4}\int_{\mathbb{R}^4} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\mathrm{d}^4x = -\frac{1}{4}\int_{\mathbb{R}^4} F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}^4x. \tag{10.85a}$$

Exercise 10.35

(*a*) Let $*\mathcal{F}_{\mu\nu} \equiv \frac{1}{2}\mathcal{F}^{\kappa\lambda}\varepsilon_{\kappa\lambda\mu\nu}$ be the dual of $\mathcal{F}_{\mu\nu}$. Show that

$$\mathcal{S}_M[\mathcal{A}] = -\frac{1}{4}\int_{\mathbb{R}^4} \mathcal{F} \wedge *\mathcal{F}. \tag{10.85b}$$

(*b*) Use (10.84) to show that

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2). \tag{10.86}$$

Show also that

$$F_{\mu\nu}{*F^{\mu\nu}} = \boldsymbol{B}\cdot\boldsymbol{E}. \tag{10.87}$$

By the variation of $\mathcal{S}_M[\mathcal{A}]$ with respect to $\mathcal{A}_\mu$ we obtain the equation of motion,

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0. \tag{10.88a}$$

We find this equation is reduced to the second set of Maxwell's equations (in the vacuum)

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = 0. \tag{10.88b}$$

10.5.2 The Dirac magnetic monopole

We have studied Maxwell's theory of electromagnetism defined on $\mathbb{R}^4$. The triviality of the base space makes the U(1) bundle trivial. Poincaré's lemma ensures that the field strength $\mathcal{F}$ is globally exact; $\mathcal{F} = \mathrm{d}\mathcal{A}$. It is interesting to extend our analysis to U(1) bundles over a non-trivial base space. We assume everything is independent of time for simplicity.

The Dirac monopole is defined in $\mathbb{R}^3$ with the origin 0 removed. $\mathbb{R}^3 - \{0\}$ and S^2 are of the same homotopy type and the relevant bundle is a U(1) bundle $P(S^2, \mathrm{U}(1))$. S^2 is covered by two charts

$$U_\mathrm{N} \equiv \{(\theta, \phi)|0 \leq \theta \leq \tfrac{1}{2}\pi + \varepsilon\},\ U_\mathrm{S} \equiv \{(\theta, \phi)|\tfrac{1}{2}\pi - \varepsilon \leq \theta \leq \pi\}$$

where θ and ϕ are polar coordinates. Let ω be an Ehresmann connection on P. Take a local section $\sigma_{\mathrm{N}}(\sigma_{\mathrm{S}})$ on U_{N} (U_{S}) and define the local gauge potentials

$$\mathcal{A}_{\mathrm{N}} = \sigma_{\mathrm{N}}^{*}\omega \qquad \mathcal{A}_{\mathrm{S}} = \sigma_{\mathrm{S}}^{*}\omega.$$

We take $\mathcal{A}_{\mathrm{N}}$ and $\mathcal{A}_{\mathrm{S}}$ to be of the Wu–Yang form (§1.3),

$$\mathcal{A}_{\mathrm{N}} = \mathrm{i}g(1 - \cos\theta)\,\mathrm{d}\phi \qquad \mathcal{A}_{\mathrm{S}} = -\mathrm{i}g(1 + \cos\theta)\,\mathrm{d}\phi \tag{10.89}$$

where g is the strength of the monopole.

Let t_{NS} be the transition function defined on the equator $U_{\mathrm{N}} \cap U_{\mathrm{S}}$. t_{NS} defines a map from S^1 (equator) to U(1) (structure group), which is classified by $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$, see example 9.11. Let us write

$$t_{\mathrm{NS}}(\phi) = \exp[\mathrm{i}\varphi(\phi)] \qquad \varphi \in \mathbb{R}. \tag{10.90}$$

On $U_{\mathrm{N}} \cap U_{\mathrm{S}}$, the gauge potentials $\mathcal{A}_{\mathrm{N}}$ and $\mathcal{A}_{\mathrm{S}}$ are related by

$$\mathcal{A}_{\mathrm{N}} = t_{\mathrm{NS}}^{-1}\mathcal{A}_{\mathrm{S}}t_{\mathrm{NS}} + t_{\mathrm{NS}}^{-1}\mathrm{d}t_{\mathrm{NS}} = \mathcal{A}_{\mathrm{S}} + \mathrm{i}\,\mathrm{d}\varphi. \tag{10.91}$$

For the gauge potentials (10.89), we find

$$\mathrm{d}\varphi = -\mathrm{i}(\mathcal{A}_{\mathrm{N}} - \mathcal{A}_{\mathrm{S}}) = 2g\,\mathrm{d}\phi.$$

While ϕ runs from 0 to 2π around the equator, $\varphi(\phi)$ takes the range

$$\Delta\varphi \equiv \int \mathrm{d}\varphi = \int_0^{2\pi} 2g\,\mathrm{d}\phi = 4\pi g. \tag{10.92}$$

For t_{NS} to be defined uniquely, $\Delta\varphi$ must be a multiple of 2π, .

$$\Delta\varphi/2\pi = 2g \in \mathbb{Z} \tag{10.93}$$

which is the quantisation condition of the magnetic monopole. The integer $2g$ represents the homotopy class to which this bundle belongs. This number is also obtained by considering $F_{\mathrm{N}} = \mathrm{d}A_{\mathrm{N}}$ and $F_{\mathrm{S}} = \mathrm{d}A_{\mathrm{S}}$ ($\mathcal{F}_{\mathrm{N}} = \mathrm{i}F_{\mathrm{N}}$ etc). The total flux Φ is

$$\begin{aligned}\Phi &= \int_{S^2} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \int_{U_{\mathrm{N}}} \mathrm{d}A_{\mathrm{N}} + \int_{U_{\mathrm{S}}} \mathrm{d}A_{\mathrm{S}} \\ &= \int_{S^1} A_{\mathrm{N}} - \int_{S^1} A_{\mathrm{S}} = 2g\int_0^{2\pi} \mathrm{d}\phi = 4\pi g.\end{aligned} \tag{10.94}$$

Thus the curvature, that is the pair of the field strengths $\mathrm{d}A_{\mathrm{N}}$ and $\mathrm{d}A_{\mathrm{S}}$, characterises the twisting of the bundle. We discuss this further in Chapter 11.

10.5.3 The Aharonov–Bohm effect

In the elementary study of electromagnetism, the electric and magnetic fields (that is $F_{\mu\nu}$) are of central interest. The vector potential A and the scalar potential $\phi = A_0$ are considered to be of secondary importance.

In quantum mechanics, however, there are a variety of situations in which $F_{\mu\nu}$ are not sufficient to describe the phenomena and the use of $A_\mu = (\boldsymbol{A}, A_0)$ is essential. One of these examples is the **Aharonov–Bohm effect**.

The Aharonov–Bohm (AB) experiment is schematically described in figure 10.4. A beam of electrons with charge e is incoming from $x = -\infty$ and forms an interference pattern on the screen C. A solenoid of infinite length is placed in the middle of the beam. A shield S prevents electrons from penetrating into the solenoid. Accordingly the electrons do not feel the magnetic field at all. What about the gauge field A_μ?

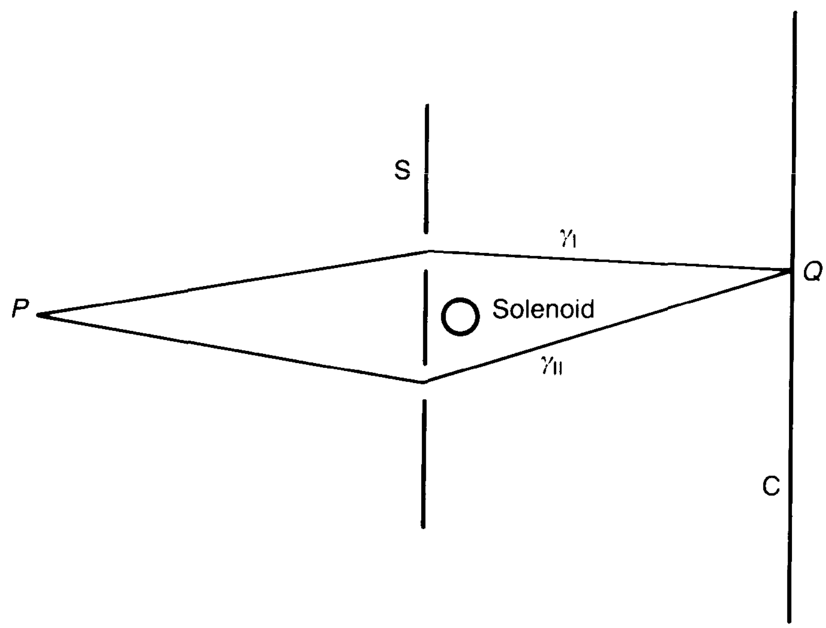


Figure 10.4 The Aharonov–Bohm experiment. $\boldsymbol{B} = 0$ outside the solenoid.

For simplicity, we make the radius of the solenoid infinitesimally small, keeping the total flux $\Phi = \int_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S}$ fixed. It is easy to verify that

$$\boldsymbol{A}(\boldsymbol{r}) = \left(-\frac{y\Phi}{2\pi r^2}, \frac{x\Phi}{2\pi r^2}, 0\right) \qquad A_0 = 0 \tag{10.95}$$

satisfies $\int (\nabla \times \boldsymbol{A})\cdot \mathrm{d}\boldsymbol{S} = \Phi$ and $\nabla \times \boldsymbol{A} = \boldsymbol{0}$ for $r \neq 0$. The vector potential does not vanish outside the solenoid. Classically the solenoid cannot have any influence on electrons since the Lorentz force $e(\boldsymbol{v} \times \boldsymbol{B})$ vanishes on the path of the beam.

In quantum mechanics, the Hamiltonian $\mathscr{H}$ of this system is

$$\mathscr{H} = -\frac{1}{2m}\left(\frac{\partial}{\partial x^\mu} - \mathrm{i}eA_\mu\right)^2 + V(\boldsymbol{r}) \tag{10.96}$$

where $V(\boldsymbol{r})$ represents the effect of the experimental apparatus. Semiclassically, we can distinguish between the paths γ_{I} and γ_{II} in figure 10.4. We write the wavefunction corresponding to γ_{I} (γ_{II}) as ψ_{I} (ψ_{II}) when $\boldsymbol{A} = 0$. If $\boldsymbol{A} \neq 0$, the wavefunction is given by the gauge-transformed form,

$$\psi_i^A(\boldsymbol{r}) \equiv \exp\left(ie \int_P^{\boldsymbol{r}} \boldsymbol{A}(\boldsymbol{r}')\cdot \mathrm{d}\boldsymbol{r}'\right)\psi_i(\boldsymbol{r}) \qquad (i = \mathrm{I}, \mathrm{II}) \tag{10.97}$$

where P is a reference point far from the apparatus. Let us consider a superposition $\psi_{\mathrm{I}}^A + \psi_{\mathrm{II}}^A$ of wavefunctions ψ_{I} and ψ_{II} such that $\psi_{\mathrm{I}}^A(P) = \psi_{\mathrm{II}}^A(P)$. Its amplitude at a point Q on the screen is

$$\begin{aligned}\psi_{\mathrm{I}}^A(Q) + \psi_{\mathrm{II}}^A(Q) &= \exp\left(ie \int_{\gamma_{\mathrm{I}}} \boldsymbol{A}(\boldsymbol{r}')\cdot \mathrm{d}\boldsymbol{r}'\right)\psi_{\mathrm{I}}(Q) \\ &\quad + \exp\left(ie \int_{\gamma_{\mathrm{II}}} \boldsymbol{A}(\boldsymbol{r}')\cdot \mathrm{d}\boldsymbol{r}'\right)\psi_{\mathrm{II}}(Q) \\ &= \exp\left(ie \int_{\gamma_{\mathrm{II}}} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{r}'\right)\left[\exp\left(ie \oint_{\gamma} \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{r}'\right)\psi_{\mathrm{I}}(Q) + \psi_{\mathrm{II}}(Q)\right]\end{aligned} \tag{10.98}$$

where $\gamma \equiv \gamma_{\mathrm{I}} - \gamma_{\mathrm{II}}$. It is evident that even though $\boldsymbol{B} = \boldsymbol{0}$ at the points in space through which the electrons travel, the wavefunction depends on the vector potential $\boldsymbol{A}$. From Stokes' theorem, we find

$$\oint_\gamma \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{r}' = \int_S (\boldsymbol{\nabla} \times \boldsymbol{A})\cdot \mathrm{d}\boldsymbol{S} = \int_S \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = \Phi \tag{10.99}$$

where S is a surface bounded by γ. From this and (10.98), we find the interference pattern should be the same for two values of the fluxes Φ_{a} and Φ_{b} if

$$e(\Phi_{\mathrm{a}} - \Phi_{\mathrm{b}}) = 2\pi n \qquad n \in \mathbb{Z}. \tag{10.100}$$

What is the geometry underlying the Aharonov–Bohm effect? Since the problem is essentially two dimensional, we consider a region $M = \mathbb{R}^2$, where the solenoid is assumed to be at the origin. The relevant bundles are the principal bundle $P(M, \mathrm{U}(1))$ and its associated bundle $E = P \times_\rho \mathbb{C}$, where $\mathrm{U}(1)$ acts on $\mathbb{C}$ in an obvious way. E is a complex line bundle over M, whose section is a wavefunction ψ.

Let us define a Lie-algebra-valued one-form $\mathcal{A} = \mathrm{i}A = \mathrm{i}A_\mu \mathrm{d}x^\mu$. The covariant derivative associated with this local connection is $\mathcal{D} = \mathrm{d} + \mathcal{A}$, where $\mathcal{A}$ is given by (10.95). Since $\mathrm{d}\mathcal{A} = \mathcal{F} = 0$, this connection is locally flat. Let us consider the unit circle S^1 which encloses the solenoid at the origin. We parametrise S^1 as $\mathrm{e}^{\mathrm{i}\theta}$ $(1 \leqslant \theta \leqslant 2\pi)$ and write the connection on S^1 as

$$\mathcal{A} = \mathrm{i}\,\frac{\Phi}{2\pi}\,\mathrm{d}\theta. \tag{10.101}$$

This is obtained from (10.95) by putting $r = 1$. We require that the wavefunction ψ is parallel transported along S^1 with respect to this local connection, namely,

$$\mathcal{D}\psi(\theta) = \left(\mathrm{d} + \mathrm{i}\,\frac{\Phi}{2\pi}\,\mathrm{d}\theta\right)\psi(\theta) = 0. \tag{10.102}$$

The solution of (10.102) is easily found to be

$$\psi(\theta) = \mathrm{e}^{-\mathrm{i}\Phi\theta/2\pi}. \tag{10.103}$$

Taking this section ψ amounts to neglecting the velocity of the electrons. The holonomy $\Gamma : \pi^{-1}(\theta = 0) \to \pi^{-1}(\theta = 0)$ is found to be

$$\Gamma : \psi(0) \mapsto \mathrm{e}^{-\mathrm{i}\Phi}\psi(0). \tag{10.104}$$

In an experiment, a toroidal permalloy (20% Fe and 80% Ni) has been used to eliminate the edge effects (Tonomura *et al* 1983). The dimensions of the permalloy are several microns and it is coated with gold to prevent electrons from penetrating into the magnetic field.

10.5.4 Yang–Mills theory

Let us consider SU(2) gauge theory defined on $\mathbb{R}^4$. The bundle which describes this gauge theory is $P(\mathbb{R}^4, \mathrm{SU}(2))$. Since $\mathbb{R}^4$ is trivial, there is just a single gauge potential

$$\mathcal{A} = A_\mu{}^\alpha T_\alpha\,\mathrm{d}x^\mu \tag{10.105}$$

where $T_\alpha \equiv \sigma_\alpha/2\mathrm{i}$ generate the algebra $\mathfrak{su}(2)$,

$$[T_\alpha, T_\beta] = \varepsilon_{\alpha\beta\gamma}T_\gamma.$$

The field strength is

$$\mathcal{F} \equiv \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = \tfrac{1}{2}\mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{10.106a}$$

where

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] = F_{\mu\nu}{}^\alpha T_\alpha \tag{10.106b}$$

$$F_{\mu\nu}{}^\alpha = \partial_\mu A_{\nu\alpha} - \partial_\nu A_{\mu\alpha} + \varepsilon_{\alpha\beta\gamma}A_{\mu\beta}A_{\nu\gamma}. \tag{10.106c}$$

The Bianchi identity is

$$\mathcal{D}\mathcal{F} = \mathrm{d}\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0. \tag{10.107}$$

The Yang–Mills action is

$$\mathcal{S}_{\mathrm{YM}}[\mathcal{A}] \equiv -\frac{1}{4}\int_M \mathrm{tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) = \frac{1}{2}\int_M \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}). \tag{10.108}$$

The variation with respect to $\mathcal{A}_\mu$ yields

$$\mathcal{D}_\mu\mathcal{F}^{\mu\nu} = 0, \text{ or } \mathcal{D}*\mathcal{F} = 0. \tag{10.109}$$

10.5.5 Instantons

A path integral is well defined only on a space with a Euclidean metric. To evaluate this integral it is important to find the local minima of the *Euclidean* action and compute the quantum fluctuations around them. Let us consider the SU(2) gauge theory on a four-dimensional Euclidean space $\mathbb{R}^4$. The local minima of this theory are known as **instantons** (or **pseudoparticles**, Belavin *et al* (1975)), see §1.4. It is easy to verify that the Euclidean action is

$$\mathcal{S}_{\mathrm{YM}}^{\mathrm{E}}[\mathcal{A}] = \frac{1}{4}\int_M \mathrm{tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) = -\frac{1}{2}\int_M \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}) \qquad (10.110)$$

where the Hodge $*$ is taken with respect to the Euclidean metric. As has been shown in §1.4 the field strength corresponding to instantons is self-dual (antiself-dual),

$$\mathcal{F}_{\mu\nu} = \pm *\mathcal{F}_{\mu\nu}. \qquad (10.111)$$

The action of a self-dual (antiself-dual) field configuration is

$$\mathcal{S}_{\mathrm{YM}}^{\mathrm{E}}[\mathcal{A}] = -\frac{1}{2}\int_M \mathrm{tr}(\mathcal{F} \wedge *\mathcal{F}) = \mp\frac{1}{2}\int_M \mathrm{tr}(\mathcal{F} \wedge \mathcal{F}). \qquad (10.112)$$

Let us consider the topological properties of an instanton. We require that

$$\mathcal{A}_\mu(x) \to g(x)^{-1}\partial_\mu g(x) \qquad \text{as } |x| \to L \qquad (10.113)$$

for the action to be finite, where L is an arbitrary positive number. Since $|x| = L$ is the sphere S^3, (10.113) defines a map $g : S^3 \to \mathrm{SU}(2)$ which is classified by $\pi_3(\mathrm{SU}(2)) \simeq \mathbb{Z}$. How is this reflected upon the transition function? We compactify $\mathbb{R}^4$ by adding the infinity. We suppose the South Pole of S^4 represents the points at infinity and the North Pole the origin. Under this compactification, we separate $\mathbb{R}^4$ into two pieces and identify them with the southern hemisphere U_{S} and the northern hemisphere U_{N} of S^4 as

$$U_{\mathrm{N}} = \{x \in \mathbb{R}^4 |\ |x| \leqslant L + \varepsilon\} \qquad (10.114a)$$

$$U_{\mathrm{S}} = \{x \in \mathbb{R}^4 |\ |x| \geqslant L - \varepsilon\} \qquad (10.114b)$$

see figure 10.5. We assume there is no 'twist' of the gauge potential on U_{S} and choose,

$$\mathcal{A}_{\mathrm{S}}(x) \equiv 0 \qquad x \in U_{\mathrm{S}}. \qquad (10.115)$$

Then all the topological information about the bundle is contained in $\mathcal{A}_{\mathrm{N}}(x)$ or the transition function $t_{\mathrm{NS}}(x)$ on the 'equator' S^3 $(= U_{\mathrm{N}} \cap U_{\mathrm{S}})$. Since $\mathcal{A}_{\mathrm{S}} = 0$, we have, for $x \in U_{\mathrm{N}} \cap U_{\mathrm{S}}$,

$$\mathcal{A}_{\mathrm{N}} = t_{\mathrm{NS}}^{-1}\mathcal{A}_{\mathrm{S}}t_{\mathrm{NS}} + t_{\mathrm{NS}}^{-1}\mathrm{d}t_{\mathrm{NS}} = t_{\mathrm{NS}}^{-1}\mathrm{d}t_{\mathrm{NS}}. \qquad (10.116)$$

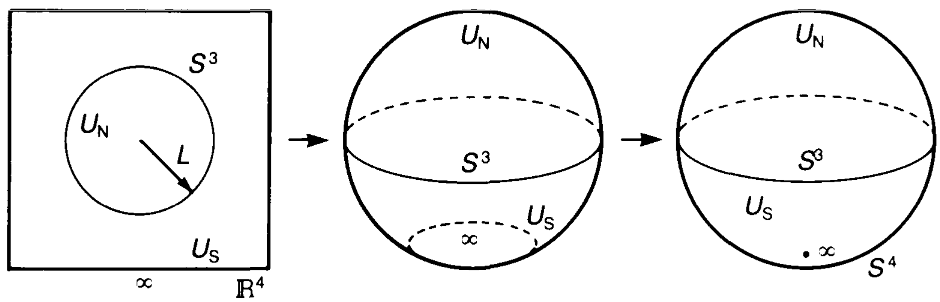


Figure 10.5 One-point compactification of $\mathbb{R}^4$ to S^4.

Thus $g(x)$ in (10.113) is identified with the transition function $t_{NS}(x)$ and classifying the maps $g : S^3 \to SU(2)$ amounts to classifying the transition functions according to $\pi_3(SU(2)) = \mathbb{Z}$; see example 9.15.

We now compute the degree of a map $g : S^3 \to SU(2)$ following Coleman (1979). First note that $SU(2) \simeq S^3$ since

$$t^4\mathbb{1} + t^i\sigma_i \in SU(2) \leftrightarrow \boldsymbol{t}^2 + (t^4)^2 = 1.$$

Thus maps $g : S^3 \to SU(2)$ are classified according to $\pi_3(SU(2)) \simeq \pi_3(S^3) \simeq \mathbb{Z}$. We easily find that:

(a) The constant map

$$g_0 : x \in S^3 \mapsto e \in SU(2) \tag{10.117a}$$

belongs to the trivial class of $\pi_3(SU(2))$.

(b) The *identity* map (this is in fact the identity map $S^3 \to S^3$)

$$g_1 : x \mapsto \frac{1}{r}[x^4\mathbb{1} + x^i\sigma_i],\ r^2 = \boldsymbol{x}^2 + (x^4)^2 \tag{10.117b}$$

defines the class 1 of $\pi_3(SU(2))$. The explicit form of the gauge potential corresponding to this homotopy class is given in §1.4.

(c) The map

$$g_n \equiv (g_1)^n : x \mapsto r^{-n}[x^4\mathbb{1} + x^i\sigma_i]^n \tag{10.117c}$$

defines the class n of $\pi_3(SU(2))$.

We recall that the strength (charge) of a magnetic monopole is given by the integral of the field strength $\mathcal{F} = \mathrm{d}\mathcal{A}$ over the sphere S^2. We expect that a similar relation exists for the instanton number. Since instantons are defined over S^4, we have to find a four-form to be integrated over S^4. A natural four-form is $\mathcal{F} \wedge \mathcal{F}$. In the following we shall omit the exterior product symbol when this does not cause confusion ($\mathcal{F}^2$ stands for $\mathcal{F} \wedge \mathcal{F}$). Observe that $\mathrm{tr}\,\mathcal{F}^2$ is closed,

$$\begin{aligned} \mathrm{d}\,\mathrm{tr}\,\mathcal{F}^2 &= \mathrm{tr}[\mathrm{d}\mathcal{F}\mathcal{F} + \mathcal{F}\mathrm{d}\mathcal{F}] \\ &= \mathrm{tr}\{-[\mathcal{A}, \mathcal{F}]\mathcal{F} - \mathcal{F}[\mathcal{A}, \mathcal{F}]\} = 0 \end{aligned} \tag{10.118}$$

where use has been made of the Bianchi identity $\mathrm{d}\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0$. [*Remarks:* In the present case, (10.118) seems to be trivial since any four-form on S^4 is closed. Note, however, that (10.118) remains true even on higher-dimensional manifolds.] By Poincaré's lemma, the closed form $\mathrm{tr}\,\mathcal{F}^2$ is *locally* exact,

$$\mathrm{tr}\,\mathcal{F}^2 = \mathrm{d}K \tag{10.119}$$

where K is a local three-form. Thus $\mathrm{tr}\,\mathcal{F}^2$ is an element of the de Rham cohomology group $H^4(S^4)$. $\mathrm{tr}\,\mathcal{F}^2$ is identified with the second Chern character and K its Chern–Simons form, see Chapter 11.

Lemma 10.36 The three-form K in (10.119) is given by

$$K = \mathrm{tr}[\mathcal{A}\mathrm{d}\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3]. \tag{10.120}$$

Proof: A straightforward computation yields

$$\begin{aligned}
\mathrm{d}K &= \mathrm{tr}[(\mathrm{d}\mathcal{A})^2 + \tfrac{2}{3}(\mathrm{d}\mathcal{A}\mathcal{A}^2 - \mathcal{A}\mathrm{d}\mathcal{A}\mathcal{A} + \mathcal{A}^2\mathrm{d}\mathcal{A})] \\
&= \mathrm{tr}[(\mathcal{F} - \mathcal{A}^2)(\mathcal{F} - \mathcal{A}^2) \\
&\quad + \tfrac{2}{3}\{(\mathcal{F} - \mathcal{A}^2)\mathcal{A}^2 - \mathcal{A}(\mathcal{F} - \mathcal{A}^2)\mathcal{A} + \mathcal{A}^2(\mathcal{F} - \mathcal{A}^2)\}] \\
&= \mathrm{tr}[\mathcal{F}^2 - \mathcal{A}^2\mathcal{F} - \mathcal{F}\mathcal{A}^2 + \mathcal{A}^4 + \tfrac{2}{3}(\mathcal{F}\mathcal{A}^2 - \mathcal{A}\mathcal{F}\mathcal{A} + \mathcal{A}^2\mathcal{F} - \mathcal{A}^4)]
\end{aligned}$$

where use has been made of the identity $\mathrm{d}\mathcal{A} = \mathcal{F} - \mathcal{A}^2$. Now we note that

$$\mathrm{tr}\,\mathcal{A}^4 = 0,\ \mathrm{tr}\,\mathcal{A}\mathcal{F}\mathcal{A} = -\mathrm{tr}\,\mathcal{A}^2\mathcal{F} = -\mathrm{tr}\,\mathcal{F}\mathcal{A}^2.$$

For example, we have

$$\begin{aligned}
\mathrm{tr}\,\mathcal{A}\mathcal{F}\mathcal{A} &= \tfrac{1}{2}\,\mathrm{tr}\,\mathcal{A}_\kappa\mathcal{F}_{\lambda\mu}\mathcal{A}_\nu\,\mathrm{d}x^\kappa \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= -\tfrac{1}{2}\mathrm{tr}\,\mathcal{A}_\nu\mathcal{A}_\kappa\mathcal{F}_{\lambda\mu}\,\mathrm{d}x^\nu \wedge \mathrm{d}x^\kappa \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu = -\mathrm{tr}\,\mathcal{A}^2\mathcal{F}
\end{aligned}$$

where the cyclicity of the trace and the anticommutativity of $\mathrm{d}x^\mu$ have been used. Then $\mathrm{d}K$ becomes

$$\begin{aligned}
\mathrm{d}K &= \mathrm{tr}[\mathcal{F}^2 - \mathcal{A}^2\mathcal{F} - \mathcal{F}\mathcal{A}^2 + \tfrac{2}{3}\{\mathcal{F}\mathcal{A}^2 + \tfrac{1}{2}(\mathcal{F}\mathcal{A}^2 + \mathcal{A}^2\mathcal{F}) + \mathcal{A}^2\mathcal{F}\}] \\
&= \mathrm{tr}\,\mathcal{F}^2
\end{aligned}$$

as has been claimed. ■

Lemma 10.37 Let $\mathcal{A}$ be the gauge potential of an instanton. Then it follows that

$$\int_{S^4}\mathrm{tr}\,\mathcal{F}^2 = -\frac{1}{3}\int_{S^3}\mathrm{tr}\,\mathcal{A}^3. \tag{10.121}$$

Proof: From Stokes' theorem, we find that

$$\int_{U_{\mathrm{N}}} \mathrm{tr}\,\mathcal{F}^2 = \int_{U_{\mathrm{N}}} \mathrm{d}K = \int_{S^3} K$$

where U_{N} is defined by (10.114) and $S^3 = \partial U_{\mathrm{N}}$. Since $\mathcal{F} = 0$ on S^3, we have

$$K = \mathrm{tr}[\mathcal{A}\,\mathrm{d}\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3] = \mathrm{tr}[\mathcal{A}(\mathcal{F} - \mathcal{A}^2) + \tfrac{2}{3}\mathcal{A}^3] = -\tfrac{1}{3}\,\mathrm{tr}\,\mathcal{A}^3$$

on S^3, from which we find

$$\int_{U_{\mathrm{N}}} \mathrm{tr}\,\mathcal{F}^2 = \int_{S^4} \mathrm{tr}\,\mathcal{F}^2 = -\frac{1}{3}\int_{S^3} \mathrm{tr}\,\mathcal{A}^3$$

where we have added $\int_{U_{\mathrm{S}}} \mathrm{tr}\,\mathcal{F}^2 = 0$ (note $\mathcal{A}_{\mathrm{S}} \equiv 0$). ■

Note that $\mathrm{tr}\,\mathcal{F}^2$ is invariant under the gauge transformation,

$$\mathrm{tr}\,\mathcal{F}^2 \longrightarrow \mathrm{tr}[g^{-1}\mathcal{F}^2 g] = \mathrm{tr}\,\mathcal{F}^2.$$

Thus it is reasonable to assume that $\mathrm{tr}\,\mathcal{F}^2$ indeed contains a certain amount of topological information about the bundle, which is independent of particular connections. Let us consider the gauge fields (10.117*a*–*c*) given before. We find:

(*a*) For $g_0(x) \equiv e$, we have $\mathcal{A} = 0$ on S^3. Since the bundle is trivial we may take $\mathcal{A} \equiv 0$ throughout S^4. Then $\mathcal{F} = 0$, hence

$$\int_{S^4} \mathrm{tr}\,\mathcal{F}^2 = -\frac{1}{3}\int_{S^3} \mathrm{tr}\,\mathcal{A}^3 = 0. \tag{10.122}$$

Note that this relation is true for any gauge potential which is obtained from $\mathcal{A} = 0$ by smooth gauge transformations, that is, for any gauge potential of the form $\mathcal{A}(x) = g(x)^{-1}\,\mathrm{d}g(x)$, $x \in S^4$.

(*b*) Next consider a gauge potential whose value on S^3 is given by (10.117*b*) as

$$\mathcal{A} = \frac{1}{r}(x^4 - \mathrm{i}x^k\sigma_k)\,\mathrm{d}\left(\frac{1}{r}(x^4 + \mathrm{i}x^l\sigma_l)\right). \tag{10.123}$$

A considerable simplification is achieved if we note that the integrand $\mathrm{tr}\,\mathcal{A}^3$ should not depend on the point on S^3 at which it is evaluated since g_1 maps S^3 onto $\mathrm{SU}(2) \cong S^3$ in a uniform way. So we may evaluate it at the North Pole ($x^4 = 1$, $\boldsymbol{x} = \boldsymbol{0}$) of the unit sphere. We then find $\mathcal{A} = \mathrm{i}\sigma_k\,\mathrm{d}x^k$ and

$$\begin{aligned}\mathrm{tr}\,\mathcal{A}^3 &= \mathrm{i}^3\mathrm{tr}[\sigma_i\sigma_j\sigma_k]\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \\ &= 2\varepsilon_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k = 12\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3.\end{aligned} \tag{10.124}$$

Next we note that (x^1, x^2, x^3) is a good coordinate system on *each* hemisphere of S^3 and $\omega \equiv \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ is a volume element at the North Pole. We find

$$\int_{S^3} \operatorname{tr} \mathcal{A}^3 = 12 \int_{S^3} \omega = 12(2\pi^2) = 24\pi^2$$

where $2\pi^2$ is the area of the unit sphere S^3. We finally obtain

$$-\frac{1}{8\pi^2} \int_{S^4} \operatorname{tr} \mathcal{F}^2 = \frac{1}{24\pi^2} \int_{S^3} \operatorname{tr} \mathcal{A}^3 = 1. \tag{10.125}$$

(*c*) Next we consider the map $g_n : S^3 \to \mathrm{SU}(2)$ given by (10.117*c*). We show that $g_2 = g_1 g_1$ has a winding number 2. We divide S^3 into the northern hemisphere U_N and the southern hemisphere U_S. Given a map $g_1 : S^3 \to \mathrm{SU}(2)$, it is always possible to transform g_1 smoothly to $g_{1\mathrm{N}}$ which has the winding number one and $g_{1\mathrm{N}}(x) = e$ for $x \in U_\mathrm{S}$. All the variation takes place on U_N. Similarly, g_1 may be deformed to $g_{1\mathrm{S}}$ with the same winding number and $g_{1\mathrm{S}}(x) = e$ for $x \in U_\mathrm{N}$. Under this deformation, g_2 becomes

$$g_2(x) \to g_2'(x) = \begin{cases} g_{1\mathrm{N}}(x) & x \in U_\mathrm{N} \\ g_{1\mathrm{S}}(x) & x \in U_\mathrm{S}. \end{cases}$$

For $\mathcal{A}(x) = g_2'(x)^{-1} \mathrm{d} g_2'(x)$ $(x \in S^3)$, we have

$$\begin{aligned} \frac{1}{24\pi^3} \int_{S^3} \operatorname{tr} \mathcal{A}^3 &= \frac{1}{24\pi^2} \left(\int_{U_\mathrm{N}} \operatorname{tr}(g_{1\mathrm{N}}^{-1} \mathrm{d} g_{1\mathrm{N}})^3 + \int_{U_\mathrm{S}} \operatorname{tr}(g_{1\mathrm{S}}^{-1} \mathrm{d} g_{1\mathrm{S}})^3 \right) \\ &= 1 + 1 = 2. \end{aligned} \tag{10.126}$$

Repeating the same procedure we find for $\mathcal{A} = g_n^{-1} \mathrm{d} g_n$ that

$$-\frac{1}{8\pi^2} \int_{S^4} \operatorname{tr} \mathcal{F}^2 = \frac{1}{24\pi^2} \int_{S^3} \operatorname{tr} \mathcal{A}^3 = n. \tag{10.127}$$

Collecting these results we establish the following theorem.

Theorem 10.38 The degree of mapping $g : S^3 \to \mathrm{SU}(2)$ is given by

$$n = \frac{1}{24\pi^2} \int_{S^3} \operatorname{tr}(g^{-1} \mathrm{d} g)^3 = \frac{1}{2} \int_{S^4} \operatorname{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi} \right)^2. \tag{10.128}$$

10.6 Berry's phase

Let $H(\boldsymbol{R})$ be a Hamiltonian of a quantum system which depends on k parameters $\boldsymbol{R} = (R_1, \ldots, R_k)$. Let $|n; \boldsymbol{R}\rangle$ be the nth normalised eigenstate of $H(\boldsymbol{R})$,

$$H(\boldsymbol{R})\,|n; \boldsymbol{R}\rangle = E_n(\boldsymbol{R})|n; \boldsymbol{R}\rangle \qquad \langle n; \boldsymbol{R}|n; \boldsymbol{R}\rangle = 1. \tag{10.129}$$

$E_n(\boldsymbol{R})$ is assumed to be isolated and non-degenerate. Suppose $\boldsymbol{R}$ executes a *closed loop* in the parameter space: $\boldsymbol{R} = \boldsymbol{R}(t)$ where $\boldsymbol{R}(0) = \boldsymbol{R}(1)$. We assume $\boldsymbol{R}$ changes adiabatically and the state always remains in the nth eigenstate. After a round trip along a loop γ in the parameter space M, the state gains an extra phase factor,

$$\eta_n \equiv \mathrm{i}\oint_\gamma \left\langle n;\, \boldsymbol{R} \left| \frac{\partial}{\partial R^\mu} \right| n;\, \boldsymbol{R} \right\rangle \mathrm{d}R^\mu \tag{10.130}$$

known as **Berry's phase**; see §1.7. It was Simon (1983) who first recognised the deep geometrical meaning underlying Berry's phase. He observed that the origin of Berry's phase is attributed to the holonomy in the parameter space. We shall work out this point of view following Berry (1984), Simon (1983), Aitchison (1987) and Zumino (1987).

10.6.1 Berry's phase, Berry's connection and Berry's curvature

Let M be a manifold describing the parameter space and let $\boldsymbol{R} = (R^1, \ldots, R^k)$ be the local coordinate. At each point $\boldsymbol{R}$ of M, we consider the normalised nth eigenstate of the Hamiltonian $H(\boldsymbol{R})$. Since a quantum state $|n;\boldsymbol{R}\rangle$ cannot be distinguished from $\mathrm{e}^{\mathrm{i}\varphi}|n;\boldsymbol{R}\rangle$, a physical state is expressed by an equivalence class

$$[|\boldsymbol{R}\rangle] \equiv \{g|\boldsymbol{R}\rangle | g \in \mathrm{U}(1)\} \tag{10.131}$$

where we omit the index n since we are interested only in the nth eigenvector (figure 10.6). At each point $\boldsymbol{R}$ of M, we have a U(1) degree of freedom and we have a U(1) bundle $P(M, \mathrm{U}(1))$ over the parameter space M. The projection is given by $\pi(g|\boldsymbol{R}\rangle) = \boldsymbol{R}$.

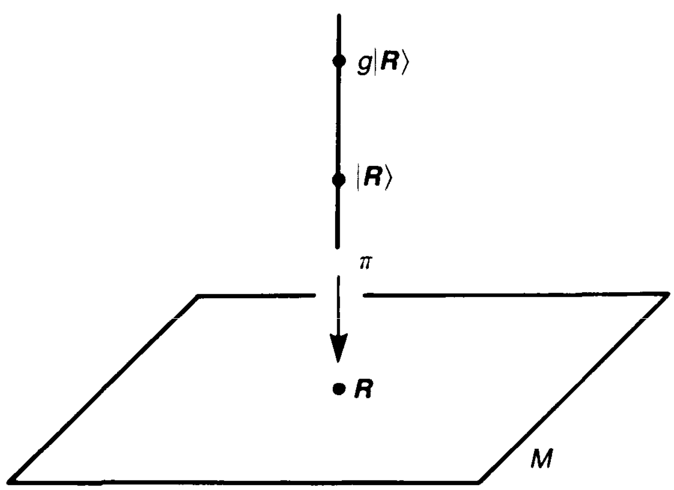


Figure 10.6 The fibre of a quantum mechanical system which depends on adiabatic parameters $\boldsymbol{R}$.

Fixing the phase of $|\boldsymbol{R}\rangle$ at each point $\boldsymbol{R} \in M$ amounts to choosing a section. Let $\sigma(\boldsymbol{R}) = |\boldsymbol{R}\rangle$ be a local section over a chart U of M. The canonical local trivialisation is given by

$$\phi^{-1}(|\boldsymbol{R}\rangle) = (\boldsymbol{R}, e). \tag{10.132}$$

The 'right' action yields

$$\phi^{-1}(|\boldsymbol{R}\rangle \cdot g) = (\boldsymbol{R}, e)g = (\boldsymbol{R}, g). \tag{10.133}$$

Now that the bundle structure is defined, we provide it with a connection. Let us define **Berry's connection** by

$$\mathcal{A} = \mathcal{A}_\mu \, \mathrm{d}R^\mu \equiv \langle \boldsymbol{R}|(\mathrm{d}|\boldsymbol{R}\rangle) = -(\mathrm{d}\langle \boldsymbol{R}|)|\boldsymbol{R}\rangle \tag{10.134}$$

where $\mathrm{d} = (\partial/\partial R^\mu)\mathrm{d}R^\mu$ is the exterior derivative in $\boldsymbol{R}$-space. Note that $\mathcal{A}$ is anti-Hermitian since,

$$0 = \mathrm{d}(\langle \boldsymbol{R}|\boldsymbol{R}\rangle) = (\mathrm{d}\langle \boldsymbol{R}|)|\boldsymbol{R}\rangle + \langle \boldsymbol{R}|\mathrm{d}|\boldsymbol{R}\rangle = \langle \boldsymbol{R}|\mathrm{d}|\boldsymbol{R}\rangle^\dagger + \langle \boldsymbol{R}|\mathrm{d}|\boldsymbol{R}\rangle.$$

To see (10.134) is indeed a local form of a connection, we have to check the compatibility condition. Let U_i and U_j be overlapping charts of M and let $\sigma_i(\boldsymbol{R}) = |\boldsymbol{R}\rangle_i$ and $\sigma_j(\boldsymbol{R}) = |\boldsymbol{R}\rangle_j$ be the respective local sections. They are related by the transition function as $|\boldsymbol{R}\rangle_j = |\boldsymbol{R}\rangle_i t_{ij}(\boldsymbol{R})$. We then find that

$$\begin{aligned}\mathcal{A}_j(\boldsymbol{R}) &= {}_j\langle \boldsymbol{R}|\mathrm{d}|\boldsymbol{R}\rangle_j = t_{ij}(\boldsymbol{R})^{-1}{}_i\langle \boldsymbol{R}|[\mathrm{d}|\boldsymbol{R}\rangle_i t_{ij}(\boldsymbol{R}) + |\boldsymbol{R}\rangle \, \mathrm{d}t_{ij}(\boldsymbol{R})] \\ &= \mathcal{A}_i(\boldsymbol{R}) + t_{ij}(\boldsymbol{R})^{-1}\mathrm{d}t_{ij}(\boldsymbol{R}).\end{aligned} \tag{10.135}$$

The set of one-forms $\{\mathcal{A}_i\}$ satisfying (10.135) defines an Ehresmann connection on $P(M, \mathrm{U}(1))$.

The field strength $\mathcal{F}$ of $\mathcal{A}$ is called **Berry's curvature** and is given by

$$\mathcal{F} = \mathrm{d}\mathcal{A} = (\mathrm{d}\langle \boldsymbol{R}|) \wedge (\mathrm{d}|\boldsymbol{R}\rangle) = \left(\frac{\partial\langle \boldsymbol{R}|}{\partial R^\mu}\right)\left(\frac{\partial|\boldsymbol{R}\rangle}{\partial R^\nu}\right) \mathrm{d}R^\mu \wedge \mathrm{d}R^\nu. \tag{10.136}$$

After an example from atomic physics, we shall clarify how this geometrical structure is reflected in Berry's phase.

Example 10.39 Let us consider a quantum mechanical system which contains 'fast' degrees of freedom $\boldsymbol{r}$ and 'slow' degrees of freedom $\boldsymbol{R}$. For example, we may imagine an electron moving under the potential of slowly vibrating ions. Suppose the Hamiltonian is given by

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{\boldsymbol{P}^2}{2M} + v(\boldsymbol{r}; \boldsymbol{R}) \tag{10.137}$$

where $\boldsymbol{p}(\boldsymbol{P})$ is the canonical conjugate of $\boldsymbol{r}(\boldsymbol{R})$. As a first approximation, we may consider the slow degrees of freedom are 'frozen' at some value $\boldsymbol{R}$ and consider an instantaneous sub-Hamiltonian

$$h(\boldsymbol{R}) = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r}; \boldsymbol{R}) \tag{10.138}$$

and the eigenvalue problem

$$h(\boldsymbol{R})|\boldsymbol{R}\rangle = \varepsilon_n(\boldsymbol{R})|\boldsymbol{R}\rangle \tag{10.139}$$

where $|\boldsymbol{R}\rangle$ stands for the nth eigenvector $|n; \boldsymbol{R}\rangle$. We assume that the eigenvalue is isolated and non-degenerate. Berry's connection is $\mathcal{A}(\boldsymbol{R}) = \langle \boldsymbol{R}|\mathrm{d}|\boldsymbol{R}\rangle$ while the curvature is $\mathcal{F} = (\mathrm{d}\langle \boldsymbol{R}|) \wedge (\mathrm{d}|\boldsymbol{R}\rangle)$.

It is interesting to see how the fast degrees of freedom affect the slow degrees of freedom. We assume the total wavefunction is written in the form

$$\Psi(\boldsymbol{r}; \boldsymbol{R}) = \Phi(\boldsymbol{R})|\boldsymbol{R}\rangle \tag{10.140}$$

and find the 'effective' Schrödinger equation which $\Phi(\boldsymbol{R})$, the wavefunction of the 'slow' degrees of freedom, satisfies. The eigenvalue problem of the Hamiltonian (10.137) is

$$\begin{aligned} H\Psi(\boldsymbol{r}; \boldsymbol{R}) &= -\frac{1}{2M}[\nabla_{\boldsymbol{R}}^2\Phi(\boldsymbol{R})|\boldsymbol{R}\rangle + 2\nabla_{\boldsymbol{R}}\Phi(\boldsymbol{R})\cdot\nabla_{\boldsymbol{R}}|\boldsymbol{R}\rangle + \Phi(\boldsymbol{R})\nabla_{\boldsymbol{R}}^2|\boldsymbol{R}\rangle] \\ &\quad - \Phi(\boldsymbol{R})\frac{1}{2m}\nabla_{\boldsymbol{r}}^2|\boldsymbol{R}\rangle + \Phi(\boldsymbol{R})V(\boldsymbol{r}; \boldsymbol{R})|\boldsymbol{R}\rangle \\ &= E_n(\boldsymbol{R})\Phi(\boldsymbol{R})|\boldsymbol{R}\rangle. \end{aligned}$$

If we multiply $\langle \boldsymbol{R}|$ on the left and use the Schrödinger equation (10.139), the above equation becomes

$$\begin{aligned} &-\frac{1}{2M}[\nabla_{\boldsymbol{R}}^2\Phi(\boldsymbol{R}) + 2\nabla_{\boldsymbol{R}}\Phi(\boldsymbol{R})\cdot\langle \boldsymbol{R}|\nabla_{\boldsymbol{R}}|\boldsymbol{R}\rangle + \Phi(\boldsymbol{R})(\langle \boldsymbol{R}|\nabla_{\boldsymbol{R}}|\boldsymbol{R}\rangle)^2] \\ &+ \varepsilon_n(\boldsymbol{R})\Phi(\boldsymbol{R}) = E_n(\boldsymbol{R})\Phi(\boldsymbol{R}) \end{aligned} \tag{10.141}$$

where we have employed the Born–Oppenheimer approximation, in which all the matrix elements except the diagonal ones are neglected,

$$\langle n; \boldsymbol{R}|\nabla_{\boldsymbol{R}}|n'; \boldsymbol{R}\rangle = 0 \qquad n' \neq n. \tag{10.142}$$

Now the effective Hamiltonian for $|\Phi(\boldsymbol{R})\rangle$ is given by

$$H_{\mathrm{eff}}(n) \equiv -\frac{1}{2M}\left(\frac{\partial}{\partial R^\mu} + \mathcal{A}_\mu(\boldsymbol{R})\right)^2 + \varepsilon_n(\boldsymbol{R}) \tag{10.143}$$

where $\mathcal{A}_\mu$ is a component of Berry's connection,

$$\mathcal{A}_\mu(\boldsymbol{R}) = \left\langle \boldsymbol{R}\left|\frac{\partial}{\partial R^\mu}\right|\boldsymbol{R}\right\rangle. \tag{10.144}$$

It is remarkable that the fast degrees of freedom have induced a *vector potential* coupled to the slow degrees of freedom. Note also that the eigenvalue $\varepsilon_n(\boldsymbol{R})$ behaves as a *potential energy* in H_{eff}. This 'spontaneous creation' of the gauge symmetry reflects the phase degree of freedom of the wavefunction $|\boldsymbol{R}\rangle$.

The Schrödinger equation describing the adiabatic change is

$$H(\boldsymbol{R}(t))|\boldsymbol{R}(t), t\rangle = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{R}(t), t\rangle \tag{10.145a}$$

where we note that $|\boldsymbol{R}(t), t\rangle$ has an explicit t-dependence as well as an implicit one through $\boldsymbol{R}(t)$. Berry assumes that

$$|\boldsymbol{R}(t), t\rangle = \exp\left(-\mathrm{i}\int_0^t E_n(t)\,\mathrm{d}t\right)\mathrm{e}^{\mathrm{i}\eta(t)}|\boldsymbol{R}(t)\rangle \tag{10.146a}$$

where $|\boldsymbol{R}\rangle$ is an *instantaneous* normalised eigenstate of $H(\boldsymbol{R})$,

$$H(\boldsymbol{R})|\boldsymbol{R}\rangle = E_n(\boldsymbol{R})|\boldsymbol{R}\rangle \qquad \langle\boldsymbol{R}|\boldsymbol{R}\rangle = 1. \tag{10.147}$$

The first exponential is the ordinary dynamical phase while the second one is Berry's phase. It is convenient for our purpose to define an operator

$$\mathscr{H}(\boldsymbol{R}) \equiv H(\boldsymbol{R}) - E_n(\boldsymbol{R}) \tag{10.148}$$

to dispose of the dynamical phase. The state $|\boldsymbol{R}\rangle$ is the zero-energy eigenstate of $\mathscr{H}(\boldsymbol{R})$; $\mathscr{H}(\boldsymbol{R})|\boldsymbol{R}\rangle = 0$. The solution of the modified Schrödinger equation,

$$\mathscr{H}(\boldsymbol{R})|\boldsymbol{R}(t), t\rangle = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{R}(t), t\rangle \tag{10.145b}$$

is given by

$$|\boldsymbol{R}(t), t\rangle = \mathrm{e}^{\mathrm{i}\eta(t)}|\boldsymbol{R}(t)\rangle. \tag{10.146b}$$

We found in §1.7 that η is given by

$$\eta(t) = \mathrm{i}\int_0^t \mathrm{d}t\,\frac{\mathrm{d}R^\mu}{\mathrm{d}t}\left\langle\boldsymbol{R}(t)\left|\frac{\partial}{\partial R^\mu}\right|\boldsymbol{R}(t)\right\rangle = \mathrm{i}\int_{\boldsymbol{R}(0)}^{\boldsymbol{R}(t)}\langle\boldsymbol{R}(t)|\mathrm{d}|\boldsymbol{R}(t)\rangle. \tag{10.149}$$

We show that Berry's phase is a *holonomy* associated with the connection (10.134) on $P(M, \mathrm{U}(1))$. Take a section $\sigma(\boldsymbol{R}) = |\boldsymbol{R}\rangle$ over a chart U of M. Let $\boldsymbol{R}: [0, 1] \to M$ be a loop in U. [We shall be a little sloppy in our notation.] We write a horizontal lift of $\boldsymbol{R}(t)$ with respect to the connection (10.134) as

$$\widetilde{\boldsymbol{R}}(t) = \sigma(\boldsymbol{R}(t))g(\boldsymbol{R}(t)) \tag{10.150}$$

where $g(\boldsymbol{R}(0))$ is taken to be the unit element of U(1). The group element $g(t)$ satisfies (10.13b),

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t}g(t)^{-1} = -\mathscr{A}(\mathrm{d}/\mathrm{d}t) = -\left\langle\boldsymbol{R}(t)\left|\frac{\mathrm{d}}{\mathrm{d}t}\right|\boldsymbol{R}(t)\right\rangle \tag{10.151}$$

where $g(t)$ stands for $g(\boldsymbol{R}(t))$. Since $g(t) = \exp(\mathrm{i}\eta(t))$, we have

$$\mathrm{i}\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = -\left\langle\boldsymbol{R}(t)\left|\frac{\mathrm{d}}{\mathrm{d}t}\right|\boldsymbol{R}(t)\right\rangle$$

which is easily integrated to yield

$$\eta(1) = \mathrm{i}\int_0^1 \left\langle \boldsymbol{R}(s) \left| \frac{\mathrm{d}}{\mathrm{d}s} \right| \boldsymbol{R}(s) \right\rangle \mathrm{d}s = \mathrm{i}\oint \langle \boldsymbol{R}|\mathrm{d}|\boldsymbol{R}\rangle. \qquad (10.152)$$

Let us note that $\boldsymbol{R}(0) = \boldsymbol{R}(1)$, hence $|\boldsymbol{R}(0)\rangle = |\boldsymbol{R}(1)\rangle$. Then $\exp[\mathrm{i}\eta(1)]$ is regarded as a holonomy (figure 10.7)

$$\widetilde{\boldsymbol{R}}(1) = \exp\left(-\oint \langle \boldsymbol{R}|\mathrm{d}|\boldsymbol{R}\rangle\right)\cdot|\boldsymbol{R}(0)\rangle. \qquad (10.153a)$$

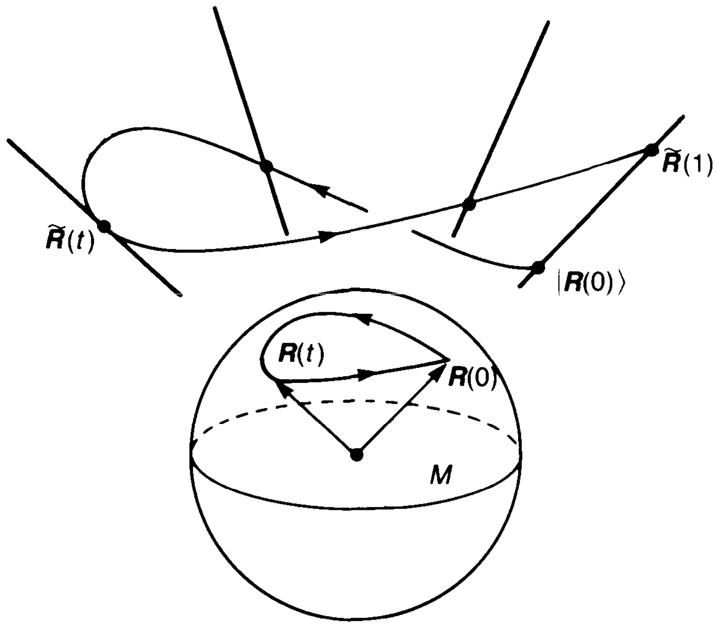


Figure 10.7 If the parameter changes adiabatically along a loop $\boldsymbol{R}(t)$, the state with initial condition $|\boldsymbol{R}(0)\rangle$ becomes $|\widetilde{\boldsymbol{R}}(1)\rangle$ which is different from $|\boldsymbol{R}(0)\rangle$ in general. The difference is the holonomy and is identified with Berry's phase.

Exercise 10.40 Let S be a surface in M, which is bounded by the loop $\boldsymbol{R}(t)$. Show that

$$\widetilde{\boldsymbol{R}}(1) = \exp\left(-\int_S \mathcal{F}\right)\cdot|\boldsymbol{R}(0)\rangle \qquad (10.153b)$$

where $\mathcal{F}$ is given by (10.136).

Example 10.41 Let us consider a spin-$\frac{1}{2}$ particle in a magnetic field with the Hamiltonian

$$H(\boldsymbol{R}) = \boldsymbol{R}\cdot\boldsymbol{\sigma} = \begin{pmatrix} R_3 & R_1 - \mathrm{i}R_2 \\ R_1 + \mathrm{i}R_2 & -R_3 \end{pmatrix}. \qquad (10.154)$$

The parameter $\boldsymbol{R}$ corresponds to the applied magnetic field. This is a two-level system taking eigenvalues $\pm|R|$. Let us consider the eigenvalue $R = +|\boldsymbol{R}|$. According to the prescription above, we introduce a

Hamiltonian $\mathcal{H}(\boldsymbol{R}) \equiv H(\boldsymbol{R}) - |\boldsymbol{R}|$ and consider the zero-energy eigenstate of $\mathcal{H}(\boldsymbol{R})$ given by

$$|\boldsymbol{R}\rangle_{\mathrm{N}} = [2R(R + R_3)]^{-1/2} \begin{pmatrix} R + R_3 \\ R_1 + \mathrm{i}R_2 \end{pmatrix}. \tag{10.155}$$

The gauge potential is obtained after a straightforward but tedious calculation as

$$\mathcal{A}_{\mathrm{N}} = {}_{\mathrm{N}}\langle \boldsymbol{R}|\mathrm{d}|\boldsymbol{R}\rangle_{\mathrm{N}} = -\mathrm{i}\,\frac{R_2\,\mathrm{d}R_1 - R_1\,\mathrm{d}R_2}{2R(R + R_3)}. \tag{10.156}$$

As for the field strength, we have

$$\mathcal{F} = \mathrm{d}\mathcal{A} = \frac{\mathrm{i}}{2}\,\frac{R_1\,\mathrm{d}R_2 \wedge \mathrm{d}R_3 + R_2\,\mathrm{d}R_3 \wedge \mathrm{d}R_1 + R_3\,\mathrm{d}R_1 \wedge \mathrm{d}R_2}{R^3}. \tag{10.157}$$

So far we have assumed that the state $|\boldsymbol{R}\rangle$ is isolated. However, this assumption breaks down if $\boldsymbol{R} = 0$, in which case two eigenstates are degenerate. Surprisingly this singularity behaves like a *magnetic monopole* in $\boldsymbol{R}$-space. To see this, we introduce polar coordinates θ and ϕ in $\boldsymbol{R}$-space,

$$R_1 = R\sin\theta\cos\phi,\ R_2 = R\sin\theta\sin\phi,\ R_3 = R\cos\theta.$$

The state (10.155) is expressed as

$$|\boldsymbol{R}\rangle_{\mathrm{N}} = \begin{pmatrix} \cos(\theta/2) \\ \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2) \end{pmatrix}. \tag{10.158}$$

This state is singular at $\theta = \pi$, reflecting that $|\boldsymbol{R}\rangle_{\mathrm{N}}$ is not defined for $R_3 = -R$. Consider another eigenvector

$$\begin{aligned} |\boldsymbol{R}\rangle_{\mathrm{S}} \equiv \mathrm{e}^{-\mathrm{i}\phi}|\boldsymbol{R}\rangle_{\mathrm{N}} &= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi}\cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \\ &= [2R(R - R_3)]^{-1/2} \begin{pmatrix} R_1 - \mathrm{i}R_2 \\ R - R_3 \end{pmatrix} \end{aligned} \tag{10.159}$$

with the same eigenvalue. This eigenvector is singular at $\theta = 0$, that is at $R_3 = R$. Corresponding to these vectors we have Berry's gauge potentials in polar coordinates,

$$\mathcal{A}_{\mathrm{N}} = \tfrac{1}{2}\mathrm{i}(1 - \cos\theta)\,\mathrm{d}\phi \qquad \theta \neq \pi \tag{10.160a}$$

$$\mathcal{A}_{\mathrm{S}} = -\tfrac{1}{2}\mathrm{i}(1 + \cos\theta)\,\mathrm{d}\phi \qquad \theta \neq 0. \tag{10.160b}$$

They are related by the gauge transformation,

$$\mathcal{A}_{\mathrm{S}} = \mathcal{A}_{\mathrm{N}} - \mathrm{i}\mathrm{d}\phi = \mathcal{A}_{\mathrm{N}} + \mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\mathrm{e}^{-\mathrm{i}\phi} \tag{10.161}$$

where $g(\pi/2, \phi) = \exp(-\mathrm{i}\phi)$ is identified with the transition function

t_{NS}. Equation (10.160) is simply the vector potential of the Wu–Yang monopole of strength $-\frac{1}{2}$, see §§1.3 and 10.5. The total flux of the monopole is $\Phi = 4\pi(-\frac{1}{2}) = -2\pi$.

The analogy between the present problem and the magnetic monopole is evident by now. If we fix the amplitude R of the magnetic field, the restricted parameter space is S^2. At each point $\boldsymbol{R}$ of S^2, the state has a phase degree of freedom. Thus we are dealing with a U(1) bundle $P(S^2, \mathrm{U}(1))$, which also describes a magnetic monopole. For each choice of the parameters $\boldsymbol{R}$, we have a fibre corresponding to the nth eigenstate $|n; \boldsymbol{R}\rangle$. The fibre at $\boldsymbol{R}$ consists of the equivalence class $[|\boldsymbol{R}\rangle]$ given by (10.131). The projection π maps a state to the parameter on which it is defined; $\pi : \mathrm{e}^{\mathrm{i}\alpha}|\boldsymbol{R}\rangle \to \boldsymbol{R} \in S^2$. As we have seen above, this bundle is non-trivial since it cannot be described by a single connection. The non-triviality of the bundle implies the existence of a monopole at the origin. Note that $\boldsymbol{R} = 0$ (that is, $\boldsymbol{B} = 0$) is a singular point at which all the eigenvalues are degenerate.

Next we turn to the problem of holonomy. Take a standard point $\boldsymbol{R}(0)$ on S^2 and choose a vector $|\boldsymbol{R}(0)\rangle$. We choose a loop $\boldsymbol{R}(t)$ on S^2 and execute a parallel transport $|\boldsymbol{R}(0)\rangle$ along γ, after which it comes back as a vector $\exp[\mathrm{i}\eta(1)]|\boldsymbol{R}\rangle$. The additional phase η represents the holonomy $\pi^{-1}(\boldsymbol{R}) \to \pi^{-1}(\boldsymbol{R})$ and corresponds to Berry's phase. From (10.152), $\eta(1)$ is given by

$$\eta(1) = \mathrm{i}\oint_{\boldsymbol{R}} \mathcal{A} = \mathrm{i}\int_S \mathcal{F} \tag{10.162}$$

where $\mathcal{F} = \mathrm{d}\mathcal{A}$ is the field strength and S is the surface bounded by the loop $\boldsymbol{R}(t)$. It follows from (10.162) that Berry's phase $\eta(1)$ represents the 'magnetic flux' through the area S.

Exercise 10.42 Use (10.159) to show that

$$\mathcal{A}_S = \frac{\mathrm{i}}{2}\frac{R_2\,\mathrm{d}R_1 - R_1\,\mathrm{d}R_2}{R(R - R_3)}. \tag{10.163}$$

Show also that

$$\mathrm{d}\phi = -\frac{R_2\,\mathrm{d}R_1 - R_1\,\mathrm{d}R_2}{(R + R_3)(R - R_3)}. \tag{10.164}$$

Observe that $\mathrm{d}\phi$ is singular at $R_3 = \pm R$.

Problems 10

1 Consider a two-dimensional plane M with coordinate $\boldsymbol{R}$ and a wavefunction ψ which depends on $\boldsymbol{R}$ adiabatically as $\psi = \psi(\boldsymbol{r}, \boldsymbol{R})$. Let $\boldsymbol{R} : [0, 1] \to M$ be a loop in M and suppose $\psi(\boldsymbol{r}, \boldsymbol{R}(1)) = -\psi(\boldsymbol{r}, \boldsymbol{R}(0))$,

that is the phase of ψ changes by π after an adiabatic change along the loop. Show that there is a point within the loop at which the adiabatic assumption breaks down. See Longuet-Higgins (1975).

11

CHARACTERISTIC CLASSES

Given a fibre F, a structure group G and a base space M, we may construct many fibre bundles over M, depending on the choice of the transition functions. Natural questions we may ask ourselves are how many bundles there are over M with given F and G, and how much they are different from a trivial bundle $M \times F$. For example, in §10.5, we observed that an SU(2) bundle over S^4 is classified by the homotopy group $\pi_3(\mathrm{SU}(2)) \simeq \mathbb{Z}$. The number $n \in \mathbb{Z}$ tells us how the transition functions twist the local pieces of the bundle when glued together. We have also observed that this homotopy group is evaluated by integrating $\mathrm{tr}\,\mathcal{F}^2 \in H^4(S^4)$ over S^4, see theorem 10.38.

Characteristic classes are subsets of the cohomology classes of the base space, and measure the *non-triviality* or *twisting* of a bundle. In this sense, they are *obstructions* which prevent a bundle from being a trivial bundle. Most of the characteristic classes are given by the de Rham cohomology classes. Besides their importance in classifications of fibre bundles, characteristic classes play central roles in index theorems.

Here we follow Alvalez-Gaumé and Ginsparg (1984), Eguchi *et al* (1980), Gilkey (1984) and Wells (1980).

11.1 Invariant polynomials and the Chern–Weil homomorphism

We give here a brief summary of the de Rham cohomology group (see Chapter 6 for details). Let M be an m-dimensional manifold. An r-form $\omega \in \Omega^r(M)$ is *closed* if $\mathrm{d}\omega = 0$ and *exact* if $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{r-1}(M)$. The set of closed r-forms is denoted by $Z^r(M)$ and the set of exact r-forms by $B^r(M)$. Since $\mathrm{d}^2 = 0$, it follows that $Z^r(M) \supset B^r(M)$. We define the rth de Rham cohomology group $H^r(M)$ by

$$H^r(M) \equiv Z^r(M)/B^r(M).$$

In $H^r(M)$, two closed r-forms ω_1 and ω_2 are identified if $\omega_1 - \omega_2 = \mathrm{d}\eta$ for some $\eta \in \Omega^{r-1}(M)$. Let M be an m-dimensional manifold. The formal sum

$$H^*(M) \equiv H^0(M) + H^1(M) + \ldots + H^m(M)$$

is the cohomology ring with the product $\wedge : H^*(M) \times H^*(M) \to H^*(M)$

induced by $\wedge : H^p(M) \times H^q(M) \to H^{p+q}(M)$. Let $f : M \to N$ be a smooth map. The pullback $f^* : \Omega^r(N) \to \Omega^r(M)$ naturally induces a linear map $f^* : H^r(N) \to H^r(M)$ since f^* commutes with the exterior derivative: $f^* \mathrm{d}\omega = \mathrm{d}f^*\omega$. f^* preserves the algebraic structure of the cohomology ring since $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

11.1.1 Invariant polynomials

Let $M(k, \mathbb{C})$ be the set of complex $k \times k$ matrices. Let $S^r(M(k, \mathbb{C}))$ denote the vector space of *symmetric r-linear* $\mathbb{C}$*-valued* functions on $M(k, \mathbb{C})$. In other words, a map

$$\widetilde{P} : \overset{r}{\otimes} M(k, \mathbb{C}) \to \mathbb{C}$$

is an element of $S^r(M(k, \mathbb{C}))$ if it satisfies, in addition to linearity in each entry, the symmetry

$$\widetilde{P}(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_r) = \widetilde{P}(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_r) \qquad 1 \leqslant i, j \leqslant r. \tag{11.1}$$

Let

$$S^*(M(k, \mathbb{C})) \equiv \bigoplus_{r=0}^{\infty} S^r(M(k, \mathbb{C}))$$

denote the formal sum of symmetric multilinear $\mathbb{C}$-valued functions. We define a product of $\widetilde{P} \in S^p(M(k, \mathbb{C}))$ and $\widetilde{Q} \in S^q(M(k, \mathbb{C}))$ by

$$\widetilde{P}\widetilde{Q}(X_1, \ldots, X_{p+q}) = \frac{1}{(p+q)!} \sum_P \widetilde{P}(X_{P(1)}, \ldots, X_{P(p)})\widetilde{Q}(X_{P(p+1)}, \ldots, X_{P(p+q)}) \tag{11.2}$$

where P is the permutation of $(1, \ldots, p+q)$. $S^*(M(k, \mathbb{C}))$ is an algebra with this multiplication.

Let G be a matrix group and $\mathfrak{g}$ its Lie algebra. In practice we take $G = \mathrm{GL}(k, \mathbb{C})$, $\mathrm{U}(k)$ or $\mathrm{SU}(k)$. The Lie algebra $\mathfrak{g}$ is a subspace of $M(k, \mathbb{C})$ and we may consider the restrictions $S^r(\mathfrak{g})$ and $S^*(\mathfrak{g}) \equiv \oplus_{r \geqslant 0} S^r(\mathfrak{g})$. $\widetilde{P} \in S^r(\mathfrak{g})$ is said to be **invariant** if, for any $g \in G$ and $A_i \in \mathfrak{g}$, $\widetilde{P}$ satisfies

$$\widetilde{P}(\mathrm{Ad}_g A_1, \ldots, \mathrm{Ad}_g A_r) = \widetilde{P}(A_1, \ldots, A_r) \tag{11.3}$$

where $\mathrm{Ad}_g A_i = g^{-1} A_i g$. For example,

$$\widetilde{P}(A_1, A_2, \ldots, A_r) = \mathrm{str}(A_1, A_2, \ldots, A_r) \equiv \frac{1}{r!} \sum_P \mathrm{tr}(A_{P(1)} A_{P(2)} \ldots A_{P(r)}) \tag{11.4}$$

is symmetric, r-linear and invariant, where 'str' stands for the **symmetrised trace** and is defined by the last equality. The set of G-invariant members of $S^r(\mathfrak{g})$ is denoted by $I^r(G)$. Note that $\mathfrak{g}_1 = \mathfrak{g}_2$ does not necessarily imply $I^r(G_1) = I^r(G_2)$. The product defined by (11.2) naturally induces a multiplication

$$I^p(G) \otimes I^q(G) \to I^{p+q}(G). \tag{11.5}$$

The sum $I^*(G) \equiv \oplus_{r\geq 0} I^r(G)$ is an algebra with this product.

Take $\widetilde{P} \in I^r(G)$. The shorthand notation for the diagonal combination is

$$P(A) \equiv \widetilde{P}(\underbrace{A, A, \ldots, A}_{r}) \qquad A \in \mathfrak{g}. \tag{11.6}$$

Clearly, P is a polynomial of degree r. P is said to be an **invariant polynomial**. P is also Ad G-invariant,

$$P(\mathrm{Ad}_g A) = P(g^{-1}Ag) = P(A) \qquad A \in \mathfrak{g},\ g \in G. \tag{11.7}$$

For example, $\mathrm{tr}\,(A^r)$ is an invariant polynomial obtained from (11.4). In general, an invariant polynomial may be written in terms of a sum of products of $P_r \equiv \mathrm{tr}\,(A^r)$.

Conversely, any invariant polynomial P defines an invariant and symmetric r-linear form $\widetilde{P}$ by expanding $P(t_1A_1 + \ldots + t_rA_r)$ as a polynomial in t_i. $1/r!$ times the coefficient of $t_1t_2\ldots t_r$ is invariant and symmetric by construction and is called the **polarisation** of P. Take $P(A) \equiv \mathrm{tr}\,(A^3)$, for example. Following the prescription above, we expand $\mathrm{tr}\,(t_1A_1 + t_2A_2 + t_3A_3)^3$ in powers of t_1, t_2 and t_3. The coefficient of $t_1t_2t_3$ is

$$\begin{aligned}&\mathrm{tr}\,(A_1A_2A_3 + A_1A_3A_2 + A_2A_1A_3 + A_2A_3A_1 + A_3A_1A_2 + A_3A_2A_1)\\ &\quad = 3\,\mathrm{tr}\,(A_1A_2A_3 + A_2A_1A_3)\end{aligned}$$

where the cyclicity of the trace has been used. The polarisation is

$$\widetilde{P}(A_1, A_2, A_3) = \tfrac{1}{2}\,\mathrm{tr}\,(A_1A_2A_3 + A_2A_1A_3) = \mathrm{str}\,(A_1, A_2, A_3).$$

In the previous chapter, we introduced the local gauge potential $\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu$ and the field strength $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ on a principal bundle. We have shown that these geometrical objects describe the associated vector bundles as well. Since the set of connections $\{\mathcal{A}_i\}$ describes the twisting of a fibre bundle, the non-triviality of a principal bundle is equally shared by its associated bundle. In fact, if (10.57) is employed as a definition of the local connection in a vector bundle, it can be defined even without reference to the principal bundle with which it is originally associated. Later we encounter situations in which use of vector bundles is essential (the Whitney sum bundle, the splitting principle and so on).

Let $P(M, G)$ be a principal bundle. We extend the domain of invariant polynomials from $\mathfrak{g}$ to $\mathfrak{g}$-valued p-forms on M. For $A_i\eta_i$ ($A_i \in \mathfrak{g}$, $\eta_i \in \Omega^{p_i}(M)$; $1 \leq i \leq r$) we define

$$\widetilde{P}(A_1\eta_1, \ldots, A_r\eta_r) \equiv \eta_1 \wedge \ldots \wedge \eta_r \widetilde{P}(A_1, \ldots, A_r). \quad (11.8)$$

For example, corresponding to (11.4), we have

$$\operatorname{str}(A_1\eta_1, \ldots, A_r\eta_r) = \eta_1 \wedge \ldots \wedge \eta_r \operatorname{str}(A_1, \ldots, A_r).$$

The diagonal combination is

$$P(A\eta) \equiv \underbrace{\eta \wedge \ldots \wedge \eta}_{r} P(A). \quad (11.9)$$

The action $\widetilde{P}$ or P on general elements is given by the r-linearity. In particular, we are interested in the invariant polynomial of the form $P(\mathcal{F})$ in the following. The importance of invariant polynomials resides in the following fundamental theorem.

Theorem 11.1 **(Chern–Weil)** Let P be an invariant polynomial. Then $P(\mathcal{F})$ satisfies:

(*a*) $\mathrm{d}P(\mathcal{F}) = 0$.

(*b*) Let $\mathcal{F}$ and $\mathcal{F}'$ be curvature two-forms corresponding to different connections $\mathcal{A}$ and $\mathcal{A}'$ on E. Then the difference $P(\mathcal{F}') - P(\mathcal{F})$ is exact.

Proof: (*a*) It is sufficient to prove that $\mathrm{d}P(\mathcal{F}) = 0$ for an invariant polynomial $P_r(\mathcal{F})$ which is homogeneous of degree r, since any invariant polynomial can be decomposed into homogeneous polynomials. First consider the identity,

$$\widetilde{P}_r(g_t^{-1}X_1g_t, \ldots g_t^{-1}X_rg_t) = \widetilde{P}_r(X_1, \ldots, X_r)$$

where $g_t \equiv \exp(tX)$ and $X, X_i \in \mathfrak{g}$. By putting $t = 0$ after differentiation with respect to t, we find

$$\sum_{i=1}^{r} \widetilde{P}_r(X_1, \ldots, [X_i, X], \ldots, X_r) = 0. \quad (11.10)$$

Next let A be a $\mathfrak{g}$-valued p-form and Ω_i be a $\mathfrak{g}$-valued p_i-form ($1 \leq i \leq r$). Without loss of generality, we may take $A = X\eta$ and $\Omega_i = X_i\eta_i$ where X, $X_i \in \mathfrak{g}$ and $\eta(\eta_i)$ is a p-form (p_i-form). Define

$$\begin{aligned}[\Omega_i, A] &\equiv \eta_i \wedge \eta[X_i, X] \\ &= X_iX(\eta_i \wedge \eta) - (-1)^{pp_i}XX_i(\eta \wedge \eta_i). \quad (11.11)\end{aligned}$$

Let us note that

$$\begin{aligned}
&\widetilde{P}_r(\Omega_1, \ldots, [\Omega_i, A], \ldots, \Omega_r) \\
&\quad = \eta_1 \wedge \ldots \wedge \eta_i \wedge \eta \wedge \ldots \wedge \eta_r \widetilde{P}_r(X_1, \ldots, X_i X, \ldots, X_r) \\
&\qquad - (-1)^{p \cdot p_i} \eta_1 \wedge \ldots \wedge \eta \wedge \eta_i \wedge \ldots \\
&\qquad \wedge \eta_r \widetilde{P}_r(X_1, \ldots, XX_i, \ldots, X_r) \\
&\quad = \eta \wedge \eta_1 \wedge \ldots \wedge \eta_r (-1)^{p(p_1 + \ldots + p_i)} \\
&\qquad \times \widetilde{P}_r(X_1, \ldots, [X_i, X], \ldots, X_r).
\end{aligned}$$

From this and (11.10), we find

$$\sum_{i=1}^{r} (-1)^{p(p_1 + \ldots + p_i)} \widetilde{P}_r(\Omega_1, \ldots, [\Omega_i, A], \ldots, \Omega_r) = 0. \qquad (11.12)$$

Next consider the derivative,

$$\begin{aligned}
\mathrm{d}\widetilde{P}_r(\Omega_1, \ldots, \Omega_r) &= \mathrm{d}(\eta_1 \wedge \ldots \wedge \eta_r) \widetilde{P}_r(X_1, \ldots, X_r) \\
&= \sum_{i=1}^{r} (-1)^{(p_1 + \ldots + p_{i-1})} (\eta_1 \wedge \ldots \wedge \mathrm{d}\eta_i \wedge \ldots \wedge \eta_r) \\
&\quad \times \widetilde{P}_r(X_1, \ldots, X_i, \ldots, X_r) \\
&= \sum_{i=1}^{r} (-1)^{(p_1 + \ldots + p_{i-1})} \widetilde{P}_r(\Omega_1, \ldots, \mathrm{d}\Omega_i, \ldots, \Omega_r). \qquad (11.13)
\end{aligned}$$

Let $A = \mathcal{A}$ and $\Omega_i = \mathcal{F}$ in (11.12) and (11.13) for which $p = 1$ and $p_i = 2$. By adding 0 of the form (11.12) to (11.13) we have

$$\begin{aligned}
&\mathrm{d}\widetilde{P}_r(\mathcal{F}, \ldots, \mathcal{F}) \\
&\quad = \sum_{i=1}^{r} [\widetilde{P}_r(\mathcal{F}, \ldots, \mathrm{d}\mathcal{F}, \ldots, \mathcal{F}) + \widetilde{P}_r(\mathcal{F}, \ldots, [\mathcal{A}, \mathcal{F}], \ldots, \mathcal{F})] \\
&\quad = \sum_{i=1}^{r} \widetilde{P}_r(\mathcal{F}, \ldots, \mathcal{D}\mathcal{F}, \ldots, \mathcal{F}) = 0 \qquad (11.14)
\end{aligned}$$

since $\mathcal{D}\mathcal{F} = \mathrm{d}\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0$ (the Bianchi identity). We have proved

$$\mathrm{d}P_r(\mathcal{F}) = \mathrm{d}\widetilde{P}_r(\mathcal{F}, \ldots, \mathcal{F}) = 0.$$

(*b*) Let $\mathcal{A}$ and $\mathcal{A}'$ be two connections on E and let $\mathcal{F}$ and $\mathcal{F}'$ be the respective field strengths. Define an interpolating gauge potential $\mathcal{A}_t$ by

$$\mathcal{A}_t \equiv \mathcal{A} + t\theta \qquad \theta \equiv (\mathcal{A}' - \mathcal{A}), 0 \leqslant t \leqslant 1 \qquad (11.15)$$

so that $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_1 = \mathcal{A}'$. The corresponding field strength is

$$\mathcal{F}_t \equiv \mathrm{d}\mathcal{A}_t + \mathcal{A}_t \wedge \mathcal{A}_t = \mathcal{F} + t\mathcal{D}\theta + t^2\theta^2 \qquad (11.16)$$

where $\mathcal{D}\theta = \mathrm{d}\theta + [\mathcal{A}, \theta] = \mathrm{d}\theta + \mathcal{A} \wedge \theta + \theta \wedge \mathcal{A}$. We first note that

$$P_r(\mathcal{F}') - P_r(\mathcal{F}) = P_r(\mathcal{F}_1) - P_r(\mathcal{F}_0) = \int_0^1 \mathrm{d}t \, \frac{\mathrm{d}}{\mathrm{d}t} P_r(\mathcal{F}_t)$$
$$= r \int_0^1 \mathrm{d}t \, \widetilde{P}_r \left(\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_t, \mathcal{F}_t, \ldots, \mathcal{F}_t \right). \qquad (11.17)$$

From (11.16), we find

$$\frac{\mathrm{d}}{\mathrm{d}t} P_r(\mathcal{F}_t) = r \widetilde{P}_r(\mathcal{D}\theta + 2t\theta^2, \mathcal{F}_t, \ldots, \mathcal{F}_t)$$
$$= r \widetilde{P}_r(\mathcal{D}\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t) + 2rt \widetilde{P}_r(\theta^2, \mathcal{F}_t, \ldots, \mathcal{F}_t). \qquad (11.18)$$

On the other hand, we have

$$\mathcal{D}\mathcal{F}_t = \mathrm{d}\mathcal{F}_t + [\mathcal{A}, \mathcal{F}_t] = -[\mathcal{A}_t, \mathcal{F}_t] + [\mathcal{A}, \mathcal{F}_t] = t[\mathcal{F}_t, \theta]$$

where use has been made of the Bianchi identity $\mathcal{D}_t \mathcal{F}_t = \mathrm{d}\mathcal{F}_t + [\mathcal{A}_t, \mathcal{F}_t] = 0$. [$\mathcal{D}$ is the covariant derivative with respect to $\mathcal{A}$ while $\mathcal{D}_t$ is that with respect to $\mathcal{A}_t$.] It then follows that

$$\begin{aligned} \mathrm{d}[\widetilde{P}_r(\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t)] & \\ = {} & \widetilde{P}_r(\mathrm{d}\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t) - (r-1) \widetilde{P}_r(\theta, \mathrm{d}\mathcal{F}_t, \ldots, \mathcal{F}_t) \\ = {} & \widetilde{P}_r(\mathcal{D}\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t) - (r-1) \widetilde{P}_r(\theta, \mathcal{D}\mathcal{F}_t, \ldots, \mathcal{F}_t) \\ = {} & \widetilde{P}_r(\mathcal{D}\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t) - (r-1)t \widetilde{P}_r(\theta, [\mathcal{F}_t, \theta], \mathcal{F}_t, \ldots, \mathcal{F}_t) \end{aligned} \qquad (11.19)$$

where we have added a 0 of the form (11.12) to change d to $\mathcal{D}$. If we take $\Omega_1 = A = \theta$ and $\Omega_2 = \ldots = \Omega_m = \mathcal{F}_t$ in (11.12), we have

$$2 \widetilde{P}_r(\theta^2, \mathcal{F}_t, \ldots, \mathcal{F}_t) + (r-1) \widetilde{P}_r(\theta, [\mathcal{F}_t, \theta], \mathcal{F}_t, \ldots, \mathcal{F}_t) = 0.$$

From (11.18), (11.19) and the above identity, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} P_r(\mathcal{F}_t) = r \, \mathrm{d}[\widetilde{P}_r(\theta, \mathcal{F}_t, \ldots, \mathcal{F}_t)].$$

We finally find

$$P_r(\mathcal{F}') - P_r(\mathcal{F}) = \mathrm{d} \left(r \int_0^1 \widetilde{P}_r(\mathcal{A}' - \mathcal{A}, \mathcal{F}_t, \ldots, \mathcal{F}_t) \, \mathrm{d}t \right). \qquad (11.20)$$

This shows that $P_r(\mathcal{F}')$ differs from $P_r(\mathcal{F})$ by an exact form. ■

We define the **transgression** $TP_r(\mathcal{A}', \mathcal{A})$ of P_r by

$$TP_r(\mathcal{A}', \mathcal{A}) \equiv r \int_0^1 \mathrm{d}t \, \widetilde{P}_r(\mathcal{A}' - \mathcal{A}, \mathcal{F}_t, \ldots, \mathcal{F}_t) \qquad (11.21)$$

where $\widetilde{P}_r$ is the polarisation of P_r. Transgressions will play an important role when we discuss Chern–Simons forms in §11.5. Let

$\dim M = m$. Since $P_m(\mathcal{F}')$ differs from $P_m(\mathcal{F})$ by an exact form, their integrals over a manifold M without boundary should agree:

$$\int_M P_m(\mathcal{F}') - \int_M P_m(\mathcal{F}) = \int_M \mathrm{d}TP_m(\mathcal{A}', \mathcal{A}) = \int_{\partial M} TP_m(\mathcal{A}', \mathcal{A}) = 0. \tag{11.22}$$

As has been proved, an invariant polynomial is closed and, in general, non-trivial. Accordingly it defines a cohomology class of M. Theorem 11.1(b) ensures that this cohomology class is independent of the gauge potential chosen. The cohomology class thus defined is called the **characteristic class**. The characteristic class defined by an invariant polynomial P is denoted by $\chi_E(P)$ where E is a fibre bundle on which connections and curvatures are defined. [*Remark:* Since a principal bundle and its associated bundles share the same gauge potentials and field strengths, the above Chern–Weil theorem applies equally to both bundles. Accordingly E can be either a principal bundle or a vector bundle.]

Theorem 11.2 Let P be an invariant polynomial in $I^*(G)$ and E be a fibre bundle over M with structure group G.

(a) The map

$$\chi_E : I^*(G) \to H^*(M) \tag{11.23}$$

defined by $P \to \chi_E(P)$ is a homomorphism (**Weil homomorphism**).

(b) Let $f : N \to M$ be a differentiable map. For the pullback bundle f^*E of E, we have the so called **naturality**

$$\chi_{f^*E} = f^*\chi_E. \tag{11.24}$$

Proof: (a) Take $P_r \in I_r(G)$ and $P_s \in I_s(G)$. If we write $\mathcal{F} = \mathcal{F}^\alpha T_\alpha$, we have

$$\begin{aligned}(P_rP_s)(\mathcal{F}) &= \mathcal{F}^{\alpha_1} \wedge \ldots \wedge \mathcal{F}^{\alpha_r} \wedge \mathcal{F}^{\beta_1} \wedge \ldots \wedge \mathcal{F}^{\beta_s} \\ &\quad \times \frac{1}{(r+s)!} \widetilde{P}_r(T_{\alpha_1}, \ldots, T_{\alpha_r}) \widetilde{P}_n(T_{\beta_1}, \ldots, T_{\beta_s}) \\ &= P_r(\mathcal{F}) \wedge P_s(\mathcal{F}).\end{aligned}$$

(a) follows since $P_r(\mathcal{F}), P_s(\mathcal{F}) \in H^*(M)$.

(b) Let $\mathcal{A}$ be a gauge potential of E and $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. It is easy to verify that the pullback $f^*\mathcal{A}$ is a connection in f^*E. In fact let $\mathcal{A}_i$ and $\mathcal{A}_j$ be local connections in overlapping charts U_i and U_j of M. If t_{ij} is a transition function on $U_i \cap U_j$, the transition function on f^*E is given by $f^*t_{ij} = t_{ij} \cdot f$. The pullback $f^*\mathcal{A}_i$ and $f^*\mathcal{A}_j$ are related as

$$\begin{aligned}f^*\mathcal{A}_j &= f^*(t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}\mathrm{d}t_{ij}) \\ &= (f^*t_{ij}^{-1})(f^*\mathcal{A}_i)(f^*t_{ij}) + (f^*t_{ij}^{-1})(\mathrm{d}f^*t_{ij}).\end{aligned}$$

This shows that $f^*\mathcal{A}$ is indeed a local connection on f^*E. The corresponding field strength on f^*E is

$$\mathrm{d}(f^*\mathcal{A}_i) + f^*\mathcal{A}_i \wedge f^*\mathcal{A}_i = f^*[\mathrm{d}\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i] = f^*\mathcal{F}_i.$$

Hence $f^*P(\mathcal{F}_i) = P(f^*\mathcal{F}_i)$, that is $f^*\chi_E(P) = \chi_{f^*E}(P)$. ■

Corollary 11.3 Characteristic classes of a trivial bundle are trivial.

Proof: Let $F \xrightarrow{\pi} M$ be a trivial bundle. Since F is trivial, there exists a map $f : M \to \{p\}$ such that $F = f^*E$ where $E \xrightarrow{\pi'} \{p\}$ is a bundle over a point p. All the de Rham cohomology groups of a point are trivial and so are the characteristic classes. Theorem 11.2(b) ensures that the characteristic classes χ_F $(= f^*\chi_E)$ of F are also trivial. ■

11.2 Chern classes

11.2.1 Definitions

Let $E \xrightarrow{\pi} M$ be a complex vector bundle whose fibre is $\mathbb{C}^k$. The structure group G is a subgroup of $\mathrm{GL}(k, \mathbb{C})$, and the gauge potential $\mathcal{A}$ and the field strength $\mathcal{F}$ take their values in $\mathfrak{g}$. Define the **total Chern class** by

$$c(\mathcal{F}) \equiv \det\left(I + \frac{\mathrm{i}\mathcal{F}}{2\pi}\right). \tag{11.25}$$

Since $\mathcal{F}$ is a two-form, $c(\mathcal{F})$ is a direct sum of forms of even degrees,

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F}) + c_2(\mathcal{F}) + \ldots \tag{11.26}$$

where $c_j(\mathcal{F}) \in \Omega^{2j}(M)$ is called the jth **Chern class**. In an m-dimensional manifold M, the Chern class $c_j(\mathcal{F})$ with $2j > m$ vanishes trivially. Irrespective of $\dim M$, the series terminates at $c_k(\mathcal{F}) = \det(\mathrm{i}\mathcal{F}/2\pi)$ and $c_j(\mathcal{F}) = 0$ for $j > k$. Since $c_j(\mathcal{F})$ is closed, it defines an element $[c_j(\mathcal{F})]$ of $H^{2j}(M)$.

Example 11.4 Let E be a complex vector bundle with fibre $\mathbb{C}^2$ over M, where $G = \mathrm{SU}(2)$ and $\dim M = 4$. If we write the field strength as $\mathcal{F} = \mathcal{F}^\alpha(\sigma_\alpha/2\mathrm{i})$, $\mathcal{F}^\alpha = \frac{1}{2}\mathcal{F}^\alpha{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, we have

$$\begin{aligned}
c(\mathcal{F}) &= \det\left(I + \frac{\mathrm{i}}{2\pi}\mathcal{F}^\alpha(\sigma_\alpha/2\mathrm{i})\right) \\
&= \det\begin{pmatrix} 1 + (\mathrm{i}/2\pi)(\mathcal{F}^3/2\mathrm{i}) & (\mathrm{i}/2\pi)(\mathcal{F}^1 - \mathrm{i}\mathcal{F}^2)/2\mathrm{i} \\ (\mathrm{i}/2\pi)(\mathcal{F}^1 + \mathrm{i}\mathcal{F}^2)/2\mathrm{i} & 1 - (\mathrm{i}/2\pi)(\mathcal{F}^3/2\mathrm{i}) \end{pmatrix} \\
&= 1 + \tfrac{1}{4}(\mathrm{i}/2\pi)^2(\mathcal{F}^3 \wedge \mathcal{F}^3 + \mathcal{F}^1 \wedge \mathcal{F}^1 + \mathcal{F}^2 \wedge \mathcal{F}^2).
\end{aligned} \tag{11.27}$$

Individual Chern classes are

$$c_0(\mathcal{F}) = 1 \tag{11.28a}$$

$$c_1(\mathcal{F}) = 0 \tag{11.28b}$$

$$c_2(\mathcal{F}) = \left(\frac{\mathrm{i}}{2\pi}\right)^2 \sum \frac{\mathcal{F}^\alpha \wedge \mathcal{F}^\alpha}{4} = \det\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right). \tag{11.28c}$$

Higher Chern classes vanish identically.

For general fibre bundles, it is rather cumbersome to compute the Chern classes by expanding the determinant and it is desirable to find a formula which yields them more easily. This is done by diagonalising the curvature form. $\mathcal{F}$ is diagonalised by an appropriate matrix $g \in \mathrm{GL}(k, \mathbb{C})$ as $g^{-1}(\mathrm{i}\mathcal{F}/2\pi)g = \mathrm{diag}(x_1, \ldots, x_k)$, where x_i is a two-form. This diagonal matrix will be denoted by A. For example, if $G = \mathrm{SU}(k)$, the generators are chosen to be anti-Hermitian and a Hermitian matrix $\mathrm{i}\mathcal{F}/2\pi$ can be diagonalised by $g \in \mathrm{SU}(k)$. We have

$$\begin{aligned}
\det(\mathbb{1} + A) &= \det[\mathrm{diag}(1 + x_1, 1 + x_2, \ldots, 1 + x_k)] \\
&= \prod_{j=1}^{k} (1 + x_j) \\
&= 1 + (x_1 + \ldots + x_k) + (x_1x_2 + \ldots + x_{k-1}x_k) \\
&\quad + \ldots + (x_1x_2 \ldots x_k) \\
&= 1 + \operatorname{tr} A + \tfrac{1}{2}\{(\operatorname{tr} A)^2 - \operatorname{tr} A^2\} + \ldots + \det A.
\end{aligned} \tag{11.29}$$

Observe that each term of (11.29) is an elementary symmetric function of $\{x_j\}$,

$$S_0(x_j) \equiv 1 \tag{11.30a}$$

$$S_1(x_j) \equiv \sum_{j=1}^{k} x_j \tag{11.30b}$$

$$S_2(x_j) \equiv \sum_{i<j} x_i x_j \tag{11.30c}$$

$$\vdots$$

$$S_k(x_j) \equiv x_1 x_2 \ldots x_k. \tag{11.30d}$$

Since $\det(\mathbb{1} + A)$ is an invariant polynomial, we have $P(\mathcal{F}) = P(g\mathcal{F}g^{-1}) = P(2\pi A/\mathrm{i})$, see (11.7). Accordingly, we have, for general $\mathcal{F}$,

$$c_0(\mathcal{F}) = 1 \tag{11.31a}$$

$$c_1(\mathcal{F}) = \operatorname{tr} A = \operatorname{tr}\left(g\, \frac{\mathrm{i}\mathcal{F}}{2\pi}\, g^{-1}\right) = \frac{\mathrm{i}}{2\pi} \operatorname{tr} \mathcal{F} \tag{11.31b}$$

$$\begin{aligned}
c_2(\mathcal{F}) &= \tfrac{1}{2}[(\operatorname{tr} A)^2 - \operatorname{tr} A^2] \\
&= \tfrac{1}{2}\, (\mathrm{i}/2\pi)^2 [\operatorname{tr} \mathcal{F} \wedge \operatorname{tr} \mathcal{F} - \operatorname{tr}(\mathcal{F} \wedge \mathcal{F})]
\end{aligned} \tag{11.31c}$$

$$\vdots$$

$$c_k(\mathcal{F}) = \det A = (\mathrm{i}/2\pi)^k \det \mathcal{F}. \tag{11.31d}$$

Example 11.4 is easily verified from (11.31). [Note that the Pauli matrices (in general, any element of the Lie algebra $\mathfrak{su}(n)$ of $\mathrm{SU}(n)$) are traceless, $\operatorname{tr}\sigma_\alpha = 0$.]

11.2.2 Properties of Chern classes

We will deal with several vector bundles in the following. We often denote the Chern class of a vector bundle E by $c(E)$. If the specification of the curvature is required, we write $c(\mathcal{F}_E)$.

Theorem 11.5 Let $E \xrightarrow{\pi} M$ be a vector bundle with $G = \mathrm{GL}(k, \mathbb{C})$ and $F = \mathbb{C}^k$.

(a) (Naturality) Let $f : N \to M$ be a smooth map. Then

$$c(f^* E) = f^* c(E). \tag{11.32}$$

(b) Let $F \xrightarrow{\pi'} M$ be another vector bundle with $F = \mathbb{C}^l$ and $G = \mathrm{GL}(l, \mathbb{C})$. The total Chern class of a Whitney sum bundle $E \oplus F$ is

$$c(E \oplus F) = c(E) \wedge c(F). \tag{11.33}$$

Proof: (a) The naturality follows directly from theorem 11.2(a). Since the curvature of f^*E is $\mathcal{F}_{f^*E} = f^*\mathcal{F}_E$, the total Chern class of f^*E is

$$\begin{aligned} c(f^*E) &= \det\left(\mathbb{1} + \frac{\mathrm{i}}{2\pi}\mathcal{F}_{f^*E}\right) = \det\left(\mathbb{1} + \frac{\mathrm{i}}{2\pi} f^*\mathcal{F}_E\right) \\ &= f^*\det\left(\mathbb{1} + \frac{\mathrm{i}}{2\pi}\mathcal{F}_E\right) = f^*c(E). \end{aligned}$$

(b) Let us consider the Chern polynomial of a matrix

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}.$$

[Note that the curvature of a Whitney sum bundle is block diagonal: $\mathcal{F}_{E\oplus F} = \operatorname{diag}(\mathcal{F}_E, \mathcal{F}_F)$.] We find

$$\begin{aligned} \det\left(\mathbb{1} + \frac{\mathrm{i}A}{2\pi}\right) &= \det\begin{pmatrix} \mathbb{1} + \dfrac{\mathrm{i}B}{2\pi} & 0 \\ 0 & \mathbb{1} + \dfrac{\mathrm{i}C}{2\pi} \end{pmatrix} \\ &= \det\left(\mathbb{1} + \frac{\mathrm{i}B}{2\pi}\right)\det\left(\mathbb{1} + \frac{\mathrm{i}C}{2\pi}\right) = c(B)c(C). \end{aligned}$$

This relation remains true when B and C are replaced by $\mathcal{F}_E$ and $\mathcal{F}_F$, namely

$$c(\mathcal{F}_{E\oplus F}) = c(\mathcal{F}_E) \wedge c(\mathcal{F}_F)$$

which proves (11.33). ■

Exercise 11.6

(*a*) Let E be a trivial bundle. Use corollary 11.3 to show that

$$c(E) = 1. \tag{11.34}$$

(*b*) Let E be a vector bundle such that $E = E_1 \oplus E_2$ where E_1 is a vector bundle of dimension k_1 and E_2 is a *trivial* vector bundle of dimension k_2. Show that

$$c_i(E) = 0 \qquad k_1 + 1 \leqslant i \leqslant k_1 + k_2. \tag{11.35}$$

11.2.3 Splitting principle

Let E be a Whitney sum of n complex line bundles,

$$E = L_1 \oplus L_2 \oplus \ldots \oplus L_n. \tag{11.36}$$

From (11.33) above, we have

$$c(E) = c(L_1)c(L_2) \ldots c(L_n) \tag{11.37}$$

where the product is the exterior product of differential forms. Since $c_r(L) = 0$ for $r \geqslant 2$, we write

$$c(L_i) = 1 + c_1(L_i) \equiv 1 + x_i. \tag{11.38}$$

Then (11.37) becomes

$$c(E) = \prod_{i=1}^{n} (1 + x_i). \tag{11.39}$$

Comparing this with (11.29), we find that the Chern class of an n-dimensional vector bundle E is identical with that of the Whitney sum of n complex line bundles. Although E is not a Whitney sum of complex line bundles in general, as far as the Chern classes are concerned, we may pretend that this is the case. This is called the **splitting principle** and we accept this fact without proof. The general proof is found in Shanahan (1978) and Hirzebruch (1966), for example.

Intuitively speaking, if the curvature $\mathcal{F}$ is diagonalised, the complex vector space on which $\mathfrak{g}$ acts splits into k independent pieces: $\mathbb{C}^k \to \mathbb{C} \oplus \ldots \oplus \mathbb{C}$. An eigenvalue x_i is a curvature in each complex line bundle. Since diagonalisable matrices are *dense* in $M(n, \mathbb{C})$, any matrix may be approximated by a diagonal one as closely as we wish. Hence the splitting principle applies to any matrix. As an exercise, the reader may prove (11.33) using the splitting principle.

11.2.4 Universal bundles and classifying spaces

By now the reader must have some acquaintance with characteristic classes. Before we close this section, we examine these from a slightly different point of view emphasising their role in the classification of fibre bundles. Let $E \xrightarrow{\pi} M$ be a vector bundle with fibre $\mathbb{C}^k$. It is known that we can always find a bundle $\bar{E} \xrightarrow{\pi'} M$ such that

$$E \oplus \bar{E} \cong M \times \mathbb{C}^n \tag{11.40}$$

for some $n \geq k$. The fibre F_p of E at $p \in M$ is a k-plane lying in $\mathbb{C}^n$. Let $G_{k,n}(\mathbb{C})$ be the Grassmann manifold defined in example 8.5. $G_{k,n}(\mathbb{C})$ is the set of k-planes in $\mathbb{C}^n$. Similarly to the canonical line bundle, we define the canonical k-plane bundle $L_{k,n}(\mathbb{C})$ over $G_{k,n}(\mathbb{C})$ with the fibre $\mathbb{C}^k$. Consider a map $f: M \to G_{k,n}(\mathbb{C})$ which maps a point p to the k-plane F_p in $\mathbb{C}^n$.

Theorem 11.7 Let M be a manifold with $\dim M = m$ and let $E \xrightarrow{\pi} M$ be a complex vector bundle with the fibre $\mathbb{C}^k$. Then there exists a number N such that for $n > N$:

(a) there exists a map $f: M \to G_{k,n}(\mathbb{C})$ such that

$$E \cong f^* L_{k,n}(\mathbb{C}) \tag{11.41}$$

(b) $f^* L_{k,n}(\mathbb{C}) \cong g^* L_{k,n}(\mathbb{C})$ if and only if $f,\ g: M \to G_{k,n}(\mathbb{C})$ are homotopic.

The proof is found in Chern (1979). For example, if $E \xrightarrow{\pi} M$ is a complex line bundle, then there exists a bundle $\bar{E} \xrightarrow{\pi'} M$ such that $E \oplus \bar{E} \cong M \times \mathbb{C}^n$ and a map $f: M \to G_{1,n}(\mathbb{C}) = \mathbb{C}P^{n-1}$ such that $E = f^* L$, L being the canonical line bundle over $\mathbb{C}P^{n-1}$. Moreover, if $f \sim g$, then $f^* L$ is equivalent to $g^* L$. Theorem 11.7 shows that the classification of vector bundles reduces to that of the *homotopy classes* of the maps $M \to G_{k,n}(\mathbb{C})$.

It is convenient to define the **classifying space** $G_k(\mathbb{C})$. Regarding a k-plane in $\mathbb{C}^n$ as that in $\mathbb{C}^{n+1}$, we have natural inclusions,

$$G_{k,k}(\mathbb{C}) \hookrightarrow G_{k,k+1}(\mathbb{C}) \hookrightarrow \ldots \hookrightarrow G_k(\mathbb{C}) \tag{11.42}$$

where

$$G_k(\mathbb{C}) \equiv \bigcup_{n=k}^{\infty} G_{k,n}(\mathbb{C}). \tag{11.43}$$

Correspondingly, we have the **universal bundle** $L_k \to G_k(\mathbb{C})$ whose fibre is $\mathbb{C}^k$. For *any* complex vector bundle $E \xrightarrow{\pi} M$ with fibre $\mathbb{C}^k$, there exists a map $f: M \to G_k(\mathbb{C})$ such that $E = f^* L_k(\mathbb{C})$.

Let $E \xrightarrow{\pi} M$ be a vector bundle. A characteristic class χ is defined as a map $\chi: E \to \chi(E) \in H^*(M)$ such that

$$\chi(f^* E) = f^* \chi(E) \qquad \text{(naturality)} \tag{11.44a}$$

$$\chi(E) = \chi(E') \qquad \text{if } E \text{ is equivalent to } E'. \tag{11.44b}$$

In (11.44a), f^* on the LHS is a pullback of the bundle while f^* on the

RHS is that of the cohomology class. Since the homotopy class $[f]$ of $f : M \to G_k(\mathbb{C})$ *uniquely* defines the pullback

$$f^* : H^*(G_k) \to H^*(M) \tag{11.45}$$

an element $\chi(E) = f^*\chi(G_k)$ proves to be useful in classifying complex vector bundles over M with $\dim E = k$. For each choice of $\chi(G_k)$, there exists a characteristic class in E.

The Chern class $c(E)$ is also defined axiomatically by

(i) $c(f^*E) = f^*c(E)$ (naturality) (11.46*a*)

(ii) $c(E) = c_0(E) \oplus c_1(E) \oplus \ldots \oplus c_k(E)$,

$c_i(E) \in H^{2i}(M)$; $c_i(E) = 0,\ i > k$ (11.46*b*)

(iii) $c(E \oplus F) = c(E)c(E)$ (Whitney sum) (11.46*c*)

(iv) $c(L) = 1 + x$ (normalisation) (11.46*d*)

L being the canonical line bundle over $\mathbb{C}P^n$. It can be shown that these axioms uniquely define the Chern class as (11.25).

11.3 Chern characters

11.3.1. Definitions

Among the characteristic classes, the Chern characters are of special importance due to their appearance in the Atiyah–Singer index theorem. The **total Chern character** is defined by

$$\operatorname{ch}(\mathcal{F}) \equiv \operatorname{tr}\exp\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right) = \sum_{j=1} \frac{1}{j!}\operatorname{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^j. \tag{11.47}$$

The ***j*th Chern character** $\operatorname{ch}_j(\mathcal{F})$ is

$$\operatorname{ch}_j(\mathcal{F}) \equiv \frac{1}{j!}\operatorname{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^j. \tag{11.48}$$

If $2j > m = \dim M$, $\operatorname{ch}_j(\mathcal{F})$ vanishes, hence $\operatorname{ch}(\mathcal{F})$ is a polynomial of finite order.

Let us diagonalise $\mathcal{F}$ as

$$\frac{\mathrm{i}\mathcal{F}}{2\pi} \to g^{-1}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)g = A \equiv \operatorname{diag}(x_1, \ldots, x_k) \qquad g \in \mathrm{GL}(k, \mathbb{C}).$$

The total Chern character is expressed as

$$\operatorname{tr}[\exp(A)] = \sum_{j=1}^{k} \exp(x_j). \tag{11.49}$$

In terms of the elementary symmetric functions $S_r(x_j)$, the total Chern character becomes

$$\sum_{j=1}^{k} \exp(x_j) = \sum_{j=1}^{k} \left(1 + x_j + \frac{1}{2!} x_j^2 + \frac{1}{3!} x_j^3 + \dots\right)$$
$$= k + S_1(x_j) + \frac{1}{2!}[S_1(x_j)^2 - 2S_2(x_j)] + \dots . \quad (11.50)$$

Accordingly each Chern character is expressed in terms of the Chern classes as

$$\mathrm{ch}_0(\mathcal{F}) = k \quad (11.51a)$$
$$\mathrm{ch}_1(\mathcal{F}) = c_1(\mathcal{F}) \quad (11.51b)$$
$$\mathrm{ch}_2(\mathcal{F}) = \tfrac{1}{2}[c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})] \quad (11.51c)$$
$$\vdots$$

In (11.51a), k is the fibre dimension of the bundle.

Example 11.8 Let P be a U(1) bundle over S^2. If $\mathcal{A}_{\mathrm{N}}$ and $\mathcal{A}_{\mathrm{S}}$ are the local connections on U_{N} and U_{S} defined in §10.5, the field strength is given by $\mathcal{F}_i = \mathrm{d}\mathcal{A}_i$ ($i = \mathrm{N, S}$). We have

$$\mathrm{ch}(\mathcal{F}) = 1 + \frac{\mathrm{i}\mathcal{F}}{2\pi} \quad (11.52)$$

where we have noted that $\mathcal{F}^n = 0$ ($n \geqslant 2$) on S^2. This bundle describes the magnetic monopole. The magnetic charge $2g$ given by (10.94) is an *integer* expressed in terms of the Chern character as

$$N = \frac{\mathrm{i}}{2\pi} \int_{S^2} \mathcal{F} = \int_{S^2} \mathrm{ch}_1(\mathcal{F}). \quad (11.53)$$

Let P be an SU(2) bundle over S^4. The total Chern class of P is given by (11.27). The total Chern character is

$$\mathrm{ch}(\mathcal{F}) = 2 + \mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right) + \tfrac{1}{2}\,\mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^2. \quad (11.54)$$

$\mathrm{ch}(\mathcal{F})$ terminates at $\mathrm{ch}_2(\mathcal{F})$ since $\mathcal{F}^n = 0$ for $n \geqslant 3$. Moreover, $\mathrm{tr}\,\mathcal{F} = 0$ for $G = \mathrm{SU}(n)$, $n \geqslant 2$. As we found in §10.5, the instanton number is given by

$$\tfrac{1}{2} \int_{S^4} \mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^2 = \int_{S^4} \mathrm{ch}_2(\mathcal{F}). \quad (11.55)$$

In both cases, ch_j measures how the bundle is twisted when local pieces are patched together.

Example 11.9 Let P be a U(1) bundle over $2m$-dimensional space M. The mth Chern character is given by

$$\frac{1}{m!}\operatorname{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^m$$
$$= \frac{1}{m!}\left(\frac{\mathrm{i}}{2\pi}\right)^m [\tfrac{1}{2}\mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu]^m$$
$$= \frac{1}{m!}\left(\frac{\mathrm{i}}{4\pi}\right)^m \mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_m\nu_m}\,\mathrm{d}x^{\mu_1}\wedge \mathrm{d}x^{\nu_1}\wedge\ldots\wedge \mathrm{d}x^{\mu_m}\wedge \mathrm{d}x^{\nu_m}$$
$$= \left(\frac{\mathrm{i}}{4\pi}\right)^m \varepsilon^{\mu_1\nu_1\ldots\mu_m\nu_m}\mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_m\nu_m}\,\mathrm{d}x^1\wedge\ldots\wedge \mathrm{d}x^{2m}$$

which describes the U(1) anomaly in $2m$-dimensional space, see Chapter 13.

Example 11.10 Let L be a complex line bundle. It then follows that

$$\operatorname{ch}(L) = \operatorname{tr}\exp\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right) = \mathrm{e}^x = 1 + x \qquad x \equiv \frac{\mathrm{i}\mathcal{F}}{2\pi}. \tag{11.56}$$

For example, let $L \xrightarrow{\pi} \mathbb{C}P^1$ be the canonical line bundle over $\mathbb{C}P^1 = S^2$. The Fubini–Study metric yields the curvature

$$\mathcal{F} = -\partial\bar\partial\ln(1 + |z|^2) = -\frac{\mathrm{d}z\wedge \mathrm{d}\bar z}{(1 + z\bar z)^2} \tag{11.57}$$

see example 8.28. In real coordinates $z = x + \mathrm{i}y = r\exp(\mathrm{i}\theta)$, we have

$$\mathcal{F} = 2\mathrm{i}\frac{\mathrm{d}x\wedge \mathrm{d}y}{(1 + x^2 + y^2)^2} = 2\mathrm{i}\frac{r\mathrm{d}r\wedge \mathrm{d}\theta}{(1 + r^2)^2}. \tag{11.58}$$

From $\operatorname{ch}(\mathcal{F}) = 1 + \operatorname{tr}(\mathrm{i}\mathcal{F}/2\pi)$, we have

$$\operatorname{ch}_1(\mathcal{F}) = -\frac{1}{\pi}\frac{r\mathrm{d}r\wedge \mathrm{d}\theta}{(1 + r^2)^2}. \tag{11.59}$$

$\operatorname{Ch}_1(L)$, the integral of $\operatorname{ch}_1(\mathcal{F})$ over S^2, is an integer,

$$\operatorname{Ch}_1(L) = -\frac{1}{\pi}\int\frac{r\mathrm{d}r\,\mathrm{d}\theta}{(1 + r^2)^2} = -\int_1^\infty t^{-2}\,\mathrm{d}t = -1. \tag{11.60}$$

11.3.2 Properties of the Chern characters

Theorem 11.11

(*a*) (Naturality) Let $E \xrightarrow{\pi} M$ be a vector bundle with $F = \mathbb{C}^k$. Let $f: N \to M$ be a smooth map. Then

$$\operatorname{ch}(f^*E) = f^*\operatorname{ch}(E). \tag{11.61}$$

(*b*) Let E and F be vector bundles over a manifold M. The Chern characters of $E \otimes F$ and $E \oplus F$ are given by

$$\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \wedge \mathrm{ch}(F) \tag{11.62a}$$

$$\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) \oplus \mathrm{ch}(F). \tag{11.62b}$$

Proof: (*a*) follows from theorem 11.2(*a*).

(*b*) These results are immediate from the definition of the ch-polynomial. Let

$$\mathrm{ch}(A) = \sum (1/j!)\,\mathrm{tr}\,(\mathrm{i}A/2\pi)^j$$

be a polynomial of a *matrix* A. Suppose A is a tensor product of B and C, $A = B \otimes C = B \otimes \mathbb{1} + \mathbb{1} \otimes C$ (note that $\mathcal{F}_{E\otimes F} = \mathcal{F}_E \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{F}_F$). Then we find that

$$\begin{aligned}
\mathrm{ch}(B \otimes C) &= \sum_j \left(\frac{1}{j!}\right)\left(\frac{\mathrm{i}}{2\pi}\right)^j \mathrm{tr}\,(B \otimes \mathbb{1} + \mathbb{1} \otimes C)^j \\
&= \sum_j \left(\frac{1}{j!}\right)\left(\frac{\mathrm{i}}{2\pi}\right)^j \sum_{m=1}^{j} \binom{j}{m} \mathrm{tr}\,(B^m)\,\mathrm{tr}\,(C^{j-m}) \\
&= \sum_m \left(\frac{1}{m!}\right) \mathrm{tr}\left(\frac{\mathrm{i}B}{2\pi}\right)^m \sum_n \left(\frac{1}{n!}\right) \mathrm{tr}\left(\frac{\mathrm{i}C}{2\pi}\right)^n = \mathrm{ch}(B)\,\mathrm{ch}(C).
\end{aligned}$$

(11.62*a*) is proved if B is replaced by $\mathcal{F}_E$ and C by $\mathcal{F}_F$.

If A is block diagonal,

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} = B \oplus C$$

we have

$$\begin{aligned}
\mathrm{ch}(B \oplus C) &= \sum \left(\frac{1}{j!}\right)\left(\frac{\mathrm{i}}{2\pi}\right)^j \mathrm{tr}\,(B \oplus C)^j \\
&= \sum \left(\frac{1}{j!}\right)\left(\frac{1}{2\pi}\right)^j [\mathrm{tr}\,(B^j) + \mathrm{tr}\,(C^j)] = \mathrm{ch}(B) + \mathrm{ch}(C).
\end{aligned}$$

This relation remains true when A, B and C are replaced by $\mathcal{F}_{E\oplus F}$, $\mathcal{F}_E$ and $\mathcal{F}_F$ respectively. ■

Let us see how the splitting principle works in this case. Let L_j, $1 \leqslant j \leqslant k$, be complex line bundles. From (11.62*b*) we have, for $E = L_1 \oplus L_2 \oplus \ldots \oplus L_k$,

$$\mathrm{ch}(E) = \mathrm{ch}(L_1) \oplus \mathrm{ch}(L_2) \oplus \ldots \oplus \mathrm{ch}(L_k). \tag{11.63}$$

Since $\mathrm{ch}(L_i) = \exp(x_i)$, we find

$$\mathrm{ch}(E) = \prod_{j=1}^{k} \exp(x_j) \tag{11.64}$$

which is simply (11.50). Hence the Chern character of a general vector bundle E is given by that of a Whitney sum of k complex line bundles. The characteristic classes themselves cannot differentiate between two vector bundles of the same base space and the same fibre dimension. What is important is their *integral* over the base space.

11.3.3 Todd classes

Another useful characteristic class associated with a complex vector bundle is the **Todd class** defined by

$$\mathrm{Td}(\mathcal{F}) = \prod_j \frac{x_j}{1 - \mathrm{e}^{-x_j}} \tag{11.65}$$

where the splitting principle is understood. If expanded in powers of x_j, $\mathrm{Td}(\mathcal{F})$ becomes,

$$\begin{aligned}\mathrm{Td}(\mathcal{F}) &= \prod_j \left(1 + \tfrac{1}{2}x_j + \sum_{k\geq 1} (-1)^{k-1} \frac{B_k}{(2k)!} x_j^{2k}\right) \\ &= 1 + \tfrac{1}{2}\sum_j x_j + \tfrac{1}{12}\sum_j x_j^2 + \tfrac{1}{4}\sum_{j<k} x_j x_k + \ldots \\ &= 1 + \tfrac{1}{2}c_1(\mathcal{F}) + \tfrac{1}{12}[c_1(\mathcal{F})^2 + c_2(\mathcal{F})] + \ldots \end{aligned} \tag{11.66}$$

where the B_k are the **Bernoulli numbers** given by

$$B_1 = \tfrac{1}{6},\ B_2 = \tfrac{1}{30},\ B_3 = \tfrac{1}{42},\ B_4 = \tfrac{1}{30},\ B_5 = \tfrac{5}{66},\ \ldots .$$

The first few terms of (11.66) are

$$\mathrm{Td}_0(\mathcal{F}) = 1 \tag{11.67a}$$

$$\mathrm{Td}_1(\mathcal{F}) = \tfrac{1}{2}c_1 \tag{11.67b}$$

$$\mathrm{Td}_2(\mathcal{F}) = \tfrac{1}{12}(c_1^2 + c_2) \tag{11.67c}$$

$$\mathrm{Td}_3(\mathcal{F}) = \tfrac{1}{24}c_1c_2 \tag{11.67d}$$

$$\mathrm{Td}_4(\mathcal{F}) = \tfrac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) \tag{11.67e}$$

$$\mathrm{Td}_5(\mathcal{F}) = \tfrac{1}{1440}(-c_1^3c_2 + 3c_1c_2^2 + c_1^2c_3 - c_1c_4) \tag{11.67f}$$

where c_i stands for $c_i(\mathcal{F})$.

Exercise 11.12 Let E and F be complex vector bundles over M. Show that

$$\mathrm{Td}(E \oplus F) = \mathrm{Td}(E) \wedge \mathrm{Td}(F). \tag{10.68}$$

11.4 Pontrjagin and Euler classes

In the present section we will be concerned with the characteristic classes associated with a real vector bundle.

11.4.1 Pontrjagin classes

Let E be a real vector bundle over an m-dimensional manifold M with $\dim_{\mathbb{R}} E = k$. If E is endowed with the fibre metric, we may introduce orthonormal frames at each fibre. The structure group may be reduced to $O(k)$ from $GL(k, \mathbb{R})$. Since the generators of $\mathfrak{o}(k)$ are skew symmetric, the field strength $\mathcal{F}$ of E is also skew symmetric. A skew-symmetric matrix A is not diagonalisable by an element of a subgroup of $GL(k, \mathbb{R})$. It is, however, reducible to block diagonal form as

$$A \to \begin{pmatrix} 0 & \lambda_1 & & & 0 \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ 0 & & & & \ddots \end{pmatrix}$$

$$\to \begin{pmatrix} i\lambda_1 & & & & \\ & -i\lambda_1 & & 0 & \\ & & i\lambda_2 & & \\ & & & -i\lambda_2 & \\ & 0 & & & \ddots \end{pmatrix} \tag{11.69}$$

where the second diagonalisation is achieved only by an element of $GL(k, \mathbb{C})$. If k is odd, the last diagonal element is set to zero. For example, the generator of $\mathfrak{o}(3) = \mathfrak{so}(3)$ generating rotations around the z-axis is

$$T_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The **total Pontrjagin class** is defined by

$$p(\mathcal{F}) = \det\left(I + \frac{\mathcal{F}}{2\pi}\right). \tag{11.70}$$

From the skew symmetry $\mathcal{F}^t = -\mathcal{F}$, it follows that

$$\det\left(I + \frac{\mathcal{F}}{2\pi}\right) = \det\left(I + \frac{\mathcal{F}^t}{2\pi}\right) = \det\left(I - \frac{\mathcal{F}}{2\pi}\right).$$

This shows that $p(\mathcal{F})$ is an *even* function in $\mathcal{F}$. The expansion of $p(\mathcal{F})$ is

$$p(\mathcal{F}) = 1 + p_1(\mathcal{F}) + p_2(\mathcal{F}) + \ldots \tag{11.71}$$

where $p_j(\mathcal{F})$ is a polynomial of order $2j$ and is an element of $H^{4j}(M; \mathbb{R})$. We note that $p_j(\mathcal{F}) = 0$ for either $2j > k = \dim E$ or

$4j > \dim M$. [*Remark:* Although $p_m(\mathcal{F}) = 0$, $p_m(B)$ need not vanish for a *matrix* B. p_m will be used to define the Euler class later.]

Let us diagonalise $\mathcal{F}/2\pi$ as

$$\frac{\mathcal{F}}{2\pi} \to A \equiv \begin{pmatrix} -\mathrm{i}x_1 & & & & \\ & \mathrm{i}x_1 & & 0 & \\ & & -\mathrm{i}x_2 & & \\ & 0 & & \mathrm{i}x_2 & \\ & & & & \ddots \end{pmatrix} \tag{11.72}$$

where $x_k \equiv -\lambda_k/2\pi$, λ_k being the eigenvalues of $\mathcal{F}$. The sign has been chosen to simplify the Euler class defined below. The generating function of $p(\mathcal{F})$ is given by

$$p(\mathcal{F}) = \det(\mathbb{1} + A) = \prod_{i=1}^{[k/2]} (1 + x_i^2) \tag{11.73}$$

where

$$[k/2] = \begin{cases} k/2 & \text{if } k \text{ is even} \\ (k-1)/2 & \text{if } k \text{ is odd.} \end{cases}$$

In (11.73) only *even* powers appear, reflecting the skew symmetry. Each Pontrjagin class is computed from (11.73) as

$$p_j(\mathcal{F}) = \sum_{i_1 < i_2 < \ldots < i_j}^{[k/2]} x_{i_1}^2 x_{i_2}^2 \ldots x_{i_j}^2. \tag{11.74}$$

To write $p_j(\mathcal{F})$ in terms of the curvature two-form $\mathcal{F}/2\pi$, we first note that

$$\operatorname{tr}\left(\frac{\mathcal{F}}{2\pi}\right)^{2j} = \operatorname{tr} A^{2j} = 2(-1)^j \sum_{i=1}^{[k/2]} x_i^{2j}.$$

It then follows that

$$p_1(\mathcal{F}) = \sum_i x_i^2 = -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \operatorname{tr} \mathcal{F}^2 \tag{11.75a}$$

$$\begin{aligned} p_2(\mathcal{F}) &= \sum_{i<j} x_i^2 x_j^2 = \frac{1}{2}\left[\left(\sum_i x_i^2\right)^2 - \sum_i x_i^4\right] \\ &= \frac{1}{8}\left(\frac{1}{2\pi}\right)^4 [(\operatorname{tr} \mathcal{F}^2)^2 - 2 \operatorname{tr} \mathcal{F}^4] \end{aligned} \tag{11.75b}$$

$$\begin{aligned} p_3(\mathcal{F}) &= \sum_{i<j<k} x_i^2 x_j^2 x_k^2 \\ &= \frac{1}{48}\left(\frac{1}{2\pi}\right)^6 [-(\operatorname{tr} \mathcal{F}^2)^3 + 6 \operatorname{tr} \mathcal{F}^2 \operatorname{tr} \mathcal{F}^4 - 8 \operatorname{tr} \mathcal{F}^6] \end{aligned} \tag{11.75c}$$

$$p_4(\mathcal{F}) = \sum_{i<j<k<l} x_i^2 x_j^2 x_k^2 x_l^2$$

$$= \frac{1}{384}\left(\frac{1}{2\pi}\right)^8 [(\operatorname{tr}\mathcal{F}^2)^4 - 12(\operatorname{tr}\mathcal{F}^2)^2 \operatorname{tr}\mathcal{F}^4 + 32\operatorname{tr}\mathcal{F}^2 \operatorname{tr}\mathcal{F}^6$$
$$+ 12(\operatorname{tr}\mathcal{F}^4)^2 - 48\operatorname{tr}\mathcal{F}^8]. \tag{11.75d}$$

$$\vdots$$

$$p_{[k/2]}(\mathcal{F}) = x_1^2 x_2^2 \ldots x_{[k/2]}^2 = \left(\frac{1}{2\pi}\right)^k \det\mathcal{F}. \tag{11.75e}$$

The reader should verify that

$$p(E \oplus F) = p(E) \wedge p(F). \tag{11.76}$$

It is easy to guess that the Pontrjagin classes are written in terms of Chern classes. Since Chern classes are defined only for complex vector bundles, we must complexify the fibre of E so that complex numbers make sense. The resulting vector bundle is denoted by $E^{\mathbb{C}}$. Let A be a skew-symmetric *real* matrix. We find that

$$\det(\mathbb{1} + \mathrm{i}A) = \det\begin{pmatrix} 1 + x_1 & & & & 0 \\ & 1 - x_1 & & & \\ & & 1 + x_2 & & \\ & 0 & & 1 - x_2 & \\ & & & & \ddots \end{pmatrix}$$

$$= \prod_{i=1}^{[k/2]} (1 - x_i^2) = 1 - p_1(A) + p_2(A) - \ldots$$

from which it follows that

$$p_j(E) = (-1)^j c_{2j}(E^{\mathbb{C}}). \tag{11.77}$$

Example 11.13 Let M be a four-dimensional Riemannian manifold. When the orthonormal frame $\{\hat{e}_\alpha\}$ is employed, the structure group of the tangent bundle TM may be reduced to O(4). Let $\mathcal{R} = \frac{1}{2}\mathcal{R}_{\alpha\beta}\theta^\alpha \wedge \theta^\beta$ be the curvature two-form ($\mathcal{R}$ should not be confused with the scalar curvature). For the tangent bundle, it is common to write $p(M)$ instead of $p(\mathcal{R})$. We have

$$\det\left(\mathbb{1} + \frac{\mathcal{R}}{2\pi}\right) = 1 - \frac{1}{8\pi^2}\operatorname{tr}\mathcal{R}^2 + \frac{1}{128\pi^4}[(\operatorname{tr}\mathcal{R}^2)^2 - 2\operatorname{tr}\mathcal{R}^4]. \tag{11.78}$$

Each Pontrjagin class is given by

$$p_0(M) = 1 \tag{11.79a}$$

$$p_1(M) = -\frac{1}{8\pi^2}\operatorname{tr}\mathcal{R}^2 = -\frac{1}{8\pi^2}\mathcal{R}_{\alpha\beta}\mathcal{R}_{\beta\alpha} \tag{11.79b}$$

$$p_2(M) = \frac{1}{128\pi^4}\,[(\mathrm{tr}\,\mathcal{R}^2)^2 - 2\,\mathrm{tr}\,\mathcal{R}^4] = \left(\frac{1}{2\pi}\right)^4 \det\mathcal{R}. \qquad (11.79c)$$

Although $p_2(M)$ vanishes as a differential form, we need it in the next subsection to compute the Euler class.

11.4.2 Euler classes

Let M be a $2l$-dimensional *orientable* Riemannian manifold and let TM be the tangent bundle of M. We denote the curvature by $\mathcal{R}$. It is always possible to reduce the structure group of TM down to $\mathrm{SO}(2l)$ by employing an orthonormal frame. The **Euler class** e of M is defined by the *square root* of the $4l$-form p_l,

$$e(A)e(A) = p_l(A). \qquad (11.80)$$

Both sides should be understood as functions of a $2l \times 2l$ matrix A and not of the curvature $\mathcal{R}$, since $p_l(\mathcal{R})$ vanishes identically. However, $e(M) \equiv e(\mathcal{R})$ thus defined is a $2l$-form and indeed gives a volume element of M. If M is an odd-dimensional manifold we define $e(M) = 0$, see below.

Example 11.14 Let $M = S^2$ and consider the tangent bundle TS^2. From example 7.42, we find the curvature two-form,

$$\mathcal{R}_{\theta\phi} = -\mathcal{R}_{\phi\theta} = \sin^2\theta\,\frac{\mathrm{d}\theta \wedge \mathrm{d}\phi}{\sin\theta} = \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi$$

where we have noted that $g_{\theta\theta} = \sin^2\theta$. Although $p_1(S^2) = 0$ as a differential form, we compute it to find the Euler form. We have

$$\begin{aligned} p_1(S^2) &= -\frac{1}{8\pi^2}\,\mathrm{tr}\,\mathcal{R}^2 = -\frac{1}{8\pi^2}\,[\mathcal{R}_{\theta\phi}\mathcal{R}_{\phi\theta} + \mathcal{R}_{\phi\theta}\mathcal{R}_{\theta\phi}] \\ &= \left(\frac{1}{2\pi}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi\right)^2 \end{aligned}$$

from which we read off

$$e(S^2) = \frac{1}{2\pi}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi. \qquad (11.81)$$

It is interesting to note that

$$\int_{S^2} e(S^2) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\sin\theta = 2 \qquad (11.82)$$

which is the Euler characteristic of S^2, see §2.4. This is not just a coincidence. Let us take another convincing example, a torus T^2. Since T^2 admits a flat connection, the curvature vanishes identically. It then follows that $e(T^2) \equiv 0$ and $\chi(T^2) = 0$. These are special cases of the

Gauss–Bonnet theorem,

$$\int_M e(M) = \chi(M) \tag{11.83}$$

for a compact orientable manifold M. If M is odd-dimensional both e and χ vanish, see (6.38).

In general, the determinant of a $2l \times 2l$ skew-symmetric matrix A is a square of a polynomial called the **Pfaffian** $\mathrm{Pf}(A)$,

$$\det A = \mathrm{Pf}(A)^2. \tag{11.84}$$

We show that the Pfaffian is given by

$$\mathrm{Pf}(A) = \frac{(-1)^l}{2^l l!} \sum_P \operatorname{sgn}(P) A_{P(1)P(2)} A_{P(3)P(4)} \ldots A_{P(2l-1)P(2l)} \tag{11.85}$$

where the phase has been chosen for later convenience. We first note that a skew-symmetric matrix A can be block diagonalised by an element of $\mathrm{O}(2l)$ as

$$S^{\mathrm{t}} A S = \Lambda = \begin{pmatrix} 0 & \lambda_1 & & & & & \\ -\lambda_1 & 0 & & & & 0 & \\ & & 0 & \lambda_2 & & & \\ & & -\lambda_2 & 0 & & & \\ & & & & \ddots & & \\ & 0 & & & & 0 & \lambda_l \\ & & & & & -\lambda_l & 0 \end{pmatrix}. \tag{11.86}$$

It is easy to see that

$$\det A = \det \Lambda = \prod_{i=1}^{l} \lambda_i^2.$$

To compute $\mathrm{Pf}(\Lambda)$, we note that the non-vanishing terms in (11.85) are of the form $A_{12}A_{34} \ldots A_{2l-1,2l}$. Moreover, there are 2^l ways of changing the suffices as $A_{ij} \to A_{ji}$ such as

$$A_{12}A_{34} \ldots A_{2l-1,2l} \to A_{21}A_{34} \ldots A_{2l-1,2l}$$

and $l!$ permutations of the pairs of indices, for example,

$$A_{12}A_{34} \ldots A_{2l-1,2l} \to A_{34}A_{12} \ldots A_{2l-1,2l}.$$

Hence we have

$$\mathrm{Pf}(\Lambda) = (-1)^l A_{12}A_{34} \ldots A_{2l-1,2l} = (-1)^l \prod_{i=1}^{l} \lambda_i.$$

Thus we conclude that a block diagonal matrix Λ satisfies

$$\det \Lambda = \mathrm{Pf}(\Lambda)^2.$$

To show that (11.84) is true for any skew-symmetric matrices (not necessarily block diagonal) we use the following lemma,

$$\mathrm{Pf}(X^{\mathrm{t}}AX) = \mathrm{Pf}(A)\det X. \tag{11.87}$$

[*Proof:* Since $\det(X^{\mathrm{t}}AX) = (\det X)^2 \det A$, we have $\mathrm{Pf}(X^{\mathrm{t}}AX) = \pm\mathrm{Pf}(A)\det X$. Here the plus sign should be chosen since $\mathrm{Pf}(I^{\mathrm{t}}AI) = \mathrm{Pf}(A)$.] If $S^{\mathrm{t}}AS = \Lambda$ for $S \in \mathrm{O}(2l)$, we have $A = S\Lambda S^{\mathrm{t}}$, hence

$$\mathrm{Pf}(S\Lambda S^{\mathrm{t}}) = \mathrm{Pf}(\Lambda)\det S = (-1)^l \prod_{i=1}^{l} \lambda_i \det S.$$

We finally find $\det A = \mathrm{Pf}(A)^2$ for a skew-symmetric matrix A.

Note that $\mathrm{Pf}(A)$ is $\mathrm{SO}(2l)$ invariant but changes sign under an improper rotation S ($\det S = -1$) of $\mathrm{O}(2l)$.

Exercise 11.15 Show that the determinant of an odd-dimensional skew-symmetric matrix vanishes. This is why we put $e(M) = 0$ for an odd-dimensional manifold.

In terms of the curvature $\mathcal{R}$, we have

$$\begin{aligned} e(M) &= \mathrm{Pf}(\mathcal{R}) \\ &= \frac{(-1)^l}{(4\pi)^l l!} \sum_P \mathrm{sgn}\,(P)\mathcal{R}_{P(1)P(2)} \ldots \mathcal{R}_{P(2l-1)P(2l)}. \end{aligned} \tag{11.88}$$

The generating function is obtained by taking $x_i = -\lambda_i/2\pi$,

$$e(x) = x_1 x_2 \ldots x_l = \prod_{i=1}^{l} x_i. \tag{11.89}$$

The phase $(-1)^l$ has been chosen to simplify the RHS.

Example 11.16 Let M be a four-dimensional orientable manifold. The structure group of TM is $\mathrm{SO}(4)$; see example 11.13. The Euler class is obtained from (11.88) as

$$e(M) = \frac{1}{2(4\pi)^2}\, \varepsilon^{ijkl}\mathcal{R}_{ij} \wedge \mathcal{R}_{kl}. \tag{11.90}$$

This is in agreement with the result of example 11.13. The relevant Pontrjagin class is

$$p_2(M) = \frac{1}{128\pi^4}\,[(\mathrm{tr}\,\mathcal{R}^2)^2 - 2\,\mathrm{tr}\,\mathcal{R}^4] = x_1^2 x_2^2.$$

Since $e(M) = x_1 x_2$, we have $p_2(M) = e(M) \wedge e(M)$. This is written as a matrix identity,

$$\frac{1}{128\pi^4}\,[(\mathrm{tr}\,A^2)^2 - 2\,\mathrm{tr}\,A^4] = \left(\frac{1}{2(4\pi)^4}\, \varepsilon^{ijkl} A_{ij} A_{kl}\right)^2.$$

11.4.3 Hirzebruch L-polynomial and $\hat{A}$-genus

The **Hirzebruch L-polynomial** is defined by

$$L(x) = \prod_{j=1}^{k} \frac{x_j}{\tanh x_j}$$

$$= \prod_{j=1}^{k} \left(1 + \sum_{n \geq 1} (-1)^{n-1} \frac{2^{2n}}{(2n)!} B_n x_j^{2n}\right) \tag{11.91}$$

where the B_n are Bernoulli numbers, see (11.66). $L(x)$ is an even function of x_j and can be written in terms of the Pontrjagin classes,

$$L(\mathcal{F}) = 1 + \tfrac{1}{3}p_1 + \tfrac{1}{45}(-p_1^2 + 7p_2) + \tfrac{1}{945}(2p_1^3 - 13p_1p_2 + 62p_3) + \ldots \tag{11.92}$$

where p_j stands for $p_j(\mathcal{F})$. From the splitting principle, we find that

$$L(E \oplus F) = L(E) \wedge L(F). \tag{11.93}$$

The $\hat{A}$ (**A-roof) genus** $\hat{A}(\mathcal{F})$ is defined by

$$\hat{A}(\mathcal{F}) = \prod_{j=1}^{k} \frac{x_j/2}{\sinh(x_j/2)}$$

$$= \prod_{j=1}^{k} \left(1 + \sum_{n \geq 1} (-1)^{n} \frac{(2^{2n} - 2)}{(2n)!} B_n x_j^{2n}\right). \tag{11.94}$$

This is an even function of x_j and can be expanded in p_j. $\hat{A}$ is also called the **Dirac genus** by physicists. It satisfies

$$\hat{A}(E \oplus F) = \hat{A}(E) \wedge \hat{A}(F). \tag{11.95}$$

$\hat{A}$ is written in terms of the Pontrjagin classes as

$$A(\mathcal{F}) = 1 - \tfrac{1}{24}p_1 + \tfrac{1}{5760}(7p_1^2 - 4p_2)$$
$$+ \tfrac{1}{967680}(-31p_1^3 + 44p_1p_2 - 16p_3) + \ldots \tag{11.96}$$

Example 11.17 Let M be a compact connected and orientable four-dimensional manifold. Let us consider the symmetric bilinear form $\sigma : H^2(M; \mathbb{R}) \times H^2(M; \mathbb{R}) \to \mathbb{R}$ defined by

$$\sigma([\alpha], [\beta]) = \int_M \alpha \wedge \beta. \tag{11.97}$$

σ is a $b^2 \times b^2$ symmetric matrix where $b^2 = \dim H^2(M; \mathbb{R})$. Clearly σ is non-degenerate since $\sigma([\alpha], [\beta]) = 0$ for any $[\alpha] \in H^2(M; \mathbb{R})$ implies $[\beta] = 0$. Let $p(q)$ be the number of positive (negative) eigenvalues of σ.

The **Hirzebruch signature** of M is defined by

$$\tau(M) \equiv p - q. \tag{11.98}$$

According to the **Hirzebruch signature theorem** (see §12.5), this number is also given by the L-polynomial as

$$\tau(M) = \int_M L_1(M) = \tfrac{1}{3}\int_M p_1(M). \tag{11.99}$$

11.5 Chern–Simons forms

11.5.1 Definition

Let $P_j(\mathcal{F})$ be an arbitrary $2j$-form characteristic class. Since $P_j(\mathcal{F})$ is closed, it can be written *locally* as an exact form by Poincaré's lemma. Let us write

$$P_j(\mathcal{F}) = \mathrm{d}Q_{2j-1}(\mathcal{A}, \mathcal{F}) \tag{11.100}$$

where $Q_{2j-1}(\mathcal{A}, \mathcal{F}) \in \mathfrak{g} \otimes \Omega^{2j-1}(M)$. [*Warning:* This cannot be true *globally*. If $P_j = \mathrm{d}Q_{2j-1}$ globally on a manifold M without boundary, we would have

$$\int_M P_{m/2} = \int_M \mathrm{d}Q_{m-1} = \int_{\partial M} Q_{m-1} = 0$$

where $m = \dim M$.] $Q_{2j-1}(\mathcal{A}, \mathcal{F})$ is called the **Chern–Simons form** of $P_j(\mathcal{F})$. From the proof of theorem 11.2(b), we find that Q is given by the transgression of P_j,

$$Q_{2j-1}(\mathcal{A}, \mathcal{F}) = TP_j(\mathcal{A}, 0) = j\int_0^1 \widetilde{P}_j(\mathcal{A}, \mathcal{F}_t, \ldots, \mathcal{F}_t)\,\mathrm{d}t \tag{11.101}$$

where $\widetilde{P}_j$ is the polarisation of P_j, $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}^2$ and we set $\mathcal{A}' = \mathcal{F}' = 0$. Since Q_{2j-1} depends on $\mathcal{F}$ *and* $\mathcal{A}$, we explicitly quote the $\mathcal{A}$-dependence. Of course $\mathcal{A}'$ can be put equal to zero only on a local chart over which the bundle is trivial.

Suppose M is an even-dimensional manifold ($\dim M = m = 2l$) such that $\partial M \neq 0$. Then Stokes' theorem implies

$$\int_M P_l(\mathcal{F}) = \int_M \mathrm{d}Q_{m-1}(\mathcal{A}, \mathcal{F}) = \int_{\partial M} Q_{m-1}(\mathcal{A}, \mathcal{F}). \tag{11.102}$$

The LHS takes its value in integers, and so does the RHS. Thus Q_{m-1} is a characteristic class in its own right. Q_{m-1} describes the topology of the boundary ∂M.

11.5.2 The Chern–Simons form of the Chern character

As an example, let us work out the Chern–Simons form of a Chern character $\mathrm{ch}_j(\mathcal{F})$. The connection $\mathcal{A}_t$ which interpolates between 0 and

$\mathcal{A}$ is

$$\mathcal{A}_t = t\mathcal{A} \tag{11.103}$$

the corresponding curvature being

$$\mathcal{F}_t = t\,\mathrm{d}\mathcal{A} + t^2\mathcal{A}^2 = t\mathcal{F} + (t^2 - t)\mathcal{A}^2. \tag{11.104}$$

We find from (11.21) that

$$Q_{2j-1}(\mathcal{A}, \mathcal{F}) = \frac{1}{(j-1)!}\left(\frac{\mathrm{i}}{2\pi}\right)^j \int_0^1 \mathrm{d}t\, \mathrm{str}(\mathcal{A}, \mathcal{F}_t^{j-1}). \tag{11.105}$$

For example,

$$Q_1(\mathcal{A}, \mathcal{F}) = \frac{\mathrm{i}}{2\pi}\int_0^1 \mathrm{d}t\, \mathrm{tr}\,\mathcal{A} = \frac{\mathrm{i}}{2\pi}\mathrm{tr}\,\mathcal{A} \tag{11.106a}$$

$$\begin{aligned} Q_3(\mathcal{A}, \mathcal{F}) &= \left(\frac{\mathrm{i}}{2\pi}\right)^2 \int_0^1 \mathrm{d}t\, \mathrm{str}(\mathcal{A}, t\mathrm{d}\mathcal{A} + t^2\mathcal{A}^2) \\ &= \frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^2 \mathrm{tr}(\mathcal{A}\mathrm{d}\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3). \end{aligned} \tag{11.106b}$$

$$\begin{aligned} Q_5(\mathcal{A}, \mathcal{F}) &= \frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^3 \int_0^1 \mathrm{d}t\, \mathrm{str}[\mathcal{A}, (t\mathrm{d}\mathcal{A} + t^2\mathcal{A}^2)^2] \\ &= \frac{1}{6}\left(\frac{\mathrm{i}}{2\pi}\right)^3 \mathrm{tr}[\mathcal{A}(\mathrm{d}\mathcal{A})^2 + \tfrac{3}{2}\mathcal{A}^3\mathrm{d}\mathcal{A} + \tfrac{3}{5}\mathcal{A}^5]. \end{aligned} \tag{11.106c}$$

Exercise 11.18 Let $\mathcal{F}$ be the field strength of the SU(2) gauge theory. Write down the component expression of the identity $\mathrm{ch}_2(\mathcal{F}) = \mathrm{d}Q_3(\mathcal{A}, \mathcal{F})$ to verify that (cf lemma 10.36)

$$\mathrm{tr}[\varepsilon^{\kappa\lambda\mu\nu}\mathcal{F}_{\kappa\lambda}\mathcal{F}_{\mu\nu}] = \partial_\kappa[2\varepsilon^{\kappa\lambda\mu\nu}\mathrm{tr}(\mathcal{A}_\lambda\partial_\mu\mathcal{A}_\nu + \tfrac{2}{3}\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu)]. \tag{11.107}$$

11.5.3 Cartan's homotopy operator and applications

For later purposes, we define Cartan's homotopy formula following Zumino (1985) and Alvarez-Gaumé and Ginsparg (1985). Let

$$\mathcal{A}_t = \mathcal{A}_0 + t(\mathcal{A}_1 - \mathcal{A}_0) \qquad \mathcal{F}_t = \mathrm{d}\mathcal{A}_t + \mathcal{A}_t^2 \tag{11.108}$$

as before. Define an operator l_t by

$$l_t\mathcal{A}_t = 0 \qquad l_t\mathcal{F}_t = \delta t(\mathcal{A}_1 - \mathcal{A}_0). \tag{11.109}$$

We also require that l_t be an antiderivative,

$$l_t(\eta_p\omega_q) = (l_t\eta_p)\omega_q + (-1)^p\eta_p(l_t\omega_q) \tag{11.110}$$

for $\eta_p \in \Omega^p(M)$ and $\omega_q \in \Omega^q(M)$. We verify that

$$\begin{aligned}
(\mathrm{d}l_t + l_t\mathrm{d})\mathcal{A}_t &= l_t(\mathcal{F}_t - \mathcal{A}_t^2) = \delta t(\mathcal{A}_1 - \mathcal{A}_0) = \delta t\,\frac{\partial \mathcal{A}_t}{\partial t}\\
(\mathrm{d}l_t + l_t\mathrm{d})\mathcal{F}_t &= \mathrm{d}[\delta t(\mathcal{A}_1 - \mathcal{A}_0)] + l_t[\mathcal{D}_t\mathcal{F}_t - \mathcal{A}_t\mathcal{F}_t + \mathcal{F}_t\mathcal{A}_t]\\
&= \delta t[\mathrm{d}(\mathcal{A}_1 - \mathcal{A}_0) + \mathcal{A}_t(\mathcal{A}_1 - \mathcal{A}_0) + (\mathcal{A}_1 - \mathcal{A}_0)\mathcal{A}_t]\\
&= \delta t\mathcal{D}_t(\mathcal{A}_1 - \mathcal{A}_0) = \delta t\frac{\partial \mathcal{F}_t}{\partial t}
\end{aligned}$$

where we have used the Bianchi identity $\mathcal{D}_t\mathcal{F}_t = 0$. This shows that for any polynomial $S(\mathcal{A}, \mathcal{F})$ of $\mathcal{A}$ and $\mathcal{F}$, we have

$$(\mathrm{d}l_t + l_t\mathrm{d})S(\mathcal{A}_t, \mathcal{F}_t) = \delta t\,\frac{\partial}{\partial t}\,S(\mathcal{A}_t, \mathcal{F}_t). \tag{11.111}$$

On the RHS, S should be a polynomial of $\mathcal{A}$ and $\mathcal{F}$ *only* and not of $\mathrm{d}\mathcal{A}$ or $\mathrm{d}\mathcal{F}$. If S does contain them, $\mathrm{d}\mathcal{A}$ should be replaced by $\mathcal{F} - \mathcal{A}^2$ and $\mathrm{d}\mathcal{F}$ by $\mathcal{D}\mathcal{F} - [\mathcal{A}, \mathcal{F}] = -[\mathcal{A}, \mathcal{F}]$. Integrating (11.111) over $[0, 1]$, we find **Cartan's homotopy formula**

$$S(\mathcal{A}_1, \mathcal{F}_1) - S(\mathcal{A}_0, \mathcal{F}_0) = (\mathrm{d}k_{01} + k_{01}d)S(\mathcal{A}_t, \mathcal{F}_t) \tag{11.112}$$

where the **homotopy operator** k_{01} is defined by

$$k_{01}S(\mathcal{A}_t, \mathcal{F}_t) \equiv \int_0^1 l_t S(\mathcal{A}_t, \mathcal{F}_t). \tag{11.113}$$

To operate k_{01} on $S(\mathcal{A}, \mathcal{F})$, we first replace $\mathcal{A}$ and $\mathcal{F}$ by $\mathcal{A}_t$ and $\mathcal{F}_t$ respectively then operate l_t on $S(\mathcal{A}_t, \mathcal{F}_t)$ and integrate over t.

Example 11.19 Let us compute the Chern–Simons form of the Chern character using the homotopy formula. Let $S(\mathcal{A}, \mathcal{F}) = \mathrm{ch}_{j+1}(\mathcal{F})$ and $\mathcal{A}_1 = \mathcal{A}$, $\mathcal{A}_0 = 0$. Since $\mathrm{d}\,\mathrm{ch}_{j+1}(\mathcal{F}) = 0$, we have

$$\mathrm{ch}_{j+1}(\mathcal{F}) = (\mathrm{d}k_{01} + k_{01}\mathrm{d})\,\mathrm{ch}_{j+1}(\mathcal{F}_t) = \mathrm{d}[k_{01}\,\mathrm{ch}_{j+1}(\mathcal{F}_t)].$$

Thus $k_{01}\,\mathrm{ch}_{j+1}(\mathcal{F}_t)$ is identified with the Chern–Simons form $Q_{2j+1}(\mathcal{A}, \mathcal{F})$. We find that

$$\begin{aligned}
k_{01}\,\mathrm{ch}_{j+1}(\mathcal{F}_t) &= \frac{1}{(j+1)!}\,k_{01}\,\mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^{j+1}\\
&= \frac{1}{(j+1)!}\left(\frac{\mathrm{i}}{2\pi}\right)^{j+1}\int_0^1 l_t\,\mathrm{tr}\,(\mathcal{F}_t^{j+1})\\
&= \frac{1}{j!}\left(\frac{\mathrm{i}}{2\pi}\right)^{j+1}\int_0^1 \delta t\,\mathrm{str}\,(\mathcal{A}\mathcal{F}_t^j)
\end{aligned} \tag{11.114}$$

in agreement with (11.105).

Although a characteristic class is gauge invariant, the Chern–Simons form need not be so. As an application of Cartan's homotopy formula, we compute the change in $Q_{2j+1}(\mathcal{A}, \mathcal{F})$ under $\mathcal{A} \to \mathcal{A}^g = g^{-1}(\mathcal{A} + \mathrm{d})g$, $\mathcal{F} \to \mathcal{F}^g = g^{-1}\mathcal{F}g$. Consider interpolating the families $\mathcal{A}_t^g$ and $\mathcal{F}_t^g$ defined by

$$\mathcal{A}_t^g \equiv tg^{-1}\mathcal{A}g + g^{-1}\mathrm{d}g \tag{11.115a}$$

$$\mathcal{F}_t^g \equiv \mathrm{d}\mathcal{A}_t^g + (\mathcal{A}_t^g)^2 = g^{-1}\mathcal{F}_t g \tag{11.115b}$$

where $\mathcal{F}_t \equiv t\mathcal{F} + (t^2 - t)\mathcal{A}^2$. Note that $\mathcal{A}_0^g = g^{-1}\mathrm{d}g$, $\mathcal{A}_1^g = \mathcal{A}^g$, $\mathcal{F}_0^g = 0$ and $\mathcal{F}_1^g = \mathcal{F}^g$. (11.112) yields

$$Q_{2j+1}(\mathcal{A}^g, \mathcal{F}^g) - Q_{2j+1}(g^{-1}\mathrm{d}g, 0) = (\mathrm{d}k_{01} + k_{01}\mathrm{d})Q_{2j+1}(\mathcal{A}_t^g, \mathcal{F}_t^g). \tag{11.116}$$

For example, let Q_{2j+1} be the Chern–Simons form of the Chern character $\mathrm{ch}_{j+1}(\mathcal{F})$. Since $\mathrm{d}Q_{2j+1}(\mathcal{A}_t^g, \mathcal{F}_t^g) = \mathrm{ch}_{j+1}(\mathcal{F}_t^g) = \mathrm{ch}_{j+1}(\mathcal{F}_t)$, we have

$$\begin{aligned} k_{01}\mathrm{d}Q_{2j+1}(\mathcal{A}_t^g, \mathcal{F}_t^g) &= k_{01}\,\mathrm{ch}_{j+1}(\mathcal{F}_t^g) \\ &= k_{01}\,\mathrm{ch}_{j+1}(\mathcal{F}_t) = Q_{2j+1}(\mathcal{A}, \mathcal{F}) \end{aligned} \tag{11.117}$$

where the result of example 11.19 has been used to obtain the final equality. Collecting these results, we write (11.116) as

$$Q_{2j+1}(\mathcal{A}^g, \mathcal{F}^g) - Q_{2j+1}(\mathcal{A}, \mathcal{F}) = Q_{2j+1}(g^{-1}\mathrm{d}g, 0) + \mathrm{d}\alpha_{2j} \tag{11.118}$$

where α_{2j} is a $2j$-form defined by

$$\begin{aligned} \alpha_{2j}(\mathcal{A}, \mathcal{F}, v) &\equiv k_{01}Q_{2j+1}(\mathcal{A}_t^g, \mathcal{F}_t^g) \\ &= k_{01}Q_{2j+1}(\mathcal{A}_t + v, \mathcal{F}_t) \end{aligned} \tag{11.119}$$

where $v \equiv \mathrm{d}g \cdot g^{-1}$. [Note that $Q_{2j+1}(\mathcal{A}, \mathcal{F}) = Q_{2j+1}(g\mathcal{A}g^{-1}, g\mathcal{F}g^{-1})$.] The first term on the RHS of (11.118) is

$$\begin{aligned} Q_{2j+1}(g^{-1}\mathrm{d}g, 0) &= \frac{1}{j!}\left(\frac{\mathrm{i}}{2\pi}\right)^{j+1} \int_0^1 \delta t\, \mathrm{tr}\,[g^{-1}\mathrm{d}g\{(t^2 - t)(g^{-1}\mathrm{d}g)^2\}^j] \\ &= \frac{1}{j!}\left(\frac{\mathrm{i}}{2\pi}\right)^{j+1} \mathrm{tr}\,[(g^{-1}\mathrm{d}g)^{2j+1}] \int_0^1 \delta t(t^2 - t)^j \\ &= (-1)^j \frac{j!}{(2j+1)!}\left(\frac{\mathrm{i}}{2\pi}\right)^{j+1} \mathrm{tr}\,[(g^{-1}\mathrm{d}g)^{2j+1}] \end{aligned} \tag{11.120}$$

where we have noted that $\mathcal{F}_t = (t^2 - t)(g^{-1}\mathrm{d}g)^2$ and

$$\int_0^1 \delta t(t^2 - t)^j = (-1)^j B(j+1, j+1) = (-1)^j \frac{(j!)^2}{(2j+1)!}$$

B being the beta function. $Q_{2j+1}(g^{-1}\mathrm{d}g, 0)$ is closed and hence locally exact: $\mathrm{d}Q_{2j+1}(g^{-1}\mathrm{d}g, 0) = \mathrm{ch}_{j+1}(0) = 0$.

As for α_{2j} we have, for example,

$$\alpha_2 = \frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^2 \int_0^1 l_t \operatorname{tr}[(\mathcal{A}_t + v)\mathcal{F}_t - \tfrac{1}{3}(\mathcal{A}_t + v)^3]$$
$$= \frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^2 \int_0^1 \delta t \operatorname{tr}(-t\mathcal{A}^2 - v\mathcal{A}) = -\frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^2 \operatorname{tr}(v\mathcal{A}) \qquad (11.121)$$

where we have noted that $\operatorname{tr}\mathcal{A}^2 = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \operatorname{tr}(\mathcal{A}_\mu \mathcal{A}_\nu) = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \operatorname{tr}(\mathcal{A}_\nu \mathcal{A}_\mu) = 0$.

Example 11.20 In three-dimensional spacetime, a gauge theory may have a gauge-invariant mass term given by the Chern–Simons three-form (Jackiw and Templeton (1981) and Deser *et al* (1982a, b)). Since the Chern–Simons form changes by a locally exact form under a gauge transformation, the action remains invariant. We restrict ourselves to the U(1) gauge theory for simplicity. Consider the Lagrangian (we put $\mathcal{A} = \mathrm{i}A$ and $\mathcal{F} = \mathrm{i}F$)

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{4}m\varepsilon^{\lambda\mu\nu}F_{\lambda\mu}A_\nu \qquad (11.122)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that the second term is the Chern–Simons form of the second Chern character F^2 (modulo a constant factor) of the U(1) bundle. The field equation is

$$\partial_\mu F^{\mu\nu} + m{*F^\nu} = 0 \qquad (11.123)$$

where

$${*F^\mu} = \tfrac{1}{2}\varepsilon^{\mu\kappa\lambda}F_{\kappa\lambda}, \quad F^{\mu\nu} = \varepsilon^{\mu\nu\lambda}{*F_\lambda}.$$

The Bianchi identity

$$\partial_\mu{*F^\mu} = 0 \qquad (11.124)$$

follows from (11.123) as a consequence of the skew symmetry of $F^{\mu\nu}$. It is easy to verify that the field equation is invariant under a gauge transformation,

$$A_\mu \to A_\mu + \partial_\mu\theta \qquad (11.125)$$

while the Lagrangian changes by a total derivative,

$$\mathcal{L} \to -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{4}m\varepsilon^{\lambda\mu\nu}F_{\lambda\mu}(A_\nu + \partial_\nu\theta) = \mathcal{L} + \tfrac{1}{2}m\partial_\nu({*F^\nu}\theta). \qquad (11.126)$$

(11.106*b*) shows that the last term on the RHS is identified with

$$Q_3(A^\theta, F^\theta) - Q_3(A, F) \sim (A + \mathrm{d}\theta)\,\mathrm{d}A - A\,\mathrm{d}A \sim \mathrm{d}(\theta\,\mathrm{d}A).$$

If we assume that F falls off at large spacetime distances, this term does not contribute to the action:

$$\int \mathrm{d}^3x\mathcal{L} \to \int \mathrm{d}^3x\mathcal{L} + \frac{m}{2}\int \mathrm{d}^3x\partial_\nu({*F^\nu}\theta) = \int \mathrm{d}^3x\mathcal{L}. \quad (11.127)$$

Let us show that (11.122) describes a *massive* field. We first write (11.123) as

$$\varepsilon^{\mu\nu\alpha}\partial_\mu {*F_\alpha} = -m {*F^\nu}.$$

Multiplying $\varepsilon_{\kappa\lambda\nu}$ on both sides, we have

$$\partial_\lambda {*F_\kappa} - \partial_\kappa {*F_\lambda} = -mF_{\kappa\lambda}.$$

Taking the ∂^λ-derivative and using (11.124), we find that

$$(\partial^\lambda\partial_\lambda + m^2){*F_\kappa} = 0 \tag{11.128}$$

which shows that ${*F_\kappa}$ is a massive vector field of mass m.

11.6 Stiefel–Whitney classes

The last example of the characteristic classes is the Stiefel–Whitney class. In contrast to the rest of the characteristic classes, the Stiefel–Whitney class cannot be expressed in terms of the curvature of the bundle. The Stiefel–Whitney class is important in physics since it tells us whether a manifold admits a spin or not. Let us start with a brief review of a spin bundle.

11.6.1 Spin bundles

Let $TM \xrightarrow{\pi} M$ be a tangent bundle with $\dim M = m$. TM is assumed to have a fibre metric and the structure group G is taken to be $\mathrm{O}(m)$. If, furthermore, M is orientable, G can be reduced down to $\mathrm{SO}(m)$. Let LM be the frame bundle associated with TM. Let t_{ij} be the transition function of LM which satisfies the consistency condition (9.6)

$$t_{ij}t_{jk}t_{ki} = \mathbb{1} \qquad t_{ii} = \mathbb{1}.$$

A spin structure on M is defined by the transition function $\tilde{t}_{ij} \in \mathrm{SPIN}(m)$ such that

$$\varphi(\tilde{t}_{ij}) = t_{ij}, \ \tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = \mathbb{1}, \ \tilde{t}_{ii} = \mathbb{1} \tag{11.129}$$

where φ is the double covering $\mathrm{SPIN}(m) \to \mathrm{SO}(m)$. The set of $\tilde{t}_{ij}$ defines a **spin bundle** $PS(M)$ over M, and M is said to admit a **spin structure** (of course, M may admit many spin structures depending on the choice of $\tilde{t}_{ij}$).

It is interesting to note that not all manifolds admit spin structures. Non-admittance of spin structures is measured by the second Stiefel–Whitney class which takes values in the Čech cohomology group $H^2(M, \mathbb{Z}_2)$.

11.6.2 Čech cohomology groups

Let $\mathbb{Z}_2$ be the *multiplicative group* $\{-1, +1\}$. A **Čech *r*-cochain** is a function $f(i_0, i_1, \ldots, i_r) \in \mathbb{Z}_2$, defined on $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_r} \neq \emptyset$, which is totally symmetric under an arbitrary permutation P,

$$f(i_{P(0)}, \ldots, i_{P(r)}) = f(i_0, \ldots, i_r).$$

Let $C^r(M, \mathbb{Z}_2)$ be the multiplicative group of Čech r-cochains. We define the coboundary operator $\delta : C^r(M; \mathbb{Z}_2) \to C^{r+1}(M; \mathbb{Z}_2)$ by

$$(\delta f)(i_0, \ldots, i_{r+1}) = \prod_{j=0}^{r+1} f(i_0, \ldots, \hat{i}_j, \ldots, i_{r+1}) \qquad (11.130)$$

where the variable below the ^ is omitted. For example,

$$(\delta f_0)(i_0, i_1) = f_0(i_1)f_0(i_0) \qquad f_0 \in C^0(M; \mathbb{Z}_2)$$

$$(\delta f_1)(i_0, i_1, i_2) = f_1(i_1, i_2)f_1(i_0, i_2)f_1(i_0, i_1).$$

Since we employ the multiplicative notation, the trivial element of $C^r(M; \mathbb{Z}_2)$ is denoted by 1. We verify that δ is nilpotent:

$$(\delta^2 f)(i_0, \ldots, i_{r+2}) = \prod_{j,k=1}^{r+1} f(i_0, \ldots, \hat{i}_j, \ldots, \hat{i}_k, \ldots, i_{r+2}) = 1$$

since -1 always appears an even number of times in the middle expression (for example, if $f(i_0, \ldots, \hat{i}_j, \ldots, \hat{i}_k, \ldots, i_{r+2}) = -1$, we have $f(i_0, \ldots, \hat{i}_k, \ldots, \hat{i}_j, \ldots, i_{r+2}) = -1$ from the symmetry of f). Thus we have proved, for any Čech r-cochain f, that

$$\delta^2 f = 1. \qquad (11.131)$$

The cocycle group $Z^r(M; \mathbb{Z}_2)$ and the coboundary group $B^r(M; \mathbb{Z}_2)$ are defined by

$$Z^r(M, \mathbb{Z}_2) = \{f \in C^r(M; \mathbb{Z}_2) | \delta f = 1\} \qquad (11.132)$$

$$B^r(M; \mathbb{Z}_2) = \{f \in C^r(M; \mathbb{Z}_2) | f = \delta f', f' \in C^{r-1}(M; \mathbb{Z}_2). \qquad (11.133)$$

Now the rth **Čech cohomology group** $H^r(M; \mathbb{Z}_2)$ is defined by

$$H^r(M; \mathbb{Z}_2) = \ker \delta_r / \operatorname{im} \delta_{r-1} = Z^r(M; \mathbb{Z}_2)/B^r(M; \mathbb{Z}_2). \qquad (11.134)$$

11.6.3 Stiefel–Whitney classes

The **Stiefel–Whitney** class w_r is a characteristic class which takes its values in $H^r(M, \mathbb{Z}_2)$. Let $TM \xrightarrow{\pi} M$ be a tangent bundle with a Riemannian metric. The structure group is $O(m)$, $m = \dim M$. We assume $\{U_i\}$ is a *simple* open covering of M, which means that the intersection of any number of charts is either empty or contractible. Let

$\{e_{i\alpha}\}$ $(1 \leqslant \alpha \leqslant m)$ be a local orthonormal frame of E over U_i. We have $e_{i\alpha} = t_{ij}e_{j\alpha}$ where $t_{ij} : U_i \cap U_j \to \mathrm{O}(m)$ is the transition function. Define the Čech 1-cochain $f(i, j)$ by

$$f(i, j) \equiv \det(t_{ij}) = \pm 1. \tag{11.135}$$

This is indeed an element of $C^1(M, \mathbb{Z}_2)$ since $f(i, j) = f(j, i)$. From the cocycle condition $t_{ij}t_{jk}t_{ki} = \mathbb{1}$, we verify that

$$\begin{aligned}\delta f(i, j, k) &= \det(t_{ij})\det(t_{jk})\det(t_{ki})\\ &= \det(t_{ij}t_{jk}t_{ki}) = 1.\end{aligned} \tag{11.136}$$

Hence $f \in Z^1(M, \mathbb{Z}_2)$ and this defines an element $[f]$ of $H^1(M, \mathbb{Z}_2)$. Now we show that this element is independent of the local frame chosen. Let $\{\bar{e}_{i\alpha}\}$ be another frame over U_i such that $\bar{e}_{i\alpha} = h_i e_{i\alpha}$, $h_i \in \mathrm{O}(m)$. From $\bar{e}_{i\alpha} = \bar{t}_{ij}\bar{e}_{j\alpha}$, we find $\bar{t}_{ij} = h_i t_{ij} h_j^{-1}$. If we define the 0-cochain f_0 by $f_0(i) \equiv \det h_i$, we find

$$\begin{aligned}\bar{f}(i, j) &= \det(h_i t_{ij} h_j^{-1}) = \det(h_i)\det(h_j)\det(t_{ij})\\ &= \delta f_0(i, j) f(i, j).\end{aligned}$$

Thus f changes by an exact amount and still defines the same cohomology class $[f]$ (note again that the multiplicative notation is being used). This special element $w_1(M) \equiv [f] \in H^1(M, \mathbb{Z}_2)$ is called the **first Stiefel–Whitney class**.

Theorem 11.21 Let $TM \xrightarrow{\pi} M$ be a tangent bundle with fibre metric. M is orientable if and only if $w_1(M)$ is trivial.

Proof: If M is orientable, the structure group may be reduced to $\mathrm{SO}(m)$ and $f(i, j) = \det(t_{ij}) \equiv 1$, and hence $w_1(M) = 1$, the trivial element of $\mathbb{Z}_2$. Conversely, if $w_1(M)$ is trivial, f is a coboundary; $f = \delta f_0$. Since $f_0(i) = \pm 1$, we can always choose $h_i \in \mathrm{O}(m)$ such that $\det(h_i) = f_0(i)$ for each i. If we define the new frame $\bar{e}_{i\alpha} = h_i e_{i\alpha}$, we have transition functions $\bar{t}_{ij}$ such that $\det(\bar{t}_{ij}) = 1$ for any overlapping pair (i, j) and M is orientable. [Suppose $f(i, j) = \det t_{ij} = -1$ for some pair (i, j). Then we may take $f_0(i) = -1$ and $f_0(j) = +1$, hence $\det \bar{t}_{ij} = -\det t_{ij} = +1$.] ∎

Theorem 11.21 shows that the first Stiefel–Whitney class is an obstruction to the orientability. Next we define the second Stiefel–Whitney class. Suppose M is an m-dimensional orientable manifold and TM is its tangent bundle. For the transition function $t_{ij} \in \mathrm{SO}(m)$, we consider a 'lifting' $\tilde{t}_{ij} \in \mathrm{SPIN}(m)$ such that

$$\varphi(\tilde{t}_{ij}) = t_{ij}, \quad \tilde{t}_{ji} = \tilde{t}_{ij}^{-1} \tag{11.137}$$

where $\varphi : \mathrm{SPIN}(m) \to \mathrm{SO}(m)$ is the 2 : 1 homomorphism (note that we have an option $t_{ij} \leftrightarrow \tilde{t}_{ij}$ *or* $-\tilde{t}_{ij}$). This lifting always exists locally. Since

$$\varphi(\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki}) = t_{ij}t_{jk}t_{ki} = \mathbb{1}$$

we have $\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} \in \ker \varphi = \{\pm\mathbb{1}\}$. For $\tilde{t}_{ij}$ to define a spin bundle over M, they must satisfy the cocycle condition,

$$\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = \mathbb{1}. \tag{11.138}$$

Define the Čech 2-cochain $f: U_i \cap U_j \cap U_k \to \mathbb{Z}_2$ by

$$\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = f(i, j, k)\mathbb{1}. \tag{11.139}$$

It is easy to see that f is symmetric and closed. Thus f defines an element $w_2(M) \in H^2(M, \mathbb{Z}_2)$ called the **second Stiefel–Whitney class**. It can be shown that $w_2(M)$ is independent of the local frame chosen.

Exercise 11.22 Suppose we take another lift $-\tilde{t}_{ij}$ of t_{ij}. Show that f changes by an exact amount under this change. Accordingly $[f]$ is independent of the lift. [*Hint:* Show that $f(i, j, k) \to f(i, j, k)\delta f_1(i, j, k)$ where $f_1(i, j)$ denotes the sign of $\pm\tilde{t}_{ij}$.]

Theorem 11.23 Let TM be the tangent bundle over an orientable manifold M. There exists a spin bundle over M if and only if $w_2(M)$ is trivial.

Proof: Suppose there exists a spin bundle over M. Then we define a set of transition functions $\tilde{t}_{ij}$ such that $\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = 1$ over any overlapping charts U_i, U_j and U_k, hence $w_2(M)$ is trivial. Conversely, suppose $w_2(M)$ is trivial, namely

$$f(i, j, k) = \delta f_1(i, j, k) = f_1(j, k)f_1(i, k)f_1(k, i)$$

f_1 being a 1-cochain. We consider the 1-cochain $f_1(i, j)$ defined in exercise 11.22. If we choose new transition functions $\tilde{t}'_{ij} \equiv \tilde{t}_{ij}f_1(i, j)$, we have

$$\tilde{t}'_{ij}\tilde{t}'_{jk}\tilde{t}'_{ki} = [\delta f_1(i, j, k)]^2 = \mathbb{1}$$

and hence $\{\tilde{t}'_{ij}\}$ defines a spin bundle over M. ■

We outline some useful results:

(*a*)
$$w_1(\mathbb{C}P^m) = 1,\ w_2(\mathbb{C}P^m) = \begin{cases} 1 & m \text{ odd} \\ x & m \text{ even} \end{cases} \tag{11.140}$$

x being the generator of $H^2(\mathbb{C}P^m, \mathbb{Z}_2)$.

(*b*) $w_1(S^m) = w_2(S^m) = 1$ (11.141)

(*c*) $w_1(\Sigma_g) = w_2(\Sigma_g) = 1$ (11.142)

Σ_g being the Riemann surface of genus g.

12

INDEX THEOREMS

In physics, we often consider a differential operator defined on a manifold M. Typical examples will be the Laplacian, the d'Alembertian and the Dirac operator. From the mathematical point of view, these operators are regarded as maps of sections

$$D : \Gamma(M, E) \to \Gamma(M, F)$$

where E and F are vector bundles over M. For example, the Dirac operator is a map $\Gamma(M, E) \to \Gamma(M, E)$, E being a spin bundle over M. If inner products are defined on E and F, it is possible to define the adjoint of D,

$$D^\dagger : \Gamma(M, F) \to \Gamma(M, E).$$

Since it is a differential operator, D carries analytic information on the spectrum and its degeneracy. In what follows, we are interested in the zero eigenvectors of D and $D^\dagger$,

$$\ker D \equiv \{s \in \Gamma(M, E) | Ds = 0\}$$
$$\ker D^\dagger \equiv \{s \in \Gamma(M, F) | D^\dagger s = 0\}.$$

The analytical index is defined by

$$\operatorname{ind} D = \dim \ker D - \dim \ker D^\dagger.$$

Surprisingly, this analytic quantity is a topological invariant expressed in terms of an integral of an appropriate characteristic class over M, which provides purely *topological* information on M. This interplay between *analysis* and *topology* is the main ingredient of the index theorem.

Our exposition follows Eguchi *et al* (1980), Gilkey (1984), Shanahan (1978), Kulkarni (1975) and Booss and Bleecker (1985). The reader should consult these references for details. Alvarez (1985) contains a brief summary of this subject along with applications to anomalies and strings.

12.1 Elliptic operators and Fredholm operators

In the following, we will be concerned with differential operators defined on vector bundles over a compact manifold M without a boundary. We exclusively deal with a *nice* class of differential operators called the Fredholm operators.

12.1.1 Elliptic operators

Let E and F be complex vector bundles over a manifold M. A differential operator D is a linear map

$$D: \Gamma(M, E) \to \Gamma(M, F). \tag{12.1}$$

Take a chart U of M over which E and F are trivial. The local coordinates of U are denoted by x^μ. We introduce the following multi-index notation,

$$M \equiv (\mu_1, \mu_2, \ldots, \mu_m) \qquad \mu_j \in \mathbb{Z}, \mu_j \geqslant 0$$

$$|M| \equiv \mu_1 + \mu_2 + \ldots + \mu_m$$

$$D_M = \frac{\partial^{|M|}}{\partial x^M} \equiv \frac{\partial^{\mu_1 + \ldots + \mu_m}}{\partial(x^1)^{\mu_1} \ldots \partial(x^m)^{\mu_m}}.$$

If $\dim E = k$ and $\dim F = k'$, the most general form of D is

$$[Ds(x)]^\alpha = \sum_{\substack{|M| \leqslant N \\ 1 \leqslant a \leqslant k}} A^{M\alpha}{}_a(x) D_M s^a(x) \qquad 1 \leqslant \alpha \leqslant k' \tag{12.2}$$

where $s(x)$ is a section of E. Note that x denotes a point whose coordinates are x^μ. This slight abuse simplifies the notation. $A^M \equiv (A^M)^\alpha{}_a$ is a $k \times k'$ matrix which may depend on the position x. The positive integer N in (12.2) is called the **order** of D. We are interested in the case in which $N = 1$ (the Dirac operator) and $N = 2$ (the Laplacian). For example, if E is a spin bundle over M, the Dirac operator $D \equiv \mathrm{i}\gamma^\mu \partial_\mu + m : \Gamma(M, E) \to \Gamma(M, E)$ acts on a section $\psi(x)$ of E as

$$[D\psi(x)]^\alpha = \mathrm{i}(\gamma^\mu)^\alpha{}_\beta \partial_\mu \psi^\beta(x) + m\psi^\alpha(x).$$

The **symbol** of D is a $k \times k'$ matrix

$$\sigma(D, \xi) \equiv \sum_{|M|=N} A^{M\alpha}{}_a(x) \xi_M \tag{12.3}$$

where ξ is a real m-tuple $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$. The symbol is also defined independently of the coordinates. Let $E \xrightarrow{\pi} M$ be a vector bundle and let $p \in M$, $\xi \in T_p^*M$ and $s \in \pi_E^{-1}(p)$. Take a section $\tilde{s} \in \Gamma(M, E)$ such that $\tilde{s}(p) = s$ and a function $f \in \mathcal{F}(M)$ such that $f(p) = 0$ and $\mathrm{d}f(p) = \xi \in T_p^*M$. Then the symbol may be defined by

$$\sigma(D, \xi)s = \frac{1}{N!} D(f^N \tilde{s})|_p. \tag{12.4}$$

The factor f^N automatically picks up the Nth-order term due to the condition $f(p) = 0$. (12.4) yields the same symbol as (12.3).

If the matrix $\sigma(D, \xi)$ is invertible for each $x \in M$ and each $\xi \in \mathbb{R}^m - \{0\}$, the operator D is said to be **elliptic**. Clearly this definition makes sense only when $k = k'$. It should be noted that the symbol for a composite operator $D = D_1 D_2$ is a composite of the symbols, namely $\sigma(D, \xi) = \sigma(D_1, \xi)\sigma(D_2, \xi)$. This shows that composites of elliptic operators are also elliptic. In general, powers and roots of elliptic operators are elliptic.

Example 12.1 Let x^μ be the natural coordinates in $\mathbb{R}^m$. If E and F are real line bundles over $\mathbb{R}^m$, the Laplacian $\Delta : \Gamma(\mathbb{R}^m, E) \to \Gamma(\mathbb{R}^m, F)$ is defined by

$$\Delta \equiv \frac{\partial^2}{\partial(x^1)^2} + \ldots + \frac{\partial^2}{\partial(x^m)^2}. \tag{12.5}$$

According to (12.3), the symbol is

$$\sigma(\Delta, \xi) = \sum_\mu (\xi_\mu)^2.$$

This is in agreement with the result obtained from (12.4),

$$\begin{aligned}
\sigma(\Delta, \xi)s &= \tfrac{1}{2}\Delta(f^2\tilde{s})|_p = \frac{1}{2}\sum \frac{\partial^2}{\partial(x^\mu)^2}(f^2\tilde{s})|_p \\
&= \frac{1}{2}\left(f^2\Delta\tilde{s} + 2f\Delta f\tilde{s} + 2f\sum\frac{\partial f}{\partial x^\mu}\frac{\partial\tilde{s}}{\partial x^\mu} + 2\sum\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\mu}\tilde{s}\right)\Bigg|_p \\
&= \sum(\xi_\mu)^2 s.
\end{aligned}$$

This symbol is clearly invertible for $\xi \neq 0$, and hence Δ is elliptic.

On the other hand, the d'Alembertian

$$\Box \equiv \frac{\partial^2}{\partial(x^1)^2} + \ldots + \frac{\partial^2}{\partial(x^{m-1})^2} - \frac{\partial^2}{\partial(x^m)^2} \tag{12.6}$$

is not elliptic since the symbol

$$\sigma(\Box, \xi) = (\xi^1)^2 + \ldots + (\xi^{m-1})^2 - (\xi^m)^2$$

vanishes everywhere on the light cone,

$$(\xi^m)^2 = (\xi^1)^2 + \ldots + (\xi^{m-1})^2.$$

Exercise 12.2 Let $M = \mathbb{R}^2$ and consider a differential operator D of order two. The symbol of D is of the form

$$\sigma(D, \xi) = A_{11}\xi^1\xi^1 + 2A_{12}\xi^1\xi^2 + A_{22}\xi^2\xi^2.$$

Show that D is elliptic if and only if $\sigma(D, \xi) = 1$ is an ellipse in ξ-space.

12.1.2 Fredholm operators

Let $D : \Gamma(M, E) \to \Gamma(M, F)$ be an elliptic operator. The kernel of D is the set of null eigenvectors

$$\ker D \equiv \{s \in \Gamma(M, E) | Ds = 0\}. \tag{12.7}$$

Suppose E and F are endowed with fibre metrics, which will be denoted by $\langle\ ,\ \rangle_E$ and $\langle\ ,\ \rangle_F$, respectively. The **adjoint** $D^\dagger : \Gamma(M, F) \to \Gamma(M, E)$ of D is defined by

$$\langle s', Ds \rangle_F \equiv \langle D^\dagger s', s \rangle_E \tag{12.8}$$

where $s \in \Gamma(M, E)$ and $s' \in \Gamma(M, F)$. We define the **cokernel** of D by

$$\operatorname{coker} D \equiv \Gamma(M, F)/\operatorname{im} D. \tag{12.9}$$

Among elliptic operators we are interested in a class of operators whose kernels and cokernels are finite-dimensional. An elliptic operator D which satisfies this condition is called a **Fredholm operator**. The **analytical index**

$$\operatorname{ind} D \equiv \dim\ker D - \dim\operatorname{coker} D \tag{12.10}$$

is well defined for a Fredholm operator. Henceforth, we will be concerned only with Fredholm operators. It is known from the general theory of operators that elliptic operators on a *compact* manifold are Fredholm operators. Theorem 12.3 below shows that $\operatorname{ind} D$ is also expressed as

$$\operatorname{ind} D = \dim\ker D - \dim\ker D^\dagger. \tag{12.11}$$

Theorem 12.3 Let $D : \Gamma(M, E) \to \Gamma(M, F)$ be a Fredholm operator. Then

$$\operatorname{coker} D \cong \ker D^\dagger \equiv \{s \in \Gamma(M, F) | D^\dagger s = 0\}. \tag{12.12}$$

Proof: Let $[s] \in \operatorname{coker} D$ be given by

$$[s] = \{s' \in \Gamma(M, F) | s' = s + Du,\ u \in \Gamma(M, E)\}.$$

We show that there is a surjection $\ker D^\dagger \to \operatorname{coker} D$, namely any $[s] \in \operatorname{coker} D$ has a representative $s_0 \in \ker D^\dagger$. Define s_0 by

$$s_0 \equiv s - D\frac{1}{D^\dagger D}D^\dagger s. \tag{12.13}$$

We find $s_0 \in \ker D^\dagger$ since $D^\dagger s_0 = D^\dagger s - D^\dagger D(D^\dagger D)^{-1}D^\dagger s = D^\dagger s - D^\dagger s = 0$. Next let $s_0 \neq s_0' \in \ker D^\dagger$. We show that $[s_0] \neq [s_0']$ in $\Gamma(M, F)/\operatorname{im} D$. If $[s_0] = [s_0']$, there is an element $u \in \Gamma(M, E)$ such that $s_0 - s_0' = Du$. Then $0 = \langle u, D^\dagger(s_0 - s_0')\rangle_E = \langle u, D^\dagger Du\rangle_E = \langle Du, Du\rangle_F \geqslant 0$, hence $Du = 0$, which contradicts our assumption

$s_0 \neq s'_0$. Thus the map $s_0 \mapsto [s]$ is a bijection and we have established that $\operatorname{coker} D \cong \ker D^\dagger$. ■

12.1.3 Elliptic complexes

Consider a sequence of Fredholm operators,

$$\ldots \to \Gamma(M, E_{i-1}) \xrightarrow{D_{i-1}} \Gamma(M, E_i) \xrightarrow{D_i} \Gamma(M, E_{i+1}) \xrightarrow{D_{i+1}} \ldots \quad (12.14)$$

where $\{E_i\}$ is a sequence of vector bundles over a compact manifold M. The sequence (E_i, D_i) is called an **elliptic complex** if D_i is *nilpotent* (that is, $D_i D_{i-1} = 0$) for any i. The reader may refer to $\Gamma(M, E_i) = \Omega^i(M)$ and $D_i = \mathrm{d}$ (exterior derivative) for example. The adjoint of $D_i : \Gamma(M, E_i) \to \Gamma(M, E_{i+1})$ is denoted by

$$D^\dagger_i : \Gamma(M, E_{i+1}) \to \Gamma(M, E_i).$$

The **Laplacian** $\Delta_i : \Gamma(M, E_i) \to \Gamma(M, E_i)$ is defined by

$$\Delta_i \equiv D_{i-1} D^\dagger_{i-1} + D^\dagger_i D_i. \quad (12.15)$$

The Hodge decomposition also applies to the present case,

$$s_i = D_{i-1} s_{i-1} + D^\dagger_i s_{i+1} + h_i \quad (12.16)$$

where $s_{i\pm1} \in \Gamma(M, E_{i\pm1})$ and h_i is in the kernel of Δ_i ; $\Delta_i h_i = 0$.

Analogously to the de Rham cohomology groups, we define

$$H^i(E, D) \equiv \ker D_i / \operatorname{im} D_{i-1}. \quad (12.17)$$

As in the case of the de Rham theory, it can be shown that $H^i(E, D)$ is isomorphic to the kernel of Δ_i. Accordingly we have

$$\dim H^i(E, D) = \dim \operatorname{Harm}^i(E, D) \quad (12.18)$$

where $\operatorname{Harm}^i(E, D)$ is a vector space spanned by $\{h_i\}$. The **index** of this elliptic complex is defined by

$$\operatorname{ind} D \equiv \sum_{i=0}^{m} (-1)^i \dim H^i(E, D) = \sum_{i=0}^{m} (-1)^i \dim \ker \Delta_i. \quad (12.19)$$

The index thus defined generalises the Euler characteristic; see example 12.4 below.

How is this related to (12.10)? Consider the complex $\Gamma(M, E) \xrightarrow{D} \Gamma(M, F)$. We may formally add zero on both sides,

$$0 \xrightarrow{i} \Gamma(M, E) \xrightarrow{D} \Gamma(M, F) \xrightarrow{\varphi} 0 \quad (12.20)$$

where i is the inclusion. The index according to (12.19) is

$$\dim \ker D - \{\dim \Gamma(M, F) - \dim \operatorname{im} D\} = \dim \ker D - \dim \operatorname{coker} D$$

where we have noted that $\dim \operatorname{im} i = 0$, $\ker \varphi = \Gamma(M, F)$ and

$\operatorname{coker} D = \ker \varphi/\operatorname{im} D$. Thus (12.19) yields the same index as (12.10).

It is often convenient to work with a two-term elliptic complex which has the same index as the original elliptic complex (E, D). This *rolling up* is carried out by defining

$$E_+ \equiv \bigoplus_r E_{2r}, \; E_- \equiv \bigoplus_r E_{2r+1} \tag{12.21}$$

which are called the **even bundle** and the **odd bundle**, respectively. Correspondingly we consider the operators

$$A \equiv \bigoplus_r (D_{2r} + D^\dagger_{2r-1}), \; A^\dagger \equiv \bigoplus_r (D_{2r+1} + D^\dagger_{2r}). \tag{12.22}$$

We readily verify that $A : \Gamma(M, E_+) \to \Gamma(M, E_-)$ and $A^\dagger : \Gamma(M, E_-) \to \Gamma(M, E_+)$. From A and $A^\dagger$, we construct the two Laplacians

$$\begin{aligned}\Delta_+ \equiv A^\dagger A &= \bigoplus_{r,s} (D_{2r+1} + D^\dagger_{2r})(D_{2s} + D^\dagger_{2s-1}) \\ &= \bigoplus_r (D_{2r-1}D^\dagger_{2r-1} + D^\dagger_{2r}D_{2r}) = \bigoplus_r \Delta_{2r} \end{aligned}\tag{12.23a}$$

$$\Delta_- \equiv AA^\dagger = \bigoplus_r \Delta_{2r+1}. \tag{12.23b}$$

Then we have

$$\begin{aligned}\operatorname{ind}(E_\pm, A) &= \dim\ker\Delta_+ - \dim\ker\Delta_- \\ &= \sum (-1)^r \dim\ker\Delta_r = \operatorname{ind}(E, D).\end{aligned}\tag{12.24}$$

Example 12.4 Let us consider the **de Rham complex** $\Omega^*(M)$ over a compact manifold M without a boundary,

$$0 \xrightarrow{i} \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \Omega^m(M) \xrightarrow{\mathrm{d}} 0 \tag{12.25}$$

where $m = \dim M$ and d stands for $\mathrm{d}_r : \Omega^r(M) \to \Omega^{r+1}(M)$. $H^r(E, D)$ defined by (12.25) agrees with the de Rham cohomology group $H^r(M, \mathbb{R})$. The index is identified with the Euler characteristic,

$$\operatorname{ind}(\Omega^*(M), \mathrm{d}) = \sum_{r=0}^{m} (-1)^r \dim H^r(M; \mathbb{R}) = \chi(M). \tag{12.26}$$

In Chapter 7, we found that $b^r \equiv \dim H^r(M, \mathbb{R})$ agrees with the number of linearly independent harmonic r-forms; $\dim H^r(M, \mathbb{R}) = \dim \operatorname{Harm}^r(M) = \dim\ker\Delta_r$, where Δ_r is the Laplacian

$$\Delta_r = (\mathrm{d} + \mathrm{d}^\dagger)^2 = \mathrm{d}_{r-1}\mathrm{d}^\dagger_{r-1} + \mathrm{d}^\dagger_r \mathrm{d}_r \tag{12.27}$$

$\mathrm{d}^\dagger_r : \Omega^{r+1}(M) \to \Omega^r(M)$ being the adjoint of d_r. Now we find that

$$\chi(M) = \sum_{r=0}^{m} (-1)^r \dim\ker\Delta_r. \tag{12.28}$$

This is very interesting since the LHS is a purely topological quantity which can be computed by triangulating M, for example, while the RHS is given by the solution of an analytic equation $\Delta_r u = 0$. In example 11.14, we noted that $\chi(M)$ is given by integrating the Euler character over M: $\chi(M) = \int_M e(TM)$. Now (12.28) reads

$$\sum_{r=1}^{m} (-1)^r \dim\ker\Delta_r = \int_M e(TM). \tag{12.29}$$

This is a typical form of the index theorem. The RHS is an analytic index while the LHS is a topological index given by the integral of certain characteristic classes. In §12.3, we derive (12.29) from the Atiyah–Singer index theorem.

The two-term complex is given by

$$\Omega^+(M) \equiv \bigoplus_r \Omega^{2r}(M) \qquad \Omega^-(M) \equiv \bigoplus_r \Omega^{2r+1}(M). \tag{12.30}$$

The corresponding operators are

$$A \equiv \bigoplus_r (\mathrm{d}_{2r} + \mathrm{d}^\dagger_{2r-1}) \qquad A^\dagger \equiv \bigoplus_r (\mathrm{d}_{2r-1} + \mathrm{d}^\dagger_{2r}). \tag{12.31}$$

It is left as an exercise to the reader to show that

$$\mathrm{ind}(\Omega^\pm(M), A) = \dim\ker A_+ - \dim\ker A_- = \chi(M). \tag{12.32}$$

12.2 The Atiyah–Singer index theorem

12.2.1 Statement of the theorem

Theorem 12.5 (**Atiyah–Singer index theorem**) Let (E, D) be an elliptic complex over an m-dimensional compact manifold M without a boundary. The index of this complex is given by

$$\mathrm{ind}(E, D) = (-1)^{m(m+1)/2} \int_M \mathrm{ch}\left(\bigoplus_r (-1)^r E_r\right) \frac{\mathrm{Td}(TM^{\mathbb{C}})}{e(TM)}\bigg|_{\mathrm{vol}}. \tag{12.33}$$

In the integrand of the RHS, only m-forms are picked up, so that the integration makes sense. [*Remarks:* The division by $e(TM)$ can really be carried out at the formal level. If m is an odd integer, the index vanishes identically, see below. Original references are Atiyah and Singer (1968a, b), Atiyah and Segal (1968).]

The proof of theorem 12.5 is found in Shanahan (1978), Palais (1965) and Gilkey (1984). The proof found there is based on either K-theory or the heat kernel formalism. In §13.2, we give a proof of the simplest version of the Atiyah–Singer (AS) index theorem for a spin complex.

Recently physicists have found another proof of the theorem making use of supersymmetry. Interested readers should consult Alvarez-Gaumé (1983) and Friedan and Windey (1984, 1985).

The following corollary is a direct consequence of theorem 12.5.

Corollary 12.6 Let $\Gamma(M, E) \xrightarrow{D} \Gamma(M, F)$ be a two-term elliptic complex. The index of D is given by

$$\begin{aligned} \operatorname{ind} D &= \dim\ker D - \dim\ker D^\dagger \\ &= (-1)^{m(m+1)/2} \int_M (\operatorname{ch} E - \operatorname{ch} F) \left.\frac{\operatorname{Td}(TM^{\mathbb{C}})}{e(TM)}\right|_{\text{vol}}. \end{aligned} \tag{12.34}$$

12.3 The de Rham complex

Let M be an m-dimensional compact orientable manifold with no boundary. By now we are familiar with the de Rham complex,

$$\ldots \xrightarrow{\mathrm{d}} \Omega^{r-1}(M)^{\mathbb{C}} \xrightarrow{\mathrm{d}} \Omega^r(M)^{\mathbb{C}} \xrightarrow{\mathrm{d}} \Omega^{r+1}(M)^{\mathbb{C}} \xrightarrow{\mathrm{d}} \ldots \tag{12.35}$$

where $\Omega^r(M)^{\mathbb{C}} = \Gamma(M, \wedge^r T^*M^{\mathbb{C}})$. We complexified the forms so that we may apply the AS index theorem. The exterior derivative satisfies $\mathrm{d}^2 = 0$. To show that (12.35) is an elliptic complex, we have to show d is elliptic. To find the symbol for d, we note that

$$\sigma(\mathrm{d}, \xi)\omega = \mathrm{d}(f\mathscr{T})|_p = \mathrm{d}f \wedge \mathscr{T} + f\,\mathrm{d}\mathscr{T}|_p = \xi \wedge \omega$$

where $p \in M$, $\omega \in \Omega^r_p(M)^{\mathbb{C}}$, $f(p) = 0$, $\mathrm{d}f(p) = \xi$, $\mathscr{T} \in \Omega^r(M)^{\mathbb{C}}$ and $\mathscr{T}(p) = \omega$; see (12.4). We find

$$\sigma(\mathrm{d}, \xi) = \xi \wedge. \tag{12.36}$$

This defines a map $\Omega^r(M)^{\mathbb{C}} \to \Omega^{r+1}(M)^{\mathbb{C}}$ and is non-singular if $\xi \neq 0$. Thus we have proved that $\mathrm{d} : \Omega^r(M)^{\mathbb{C}} \to \Omega^{r+1}(M)^{\mathbb{C}}$ is elliptic, and hence (12.35) is an elliptic complex.

Let us find the index theorem for this complex. We note that $\dim_{\mathbb{C}} H^r(M; \mathbb{C}) = \dim_{\mathbb{R}} H^r(M; \mathbb{R})$. Hence the analytical index is

$$\begin{aligned} \operatorname{ind} \mathrm{d} &= \sum_{r=0}^{m} (-1)^r \dim_{\mathbb{C}} H^r(M; \mathbb{C}) \\ &= \sum (-1)^r \dim_{\mathbb{R}} H^r(M; \mathbb{R}) = \chi(M) \end{aligned} \tag{12.37}$$

where $\chi(M)$ is the Euler characteristic of M. Suppose M is even dimensional, $m = 2l$. The RHS of (12.33) gives the topological index

$$(-1)^{l(2l+1)} \int_M \operatorname{ch}\left(\bigoplus_{r=0}^{m} (-1)^r \wedge^r T^*M^{\mathbb{C}}\right) \left.\frac{\operatorname{Td}(TM^{\mathbb{C}})}{e(TM)}\right|_{\text{vol}} \tag{12.38}$$

The splitting principle yields

$$\begin{aligned}\operatorname{ch}\left(\bigoplus_{r=0}^{m}(-1)^r \wedge^r T^*M^{\mathbb{C}}\right) &= 1 - \operatorname{ch}(T^*M^{\mathbb{C}}) + \operatorname{ch}(\wedge^2 T^*M^{\mathbb{C}}) + \ldots + (-1)^m \operatorname{ch}(\wedge^m T^*M^{\mathbb{C}}) \\ &= 1 - \sum_{i=1}^{m} \mathrm{e}^{-x_i}(TM^{\mathbb{C}}) + \sum_{i<j} \mathrm{e}^{-x_i}\mathrm{e}^{-x_j}(TM^{\mathbb{C}}) + \ldots \\ &\quad + (-1)^m \mathrm{e}^{-x_1}\mathrm{e}^{-x_2}\ldots \mathrm{e}^{-x_m}(TM^{\mathbb{C}}) \\ &= \prod_{i=1}^{m}(1 - \mathrm{e}^{-x_i})(TM^{\mathbb{C}})\end{aligned}$$

where we have noted that $x_i(T^*M^{\mathbb{C}}) = -x_i(TM^{\mathbb{C}})$. [Let L be a complex line bundle and L^* be its dual bundle. $L \otimes L^*$ is a bundle whose section is a map $\mathbb{C} \to \mathbb{C}$ at each fibre of L. $L \otimes L^*$ has a global section which vanishes nowhere (the identity map, for example) from which we can show $L \otimes L^*$ is a trivial bundle. We have $c_1(L \otimes L^*) = c_1(L) + c_1(L^*) = 0$, hence $x(L^*) = -x(L)$. The splitting principle yields $x_i(T^*M^{\mathbb{C}}) = -x_i(TM^{\mathbb{C}})$.] We also have

$$\operatorname{Td}(TM^{\mathbb{C}}) = \prod_{i=1}^{m} \frac{x_i}{1 - \mathrm{e}^{-x_i}}(TM^{\mathbb{C}})$$

$$e(TM) = \prod_{i=1}^{l} x_i(TM^{\mathbb{C}}).$$

Substituting these in (12.38), we have

$$\operatorname{ind} \mathrm{d} = \int_M (-1)^{l(2l+1)}(-1)^l\left(\prod_{i=1}^{l} x_i(TM^{\mathbb{C}})\right) = \int_M e(TM). \tag{12.39}$$

If m is odd, it can be shown that (Shanahan (1978), p22)

$$\operatorname{ind} \mathrm{d} = 0 \tag{12.40}$$

which is in harmony with the fact that $e(TM) = 0$ if $\dim M$ is odd. In any case, the index theorem for the de Rham complex is

$$\chi(M) = \int_M e(TM). \tag{12.41}$$

Example 12.7 Let M be a two-dimensional orientable manifold without boundary. (12.41) reads

$$\chi(M) = \frac{1}{4\pi}\int_M \varepsilon^{\alpha\beta}\mathcal{R}_{\alpha\beta} = \frac{1}{2\pi}\int_M \mathcal{R}_{12} \tag{12.42a}$$

which is the celebrated **Gauss–Bonnet theorem**. If $\dim M = 4$, we have

$$\chi(M) = \frac{1}{32\pi^2}\int_M \varepsilon^{\alpha\beta\gamma\delta}\mathcal{R}_{\alpha\beta} \wedge \mathcal{R}_{\gamma\delta}. \tag{12.42b}$$

12.4 The Dolbeault complex

We recall some elementary facts about complex manifolds (see Chapter 8 for details). Let M be a compact complex manifold of complex dimension m without a boundary. Let $z^\mu = x^\mu + iy^\mu$ be the local coordinates and $\bar{z}^\mu = x^\mu - iy^\mu$ their complex conjugates. TM^+ denotes the tangent bundle spanned by $\{\partial/\partial z^\mu\}$ and $TM^- = \overline{TM^+}$ the complex conjugate bundle spanned by $\{\partial/\partial \bar{z}^\mu\}$. The dual of TM^+ is denoted by T^*M^+ and spanned by $\{dz^\mu\}$ while that of TM^- is $T^*M^- = \overline{T^*M^+}$ spanned by $\{d\bar{z}^\mu\}$. The space $\Omega^r(M)^{\mathbb{C}}$ of complexified r-forms is decomposed as

$$\Omega^r(M)^{\mathbb{C}} = \bigoplus_{p+q=r} \Omega^{p,q}(M)$$

where $\Omega^{p,q}(M)$ is the space of the (p, q)-forms, which is spanned by a basis of the form

$$dz^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{\nu_q}.$$

The exterior derivative is decomposed as $d \equiv \partial + \bar{\partial}$ where

$$\partial = dz^\mu \wedge \partial/\partial z^\mu, \bar{\partial} = d\bar{z}^\mu \wedge \partial/\partial \bar{z}^\mu.$$

They satisfy $\partial\bar{\partial} + \bar{\partial}\partial = \partial^2 = \bar{\partial}^2 = 0$. We have the sequences

$$\ldots \xrightarrow{\bar{\partial}} \Omega^{p,q}(M) \xrightarrow{\bar{\partial}} \Omega^{p,q+1}(M) \xrightarrow{\bar{\partial}} \ldots \tag{12.43a}$$

$$\ldots \xrightarrow{\partial} \Omega^{p,q}(M) \xrightarrow{\partial} \Omega^{p+1,q}(M) \xrightarrow{\partial} \ldots. \tag{12.43b}$$

We are interested in the first sequence with $p = 0$,

$$\ldots \xrightarrow{\bar{\partial}} \Omega^{0,q}(M) \xrightarrow{\bar{\partial}} \Omega^{0,q+1}(M) \xrightarrow{\bar{\partial}} \ldots. \tag{12.44}$$

This sequence is called the **Dolbeault complex**.

To show that (12.44) is an elliptic complex, we compute the symbol for $\bar{\partial}$. Let $\xi = \xi^{0,1} + \xi^{1,0}$ be a *real* one-form at $p \in M$, where $\xi^{0,1} \in \Omega_p^{0,1}(M)$ and

$$\xi^{1,0} = \overline{\xi^{0,1}} \in \Omega_p^{1,0}(M).$$

Take an antiholomorphic r-form $\omega \in \Omega^{0,r}(M)$. We find

$$\sigma(\bar{\partial}, \xi)\omega = \bar{\partial}(f\tilde{s}) = \bar{\partial}f \wedge \tilde{s} + f\bar{\partial}\tilde{s}|_p = \xi^{0,1} \wedge \omega$$

where $f(p) = 0$, $\bar{\partial}f(p) = \xi^{0,1}$, $\tilde{s} \in \Omega^{0,r}(M)$ and $\tilde{s}(p) = \omega$. We have

$$\sigma(\bar{\partial}, \xi) = \xi^{0,1} \wedge. \tag{12.45}$$

From a similar argument to that given in the previous section, it follows that the symbol (12.45) is elliptic. Thus the Dolbeault complex (12.44) is an elliptic complex.

The AS index theorem takes the form

$$\operatorname{ind}\bar{\partial} = \int_M \operatorname{ch}\left(\sum_r (-1)^r \wedge^r T^*M^-\right) \frac{\operatorname{Td}(TM^{\mathbb{C}})}{e(TM)}\bigg|_{\text{vol}}. \tag{12.46}$$

The LHS is computed as follows. We first note that

$$\ker \bar{\partial}_r / \operatorname{im} \bar{\partial}_{r-1} = H^{0,r}(M)$$

where $H^{0,r}(M)$ is the $\bar{\partial}$-cohomology group. Then the LHS is

$$\operatorname{ind}\bar{\partial} = \sum_{r=0}^{n} (-1)^r b^{0,r} \tag{12.47}$$

where $b^{0,r} \equiv \dim_{\mathbb{C}} H^{0,r}(M)$ is the Hodge number. This index is called the **arithmetic genus** of M.

Simplification of the topological index can be carried out as in the case of the de Rham complex. We refer the reader to Shanahan (1978) for the technical details. We have

$$\sum_{r=1}^{n} (-1)^r b^{0,r} = \int_M \operatorname{Td}(TM^+) \tag{12.48}$$

where $\operatorname{Td}(TM^+)$ is the Todd class of TM^+.

12.4.1 The twisted Dolbeault complex and the Hirzebruch–Riemann–Roch theorem

In the Dolbeault complex, we may replace $\Omega^{0,r}(M)$ by the tensor product bundles $\Omega^{0,r}(M) \otimes V$, where V is a holomorphic vector bundle over M,

$$\ldots \xrightarrow{\bar{\partial}_V} \Omega^{0,r-1}(M) \otimes V \xrightarrow{\bar{\partial}_V} \Omega^{0,r}(M) \otimes V \xrightarrow{\bar{\partial}_V} \ldots. \tag{12.49}$$

The AS index theorem of this complex reduces to the **Hirzebruch–Riemann–Roch theorem**,

$$\operatorname{ind}\bar{\partial}_V = \int_M \operatorname{Td}(TM^+)\operatorname{ch}(V). \tag{12.50}$$

For example, if $m = \dim_{\mathbb{C}} M = 1$, we have

$$\begin{aligned}\operatorname{ind}\bar{\partial}_V &= \frac{1}{2}\dim V \int_M c_1(TM^+) + \int_M c_1(V) \\ &= \dim V(2-g) + \int_M \frac{\mathrm{i}\mathcal{F}}{2\pi}\end{aligned} \tag{12.51}$$

since it can be shown that

$$\int_M c_1(TM^+) = \int_M e(TM) = (2-g)$$

g being the genus of M.

12.5 The signature complex

12.5.1 The Hirzebruch signature

Let M be a compact orientable manifold of even dimension, $m = 2l$. Let $[\omega]$ and $[\eta]$ be the elements of the 'middle' cohomology group $H^l(M; \mathbb{R})$. We consider a bilinear form $H^l(M; \mathbb{R}) \times H^l(M; \mathbb{R}) \to \mathbb{R}$ defined by

$$\sigma([\omega], [\eta]) \equiv \int_M \omega \wedge \eta \tag{12.52}$$

cf example 11.17. This definition is independent of the representatives of $[\omega]$ and $[\eta]$. σ is symmetric if l is even ($m \equiv 0 \mod 4$) and antisymmetric if l is odd ($m \equiv 2 \mod 4$). Poincaré duality shows that the bilinear form σ has the maximal rank $b^l = \dim H^l(M; \mathbb{R})$ and is hence non-degenerate. If $l \equiv 2k$ is even, the symmetric form σ has real eigenvalues, b^+ of which are positive and b^- of which are negative ($b^+ + b^- = b^l$). The **Hirzebruch signature** is defined by

$$\tau(M) \equiv b^+ - b^-. \tag{12.53}$$

If l is odd, $\tau(M)$ is defined to vanish (an antisymmetric form has pure imaginary eigenvalues). In the following, we set $l = 2k$.

The Hodge $*$ satisfies $*^2 = 1$ when acting on a $2k$-form in a $4k$-dimensional manifold M and hence $*$ has eigenvalues ± 1. Let $\mathrm{Harm}^{2k}(M)$ be the set of harmonic $2k$-forms on M. We note that $\mathrm{Harm}^{2k}(M) \cong H^{2k}(M, \mathbb{R})$ and each element of $H^{2k}(M; \mathbb{R})$ has a unique harmonic representative. $\mathrm{Harm}^{2k}(M)$ is separated into disjoint subspaces,

$$\mathrm{Harm}^{2k}(M) = \mathrm{Harm}^{2k}_+(M) \oplus \mathrm{Harm}^{2k}_-(M) \tag{12.54}$$

according to the eigenvalue of $*$. This separation block diagonalises the bilinear form σ. In fact, for $\omega^\pm \in \mathrm{Harm}^{2k}_\pm(M)$,

$$\sigma(\omega^+, \omega^+) = \int_M \omega^+ \wedge \omega^+ = \int_M \omega^+ \wedge *\omega^+ = (\omega^+, \omega^+) > 0$$

where (ω^+, ω^+) is the standard positive-definite inner product defined by (7.181). We also find

$$\sigma(\omega^-, \omega^-) = -\int_M \omega^- \wedge *\omega^- = -(\omega^-, \omega^-) < 0$$

$$\sigma(\omega^+, \omega^-) = -\int_M \omega^+ \wedge *\omega^- = -\int_M \omega^- \wedge *\omega^+ = -\sigma(\omega^+, \omega^-) = 0$$

where we have noted that $\alpha \wedge *\beta = \beta \wedge *\alpha$ for any forms α and β. Hence σ is block diagonal with respect to $\mathrm{Harm}^{2k}_+(M) \oplus \mathrm{Harm}^{2k}_-(M)$ and, moreover, $b^\pm = \dim_{\mathbb{R}} \mathrm{Harm}^{2k}_\pm(M)$. Now $\tau(M)$ is expressed as

$$\tau(M) = \dim \mathrm{Harm}^{2k}_+(M) - \dim \mathrm{Harm}^{2k}_-(M). \tag{12.55}$$

Exercise 12.8 Let $\dim M = 4k$. Show that

$$\tau(M) = \chi(M) \bmod 2. \tag{12.56}$$

[*Hint:* Use the Poincaré duality to show that $\chi(M) = b^{2k} \bmod 2$.]

12.5.2 The signature complex and the Hirzebruch signature theorem

Let M be an m-dimensional compact manifold without a boundary. Consider an operator

$$\mathfrak{D} \equiv \mathrm{d} + \mathrm{d}^{\dagger}. \tag{12.57}$$

$\mathfrak{D}$ is a *square root* of the Laplacian; $\mathfrak{D}^2 = \mathrm{dd}^{\dagger} + \mathrm{d}^{\dagger}\mathrm{d} = \Delta$. To show that $\mathfrak{D}$ is elliptic, it suffices to verify that Δ is elliptic since the symbol of a product of operators is the product of symbols. Let us compute the symbol of Δ. As for d, we have $\sigma(\mathrm{d}, \xi)\omega = \xi \wedge \omega$. As for $\mathrm{d}^{\dagger}$, it can be shown that (Palais 1965, pp77–8)

$$\sigma(\mathrm{d}^{\dagger}, \xi) = -i_{\xi}. \tag{12.58}$$

Here $i_{\xi} : \Omega_p^r(M) \to \Omega_p^{r-1}(M)$ is an inner product defined by

$$i_{\xi}(\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}) \equiv \sum_{j=1}^{r} (-1)^{j+1} g^{\mu_j \mu} \xi_{\mu}\, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}\hat{x}^{\mu_j} \wedge \ldots \wedge \mathrm{d}x^{\mu_r}$$

where the one-form under $\hat{}$ is omitted and we put $\xi = \xi_{\mu}\,\mathrm{d}x^{\mu}$. g is the given Riemannian metric. Now the symbol of the Laplacian is obtained from (12.58) as

$$\begin{aligned}\sigma(\Delta, \xi)\omega &= \sigma(\mathrm{dd}^{\dagger} + \mathrm{d}^{\dagger}\mathrm{d}, \xi)\omega = -[\xi \wedge i_{\xi}(\omega) + i_{\xi}(\xi \wedge \omega)] \\ &= -i_{\xi}(\xi) \wedge \omega = -\|\xi\|^2\omega\end{aligned}$$

where ω is an arbitrary r-form and $\|\ \|$ is taken with respect to the given Riemannian metric. Finally we obtain

$$\sigma(\Delta, \xi) = -\|\xi\|^2. \tag{12.59}$$

Thus the Laplacian Δ is elliptic and so is $\mathfrak{D} = \mathrm{d} + \mathrm{d}^{\dagger}$.

Since the Laplacian $\Delta = \mathfrak{D}^2$ is self-dual on $\Omega^*(M)$, the index of Δ vanishes trivially. It is also observed that $\mathfrak{D} = \mathfrak{D}^{\dagger}$ on $\Omega^*(M)$ and hence $\mathrm{ind}\,\mathfrak{D} = 0$. To construct a non-trivial index theorem, we have to find a complex on which $\mathfrak{D} \neq \mathfrak{D}^{\dagger}$.

Exercise 12.9 Consider the restriction $\mathfrak{D}^{\mathrm{e}}$ of $\mathfrak{D}$ to even forms, $\mathfrak{D}^{\mathrm{e}} : \Omega^{\mathrm{e}}(M)^{\mathbb{C}} \to \Omega^{\mathrm{o}}(M)^{\mathbb{C}}$ where $\Omega^{\mathrm{e}}(M)^{\mathbb{C}} \equiv \oplus\Omega^{2i}(M)^{\mathbb{C}}$ and $\Omega^{\mathrm{o}}(M)^{\mathbb{C}} \equiv \oplus\Omega^{2i+1}(M)^{\mathbb{C}}$. The adjoint of $\mathfrak{D}^{\mathrm{e}}$ is $\mathfrak{D}^{\mathrm{o}} \equiv \mathfrak{D}^{\mathrm{e}\dagger} : \Omega^{\mathrm{o}}(M)^{\mathbb{C}} \to \Omega^{\mathrm{e}}(M)^{\mathbb{C}}$. Show that

$$\operatorname{ind}\mathfrak{D}^{e} = \dim\ker\mathfrak{D}^{e} - \dim\ker\mathfrak{D}^{o} = \chi(M).$$

[*Hint:* Prove $\ker\mathfrak{D}^{e} = \oplus\mathrm{Harm}^{2i}(M)$ and $\ker\mathfrak{D}^{o} = \oplus\mathrm{Harm}^{2i+1}(M)$. This complex, although non-trivial, does not yield anything new.]

If $\dim M = m = 2l$, we have $**\eta = (-1)^r\eta$ for $\eta \in \Omega^r(M)^{\mathbb{C}}$. We define an operator $\pi : \Omega^r(M)^{\mathbb{C}} \to \Omega^{m-r}(M)^{\mathbb{C}}$ by

$$\pi \equiv \mathrm{i}^{r(r-1)+l}*. \tag{12.60}$$

π is a 'square root' of $(-1)^r** = 1$. In fact, for $\omega \in \Omega^r(M)^{\mathbb{C}}$,

$$\begin{aligned}\pi^2\omega &= \mathrm{i}^{r(r-1)+l}\pi(*\omega) = \mathrm{i}^{r(r-1)+l+(2l-r)(2l-r-1)+l}**\omega \\ &= \mathrm{i}^{2r^2}**\omega = (-1)^r**\omega = \omega\end{aligned} \tag{12.61}$$

where we have noted that $r \equiv r^2 \bmod 2$. We easily verify (exercise) that

$$\{\pi, \mathfrak{D}\} = \pi\mathfrak{D} + \mathfrak{D}\pi = 0. \tag{12.62}$$

Let π act on $\Omega^*(M)^{\mathbb{C}} = \oplus\Omega^r(M)^{\mathbb{C}}$. Since $\pi^2 = 1$, the eigenvalues of π are ± 1. Then we have a decomposition of $\Omega^*(M)^{\mathbb{C}}$ into the ± 1 eigenspaces $\Omega^{\pm}(M)$ of π as

$$\Omega^*(M)^{\mathbb{C}} = \Omega^+(M) \oplus \Omega^-(M). \tag{12.63}$$

Since $\mathfrak{D}$ anticommutes with π, the restriction of $\mathfrak{D}$ to $\Omega^+(M)$ defines an elliptic complex called the **signature complex**,

$$\mathfrak{D}_+ : \Omega^+(M) \to \Omega^-(M) \tag{12.64}$$

where $\mathfrak{D}_+ \equiv \mathfrak{D}|_{\Omega^+(M)}$. The index of the signature complex is

$$\begin{aligned}\operatorname{ind}\mathfrak{D}_+ &= \dim\ker\mathfrak{D}_+ - \dim\ker\mathfrak{D}_- \\ &= \dim\mathrm{Harm}(M)^+ - \dim\mathrm{Harm}(M)^-\end{aligned} \tag{12.65}$$

where $\mathfrak{D}_- \equiv \mathfrak{D}^{\dagger}_+ : \Omega^-(M) \to \Omega^+(M)$ and $\mathrm{Harm}(M)^{\pm} \equiv \{\omega \in \Omega^{\pm}(M) | \mathfrak{D}_{\pm}\omega = 0\}$. On the RHS of (12.65), all the contributions except those from the harmonic l-forms cancel out. To see this, we separate $\ker\mathfrak{D}_+$ and $\ker\mathfrak{D}_-$ as

$$\ker\mathfrak{D}_{\pm} = \mathrm{Harm}^l(M)^{\pm} \oplus \sum_{0\le r\le l} [\mathrm{Harm}^r(M)^{\pm} \oplus \mathrm{Harm}^{m-r}(M)^{\pm}]$$

where $\mathrm{Harm}^r(M)^{\pm} \equiv \mathrm{Harm}(M)^{\pm} \cap \Omega^r(M)$. If $\omega \in \mathrm{Harm}^r(M)$, we have $\omega \pm \pi\omega \in \mathrm{Harm}^r(M)^{\pm} \oplus \mathrm{Harm}^{m-r}(M)^{\pm}$. Then a map $\omega + \pi\omega \to \omega - \pi\omega$ defines an isomorphism between $\mathrm{Harm}^r(M)^+ \oplus \mathrm{Harm}^{m-r}(M)^+$ and $\mathrm{Harm}^r(M)^- \oplus \mathrm{Harm}^{m-r}(M)^-$. Now the index simplifies as

$$\operatorname{ind}\mathfrak{D}_+ = \dim\mathrm{Harm}^{2k}(M)^+ - \dim\mathrm{Harm}^{2k}(M)^- \tag{12.66}$$

where we put $l = 2k$ as before (if l is odd, the index vanishes). It is important to note that $\mathrm{Harm}^{2k}(M)^{\pm} = \mathrm{Harm}^{2k}_{\pm}(M)$ since $\pi = *$ in

$\mathrm{Harm}^{2k}(M)$ see (12.54). Now the index (12.66) reduces to the Hirzebruch signature,

$$\operatorname{ind}\mathfrak{D}_+ = \tau(M). \tag{12.67}$$

The derivation of the topological index is rather technical and we simply quote the result from Shanahan (1978). Let $\wedge^{\pm}T^*M^{\mathbb{C}}$ be the subspace of $\wedge^*T^*M^{\mathbb{C}}$ such that $\Omega^{\pm}(M) = \Gamma(M, \wedge^{\pm}T^*M^{\mathbb{C}})$. Then we have

$$\text{topological index} = (-1)^l \int_M \operatorname{ch}(\wedge^+T^*M^{\mathbb{C}} - \wedge^-T^*M^{\mathbb{C}}) \left.\frac{\operatorname{Td}(TM^{\mathbb{C}})}{e(TM)}\right|_{\text{vol}}$$
$$= 2^l \int_M \prod_{i=1}^{l} \left.\frac{x_i/2}{\tanh x_i/2}\right|_{\text{vol}} = \int_M \prod_{i=1}^{l} \left.\frac{x_i}{\tanh x_i}\right|_{\text{vol}}$$

where the last equality is true only for the $2l$-forms in the expansion and $x_i = x_i(TM^{\mathbb{C}})$. Now we have obtained the **Hirzebruch signature theorem**

$$\tau(M) = \int_M L(TM)|_{\text{vol}} \tag{12.68}$$

where L is the Hirzebruch L-polynomial defined by (11.91). Since L is even in x_i, $\tau(M)$ vanishes if $m = 2 \bmod 4$. For example, $\tau(M) = 0$ for $m = 2$. If $m = 4$, we have

$$\tau(M) = \int_M \tfrac{1}{3}p_1(TM) = -\frac{1}{24\pi^2}\int \operatorname{tr}\mathcal{R}^2. \tag{12.69}$$

As in the case of the Dolbeault complex, we may *twist* the signature complex, see Eguchi *et al* (1980), for example.

12.6 Spin complexes

The final example of classical complexes is the spin complex. This complex is very important in physics since it describes Dirac fields interacting with gauge fields and/or gravitational fields.

12.6.1 Dirac operator

Let us consider a spin bundle $S(M)$ over an m-dimensional orientable manifold M. We shall denote the set of sections of this bundle by $\Delta(M) = \Gamma(M, S(M))$. We assume that $m = 2l$ is an even integer. The spin group $\mathrm{SPIN}(m)$ is generated by m Dirac matrices $\{\gamma^\alpha\}$, which satisfy

$$\gamma^{\alpha\dagger} = \gamma^\alpha \tag{12.70a}$$

$$\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}. \tag{12.70b}$$

Throughout this chapter we assume that the metric has the Euclidean signature. The Clifford algebra is generated by

$$1;\ \gamma^{\alpha};\ \gamma^{\alpha_1}\gamma^{\alpha_2}\ (\alpha_1 < \alpha_2);\ \ldots;$$
$$\gamma^{\alpha_1}\ldots\gamma^{\alpha_k}\ (\alpha_1 < \ldots < \alpha_k);\ \ldots;\ \gamma^1\ldots\gamma^{2l}.$$

The last generator is of particular importance and we define

$$\gamma^{m+1} \equiv \mathrm{i}^l\gamma^1\ldots\gamma^m. \tag{12.71}$$

Our convention is such that $(\gamma^{m+1})^2 = \mathbb{1}$ and $(\gamma^{m+1})^{\dagger} = \gamma^{m+1}$. It can be shown from the general theory of the Clifford algebra that the γ^{α} are represented by $2^l \times 2^l$ matrices with complex entries. It is convenient to take a representation of $\{\gamma^{\alpha}\}$ such that γ^{m+1} is diagonal,

$$\gamma^{m+1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{12.72}$$

where $\mathbb{1}$ is the $2^{l-1} \times 2^{l-1}$ unit matrix.

Example 12.10 For $m = 2$, we take

$$\gamma^0 = \sigma_2 \qquad \gamma^1 = \sigma_1 \qquad \gamma^3 = \mathrm{i}\gamma^0\gamma^1 = \sigma_3$$

σ_{α} being the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $m = 4$, we may take

$$\gamma^{\beta} = \begin{pmatrix} 0 & \mathrm{i}\alpha^{\beta} \\ -\mathrm{i}\bar{\alpha}^{\beta} & 0 \end{pmatrix} \qquad \alpha^{\beta} = (\mathbb{1}, -\mathrm{i}\boldsymbol{\sigma}),\ \bar{\alpha}^{\beta} = (\mathbb{1}, \mathrm{i}\boldsymbol{\sigma})$$

$$\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

A Dirac spinor $\psi \in \Delta(M)$ is an irreducible representation of the Clifford algebra but *not* that of SPIN($2l$). Irreducible representations of SPIN($2l$) are obtained by separating $\Delta(M)$ according to the eigenvalues of γ^{m+1}. Since $(\gamma^{m+1})^2 = \mathbb{1}$, the eigenvalues of γ^{m+1}, called the **chirality**, must be ± 1. Then $\Delta(M)$ is separated into two eigenspaces

$$\Delta(M) = \Delta^{+}(M) \oplus \Delta^{-}(M) \tag{12.73}$$

where $\gamma^{m+1}\psi^{\pm} = \pm\psi^{\pm}$ for $\psi^{\pm} \in \Delta^{\pm}(M)$. The projection operators $\mathcal{P}^{\pm}$ onto $\Delta^{\pm}$ are given by

$$\mathcal{P}^{+} \equiv \tfrac{1}{2}(\mathbb{1} + \gamma^{m+1}) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \tag{12.74a}$$

$$\mathcal{P}^{-} \equiv \tfrac{1}{2}(\mathbb{1} - \gamma^{m+1}) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{12.74b}$$

Thus we may write

$$\psi^+ = \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} \in \Delta^+(M), \quad \psi^- = \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \in \Delta^-(M). \tag{12.75}$$

[Note the minor abuse of the notation.] The reader should verify that $\mathcal{P}^+ + \mathcal{P}^- = 1$, $(\mathcal{P}^\pm)^2 = \mathcal{P}^\pm$, $\mathcal{P}^+\mathcal{P}^- = 0$, $\mathcal{P}^\pm\psi^\pm = \psi^\pm$ and $\mathcal{P}^\pm\psi^\mp = 0$.

The **Dirac operator** in a curved space is given by (§7.10),

$$\mathrm{i}\nabla\!\!\!\!/\,\psi \equiv \mathrm{i}\gamma^\mu\nabla_{\partial/\partial x^\mu}\psi = \mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu)\psi \tag{12.76}$$

where $\omega_\mu = \frac{1}{2}\mathrm{i}\omega_\mu{}^{\alpha\beta}\Sigma_{\alpha\beta}$ is the spin connection and $\gamma^\mu \equiv \gamma^\alpha e_\alpha{}^\mu$. We prove that $\mathrm{i}\nabla\!\!\!\!/$ is elliptic. Let f be a function defined near $p \in M$ such that $f(p) = 0$ and $\mathrm{i}\gamma^\mu\partial_\mu f(p) = \mathrm{i}\gamma^\mu\xi_\mu \equiv \mathrm{i}\xi\!\!\!/$ (for a vector $A = A^\mu e_\mu$, $A\!\!\!/$ denotes $\gamma^\mu A_\mu$). Take a section $\widetilde{\psi} \in \Delta(M)$ such that $\widetilde{\psi}(p) = \psi$. From (12.4), we have

$$\sigma(\mathrm{i}\nabla\!\!\!\!/, \xi)\psi = \mathrm{i}\nabla\!\!\!\!/(f\widetilde{\psi})|_p = (\mathrm{i}\nabla\!\!\!\!/ f)\widetilde{\psi}|_p = \mathrm{i}\xi\!\!\!/\psi$$

which shows that

$$\sigma(\mathrm{i}\nabla\!\!\!\!/, \xi\!\!\!/) = \mathrm{i}\xi\!\!\!/. \tag{12.77}$$

If we note that $\xi\!\!\!/\xi\!\!\!/ = \xi_\alpha\xi_\beta\gamma^\alpha\gamma^\beta = \xi^\mu\xi_\mu$, we find that (12.77) is invertible for $\mathrm{i}\xi\!\!\!/ \neq 0$, hence $\mathrm{i}\nabla\!\!\!\!/$ is an elliptic operator.

It can be shown that $\{\gamma^\alpha\}$ is taken in the form

$$\gamma^\beta = \begin{pmatrix} 0 & \mathrm{i}\alpha_\beta \\ -\mathrm{i}\bar{\alpha}_\beta & 0 \end{pmatrix} \qquad \alpha^\dagger{}_\beta = \bar{\alpha}_\beta \tag{12.78}$$

see example 12.10 for $m = 2$ and 4. Then (12.76) becomes

$$\mathrm{i}\nabla\!\!\!\!/ = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \tag{12.79}$$

where

$$D \equiv \bar{\alpha}^\beta e_\beta{}^\mu(\partial_\mu + \omega_\mu),\ D^\dagger \equiv -\alpha^\beta e_\beta{}^\mu(\partial_\mu + \omega_\mu). \tag{12.80}$$

Hence $D^\dagger$ is indeed the adjoint of D (note that $\partial_\mu + \omega_\mu$ is anti-Hermitian). For

$$\begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} \in \Delta^+(M)$$

we have

$$\mathrm{i}\nabla\!\!\!\!/ \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}\begin{pmatrix} \psi^+ \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ D\psi^+ \end{pmatrix}$$

while for

$$\begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \in \Delta^-(M)$$

we have

$$\mathrm{i}\nabla\!\!\!/ \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} = \begin{pmatrix} D^\dagger \psi^- \\ 0 \end{pmatrix}.$$

Hence $D = \mathrm{i}\nabla\!\!\!/ \mathcal{P}^+ : \Delta^+(M) \to \Delta^-(M)$ and $D^\dagger = \mathrm{i}\nabla\!\!\!/ \mathcal{P}^- : \Delta^-(M) \to \Delta^+(M)$. Now we have a two-term complex

$$\Delta^+(M) \underset{D^\dagger}{\overset{D}{\rightleftarrows}} \Delta^-(M) \tag{12.81}$$

called the **spin complex**. The analytical index of this complex is

$$\operatorname{ind} D = \dim\ker D - \dim\ker D^\dagger = \nu_+ - \nu_- \tag{12.82}$$

where ν_+ (ν_-) is the number of zero-energy modes of chirality + (−).

Let us apply the AS index theorem to this case. Without getting into the details of the Clifford algebra and the spin complex, we simply write down the result. The AS index theorem for the spin complex (12.81) is

$$\begin{aligned} \nu_+ - \nu_- &= \int_M \operatorname{ch}(\Delta^+(M) - \Delta^-(M)) \left.\frac{\operatorname{Td}(TM^{\mathbb{C}})}{e(TM)}\right|_{\mathrm{vol}} \\ &= \int_M \hat{A}(TM)|_{\mathrm{vol}} \end{aligned} \tag{12.83}$$

where $\hat{A}$ is the **Dirac genus** defined by (11.94). Since $\hat{A}$ contains only $4j$-forms, $\nu_+ - \nu_-$ vanishes unless $m = 0 \bmod 4$. Of course, this does not necessarily imply $\nu_+ = \nu_- = 0$.

12.6.2 Twisted spin complexes

In physics, a spinor field may belong to a representation of a group G. For example, the quark field in QCD belongs to the **3** of SU(3). A spinor which belongs to a representation of G is a section of the product bundle $S(M) \otimes E$, where E is an associated vector bundle of $P(M, G)$ in an appropriate representation. The Dirac operator $D_E : \Delta^+(M) \otimes E \to \Delta^-(M) \otimes E$ in this case is

$$D_E = \mathrm{i}\gamma^\alpha e_\alpha{}^\mu(\partial_\mu + \omega_\mu + \mathcal{A}_\mu)\mathcal{P}_+ \tag{12.84}$$

where $\mathcal{A}_\mu$ is the gauge potential on E. The AS index theorem for this **twisted spin complex** is

$$\nu_+ - \nu_- = \int_M \hat{A}(TM)\operatorname{ch}(E)|_{\mathrm{vol}}. \tag{12.85}$$

For $n = 2$, we have

$$\nu_+ - \nu_- = \int_M \operatorname{ch}_1(E) = \frac{\mathrm{i}}{2\pi}\int_M \operatorname{tr}\mathcal{F} \tag{12.86a}$$

while for $n = 4$,

$$\begin{aligned}\nu_+ - \nu_- &= \int_M [\mathrm{ch}_2(E) + \hat{A}_1(TM)\,\mathrm{ch}_0(E)] \\ &= \frac{-1}{8\pi^2}\int_M \mathrm{tr}\,\mathcal{F}^2 + \frac{\dim E}{192\pi^2}\int_M \mathrm{tr}\,\mathcal{R}^2. \end{aligned} \tag{12.86b}$$

Example 12.11 For

$$M = T^{2l} = \underbrace{S^1 \times \ldots \times S^1}_{\ldots\, 2l \text{ times}\, \ldots}$$

we have

$$\hat{A}(TM) = \hat{A}\left(\bigoplus_1^{2l} TS^1\right) = \prod_1^{2l} \hat{A}(TS^1) = 1.$$

We also have $\hat{A}(TS^{2l}) = 1$. For these manifolds, we have

$$\nu_+ - \nu_- = \int_M \mathrm{ch}(E)|_{\mathrm{vol}}. \tag{12.87}$$

Example 12.12 Let us consider the monopole bundle $P(S^2, \mathrm{U}(1))$. If $\mathcal{A}$ is the local gauge potential, the field strength is $\mathcal{F} = \mathrm{d}\mathcal{A}$. The index theorem is

$$\nu_+ - \nu_- = \frac{\mathrm{i}}{2\pi}\int_{S^2}\mathcal{F} = -\frac{1}{2\pi}\int_{S^2}\mathcal{F} \tag{12.88}$$

where $\mathcal{F} = \mathrm{i}F$. As was shown in §10.5, the RHS represents the winding number $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$ and analytical information (the LHS) is now expressed in a topological way (the RHS).

Let $P(S^4, \mathrm{SU}(2))$ be the instanton bundle. Expression (12.87) reads

$$\nu_+ - \nu_- = \int_{S^4}\mathrm{ch}_2(\mathcal{F}) = \frac{-1}{8\pi^2}\int_{S^4}\mathrm{tr}\,\mathcal{F}^2. \tag{12.89}$$

The RHS represents the instanton number $k \in \pi_3(\mathrm{SU}(2)) = \mathbb{Z}$. Note that $k > 0$ if $\mathcal{F} = *\mathcal{F}$ while $k < 0$ if $\mathcal{F} = -*\mathcal{F}$. It can be shown that $\nu_- = 0$ ($\nu_+ = 0$) if $k > 0$ ($k < 0$), see Jackiw and Rebbi (1977). For example, let $\mathcal{F}$ be self-dual. Suppose $\psi^- \in \ker D^\dagger = \ker DD^\dagger$. From (12.80), we find

$$DD^\dagger\psi^- = [(\partial_\mu + \mathcal{A}_\mu)^2 + 2\mathrm{i}\bar{\sigma}_{\mu\nu}\mathcal{F}^{\mu\nu}]\psi^- = 0$$

where $\bar{\sigma}_{\mu\nu} \equiv (1/4\mathrm{i})(\alpha^\mu\bar{\alpha}^\nu - \alpha^\nu\bar{\alpha}^\mu)$. It is easily verified that $\bar{\sigma}^{\mu\nu}$ is anti-self-dual ($\bar{\sigma}^{\mu\nu} = -*\bar{\sigma}^{\mu\nu}$) and hence $\bar{\sigma}_{\mu\nu}\mathcal{F}^{\mu\nu} = 0$. Since $(\partial_\mu + \mathcal{A}_\mu)^2$ is a positive-definite operator it has no normalisable bound states. This verifies that $\ker D^\dagger = \varnothing$.

12.7 The heat kernel and generalised ζ-functions

As we mentioned in §12.2, there are several methods of proving the AS index theorem. The heat kernel is relatively accessible to physicists and it also has many applications to other problems in physics. The generalised ζ-function is related to the heat kernel and also has relevance in physics.

12.7.1 The heat kernel and index theorem

Let E be a complex vector bundle over an m-dimensional compact manifold M. Let $\Delta : \Gamma(M, E) \to \Gamma(M, E)$ be an elliptic operator with eigenvectors $|n\rangle$ such that

$$\Delta|n\rangle = \lambda_n|n\rangle. \tag{12.90}$$

We denote the set of eigenvalues of Δ by Spec Δ. We assume that Δ is non-negative: all the eigenvalues are non-negative. Suppose there are n_0 modes $|0, i\rangle$, $1 \leqslant i \leqslant n_0$ with vanishing eigenvalue. In other words,

$$\dim \ker \Delta = n_0 \tag{12.91}$$

These modes are called the **zero modes**. Define the **heat kernel** $h(t)$ by

$$h(t) \equiv \mathrm{e}^{-t\Delta}. \tag{12.92}$$

It is convenient to represent $h(t)$ in the coordinate basis as

$$\begin{aligned} h(x, y; t) \equiv \langle x|h(t)|y\rangle &= \left\langle x\left|\sum_n \mathrm{e}^{-t\Delta}\right|n\right\rangle\langle n|y\rangle \\ &= \sum_n \mathrm{e}^{-t\lambda_n}\langle x|n\rangle\langle n|y\rangle. \end{aligned} \tag{12.93}$$

Multiple eigenstates should be counted as many times as they appear. We assume $\langle x|n\rangle$ is orthonormal: $\int \langle n|x\rangle\langle x|m\rangle\,\mathrm{d}x = \delta_{mn}$. The convergence of (12.92) for $t > 0$ is guaranteed since Δ is non-negative. Taking the limit $t \to \infty$, we have

$$\lim_{t\to\infty} h(x, y; t) = \sum_{i=1}^{n_0} \langle x|0, i\rangle\langle 0, i|y\rangle \tag{12.94}$$

where the summation is over the zero modes $|0, i\rangle$ only. Thus $h = \mathrm{e}^{-t\Delta}$ tends to be the projection operator onto the space of zero modes as $t \to \infty$,

$$\mathrm{e}^{-t\Delta} \xrightarrow{t\to\infty} \sum |0, i\rangle\langle 0, i|. \tag{12.95}$$

If we define

$$\widetilde{h}(t) \equiv \int h(x, x; t)\,\mathrm{d}x = \sum_n \mathrm{e}^{-t\lambda_n} \tag{12.96}$$

it follows from (12.94) that

$$n_0 = \lim_{t\to\infty} \widetilde{h}(t). \tag{12.97}$$

It is easy to verify that h satisfies the **heat equation**,

$$\left(\frac{\partial}{\partial t} + \Delta_x\right)h(x, y; t) = 0. \tag{12.98}$$

If Δ is the conventional Laplacian, (12.98) reduces to the ordinary heat equation. The initial condition is

$$h(x, y; 0) = \sum_n \langle x|n\rangle\langle n|y\rangle = \delta(x - y) \tag{12.99}$$

where the last equality follows from the completeness of the eigenvectors.

Exercise 12.13 Let $u(x, t)$ be a solution of (12.98) such that $u(x, 0) = u(x)$. Show that

$$u(x, t) = \int h(x, y; t)u(y)\,\mathrm{d}y. \tag{12.100}$$

[First verify that (12.100) satisfies the initial condition, next that it is a solution of the heat equation.]

It is known that the solution of (12.98) has an asymptotic expansion for $t \to \varepsilon$ given by

$$h(x, x; \varepsilon) = \sum_i a_i(x)\varepsilon^i \tag{12.101}$$

see Gilkey (1984). Similarly, $\widetilde{h}(t)$ has an expansion

$$\widetilde{h}(\varepsilon) \equiv \sum_i a_i\varepsilon^i \tag{12.102}$$

where $a_i \equiv \int a_i(x)\,\mathrm{d}x$.

Let E and F be complex vector bundles over M and $D : \Gamma(M, E) \to \Gamma(M, F)$ be an elliptic operator. We define two Laplacians

$$\Delta_E \equiv D^\dagger D : \Gamma(M, E) \to \Gamma(M, E) \tag{12.103a}$$

$$\Delta_F \equiv DD^\dagger : \Gamma(M, F) \to \Gamma(M, F). \tag{12.103b}$$

It is important to note that they have the same non-vanishing eigenvalues including the degeneracy. To see this, let $\Delta_E|\lambda\rangle = \lambda|\lambda\rangle$, $\lambda \neq 0$. Then there is a vector $D|\lambda\rangle \in \Gamma(M, F)$ such that

$$\Delta_F(D|\lambda\rangle) = DD^\dagger D|\lambda\rangle = D\Delta_E|\lambda\rangle = \lambda(D|\lambda\rangle).$$

Note that $D|\lambda\rangle \neq 0$ since $\ker\Delta_E = \ker D$. Conversely if $|\mu\rangle \in \Gamma(M, F)$ satisfies $\Delta_F|\mu\rangle = \mu|\mu\rangle$, then $D^\dagger|\mu\rangle \in \Gamma(M, E)$ is an eigenvector of Δ_E with the same eigenvalue μ. Thus we have found the symmetry (this is a kind of *supersymmetry*)

$$\mathrm{Spec}'\Delta_E = \mathrm{Spec}'\Delta_F \tag{12.104}$$

where the prime denotes that the zero eigenmodes are omitted.

Define two heat kernels h_E and h_F by

$$h_E(x, y, t) = \sum e^{-\lambda_n} \langle x|n\rangle\langle n|y\rangle \tag{12.105a}$$

$$h_F(x, y, t) = \sum e^{-\mu_m} \langle x|m)(m|y\rangle. \tag{12.105b}$$

We have

$$\lim_{t\to\infty} \tilde{h}_E(t) = \dim\ker\Delta_E = \dim\ker D \tag{12.106a}$$

$$\lim_{t\to\infty} \tilde{h}_F(t) = \dim\ker\Delta_F = \dim\ker D^\dagger. \tag{12.106b}$$

What is more interesting is the index of D. Since $\ker D = \ker\Delta_E$ and $\ker D^\dagger = \ker\Delta_F$, we have

$$\begin{aligned}\operatorname{ind} D &= \dim\ker D - \dim\ker D^\dagger = \dim\ker\Delta_E - \dim\ker\Delta_F \\ &= \lim_{t\to\infty}[\tilde{h}_E(t) - \tilde{h}_F(t)] = \tilde{h}_E(t) - \tilde{h}_F(t).\end{aligned} \tag{12.107}$$

The final equality follows since the t-dependent part of $\tilde{h}_E(t) - \tilde{h}_F(t)$ cancels out by the symmetry (12.104). We expand $\tilde{h}_E(t)$ and $\tilde{h}_F(t)$ as

$$\tilde{h}_E(t) = \sum a_i^E t^i \qquad \tilde{h}_F(t) = \sum a_i^F t^i.$$

Picking up t-independent terms, we have

$$\operatorname{ind} D = a_0^E - a_0^F = \int \mathrm{d}x\, [a_0^E(x) - a_0^F(x)]\,\mathrm{d}x \tag{12.108}$$

where $a_0^{E,F}(x)$ are defined in (12.101).

In general, $a_0^{E,F}(x)$ are local invariants written in terms of curvature two-forms. In §13.2, we use the heat kernel to prove the index theorem

$$\operatorname{ind} D = \nu_+ - \nu_- = \int_M \operatorname{ch}(\mathcal{F})|_{\mathrm{vol}}$$

for the twisted spin complex over a manifold with $\hat{A}(TM) = 1$.

Exercise 12.14 Let D, $D^\dagger$, Δ_E and Δ_F be as above. Show that

$$I(s) \equiv \operatorname{tr}\left[\frac{s}{\Delta_E + s} - \frac{s}{\Delta_F + s}\right] \qquad \operatorname{Re} s > 0 \tag{12.109}$$

is independent of s. Show also that $I(s) = \operatorname{ind} D$.

12.7.2 Generalised ζ-functions

Let E and F be vector bundles over M. Define a new function

$$\zeta_E(x, y; s) \equiv \sum{}' \langle x|n\rangle\langle n|y\rangle\, \lambda_n^{-s} \qquad \operatorname{Re} s > 0 \tag{12.110}$$

where $\Delta_E|n\rangle = \lambda_n|n\rangle$ and the prime denotes the omission of the zero modes ($\lambda_n = 0$). [A function $\zeta_F(x, y; s)$ may similarly be defined for Δ_F.] The functions h_E and ζ_E are related by the **Mellin transformation**. To see this we recall the definition of the Γ-function,

$$\Gamma(s) \equiv \int_0^\infty t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t = \lambda^s \int_0^\infty t^{s-1}\mathrm{e}^{-\lambda t}\,\mathrm{d}t$$

where λ is taken to be strictly positive. From this we find

$$\begin{aligned}\Gamma(s)\zeta(x, y; s) &= \sum_n{}' \int_0^\infty t^{s-1}\mathrm{e}^{-\lambda_n t}\langle x|n\rangle\langle n|y\rangle\,\mathrm{d}t \\ &= \int_0^\infty t^{s-1}\Big[h(x, y; t) - \sum_i \langle x|0, i\rangle\langle 0, i|y\rangle\Big]\,\mathrm{d}t. \qquad (12.111)\end{aligned}$$

We also note that

$$\zeta_\Delta(s) \equiv \int_M \zeta(x, x; s)\,\mathrm{d}x = \sum_n{}' \lambda_n^{-s} \qquad (12.112)$$

is the generalised ζ-function defined by (1.42).

Exercise 12.15 Verify that

$$\Delta^{-s}f(x) = \int \zeta(x, y; s)f(y)\,\mathrm{d}y \qquad (12.113)$$

where the general power of an operator may be defined in the sense of an eigenvalue, namely we put $\Delta^{-s}|n\rangle = \lambda_n^{-s}|n\rangle$. Re s is assumed to be sufficiently large so that (12.113) is well defined. [*Hint:* Use the completeness of the eigenvectors.]

Example 12.16 The following example is taken from Kulkarni (1975). Let $M = S^1 = \{\mathrm{e}^{\mathrm{i}\theta}\}$ and $E = F =$ a trivial line bundle over S^1 (a cylinder). Take an operator $\Delta \equiv -\partial^2/\partial\theta^2$. From the eigenvalue equation,

$$-\frac{\partial^2\mathrm{e}^{\mathrm{i}n\theta}}{\partial\theta^2} = n^2\mathrm{e}^{\mathrm{i}n\theta} \qquad n \in \mathbb{Z}$$

we find that

$$\lambda_n = n^2 \qquad \langle\theta|n\rangle = (2\pi)^{-1/2}\mathrm{e}^{\mathrm{i}n\theta}.$$

The heat kernel is

$$\begin{aligned}h(\theta_1, \theta_2; t) &= \sum \mathrm{e}^{-n^2t}\langle\theta_1|n\rangle\langle n|\theta_2\rangle \\ &= \frac{1}{2\pi}\Big(1 + \sum{}'\mathrm{e}^{-n^2t}\mathrm{e}^{\mathrm{i}n(\theta_1-\theta_2)}\Big) \qquad (12.114)\end{aligned}$$

while the generalised ζ-function is

$$\zeta(\theta_1, \theta_2; s) = \sum{}' n^{-2s} \langle \theta_1 | n \rangle \langle n | \theta_2 \rangle$$
$$= \frac{1}{2\pi} \sum{}' n^{-2s} \mathrm{e}^{\mathrm{i}n(\theta_1 - \theta_2)}. \tag{12.115}$$

We easily verify that $h(\theta, \theta; t) = 1 + \Sigma' \mathrm{e}^{-n^2 t}$ satisfies

$$1 + 2\int_1^\infty \mathrm{e}^{-x^2 t}\, \mathrm{d}x < h(\theta, \theta; t) < 1 + 2\int_0^\infty \mathrm{e}^{-x^2 t}\, \mathrm{d}x.$$

We now find that

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2 t}\, \mathrm{d}x - 1 < h(\theta, \theta; t) < \int_{-\infty}^{+\infty} \mathrm{e}^{-x^2 t}\, \mathrm{d}x + 1$$

or by putting the value

$$\int \mathrm{e}^{-x^2 t}\, \mathrm{d}x = \sqrt{\pi}\, t^{-1/2}$$

we have

$$\sqrt{\pi}\, t^{-1/2} - 1 < h(\theta, \theta; t) < \sqrt{\pi}\, t^{-1/2} + 1.$$

This shows that

$$\lim_{t \to 0^+} h(\theta, \theta; t) \sim \sqrt{\pi}\, t^{-1/2}. \tag{12.116}$$

In general, the asymptotic series starts with $t^{-\dim M/2}$.

12.8 The Atiyah–Patodi–Singer index theorem

So far we have been concerned with index theorems defined on a compact manifold *without a boundary*. In practical situations in physics, we often need to find an index of an operator defined over a base space *M with a boundary*. The extensions of the AS index theorem to these cases are discussed here. Our argument is restricted to the spin bundle over M since this is the only situation we shall be concerned with in Chapter 13.

12.8.1 η-invariant and spectral flow

Let $\mathrm{i}\nabla\!\!\!/$ be a Hermitian Dirac operator defined on an odd-dimensional manifold M, $\dim M = 2l + 1$. Since $\mathrm{i}\nabla\!\!\!/$ is Hermitian, the eigenvalues λ_k are real. We define the **η-invariant** of $\mathrm{i}\nabla\!\!\!/$ by the spectral asymmetry of $\mathrm{i}\nabla\!\!\!/$,

$$\eta \equiv \sum_{\lambda_k > 0} 1 - \sum_{\lambda_k < 0} 1. \tag{12.117}$$

This is not well defined and requires a proper regularisation. For example, we may define η by $\lim_{s\to 0}\eta(s)$ where

$$\eta(s) \equiv \sum_k{}' \operatorname{sgn}(\lambda_k)|\lambda_k|^{-2s} \qquad \operatorname{Re} s > 0. \tag{12.118}$$

It can be shown that, under proper boundary conditions, $\eta(s)$ has no pole at $s = 0$.

Exercise 12.17 Use the Mellin transformation

$$\frac{1}{2}\Gamma\left(\frac{s+1}{2}\right)a^{-(s+1)/2} = \int_0^\infty \mathrm{d}x\, x^s \mathrm{e}^{-ax^2} \qquad a > 0$$

to verify that

$$\eta(s) = \frac{2}{\Gamma(\frac{1}{2}(s+1))}\int_0^\infty \mathrm{d}x\, x^s \operatorname{tr} \mathrm{i}\nabla\!\!\!\!/\, \mathrm{e}^{-x^2(\mathrm{i}\nabla\!\!\!\!/)^2}. \tag{12.119}$$

Suppose a Dirac field is interacting with an external gauge potential $\mathcal{A}_t$, $t \in [0, 1]$. The Dirac operator $\mathrm{i}\nabla\!\!\!\!/(\mathcal{A}_t)$ has a t-dependent eigenvalue problem. If an eigenvalue of $\mathrm{i}\nabla\!\!\!\!/(\mathcal{A}_t)$ crosses zero, the η-invariant jumps by ± 2. This jump denotes the **spectral flow** from $\lambda \gtreqless 0$ modes to $\lambda \lesseqgtr 0$ modes; if η jumps by $+2$ (-2), there is a flow of a state from $\lambda < 0$ to $\lambda > 0$ ($\lambda > 0$ to $\lambda < 0$), see figure 12.1. In addition to the discontinuous change associated with the spectral flow, η also has a continuous variation η_c. We have

$$\eta(t=1) - \eta(t=0) = \int_0^1 \mathrm{d}t\, \frac{\mathrm{d}\eta_c}{\mathrm{d}t} + 2 \times (\text{spectral flow}). \tag{12.120}$$

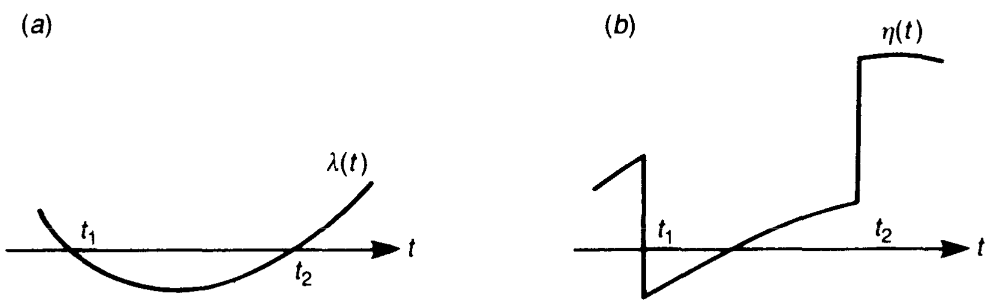


Figure 12.1 Whenever an eigenvalue λ crosses zero (a), the η-invariant jumps by ± 2. The sign depends on the way in which λ crosses zero.

12.8.2 The Atiyah–Patodi–Singer index theorem

Let us consider a $(2l + 2)$-dimensional Dirac operator

$$\mathrm{i}\hat{D}_{2l+2} = \mathrm{i}\sigma_1\frac{\partial}{\partial t} + \sigma_2 \otimes \mathrm{i}\nabla\!\!\!\!/(\mathcal{A}_t) = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} \tag{12.121a}$$

where

$$D = \mathrm{i}\partial_t - \nabla\!\!\!/(\mathcal{A}_t) \qquad D^\dagger = \mathrm{i}\partial_t + \nabla\!\!\!/(\mathcal{A}_t). \tag{12.121b}$$

[*Remark:* The positions of D and $D^\dagger$ are reversed since

$$\gamma^{2l+3} = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

for our choice of γ-matrices; cf (12.79).]

Theorem 12.18 **(Atiyah–Patodi–Singer)** Let M be an odd-dimensional manifold and $\mathrm{i}\nabla\!\!\!/(\mathcal{A}_t)$ a Dirac operator on M interacting with an external gauge field $\mathcal{A}_t$. Then

$$\begin{aligned}\operatorname{ind} D &= \dim\ker D - \dim\ker D^\dagger \\ &= \int_{M\times I} \hat{A}(\mathcal{R})\,\mathrm{ch}(\mathcal{F})|_{\mathrm{vol}} - \tfrac{1}{2}[\eta(\mathrm{i}\nabla\!\!\!/(\mathcal{A}_1)) - \eta(\mathrm{i}\nabla\!\!\!/(\mathcal{A}_0))]. \end{aligned} \tag{12.122}$$

The general argument shows that the continuous part η_{c} of the η-invariant satisfies

$$\int_0^1 \mathrm{d}t\,\frac{\mathrm{d}\eta_{\mathrm{c}}}{\mathrm{d}t} = 2\int_{M\times I} \hat{A}(\mathcal{R})\,\mathrm{ch}(\mathcal{F})|_{\mathrm{vol}}. \tag{12.123}$$

Then the RHS of (12.122) is simply the spectral flow

$$-\frac{1}{2}\,[\eta(t=1) - \eta(t=0)] + \frac{1}{2}\int_0^1 \mathrm{d}t\,\frac{\mathrm{d}\eta_{\mathrm{c}}}{\mathrm{d}t} = -\text{ spectral flow}.$$

Thus we find another expression for the APS index theorem,

$$\operatorname{ind}\mathrm{i}\hat{D}_{2l+2} = -\text{ spectral flow}. \tag{12.124}$$

The proof of the APS index theorem in its most general form is found in Atiyah *et al* (1975a,b, 1976). The physicists' proof is found in Alvarez-Gaumé *et al* (1985). We use the APS index theorem to study the odd-dimensional parity anomaly in §13.6.

Example 12.19 To see why the spectral flow appears in the index theorem, we consider an example taken from Atiyah (1985). Let $M = S^1$ and θ be its coordinate. Consider a Hermitian operator

$$\mathrm{i}\nabla_t \equiv \mathrm{i}\left(\frac{\partial}{\partial\theta} - \mathrm{i}t\right) = \mathrm{i}\partial_\theta + t \qquad t \in \mathbb{R}. \tag{12.125}$$

The term $-\mathrm{i}t$ is thought of as a U(1) gauge potential. The eigenvector and the eigenvalue of $\mathrm{i}\nabla_t$ are

$$\psi_{n,t}(\theta) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\mathrm{i}n\theta}(n \in \mathbb{Z}),\ \lambda_n(t) = n + t.$$

Since $\operatorname{Spec}\mathrm{i}\nabla_t = \operatorname{Spec}\mathrm{i}\nabla_{t+1}$, the family of operators $\mathrm{i}\nabla_t$ is periodic in t with the period 1; see figure 12.2. This periodicity manifests itself in the gauge equivalence of $\mathrm{i}\nabla_t$ and $\mathrm{i}\nabla_{t+1}$:

$$\mathrm{i}\nabla_{t+1} = \mathrm{e}^{\mathrm{i}\theta}\mathrm{i}\nabla_t \mathrm{e}^{-\mathrm{i}\theta}.$$

There is precisely unit spectral flow from $\lambda < 0$ to $\lambda > 0$ at $t = 0$ while t changes from $-\varepsilon$ to $1 - \varepsilon$, ε being a small positive number. From $\mathrm{i}\nabla_t$, we construct a two-dimensional Dirac operator

$$\mathrm{i}\not{D}_2 \equiv \mathrm{i}\sigma_1 \otimes \frac{\partial}{\partial t} + \sigma_2 \otimes \mathrm{i}\nabla_t = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} \tag{12.126a}$$

where

$$D \equiv \mathrm{i}\partial_t + \partial_\theta - \mathrm{i}t \qquad D^\dagger \equiv \mathrm{i}\partial_t - \partial_\theta + \mathrm{i}t. \tag{12.126b}$$

These operators act on functions which satisfy the boundary conditions

$$\phi(\theta + 2\pi, t) = \phi(\theta, t) \qquad \phi(\theta, t + 1) = \mathrm{e}^{\mathrm{i}\theta}\phi(\theta, t). \tag{12.127}$$

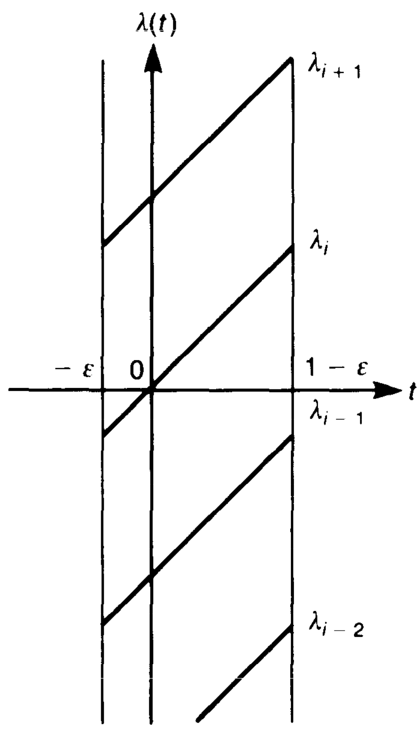


Figure 12.2 Time evolution of the eigenvalues of $\mathrm{i}\nabla_t$. $\mathrm{Spec}\,\mathrm{i}\nabla_t$ has period 1. The ith eigenvalue crosses zero at $t = 0$, and hence there is a unit spectral flow.

Let $\phi_0 \in \ker D^\dagger$. We have a Fourier expansion

$$\phi_0(\theta, t) = \sum a_n(t)\mathrm{e}^{-\mathrm{i}n\theta}.$$

From $D^\dagger\phi_0 = 0$, we have

$$a_n'(t) + (n + t)a_n(t) = 0$$

which is easily solved to yield

$$a_n(t) = c_n \exp\left(-\frac{(n+t)^2}{2}\right).$$

The boundary conditions (12.127) require that

$$\sum_n c_n \exp\left(-\frac{(n+t+1)^2}{2}\right)e^{-in\theta} = \sum_n c_n \exp\left(-\frac{(n+t)^2}{2}\right)e^{-i(n-1)\theta}$$

from which we find that c_n is independent of n. Thus $\ker D^\dagger$ is one dimensional and is spanned by the theta function,

$$\phi_0(\theta, t) = \sum \exp\left(-\frac{(n+t)^2}{2} - in\theta\right). \tag{12.128}$$

Suppose $\tilde{\phi}_0(\theta, t) \in \ker D$. If we put $\tilde{\phi}_0(\theta, t) = \Sigma\, b_n(t)e^{-in\theta}$, $b_n(t)$ satisfies

$$b_n'(t) - (n+t)b_n(t) = 0.$$

The solution of this equation is

$$b_n(t) = b_n(0) \exp\frac{(n+t)^2}{2}$$

and hence $\tilde{\phi}_0$ cannot be normalised. This shows that

$$\operatorname{ind} D = \dim\ker D - \dim\ker D^\dagger = -1$$

which agrees with − (spectral flow).

Problems 12

1 In the text, we dealt only with compact manifolds. The extension of the AS index theorem to non-compact manifolds is the **Callias–Bott–Seely index theorem** (Callias 1977, Bott and Seely 1977). Here we consider the simplest case studied by Hirayama (1983). Consider a pair of operators

$$L \equiv \frac{1}{i}\frac{d}{dx} - iW(x), \quad L^\dagger \equiv \frac{1}{i}\frac{d}{dx} + iW(x)$$

where $W(+\infty) = \mu$ and $W(-\infty) = \lambda$.

(*a*) Show that $\mathrm{Spec}' L^\dagger L = \mathrm{Spec}' LL^\dagger$, where the prime indicates that the zero eigenvalues are omitted.

(*b*) Show that

$$J(z) \equiv \mathrm{tr}\left(\frac{z}{L^\dagger L + z} - \frac{z}{LL^\dagger + z}\right) = \frac{1}{2}\left(\frac{\mu}{(\mu^2+z)^{1/2}} - \frac{\lambda}{(\lambda^2+z)^{1/2}}\right).$$

13

ANOMALIES IN GAUGE FIELD THEORIES

In particle physics, symmetry principles are some of the most important concepts in model building. Symmetries play crucial roles for the theory to be renormalisable and unitary. The Lagrangian must be chosen so that it fulfils the observed symmetry. Note, however, that the symmetry of the Lagrangian is *classical*. There is no warranty that symmetry of the Lagrangian may be elevated to a *quantum* symmetry, that is, the symmetry of the effective action. If the classical symmetry of the Lagrangian cannot be maintained in the process of quantisation, the theory is said to have an **anomaly**. There are many types of anomalies: the chiral anomaly, gauge anomaly, gravitational anomaly, supersymmetry anomaly and so on. Each adjective refers to the symmetry under consideration. In the present chapter we look at the geometrical and topological structures of the anomalies appearing in gauge theories.

We follow closely Alvarez-Gaumé (1985), Alvarez-Gaumé and Ginsparg (1985) and Sumitani (1985).

13.1 Introduction

Before we introduce topological and geometrical methods to anomalies, we give a brief survey of the subject here. Let ψ be a massless Dirac field in four-dimensional space interacting with an external gauge field $\mathcal{A}_\mu = A_\mu{}^\alpha T_\alpha$. $\{T_\alpha\}$ is the set of anti-Hermitian generators of the gauge group G which is compact and semisimple ($\mathrm{SU}(N)$, for example). The theory is described by the Lagrangian

$$\mathcal{L} = \mathrm{i}\bar{\psi}\gamma^\mu(\partial_\mu - \mathcal{A}_\mu)\psi. \tag{13.1}$$

The Lagrangian is invariant under the usual (local) gauge transformation

$$\psi(x) \to g^{-1}\psi(x) \qquad \mathcal{A}_\mu(x) \to g^{-1}[\mathcal{A}_\mu(x) + \partial_\mu]g. \tag{13.2}$$

It also has a *global* symmetry,

$$\psi(x) \to \mathrm{e}^{\mathrm{i}\gamma_5\alpha}\psi(x) \qquad \bar{\psi}(x) \to \bar{\psi}(x)\mathrm{e}^{\mathrm{i}\gamma_5\alpha} \tag{13.3}$$

called the **chiral symmetry**. The chiral current j_5 derived from this symmetry is

$$j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma_5\psi. \tag{13.4}$$

In general, whether the symmetry of a Lagrangian is retained under quantisation is not a trivial question. In fact, it has been shown that the chiral symmetry of $\mathcal{L}$ is destroyed at the quantum level. Adler (1969) and Bell and Jackiw (1969) have shown by computing the triangle diagram with an external axial current and two external vector currents that the naïve conservation law $\partial_\mu j_5^\mu = 0$ is violated,

$$\begin{aligned}\partial_\mu j_5^\mu &= \frac{1}{16\pi^2}\varepsilon^{\kappa\lambda\mu\nu}\operatorname{tr}\mathcal{F}_{\kappa\lambda}\mathcal{F}_{\mu\nu}\\ &= \frac{1}{4\pi^2}\operatorname{tr}[\varepsilon^{\kappa\lambda\mu\nu}\partial_\kappa(\mathcal{A}_\lambda\partial_\mu\mathcal{A}_\nu + \tfrac{2}{3}\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu)]\end{aligned} \tag{13.5}$$

where tr is a trace over the group indices. The current j_5^μ which appears in (13.5) has no group index, and hence (13.5) is called the **Abelian anomaly**.

It is interesting to study the behaviour of a current which carries the group index. Consider a Weyl fermion ψ which couples with an external gauge field. The non-Abelian gauge current of the theory also satisfies an anomalous conservation law which defines the **non-Abelian anomaly**. The action is given by

$$\mathcal{L} \equiv \psi^\dagger(\mathrm{i}\not{\nabla})\mathcal{P}_+\psi \qquad \mathcal{P}_\pm = \tfrac{1}{2}(1 \pm \gamma^5). \tag{13.6}$$

The Lagrangian has the gauge symmetry

$$\mathcal{A}_\mu \to g^{-1}(\mathcal{A}_\mu + \partial_\mu)g, \quad \psi \to g^{-1}\psi. \tag{13.7}$$

The corresponding non-Abelian current is

$$j^{\mu\alpha} \equiv \psi^\dagger\gamma^\mu T^\alpha\mathcal{P}_+\psi. \tag{13.8}$$

It has been shown by Bardeen (1969) and Gross and Jackiw (1972) that, up to the one-loop level, the current is not conserved,

$$(\mathcal{D}_\mu j_5^\mu)^\alpha = \frac{1}{24\pi^2}\operatorname{tr}[T^\alpha\partial_\kappa\varepsilon^{\kappa\lambda\mu\nu}(\mathcal{A}_\lambda\partial_\mu\mathcal{A}_\nu + \tfrac{1}{2}\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu)]. \tag{13.9}$$

At first sight, the RHS of (13.5) and (13.9) look very similar. However, the difference between the normalisation and the numerical factors of $\frac{2}{3}$ and $\frac{1}{2}$ have a deep topological origin. We shall see later that the Abelian anomaly in $(2l + 2)$ dimensions and the non-Abelian anomaly in $2l$ dimensions are closely related but in an unexpected manner.

13.2 Abelian anomalies

Henceforth we work in an even-dimensional manifold M ($\dim M = m = 2l$) with a Euclidean signature. Four-dimensional results will readily be obtained by putting $m = 4$. We assume our system is non-chiral, namely, the gauge field couples to the right and the left components in the same way. Our convention is

$$\gamma^{\mu\dagger} = \gamma^\mu, \quad \{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}, \quad \gamma^{m+1} = (\mathrm{i})^l\gamma^1 \ldots \gamma^m$$
$$\gamma^{m+1\dagger} = \gamma^{m+1}, \quad (\gamma^{m+1})^2 = +1.$$

The Lie group generators $\{T_\alpha\}$ satisfy

$$T^\dagger_\alpha = -T_\alpha, \quad [T_\alpha, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma, \quad \mathrm{tr}(T^\alpha T^\beta) = -\tfrac{1}{2}\delta^{\alpha\beta}.$$

13.2.1 Fujikawa's method

Among several methods of deriving anomalies, Fujikawa's way (Fujikawa 1979, 1980, 1986) reveals the topological and geometrical nature of the problem most directly. This method is equivalent to the heat kernel proof of the relevant index theorem.

Let ψ be a massless Dirac field interaction with an *external* non-Abelian gauge field $\mathcal{A}_\mu$. The effective action $W[\mathcal{A}]$ is given by

$$\mathrm{e}^{-W[\mathcal{A}]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathrm{e}^{-\int \mathrm{d}x\,\bar{\psi}\mathrm{i}\nabla\!\!\!/\psi} \tag{13.10}$$

where $\mathrm{i}\nabla\!\!\!/ = \mathrm{i}\gamma^\mu\nabla_\mu = \mathrm{i}\gamma^\mu(\partial_\mu + \omega_\mu + \mathcal{A}_\mu)$, with $\omega_\mu = \frac{1}{2}\omega_{\mu\alpha\beta}\Sigma^{\alpha\beta}$ being the spin connection of the background space. We compactify the space in such a way that the geometry (the spin connection) plays no role. For example, this can be achieved by compactifying $\mathbb{R}^4$ to $S^4 = \mathbb{R}^4 \cup \{\infty\}$, for which the Dirac genus $\hat{A}(TM)$ is trivial; see example 12.11. If this is the case, the spin connection is irrelevant and may be dropped from $\mathrm{i}\nabla\!\!\!/$. The classical action $\int \mathrm{d}x\,\bar{\psi}\mathrm{i}\nabla\!\!\!/\psi$ is invariant with respect to the chiral rotation,

$$\psi \to \mathrm{e}^{\mathrm{i}\gamma^{m+1}\alpha}\psi, \quad \bar{\psi} \to \bar{\psi}\mathrm{e}^{\mathrm{i}\gamma^{m+1}\alpha}. \tag{13.11}$$

We expand ψ and $\bar{\psi}$ as

$$\psi = \sum_i a_i\psi_i, \quad \bar{\psi} = \sum_i \bar{b}_i\psi^\dagger_i \tag{13.12}$$

where a_i and $\bar{b}_i$ are anticommuting Grassmann variables,

$$\{a_i, a_j\} = 0, \quad \{\bar{b}_i, \bar{b}_j\} = 0, \quad \{a_i, \bar{b}_j\} = 0$$

and ψ_i is an eigenvector of the Dirac operator

$$\mathrm{i}\nabla\!\!\!/\psi_i = \lambda_i\psi_i. \tag{13.13}$$

Since $\mathrm{i}\nabla\!\!\!/$ is Hermitian, λ_i is real. Since M is compact, ψ_i can be normalised as

$$\langle\psi_i|\psi_j\rangle = \int \mathrm{d}x\,\psi^\dagger_i(x)\psi_j(x) = \delta_{ij}.$$

Now the path integrals over ψ and $\bar{\psi}$ are replaced by those over a_i and $\bar{b}_i$.

Consider an infinitesimal chiral transformation,

$$\psi(x) \to \psi(x) + \mathrm{i}\alpha(x)\gamma^{m+1}\psi(x) \tag{13.14a}$$

$$\bar{\psi}(x) \to \bar{\psi}(x) + \mathrm{i}\bar{\psi}(x)\alpha(x)\gamma^{m+1}. \tag{13.14b}$$

As usual, we take $\alpha = \alpha(x)$ to be x-dependent. Under this change, the classical action transforms as

$$\begin{aligned}
\int \mathrm{d}x\bar{\psi}\mathrm{i}\nabla\!\!\!/\,\psi \to & \int \mathrm{d}x(\bar{\psi} + \mathrm{i}\bar{\psi}\alpha\gamma^{m+1})\mathrm{i}\nabla\!\!\!/\,(\psi + \mathrm{i}\alpha\gamma^{m+1}\psi) \\
&= \int \mathrm{d}x\bar{\psi}\mathrm{i}\nabla\!\!\!/\,\psi + \mathrm{i}\int \mathrm{d}x[\alpha\bar{\psi}\gamma^{m+1}\mathrm{i}\nabla\!\!\!/\,\psi + \bar{\psi}\mathrm{i}\nabla\!\!\!/\,(\alpha\gamma^{m+1}\psi)] \\
&= \int \mathrm{d}x\bar{\psi}\mathrm{i}\nabla\!\!\!/\,\psi - \int \mathrm{d}x[\alpha\bar{\psi}\gamma^{m+1}\gamma^{\mu}(\partial_\mu + \mathcal{A}_\mu)\psi \\
&\quad + \bar{\psi}\gamma^{\mu}(\partial_\mu + \mathcal{A}_\mu)(\alpha\gamma^{m+1}\psi)] \\
&= \int \mathrm{d}x\bar{\psi}\mathrm{i}\nabla\!\!\!/\,\psi + \int \mathrm{d}x\alpha(x)\partial_\mu j^{\mu}_{m+1}(x)
\end{aligned} \tag{13.15}$$

where we have used the anticommutation relations $\{\gamma^\mu, \gamma^{m+1}\} = 0$ and

$$j^{\mu}_{m+1}(x) \equiv \bar{\psi}(x)\gamma^{\mu}\gamma^{m+1}\psi(x) \tag{13.16}$$

is the **chiral current**. This is the higher-dimensional analogue of j^{μ}_5 defined previously. If (13.15) were the only change caused by (13.14), naïve application of the Ward–Takahashi relation would imply the conservation of the axial current $\partial_\mu j^{\mu}_{m+1} = 0$. In quantum theory, however, we have an additional change, namely the change of the *measure*. Define the chiral-rotated fields by

$$\psi' = \psi + \mathrm{i}\alpha\gamma^{m+1}\psi = \sum a_i'\psi_i \tag{13.17a}$$

$$\bar{\psi}' = \bar{\psi} + \mathrm{i}\bar{\psi}\alpha\gamma^{m+1} = \sum \bar{b}_i'\psi_i^{\dagger}. \tag{13.17b}$$

Now the measure changes as

$$\int \prod_i \mathrm{d}a_i\,\mathrm{d}\bar{b}_i \to \int \prod_i \mathrm{d}a_i'\,\mathrm{d}\bar{b}_i'. \tag{13.18}$$

From the orthonormality of $\{\psi_i\}$, we find

$$\begin{aligned}
a_i' &= \langle\psi_i|\psi'\rangle = \langle\psi_i|(1 + \mathrm{i}\alpha\gamma^{m+1})\psi\rangle \\
&= \sum_j \langle\psi_i|(1 + \mathrm{i}\alpha\gamma^{m+1})\psi_j\rangle a_j \equiv \sum_j C_{ij}a_j
\end{aligned} \tag{13.19a}$$

where

$$C_{ij} = \langle \psi_i | (1 + \mathrm{i}\alpha\gamma^{m+1})\psi_j \rangle = \delta_{ij} + \mathrm{i}\alpha\langle \psi_i | \gamma^{m+1}\psi_j \rangle. \tag{13.20}$$

The measure in terms of the new variables is

$$\begin{aligned}\prod \mathrm{d}a_i' &= [\det C_{ij}]^{-1} \prod \mathrm{d}a_i = \exp(-\operatorname{tr}\ln C_{ij}) \prod \mathrm{d}a_i \\ &= \exp[-\operatorname{tr}\ln(1 + \mathrm{i}\alpha\langle \psi_i | \gamma^{m+1}\psi_j \rangle)] \prod \mathrm{d}a_i \\ &\approx \exp(-\operatorname{tr}\mathrm{i}\alpha\langle \psi_i | \gamma^{m+1}\psi_j \rangle) \prod \mathrm{d}a_i \\ &= \exp\left(-\mathrm{i}\alpha \sum_i \langle \psi_i | \gamma^{m+1}\psi_i \rangle\right) \prod \mathrm{d}a_i \end{aligned} \tag{13.21}$$

where the *inverse* of the determinant appears since a_i and a_i' are Grassmann variables, see Berezin (1966). [For example, we have $\int a\,\mathrm{d}a = \int ca\,\mathrm{d}(ca) = 1$, $c \in \mathbb{R}$ and a being a real Grassmann number. This shows that $\mathrm{d}(ca) = \mathrm{d}a/c$.] As for $\bar{b}_i \to \bar{b}_i'$, we have

$$\bar{b}_i' = \sum_j \bar{b}_j \langle \psi_j | (1 + \mathrm{i}\alpha\gamma^{m+1}) | \psi_i \rangle = \sum_j C_{ji}\bar{b}_j. \tag{13.19b}$$

The Jacobian for the change $\bar{b}_i \to \bar{b}_i'$ agrees with (13.21). Thus the measure transforms under the chiral rotation (13.17) as

$$\prod_i \mathrm{d}a_i\,\mathrm{d}\bar{b}_i \to \prod_i \mathrm{d}a_i'\,\mathrm{d}\bar{b}_i' \exp\left(-2\mathrm{i}\int \mathrm{d}x\,\alpha(x) \sum \psi_n^\dagger(x)\gamma^{m+1}\psi_n(x)\right). \tag{13.22}$$

Now the effective action has two expressions,

$$\begin{aligned} \mathrm{e}^{-W[A]} &= \int \prod_i \mathrm{d}a_i\,\mathrm{d}\bar{b}_i \exp\left(-\int \mathrm{d}x\,\bar{\psi}\mathrm{i}\nabla\!\!\!\!/\,\psi\right) \\ &= \int \prod_i \mathrm{d}a_i'\,\mathrm{d}\bar{b}_i' \\ &\quad \times \exp\left(-\int \mathrm{d}x\,\bar{\psi}\mathrm{i}\nabla\!\!\!\!/\,\psi - \int \mathrm{d}x\,\alpha(x)\partial_\mu j^\mu_{m+1}(x) - 2\mathrm{i}\int \mathrm{d}x\,\alpha(x)\boldsymbol{A}(x)\right) \end{aligned} \tag{13.23}$$

where

$$\boldsymbol{A}(x) \equiv \sum_i \psi_i^\dagger(x)\gamma^{m+1}\psi_i(x). \tag{13.24}$$

Since $\alpha(x)$ is arbitrary, we have

$$\partial_\mu j^\mu_{m+1}(x) = -2\mathrm{i}\boldsymbol{A}(x). \tag{13.25}$$

Thus naïve conservation of an axial current does not hold in quantum theory. This non-conservation of the current j^μ_{m+1} is called the **Abelian anomaly** (or **chiral anomaly** or **axial anomaly**).

How is this related to the topology? Let us look at the Jacobian (13.22) and assume that $\alpha(x)$ is independent of x. [We are looking at the zero-momentum Ward–Takahashi relation.] The integral in (13.22) is not well defined and must be regularised. We introduce the Gaussian cut-off (**heat kernel regularisation**) as

$$\int \mathrm{d}x A(x) = \int \mathrm{d}x \sum_i \psi_i^\dagger(x)\gamma^{m+1}\psi_i(x)\exp[-(\lambda_i/M)^2]|_{M\to\infty}$$

$$= \sum \langle\psi_i|\gamma^{m+1}\exp[-(\mathrm{i}\nabla\!\!\!\!/\,/M)^2]|\psi_i\rangle|_{M\to\infty}. \tag{13.26}$$

In (13.26), $1/M^2$ corresponds to the 'time' parameter t in the previous chapter and $M\to\infty$ implies $t\to\varepsilon$. Let $|\psi_i\rangle$ be an eigenstate of $\mathrm{i}\nabla\!\!\!\!/$ with *non-vanishing* eigenvalue λ_i. Among the eigenstates, there exists a state $|\psi_i\rangle^\chi \equiv \gamma^{m+1}|\psi_i\rangle$ with eigenvalue $-\lambda_i$:

$$\begin{aligned}\mathrm{i}\nabla\!\!\!\!/|\psi_i\rangle^\chi &= \mathrm{i}\nabla\!\!\!\!/\gamma^{m+1}|\psi_i\rangle = -\gamma^{m+1}\mathrm{i}\nabla\!\!\!\!/|\psi_i\rangle\\ &= -\lambda_i\gamma^{m+1}|\psi_i\rangle = -\lambda_i|\psi_i\rangle^\chi\end{aligned}$$

where use has been made of the anticommutation relation $\{\gamma^{m+1},\mathrm{i}\nabla\!\!\!\!/\}=0$. Since $\mathrm{i}\nabla\!\!\!\!/$ is a Hermitian operator, eigenvectors which belong to different eigenvalues are orthogonal, hence $\langle\psi_i|\psi_i\rangle^\chi = \langle\psi_i|\gamma^{m+1}|\psi_i\rangle = 0$. This shows that

$$\langle\psi_i|\gamma^{m+1}\exp[-(\mathrm{i}\nabla\!\!\!\!/\,/M)^2]|\psi_i\rangle = \langle\psi_i|\gamma^{m+1}|\psi_i\rangle\exp[-(\lambda_i/M)^2] = 0.$$

Thus the contribution to the RHS of (13.26) comes only from the zero-energy modes. Let $|0,i\rangle$ be the zero-energy modes of $\mathrm{i}\nabla\!\!\!\!/$, $1\le i\le n_0$. They are not in an irreducible representation of the spin algebra and should be classified according to the eigenvalue of γ^{m+1}. We write

$$\gamma^{m+1}|0,i\rangle_\pm = \pm|0,i\rangle_\pm. \tag{13.27}$$

Then (13.26) becomes

$$\begin{aligned}\int \mathrm{d}x A(x) &= \sum\langle\psi_i|\gamma^{m+1}\exp[-(\mathrm{i}\nabla\!\!\!\!/\,/M)^2]|\psi_i\rangle|_{M\to\infty}\\ &= \sum_i {}_+\langle 0,i|0,i\rangle_+ - \sum_i {}_-\langle 0,i|0,i\rangle_-\\ &= \nu_+ - \nu_- = \operatorname{ind}\mathrm{i}\nabla\!\!\!\!/_+\end{aligned} \tag{13.28}$$

where ν_+ (ν_-) is the number of zero-energy modes with positive (negative) chirality $(\nu_+ + \nu_- = n_0)$ and $\mathrm{i}\nabla\!\!\!\!/_+$ is defined by

$$\mathrm{i}\nabla\!\!\!\!/ = \begin{pmatrix}0 & \mathrm{i}\nabla\!\!\!\!/_-\\ \mathrm{i}\nabla\!\!\!\!/_+ & 0\end{pmatrix},\quad \mathrm{i}\nabla\!\!\!\!/_- = (\mathrm{i}\nabla\!\!\!\!/_+)^\dagger.$$

The Atiyah–Singer index theorem now comes into the problem.

To show that (13.28) indeed represents an integral of the relevant Chern character, we first note that

$$(\mathrm{i}\nabla\!\!\!/)^2 = -\gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu = -\{\delta^{\mu\nu} + \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]\}\tfrac{1}{2}[\{\nabla_\mu, \nabla_\nu\} + \mathcal{F}_{\mu\nu}]$$
$$= -\nabla_\mu\nabla^\mu - \tfrac{1}{4}[\gamma^\mu, \gamma^\nu]\mathcal{F}_{\mu\nu} \tag{13.29}$$

where use has been made of the relation $[\nabla_\mu, \nabla_\nu] = \mathcal{F}_{\mu\nu}$. Then

$$A(x) = \sum_i \langle\psi_i|x\rangle\langle x|\gamma^{m+1}\exp[(\nabla^2 + \tfrac{1}{4}[\gamma^\mu, \gamma^\nu]\mathcal{F}_{\mu\nu})/M^2]|\psi_i\rangle|_{M\to\infty}. \tag{13.30}$$

Let us take $m = 4$ for definiteness. We introduce the plane wave basis as

$$\langle x|\psi_i\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\langle x|k\rangle\langle k|\psi_i\rangle.$$

Then (13.30) becomes

$$A(x) = \int\frac{\mathrm{d}k}{(2\pi)^4}\int\frac{\mathrm{d}k'}{(2\pi)^4}\sum_i\langle\psi_i|k'\rangle\langle k'|x\rangle$$
$$\times\, \gamma^{m+1}\exp[(\nabla^2 + \tfrac{1}{4}[\gamma^\mu, \gamma^\nu]\mathcal{F}_{\mu\nu})/M^2]\langle x|k\rangle\langle k|\psi_i\rangle\Big|_{\substack{M\to\infty\\ y\to x}}$$
$$= \int\frac{\mathrm{d}k}{(2\pi)^4}\,\mathrm{tr}\,\gamma^{m+1}\exp[(-k^2 + \tfrac{1}{4}[\gamma^\mu, \gamma^\nu]\mathcal{F}_{\mu\nu})/M^2]_{M\to\infty} \tag{13.31}$$

where use has been made of the completeness property

$$\sum_i\langle k|\psi_i\rangle\langle\psi_i|k'\rangle = (2\pi)^4\delta^4(k - k').$$

In (13.31), we have replaced ∇^2 by the symbol $-k^2$ since the residual terms containing $\mathcal{A}$ do not survive in the limit $M\to\infty$. If we put $\tilde{k}^\mu \equiv k^\mu/M$, (13.31) becomes

$$A(x) = \mathrm{tr}[\gamma^5\exp(\tfrac{1}{4}[\gamma^\mu, \gamma^\nu]\mathcal{F}_{\mu\nu}/M^2)]M^4\int\frac{\mathrm{d}\tilde{k}}{(2\pi)^4}\exp(-\tilde{k}^2).$$

We expand the first exponential and use

$$\mathrm{tr}\,\gamma^5 = \mathrm{tr}\,\gamma^5\gamma^\mu\gamma^\nu = 0,\quad \mathrm{tr}\,\gamma^5\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = -4\varepsilon^{\kappa\lambda\mu\nu},\ \int\mathrm{d}\tilde{k}\exp(-\tilde{k}^2) = \pi^2$$

to obtain

$$A(x) = \frac{1}{2}\,\mathrm{tr}\left[\gamma^5\,\frac{1}{4^2}\,\{[\gamma^\mu, \gamma^\nu]\mathcal{F}_{\mu\nu}\}^2\right]\frac{1}{16\pi^2}$$
$$= \frac{-1}{32\pi^2}\,\mathrm{tr}\,\varepsilon^{\kappa\lambda\mu\nu}\mathcal{F}_{\kappa\lambda}(x)\mathcal{F}_{\mu\nu}(x). \tag{13.32}$$

Note that the higher-order terms in the expansion of the exponential vanish in the limit $M \to \infty$. The anomalous conservation law (13.25) now becomes

$$\begin{aligned}\partial_\mu j^\mu_{m+1} &= \frac{1}{16\pi^2}\operatorname{tr}\varepsilon^{\kappa\lambda\mu\nu}\mathcal{F}_{\kappa\lambda}\mathcal{F}_{\mu\nu} \\ &= \frac{1}{4\pi^2}\operatorname{tr}[\varepsilon^{\kappa\lambda\mu\nu}\partial_\kappa(\mathcal{A}_\lambda\partial_\mu\mathcal{A}_\nu + \tfrac{2}{3}\mathcal{A}_\lambda\mathcal{A}_\mu\mathcal{A}_\nu)]. \end{aligned} \tag{13.33}$$

This is regarded as a local version of the AS index theorem. Let us write (13.33) in terms of the field strength $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. We easily verify that

$$\nu_+ - \nu_- = \int_M \mathrm{d}x\partial_\mu j^\mu_{m+1} = \int_M \mathrm{ch}_2(\mathcal{F}). \tag{13.34}$$

This is the index theorem for a twisted spinor complex with trivial background geometry ($\hat{A}(TM) = 1$).

If $\dim M = m = 2l$, we have the following identity,

$$\nu_+ - \nu_- = \int_M \mathrm{d}x\partial_\mu j^\mu_{m+1} = \int_M \mathrm{ch}_l(\mathcal{F}) = \int_M \frac{1}{l!}\operatorname{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^l. \tag{13.35}$$

13.3 Non-Abelian anomalies

In the last section we considered the chiral current which is a gauge singlet (no gauge indices). Now we turn to the study of the *gauge* current $j^\mu{}_\alpha$ where α is the gauge index. Now we consider a chiral theory in which the gauge field $\mathcal{A}$ couples only to the left-handed *Weyl* fermion ψ. Suppose ψ transforms in a complex representation $\boldsymbol{r}$ of the gauge group G. For example, suppose ψ belongs to a $\mathbf{3}$ of SU(3). The effective action $W_r[\mathcal{A}]$ is given by

$$\mathrm{e}^{-W_r[\mathcal{A}]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(-\int \mathrm{d}x\bar{\psi}\mathrm{i}\nabla\!\!\!\!/_+\psi\right) \tag{13.36}$$

where

$$\mathrm{i}\nabla\!\!\!\!/_+ = \mathrm{i}\gamma^\mu(\partial_\mu + \mathcal{A}_\mu)\mathcal{P}_+; \quad \mathcal{P}_\pm = \tfrac{1}{2}(1 \pm \gamma^{m+1}). \tag{13.37}$$

The gauge current is

$$j^\mu{}_\alpha = \mathrm{i}\bar{\psi}\gamma^\mu T_\alpha\mathcal{P}_+\psi. \tag{13.38}$$

Let $v = v^\alpha T_\alpha$ be an infinitesimal gauge transformation parameter, $g = 1 - v$, under which we have

$$\mathcal{A}_\mu \to (1 + v)(\mathcal{A}_\mu + \mathrm{d})(1 - v) = \mathcal{A}_\mu - \mathcal{D}_\mu v \tag{13.39}$$

where $\mathcal{D}_\mu v \equiv \partial_\mu v + [\mathcal{A}_\mu, v]$. The effective action transforms as

$$
\begin{aligned}
W_r[\mathcal{A}] \rightarrow\ & W_r[\mathcal{A} - \mathcal{D}v] \\
&= W_r[\mathcal{A}] - \int \mathrm{d}x \, \mathrm{tr}\left(\mathcal{D}v \frac{\delta}{\delta \mathcal{A}} W_r[\mathcal{A}]\right) \\
&= W_r[\mathcal{A}] - \int \mathrm{d}x \, \mathrm{tr}\,(\partial_\mu v^\alpha + f_{\alpha\beta\gamma} A_\mu{}^\beta v^\gamma) \frac{\delta}{\delta A_\mu{}^\alpha} W_r[\mathcal{A}] \\
&= W_r[\mathcal{A}] + \int \mathrm{d}x \, \mathrm{tr}\left(v^\alpha \mathcal{D} \frac{\delta}{\delta \mathcal{A}} W_r[\mathcal{A}]_\alpha\right).
\end{aligned} \tag{13.40}
$$

Since

$$
\frac{\delta}{\delta A_\mu{}^\alpha} W_r[\mathcal{A}] = \langle \mathrm{i}\bar{\psi}\gamma^\mu T_\alpha \tfrac{1}{2}(1 + \gamma^{m+1})\psi \rangle_{\mathcal{A}} = \langle j^\mu{}_\alpha \rangle
$$

we have

$$
W_r[\mathcal{A} - \mathcal{D}v] - W_r[\mathcal{A}] = \int \mathrm{d}x \, \mathrm{tr}\,(v^\alpha \mathcal{D}_\mu \langle j^\mu \rangle_\alpha). \tag{13.41}
$$

We are naïvely tempted to regard (13.36) as 'det$(\mathrm{i}\not\nabla_+) = \Pi\lambda_i$', λ_i being the 'eigenvalue' of $\mathrm{i}\not\nabla_+$. A subtlety arises here: $\mathrm{i}\not\nabla_+$ maps sections of $S_+ \otimes E$ to those of $S_- \otimes E$, where E is the vector bundle associated with the G bundle and $S_\pm$ are spin bundles with chirality $\pm$. Accordingly, the equation $\mathrm{i}\not\nabla_+\psi = \lambda\psi$ is meaningless. To avoid this difficulty, we formally introduce a *Dirac* spinor ψ and define

$$
\mathrm{e}^{-W_r[\mathcal{A}]} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int \mathrm{d}x \, \bar{\psi} \mathrm{i}\hat{D}\psi\right) \tag{13.42}
$$

where $\mathrm{i}\hat{D}$ is defined by

$$
\mathrm{i}\hat{D} \equiv \mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}\mathcal{A}_\mu \mathcal{P}_+) = \begin{pmatrix} 0 & \mathrm{i}\not\partial_- \\ \mathrm{i}\not\nabla_+ & 0 \end{pmatrix} \tag{13.43}
$$

where we have diagonalised γ^{m+1}. In (13.43) the gauge field $\mathcal{A}$ couples only to the positive chirality field. Now the eigenvalue problem $\mathrm{i}\hat{D}\psi_i = \lambda_i\psi_i$ is well defined. Note that $\mathrm{i}\hat{D}$ is not Hermitian and λ_i is a complex number in general. Moreover, we need to introduce right and left eigenfunctions separately by

$$
\mathrm{i}\hat{D}\psi_i = \lambda_i\psi_i \tag{13.44a}
$$

$$
\chi^\dagger{}_i(\overleftarrow{\mathrm{i}\hat{D}}) = \lambda_i\chi^\dagger{}_i \qquad (\mathrm{i}\hat{D})^\dagger\chi_i = \bar{\lambda}_i\chi_i. \tag{13.44b}
$$

Since $\int \chi^\dagger{}_i\psi_j \, \mathrm{d}x = 0$ for $i \neq j$, we may choose an orthonormal basis,

$$
\int \chi^\dagger{}_i\psi_j \, \mathrm{d}x = \delta_{ij}. \tag{13.45}
$$

It should be noted that the eigenvalue λ_i is *not* gauge invariant. This follows from the observation that

$$\begin{aligned} g(\mathrm{i}\hat{D}(\mathcal{A}^g))g^{-1} &= g\mathrm{i}\gamma^\mu[\partial_\mu + g^{-1}(\mathcal{A}_\mu + \partial_\mu)g\mathcal{P}_+]g^{-1} \\ &= \mathrm{i}\hat{D}(\mathcal{A}) - \mathrm{i}\not\partial g g^{-1} + \mathrm{i}\not\partial g g^{-1}\mathcal{P}_+ \neq \mathrm{i}\hat{D}(\mathcal{A}). \end{aligned} \tag{13.46}$$

[If the equality were to hold in (13.46), $g^{-1}\psi_i$ would satisfy $\mathrm{i}\hat{D}(\mathcal{A}^g)g^{-1}\psi_i = \lambda_i g^{-1}\psi_i$ when $\mathrm{i}\hat{D}(\mathcal{A})\psi_i = \lambda_i\psi_i$. Then $\mathrm{Spec}\,\mathrm{i}\hat{D}(\mathcal{A})$ would be gauge invariant.] Although individual eigenvalues are not gauge invariant, the absolute value of the product of eigenvalues of $\mathrm{i}\hat{D}$ is gauge invariant. In fact

$$\begin{aligned} \det(\mathrm{i}\hat{D})\det((\mathrm{i}\hat{D})^\dagger) &= \det(\mathrm{i}\hat{D}(\mathrm{i}\hat{D})^\dagger) \\ &= \det\begin{pmatrix} (\mathrm{i}\not\partial_-)(\mathrm{i}\not\partial_+) & 0 \\ 0 & (\mathrm{i}\not\nabla_+)(\mathrm{i}\not\nabla_-) \end{pmatrix} \\ &= \det(\mathrm{i}\not\partial_-\mathrm{i}\not\partial_+)\det(\mathrm{i}\not\nabla_+\mathrm{i}\not\nabla_-) \end{aligned} \tag{13.47}$$

where $\mathrm{i}\not\partial_+ = (\mathrm{i}\not\partial_-)^\dagger$ and $\mathrm{i}\not\nabla_- = (\mathrm{i}\not\nabla_+)^\dagger$. This is simply the Dirac determinant (up to an irrelevant factor $\det(\mathrm{i}\not\partial_-\mathrm{i}\not\partial_+)$),

$$[\det(\mathrm{i}\not\nabla)]^2 = \det\begin{pmatrix} \mathrm{i}\not\nabla_-\mathrm{i}\not\nabla_+ & 0 \\ 0 & \mathrm{i}\not\nabla_+\mathrm{i}\not\nabla_- \end{pmatrix} = [\det(\mathrm{i}\not\nabla_+\mathrm{i}\not\nabla_-)]^2 \tag{13.48}$$

where $\mathrm{i}\not\nabla$ is given by

$$\mathrm{i}\not\nabla = \begin{pmatrix} 0 & \mathrm{i}\not\nabla_- \\ \mathrm{i}\not\nabla_+ & 0 \end{pmatrix}. \tag{13.49}$$

The Dirac determinant is gauge invariant, hence so is $|\det(\mathrm{i}\hat{D})|$. It then follows that $\mathrm{Re}\, W_r[\mathcal{A}]$ is gauge invariant since

$$\exp(-W_r[\mathcal{A}])\exp(-\overline{W_r[\mathcal{A}]}) = \det(\mathrm{i}\hat{D})\det((\mathrm{i}\hat{D})^\dagger) \propto \det(\mathrm{i}\not\nabla_+\mathrm{i}\not\nabla_-)$$

is gauge invariant. Therefore, only the *imaginary part* of $W_r[\mathcal{A}]$, that is the *phase* of $\det(\mathrm{i}\hat{D})$, may gain an anomalous variation under gauge transformations.

The anomaly may be computed by evaluating the Jacobian as before. The functional measure is taken to be $\Pi_i\,\mathrm{d}a_i\,\mathrm{d}\bar{b}_i$. We consider an infinitesimal gauge transformation,

$$\mathcal{A} \to \mathcal{A} - \mathcal{D}v,\ \psi \to \psi + v\psi_+,\ \bar{\psi} \to \bar{\psi} - \bar{\psi}_- v \tag{13.50}$$

where the gauge transformation rotates the positive chirality parts only. The Jacobian factor is

$$\int \mathrm{d}x\,\mathrm{tr}\,v(x)\sum_n (n|x\rangle\gamma^{m+1}\langle x|n\rangle \tag{13.51}$$

where $\langle x|n\rangle = \psi_n(x)$ and $(n|x\rangle = \chi^\dagger_n(x)$ (note that $(n|$ is *not* the Hermitian conjugate of $|n\rangle$). This integral is ill defined and must be regularised. As before we employ the Gaussian regulator,

$$\int \mathrm{d}x \lim_{\substack{M\to\infty \\ x\to y}} \operatorname{tr} v(x) \sum_n (n|y\rangle \gamma^{m+1} \langle x| \mathrm{e}^{-(\mathrm{i}\hat{D})^2/M^2} |n\rangle$$

$$= \int \mathrm{d}x \lim_{\substack{M\to\infty \\ x\to y}} \operatorname{tr} v(x) \gamma^{m+1} \mathrm{e}^{-(\mathrm{i}\hat{D}_x)^2/M^2} \delta(x-y) \tag{13.52}$$

where use has been made of the completeness relation

$$\sum_n |n\rangle (n| = \mathbb{1}. \tag{13.53}$$

From (13.41) and (13.52), it follows that

$$\int v^\alpha \mathcal{D}_\mu \left(\frac{\delta}{\delta \mathcal{A}_\mu{}^\alpha} W_r[\mathcal{A}] \right) = \int \mathrm{d}x \lim_{\substack{M\to\infty \\ x\to y}} \operatorname{tr} [v \gamma^{m+1} \mathrm{e}^{-(\mathrm{i}\hat{D}_x)^2/M^2} \delta(x-y)]. \tag{13.54}$$

In the present case, W_r really changes under (13.50). The trace may be written as

$$\begin{aligned} \operatorname{tr}[v\gamma^{m+1}\mathrm{e}^{-(\mathrm{i}\hat{D}_x)^2/M^2}] &= \operatorname{tr}[v(\mathcal{P}_+ - \mathcal{P}_-)\mathrm{e}^{-(\mathrm{i}\not{\partial}\,\mathrm{i}\not{\nabla}_+)-(\mathrm{i}\not{\nabla}\,\mathrm{i}\not{\partial}_+)/M^2}] \\ &= \operatorname{tr}[v\mathcal{P}_+\mathrm{e}^{(\mathrm{i}\not{\partial}\mathrm{i}\not{\nabla})/M^2}] - \operatorname{tr}[v\mathcal{P}_-\mathrm{e}^{(\mathrm{i}\not{\nabla}\mathrm{i}\not{\partial})/M^2}]. \end{aligned} \tag{13.55}$$

(13.55) can be evaluated in the plane wave basis, which is straightforward but tedious (see Gross and Jackiw (1972), for example). We derive the non-Abelian anomaly from a topological viewpoint in the next section. For $m = 4$, the anomalous variation is

$$\begin{aligned} W_r[\mathcal{A} - \mathcal{D}v] - W_r[\mathcal{A}] &= \int \mathrm{d}x v^\alpha \mathcal{D}_\mu \langle j^\mu \rangle_\alpha \\ &= \frac{1}{24\pi^2} \int \mathrm{d}x \operatorname{tr} \{ v^\alpha T_\alpha \varepsilon^{\kappa\lambda\mu\nu} \partial_\kappa [\mathcal{A}_\lambda \partial_\mu \mathcal{A}_\nu + \tfrac{1}{2} \mathcal{A}_\lambda \mathcal{A}_\mu \mathcal{A}_\nu] \} \\ &= \frac{1}{24\pi^2} \int \operatorname{tr} \{ v \, \mathrm{d} [\mathcal{A} \mathrm{d} \mathcal{A} + \tfrac{1}{2} \mathcal{A}^3] \}. \end{aligned} \tag{13.56}$$

The anomalous divergence of the gauge current is

$$\mathcal{D}_\mu \langle j^\mu \rangle_\alpha = \frac{1}{24\pi^2} \operatorname{tr} \{ T_\alpha \varepsilon^{\kappa\lambda\mu\nu} \partial_\kappa [\mathcal{A}_\lambda \partial_\mu \mathcal{A}_\nu + \tfrac{1}{2} \mathcal{A}_\lambda \mathcal{A}_\mu \mathcal{A}_\nu] \}. \tag{13.57}$$

This should be compared with (13.33). There are two modifications: the two thirds in front of $\mathcal{A}^3$ is replaced by a half and the overall factor is different.

13.4 The Wess–Zumino consistency conditions

13.4.1 The BRS operator and the Faddeev–Popov ghost

Let $W[\mathcal{A}]$ be the effective action of the Weyl fermion in the complex representation $\boldsymbol{r}$ of the gauge group G. In the previous section we observed that the change of $W[\mathcal{A}]$ under an infinitesimal gauge transformation $\delta_v \mathcal{A} = -\mathcal{D}v$ is given by

$$\delta_v W[\mathcal{A}] = -\int (\mathcal{D}_\mu v)^\alpha \frac{\delta}{\delta \mathcal{A}_\mu{}^\alpha} W[\mathcal{A}] = \int v^\alpha \mathcal{D}_\mu \langle j^\mu \rangle_\alpha. \tag{13.58}$$

Following Stora (1984) and Zumino (1985) we introduce the BRS operator $\mathcal{S}$ and the Faddeev–Popov ghost ω. Let $\Omega^m(G)$ be the set of maps from S^m to G. [$\Omega^m(G)$ should not be confused with $\Omega^m(M)$, the set of m-forms on M. The distinction should be clear from the context.] In addition to the ordinary exterior derivative d, we introduce another exterior derivative $\mathcal{S}$ on $\Omega^m(G)$ which we call the **Becchi–Rouet–Stora (BRS) operator**. In general $\mathcal{S}$ is defined on an infinite-dimensional space, but we may also consider the restriction of $\mathcal{S}$ to a finite-dimensional compact subspace of $\Omega^m(G)$, such as S^n, parametrised by λ^α. Then $\mathcal{S}$ may be written as $\mathcal{S} \equiv \mathrm{d}\lambda^\alpha \, \partial/\partial\lambda^\alpha$. We require that d and $\mathcal{S}$ are antiderivatives,

$$\mathrm{d}^2 = \mathcal{S}^2 = \mathrm{d}\mathcal{S} + \mathcal{S}\mathrm{d} = 0. \tag{13.59}$$

If we define $\Delta \equiv \mathrm{d} + \mathcal{S}$, Δ is clearly nilpotent,

$$\Delta^2 = \mathrm{d}^2 + \mathrm{d}\mathcal{S} + \mathcal{S}\mathrm{d} + \mathcal{S}^2 = 0. \tag{13.60}$$

Under the action of $g = g(x, \lambda^\alpha)$, $\mathcal{A}$ transforms as

$$\mathcal{A} \to \boldsymbol{A} \equiv g^{-1}(\mathcal{A} + \mathrm{d})g. \tag{13.61}$$

Note that $\mathcal{A}$ is independent of λ while $\boldsymbol{A}$ depends on λ through g. Define the **Faddeev–Popov (FP) ghost** by

$$\omega \equiv g^{-1}\mathcal{S}g. \tag{13.62}$$

The actions of $\mathcal{S}$ on $\boldsymbol{A}$ and ω are found to be

$$\begin{aligned}
\mathcal{S}\boldsymbol{A} &= \mathcal{S}[g^{-1}(\mathcal{A} + \mathrm{d})g] = -g^{-1}\mathcal{S}g\boldsymbol{A} - g^{-1}\mathcal{A}\mathcal{S}g + g^{-1}\mathcal{S}(\mathrm{d}g) \\
&= -\omega\boldsymbol{A} - (\boldsymbol{A} - g^{-1}\mathrm{d}g)\omega - g^{-1}\mathrm{d}(\mathcal{S}g) \\
&= -\omega\boldsymbol{A} - \boldsymbol{A}\omega - \mathrm{d}\omega \equiv -\mathcal{D}_{\boldsymbol{A}}\omega
\end{aligned} \tag{13.63a}$$

$$\mathcal{S}\omega = -g^{-1}\mathcal{S}gg^{-1}\mathcal{S}g = -\omega^2. \tag{13.63b}$$

It is easy to verify that $\mathcal{S}$ is nilpotent on $\boldsymbol{A}$ and ω, and hence on any polynomial of $\boldsymbol{A}$ and ω as it should be; see exercise 13.1. $\boldsymbol{A}$ defines the field strength

$$\boldsymbol{F} \equiv \mathrm{d}\boldsymbol{A} + \boldsymbol{A}^2 = g^{-1}\mathcal{F}g. \tag{13.64}$$

We also define

$$\mathbb{A} \equiv g^{-1}(\mathcal{A} + \Delta)g = \boldsymbol{A} + g^{-1}\mathcal{S}g = \boldsymbol{A} + \omega \tag{13.65a}$$

$$\mathbb{F} \equiv \Delta\mathbb{A} + \mathbb{A}^2 = g^{-1}\mathcal{F}g = \boldsymbol{F} \tag{13.65b}$$

where (13.65b) follows since $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}^2 = \Delta\mathcal{A} + \mathcal{A}^2$ (note that $\mathcal{S}\mathcal{A} = 0$). From theorem 10.5, we find that $\mathbb{A}$ is an Ehresmann connection on the principal bundle and $\mathbb{F}$ its associated curvature two-form.

The existence of a non-Abelian anomaly implies that $W[\boldsymbol{A}]$ does not vanish under the action of the BRS operator $\mathcal{S}$ (ω roughly corresponds to v; see (13.39) and (13.63a)),

$$\mathcal{S}W[\boldsymbol{A}] = G[\omega, \boldsymbol{A}]. \tag{13.66}$$

Since $W[\boldsymbol{A}]$ is independent of ω, $\mathcal{S}$ acts through $\boldsymbol{A}$ only. Before we write down the Wess–Zumino consistency condition for the non-Abelian anomaly we stop here and consider the physical meaning of the BRS operator and the FP ghost.

Exercise 13.1 Verify from (13.63) that the actions of $\mathcal{S}$ on $\boldsymbol{A}$ and ω are nilpotent,

$$\mathcal{S}^2\boldsymbol{A} = 0 \qquad \mathcal{S}^2\omega = 0. \tag{13.67}$$

13.4.2 The BRS operator, FP ghost and moduli space

To find the physical meaning of $\mathcal{S}$ and ω, we need to examine the topology of the gauge fields (Atiyah and Jones 1978, Singer 1985, Sumitani 1985). Let $\mathfrak{A}$ be the space of all gauge potential configurations on S^m. For definiteness, we take $m = 4$ but the generalisation to arbitrary m is obvious. The topology of $\mathfrak{A}$ is trivial since for any gauge potential configurations $\mathcal{A}_1$ and $\mathcal{A}_2$, the combination $t\mathcal{A}_1 + (1-t)\mathcal{A}_2$ $(0 \leqslant t \leqslant 1)$ is again a gauge potential on S^4. Note, however, that $\mathfrak{A}$ does not describe the *physical* configuration space of the gauge theory. We have to identify those field configurations which are connected by G-gauge transformations. Let $\mathfrak{G}$ be the space of all gauge transformations on S^4 ($\mathfrak{G} = \Omega^4(G)$ in our previous notation). Then the physical configuration space must be identified with $\mathfrak{A}/\mathfrak{G}$, called the **moduli space** of the gauge theory. We have seen in §10.5 that the gauge field configuration on S^4 is classified by the transition function $g : S^3 \to G$, S^3 being the equator of S^4. In the present case $\mathfrak{A}/\mathfrak{G}$ is classified by the transition function on the equator $S^3 \to G$, and hence

$$\mathfrak{A}/\mathfrak{G} \simeq \Omega^3(G). \tag{13.68}$$

Thus each connected component of $\mathfrak{A}/\mathfrak{G}$ is labelled by the instanton number k. This component is denoted by $\Omega_k^4(G)$.

We note that the space $\mathfrak{A}$ has a natural projection $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{G}$ and can be made into a fibre bundle whose fibre is $\mathfrak{G}$, see figure 13.1. Let $a \in \mathfrak{A}$ be a representative of the class $[a] \in \mathfrak{A}/\mathfrak{G}$ and let

$$\mathcal{A}(x) = g^{-1}(x)(a(x) + \mathrm{d})g(x) \tag{13.69}$$

be an element of $\mathfrak{A}$ in $[a]$. We denote the exterior derivative operator in $\mathfrak{A}$ by δ, which is a *functional* variation and should not be confused with the usual derivative d; see Leinaas and Olaussen (1982). If δ is applied on (13.69) we find

$$\begin{aligned}\delta\mathcal{A} &= -g^{-1}\delta g\mathcal{A} + g^{-1}\delta a g - g^{-1}a\delta g - g^{-1}\mathrm{d}(\delta g)\\ &= g^{-1}\delta a g - \mathrm{d}(g^{-1}\delta g) - g^{-1}\delta g\mathcal{A} - \mathcal{A}g^{-1}\delta g\\ &= g^{-1}\delta a g - \mathcal{D}_{\mathcal{A}}(g^{-1}\delta g)\end{aligned} \tag{13.70}$$

where $\mathcal{D}_{\mathcal{A}} = \mathrm{d} + [\mathcal{A},\quad]$. The first term of (13.70) represents the derivative of $\mathcal{A}$ along $\mathfrak{A}/\mathfrak{G}$ while the second represents that along the fibre; see figure 13.1. The BRS transformation $\mathcal{S}$ is obtained by restricting the variation δ along the fibre,

$$\mathcal{S}\mathcal{A} \equiv \delta\mathcal{A}|_{\text{fibre}} = -\mathcal{D}_{\mathcal{A}}\omega \tag{13.71a}$$

where the FP ghost ω is $g^{-1}\mathcal{S}g \equiv g^{-1}\delta g|_{\text{fibre}}$. We also find

$$\mathcal{S}\omega = \delta\omega|_{\text{fibre}} = -g^{-1}\mathcal{S}gg^{-1}\mathcal{S}g = -\omega^2 \tag{13.71b}$$

which reproduces (13.63).

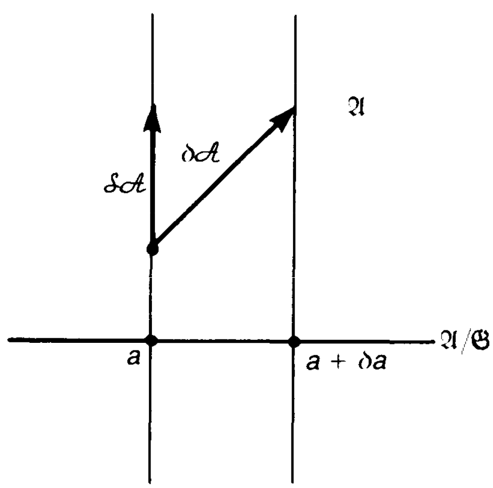


Figure 13.1 The BRS operator $\mathcal{S}$ is the restriction of δ along the fibre.

13.4.3 The Wess–Zumino conditions

Exercise 13.1 shows that $\mathcal{S}$ is *nilpotent* on any polynomial f of $\mathcal{A}$ and ω,

$$\mathcal{S}^2 f(\omega, \boldsymbol{A}) = 0. \tag{13.72}$$

The nilpotency is required by the interpretation of $\mathcal{S}$ as an exterior derivative operator. In particular, we should have

$$\mathcal{S}G[\omega, \boldsymbol{A}] = \mathcal{S}^2 W[\boldsymbol{A}] = 0. \tag{13.73}$$

This condition is called the **Wess–Zumino consistency condition** (WZ condition) and can be used to determine the non-Abelian anomaly (Wess and Zumino 1971, Stora 1984, Zumino 1985, Zumino *et al* 1984). If the anomaly G is mathematically well defined, G should satisfy the WZ condition. This condition is so strong that once the first term of $G[\omega, \boldsymbol{A}]$ is given, the anomaly is completely pinned down.

13.4.4 Descent equations and solutions of WZ conditions

Stora (1984) and Zumino (1985) constructed the solution of WZ conditions as follows. The *Abelian* anomaly in $(2l+2)$-dimensional space is given by

$$\mathrm{ch}_{l+1}(\boldsymbol{F}) = \frac{1}{(l+1)!}\,\mathrm{tr}\left(\frac{\mathrm{i}\boldsymbol{F}}{2\pi}\right)^{l+1}. \tag{13.74}$$

where $\boldsymbol{F} = \mathrm{d}\boldsymbol{A} + \boldsymbol{A}^2$, $\boldsymbol{A} = g^{-1}(\mathcal{A} + \mathrm{d})g$ as before. Note that (13.74) is *formal* since $\mathrm{ch}_{l+1}(\boldsymbol{F})$ is a $(2l+2)$-form and vanishes identically. Let $Q_{2l+1}(\boldsymbol{A}, \boldsymbol{F})$ be the Chern–Simons form of $\mathrm{ch}_{l+1}(\boldsymbol{F})$,

$$\mathrm{ch}_{l+1}(\boldsymbol{F}) = \mathrm{d}Q_{2l+1}(\boldsymbol{A}, \boldsymbol{F}). \tag{13.75}$$

Since the algebraic structure of the triplet $(\Delta, \mathbb{A}, \mathbb{F})$ is exactly the same as that of $(\mathrm{d}, \boldsymbol{A}, \boldsymbol{F})$, we also have

$$\mathrm{ch}_{l+1}(\mathbb{F}) = \Delta Q_{2l+1}(\mathbb{A}, \mathbb{F}) = \Delta Q_{2l+1}(\boldsymbol{A} + \omega, \boldsymbol{F}) \tag{13.76}$$

where we have noted that $\mathbb{A} = \boldsymbol{A} + \omega$ and $\mathbb{F} = \boldsymbol{F}$. If we expand $Q_{2l+1}(\mathbb{A}, \mathbb{F}) = Q_{2l+1}(\boldsymbol{A} + \omega, \boldsymbol{F})$ in powers of ω, we have

$$\begin{aligned} Q_{2l+1}(\mathbb{A}, \mathbb{F}) &= Q^0_{2l+1}(\boldsymbol{A}, \boldsymbol{F}) + Q^1_{2l}(\omega, \boldsymbol{A}, \boldsymbol{F}) + Q^2_{2l-1}(\omega, \boldsymbol{A}, \boldsymbol{F}) \\ &\quad + \ldots + Q^{2l+1}_0(\omega, \boldsymbol{A}, \boldsymbol{F}) \end{aligned} \tag{13.77}$$

where Q^s_r is sth order in ω and $r + s = 2l + 1$.

We now note that $\mathrm{ch}_{l+1}(\mathbb{F}) = \mathrm{ch}_{l+1}(\boldsymbol{F})$ since $\mathbb{F} = \boldsymbol{F} = g^{-1}\mathcal{F}g$. In terms of the Chern–Simons forms, this can be expressed as

$$\Delta Q_{2l+1}(\mathbb{A}, \mathbb{F}) = \mathrm{d}Q_{2l+1}(\boldsymbol{A}, \boldsymbol{F}). \tag{13.78}$$

Substituting (13.77) into (13.78), we have

$$\begin{aligned} &(\mathrm{d} + \mathcal{S})[Q^0_{2l+1}(\boldsymbol{A}, \boldsymbol{F}) + Q^1_{2l}(\omega, \boldsymbol{A}, \boldsymbol{F}) \\ &\qquad + \ldots + Q^{2l+1}_0(\omega, \boldsymbol{A}, \boldsymbol{F})] = \mathrm{d}Q^0_{2l+1}(\boldsymbol{A}, \boldsymbol{F}). \end{aligned} \tag{13.79}$$

If we collect terms of the same order in ω, we have the '**descent equations**'

$$\mathcal{S}Q^0_{2l+1}(A, F) + dQ^1_{2l}(\omega, A, F) = 0 \tag{13.80a}$$

$$\mathcal{S}Q^1_{2l}(\omega, A, F) + dQ^2_{2l-1}(\omega, A, F) = 0 \tag{13.80b}$$

$$\vdots$$

$$\mathcal{S}Q^{2l}_1(\omega, A, F) + dQ^{2l+1}_0(\omega, A, F) = 0 \tag{13.80c}$$

$$\mathcal{S}Q^{2l+1}_0(\omega, A, F) = 0. \tag{13.80d}$$

[Note that $\mathcal{S}$ increases the degree of ω by one, see (13.63).] Let us look at the $2l$-form $Q^1_{2l}(\omega, A, F)$. If we put

$$G[\omega, A, F] \equiv \int_M Q^1_{2l}(\omega, A, F) \tag{13.81}$$

$G[\omega, A, F]$ satisfies the WZ condition,

$$\mathcal{S}G[\omega, A, F] = \int_M \mathcal{S}Q^1_{2l}(\omega, A, F) = -\int_M dQ^2_{2l-1}(\omega, A, F)$$

$$= -\int_{\partial M} Q^2_{2l-1}(\omega, A, F) = 0$$

where we assumed that M has no boundary and use has been made of (13.80b). This shows that once $Q^1_{2l}(\omega, A, F)$ is obtained, the anomaly $G[\omega, A, F]$ is easily found.

Proposition 13.2 Q^1_{2l} defined above is given by

$$Q^1_{2l}(\omega, \mathcal{A}, \mathcal{F}) = \left(\frac{i}{2\pi}\right)^{l+1} \frac{1}{(l-1)!} \int_0^1 \delta t(1-t)\,\mathrm{str}\,[\omega\, d(\mathcal{A}\mathcal{F}_t^{l-1})]. \tag{13.82}$$

[*Remark:* In the proof, we tentatively drop the normalisation factor $(i/2\pi)^{l+1}$. This factor will be recovered at the very end.]

Proof: We start with (11.105),

$$Q_{2l+1}(\mathcal{A} + \omega, \mathcal{F}) = \frac{1}{l!} \int \delta t\, \mathrm{tr}\,[(\mathcal{A} + \omega)\hat{\mathcal{F}}_t^l]$$

where

$$\hat{\mathcal{F}}_t \equiv t\mathcal{F} + (t^2 - t)(\mathcal{A} + \omega)^2$$
$$= \mathcal{F}_t + (t^2 - t)\{\mathcal{A}, \omega\} + (t^2 - t)\omega^2$$
$$\mathcal{F}_t \equiv d(t\mathcal{A}) + (t\mathcal{A})^2.$$

If we substitute $\hat{\mathcal{F}}_t$ into Q_{2l+1} and collect terms of first order in ω, we have

$$\frac{1}{l!}\int_0^1 \delta t\,\mathrm{tr}\,[\omega\mathcal{F}_t^l + (t^2 - t)(\mathcal{A}[\mathcal{A}, \omega]\mathcal{F}_t^{l-1} + \mathcal{A}\mathcal{F}_t[\mathcal{A}, \omega]\mathcal{F}_t^{l-2}$$
$$+ \ldots + \mathcal{A}\mathcal{F}_t^{l-1}[\mathcal{A}, \omega])]$$
$$= \frac{1}{l!}\int \delta t\,\mathrm{str}\,[\omega\mathcal{F}_t^l + (t^2 - t)\mathcal{A}(\mathcal{F}_t^{l-1}[\mathcal{A}, \omega]$$
$$+ \mathcal{F}_t^{l-2}[\mathcal{A}, \omega]\mathcal{F}_t + \ldots)]$$
$$= \frac{1}{l!}\int \delta t\,\mathrm{str}\,[\omega\mathcal{F}_t^l + (t^2 - t)l\mathcal{A}[\mathcal{A}, \omega]\mathcal{F}_t^{l-1}]$$
$$= \frac{1}{l!}\int \delta t\,\mathrm{str}\,[\omega\mathcal{F}_t^l + l(t^2 - t)([\mathcal{A}, \mathcal{A}]\omega\mathcal{F}_t^{l-1} + \mathcal{A}\omega[\mathcal{A}, \mathcal{F}_t^{l-1}])]$$
$$= \frac{1}{l!}\int \delta t\,\mathrm{str}\,[\omega\{\mathcal{F}_t^l + l(t - 1)(t[\mathcal{A}, \mathcal{A}]\mathcal{F}_t^{l-1} - \mathcal{A}[\mathcal{A}_t, \mathcal{F}_t^{l-1}])\}]$$

where str is the symmetrised trace defined by (11.8). Now we use

$$\mathcal{D}_t\mathcal{F}_t^{l-1} \equiv \mathrm{d}\mathcal{F}_t^{l-1} + [\mathcal{A}_t, \mathcal{F}_t^{l-1}] = 0$$
$$\frac{\partial\mathcal{F}_t}{\partial t} = \mathrm{d}\mathcal{A} + t[\mathcal{A}, \mathcal{A}]$$

to change the final line of the above equation to

$$\frac{1}{l!}\int \delta t\,\mathrm{str}\left[\omega\left\{\mathcal{F}_t^l + l(t - 1)\left[\left(\frac{\partial\mathcal{F}_t}{\partial t} - \mathrm{d}\mathcal{A}\right)\mathcal{F}_t^{l-1} + \mathcal{A}\,\mathrm{d}\mathcal{F}_t^{l-1}\right]\right\}\right]$$
$$= \frac{1}{l!}\int \delta t\,\mathrm{str}\left[\omega\left(\mathcal{F}_t^l + l(1 - t)\mathrm{d}(\mathcal{A}\mathcal{F}_t^{l-1}) + (t - 1)\frac{\partial\mathcal{F}_t^l}{\partial t}\right)\right].$$

Integrating by parts, we find that

$$Q_{2l}^1(\omega, \mathcal{A}, \mathcal{F}) = \frac{1}{(l-1)!}\int \delta t(1 - t)\,\mathrm{str}\,[\omega\,\mathrm{d}(\mathcal{A}\mathcal{F}_t^{l-1})].$$

If we recover the normalisation, we finally have

$$Q_{2l}^1(\omega, \mathcal{A}, \mathcal{F}) = \left(\frac{\mathrm{i}}{2\pi}\right)^{l+1}\frac{1}{(l-1)!}\int_0^1 \delta t(1 - t)\,\mathrm{str}\,[\omega\,\mathrm{d}(\mathcal{A}\mathcal{F}_t^{l-1})]. \blacksquare$$

For $m = 2l = 2$ and $m = 4$, we have

$$Q_2^1(\omega, \boldsymbol{A}, \boldsymbol{F}) = \left(\frac{\mathrm{i}}{2\pi}\right)^2 \mathrm{tr}\,(\omega\,\mathrm{d}\boldsymbol{A}) \tag{13.83a}$$

$$Q_4^1(\omega, \boldsymbol{A}, \boldsymbol{F}) = \frac{1}{6}\left(\frac{\mathrm{i}}{2\pi}\right)^3 \mathrm{str}\,(\omega\,\mathrm{d}(\boldsymbol{A}\,\mathrm{d}\boldsymbol{A} + \tfrac{1}{2}\boldsymbol{A}^3)). \tag{13.83b}$$

These results are also verified by direct computations. Up to the normalisation factor (13.83b) yields the non-Abelian anomaly in four-dimensional space; see (13.56).

Sumitani (1984) pointed out that the approach to the non-Abelian anomalies here is *ad hoc* and does not clarify the following points:

(1) The WZ condition (13.73) does not fix the normalisation of the anomaly and, moreover, the uniqueness of the solution is far from trivial.

(2) It is not clear why we should start from the *Abelian* anomaly in $(m + 2)$-dimensional space.

To answer these questions we need to develop a more elaborate index theorem called the family index theorem; see Atiyah and Singer (1984), Singer (1985) and Sumitani (1984, 1985). In the next section, we outline the physicists' approach to this problem, closely following the work of Alvarez-Gaumé and Ginsparg (1984).

13.5 Abelian anomalies versus non-Abelian anomalies

Let us consider an m-dimensional Euclidean space ($m = 2l$) which is compactified to $S^m = \mathbb{R}^m \cup \{\infty\}$ and let G be a semisimple gauge group which is simply connected (like $\mathrm{SU}(N)$ for which $\pi_1(\mathrm{SU}(N)) = 0$). Consider a one-parameter family of gauge transformations $g(\theta, x)$ $(0 \leqslant \theta \leqslant 2\pi)$ such that

$$g(0, x) = g(2\pi, x) = e. \tag{13.84}$$

Without loss of generality we may normalise g so that $g(\theta, x_0) = e$ at a point $x_0 \in S^m$. The map $g : S^1 \times S^m \to G$ is classified according to the homotopy class $\pi_{m+1}(G)$. To see this we define the **smash product** $X \wedge Y$ of topological spaces X and Y by the direct product $X \times Y$ with $X \vee Y \equiv (x_0 \times X) \cup (X \times y_0)$ shrunk to a point. From figure 13.2, we easily find that $S^1 \wedge S^m = S^m \wedge S^1 = S^{m+1}$. [The readers may convince themselves by explicitly drawing $S^1 \wedge S^1 = S^2$.] Repeated applications of this yield

$$S^m \wedge S^n = S^{m+n}. \tag{13.85}$$

In the case which interests us, the conditions (13.84) make the direct product $S^1 \times S^m$ look topologically like $S^1 \wedge S^m = S^{m+1}$. Thus g is regarded as a map from S^{m+1} to G and is classified by $\pi_{m+1}(G)$. Since we have a one-parameter family in the space $\mathfrak{G} = \Omega^m(G)$, we also have $\pi_{m+1}(G) = \pi_1(\mathfrak{G})$. In practice we take $G = \mathrm{SU}(N)$ for which we have

$$\pi_{m+1}(\mathrm{SU}(N)) = \mathbb{Z} \qquad N \geqslant \tfrac{1}{2}m + 1. \tag{13.86}$$

Now we take a 'reference' gauge field $\mathcal{A}$ in the zero instanton sector $\Omega_0^m(G)$ for which we may assume, without loss of generality, that the Dirac operator (13.49) has no zero modes. Consider a one-parameter family of gauge potentials

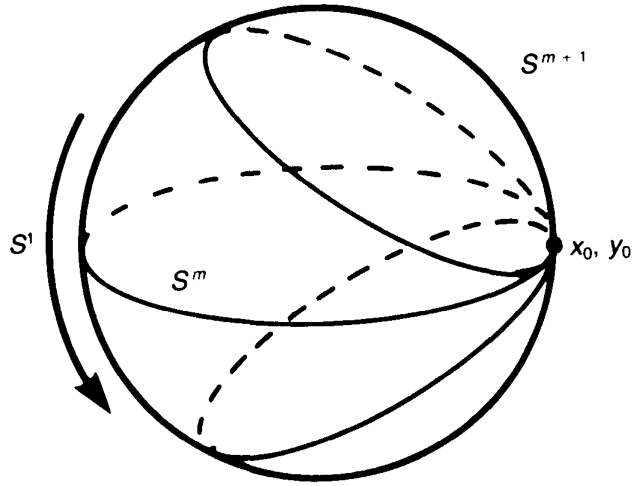


Figure 13.2 The smash product $S^1 \wedge S^m \cong S^{m+1}$.

$$\mathcal{A}^{g(\theta)}(x) \equiv g^{-1}(\theta, x)(\mathcal{A}(x) + \mathrm{d})g(\theta, x) \tag{13.87}$$

where θ parametrises S^1. In §13.3, we observed that $|\det \mathrm{i}\hat{D}|$ is gauge invariant (see (13.47)) and only the *phase* of $\det \mathrm{i}\hat{D}$ may gain an anomalous variation under a gauge transformation. This, in particular, implies that $\det \mathrm{i}\hat{D}$ does not vanish for any θ. We write

$$\exp\{-W_r[\mathcal{A}^{g(\theta)}]\} = \det \mathrm{i}\hat{D}(\mathcal{A}^{g(\theta)}) = [\det \mathrm{i}\nabla\!\!\!/(\mathcal{A})]^{1/2} \exp[\mathrm{i}w(\mathcal{A}, \theta)] \tag{13.88}$$

where $\mathrm{i}\nabla\!\!\!/$ is the Dirac operator (13.49) and $\mathrm{i}w(\mathcal{A}, \theta)$ is the anomalous phase associated with the gauge transformation (13.87). Next we consider a *two*-parameter family of gauge fields $\mathcal{A}^{t,\theta}$ $(0 \leqslant t \leqslant 1)$ which interpolate between $\mathcal{A} = 0$ and $\mathcal{A}^{g(\theta)}$,

$$\mathcal{A}^{t,\theta} \equiv t\mathcal{A}^{g(\theta)}. \tag{13.89}$$

The parameter space specified by (t, θ) is considered to be a two-dimensional unit disc D^2 with polar coordinates (t, θ). On the boundary of the disc, $\partial D^2 = S^1$, the modulus of $\det \mathrm{i}\hat{D}(\mathcal{A}^{1,\theta})$ is a non-vanishing constant. The phase $\mathrm{e}^{\mathrm{i}w(\mathcal{A},\theta)}$ now defines a map $S^1(=\partial D^2) \to S^1(=\mathrm{U}(1))$; see figure 13.3. As we move around the boundary of the disc, the phase winds around the unit circle. The winding number of this map is an integer

$$\mathcal{N} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial w(\mathcal{A}, \theta)}{\partial \theta}\,\mathrm{d}\theta. \tag{13.90}$$

We find below that $\mathcal{N}$ is derived from the Abelian anomaly in $(m + 2)$ dimensions.

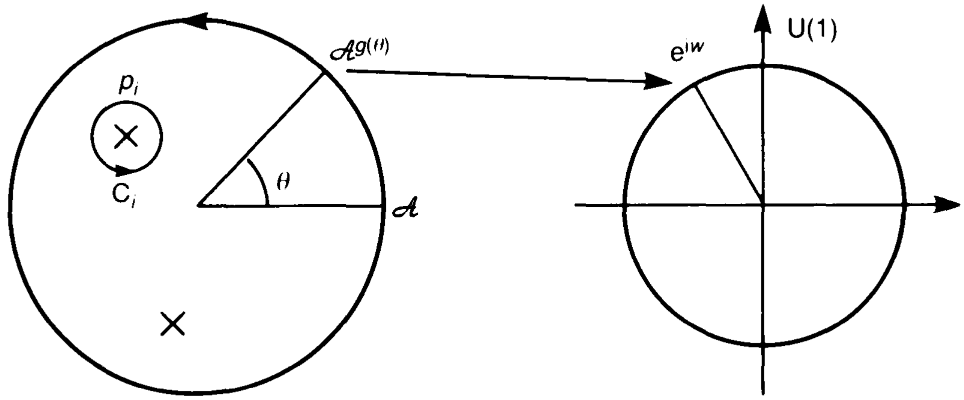


Figure 13.3 The phase of the effective action $W[\mathcal{A}^{g(\theta)}]$ defines a map $S^1 \to \mathrm{U}(1)$ by $\theta \mapsto \mathrm{e}^{\mathrm{i}w(\mathcal{A},\theta)}$. On the disc, there are points $\{p_i\}$ at which $\det \mathrm{i}\hat{D}(\mathcal{A}^{t,\theta})$ vanishes. The winding number of the map $S^1 \to \mathrm{U}(1)$ is obtained by summing a winding number along C_i.

Exercise 13.3 Show that

$$W[\mathcal{A}^{g(2\pi)}] - W[\mathcal{A}^{g(0)}] = -2\pi\mathrm{i}\mathcal{N}. \tag{13.91}$$

Since $g(2\pi) = g(0)$, (13.91) may be regarded as a Berry phase.

13.5.1 m dimensions versus m + 2 dimensions

We recall that our reference gauge field $\mathcal{A}$ yields no zero modes of the operator $\mathrm{i}\hat{D}(\mathcal{A})$. Since $|\det \mathrm{i}\hat{D}(\mathcal{A}^{g(\theta)})| = |\det \mathrm{i}\hat{D}(\mathcal{A})| \neq 0$, the operator $\mathrm{i}\hat{D}(\mathcal{A}^{g(\theta)})$ does not admit zero modes either. Of course, $\mathrm{i}\hat{D}(\mathcal{A}^{t,\theta})$ may have zero modes since $\mathcal{A}^{t,\theta}$ is *not* obtained from $\mathcal{A}$ by a gauge transformation in general. Suppose it has a zero mode at $p_i = (t_i, \theta_i)$. We assume they are isolated points. Since $\det \mathrm{i}\hat{D}(\mathcal{A}^{t,\theta})$ is a regularised product of eigenvalues, it vanishes at p_i. The phase of $\det \mathrm{i}\hat{D}(\mathcal{A}^{t,\theta})$ may be homotopically non-trivial only around these points. Moreover, the winding number at p_i is determined by the eigenvalue which vanishes at p_i. For example, if $\lambda_n(t, \theta)$ vanishes at p_i, it should be of the form,

$$\lambda_n(t, \theta) = f(t, \theta)\mathrm{e}^{\mathrm{i}w_i(t,\theta)} \tag{13.92}$$

where $f(t_i, \theta_i) = 0$. The winding number at p_i is

$$m_i = \frac{1}{2\pi}\int_{C_i} \frac{\mathrm{d}}{\mathrm{d}s} w_i(t, \theta)\, \mathrm{d}s \tag{13.93}$$

where C_i is a small contour surrounding p_i, see figure 13.3. Continuously deforming the loop $S^1 = \partial D^2$ into a sum of small circles C_i enclosing p_i, we find that the total winding number is

$$\mathcal{W} = \frac{1}{2\pi}\int_{S^1} \mathrm{d}\theta \frac{\partial}{\partial\theta} w(\mathcal{A}, \theta) = \sum m_i. \tag{13.94}$$

Now we show that the winding number $\mathcal{W}$ is related to the index theorem in $(m+2)$-dimensional space $(m = 2l)$; $\mathcal{W} = \operatorname{ind} \mathrm{i}\nabla\!\!\!/_{m+2}$ where $\mathrm{i}\nabla\!\!\!/_{m+2}$ is the Dirac operator on $S^2 \times S^m$ defined below. Let us consider a gauge theory defined on $D^2 \times S^m$ whose coordinates are (t, θ, x). To avoid the boundary term, we add another piece, $D^2 \times S^m$, with coordinates (s, θ, x), to form a manifold $S^2 \times S^m$ without a boundary; see figure 13.4. We call the patch (t, θ) the northern hemisphere U_{N} and (s, θ) the southern hemisphere U_{S}. On the equator S^1 of S^2, we have $t = s = 1$. We choose the following local gauge potentials

$$\mathcal{A}_{\mathrm{N}}(t, \theta, x) = \mathcal{A}^{t,\theta} + g^{-1}\mathrm{d}_\theta g \qquad (t, \theta) \in U_{\mathrm{N}} \tag{13.95a}$$

$$\mathcal{A}_{\mathrm{S}}(s, \theta, x) = \mathcal{A} \qquad (s, \theta) \in U_{\mathrm{S}} \tag{13.95b}$$

where $\mathcal{A}$ is the reference gauge field introduced previously. To elevate $\mathcal{A}_{\mathrm{N}} = \mathcal{A}_{\mathrm{N}\mu}\,\mathrm{d}x^\mu$ and $\mathcal{A}_{\mathrm{S}} = \mathcal{A}_{\mathrm{S}\mu}\,\mathrm{d}x^\mu$ to the globally defined connection on the G bundle over $S^2 \times S^m$, we define the $(m+2)$-dimensional gauge potentials

$$\mathbb{A}_{\mathrm{N}}(t, \theta, x) = (\mathcal{A}_t, \mathcal{A}_\theta, \mathcal{A}_\mu) = (0, 0, \mathcal{A}_{\mathrm{N}}) \tag{13.96a}$$

$$\mathbb{A}_{\mathrm{S}}(s, \theta, x) = (\mathcal{A}_s, \mathcal{A}_\theta, \mathcal{A}_\mu) = (0, 0, \mathcal{A}_{\mathrm{S}}). \tag{13.96b}$$

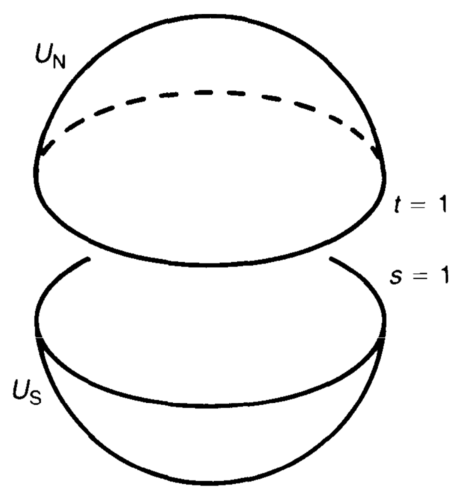


Figure 13.4 U_{N} parametrised by (t, θ) and U_{S} by (s, θ) are glued together at $t = s = 1$ to form a sphere S^2. This is a sphere in the moduli space $\mathfrak{A}/\mathfrak{G}$.

On the equator $(t = s = 1)$, we have $\mathbb{A}_{\mathrm{N}} = g^{-1}(\mathbb{A}_{\mathrm{S}} + \Delta)g$, where $\Delta \equiv \mathrm{d} + \mathrm{d}_\theta + \mathrm{d}_t$ (note that $\mathrm{d}_t g = 0$). Thus $\mathbb{A} = \{\mathbb{A}_{\mathrm{N}}, \mathbb{A}_{\mathrm{S}}\}$ defines a global connection on $S^2 \times S^m$. Consider a *Dirac* operator $\mathrm{i}\nabla\!\!\!/_{m+2}$ which couples to $\mathbb{A}$. The index theorem for $\mathrm{i}\nabla\!\!\!/_{m+2}$ is given by

$$\operatorname{ind} \mathrm{i}\nabla\!\!\!\!/_{m+2} = \mathcal{N}_+ - \mathcal{N}_- = \int_{S^2\times S^m} \mathrm{ch}_{l+1}(\mathbb{F}) \tag{13.97}$$

where $\mathbb{F} = \Delta\mathbb{A} + \mathbb{A}^2$ and $\mathcal{N}_+(\mathcal{N}_-)$ is the number of $+$ $(-)$ chirality zero modes of $\mathrm{i}\nabla\!\!\!\!/_{m+2}$ (chirality is defined in an $(m+2)$-space).

Alvarez-Gaumé and Ginsparg (1984) have shown, using an adiabatic perturbative computation, that each winding number m_i must be ± 1. Moreover, the Dirac operator $\mathrm{i}\nabla\!\!\!\!/_{m+2}$ has a zero mode at $p_i = (t_i, \theta_i)$ with $(m+2)$-dimensional chirality $\chi = m_i = \pm 1$. Then the total winding number $\mathcal{N} = \Sigma m_i$ is given by the index $\mathcal{N}_+ - \mathcal{N}_-$. Now we have

$$\operatorname{ind} \mathrm{i}\nabla\!\!\!\!/_{m+2} = \int_{S^2\times S^m} \mathrm{ch}_{l+1}(\mathbb{F}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \, \frac{\partial w(\mathcal{A}, \theta)}{\partial\theta}. \tag{13.98}$$

We easily find the non-Abelian anomaly from (13.98) including the normalisation. Since $\mathrm{ch}_{l+1}(\mathbb{F}) = \mathrm{d}Q_{m+1}(\mathbb{A}, \mathbb{F})$, we have

$$\begin{aligned}\int_{S^2\times S^m} \mathrm{ch}_{l+1}(\mathbb{F}) &= \int_{D^2\times S^m} \mathrm{ch}_{l+1}(\mathbb{F}_{\mathrm{N}}) + \int_{D^2\times S^m} \mathrm{ch}_{l+1}(\mathbb{F}_{\mathrm{S}}) \\ &= \int_{S^1\times S^m} [Q_{m+1}(\mathbb{A}_{\mathrm{N}}, \mathbb{F}_{\mathrm{N}})|_{t=1} - Q_{m+1}(\mathbb{A}_{\mathrm{S}}, \mathbb{F}_{\mathrm{S}})|_{s=1}]. \end{aligned}\tag{13.99}$$

From (11.118), we find

$$\begin{aligned} Q_{m+1}(\mathbb{A}_{\mathrm{N}}, \mathbb{F}_{\mathrm{N}})|_{t=1} &- Q_{m+1}(\mathbb{A}_{\mathrm{S}}, \mathbb{F}_{\mathrm{S}})|_{s=1} \\ &= Q_{m+1}(g^{-1}\Delta g, 0) + \Delta\alpha_m \\ &= (-1)^l \left(\frac{\mathrm{i}}{2\pi}\right)^{l+1} \frac{l!}{(m+1)!} \operatorname{tr}(g^{-1}\Delta g)^{m+1} + \Delta\alpha_m. \end{aligned}\tag{13.100}$$

The index theorem is now given by

$$\operatorname{ind} \mathrm{i}\nabla\!\!\!\!/_{m+2} = (-1)^l \left(\frac{\mathrm{i}}{2\pi}\right)^{l+1} \frac{l!}{(m+1)!} \int_{S^1\times S^m} \operatorname{tr}(g^{-1}\Delta g)^{m+1}. \tag{13.101}$$

Theorem 10.38 states that $\int_{S^3} \operatorname{tr}(g^{-1}\,\mathrm{d}g)^3$ yields the winding number of the map $g: S^3 \to \mathrm{SU}(2)$. In the same manner, (13.101) represents the winding number of the map $g: S^{m+1} \to G$ and is classified by $\pi_{m+1}(G)$ (note that $S^1 \wedge S^m = S^{m+1}$).

Finally we show that the non-Abelian anomaly should be identified with Q_m^1. We first note that

$$\int_{S^1\times S^m} Q_{m+1}(\mathbb{A}_{\mathrm{S}}, \mathbb{F}_{\mathrm{S}}) = 0$$

since the integrand is independent of $\mathrm{d}\theta$, and thus cannot be a volume element of $S^1 \times S^m$. Then we have

$$\operatorname{ind} \mathrm{i}\nabla\!\!\!\!/_{m+2} = \int_{S^1\times S^m} Q_{m+1}(\mathcal{A}^{g(\theta)} + \omega, \mathcal{F}^{g(\theta)}) \tag{13.102}$$

where $\omega = g^{-1}\mathrm{d}_\theta g$ and $\mathcal{F}^{g(\theta)} = \mathrm{d}\mathcal{A}^{g(\theta)} + (\mathcal{A}^{g(\theta)})^2 = g(\theta)^{-1}\mathcal{F}g(\theta)$. If the integrand in (13.102) is expanded in ω, only the term *linear* in $\mathrm{d}\theta$ contributes to the integral. This term $Q_m^1(\omega, \mathcal{A}^{g(\theta)}, \mathcal{F}^{g(\theta)})$ is proportional to $\mathrm{d}\theta \wedge$ (volume element in S^m), and hence is a volume element of $S^1 \times S^m$. We now have

$$\begin{aligned}\delta_\omega W[\mathcal{A}] &= \int_{S^m} \operatorname{tr} \omega \mathcal{D}_\mu \frac{\delta W[\mathcal{A}]}{\delta \mathcal{A}_\mu} \\ &= \mathrm{i}\mathrm{d}_\theta w(\theta, \mathcal{A}) = 2\pi \mathrm{i} \int_{S^m} Q_m^1(\omega, \mathcal{A}^{g(\theta)}, \mathcal{F}^{g(\theta)}).\end{aligned} \tag{13.103}$$

The explicit form of Q_m^1 is given by (13.82). For $m = 4$, we find that

$$\begin{aligned}\int \operatorname{tr} \omega \mathcal{D}_\mu \frac{\delta W[\mathcal{A}]}{\delta \mathcal{A}_\mu} &= 2\pi \mathrm{i} \int_{S^4} Q_4^1(\omega, \mathcal{A}^{g(\theta)}, \mathcal{F}^{g(\theta)}) \\ &= \frac{1}{24\pi^2} \int_{S^4} \operatorname{tr} \omega \,\mathrm{d}[\mathcal{A}^{g(\theta)} \mathrm{d}\mathcal{A}^{g(\theta)} + \tfrac{1}{2}(\mathcal{A}^{g(\theta)})^3].\end{aligned} \tag{13.104}$$

Putting $\theta = 0$ $(g = e)$, we reproduce the anomalous divergence

$$\mathcal{D}_\mu \langle j^\mu \rangle_\alpha = \frac{1}{24\pi^2} \operatorname{tr} T_\alpha \varepsilon^{\kappa\lambda\mu\nu} \partial_\kappa [\mathcal{A}_\lambda \partial_\mu \mathcal{A}_\nu + \tfrac{1}{2} \mathcal{A}_\lambda \mathcal{A}_\mu \mathcal{A}_\nu] \tag{13.105}$$

which is in agreement with (13.56). The present method guarantees that the WZ condition yields the correct answer. Moreover, it reproduces the anomalous divergence including the normalisation which cannot be fixed by the WZ condition alone.

13.6 The parity anomaly in odd-dimensional spaces

So far, we have been working in even-dimensional spaces. One of the reasons for this is that SO$(2l + 1)$ has real or pseudo-real spinor representations but no *complex* representations, hence no gauge anomaly is expected. However, we can show that gauge theories in odd-dimensional spaces have a different kind of anomaly called the 'parity anomaly', in which the parity symmetry of the classical action is not maintained through quantisation. It should be noted that the parity anomaly in $2l + 1$ dimensions is related to the Abelian anomaly in $2l + 2$ dimensions as was pointed out by Alvarez-Gaumé *et al* (1985).

13.6.1 The parity anomaly

Let M be a $(2l + 1)$-dimensional Riemannian manifold. We distinguish one dimension from the others; namely, we assume that M is of the form $\mathbb{R} \times \mathcal{M}$ or $S^1 \times \mathcal{M}$, where $\mathcal{M}$ is a $2l$-dimensional compact manifold without a boundary. We denote the coordinate of $\mathbb{R}$ or S^1 by t

while that of $\mathcal{M}$ is denoted by x. The index 0 denotes the component in t-space while μ denotes that in x-space. For example, the components of the γ-matrices are $\{\gamma^0, \gamma^\mu\ (1 \leqslant \mu \leqslant 2l)\}$.

Define the *'parity'* operation P by

$$\begin{aligned}
\mathcal{A}_0(t, x) &\to \mathcal{A}_0^{\mathrm{P}}(t, x) = -\mathcal{A}_0(-t, x)\\
\mathcal{A}_\mu(t, x) &\to \mathcal{A}_\mu^{\mathrm{P}}(t, x) = \mathcal{A}_\mu(-t, x)\\
\psi(t, x) &\to \psi^{\mathrm{P}}(t, x) = \mathrm{i}\gamma_0\psi(-t, x)\\
\bar{\psi}(t, x) &\to \bar{\psi}^{\mathrm{P}}(t, x) = \mathrm{i}\bar{\psi}(-t, x)\gamma_0.
\end{aligned}$$

The classical action is invariant under the parity operation,

$$\begin{aligned}
\int \mathrm{d}t\,\mathrm{d}x\bar{\psi}\mathrm{i}\not{\nabla}\psi \to -\int &\mathrm{d}t\,\mathrm{d}x\bar{\psi}(-t, x)\gamma^0\mathrm{i}[\gamma^0(\partial_0 - \mathcal{A}_0(-t, x))\\
&+ \gamma^\mu(\partial_\mu + \mathcal{A}_\mu(-t, x))]\gamma^0\psi(-t, x)\\
= &\int \mathrm{d}t\,\mathrm{d}x\bar{\psi}(t, x)\mathrm{i}[\gamma^0(\partial_0 + \mathcal{A}_0(t, x))\\
&+ \gamma^\mu(\partial_\mu + \mathcal{A}_\mu(t, x))]\psi(t, x)
\end{aligned}$$

where we put $t \to -t$ in the final line. Let us see whether this invariance is observed by the effective action. The effective action is given by the regularised product of the eigenvalues of $\mathrm{i}\not{\nabla}$. We employ the **Pauli–Villars regularisation** to regulate the product, namely

$$\mathcal{L}_{\mathrm{reg}} \equiv \bar{\chi}\mathrm{i}\not{\nabla}\chi + \mathrm{i}M\bar{\chi}\chi \tag{13.106}$$

is added to the original Lagrangian. The Pauli–Villars regulator χ is a spinor which obeys *bosonic* statistics and the limit $M \to \infty$ is understood. The regularised determinant is

$$\mathrm{e}^{-W[\mathcal{A}]} = \frac{\det \mathrm{i}\not{\nabla}}{\det(\mathrm{i}\not{\nabla} + \mathrm{i}M)} = \prod_i \frac{\lambda_i}{\lambda_i + \mathrm{i}M} \tag{13.107}$$

where we noted that χ is bosonic. λ_i is the ith eigenvalue of $\mathrm{i}\not{\nabla}$; $\mathrm{i}\not{\nabla}\psi_i = \lambda_i\psi_i$. Under the parity operation, eigenvalues change sign,

$$\begin{aligned}
&\mathrm{i}[\gamma^0(\partial_0 - \mathcal{A}_0(-t, x)) + \gamma^i(\partial_i + \mathcal{A}_i(-t, x))]\mathrm{i}\gamma^0\psi_i(-t, x)\\
&\quad = \mathrm{i}\gamma^0[\gamma^0(-\partial_\tau - \mathcal{A}_0(\tau, x)) - \gamma^i(\partial_i + \mathcal{A}_i(\tau, x))]\mathrm{i}\psi(\tau, x)\\
&\quad = -\lambda_i\mathrm{i}\gamma^0\psi_i(\tau, x)
\end{aligned}$$

where $\tau = -t$. This shows that the effective action $W[\mathcal{A}]$ transforms under the parity operation P as

$$W[\mathcal{A}] \to W[\mathcal{A}^{\mathrm{P}}] = -\ln\prod \frac{-\lambda_i}{-\lambda_i + \mathrm{i}M} = \overline{W[\mathcal{A}]} \tag{13.108}$$

where the bar denotes complex conjugation. (13.108) shows that the imaginary part of W is identified with the parity-violating part

$$W[\mathcal{A}] - W[\mathcal{A}^{\mathrm{P}}] = 2\,\mathrm{Im}\,W[\mathcal{A}]. \tag{13.109}$$

$\mathrm{Im}\,W[\mathcal{A}]$ is given by the η-invariant defined in §12.8. In fact,

$$\begin{aligned}\mathrm{Im}\,W[\mathcal{A}] &= \lim_{M\to\infty} \mathrm{Im}\left(-\sum_i \ln \frac{\lambda_i}{\lambda_i + \mathrm{i}M}\right) = \lim_{M\to\infty} \sum_i \tan^{-1}(M/\lambda_i) \\ &= \frac{\pi}{2}\left(\sum_{\lambda>0} 1 - \sum_{\lambda<0} 1\right) = \frac{\pi}{2}\eta. \end{aligned} \tag{13.110}$$

Thus the Pauli–Villars regulator gives a regularised form for the η-invariant. We finally have

$$\mathrm{Im}\,W[\mathcal{A}] = \frac{\pi}{2}\,\eta = \frac{\pi}{2} \lim_{s\to 0} \sum_i{}' \operatorname{sgn} \lambda_i\, |\lambda_i|^{-2s}. \tag{13.111}$$

13.6.2 The dimensional ladder: 4–3–2

It is remarkable that the parity anomaly (13.110) is closely related to the chiral anomaly in a $(2l + 2)$-dimensional space (Alvarez-Gaumé *et al* 1985). Following Forte (1987), we look at the **dimensional ladder**,

$$\begin{gathered}\text{four-dimensional Abelian anomaly}\\ \downarrow \\ \text{three-dimensional parity anomaly} \\ \downarrow \\ \text{two-dimensional non-Abelian anomaly.}\end{gathered} \tag{13.112}$$

We take $M_4 = S^2 \times S^2$ as four-dimensional space. The Abelian anomaly is given by the index

$$\operatorname{ind} \mathrm{i}\nabla\!\!\!\!/_4 = \mathcal{N}_+ - \mathcal{N}_- = \int_{S^2\times S^2} \partial_\mu j_5^\mu = \int_{S^2\times S^2} \mathrm{ch}_2(\mathbb{F}). \tag{13.113}$$

As before, $\mathcal{N}_+$ ($\mathcal{N}_-$) is the number of positive (negative) chirality zero modes. Let Q_3 be the Chern–Simons form of $\mathrm{ch}_2(\mathbb{F})$; $\mathrm{ch}_2(\mathbb{F}) = \mathrm{d}Q_3(\mathbb{A}, \mathbb{F})$. Then $\mathcal{N} \equiv \mathcal{N}_+ - \mathcal{N}_-$ is given by

$$\begin{aligned}\mathcal{N} &= \int_{S^2\times S^2} \mathrm{ch}_2(\mathbb{F}) = \int_{U_{\mathrm{N}}\times S^2} \mathrm{d}Q_3(\mathbb{A}_{\mathrm{N}}, \mathbb{F}_{\mathrm{N}}) + \int_{U_{\mathrm{S}}\times S^2} \mathrm{d}Q_3(\mathbb{A}_{\mathrm{S}}, \mathbb{F}_{\mathrm{S}}) \\ &= \int_{S^1\times S^2} [Q_3(\mathbb{A}_{\mathrm{N}}, \mathbb{F}_{\mathrm{N}}) - Q_3(\mathbb{A}_{\mathrm{S}}, \mathbb{F}_{\mathrm{S}})] \\ &= \frac{1}{24\pi^2} \int_{S^1\times S^2} \mathrm{tr}\,(g^{-1}\,\mathrm{d}g)^3 \end{aligned} \tag{13.114}$$

where g is the gauge transformation connecting $\mathbb{A}_{\mathrm{N}}$ and $\mathbb{A}_{\mathrm{S}}$; $\mathbb{A}_{\mathrm{N}} =$

$g^{-1}(\mathbb{A}_S + \mathrm{d} + \mathrm{d}_\theta)g$. In the previous section, we have shown that $\mathcal{N}$ also represents the non-Abelian anomaly

$$\mathcal{N} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\, \frac{\partial w(\mathcal{A}, \theta)}{\partial \theta} \tag{13.115a}$$

where w is defined by

$$\det \mathrm{i}\hat{D}(\mathcal{A}^{g(\theta)}) = \mathrm{e}^{\mathrm{i}w(\mathcal{A},\theta)} \det \mathrm{i}\hat{D}(\mathcal{A}). \tag{13.115b}$$

Here $\mathcal{A}$ is the reference gauge potential and

$$\mathcal{A}^{g(\theta)} = g^{-1}(x, \theta)(\mathcal{A} + \mathrm{d})g(x, \theta) \qquad \mathrm{i}\hat{D} = \not{\partial} + \not{\mathcal{A}}\mathcal{P}_+.$$

Next we show that $\mathcal{N}$ is also related to the parity anomaly in three-dimensional space. Let $\mathrm{i}\not{\nabla}_3$ be a three-dimensional Dirac operator and define a four-dimensional Dirac operator by

$$\mathrm{i}\not{D}_4[\mathcal{A}] \equiv \mathrm{i}\sigma_1 \otimes \mathbb{1}\, \frac{\partial}{\partial t} + \sigma_2 \otimes \mathrm{i}\not{\nabla}_3[\mathcal{A}_t] \tag{13.116}$$

where $\mathcal{A}_t$ is a one-parameter family of gauge potentials interpolating $\mathcal{A}_0 = \mathcal{A}_{t=0}$ and $\mathcal{A}_1 = \mathcal{A}_{t=1}$. The Atiyah–Patodi–Singer index theorem (§12.8) is

$$\operatorname{ind} \mathrm{i}\not{D}_4 = -\int_{S^2\times S^1\times I} \mathrm{ch}_2(\mathcal{F}) + \tfrac{1}{2}[\eta(t=1) - \eta(t=0)] \tag{13.117}$$

where we have noted that the Dirac genus $\hat{A}$ is trivial on $S^2 \times S^1 \times I$. Suppose $\mathcal{A}_0$ and $\mathcal{A}_1$ are related by a gauge transformation,

$$\mathcal{A}_1 = g^{-1}(\mathcal{A}_0 + \mathrm{d})g \tag{13.118a}$$

and consider an interpolating potential

$$\mathcal{A}_t \equiv t\mathcal{A}_1 + (1-t)\mathcal{A}_0. \tag{13.118b}$$

Since the spectrum of $\mathrm{i}\not{\nabla}_3$ is gauge invariant, in particular $\operatorname{Spec} \mathrm{i}\not{\nabla}_3(\mathcal{A}_0) = \operatorname{Spec} \mathrm{i}\not{\nabla}_3(\mathcal{A}_1)$, the η-invariant is also gauge invariant. [Note that there is no gauge anomaly in odd-dimensional spaces.] Then $\eta(t=0) = \eta(t=1)$ and the APS index theorem (13.117) yields

$$\begin{aligned}\text{spectral flow} &= \operatorname{ind} \mathrm{i}\not{D}_4(\mathcal{A}_t)\\ &= \int_{S^2\times S^2} \mathrm{ch}_2(\mathcal{F}) = \int_{S^1\times S^2} [Q_3(\mathcal{A}_1, \mathcal{F}_1) - Q_3(\mathcal{A}_0, \mathcal{F}_0)]\\ &= \int_{S^1\times S^2} Q_3(g^{-1}\mathrm{d}g, 0) = \mathcal{N}.\end{aligned} \tag{13.119}$$

Thus the spectral flow of the three-dimensional theory is given by the index $\mathcal{N}$.

In summary, the map $g : S^2 \times S^1 \to G$ is understood in three different ways:

(1) g is a transition function at the boundary of two patches of a G bundle over $S^2 \times S^2$. It yields the index $\mathcal{N}$ of the four-dimensional Abelian anomaly.

(2) Suppose $\mathcal{A}_0$ and $\mathcal{A}_1 = g^{-1}(\mathcal{A}_0 + \mathrm{d})g$ are gauge potentials on $S^2 \times S^1$. The gauge transformation function g measures the spectral flow $\mathcal{N}$ between $\operatorname{Spec} \mathrm{i}\nabla\!\!\!/_3(\mathcal{A}_0)$ and $\operatorname{Spec} \mathrm{i}\nabla\!\!\!/_3(\mathcal{A}_1)$.

(3) $g : S^2 \times S^1 \to G$ induces a map $S^1 \to \mathcal{G}$, the winding number $\mathcal{N}$ of which is identified with the non-Abelian anomaly in two-dimensional space.

Thus we have obtained the **dimensional ladder** 4–3–2. The extension to higher dimensions is obvious.

14

BOSONIC STRING THEORY

In the present chapter, we study the one-loop amplitude of bosonic string theory. Our example is the simplest one; closed, oriented bosonic strings in 26-dimensional Euclidean space. [The reason for $D = 26$ will be clarified in §14.2.] The action is the Polyakov action

$$S = \frac{1}{2\pi}\int_{\Sigma_g} \mathrm{d}^2\xi\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{\lambda}{4\pi}\int_{\Sigma_g}\mathrm{d}^2\xi\sqrt{\gamma}\mathcal{R} \qquad (14.1)$$

where Σ_g is a Riemann surface with genus g. The second term is proportional to the Euler characteristic $\chi = 2 - 2g$ and hence determines the relative ratio of multiloop amplitudes; the g-loop amplitude is proportional to $\exp(-\lambda g)$. We have not written down the possible counter terms explicitly.

In the following sections we work out the path integral formalism of bosonic strings. We first develop the necessary mathematical tools, namely differential geometry on Riemann surfaces. Then we write down the path integral expression for the vacuum amplitude. As an example, we compute the one-loop vacuum amplitude. Our exposition is based on D'Hoker and Phong (1986), Polchinski (1986) and Moore and Nelson (1986). There are many surveys of these topics, for example, Alvarez-Gaumé and Nelson (1986), Bagger (1987), D'Hoker and Phong (1988) and Weinberg (1987).

14.1 Differential geometry on Riemann surfaces

Riemann surfaces are real two-dimensional manifolds without boundary. In our study of topology and geometry, we referred to them in various places. Here we summarise the basic facts on Riemann surfaces, which will make this chapter self-contained. We also introduce several new aspects of Riemann surfaces, which provide enough background for the study of bosonic string amplitudes.

14.1.1 Metric and complex structure

Let Σ_g be a Riemann surface of genus g. As was shown in example 7.32, we may introduce, in any chart U, the **isothermal coordinates** (ξ^1, ξ^2) in which the metric is conformally flat

$$g = \mathrm{e}^{2\sigma(\xi)}(\mathrm{d}\xi^1 \otimes \mathrm{d}\xi^1 + \mathrm{d}\xi^2 \otimes \mathrm{d}\xi^2). \tag{14.2}$$

Introduce the complex coordinates

$$z = \xi^1 + \mathrm{i}\xi^2 \qquad \bar{z} = \xi^1 - \mathrm{i}\xi^2. \tag{14.3}$$

Forms and vectors are spanned by

$$\mathrm{d}z = \mathrm{d}\xi^1 + \mathrm{i}\mathrm{d}\xi^2 \qquad \mathrm{d}\bar{z} = \mathrm{d}\xi^1 - \mathrm{i}\mathrm{d}\xi^2 \tag{14.4a}$$

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial\xi^1} - \mathrm{i}\,\frac{\partial}{\partial\xi^2}\right) \qquad \partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial\xi^1} + \mathrm{i}\,\frac{\partial}{\partial\xi^2}\right). \tag{14.4b}$$

In terms of the complex coordinates, the metric takes the form

$$g = \tfrac{1}{2}\mathrm{e}^{2\sigma(z,\bar{z})}[\mathrm{d}z \otimes \mathrm{d}\bar{z} + \mathrm{d}\bar{z} \otimes \mathrm{d}z]. \tag{14.5}$$

The components of g are

$$g_{z\bar{z}} = g_{\bar{z}z} = \tfrac{1}{2}\mathrm{e}^{2\sigma} \qquad g_{zz} = g_{\bar{z}\bar{z}} = 0 \tag{14.6a}$$

$$g^{z\bar{z}} = g^{\bar{z}z} = 2\mathrm{e}^{-2\sigma} \qquad g^{zz} = g^{\bar{z}\bar{z}} = 0. \tag{14.6b}$$

Let V be another chart of Σ_g such that $U \cap V \neq \varnothing$. Let $(w, \bar{w})$ be the complex coordinates in V. The metric in V is

$$g = \mathrm{e}^{2\sigma'(w,\bar{w})}\mathrm{d}w \otimes \mathrm{d}\bar{w}. \tag{14.7}$$

The two expressions (14.5) and (14.7) should agree on $U \cap V$,

$$\mathrm{e}^{2\sigma(z,\bar{z})}\mathrm{d}z \otimes \mathrm{d}\bar{z} = \mathrm{e}^{2\sigma'(w,\bar{w})}\mathrm{d}w \otimes \mathrm{d}\bar{w}.$$

Since

$$\begin{aligned}\mathrm{d}w \otimes \mathrm{d}\bar{w} &= [(\partial w/\partial z)\mathrm{d}z + (\partial w/\partial \bar{z})\mathrm{d}\bar{z}] \otimes [(\partial \bar{w}/\partial z)\mathrm{d}z + (\partial \bar{w}/\partial \bar{z})\mathrm{d}\bar{z}] \\ &\propto \mathrm{d}z \otimes \mathrm{d}\bar{z}\end{aligned}$$

we must have $\partial w/\partial \bar{z} = \partial \bar{w}/\partial z = 0$. [Another possibility, $\partial w/\partial z = \partial \bar{w}/\partial \bar{z} = 0$ is ruled out if $(z, \bar{z})$ and $(w, \bar{w})$ define the same orientation.] Thus it follows that

$$w = w(z), \quad \bar{w} = \bar{w}(\bar{z}) \tag{14.8}$$

which verifies that Σ_g is a complex manifold. We also have

$$\mathrm{e}^{2\sigma(z,\bar{z})} = \mathrm{e}^{2\sigma'(w,\bar{w})}|\partial w/\partial z|^2. \tag{14.9}$$

14.1.2 Vectors, forms and tensors

The components of vector fields $V^z\partial/\partial z \in TM^+$ and $V^{\bar{z}}\partial/\partial \bar{z} \in TM^-$ transform as

$$V^w = (\partial w/\partial z)V^z \qquad V^{\bar{w}} = (\partial \bar{w}/\partial \bar{z})V^{\bar{z}}. \tag{14.10}$$

The components of differential forms $w_z \, dz \in \Omega^{1,0}(M)$ and $\omega_{\bar{z}} \, d\bar{z} \in \Omega^{0,1}(M)$ transform as

$$\omega_w = (\partial w/\partial z)^{-1}\omega_z \qquad \omega_{\bar{w}} = (\partial \bar{w}/\partial \bar{z})^{-1}\omega_{\bar{z}}. \tag{14.11}$$

These are identified with sections of the holomorphic (antiholomorphic) line bundles over Σ_g, for which the transition functions are holomorphic (antiholomorphic). The metric provides a natural isomorphism between TM^+ and $\Omega^{0,1}(M)$ through

$$\omega_{\bar{z}} = g_{\bar{z}z}V^z, \quad V^z = g^{z\bar{z}}\omega_{\bar{z}}. \tag{14.12}$$

Similarly TM^- is isomorphic to $\Omega^{1,0}(M)$:

$$\omega_z = g_{z\bar{z}}V^{\bar{z}}, \quad V^{\bar{z}} = g^{\bar{z}z}\,\omega_z. \tag{14.13}$$

In general, given an arbitrary tensor, the metric allows us to trade all the $\bar{z}$-indices for z-indices. It is easy to see that

$$T^{\overbrace{z\ldots z}^{q_1}\overbrace{\bar{z}\ldots\bar{z}}^{q_2}}{}_{\underbrace{z\ldots z}_{p_1}\underbrace{\bar{z}\ldots\bar{z}}_{p_2}} \longrightarrow T^{\overbrace{z\ldots z}^{q_1+p_2}}{}_{\underbrace{z\ldots z}_{p_1+q_2}} = (g_{z\bar{z}})^{q_2}(g^{z\bar{z}})^{p_2}\, T^{\overbrace{z\ldots z}^{q_1}\overbrace{\bar{z}\ldots\bar{z}}^{q_2}}{}_{\underbrace{z\ldots z}_{p_1}\underbrace{\bar{z}\ldots\bar{z}}_{p_2}}. \tag{14.14}$$

This correspondence is an isomorphism. For example, we have

$$T_{z\bar{z}}{}^{\bar{z}} \longrightarrow g^{z\bar{z}}g_{z\bar{z}}T_{z\bar{z}}{}^{\bar{z}} = T_z{}^z{}_z.$$

Thus it is only necessary to consider tensors with pure z-indices. For these tensors, we assign the helicity. Since T has z-indices only, it transforms under $z \longrightarrow w$ as

$$T \longrightarrow \left(\frac{\partial w}{\partial z}\right)^n T \tag{14.15}$$

where $n \in \mathbb{Z}$ is given by the number of upper z-indices minus the number of lower z-indices. For example,

$$T^{zz}{}_z \longrightarrow T^{ww}{}_w = \left(\frac{\partial w}{\partial z}\right)T^{zz}{}_z.$$

All that matters is the *difference* between the number of upper indices and the number of lower indices. $T^z{}_z$ is left invariant under $z \longrightarrow w$ and is regarded as a scalar. The number n is called the **helicity**. The space of helicity-n tensors is denoted by $\mathcal{T}^n$;

$$\mathcal{T}^n \equiv \{T^{\overbrace{z\ldots z}^{q}}{}_{\underbrace{z\ldots z}_{p}} \,|\, q - p = n\}. \tag{14.16}$$

The helicity characterises the irreducible representation of $\mathrm{U}(1) = \mathrm{SO}(2)$.

[*Remarks:* So far we have assumed n is an integer. It can be shown that

$n = \frac{1}{2}$ corresponds to the spinor field on Σ_g. In fact the existence of spinors on the Riemann surfaces is guaranteed by the triviality of the second Stiefel–Whitney class of Σ_g. $\mathcal{T}^1$ is identified with the holomorphic line bundle K over Σ_g. Then $\mathcal{T}^{1/2}$ is the *square root* of K: $S_+^2 = K = \mathcal{T}^1$, where S_+ is the positive-chirality spin bundle. Similarly we have $\mathcal{T}^{-1} = \bar{K} = S_-^2$ where S_- is the negative-chirality spin bundle.]

Example 14.1 In real indices, the helicity ± 1 vectors are given by $V^1 \pm \mathrm{i}V^2$. This follows since

$$V^1 \frac{\partial}{\partial \xi^1} + V^2 \frac{\partial}{\partial \xi^2} = (V^1 + \mathrm{i}V^2)\partial_z + (V^1 - \mathrm{i}V^2)\partial_{\bar{z}}.$$

We put $V^z = V^1 + \mathrm{i}V^2$ and $V^{\bar{z}} = V^1 - \mathrm{i}V^2 \simeq V_z$. The helicity ± 2 tensors are $T^{11} \pm \mathrm{i}T^{12}$, where T is a *symmetric traceless* tensor of rank two. In fact, we find

$$T^{11}\left(\frac{\partial}{\partial \xi^1} \otimes \frac{\partial}{\partial \xi^1} - \frac{\partial}{\partial \xi^2} \otimes \frac{\partial}{\partial \xi^2}\right) + T^{12}\left(\frac{\partial}{\partial \xi^1} \otimes \frac{\partial}{\partial \xi^2} + \frac{\partial}{\partial \xi^2} \otimes \frac{\partial}{\partial \xi^1}\right)$$

$$= 2(T^{11} + \mathrm{i}T^{12})\partial_z \otimes \partial_z + 2(T^{11} - \mathrm{i}T^{12})\partial_{\bar{z}} \otimes \partial_{\bar{z}}.$$

Clearly $T^{zz} = 2(T^{11} + \mathrm{i}T^{12})$ has helicity $+2$ and $T^{\bar{z}\bar{z}} = 2(T^{11} - \mathrm{i}T^{12})$ has helicity -2 (note that $g_{z\bar{z}}g_{z\bar{z}}T^{\bar{z}\bar{z}} = T_{zz}$).

14.1.3 Covariant derivatives

The only non-vanishing Christoffel symbols of Σ_g are (see (8.69))

$$\Gamma^z{}_{zz} = g^{z\bar{z}}\partial_z g_{z\bar{z}} = 2\partial_z\sigma, \quad \Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} = g^{\bar{z}z}\partial_{\bar{z}} g_{\bar{z}z} = 2\partial_{\bar{z}}\sigma. \tag{14.17}$$

For tensors in $\mathcal{T}^n$, we define two kinds of covariant derivatives: $\nabla^z_{(n)} : \mathcal{T}^n \to \mathcal{T}^{n+1}$ and $\nabla^{(n)}_z : \mathcal{T}^n \to \mathcal{T}^{n-1}$. Let

$$T^{\overbrace{z\ldots z}^{q}}{}_{\underbrace{z\ldots z}_{p}} \in \mathcal{T}^n (q - p = n).$$

We define

$$\begin{aligned}
\nabla^z_{(n)} T^{z\ldots z}{}_{z\ldots z} &= g^{z\bar{z}}\nabla_{\bar{z}} T^{z\ldots z}{}_{z\ldots z} \\
&= g^{z\bar{z}}[\partial_{\bar{z}} + (q-p)\Gamma^z{}_{\bar{z}z}]T^{z\ldots z}{}_{z\ldots z} \\
&= g^{z\bar{z}}\partial_{\bar{z}} T^{z\ldots z}{}_{z\ldots z}
\end{aligned} \tag{14.18a}$$

$$\begin{aligned}
\nabla^{(n)}_z T^{z\ldots z}{}_{z\ldots z} &= \nabla_z T^{z\ldots z}{}_{z\ldots z} \\
&= [\partial_z + (q-p)\Gamma^z{}_{zz}]T^{z\ldots z}{}_{z\ldots z} \\
&= (\partial_z + 2n\partial_z\sigma)T^{z\ldots z}{}_{z\ldots z}.
\end{aligned} \tag{14.18b}$$

In (14.18*b*), $2n\partial_z\sigma$ acts like a gauge potential $\mathcal{A}$. We also define covariant derivatives with respect to $\bar{z}$,

$$\nabla^{\bar{z}}_{(n)} = g^{\bar{z}z}\nabla^{(n)}_{z}, \quad \nabla^{(n)}_{\bar{z}} = g_{\bar{z}z}\nabla^{z}_{(n)}. \tag{14.19}$$

The curvature two-form of K and the scalar curvature associated with the Christoffel symbols are

$$\mathcal{F} = R^{z}{}_{zz\bar{z}}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} = -\partial_{\bar{z}}(2\partial_z\sigma)\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

$$= -2\partial_z\partial_{\bar{z}}\sigma\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{14.20a}$$

$$\mathcal{R} = g^{\bar{z}z}Ric_{\bar{z}z} + g^{z\bar{z}}Ric_{z\bar{z}} = -8\mathrm{e}^{-2\sigma}\partial_z\partial_{\bar{z}}\sigma. \tag{14.20b}$$

Exercise 14.2 Verify that

$$\nabla^{z}_{(n)} = 2\mathrm{e}^{-2\sigma}\partial_{\bar{z}}, \quad \nabla^{(n)}_{z} = \mathrm{e}^{-2n\sigma}\partial_z\mathrm{e}^{2n\sigma} \tag{14.21a}$$

$$\nabla^{\bar{z}}_{(n)} = 2\mathrm{e}^{-2(n+1)\sigma}\partial_z\mathrm{e}^{2n\sigma}, \quad \nabla^{(n)}_{\bar{z}} = \partial_{\bar{z}}. \tag{14.21b}$$

$\nabla^{z}_{(n)}$ and $\nabla^{(n)}_{z}$ are mutual adjoints with respect to a properly defined inner product. Let $T, U \in \mathcal{T}^n$. We require that the inner product be invariant under a holomorphic change of the coordinate $z \to w$. Since

$$g_{z\bar{z}} \to |\mathrm{d}w/\mathrm{d}z|^{-2}g_{z\bar{z}}, \qquad \mathrm{d}^2z\sqrt{g} \to \mathrm{d}^2w\sqrt{g}$$

$$\bar{T} \to \overline{(\mathrm{d}w/\mathrm{d}z)}^n\bar{T}, \qquad U \to (\mathrm{d}w/\mathrm{d}z)^nU$$

we find the combination

$$(T, U) \equiv \int \mathrm{d}^2z\sqrt{g}\,(g_{z\bar{z}})^n\bar{T}U \tag{14.22}$$

is invariant under holomorphic coordinate transformations. Take $T \in \mathcal{T}^n$ and $U \in \mathcal{T}^{n+1}$. We find that

$$(U, \nabla^{z}_{(n)}T) = \int \mathrm{d}^2z\mathrm{e}^{2\sigma}2^{-n-1}\mathrm{e}^{2(n+1)\sigma}\bar{U}2\mathrm{e}^{-2\sigma}\partial_{\bar{z}}T$$

$$= -2^{-n}\int \mathrm{d}^2zT\partial_{\bar{z}}[\mathrm{e}^{(2n+1)\sigma}\bar{U}] \qquad \text{(partial integration)}$$

$$= -2^{-n}\int \mathrm{d}^2zT\mathrm{e}^{(2n+1)\sigma}\,\overline{[\partial_zU + (2n+1)(\partial_z\sigma)U]}$$

$$= -\overline{\int \mathrm{d}^2z\sqrt{g}\,(g_{z\bar{z}})^n[\nabla^{(n+1)}_{z}U]\bar{T}} = \overline{(-\nabla^{(n+1)}_{z}U, T)}.$$

This shows that

$$(\nabla^{z}_{(n)})^{\dagger} = -\nabla^{(n+1)}_{z}. \tag{14.23a}$$

Exercise 14.3 Show that

$$(\nabla^{(n)}_{z})^{\dagger} = -\nabla^{z}_{(n-1)}. \tag{14.23b}$$

We define two kinds of Laplacians $\Delta^{\pm}_{(n)} : \mathcal{T}^n \to \mathcal{T}^{n\pm1} \to \mathcal{T}^n$ by

$$\Delta^{+}_{(n)} \equiv -\nabla^{(n+1)}_{z}\nabla^{z}_{(n)} = -2\mathrm{e}^{-2\sigma}[\partial_z\partial_{\bar{z}} + 2n(\partial_z\sigma)\partial_{\bar{z}}] \tag{14.24a}$$

$$\Delta^{-}_{(n)} \equiv -\nabla^{z}_{(n-1)}\nabla^{(n)}_{z} = -2\mathrm{e}^{-2\sigma}[\partial_z\partial_{\bar{z}} + 2n(\partial_z\sigma)\partial_{\bar{z}} + 2n(\partial_z\partial_{\bar{z}}\sigma)]. \tag{14.24b}$$

Then it follows that

$$\Delta^+_{(n)} - \Delta^-_{(n)} = 4n\mathrm{e}^{-2\sigma}(\partial_z\partial_{\bar z}\sigma) = -\tfrac{1}{2}n\mathcal{R}. \tag{14.25}$$

This shows, in particular, that

$$\Delta^+_{(0)} = \Delta^-_{(0)} \ (\equiv \Delta_{(0)}). \tag{14.26}$$

14.1.4 The Riemann–Roch theorem

Here we derive a version of the Riemann–Roch theorem from the Atiyah–Singer index theorem following D'Hoker and Phong (1988).

Theorem 14.4 (**Riemann–Roch**) Let Σ_g be a Riemann surface of genus g. Then the index of the operator $\nabla^{(n)}_z$ is

$$\dim_{\mathbb{C}} \ker \nabla^{(n)}_z - \dim_{\mathbb{C}} \ker \nabla^z_{(n-1)} = (2n-1)(g-1). \tag{14.27}$$

Proof: We use the heat kernel to evaluate the index. We first note that $\ker \nabla^{(n)}_z = \ker \Delta^-_{(n)}$ and $\ker \nabla^z_{(n-1)} = \ker \Delta^+_{(n-1)}$ (see (14.24)). The heat kernel $\mathcal{K}^+_n$ of $\Delta^+_{(n)}$ satisfies

$$\left(\frac{\partial}{\partial t} + \Delta^+_{(n)}\right) \mathcal{K}^+_n(z, w; t) = \left(\frac{\partial}{\partial t} + \Delta - V_n\right) \mathcal{K}^+_n(z, w; t) = 0$$

where $\Delta \equiv -2\partial_z\partial_{\bar z}$ is the flat-space Laplacian and

$$V_n \equiv \Delta - \Delta^+_{(n)} = (1 - \mathrm{e}^{-2\sigma})\Delta + 4n\mathrm{e}^{-2\sigma}\partial_z\sigma\partial_{\bar z}.$$

Δ also defines a heat kernel by

$$\left(\frac{\partial}{\partial t} + \Delta\right)K(z, w; t) = 0$$

which is easily solved to yield

$$K(z, w; t) = \frac{1}{4\pi t}\mathrm{e}^{-|z-w|^2/2t}.$$

The perturbative computation and iteration yield

$$\begin{aligned}
\mathcal{K}^+_n(z, z'; t) &= K(z, z'; t) \\
&\quad + \int_0^t \mathrm{d}s \int \mathrm{d}w K(z, w; t-s)V_n(w)\mathcal{K}^+_n(w, z'; s) \\
&= K(z, z'; t) + \int \mathrm{d}s \int \mathrm{d}w K(z, w; t-s)V_n(w)K(w, z'; s) \\
&\quad + \int \mathrm{d}s \int \mathrm{d}s' \int \mathrm{d}v \int \mathrm{d}w K(z, v; t-s)V_n(v) \\
&\quad \times K(v, w; s-s')V_n(w)K(w, z'; s') \\
&\quad + \ldots.
\end{aligned}$$

We are particularly interested in $\mathcal{K}_n^+(z, z; t)$, t being small,

$$\mathcal{K}_n^+(z, z; t) = \frac{1}{4\pi t} + \int_0^t ds \int dw K(z, w; t-s) V_n(w) K(w, z; s) + O(t). \tag{14.28}$$

If we take a coordinate system in which $\sigma = 0$ at z, we have

$$\sigma(w) \simeq 0 + \partial_z\sigma(w - z) + \partial_{\bar z}\sigma(\bar w - \bar z) \\ + \tfrac{1}{2}[\partial_z^2\sigma(w-z)^2 + \partial_{\bar z}^2\sigma(\bar w - \bar z)^2 + 2\partial_z\partial_{\bar z}\sigma|w - z|^2] + \dots.$$

Due to rotational symmetry in two-dimensional space, only those terms with one z-derivative and one $\bar z$-derivative survive in the integral of (14.28). Terms proportional to $\partial_z\sigma\partial_{\bar z}\sigma$ cancel between the second and third terms in the expansion and we are left with terms proportional to $\partial_z\partial_{\bar z}\sigma$. Now we have to evaluate

$$\int_0^t ds \int d^2w K(z, w; t-s) \\ \times [2\partial_z\partial_{\bar z}\sigma|\bar w - \bar z|^2\Delta_w + 4n(\bar w - \bar z)\partial_z\partial_{\bar z}\sigma\partial_{\bar w}]K(w, z; s).$$

From the identities

$$\int d^2w K(z, w; t-s)|w - z|^2\Delta_w K(w, z; s) \\ = \frac{1}{16\pi^2 s^2(t-s)}\int d^2w|w|^2\exp\left(-\frac{t}{2s(t-s)}|w|^2\right) \\ - \frac{1}{32\pi^2 s^3(t-s)}\int d^2w|w|^4\exp\left(-\frac{t}{2s(t-s)}|w|^2\right) \\ = \frac{(t-s)(2s-t)}{2\pi t^3}$$

$$\int d^2w K(z, w; t-s)(\bar z - \bar w)\partial_{\bar w}K(w, z; s) \\ = \frac{1}{32\pi^2 s^2(t-s)}\int d^2w \exp\left(-\frac{t}{2s(t-s)}|w|^2\right) = \frac{t-s}{4\pi t^2}$$

we find that

$$\mathcal{K}_n^+(z, z; t) = \frac{1}{4\pi t} + \frac{1+3n}{12\pi}\Delta\sigma + O(t). \tag{14.29a}$$

We also have the diagonal part of the heat kernel $\mathcal{K}_n^-$ for $\Delta_{(n)}^-$,

$$\mathcal{K}_n^-(z, z; t) = \frac{1}{4\pi t} + \frac{1-3n}{12\pi}\Delta\sigma + O(t). \tag{14.29b}$$

From (14.29) and (14.20b), we have

$$
\begin{aligned}
\operatorname{ind}\nabla_z^{(n)} &= \int \mathrm{d}^2 z\left(\frac{1-3n}{12\pi} - \frac{1+3(n-1)}{12\pi}\right)\Delta\sigma = \frac{1-2n}{8\pi}\int \mathrm{d}^2 x\sqrt{g}\mathcal{R} \\
&= -\frac{2n-1}{2}\chi(\Sigma_g) = (2n-1)(g-1)
\end{aligned}
$$

where

$$\chi = \frac{1}{4\pi}\int \mathrm{d}^2 x\sqrt{g}\mathcal{R} = 2 - 2g$$

is the Euler characteristic of Σ_g. ∎

14.2 Quantum theory of bosonic strings

Now we are ready to introduce Polyakov's formulation of bosonic strings, which is based on the path integral over *geometries*. Since the string action contains an enormous symmetry, we have to pay special attention to counting independent geometries once and only once. This is achieved by the *Faddeev–Popov* trick. Our argument will be restricted to the simplest case, namely *closed orientable bosonic* strings; the theory is defined on Riemann surfaces.

14.2.1 Vacuum amplitude of Polyakov strings

According to the general prescription of the path integral formalism, the partition function (vacuum-to-vacuum amplitude) of the string theory is given by

$$Z = \sum_{g=0}^{\infty} Z_g = \sum_{g=0}^{\infty}\int \mathcal{D}X\mathcal{D}\gamma\, \mathrm{e}^{-S[X,\gamma]} \tag{14.30}$$

see figure 14.1. To avoid confusion, we denote the genus by g and the metric by γ. The sum over genera amounts to the sum over the topologies. Z_g is the g-loop amplitude and is obtained by integrating over all metrics γ and all embeddings X. As we shall see below, the measure $\mathcal{D}X\mathcal{D}\gamma$ is not well defined and we need some modifications.

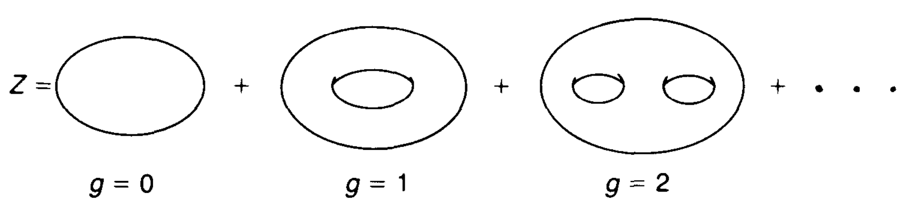


Figure 14.1 The total vacuum amplitude is given by summing over g-loop amplitudes.

The string action $S[X,\gamma]$ is taken to be

$$S[X,\gamma] \equiv \frac{1}{2}\int \mathrm{d}^2\xi\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + \frac{\lambda}{4\pi}\int \mathrm{d}^2\xi\sqrt{\gamma}\mathcal{R}. \tag{14.31}$$

The first term is the Polyakov action. The second term is proportional to the Euler characteristic

$$\chi = \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma} \mathcal{R} = 2 - 2g$$

and serves as the *string coupling constant*; the amplitude of a loop with genus g is suppressed by the factor $e^{-2\lambda g}$. Since this term is a topological invariant, it does not affect the dynamics of the string. We are interested in Riemann surfaces of a fixed genus g and drop this term. The first term of the action has the following symmetries (§7.10):

(A) $\mathrm{Diff}(\Sigma_g)$, the group of diffeomorphisms $f : \Sigma_g \to \Sigma_g$. Let $\xi^\alpha \to \xi'^\alpha(\xi)$ be the coordinate expression for f. The new metric is the pullback of the old one whose coordinate component expression is

$$\gamma_{\alpha\beta} \to f^*\gamma_{\alpha\beta} = \frac{\partial \xi^\gamma}{\partial \xi'^\alpha} \frac{\partial \xi^\delta}{\partial \xi'^\beta} \gamma_{\gamma\delta}. \tag{14.32}$$

The embedding also gets transformed as

$$X^\mu \to f^*X^\mu = X^\mu f. \tag{14.33}$$

The invariance of the classical action takes the form

$$S[X, \gamma] = S[f^*X, f^*\gamma]. \tag{14.34}$$

(B) $\mathrm{Weyl}(\Sigma_g)$, the group of two-dimensional Weyl rescalings

$$\gamma_{\alpha\beta} \to \hat{\gamma}_{\alpha\beta} \equiv e^\phi \gamma_{\alpha\beta} \tag{14.35}$$

where $\phi \in \mathcal{F}(\Sigma_g)$. The conformal invariance of S takes the form

$$S[X, \gamma] = S[X, \hat{\gamma}]. \tag{14.36}$$

The symmetries (A) and (B) must be preserved under quantisation, otherwise the theory has anomalies.

According to the standard Faddeev–Popov formalism, the degrees of freedom corresponding to the above symmetries have to be omitted when we define Z_g. For example, the string geometry specified by the pairs (X_1, γ_1) and (X_2, γ_2) should not be counted independently if they are related by an element of $\mathrm{Diff}(\Sigma_g)$. Similarly (X, γ) and $(X, e^\phi\gamma)$ should not be counted as independent configurations. Unless special attention is paid, we would count the same configurations infinitely many times, which leads to disastrous divergences. It turns out that the space of all the geometries (X, γ) can be separated into equivalence classes (the **gauge slice**), any two points of which cannot be connected by the above symmetries, see figure 14.2.

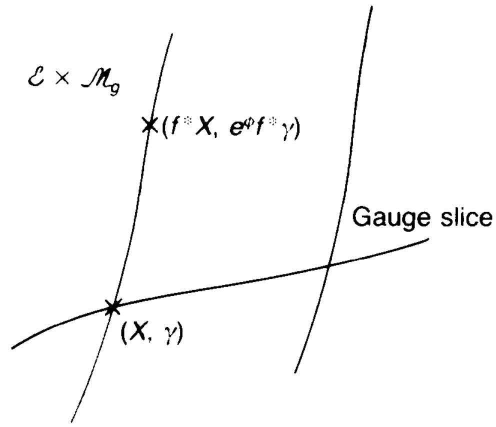


Figure 14.2 An element of $\mathcal{E} \times \mathcal{M}_g$ is obtained by the action of $\mathrm{Diff}(\Sigma_g)*\mathrm{Weyl}(\Sigma_g)$ on an element (X, γ) in the gauge slice.

To be more mathematical, let $\mathcal{E}$ be the space of all the embeddings $X : \Sigma_g \to \mathbb{R}^D$ and let $\mathcal{M}_g$ be the space of all the metrics defined on Σ_g. Naïvely, the path integral is defined over $\mathcal{E} \times \mathcal{M}_g$. Because of the symmetries (A) and (B), however, the integral should be restricted to the quotient space $(\mathcal{E} \times \mathcal{M}_g)/G$ where $G = \mathrm{Diff}(\Sigma_g)*\mathrm{Weyl}(\Sigma_g)$ is the gauge group. [$*$ is the semi-direct product. Note that $\mathrm{Diff}(\Sigma_g) \cap \mathrm{Weyl}(\Sigma_g) \neq \emptyset$. We shall come back to this point later.] The action of $(f, e^{\phi}) \in G$ on $(X, \gamma) \in \mathcal{E} \times \mathcal{M}_g$ is

$$(f, e^{\phi})(X, \gamma) = (f^*X, e^{\phi}f^*\gamma). \tag{14.37}$$

The quotient $\mathcal{M}_g/G$ is called the **moduli space** of Σ_g and is denoted by $\mathrm{Mod}(\Sigma_g)$. We are also interested in the subgroup $\mathrm{Diff}_0(\Sigma_g)$ of $\mathrm{Diff}(\Sigma_g)$, which is a connected component of the identity map. $\mathrm{Teich}(\Sigma_g) \equiv \mathcal{M}_g/\mathrm{Diff}_0(\Sigma_g)*\mathrm{Weyl}(\Sigma_g)$ is called the **Teichmüller space** of Σ_g. The general theory of Riemann surfaces shows that $\mathrm{Teich}(\Sigma_g)$ is a finite-dimensional universal covering space of $\mathrm{Mod}(\Sigma_g)$. Explicitly we have

$$\dim_{\mathbb{R}} \mathrm{Teich}(\Sigma_g) = \begin{cases} 0 & g = 0 \\ 2 & g = 1 \\ 6g - 6 & g \geqslant 2. \end{cases} \tag{14.38}$$

The group $\mathrm{Diff}(\Sigma_g)/\mathrm{Diff}_0(\Sigma_g)$ is known as the **modular group** (MG) or the **mapping class group** (MCG). The MCG is generated by the **Dehn twists** defined in example 8.3. For the torus with genus g, the MCG is generated by $3g - 1$ Dehn twists around α_i, β_i and γ_i in figure 14.3. Unfortunately, these $3g - 1$ Dehn twists are *not* the minimal set of the generators. The general form of MCG for $g \geqslant 2$ is not well understood.

From the above arguments, the meaningful partition function turns out to be

$$Z_g \equiv \int_{\mathcal{X} \times \mathcal{M}_g} \frac{\mathcal{D}X\mathcal{D}\gamma}{V(\text{Diff}*\text{Weyl})} \mathrm{e}^{-S[X,\gamma]} \tag{14.39}$$

where $V(\text{Diff}*\text{Weyl})$ is the (infinite) volume of the space of $\text{Diff}(\Sigma_g)*\text{Weyl}(\Sigma_g)$ and takes care of the infinite overcounting of the same geometry. The order (the number of elements) of MCG is denoted by $|\text{MCG}|$. Clearly

$$V(\text{Diff}*\text{Weyl}) = |\text{MCG}|\, V(\text{Diff}_0*\text{Weyl}). \tag{14.40}$$

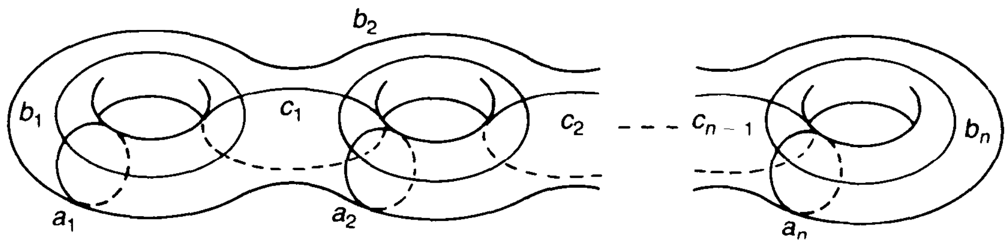


Figure 14.3 The mapping class group (MCG) is generated by Dehn twists around a_i, b_i and c_i $(1 \leqslant i \leqslant g)$.

14.2.2 Measures of integration

To carry out the integration (14.39) we have to define a sensible measure so that the physical degrees of freedom and the gauge degrees of freedom are separated. This separation of degrees of freedom requires the Jacobian,

$$\mathcal{D}\gamma\mathcal{D}X \to J(\mathcal{D}\text{ physical})(\mathcal{D}\text{ gauge}). \tag{14.41}$$

To find this Jacobian, we note that the Jacobian on a manifold M agrees with that on TM. To see this, let $x^\mu(y^\mu)$ be a coordinate of a chart $U(V)$ of M such that $U \cap V \neq \varnothing$. The Jacobian of the coordinate change is $J = \det(\partial y^\mu/\partial x^\nu)$. Take $V \in T_pM$. In components we have $V = u^\mu\partial/\partial x^\mu = v^\mu\partial/\partial y^\mu$, where

$$v^\mu = u^\nu(\partial y^\mu/\partial x^\nu). \tag{14.42}$$

$\{u^\mu\}$ and $\{v^\mu\}$ are coordinates of T_pM. The Jacobian $\hat{J}$ associated with this coordinate change is

$$\hat{J} = \det(\partial v^\mu/\partial u^\nu) = \det(\partial y^\mu/\partial x^\nu) = J. \tag{14.43}$$

This shows that the *Jacobian* at $p \in M$ is the *same* as that on T_pM. $\hat{J}$ depends on p but *not* on the vector itself, since J depends only on p.

Example 14.5 Let (x, y) and (r, θ) be coordinates of $\mathbb{R}^2$, where $x = r\cos\theta$ and $y = r\sin\theta$. The Jacobian of the coordinate change is

$$J = \det\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$

Let us take

$$V = v_x \partial/\partial x + v_y \partial/\partial y = v_r \partial/\partial r + v_\theta \partial/\partial \theta \in T_p\mathbb{R}^2.$$

(v_x, v_y) and (v_r, v_θ) serve as coordinates of $T_p\mathbb{R}^2$. Since

$$v_x = v_r \partial x/\partial r + v_\theta \partial x/\partial \theta \qquad v_y = v_r \partial y/\partial r + v_\theta \partial y/\partial \theta$$

the associated Jacobian $\hat{J}$ is easily calculated to be

$$\hat{J} = \det[\partial(v_x, v_y)/\partial(v_r, v_\theta)] = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = J.$$

Let us derive this Jacobian in an indirect but suggestive way. We normalise the measure $\mathrm{d}^2 v$ as

$$1 = \int \mathrm{d}^2 v \exp(-\tfrac{1}{2}\|v\|^2) = \int \mathrm{d}v_x \mathrm{d}v_y \exp[-\tfrac{1}{2}(v_x^2 + v_y^2)].$$

[This normalisation of the measure differs by a constant factor from the conventional one.] We also have $\|v\|^2 = v_r^2 + r^2 v_\theta^2$. Noting that the Jacobian is independent of v_r and v_θ, we have

$$1 = J \int \mathrm{d}v_r \mathrm{d}v_\theta \exp[-\tfrac{1}{2}(v_r^2 + r^2 v_\theta^2)] = J r^{-1}$$

from which we find $J = r$. We use this procedure to find the functional measure of string theory. [It should be kept in mind that we introduce the tangent space only to obtain the Jacobian. The tangent space itself has no physical relevance.]

The analysis above enables us to write

$$\mathcal{D}\delta\gamma \mathcal{D}\delta X = J \mathcal{D}\delta(\text{physical}) \mathcal{D}\delta(\text{gauge}) \tag{14.44}$$

where $\delta\gamma$ (δX) is a small variation of the metric γ (the embedding X) and is regarded as an element of $T_\gamma \mathcal{M}_g$ ($T_X \mathcal{E}$). The meaning of the RHS becomes clear in a moment.

Consider the diffeomorphism generated by an infinitesimal vector field δv on Σ_g. Since δv is infinitesimal, it belongs to $\mathrm{Diff}_0(\Sigma_g)$ rather than the full group $\mathrm{Diff}(\Sigma_g)$. The changes of the metric and the embedding under δv are (see (7.120))

$$\delta_{\mathrm{D}} \gamma_{\alpha\beta} = (\mathcal{L}_{\delta v}\gamma)_{\alpha\beta} = \nabla_\alpha \delta v_\beta + \nabla_\beta \delta v_\alpha \qquad \delta_{\mathrm{D}} X = \delta v^\alpha \partial_\alpha X. \tag{14.45}$$

The changes of γ and X under an infinitesimal Weyl rescaling $\mathrm{e}^{\delta\phi}$ are

$$\delta_{\mathrm{W}} \gamma_{\alpha\beta} = \delta\phi \gamma_{\alpha\beta} \qquad \delta_{\mathrm{W}} X = 0. \tag{14.46}$$

These changes belong to unphysical (gauge) degrees of freedom. In general, a small change of metric is given by

$$\begin{aligned} \delta\gamma_{\alpha\beta} &= \delta_{\mathrm{W}}\gamma_{\alpha\beta} + \delta_{\mathrm{D}}\gamma_{\alpha\beta} + (\text{physical change}) \\ &= \delta\phi\gamma_{\alpha\beta} + \nabla_\alpha \delta v_\beta + \nabla_\beta \delta v_\alpha + \delta t^i \frac{\partial}{\partial t^i} \gamma_{\alpha\beta}(t) \end{aligned} \tag{14.47}$$

where the last term is called the **Teichmüller deformation** of the metric, which can neither be described by a diffeomorphism nor by a Weyl rescaling. As mentioned before, $\{i\}$ is a finite set, $1 \leqslant i \leqslant n = \dim_{\mathbb{R}} \mathrm{Teich}(\Sigma_g)$. It is convenient for later purposes to separate $\delta\gamma$ into a traceless part and a part with a non-zero trace. We write

$$\delta\gamma_{\alpha\beta} = \delta\bar{\phi}\gamma_{\alpha\beta}+(P_1\delta v)_{\alpha\beta} + \delta t^i T_{i\alpha\beta}(t) \tag{14.48}$$

where $T_{i\alpha\beta}$ is the traceless part of the Teichmüller deformation,

$$T_{i\alpha\beta} \equiv \frac{\partial\gamma_{\alpha\beta}}{\partial t^i} - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\gamma\delta}\frac{\partial\gamma_{\gamma\delta}}{\partial t^i}. \tag{14.49}$$

The operator P_1 is defined by

$$(P_1\delta v)_{\alpha\beta} \equiv \nabla_\alpha\delta v_\beta + \nabla_\beta\delta v_\alpha - \gamma_{\alpha\beta}(\nabla_\gamma\delta v^\gamma) \tag{14.50}$$

and picks up the traceless part of $\delta_{\mathrm{D}}\gamma_{\alpha\beta}$. $\delta\bar{\phi}$ is defined by

$$\delta\bar{\phi} = \delta\phi + \left(\nabla_\gamma\delta v^\gamma + \text{trace part of } \delta t\,\frac{\partial\gamma}{\partial t}\right) \tag{14.51}$$

where we do not need the explicit form in the parenthesis.

As for the embeddings, we consider the quotient $\mathcal{E}/\mathrm{Diff}(\Sigma_g)$. An arbitrary embedding X is obtained by the action of $\mathrm{Diff}(\Sigma_g)$ on some $\widetilde{X} \in \mathcal{E}/\mathrm{Diff}(\Sigma_g)$. Then a small change of the embedding is expressed as

$$\delta X = \delta v^\alpha\partial_\alpha\widetilde{X} + \delta\widetilde{X} \tag{14.52}$$

where the first term represents the change of X generated by δv while the second is not associated with diffeomorphisms. Now the measure should look like

$$\mathcal{D}\delta\gamma\mathcal{D}\delta X = J\mathrm{d}^n t\,\mathcal{D}\delta v\mathcal{D}\delta\phi\mathcal{D}\delta\widetilde{X}. \tag{14.53}$$

To define the measure, we need to specify a *metric* on the tangent space, see example 14.5. We restrict ourselves to the so called *ultralocal* metric which is quadratic and depends on $\gamma_{\alpha\beta}$ but not on $\partial\gamma_{\alpha\beta}$. Define a metric for symmetric second-rank tensors by

$$||\delta h||_\gamma^2 \equiv \int \mathrm{d}^2\xi\sqrt{\gamma}(G^{\alpha\beta\gamma\delta} + u\gamma^{\alpha\beta}\gamma^{\gamma\delta})\delta h_{\alpha\beta}\delta h_{\gamma\delta} \tag{14.54a}$$

where $u > 0$ is an arbitrary constant and

$$G^{\alpha\beta\gamma\delta} \equiv \gamma^{\alpha\gamma}\gamma^{\beta\delta} + \gamma^{\alpha\delta}\gamma^{\beta\gamma} - \gamma^{\alpha\beta}\gamma^{\gamma\delta}. \tag{14.55}$$

It is readily verified that G is the projection operator to the traceless part ($\mathrm{tr}\, G^{\alpha\beta\gamma\delta}\delta h_{\gamma\delta} = \gamma_{\alpha\beta}G^{\alpha\beta\gamma\delta}\delta h_{\gamma\delta} = 0$) while $u\gamma^{\alpha\beta}\gamma^{\gamma\delta}$ is that to the trace part. In a finite-dimensional manifold, a metric defines a natural volume element. In the present case, however, the measure cannot be defined explicitly and we have to define it implicitly in terms of the Gaussian integral (see example 14.5),

$$\int \mathcal{D}\delta h \exp(-\tfrac{1}{2}||\delta h||_\gamma^2) = 1. \tag{14.56a}$$

Similarly the metrics for a scalar $\delta\phi$, a vector δv and a map δX^μ are defined by

$$||\delta\phi||_\gamma^2 = \int \mathrm{d}^2\xi\sqrt{\gamma}\delta\phi^2 \tag{14.54b}$$

$$||\delta v||_\gamma^2 = \int \mathrm{d}^2\xi\sqrt{\gamma}\gamma_{\alpha\beta}\delta v^\alpha\delta v^\beta \tag{14.54c}$$

$$||\delta X||_\gamma^2 = \int \mathrm{d}^2\xi\sqrt{\gamma}\delta X^\mu\delta X_\mu. \tag{14.54d}$$

With these metrics, the measures are defined by

$$\int \mathcal{D}\delta\phi \exp(-\tfrac{1}{2}||\delta\phi||_\gamma^2) = 1 \tag{14.56b}$$

$$\int \mathcal{D}\delta v \exp(-\tfrac{1}{2}||\delta v||_\gamma^2) = 1 \tag{14.56c}$$

$$\int \mathcal{D}\delta X \exp(-\tfrac{1}{2}||\delta X||_\gamma^2) = 1. \tag{14.56d}$$

Exercise 14.6 Show that $||\delta\gamma||_\gamma^2$ and $||\delta X||_\gamma^2$ are invariant under $\mathrm{Diff}(\Sigma_g)$ but *not* under $\mathrm{Weyl}(\Sigma_g)$. This is the possible origin of conformal anomalies, see (14.84).

Before we proceed further, we need to clarify the overlap between $\mathrm{Diff}_0(\Sigma_g)$ and $\mathrm{Weyl}(\Sigma_g)$. Suppose $\delta v \in \ker P_1$, namely,

$$P_1\delta v = \nabla_\alpha\delta v_\beta + \nabla_\beta\delta v_\alpha - \gamma_{\alpha\beta}(\nabla_\gamma\delta v^\gamma) = 0. \tag{14.57}$$

We find, for such δv, that $\delta_{\mathrm{D}}\gamma_{\alpha\beta} = (\nabla_\gamma\delta v^\gamma)\gamma_{\alpha\beta}$. A vector $\delta v \in \ker P_1$ is identified with the **conformal Killing vector** (CKV), see §7.7. It is important to note that δ_{D} and δ_{W} yield the same metric deformations if $\delta\phi$ is taken to be $\nabla_\gamma\delta v^\gamma$. Thus the set of the CKV is identified with the overlap between $\mathrm{Diff}_0(\Sigma_g)$ and $\mathrm{Weyl}(\Sigma_g)$. Let there be k independent CKV on Σ_g and denote these by Φ_s^α, $(1 \leqslant s \leqslant k)$. It is known from the theory of Riemann surfaces that

$$k = \begin{cases} 6 & g = 0 \\ 2 & g = 1 \\ 0 & g \geqslant 2. \end{cases} \tag{14.58}$$

We separate δv into a part generated by the CKV, and its orthogonal complement, which we write as $\delta\tilde{v}$,

$$\delta v^\alpha = \delta\tilde{v}^\alpha + \delta a^s\Phi_s^\alpha. \tag{14.59}$$

The tangent vector δX is also decomposed as

$$\delta X = \delta\tilde{X} + \delta\tilde{v}^\alpha\partial_\alpha\tilde{X}^\mu + \delta a^s\Phi_s^\alpha\partial_\alpha\tilde{X}^\mu. \tag{14.60}$$

The functional measures now become

$$\mathcal{D}\delta\gamma\mathcal{D}\delta X \to J\mathrm{d}^n\delta t\mathcal{D}\delta\phi\mathcal{D}\delta\tilde{v}\,\mathrm{d}^k\delta a\mathcal{D}\delta\tilde{X} \tag{14.61}$$

where we noted that the t- and a-parameters are finite-dimensional.

Let $\mathrm{Diff}_0^\perp(\Sigma_g)$ be the subspace of $\mathrm{Diff}_0(\Sigma_g)$, which is orthogonal to the CKV. We have

$$V(\mathrm{Diff}_0) = V(\mathrm{Diff}_0^\perp)\cdot V(\text{CKV}) \tag{14.62}$$

$$\begin{aligned} V(\mathrm{Diff}_0 * \mathrm{Weyl}) &= V(\mathrm{Diff}_0^\perp)V(\mathrm{Weyl}) \\ &= V(\mathrm{Diff}_0)V(\mathrm{Weyl})/V(\text{CKV}). \end{aligned} \tag{14.63}$$

Take a slice $\hat{\gamma}(t)$ of $\mathcal{M}_g$. The slice is parametrised by n Teichmüller parameters. Any metric $\tilde{\gamma}$ related to $\hat{\gamma}$ by $G = \mathrm{Diff}_0(\Sigma_g)*\mathrm{Weyl}(\Sigma_g)$ is written as

$$\tilde{\gamma} = f^*(\mathrm{e}^\phi \hat{\gamma}) \qquad f \in \mathrm{Diff}(\Sigma_g),\ \mathrm{e}^\phi \in \mathrm{Weyl}(\Sigma_g). \tag{14.64}$$

We express a small deformation $\delta\tilde{\gamma}$ at $\tilde{\gamma}$ as a pullback of a deformation $\delta\gamma$ at $\gamma \equiv \mathrm{e}^\phi\hat{\gamma}$: $\delta\tilde{\gamma} = f^*(\delta\gamma)$. Note that $\delta\gamma$ is a small diffeomorphism at the *origin* of $\mathrm{Diff}_0(\Sigma_g)$, and hence can be described by a vector field δv. As was shown in exercise 14.6, $\mathrm{Diff}(\Sigma_g)$ is the isometry of the relevant vector spaces. It then follows that

$$\|\delta\tilde{\gamma}\|^2_{\tilde{\gamma}} = \|f^*(\delta\gamma)\|^2_{f^*\gamma} = \|\delta\gamma\|^2_\gamma \qquad \gamma = \mathrm{e}^\phi\hat{\gamma}. \tag{14.65}$$

At the point γ, we decompose $\delta\gamma$ as

$$\delta\gamma_{\alpha\beta} = \delta\phi\gamma_{\alpha\beta} + (P_1\delta\tilde{v})_{\alpha\beta} + \delta\tau^i T_{i\alpha\beta} \tag{14.66}$$

where $\delta\phi$ has been redefined so that it includes the trace parts of the Teichmüller deformation and $\nabla_\alpha\delta v_\beta + \nabla_\beta\delta v_\alpha$, see (14.51).

Exercise 14.7 Show that $T_{i\alpha\beta}$ at γ is related to $\hat{T}_{i\alpha\beta}$ at $\hat{\gamma}$ as

$$T_{i\alpha\beta} = \mathrm{e}^\phi\hat{T}_{i\alpha\beta}. \tag{14.67}$$

Now we are ready to give the explicit form of the measure. We first find the Jacobian associated with the change $\mathcal{D}\delta v \to \mathcal{D}\delta\tilde{v}\,\mathrm{d}^k\delta a$. We have

$$\begin{aligned} 1 &= \int \mathcal{D}\delta v \exp(-\tfrac{1}{2}\|\delta v\|^2_\gamma) \\ &= J\int \mathcal{D}\delta\tilde{v}\,\mathrm{d}^k\delta a \exp(-\tfrac{1}{2}\|\delta\tilde{v}\|^2_\gamma - \tfrac{1}{2}\|\delta a^s\Phi_s\|^2_\gamma) \\ &= J[\det(\Phi_s, \Phi_r)]^{-1/2} \end{aligned} \tag{14.68a}$$

where

$$(\Phi_s, \Phi_r) = \int \mathrm{d}^2\xi\sqrt{\gamma}\gamma_{\alpha\beta}\Phi_s^\alpha\Phi_r^\beta. \tag{14.68b}$$

[Although the matrix element (14.68b) is defined for $\gamma = \mathrm{e}^\phi\hat{\gamma}$, we can show that it is independent of e^ϕ. To see this, let us take a CKV $\hat{\Phi}_s^\alpha$ of the metric $\hat{\gamma}$; $\hat{\nabla}_\alpha\hat{\Phi}_{s\beta} + \hat{\nabla}_\beta\hat{\Phi}_{s\alpha} = \hat{\gamma}_{\alpha\beta}\hat{\nabla}_\gamma\hat{\Phi}_s^\gamma$, where $\hat{\nabla}$ is the covariant derivative with respect to $\hat{\gamma}$ and $\hat{\Phi}_{s\alpha} \equiv \hat{\gamma}_{\alpha\beta}\hat{\Phi}^{\beta}_{s}$. A simple calculation shows that $\Phi_{s\alpha} = \gamma_{\alpha\beta}\hat{\Phi}^{\beta}_{s} = e^{\phi}\hat{\Phi}_{s\alpha}$ satisfies

$$\begin{aligned}\nabla_\alpha \Phi_{s\beta} + \nabla_\beta \Phi_{s\alpha} &= e^{\phi}(\hat{\nabla}_\alpha \hat{\Phi}_{s\beta} + \hat{\nabla}_\beta \hat{\Phi}_{s\alpha} + \hat{\gamma}_{\alpha\beta}\Phi^{\gamma}_{s}\partial_\gamma \phi)\\ &= e^{\phi}\hat{\gamma}_{\alpha\beta}(\hat{\nabla}_\gamma \Phi^{\gamma}_{s} + \Phi^{\gamma}_{s}\partial_\gamma\phi) = \gamma_{\alpha\beta}\nabla_\gamma\Phi^{\gamma}_{s}\end{aligned}$$

∇ being the covariant derivative with respect to γ. Thus $\Phi^{\alpha}_{s} = \hat{\Phi}^{\alpha}_{s}$ is a CKV of the metric $\gamma = e^{\phi}\hat{\gamma}$ and the CKV are taken to be ϕ-independent.] Equation (14.68a) shows that

$$\mathcal{D}\delta v = [\det(\Phi_r, \Phi_s)]^{1/2}\mathcal{D}\delta\tilde{v}\,d^k\delta a. \tag{14.69}$$

Now the total measure is written as

$$J[\det(\Phi_r, \Phi_s)]^{1/2}d^n t\mathcal{D}\delta\phi\mathcal{D}\delta\tilde{v}\,d^k\delta a\mathcal{D}\delta\tilde{X} \tag{14.70}$$

where J takes care of the rest of the variable changes.

The Jacobian J is now obtained from (14.60), (14.66), (14.70) and the definition of the measures (14.56). We have

$$\begin{aligned}1 &= \int \mathcal{D}\delta\gamma\mathcal{D}\delta X \exp(-\tfrac{1}{2}\|\delta\gamma\|^2_\gamma - \tfrac{1}{2}\|\delta X\|^2_\gamma)\\ &= J\det{}^{1/2}(\Phi,\Phi)\int d^n\delta t\mathcal{D}\delta\tilde{v}\mathcal{D}\delta\phi\,d^k\delta a\mathcal{D}\delta\tilde{X}\\ &\quad\times \exp\Bigg[-\frac{1}{2}\left\|\delta\phi\gamma_{\alpha\beta} + (P_1\delta\tilde{v})_{\alpha\beta} + \delta t^i\frac{\partial\gamma_{\alpha\beta}}{\partial t^i}\right\|^2\\ &\quad -\tfrac{1}{2}\|\delta\tilde{X} + \delta\tilde{v}^\alpha\partial_\alpha\tilde{X} + \delta a^s\Phi^\alpha_s\partial_\alpha\tilde{X}\|^2\Bigg]\\ &= J\det{}^{1/2}(\Phi,\Phi)\int d^n\delta t\mathcal{D}\delta\tilde{v}\ldots\exp(-\tfrac{1}{2}\|MV\|^2)\end{aligned} \tag{14.71}$$

where

$$V = \begin{pmatrix}\delta t\\ \delta\phi\\ \delta\tilde{v}\\ \delta a\\ \delta\tilde{X}\end{pmatrix}\qquad M = \left(\begin{array}{ccc|cc}\partial\gamma/\partial t & \gamma & P_1 & 0 & 0\\ \hline 0 & 0 & \partial\tilde{X} & \Phi\cdot\partial\tilde{X} & 1\end{array}\right) \equiv \begin{pmatrix}A & 0\\ C & B\end{pmatrix}. \tag{14.72}$$

The matrix in the exponent of (14.71) is

$$\begin{aligned}M^\dagger M &= \begin{pmatrix}A^\dagger & C^\dagger\\ 0 & B^\dagger\end{pmatrix}\begin{pmatrix}A & 0\\ C & B\end{pmatrix} = \begin{pmatrix}A^\dagger A + C^\dagger C & C^\dagger B\\ B^\dagger C & B^\dagger B\end{pmatrix}\\ &= \begin{pmatrix}I & *\\ 0 & B^\dagger B\end{pmatrix}\begin{pmatrix}A^\dagger A & 0\\ ** & I\end{pmatrix}\end{aligned} \tag{14.73}$$

where * and ** are irrelevant. The last expression has been obtained from the identity,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{pmatrix}.$$

The Gaussian integrals in (14.71) are readily evaluated to yield

$$\begin{aligned} 1 &= \boldsymbol{J} \det{}^{1/2}(\Phi, \Phi) \det{}^{-1/2}(M^{\dagger}M) \\ &= \boldsymbol{J} \det{}^{1/2}(\Phi, \Phi)[\det(A^{\dagger}A)\det(B^{\dagger}B)]^{-1/2}. \end{aligned} \tag{14.74}$$

To compute $\det^{1/2}(A^{\dagger}A)$, we have to evaluate $||\delta\gamma||_{\gamma}^{2}$. We have

$$\begin{aligned} ||\delta\gamma||_{\gamma}^{2} &= \int \mathrm{d}^2\xi \sqrt{\gamma}(G^{\alpha\beta\gamma\delta} + u\gamma^{\alpha\beta}\gamma^{\gamma\delta}) \\ &\quad \times [\delta\phi\gamma_{\alpha\beta} + (P_1\delta\widetilde{v})_{\alpha\beta} + \delta t^i T_{i\alpha\beta}][\delta\phi\gamma_{\gamma\delta} + (P_1\delta\widetilde{v})_{\gamma\delta} + \delta t^j T_{j\gamma\delta}] \\ &= 4u||\delta\phi||_{\gamma}^{2} + ||P_1\delta\widetilde{v}||^2 + \delta t^i \delta t^j (T_i, T_j) + 2\delta t^i (P_1\delta\widetilde{v}, T_i). \end{aligned} \tag{14.75}$$

In general T_i is not orthogonal to $P_1\delta v$. To separate T_i into parts orthogonal to $P_1\delta v$ and parallel to $P_1\delta v$, we need to define the adjoint $P^{\dagger}_1$ of P_1. P_1 is an elliptic operator which takes a vector field into a traceless symmetric tensor field. Thus $P^{\dagger}_1$ maps symmetric traceless tensors to vectors. For a symmetric traceless tensor δh, we have

$$\begin{aligned} (P_1\delta v, \delta h) &= \int \mathrm{d}^2\xi \sqrt{\gamma} G^{\alpha\beta\gamma\delta}(P_1\delta v)_{\alpha\beta}\delta h_{\gamma\delta} \\ &= \int \mathrm{d}^2\xi \sqrt{\gamma}(\nabla^{\alpha}\delta v^{\beta} + \nabla^{\beta}\delta v^{\alpha})\delta h_{\alpha\beta} \\ &= \int \mathrm{d}^2\xi \sqrt{\gamma}\delta v^{\alpha}(-2\nabla^{\beta})\delta h_{\alpha\beta} \equiv (\delta v, P^{\dagger}_1\delta h) \end{aligned}$$

where the inner product in the last expression is defined by (14.54c). Thus it follows that

$$(P^{\dagger}_1\delta h)_{\alpha} = -2\nabla^{\beta}\delta h_{\alpha\beta}. \tag{14.76}$$

Suppose δh is orthogonal to $P_1\delta v$. From above, we have $(P_1\delta v, \delta h) = (\delta v, P^{\dagger}_1\delta h) = 0$. Since δv is arbitrary, δh must be an element of $\ker P^{\dagger}_1$, see figure 14.4. Now T_i may be separated as

$$T_i = \mathcal{P}_0 T_i + \mathcal{P}_{\perp} T_i \tag{14.77a}$$

where the projection operators $\mathcal{P}_0$ and $\mathcal{P}_{\perp}$ are defined by

$$\mathcal{P}_0 \equiv 1 - P_1 \frac{1}{P^{\dagger}_1 P_1} P^{\dagger}_1 \qquad \mathcal{P}_{\perp} \equiv P_1 \frac{1}{P^{\dagger}_1 P_1} P^{\dagger}_1. \tag{14.77b}$$

It is easy to verify that $\mathcal{P}_0 + \mathcal{P}_{\perp} = 1$, $\mathcal{P}_0\mathcal{P}_{\perp} = 0$, $P^{\dagger}_1\mathcal{P}_0 = 0$, $P^{\dagger}_1\mathcal{P}_{\perp} = P^{\dagger}_1$, $\mathcal{P}_0 T_i = T_i$ and $\mathcal{P}_{\perp} T_i = 0$ for $T_i \in \ker P^{\dagger}_1$ etc. Thus (14.77a) is an orthogonal decomposition of T_i. We write $\mathcal{P}_{\perp} T_i = P_1 u_i$, where

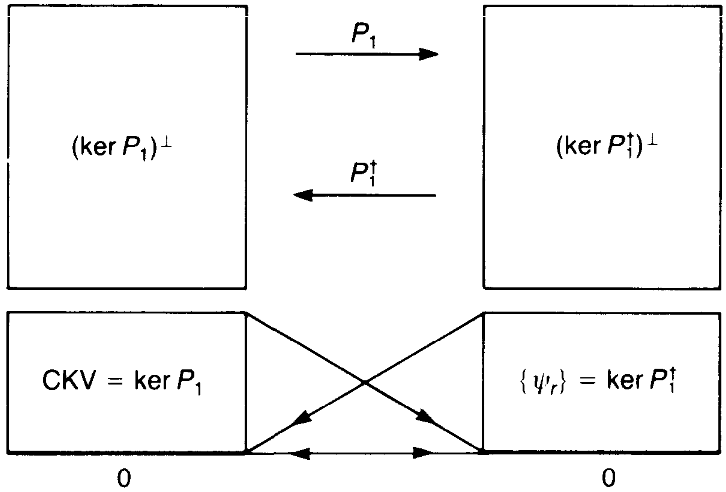


Figure 14.4 The map P_1 and its adjoint $P^\dagger_1$.

$$u_i = \frac{1}{P^\dagger_1 P_1} P^\dagger_1 T_i.$$

Let $\{\psi_r\}$ $(1 \leq r \leq n)$ be a real basis of $\ker P^\dagger_1$, which is not necessarily orthonormal. Then T_i can be expanded as (figure 14.5)

$$T_i = \sum_r \psi_r Q_{ri} + P_1 u_i. \tag{14.78}$$

Taking an inner product between T_i and ψ_r, we find

$$Q_{ri} = \sum_s [(\psi, \psi)^{-1}]_{rs}(\psi_s, T_i). \tag{14.79}$$

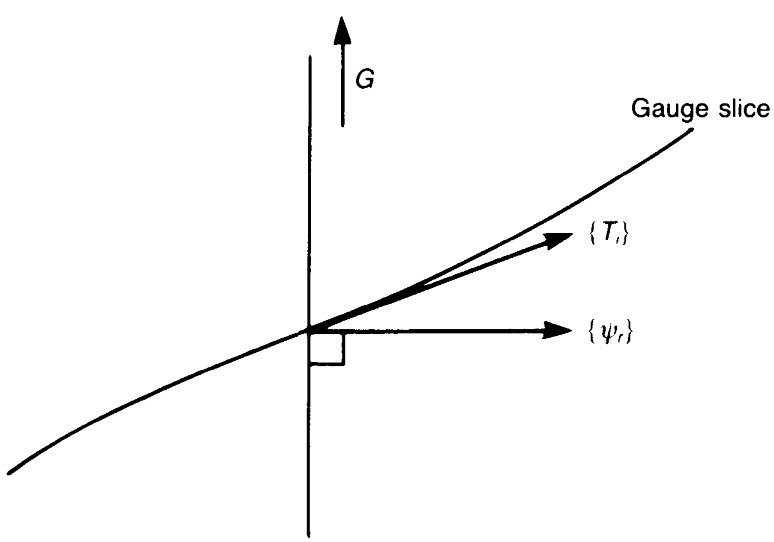


Figure 14.5 $\{T_i\}$ spans the deformation tangent to the gauge slice while $\{\psi_r\}$ spans $\ker P^\dagger_1$.

Finally $\delta\gamma$ is decomposed into mutually orthogonal pieces as

$$\delta\gamma = \delta\phi\gamma + P_1(\delta\tilde{v} + \delta t^i u_i) + \delta t^i \psi_r Q_{ri}. \tag{14.80a}$$

Correspondingly the space of the metric deformation $\{\delta\gamma\}$ separates into the direct sum

$$\{\delta\gamma\} = \{\text{conf}\} \oplus \{\text{image } P_1\} \oplus \{\ker P_1^\dagger\}. \tag{14.80b}$$

Substituting (14.80a) into (14.75), we have

$$\begin{aligned} \|\delta\gamma\|^2 &= 4u\|\delta\phi\|^2 + \|P_1\delta\bar{v}\|^2 \\ &\quad + \delta t^i \delta t^j (T_i, \psi_r)_\gamma [(\psi, \psi)_\gamma^{-1}]_{rs} (\psi_s, T_j)_\gamma \end{aligned} \tag{14.81}$$

where $\delta\bar{v} \equiv \delta\tilde{v} + \delta t^i u_i$ and the inverse in the last term refers to the inverse of the matrix $(a_{rs}) = ((\psi_r, \psi_s))$. If we put $\mathcal{D}_1^t = (\delta t, \delta\phi, \delta\bar{v})$, we have

$$\begin{aligned} \det{}^{-1/2}(A^\dagger A) &= \int \mathrm{d}^n\delta t \mathcal{D}\delta\phi \mathcal{D}\delta\bar{v} \exp(-\tfrac{1}{2}\mathcal{D}_1^t A^\dagger A \mathcal{D}_1) \\ &= \int \mathcal{D}\delta\phi \exp(-2u\|\delta\phi\|^2) \int \mathcal{D}\delta\bar{v} \exp(-\tfrac{1}{2}\|P_1\bar{v}\|^2) \\ &\quad \times \int \mathrm{d}^n\delta t \exp\{-\tfrac{1}{2}\delta t^i (T_i, \psi_r)[(\psi, \psi)^{-1}]_{rs}(\psi_s, T_i)\delta t^j\} \\ &\propto (\det P_1^\dagger P_1)^{-1/2} \left(\frac{\det(T, \psi)^2}{\det(\psi, \psi)}\right)^{-1/2}. \end{aligned} \tag{14.82}$$

Collecting the results (14.71) and (14.82), we have

$$1 = J \det{}^{1/2}(\Phi, \Phi) \det{}^{-1/2} B^\dagger B (\det P_1^\dagger P_1)^{-1/2} \left(\frac{\det(T, \psi)^2}{\det(\psi, \psi)}\right)^{-1/2}.$$

The g-loop partition function is then given by

$$\begin{aligned} Z_g &= \int \frac{\mathrm{d}^n t \mathcal{D}\bar{v} \mathcal{D}\phi \mathcal{D}\tilde{X}}{V(\text{Diff}*\text{Weyl})} \det{}^{1/2} B^\dagger B \det{}^{-1/2}(\Phi, \Phi) \\ &\quad \times \left(\det P_1^\dagger P_1 \frac{\det(T, \psi)^2}{\det(\psi, \psi)}\right)^{1/2} \mathrm{e}^{-S}. \end{aligned} \tag{14.83}$$

The integral over a (the CKV) has been omitted since it is already included in the ϕ-integration. Naïvely the integral over $\bar{v}$ yields $V(\text{Diff}_0^\perp)$ and that over ϕ yields $V(\text{Weyl})$. However, as exercise 14.6 shows, the measures $\mathcal{D}X$ and $\mathcal{D}\gamma$ depend on the conformal factor. Polyakov (1981) has shown that, under the conformal transformation $\gamma \to \mathrm{e}^{2\phi}\gamma$, the measures transform as

$$\mathcal{D}X \to \exp\left(\frac{D}{24\pi^2} \int \mathrm{d}^2\xi \sqrt{\gamma}(\gamma^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \mathcal{R}\phi)\right)\mathcal{D}X \tag{14.84a}$$

$$\mathcal{D}\gamma \to \exp\left(\frac{-26}{24\pi^2} \int \mathrm{d}^2\xi \sqrt{\gamma}(\gamma^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \mathcal{R}\phi)\right)\mathcal{D}\gamma. \tag{14.84b}$$

Thus the measure $\mathcal{D}X\mathcal{D}\gamma$ is conformally invariant if and only if $D = 26$. This number 26 is called the **critical dimension**. Henceforth we always assume $D = 26$. Now (14.83) simplifies as

$$Z_g = \frac{1}{|\mathrm{MCG}|} \int \mathrm{d}^n t \mathcal{D}\widetilde{X} \det{}^{1/2} B^\dagger B \det{}^{-1/2}(\Phi, \Phi) \times \left(\det P^\dagger_1 P_1 \frac{\det(T, \psi)^2}{\det(\psi, \psi)}\right)^{1/2} \mathrm{e}^{-S}. \tag{14.85}$$

We perform the X-integration to eliminate $\det^{1/2} B^\dagger B$. We have

$$\begin{aligned}
1 &= \int \mathcal{D}\delta X \exp(-\tfrac{1}{2}||\delta X||^2) \\
&= J \int \mathcal{D}\delta \widetilde{X} \mathrm{d}^k \delta a \exp(-\tfrac{1}{2}||\delta \widetilde{X} + \delta a^s \Phi_s^\alpha \partial_\alpha \widetilde{X}||^2) \\
&= J \int \mathcal{D}\delta \widetilde{X} \exp(-\tfrac{1}{2}||\delta \widetilde{X}||^2) \int \mathrm{d}^k \delta a \exp(-\tfrac{1}{2}||\delta a^s \Phi_s^\alpha \partial_\alpha \widetilde{X}||^2) \\
&= J \det{}^{-1/2}(B^\dagger B)
\end{aligned}$$

and hence $\det^{1/2}(B^\dagger B)$ is identified with the Jacobian of the transformation $X \to (\widetilde{X}, a)$. Thus it follows that

$$\int \mathcal{D}\widetilde{X} \det{}^{1/2} B^\dagger B \mathrm{e}^{-S} = \int \frac{\mathcal{D}X}{V(\textsc{ckv})} \mathrm{e}^{-S} \tag{14.86}$$

where $V(\textsc{ckv}) = \int \mathrm{d}^k a$ is the volume of the CKV.

The integration over X is readily carried out. Let us write

$$\int_{\mathcal{C}} \mathcal{D}X \mathrm{e}^{-S} = \int_{\mathcal{C}} \mathcal{D}X \exp[-\tfrac{1}{2}(X, \Delta X)] \tag{14.87a}$$

where

$$\Delta = -\frac{1}{\sqrt{\gamma}} \partial_\alpha \sqrt{\gamma}\gamma^{\alpha\beta}\partial_\beta \tag{14.87b}$$

is the Laplacian acting on 0-forms, see (7.188). We write down the explicit form of the path integral (14.87a). Let ψ_n be the eigenfunction of Δ,

$$\Delta\psi_n = \lambda_n \psi_n \qquad \lambda_n \in [0, \infty) \tag{14.88}$$

where ψ_n are normalised as

$$(\psi_n, \psi_m) = \int \mathrm{d}^2\xi \sqrt{\gamma}\psi_n \psi_m = \delta_{nm}.$$

λ is non-negative since Δ is positive definite. Let us expand X^μ in ψ_n as

$$X^\mu = \sum_{n=0}^{\infty} a_n^\mu \psi_n = X_0^\mu + X'^\mu \qquad a_n^\mu \in \mathbb{R} \tag{14.89}$$

where $X_0^\mu = a_0^\mu \psi_0$ is the zero eigenfunction of Δ and X'^μ are the remaining degrees of freedom. Correspondingly, the path integral (14.87a) is written as

$$\begin{aligned}\int \mathcal{D}X \exp[-\tfrac{1}{2}(X, \Delta X)] &= \int \prod_{n,\mu} \mathrm{d}a_n^\mu \exp\Big(-\frac{1}{2}\sum_{n,\mu}\lambda_n (a_n^\mu)^2\Big)\\ &= \int \prod_\mu \mathrm{d}a_0^\mu \int \prod_{n\neq 0}\prod_\mu \mathrm{d}a_n^\mu \exp\Big(-\frac{1}{2}\sum_{n,\mu}\lambda_n (a_n^\mu)^2\Big)\\ &= \Big(\int \prod_\mu \mathrm{d}a_0^\mu\Big)(\det{}'\Delta)^{-13} \qquad (14.90)\end{aligned}$$

where the prime indicates that the zero mode is omitted. To integrate over the zero mode, we note that the *normalised* eigenvector ψ_0 is given by

$$\psi_0 = \left(\frac{1}{\int \mathrm{d}^2\xi \sqrt{\gamma}}\right)^{1/2}. \qquad (14.91)$$

[Since ψ_0 satisfies $\Delta\psi_0 = 0$ it is a harmonic function. Any harmonic function on a Riemann surface must be a *constant* by the maximum principle.] From $X_0^\mu = a_0^\mu \psi_0$ we have

$$\int \prod_\mu \mathrm{d}a_0^\mu = \int \prod_\mu \mathrm{d}X_0^\mu (\psi_0)^{-26} = V\left(\frac{1}{\int \mathrm{d}^2\xi\sqrt{\gamma}}\right)^{-13} \qquad (14.92)$$

where $V = \int \Pi\, \mathrm{d}X_0^\mu$ is the spacetime volume. Collecting the results (14.90) and (14.92), we find that

$$\int \mathcal{D}X \mathrm{e}^{-S} = \left(\frac{\det{}'\Delta}{\int \mathrm{d}^2\xi\sqrt{\gamma}}\right)^{-13} \qquad (14.93)$$

where we have dropped V and other irrelevant constants.

Finally we have obtained the expression for the g-loop partition function

$$\begin{aligned}Z_g = \int_{\mathrm{Mod}} &\frac{\mathrm{d}^n t}{V(\mathrm{CKV})} \frac{\det(T, \psi)}{\det^{1/2}(\psi, \psi)\det^{1/2}(\Phi, \Phi)}\\ &\times [\det{}' P_1^\dagger P_1]^{1/2}\left(\frac{\det{}'\Delta}{\int \mathrm{d}^2\xi\sqrt{\gamma}}\right)^{-13} \qquad (14.94)\end{aligned}$$

where we have noted that

$$\frac{1}{|\mathrm{MCG}|}\int_{\mathrm{Teich}} \mathrm{d}^n t = \int_{\mathrm{Mod}} \mathrm{d}^n t. \qquad (14.95)$$

If $g \geqslant 2$, the Riemann surfaces have no CKV and (14.95) reduces to

$$Z_g = \int_{\mathrm{Mod}} \mathrm{d}^n t \frac{\det(T, \psi)}{\det^{1/2}(\psi, \psi)} (\det P_1^\dagger P_1)^{1/2}\left(\frac{\det{}'\Delta}{\int \mathrm{d}^2\xi\sqrt{\gamma}}\right)^{-13}. \qquad (14.96)$$

14.2.3 Complex tensor calculus and string measure

Since any Riemann surface admits complex structures, we may take advantage of this fact to compute string amplitudes. Many beautiful aspects of string theory are revealed only when these complex structures are explicitly taken into account. Here we rewrite the partition function in the language of complex differential geometry.

We first fix the *gauge* in $\mathcal{M}_g$ by choosing the isothermal coordinate system

$$\gamma = \tfrac{1}{2}e^{2\sigma}[dz \otimes d\bar{z} + d\bar{z} \otimes dz]$$

where $\gamma_{z\bar{z}} = \gamma_{\bar{z}z} = \frac{1}{2}\exp 2\sigma$. [In fact, the gauge is not uniquely fixed with this choice. We will invoke the *uniformisation theorem* later to fix the gauge completely.] Then the deformation of γ under a diffeomorphism generated by δv is (cf (14.45))

$$\delta_D\gamma_{zz} = 2\nabla_z^{(-1)}\delta v_z \tag{14.97}$$

$$\delta_D\gamma_{z\bar{z}} = \nabla_z\delta v_{\bar{z}} + \nabla_{\bar{z}}\delta v_z = \gamma_{\bar{z}z}(\nabla_z^{(1)}\delta v^z + \nabla^z_{(-1)}\delta v_z).$$

Similarly $\delta_W\gamma$ generated by an infinitesimal conformal change is (cf (14.46))

$$\delta_W\gamma_{z\bar{z}} = \delta\phi\gamma_{z\bar{z}} \qquad \delta_W\gamma_{zz} = 0. \tag{14.98}$$

To see the action of the operator P_1 on vectors, we take $\delta v^z \in \mathcal{T}^1$ and $\delta v_z \in \mathcal{T}^{-1}$. From (14.50), we find

$$(P_1\delta v)^{zz} = 2\nabla^z_{(1)}\delta v^z \in \mathcal{T}^2 \tag{14.99a}$$

$$(P_1\delta v)_{zz} = 2\nabla_z^{(-1)}\delta v_z \in \mathcal{T}^{-2}. \tag{14.99b}$$

This shows that P_1 is a map

$$P_1 = \begin{pmatrix} \nabla^z_{(1)} & 0 \\ 0 & \nabla_z^{(-1)} \end{pmatrix} : \mathcal{T}^1 \oplus \mathcal{T}^{-1} \to \mathcal{T}^2 \oplus \mathcal{T}^{-2}. \tag{14.100}$$

Similarly $P^\dagger_1$ maps traceless symmetric tensors to vectors. For $\delta h^{zz} \in \mathcal{T}^2$ and $\delta h_{zz} \in \mathcal{T}^{-2}$, we have

$$(P^\dagger_1\delta h)^z = \nabla_z^{(2)}\delta h^{zz} \in \mathcal{T}^1 \tag{14.101a}$$

$$(P^\dagger_1\delta h)_z = \nabla^z_{(-2)}\delta h_{zz} \in \mathcal{T}^{-1}. \tag{14.101b}$$

Thus $P^\dagger_1$ is a map

$$P^\dagger_1 = \begin{pmatrix} \nabla_z^{(2)} & 0 \\ 0 & \nabla^z_{(-2)} \end{pmatrix} : \mathcal{T}^2 \oplus \mathcal{T}^{-2} \to \mathcal{T}^1 \oplus \mathcal{T}^{-1}. \tag{14.102}$$

The product $P^\dagger_1 P_1$ is

$$P^\dagger_1 P_1 = \begin{pmatrix} \nabla_z^{(2)}\nabla^z_{(1)} & 0 \\ 0 & \nabla^z_{(-2)}\nabla_z^{(-1)} \end{pmatrix} : \mathcal{T}^1 \oplus \mathcal{T}^{-1} \to \mathcal{T}^1 \oplus \mathcal{T}^{-1}. \tag{14.103}$$

Accordingly, the determinant in (14.96) becomes

$$\begin{aligned}(\det{}' P_1^\dagger P_1)^{1/2} &= (\det{}' \nabla_z^{(2)} \nabla_{(1)}^z \det{}' \nabla_{(-2)}^{\bar z} \nabla_{\bar z}^{(-1)})^{1/2} \\ &= (\det{}' \Delta_{(1)}^+ \Delta_{(-1)}^-)^{1/2} \qquad (14.104)\end{aligned}$$

where $\Delta_{(n)}^{\pm}$ are the Laplacians. We show that the spectrum of $\Delta_{(1)}^+$ is the same as that of $\Delta_{(-1)}^-$. Take an eigenfunction δv^z of $\Delta_{(1)}^+$,

$$\Delta_{(1)}^+ \delta v^z = -2e^{-4\sigma} \partial_z e^{2\sigma} \partial_{\bar z} \delta v^z = \lambda \delta v^z \qquad (14.105)$$

where (14.21a) has been used. The eigenvalue λ is a non-negative real number (note $\Delta_{(n)}^{\pm}$ are positive-definite Hermitian operators). Then we find

$$\begin{aligned}\Delta_{(-1)}^-(\gamma_{z\bar z}\overline{\delta v^z}) &= -e^{-2\sigma}\partial_{\bar z} e^{2\sigma} \partial_z \overline{\delta v^z} = -e^{-2\sigma}\, \overline{\partial_z e^{2\sigma} \partial_{\bar z} \delta v^z} \\ &= -\ \gamma_{z\bar z} \overline{2e^{-4\sigma}\partial_z e^{2\sigma}\partial_{\bar z}\delta v^z} = \lambda \gamma_{z\bar z}\overline{\delta v^z} \qquad (14.106)\end{aligned}$$

which shows that $\gamma_{z\bar z}\overline{\delta v^z}$ is an eigenfunction of $\Delta_{(-1)}^-$ with the same eigenvalue λ. It is easy to see that the converse is also true, see exercise 14.8. Thus $\Delta_{(1)}^+$ and $\Delta_{(-1)}^-$ share the same eigenvalues and $\det{}'\Delta_{(1)}^+ = \det{}'\Delta_{(-1)}^-$. Now (14.104) becomes

$$(\det{}' P_1^\dagger P_1)^{1/2} = \det{}' \Delta_{(-1)}^- = \det{}' \Delta_{(1)}^+. \qquad (14.107)$$

Exercise 14.8 Let δv_z be an eigenvector of $\Delta_{(-1)}^-$ with an eigenvalue λ. Show that $\gamma^{z\bar z}\overline{\delta v_z}$ is an eigenvector of $\Delta_{(1)}^+$ with the same eigenvalue.

The physical change of the metric is the Teichmüller deformation $\delta\tau^i\mu_i$, where τ^i (μ_i) is the complex counterpart of $t^i(T_i)$. From our experience we know that the relevant part of the Teichmüller deformation is *symmetric* and *traceless* in the real basis. In the complex basis this amounts to $\mu_{iz\bar z} = \mu_{i\bar z z} = 0$. Accordingly the general variation of the metric is given by

$$\delta\gamma_{zz} = \nabla_z^{(-1)} \delta \tilde v_z + \delta\tau^i \mu_{izz} \qquad (14.108a)$$

$$\delta\gamma_{z\bar z} = \delta\phi\gamma_{z\bar z} \qquad (14.108b)$$

where we have redefined $\delta\phi$ so that it includes the variation of $\delta\gamma_{z\bar z}$ due to δv (note that $\delta_D\gamma_{z\bar z} \propto \gamma_{z\bar z}$). In (14.108$a$), $\delta\tilde v$ does not contain the CKV, that is, $\delta\tilde v \in (\ker \nabla_z^{(-1)})^\perp$.

To carry out the orthogonal decomposition of $\{\delta\gamma\}$, we need to define the inner products in various spaces. The most natural choices are

$$||\delta\gamma_{zz}||^2 = \int d^2z\, \sqrt{\gamma}\, \overline{\delta\gamma_{zz}}\delta\gamma^{zz} \qquad (14.109a)$$

$$||\delta\gamma_{z\bar z}||^2 = \int d^2z \sqrt{\gamma}\, \overline{\delta\gamma_{z\bar z}}\delta\gamma^{z\bar z} \qquad (14.109b)$$

and

$$||\delta v_z||^2 = \int \mathrm{d}^2 z \, \sqrt{\gamma}\, \gamma_{z\bar{z}} \overline{\delta v^z} \delta v^z. \tag{14.109c}$$

[Since $\delta\gamma_{zz}\, \mathrm{d}z \otimes \mathrm{d}z$ and $\delta\gamma_{z\bar{z}}\, \mathrm{d}z \otimes \mathrm{d}\bar{z}$ are different tensors, we have to specify the inner product separately.]

Following the argument in the previous subsection, we introduce the orthogonal decomposition,

$$\delta\gamma_{zz} = \nabla_z^{(-1)}\delta\tilde{v}_z + \delta\tau^i\mu_{izz} = \nabla_z^{(-1)}\delta\bar{v}_z + \delta\tau^i\phi_{izz} \tag{14.110}$$

where $\delta\bar{v} = \delta\tilde{v}$ + (projection of $\delta\tau^i\mu_{izz}$ into {image $\nabla_z^{(-1)}$}). The orthogonality of $\nabla_z^{(-1)}\delta\bar{v}_z$ and ϕ_{izz} implies

$$0 = (\nabla_z^{(-1)}\delta v_z, \phi_{izz}) = \int \mathrm{d}^2 z \sqrt{\gamma}\, \overline{\delta v_z}\, (-\nabla^z_{(-2)}\phi_{izz})$$

where we have noted that $\nabla_z^{(-1)\dagger} = -\nabla^z_{(-2)}$. Thus we find (figure 14.6)

$$\phi_{izz} \in \ker \nabla^z_{(-2)}. \tag{14.111}$$

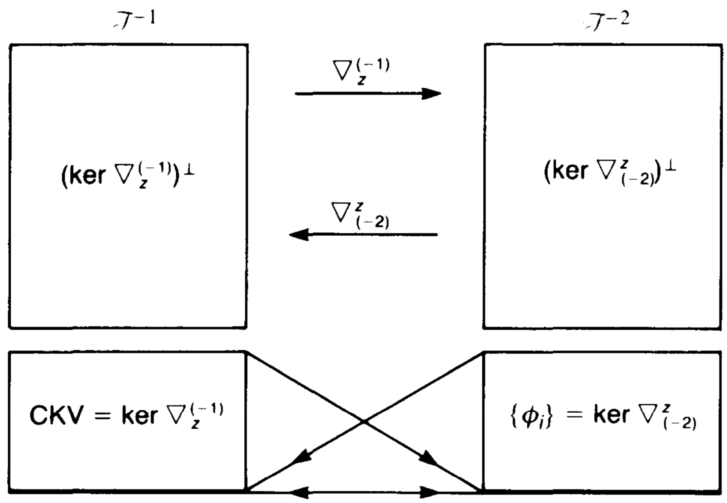


Figure 14.6 The map $\nabla_z^{(-1)}$ and its adjoint $\nabla^z_{(-2)}$.

The explicit form of $\nabla^z_{(-2)}$ shows that $\partial_{\bar{z}}\phi_{izz} = 0$, namely, $\ker \nabla^z_{(-2)}$ is the set of holomorphic tensors of helicity -2. The tensor $\phi_i = \phi_{izz}\, \mathrm{d}z \otimes \mathrm{d}z$ is called the **quadratic differential** while $\mu_i = \mu_{izz}\, \mathrm{d}z \otimes \mathrm{d}z$ is the **Beltrami differential**, see figure 14.7. In practical computations, it is often convenient to specify the gauge slice by the Beltrami differential, see below. Now we have established that

$$\{\ker P^\dagger_1\} = \{\text{Quadratic differential}\} = \{\ker \nabla^z_{(-2)}\}. \tag{14.112}$$

The Riemann–Roch theorem (14.27) takes the form,

$$\dim_{\mathbf{C}} \ker \nabla_z^{(-1)} - \dim_{\mathbf{C}} \ker \nabla^z_{(-2)} = 3 - 3g. \tag{14.113}$$

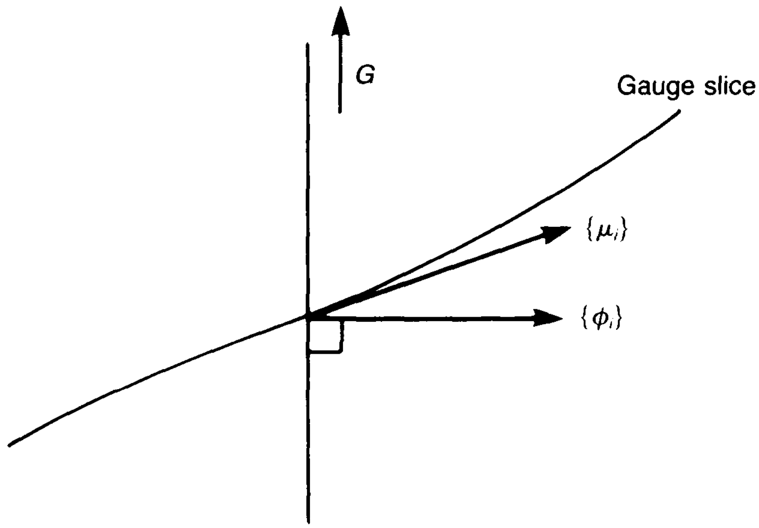


Figure 14.7 The Beltrami differential $\{\mu_i\}$ spans the deformation tangent to the gauge slice while $\{\phi_i\}$ spans ker $\nabla^z_{(-2)}$.

Now we have separated $\{\delta\gamma\}$ into mutually orthogonal pieces

$$\{\delta\gamma\} = \{\text{conf}\} \oplus \{\text{image } \nabla_z^{(-1)}\} \oplus \{\ker \nabla^z_{(-2)}\} + \text{cc} \qquad (14.114)$$

which should be compared with (14.80b). The measure becomes

$$\mathcal{D}\delta\gamma\mathcal{D}\delta X \to J \mathrm{d}^n \delta\tau \mathcal{D}\delta v \mathcal{D}\delta\phi \mathcal{D}\delta \tilde{X} \mathrm{d}^k \delta a \qquad (14.115)$$

where n and k are the complex dimensions of the Teichmüller space and the CKV, respectively. The Jacobian is obtained by repeating the argument in the previous subsection and we find

$$\begin{aligned} Z_g &= \int \mathcal{D}\gamma\mathcal{D}X \frac{1}{V(\text{Diff}*\text{Weyl})} \mathrm{e}^{-S} \\ &= \int_{\text{Mod}} \mathrm{d}^n\tau\mathcal{D}X \frac{\det{}'\Delta^+_{(1)}}{V(\text{CKV})} \frac{|\det(\mu,\phi)|^2}{\det(\phi,\phi)\det(\Phi,\Phi)} \mathrm{e}^{-S}. \end{aligned} \qquad (14.116)$$

Since we are integrating over complex variables, the power of a half in (14.96) does not appear in (14.116). The X-integration yields

$$\begin{aligned} Z_g = \int_{\text{Mod}} & \frac{\mathrm{d}^n\tau}{V(\text{CKV})} \frac{|\det(\mu,\phi)|^2}{\det(\phi,\phi)\det(\Phi,\Phi)} \\ & \times \det{}'\Delta^+_{(1)} \left(\frac{\det{}'\Delta}{\int \mathrm{d}^2 z\sqrt{\gamma}} \right)^{-13}. \end{aligned} \qquad (14.117)$$

14.2.4 Moduli spaces of Riemann surfaces

The spaces Mod(Σ_g) and Teich(Σ_g) have been defined by

$$\text{Mod}(\Sigma_g) \equiv \mathcal{M}_g/\text{Diff}(\Sigma_g) \qquad \text{Teich}(\Sigma_g) \equiv \mathcal{M}_g/\text{Diff}_0(\Sigma_g).$$

They are related through MCG $\equiv \text{Diff}(\Sigma_g)/\text{Diff}_0(\Sigma_g)$ as $\text{Mod}(\Sigma_g) = \text{Teich}(\Sigma_g)/\text{MCG}$. We look at these objects more closely here. We first note

g	$\dim_{\mathbb{C}}$ CKV	CKV	$\dim_{\mathbb{C}} \text{Teich}(\Sigma_g)$	MCG
0	3	$SL(2, \mathbb{C})$	0	$SL(2, \mathbb{R})$
1	1	$U(1) \times U(1)$	1	$SL(2, \mathbb{Z})$
$g \geqslant 2$	0	empty	$3g - 3$	?

(14.118)

[*Remark:* MCG for $g \geqslant 2$ can be expressed by $3g - 1$ Dehn twists which are, however, not minimal.] From (14.118), we immediately conclude that $Z_0 = 0$ since the Teichmüller space is a single point and the volume of $SL(2, \mathbb{C})$ is infinite. [Of course this does not imply that the tree amplitudes with vertex operators vanish.] In general, $\text{Mod}(\Sigma_g)$ is topologically non-trivial although $\text{Teich}(\Sigma_g)$ is. $\text{Teich}(\Sigma_g)$ is a universal covering space of $\text{Mod}(\Sigma_g)$ and the topological non-triviality comes from MCG.

In actual computations, the uniformisation theorem is very useful. In the previous subsection, we first chose the Beltrami differential μ_i then changed the basis to $\phi_i \in \ker P_1^\dagger$. Our initial choice μ_i is motivated by the uniformisation theorem.

Theorem 14.9 (**Uniformisation theorem**) Let Σ_g be a torus with genus g. Then it is conformally related to the constant-curvature Riemann surface, which is given by

g	Riemann surface	Metric	sign $\mathcal{R}$
0	$\mathbb{C} \cup \{\infty\}$	$ds^2 = dz \otimes d\bar{z}/(1 + z\bar{z})^2$	+
1	$\mathbb{C}/L$	$ds^2 = dz \otimes d\bar{z}$	0
$g \geqslant 2$	H/G	$ds^2 = dz \otimes d\bar{z}/(\text{Im}\, z)^2$	−

(14.119)

where L is a lattice in $\mathbb{C}$ (see example 8.3), H the upper half plane and $G \subset SL(2, \mathbb{R})$ is called the **Fuchsian group**. The metric for $g \geqslant 2$ is the **Poincaré metric**, see example 7.18.

The proof of this theorem is found in Farkas and Kra (1980), for example. Thanks to this theorem, we may always take constant-curvature metrics to form the gauge slice in $\mathcal{M}_g$. This corresponds to a special choice of the Beltrami differential μ_i. This slice defines the

Weil–Petersson measure

$$\int d''\tau \frac{|\det(\mu, \phi)|^2}{\det(\phi, \phi)} = \int d(\text{Weil–Petersson}) \qquad (14.120)$$

see D'Hoker and Phong (1986).

Exercise 14.10 Compute the scalar curvature of the metrics given in (14.119). Verify that they are independent of z and $\bar{z}$.

14.3 One-loop amplitudes

As an illustration of the formalism developed in the previous section we compute the one-loop vacuum-to-vacuum amplitude of the closed orientable bosonic string theory. Since $\dim_{\mathbb{C}} \text{Teich}(\Sigma_1) = 1$ and $\dim_{\mathbb{C}} \ker \nabla_z^{(-1)} = 1$, we have

$$Z_1 = \int_{\text{Mod}} \frac{d\tau}{V(\text{CKV})} \frac{|(\mu, \phi)|^2}{(\phi, \phi)\cdot(\Phi, \Phi)} \det{}'\Delta^{+}_{(1)} \left(\frac{\det{}'\Delta}{\int d^2\xi\sqrt{\gamma}}\right)^{-13}. \qquad (14.121)$$

To evaluate (14.121) we need to take several steps.

14.3.1 Moduli spaces, CKV, Beltrami and quadratic differentials

In example 8.3, we have shown that the complex structure, namely the conformal structure of the torus, is specified by a complex parameter τ ($\text{Im}\,\tau > 0$). Figure 8.3 shows the moduli space

$$\text{Mod}(\Sigma_1) = \mathcal{M}_1/G = \text{Teich}(\Sigma_g)/\text{SL}(2, \mathbb{Z}) = H/\text{SL}(2, \mathbb{Z}).$$

Take the torus T_τ specified by the Teichmüller parameter $\tau = \tau_1 + i\tau_2$ ($\tau_2 > 0$). As a representative, we take a torus in figure 14.8. The metric in $\mathbb{C}$ naturally induces a flat metric (as guaranteed by the uniformisation theorem)

$$\gamma = \tfrac{1}{2}[dz \otimes d\bar{z} + d\bar{z} \otimes dz]. \qquad (14.122)$$

The CKV are globally defined holomorphic vectors. We take $\Phi = \alpha\partial/\partial z$ as the normalised basis of the CKV. The condition $(\Phi, \Phi) = 1$ yields $\int d^2z|\alpha|^2 = \tau_2|\alpha|^2 = 1$, that is $\alpha = \tau_2^{-1/2}$ (we have dropped the phase). Φ generates translations in the complex plane,

$$z \to z' = z + \tau_2^{-1/2}(v^1 + iv^2). \qquad (14.123)$$

We must note, however, that the translation is defined modulo the lattice; $\tau_2^{-1/2}(v^1 + iv^2)$ and $\tau_2^{-1/2}(v^1 + iv^2) + (m + \tau n)$ yield the identical translation. This forces $\tau_2^{-1/2}(v_1 + iv_2)$ to lie within the parallelogram of figure 14.8. Since

$$\tau_2 = \int d^2z = \tau_2^{-1} \int d^2v$$

V(CKV) is found to be

$$V(\text{CKV}) = \int d^2v = \tau_2^2. \tag{14.124}$$

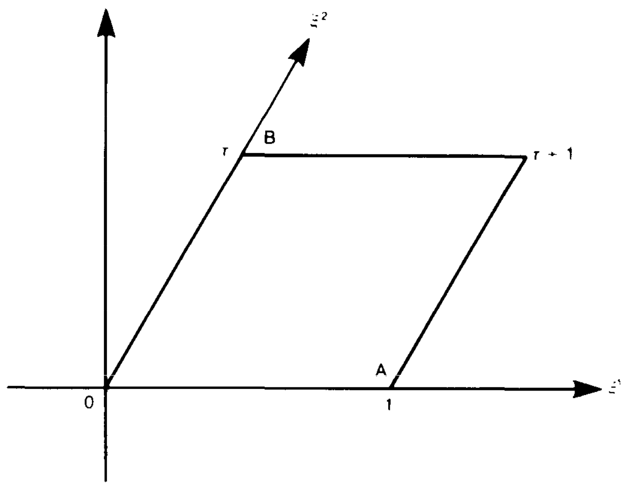


Figure 14.8 The parallelogram whose complex structure is parametrised by τ.

Our next task is to evaluate the Weil–Petersson measure. On the torus there is one quadratic differential ϕ. Since $\phi \in \mathcal{T}^{-2}$ is a globally defined holomorphic differential, it is of the form,

$$\phi = a\,dz \otimes dz \qquad a \in \mathbb{C}. \tag{14.125}$$

To find the Beltrami differential, we evaluate the change of the metric under a small variation of τ. For this purpose it is convenient to introduce the ξ^α-coordinate system in figure 14.8. The point A corresponds to (1, 0) and B to (0, 1). Accordingly, we have $z = \xi^1 + \tau\xi^2$. Under a small change $\delta\tau$ of the Teichmüller parameter, we have, up to a conformal factor,

$$|dz|^2 \to |d\xi^1 + (\tau + \delta\tau)d\xi^2|^2 = |dz + \delta\tau d\xi^2|^2$$

$$= \left| dz + \delta\tau \frac{dz - d\bar{z}}{2i\tau_2} \right|^2 = \left| dz + \delta\tau \frac{id\bar{z}}{2\tau_2} \right|^2.$$

Comparing this with (14.110), we find that

$$\mu_{zz} = i/2\tau_2. \tag{14.126}$$

[$(\delta\tau)\mu$ here is the cc of $(\delta\tau)\mu$ in (14.110). Of course this is a reparametrisation of the Teichmüller space and does not affect the results. If the reader feels awkward with this, he may choose $\bar{\tau}$ as the

Teichmüller parameter.] From (14.125) and (14.126), we have, up to irrelevant constants,

$$(\mu, \phi) = \int \mathrm{d}^2 z \overline{\mu^{zz}} \phi_{zz} = \frac{\mathrm{i}}{2\tau_2} a\tau_2 \propto a$$

$$(\phi, \phi) = \int \mathrm{d}^2 z\, \overline{\phi^{zz}} \phi_{zz} = a^2 \tau_2.$$

Finally we have obtained

$$\frac{|(\mu, \phi)|^2}{(\phi, \phi)} = \tau_2^{-1}. \tag{14.127}$$

14.3.2 The evaluation of determinants

We first consider $\det{}' P^\dagger_1 P_1 = \det{}' \Delta^+_{(1)}$. Since we take a flat metric, the Laplacian takes quite a simple form,

$$\Delta^+_{(1)} = -2\partial_z \partial_{\bar{z}} = \Delta \tag{14.128}$$

where Δ is the Laplacian defined by (14.87*b*). Since

$$\int \mathrm{d}^2 \xi \sqrt{\gamma} = \int \mathrm{d}^2 z = \tau_2$$

the amplitude (14.121) reduces to

$$Z_1 = \int_{\mathrm{Mod}} \underbrace{\frac{\mathrm{d}\tau}{\tau_2^2}}_{\substack{\uparrow \\ V(\text{CKV})}} \underbrace{\frac{\det{}'\Delta}{\tau_2}}_{\substack{\uparrow \\ \text{W–P}}} \Bigg(\underbrace{\frac{\det{}'\Delta}{\tau_2}}_{\substack{\uparrow \\ \int \mathrm{d}^2 z}} \Bigg)^{-13} \tag{14.129}$$

where we have used (14.124) and (14.127). We have factorised the integrand so that the modular invariance is manifest, see exercise 14.11.

Let us compute the spectrum of Δ. It is convenient to express the Laplacian in ξ^α-coordinates. From

$$\xi^1 = \mathrm{i}(\bar{\tau} z - \tau \bar{z})/2\tau_2 \qquad \xi^2 = (z - \bar{z})/2\mathrm{i}\tau_2 \tag{14.130}$$

we readily find that

$$\Delta = -\frac{1}{2\tau_2^2} [|\tau|^2 (\partial_1)^2 - 2\tau_1 \partial_1 \partial_2 + (\partial_2)^2] \tag{14.131}$$

where $\partial_1 = \partial/\partial \xi^1$ etc. The eigenfunction satisfying the periodic boundary condition on the torus is

$$\psi_{m,n}(\xi) = \exp[2\pi \mathrm{i}(n\xi^1 + m\xi^2)] \qquad (m, n) \in \mathbb{Z}^2. \tag{14.132}$$

Substituting this into (14.131), we find the eigenvalue

$$\lambda_{m,n} = \frac{2\pi^2}{\tau_2^2} (m - \tau n)(m - \bar{\tau} n). \tag{14.133}$$

The determinant is expressed as an infinite product

$$\det{}'\Delta = \prod_{m,n}{}' \frac{2\pi^2}{\tau_2^2} |m + \tau n|^2 \tag{14.134}$$

the product being taken for all integers $(m, n) \neq (0, 0)$.

Clearly $\det{}'\Delta$ is ill defined and needs to be regularised. Let us introduce the **Eisenstein series** (Siegel 1980, Lang 1987) defined by

$$E(\tau, s) \equiv \sum_{m,n}{}' \frac{\tau_2^s}{|m + \tau n|^{2s}} \tag{14.135}$$

the summation being taken for all integers $(m, n) \neq (0, 0)$. This series converges for $\mathrm{Re}\, s > 1$ and can be analytically continued to the complex s-plane. $E(\tau, s)$ has a simple pole at $s = 1$ where we have a Laurent expansion,

$$E(\tau, s) = \frac{\pi}{s - 1} + 2\pi[\gamma - \ln 2 - \ln(\sqrt{\tau_2}|\eta(\tau)|^2)] + O(s - 1). \tag{14.136}$$

This expression is known as the **Kronecker first limit formula** and is essential for our purposes. In (14.136), $\gamma = 0.577\,21 \ldots$ is Euler's constant and $\eta(\tau)$ is the **Dedekind $\boldsymbol{\eta}$-function** defined by

$$\eta(\tau) \equiv \mathrm{e}^{\mathrm{i}\pi\tau/12} \prod_{n \geqslant 1} (1 - \mathrm{e}^{2\mathrm{i}\pi n\tau}). \tag{14.137}$$

Neglecting constant factors, we have

$$\begin{aligned} \frac{\det{}'\Delta}{\tau_2} &= \exp\left(- \ln \tau_2 + \sum{}' \ln \frac{|m + \tau n|^2}{\tau_2^2} \right) \\ &= \exp\left(- \ln \tau_2 - \frac{\partial}{\partial s} [\tau_2^s E(\tau, s)] \bigg|_{s=0} \right) \\ &= \exp\{-\ln \tau_2 \, [1 + E(\tau, 0)] - E'(\tau, 0)\}. \end{aligned} \tag{14.138}$$

To evaluate the exponent we note the functional equation,

$$\pi^{-s}\Gamma(s)E(\tau, s) = \pi^{-(1-s)}\Gamma(1 - s)E(\tau, 1 - s). \tag{14.139}$$

Taking the limit $s \to 0$ in (14.139), we have

$$\begin{aligned} &sE(\tau, 1 - s) \\ &\quad = \pi^{1-2s} \frac{\Gamma(1 + s)}{\Gamma(1 - s)} E(\tau, s) \\ &\quad = \pi(1 - 2s \ln \pi + \ldots) \frac{(1 - \gamma s + \ldots)}{(1 + \gamma s + \ldots)} [E(\tau, 0) + E'(\tau, 0)\, s + \ldots] \\ &\quad = \pi E(\tau, 0) + [-2(\ln \pi + \gamma)E(\tau, 0) + E'(\tau, 0)]\pi s + \ldots . \end{aligned}$$

From (14.136), we also have

$$sE(\tau, 1-s) = -\pi + 2\pi s[\gamma - \ln 2 - \ln(\sqrt{\tau_2}|\eta(\tau)|^2)] + \ldots.$$

Equating the coefficients of s^0 and s^1, we find

$$E(\tau, 0) = -1 \tag{14.140a}$$

$$E'(\tau, 0) = -2[\ln 2\pi + \ln(\sqrt{\tau_2}|\eta(\tau)|^2)]. \tag{14.140b}$$

Substituting (14.140) into (14.138), we obtain (Nakahara and Asai 1989)

$$\frac{\det{}'\Delta}{\tau_2} = \exp[-E'(\tau, 0)] = \tau_2|\eta(\tau)|^4. \tag{14.141}$$

From (14.129) and (14.141) we finally have

$$Z_1 = \int_{\text{Mod}} \frac{d\tau}{\tau_2^2}\, \tau_2^{-12}|\eta(\tau)|^{-48}. \tag{14.142}$$

A neat form of Z_1 is obtained if we define the **discriminant**

$$\Delta(\tau) \equiv (2\pi)^{12}\eta(\tau)^{24}. \tag{14.143}$$

Up to an irrelevant constant, the one-loop amplitude is

$$Z_1 = \int_{\text{Mod}} \frac{d\tau}{\tau_2^2}\, \tau_2^{-12}|\Delta(\tau)|^{-2}. \tag{14.144}$$

$\Delta(\tau)$ is known as the **cusp form** of weight 12, implying

$$\Delta\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{12}\Delta(\tau) \tag{14.145}$$

and $c(0) = 0$, where the $c(n)$ are the Fourier coefficients,

$$\Delta(\tau) = \sum_{n\geqslant 0} c(n)e^{2\pi i n\tau}. \tag{14.146}$$

Higher genus amplitudes are given by the cusp forms of other weights, see Belavin and Knizhnik (1986), Moore (1986), Gilbert (1986) and Morozov (1987).

Exercise 14.11 Show that

$$\eta(\tau + 1) = e^{\pi i/12}\eta(\tau) \qquad \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \tag{14.147}$$

where the branch is chosen so that $\sqrt{z} > 0$ if $z > 0$. Use this result to show that $d\tau/\tau_2^2$ and $\tau_2^{-12}|\eta(\tau)|^{-48}$ are independently invariant under $\tau \to \tau + 1$ and $\tau \to -1/\tau$.

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