2 Multifactorial

2.1 Double Factorial

2.1.1 The definition of a Double Factorial

Definition 2.1.1

When m, n denote a natural number,

$$m!! \equiv n\cdot(n-2)\cdot(n-4)\cdot\ \cdots\ \cdot 5\cdot 3\cdot 1 \qquad m=2n-1 \tag{1^1}$$

$$\equiv n\cdot(n-2)\cdot(n-4)\cdot\ \cdots\ \cdot 6\cdot 4\cdot 2 \qquad m=2n \tag{1^0}$$

$$\equiv 1 \qquad m=0,\ -1 \tag{1^-}$$

Example.

$$2!! = 2,\quad 4!! = 2\cdot 4,\quad 6!! = 2\cdot 4\cdot 6,\ \cdots$$

$$1!! = 1,\quad 3!! = 1\cdot 3,\quad 5!! = 1\cdot 3\cdot 5,\ \cdots$$

$$0!! = (-1)!! = 1$$

2.1.2 The basic formulas of a double factorial

Formula 2.1.2

When n , π and $\Gamma(z)$ denote a natural number, a circular constant and a gamma function respectively, the following expressions hold.

$$n! = n!!\cdot(n-1)!! \tag{2}$$

$$(2n-1)!! = 2^n\Gamma\left(n+\frac{1}{2}\right) \Big/ \Gamma\left(\frac{1}{2}\right) = 2^n\Gamma\left(n+\frac{1}{2}\right) \Big/ \sqrt{\pi} \tag{3^1}$$

$$(2n)!! = 2^n\Gamma\left(n+\frac{2}{2}\right) \Big/ \Gamma\left(\frac{2}{2}\right) = 2^n\Gamma(n+1) = 2^n n! \tag{3^0}$$

$$(-2n)!! = \infty , \qquad 0!! = 1 \tag{4}$$

$$\{-(2n+1)\}!! = (-1)^{-n}\frac{2}{(2n-1)!!} , \qquad (-1)!! = 1 \tag{5}$$

Proof

First, calculate (1^1) × (1^0), as follows.

$$(2n)!!\cdot(2n-1)!! = (2n-1)\cdot(2n-3)\cdot(2n-5)\cdot\ \cdots\ \cdot 5\cdot 3\cdot 1$$
$$\times\ 2n\ \cdot(2n-2)\cdot(2n-4)\cdot\ \cdots\ \cdot 6\cdot 4\cdot 2 = (2n)! \tag{2}$$

Replacing $2n$ with n, we obtain (2).

Next ,since $\Gamma(1+z) = z\Gamma(z)$,

$$\Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(2+\frac{1}{2}\right) = \Gamma\left(1+\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(3+\frac{1}{2}\right) = \Gamma\left(1+\frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\vdots$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\Gamma\left(\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\Gamma\left(\frac{1}{2}\right)$$

$$\therefore \quad (2n-1)!! = 2^n\Gamma\left(n+\frac{1}{2}\right) / \Gamma\left(\frac{1}{2}\right) = 2^n\Gamma\left(n+\frac{1}{2}\right) / \sqrt{\pi} \qquad (3^1)$$

In a similar way, we obtain (3^0).

Substitute $n=0$ for (3^0) ; then

$$0!! = 2^0\Gamma(1) = 1$$

Replace n with $-n$ in (3^0) ; then

$$(-2n)!! = \Gamma(1-n)2^{-n} = \infty \qquad \text{(approach from +)} \qquad (4)$$

Next, from (2) and (3^0)

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{\Gamma(1+2n)}{2^n\Gamma(1+n)} \qquad (w)$$

Substitute $n=0$ for (w) , then

$$(-1)!! = \frac{\Gamma(1)}{\Gamma(1)2^0} = \frac{0!}{0!\cdot 1} = 1$$

This secures the justification of the definition (1^-) conjointly with $0!!=1$.

Finally, replace n with $-n$ in (w) , then

$$\{-(2n+1)\}!! = \frac{\Gamma(1-2n)}{2^{-n}\Gamma(1-n)} = \frac{2^n\Gamma\{-(2n\,1)\}}{\Gamma\{-(n-1)\}}$$

According to the Formula 1.3.1 in **1.3** (Singular Point Formulas) ,

$$\frac{\Gamma\{-(2n\,1)\}}{\Gamma\{-(n-1)\}} = (-1)^{(n-1)-(2n-1)}\frac{(n-1)!}{(2n-1)!} = (-1)^{-n}\frac{(n-1)!}{(2n-1)!}$$

Then

$$\{-(2n+1)\}!! = (-1)^{-n}\frac{(n-1)!\,2^n}{(2n-1)!} = (-1)^{-n}\frac{2\,n!\,2^n}{(2n)!}$$

$$= (-1)^{-n}\frac{2}{(2n-1)!!} \qquad \left(\because \quad \frac{(2n)!}{2^n n!} = (2n-1)!!\right) \qquad (5)$$

Example 1

$$13!! = (2\cdot 7-1)!! = 2^7\Gamma\left(7+\frac{1}{2}\right) / \Gamma\left(\frac{1}{2}\right) = 2^7\Gamma\left(\frac{15}{2}\right) / \sqrt{\pi}$$

$$= 128\times 1871.25430543/1.77245385 = 135135.0000$$

Example 2

$$(-3)!! = (-1)^1\frac{2}{1!!} = -2\,, \qquad (-5)!! = (-1)^2\frac{2}{3!!} = \frac{2}{3}$$

2.1.3 Expression by the double factorial of Maclaurin expansion

Formula 2.1.3

$$f\left(\frac{x}{2}\right) = \frac{f(0)}{0!!}x^0 + \frac{f'(0)}{2!!}x^1 + \frac{f''(0)}{4!!}x^2 + \frac{f^{(3)}(0)}{6!!}x^3 + \cdots \qquad (6)$$

Calculation

$$f(x) = f(0)+\frac{f'(0)}{1!}x^1+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\cdots$$

$$= \frac{f(0)}{2^0\cdot 0!}2x^0+\frac{f'(0)}{2^1\cdot 1!}2^1x^1+\frac{f''(0)}{2^2\cdot 2!}2^2x^2+\frac{f^{(3)}(0)}{2^3\cdot 3!}2^3x^3+\cdots$$

$$= \frac{f(0)}{0!!}(2x)^0+\frac{f'(0)}{2!!}(2x)^1+\frac{f''(0)}{4!!}(2x)^2+\frac{f^{(3)}(0)}{6!!}(2x)^3+\cdots$$

Replacing x with $x/2$, we obtain (6).

2.1.4 Expansion by the double factorial of an elementary function

Applying Formula 2.1.3 to an elementary function, we obtain the following expressions.

(1) Expansion by the double factorial of the exponental function

$$e^{\pm\frac{x}{2}} = \frac{x^0}{0!!} \pm \frac{x^1}{2!!} + \frac{x^2}{4!!} \pm \frac{x^3}{6!!} + \frac{x^4}{8!!} \pm \cdots$$

(2) Expansions by the double factorial of the trigonometric function

$$\cos\frac{x}{2} = \frac{x^0}{0!!} - \frac{x^2}{4!!} + \frac{x^4}{8!!} - \frac{x^6}{12!!} + \frac{x^8}{16!!} - \cdots$$

$$\sin\frac{x}{2} = \frac{x^1}{2!!} - \frac{x^3}{6!!} + \frac{x^5}{10!!} - \frac{x^7}{14!!} + \frac{x^9}{18!!} - \cdots$$

(3) Expansions by the double factorial of the hyperbolic function

$$\cosh\frac{x}{2} = \frac{x^0}{0!!} + \frac{x^2}{4!!} + \frac{x^4}{8!!} + \frac{x^6}{12!!} + \frac{x^8}{16!!} + \cdots$$

$$\sinh\frac{x}{2} = \frac{x^1}{2!!} + \frac{x^3}{6!!} + \frac{x^5}{10!!} + \frac{x^7}{14!!} + \frac{x^9}{18!!} + \cdots$$

(4) Expansions by the double factorial of the logarithmic function

$$\log\left(1\pm\frac{x}{2}\right) = \pm\frac{0!}{2!!}x - \frac{1!}{4!!}x^2 \pm \frac{2!}{6!!}x^3 - \frac{3!}{8!!}x^4 \pm \cdots$$

$$\log\sqrt{\frac{2+x}{2-x}} = \frac{0!}{2!!}x + \frac{2!}{6!!}x^3 + \frac{4!}{10!!}x^5 + \frac{6!}{14!!}x^7 + \cdots$$

$$\frac{1}{2i}\log\frac{2+xi}{2-xi} = \frac{0!}{2!!}x - \frac{2!}{6!!}x^3 + \frac{4!}{10!!}x^5 - \frac{6!}{14!!}x^7 + \cdots$$

(5) Expansions by the double factorial of the inverse trigonometric function

$$\sin^{-1}x = \frac{(-1)!!}{0!!}\frac{x^1}{1} + \frac{1!!}{2!!}\frac{x^3}{3} + \frac{3!!}{4!!}\frac{x^5}{5} + \frac{5!!}{6!!}\frac{x^7}{7} + \cdots$$

$$\tan^{-1}\frac{x}{2} = \frac{0!}{2!!}x - \frac{2!}{6!!}x^3 + \frac{4!}{10!!}x^5 - \frac{6!}{14!!}x^7 + \cdots$$

$$\therefore \quad tan^{-1}\frac{x}{2} = \frac{1}{2i}log\frac{2+xi}{2-xi}$$

(6) Expansions by the double factorial of the inverse hyperbolic function

$$sinh^{-1}x = \frac{(-1)!!}{0!!}\frac{x^1}{1} - \frac{1!!}{2!!}\frac{x^3}{3} + \frac{3!!}{4!!}\frac{x^5}{5} - \frac{5!!}{6!!}\frac{x^7}{7} + \cdots$$

$$tanh^{-1}\frac{x}{2} = \frac{0!}{2!!}x + \frac{2!}{6!!}x^3 + \frac{4!}{10!!}x^5 + \frac{6!}{14!!}x^7 + \cdots$$

$$\therefore \quad tanh^{-1}\frac{x}{2} = log\sqrt{\frac{2+x}{2-x}} \qquad |x|<2$$

(7) Expansions by the double factorial of the irrational function

$$(1\pm x)^{\frac{1}{2}} = 1 \pm \frac{(-1)!!}{2!!}x - \frac{1!!}{4!!}x^2 \pm \frac{3!!}{6!!}x^3 - \frac{5!!}{8!!}x^4 \pm \cdots$$

$$(1\pm x)^{-\frac{1}{2}} = 1 \mp \frac{1!!}{2!!}x + \frac{3!!}{4!!}x^2 \mp \frac{5!!}{6!!}x^3 + \frac{7!!}{8!!}x^4 \mp \cdots$$

$$\int_0^x \sqrt{1\pm x^2}dx = x \pm \frac{(-1)!!}{2!!}\frac{1}{3}x^3 - \frac{1!!}{4!!}\frac{1}{5}x^5 \pm \frac{3!!}{6!!}\frac{1}{7}x^7 - \cdots$$

$$\int_0^x \frac{dx}{\sqrt{1\pm x^2}} = x \mp \frac{1!!}{2!!}\frac{1}{3}x^3 + \frac{3!!}{4!!}\frac{1}{5}x^5 \mp \frac{5!!}{6!!}\frac{1}{7}x^7 + \cdots$$

2.1.5 The double factorial series of the elementary function

The following series are obtained as the special values of **2.1.4** .

$$e^{\frac{1}{2}} = \frac{1}{0!!} + \frac{1}{2!!} + \frac{1}{4!!} + \frac{1}{6!!} + \frac{1}{8!!} + \cdots$$

$$e^{-\frac{1}{2}} = \frac{1}{0!!} - \frac{1}{2!!} + \frac{1}{4!!} - \frac{1}{6!!} + \frac{1}{8!!} - \cdots$$

$$cos\frac{1}{2} = \frac{1}{0!!} - \frac{1}{4!!} + \frac{1}{8!!} - \frac{1}{12!!} + \frac{1}{16!!} - \cdots$$

$$sin\frac{1}{2} = \frac{1}{2!!} - \frac{1}{6!!} + \frac{1}{10!!} - \frac{1}{14!!} + \frac{1}{18!!} - \cdots$$

$$cosh\frac{1}{2} = \frac{1}{0!!} + \frac{1}{4!!} + \frac{1}{8!!} + \frac{1}{12!!} + \frac{1}{16!!} + \cdots$$

$$sinh\frac{1}{2} = \frac{1}{2!!} + \frac{1}{6!!} + \frac{1}{10!!} + \frac{1}{14!!} + \frac{1}{18!!} + \cdots$$

$$log\frac{3}{2} = \frac{0!}{2!!} - \frac{1!}{4!!} + \frac{2!}{6!!} - \frac{3!}{8!!} + \cdots$$

$$log 2 = \frac{0!}{2!!} + \frac{1!}{4!!} + \frac{2!}{6!!} + \frac{3!}{8!!} + \cdots$$

$$log\sqrt{3} = \frac{0!}{2!!} + \frac{2!}{6!!} + \frac{4!}{10!!} + \frac{6!}{14!!} + \cdots$$

$$\tan^{-1}\frac{1}{2} = \frac{0!}{2!!} - \frac{2!}{6!!} + \frac{4!}{10!!} - \frac{6!}{14!!} + \cdots$$

$$\tanh^{-1}\frac{1}{2} = \frac{0!}{2!!} + \frac{2!}{6!!} + \frac{4!}{10!!} + \frac{6!}{14!!} + \cdots$$

$$\sqrt{2} = 1 + \frac{(-1)!!}{2!!} - \frac{1!!}{4!!} + \frac{3!!}{6!!} - \frac{5!!}{8!!} + \cdots$$

$$\frac{1}{\sqrt{2}} = 1 - \frac{1!!}{2!!} + \frac{3!!}{4!!} - \frac{5!!}{6!!} + \frac{7!!}{8!!} - \cdots$$

$$1 = \frac{(-1)!!}{2!!} + \frac{1!!}{4!!} + \frac{3!!}{6!!} + \frac{5!!}{8!!} + \cdots$$

2.2 Triple factorial

2.2.1 The definition of a Triple Factorial

Definition 2.2.1

When m, n denote a natural number,

$$m!!! \equiv m\cdot(m-3)\cdot(m-6)\cdot \cdots \cdot 7\cdot 4\cdot 1 \qquad m=3n-2 \tag{1^2}$$

$$\equiv m\cdot(m-3)\cdot(m-6)\cdot \cdots \cdot 8\cdot 5\cdot 2 \qquad m=3n-1 \tag{1^1}$$

$$\equiv m\cdot(m-3)\cdot(m-6)\cdot \cdots \cdot 9\cdot 6\cdot 3 \qquad m=3n \tag{1^0}$$

$$\equiv 1 \qquad m=0,\ -1,\ -2 \tag{1^-}$$

Example.

$$1!!! = 1,\quad 4!!! = 1\cdot 4,\quad 7!!! = 1\cdot 4\cdot 7,\ \cdots$$

$$2!!! = 2,\quad 5!!! = 2\cdot 5,\quad 8!!! = 2\cdot 5\cdot 8,\ \cdots$$

$$3!!! = 3,\quad 6!!! = 3\cdot 6,\quad 9!!! = 3\cdot 6\cdot 9,\ \cdots$$

$$0!!! = (-1)!!! = (-2)!!! = 1$$

2.2.2 The basic formulas of a triple factorial

Formula 2.2.2

When n and $\Gamma(z)$ denote a natural number and a gamma function respectively,

$$n! = n!!!\cdot(n-1)!!!\cdot(n-2)!!! \tag{2}$$

$$(3n-2)!!! = 3^n\Gamma\left(n+\frac{1}{3}\right) / \Gamma\left(\frac{1}{3}\right) \tag{3^2}$$

$$(3n-1)!!! = 3^n\Gamma\left(n+\frac{2}{3}\right) / \Gamma\left(\frac{2}{3}\right) \tag{3^1}$$

$$(3n)!!! = 3^n\Gamma\left(n+\frac{3}{3}\right) / \Gamma\left(\frac{3}{3}\right) = 3^n\Gamma(n+1) = 3^n n! \tag{3^0}$$

$$(-3n)!!! = \infty, \qquad 0!!! = 1 \tag{4}$$

$$(-1)!!!\cdot(-2)!!! = 1 \tag{5}$$

$$\{-(3n+1)\}!!!\cdot\{-(3n+2)\}!!! = \frac{3}{(3n-1)!!!\cdot(3n-2)!!!} \tag{6}$$

Proof

First, calculate $(1^2)\times(1^1)\times(1^0)$, as follows.

$$\begin{aligned}(3n-2)!!!\cdot(3n-1)!!!\cdot(3n)!!! \\ &= (3n-2)\cdot(3n-5)\cdot(3n-8)\cdot \cdots \cdot 7\cdot 4\cdot 1 \\ &\times(3n-1)\cdot(3n-4)\cdot(3n-7)\cdot \cdots \cdot 8\cdot 5\cdot 2 \\ &\times\ 3n \cdot(3n-3)\cdot(3n-6)\cdot \cdots \cdot 9\cdot 6\cdot 3 = (3n)!\end{aligned}$$

Replacing $2n$ with n, we obtain (2).

Next, since $\Gamma(1+z) = z\Gamma(z)$,

$$\Gamma\left(1+\frac{1}{3}\right) = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

$$\Gamma\left(2+\frac{1}{3}\right) = \Gamma\left(1+\frac{4}{3}\right) = \frac{4}{3}\Gamma\left(\frac{4}{3}\right) = \frac{4}{3}\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

$$\Gamma\left(3+\frac{1}{3}\right) = \Gamma\left(1+\frac{7}{3}\right) = \frac{7}{3}\Gamma\left(\frac{7}{3}\right) = \frac{7}{3}\frac{4}{3}\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

$$\vdots$$

$$\Gamma\left(n+\frac{1}{3}\right) = \frac{1\cdot 4\cdot 7\cdots(3n-2)}{3^n}\Gamma\left(\frac{1}{3}\right) = \frac{(3n-2)!!!}{3^n}\Gamma\left(\frac{1}{3}\right)$$

$$\therefore\quad (3n-2)!!! = 3^n\Gamma\left(n+\frac{1}{3}\right) / \Gamma\left(\frac{1}{3}\right) \tag{3^2}$$

In a similar way, we obtain (3^1),(3^0).

Substitute $n=0$ for (3^0) ; then

$$0!!! = 3^0\Gamma(1) = 1$$

Replace n with $-n$ in (3^0) ; then

$$(-3n)!!! = 3^{-n}\Gamma(1-n) = \infty \qquad \text{(approach from +)} \tag{4}$$

Next, from (2) and (3^0)

$$(3n-1)!!!\cdot(3n-2)!!! = \frac{(3n)!}{(3n)!!!} = \frac{\Gamma(1+3n)}{3^n\Gamma(1+n)} \tag{w}$$

Substitute $n=0$ for (w) ; then

$$(-1)!!!\cdot(-2)!!! = \frac{\Gamma(1)}{\Gamma(1)3^0} = \frac{0!}{0!\cdot 1} = 1 \tag{5}$$

This secures the justification of a definition (1^-) conjointly with $0!!!=1$.

Finally, replace n with $-n$ in (w) , then

$$\{-(3n+1)\}!!!\cdot\{-(3n+2)!!!\} = \frac{\Gamma(1-3n)}{3^{-n}\Gamma(1-n)} = \frac{3^n\Gamma\{-(3n-1)\}}{\Gamma\{-(n-1)\}}$$

According to Formulas1.1 in **1.3** (Singular Point Formulas) ,

$$\frac{\Gamma\{-(3n-1)\}}{\Gamma\{-(n-1)\}} = (-1)^{(n-1)-(3n-1)}\frac{(n-1)!}{(3n-1)!} = \frac{(n-1)!}{(3n-1)!}$$

Then

$$\{-(3n+1)\}!!!\cdot\{-(3n+2)!!!\} = \frac{(n-1)!\,3^n}{(3n-1)!} = \frac{3n(n-1)!\,3^n}{(3n)!}$$

$$= \frac{3\,n!\,3^n}{(3n)!!!\cdot(3n-1)!!!\cdot(3n-2)!!!} \qquad \{\text{ from (2) }\}$$

$$= \frac{3\,n!\,3^n}{3^n n!\cdot(3n-1)!!!\cdot(3n-2)!!!} \qquad \{\text{ from } (3^0)\ \}$$

$$= \frac{3}{(3n-1)!!!\cdot(3n-2)!!!} \tag{6}$$

Example 1

$$17!!! = (3\cdot 6-1)!!! = 3^6\Gamma\left(6+\frac{2}{3}\right) / \Gamma\left(\frac{2}{3}\right) = 3^6\Gamma\left(\frac{20}{3}\right) / \Gamma\left(\frac{2}{3}\right)$$
$$= 729\times 389.03492617/1.35411794 = 209439.9998$$

Example 2

$$(-4)!!!\cdot(-5)!!! = \frac{3}{1!!!\cdot 2!!!} = \frac{3}{2}$$

2.2.3 Expression by the triple factorial of Maclaurin expansion

Formula 2.2.3

$$f\left(\frac{x}{3}\right) = \frac{f(0)}{0!!!}x^0+\frac{f'(0)}{3!!!}x^1+\frac{f''(0)}{6!!!}x^2+\frac{f^{(3)}(0)}{9!!!}x^3+\cdots \tag{7}$$

Calculation

$$f(x) = f(0)+\frac{f'(0)}{1!}x^1+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\cdots$$
$$= \frac{f(0)}{3^0\cdot 0!}3x^0+\frac{f'(0)}{3^1\cdot 1!}3^1x^1+\frac{f''(0)}{3^2\cdot 2!}3^2x^2+\frac{f^{(3)}(0)}{3^3\cdot 3!}3^3x^3+\cdots$$
$$= \frac{f(0)}{0!!!}(3x)^0+\frac{f'(0)}{3!!!}(3x)^1+\frac{f''(0)}{6!!!}(3x)^2+\frac{f^{(3)}(0)}{9!!!}(3x)^3+\cdots$$

Replacing x with $x/3$, we obtain (7).

2.2.4 Expansion by the triple factorial of an elementary function

Applying Formula 2.2.3 to an elementary function, we obtain the following expressions.

(1) Expansion by the triple factorial of the exponental function

$$e^{\pm\frac{x}{3}} = \frac{x^0}{0!!!} \pm \frac{x^1}{3!!!} + \frac{x^2}{6!!!} \pm \frac{x^3}{9!!!} + \frac{x^4}{12!!!} \pm \cdots$$

(2) Expansions by the triple factorial of the trigonometric function

$$\cos\frac{x}{3} = \frac{x^0}{0!!!} - \frac{x^2}{6!!!} + \frac{x^4}{12!!!} - \frac{x^6}{18!!!} + \frac{x^8}{24!!!} - \cdots$$
$$\sin\frac{x}{3} = \frac{x^1}{3!!!} - \frac{x^3}{9!!!} + \frac{x^5}{15!!!} - \frac{x^7}{21!!!} + \frac{x^9}{27!!!} - \cdots$$

(3) Expansions by the triple factorial of the hyperbolic function

$$\cosh\frac{x}{3} = \frac{x^0}{0!!!} + \frac{x^2}{6!!!} + \frac{x^4}{12!!!} + \frac{x^6}{18!!!} + \frac{x^8}{24!!!} + \cdots$$
$$\sinh\frac{x}{3} = \frac{x^1}{3!!!} + \frac{x^3}{9!!!} + \frac{x^5}{15!!!} + \frac{x^7}{21!!!} + \frac{x^9}{27!!!} + \cdots$$

(4) Expansions by the triple factorial of the logarithmic function

$$log\left(1\pm\frac{x}{3}\right) = \pm\frac{0!}{3!!!}x - \frac{1!}{6!!!}x^2 \pm \frac{2!}{9!!!}x^3 - \frac{3!}{12!!!}x^4 \pm \cdots$$

$$log\sqrt{\frac{3+x}{3-x}} = \frac{0!}{3!!!}x + \frac{2!}{9!!!}x^3 + \frac{4!}{15!!!}x^5 + \frac{6!}{21!!!}x^7 + \cdots$$

$$\frac{1}{2i}log\frac{3+xi}{3-xi} = \frac{0!}{3!!!}x - \frac{2!}{9!!!}x^3 + \frac{4!}{15!!!}x^5 - \frac{6!}{21!!!}x^7 + \cdots$$

(5) Expansions by the triple factorial of the inverse trigonometric function

$$tan^{-1}\frac{x}{3} = \frac{0!}{3!!!}x - \frac{2!}{9!!!}x^3 + \frac{4!}{15!!!}x^5 - \frac{6!}{21!!!}x^7 + \cdots$$

$$\therefore\ tan^{-1}\frac{x}{3} = \frac{1}{2i}log\frac{3+xi}{3-xi}$$

(6) Expansions by the triple factorial of the inverse hyperbolic function

$$tanh^{-1}\frac{x}{3} = \frac{0!}{3!!!}x + \frac{2!}{9!!!}x^3 + \frac{4!}{15!!!}x^5 + \frac{6!}{21!!!}x^7 + \cdots$$

$$\therefore\ tanh^{-1}\frac{x}{3} = log\sqrt{\frac{3+x}{3-x}} \qquad |x|<3$$

(7) Expansions by the triple factorial of the irrational function

$$(1\pm x)^{\frac{1}{3}} = 1 \pm \frac{(-1)!!!}{3!!!}x - \frac{2!!!}{6!!!}x^2 \pm \frac{5!!!}{9!!!}x^3 - \frac{8!!!}{12!!!}x^4 \pm \cdots$$

$$(1\pm x)^{-\frac{1}{3}} = 1 \mp \frac{1!!!}{3!!!}x + \frac{4!!!}{6!!!}x^2 \mp \frac{7!!!}{9!!!}x^3 + \frac{10!!!}{12!!!}x^4 \mp \cdots$$

$$\int_0^x \sqrt[3]{1\pm x^3}dx = x \pm \frac{(-1)!!!}{3!!!}\frac{1}{4}x^4 - \frac{2!!!}{6!!!}\frac{1}{7}x^7 \pm \frac{5!!!}{9!!!}\frac{1}{10}x^{10} - \cdots$$

$$\int_0^x \frac{dx}{\sqrt[3]{1\pm x^3}} = x \mp \frac{1!!!}{3!!!}\frac{1}{4}x^4 + \frac{4!!!}{6!!!}\frac{1}{7}x^7 \mp \frac{7!!!}{9!!!}\frac{1}{10}x^{10} + \cdots$$

2.2.5 The triple factorial series of the elementary function

The following series are obtained as the special values of 2.2.4 .

$$e^{\frac{1}{3}} = \frac{1}{0!!!} + \frac{1}{3!!!} + \frac{1}{6!!!} + \frac{1}{9!!!} + \frac{1}{12!!!} + \cdots$$

$$e^{-\frac{1}{3}} = \frac{1}{0!!!} - \frac{1}{3!!!} + \frac{1}{6!!!} - \frac{1}{9!!!} + \frac{1}{12!!!} - \cdots$$

$$cos\frac{1}{3} = \frac{1}{0!!!} - \frac{1}{6!!!} + \frac{1}{12!!!} - \frac{1}{18!!!} + \frac{1}{24!!!} - \cdots$$

$$sin\frac{1}{3} = \frac{1}{3!!!} - \frac{1}{9!!!} + \frac{1}{15!!!} - \frac{1}{21!!!} + \frac{1}{27!!!} - \cdots$$

$$\cosh\frac{1}{3} = \frac{1}{0!!!} + \frac{1}{6!!!} + \frac{1}{12!!!} + \frac{1}{18!!!} + \frac{1}{24!!!} + \cdots$$

$$\sinh\frac{1}{3} = \frac{1}{3!!!} + \frac{1}{9!!!} + \frac{1}{15!!!} + \frac{1}{21!!!} + \frac{1}{27!!!} + \cdots$$

$$\log\frac{4}{3} = \frac{0!}{3!!!} - \frac{1!}{6!!!} + \frac{2!}{9!!!} - \frac{3!}{12!!!} + \cdots$$

$$-\log\frac{2}{3} = \frac{0!}{3!!!} + \frac{1!}{6!!!} + \frac{2!}{9!!!} + \frac{3!}{12!!!} + \cdots$$

$$\log\sqrt{2} = \frac{0!}{3!!!} + \frac{2!}{9!!!} + \frac{4!}{15!!!} + \frac{6!}{21!!!} + \cdots$$

$$\tan^{-1}\frac{1}{3} = \frac{0!}{3!!!} - \frac{2!}{9!!!} + \frac{4!}{15!!!} - \frac{6!}{21!!!} + \cdots$$

$$\tanh^{-1}\frac{1}{3} = \frac{0!}{3!!!} + \frac{2!}{9!!!} + \frac{4!}{15!!!} + \frac{6!}{21!!!} + \cdots$$

$$\sqrt[3]{2} = 1 + \frac{(-1)!!!}{3!!!} - \frac{2!!!}{6!!!} + \frac{5!!!}{9!!!} - \frac{8!!!}{12!!!} + \cdots$$

$$\frac{1}{\sqrt[3]{2}} = 1 - \frac{1!!!}{3!!!} + \frac{4!!!}{6!!!} - \frac{7!!!}{9!!!} + \frac{10!!!}{12!!!} - \cdots$$

$$1 = \frac{(-1)!!!}{3!!!} + \frac{2!!!}{6!!!} + \frac{5!!!}{9!!!} + \frac{8!!!}{12!!!} + \cdots$$

2.3 Multi factorial

2.3.1 The definition of a Multi factorial

Definition 2.3.1

When m, n denote natural number and $!! \cdots !$ (k pieces) $\equiv !_k$,

$$m!_k \equiv n\cdot(n-k)\cdot(n-2k)\ \cdots\ (1+k)\cdot 1 \qquad m=kn-(k-1) \qquad (1^{k-1})$$

$$\vdots$$

$$\equiv n\cdot(n-k)\cdot(n-2k)\ \cdots\ \{(k-1)+k\}\cdot(k-1) \qquad m=kn-1 \qquad (1^{1})$$

$$\equiv n\cdot(n-k)\cdot(n-2k)\ \cdots\ (k+k)\cdot k \qquad m=kn \qquad (1^{0})$$

$$\equiv 1 \qquad m=0,-1,\cdots,-(k-1) \qquad (1^{-})$$

Example

$$1!_4 = 1,\quad 5!_4 = 1\cdot 5,\quad 9!_4 = 1\cdot 5\ \cdot 9,\quad 13!_4 = 1\cdot 5\ \cdot 9\cdot 13,\quad \cdots$$

$$2!_4 = 2,\quad 6!_4 = 2\cdot 6,\quad 10!_4 = 2\cdot 6\cdot 10,\quad 14!_4 = 2\cdot 6\cdot 10\cdot 14,\quad \cdots$$

$$3!_4 = 3,\quad 7!_4 = 3\cdot 7,\quad 11!_4 = 3\cdot 7\cdot 11,\quad 15!_4 = 3\cdot 7\cdot 11\cdot 15,\quad \cdots$$

$$4!_4 = 4,\quad 8!_4 = 4\cdot 8,\quad 12!_4 = 4\cdot 8\cdot 12,\quad 16!_4 = 4\cdot 8\cdot 12, 16,\quad \cdots$$

$$0!_4 = (-1)!_4 = (-2)!_4 = (-3)!_4 = 1$$

2.3.2 The basic formulas of a multifactorial

Formula 2.3.2

When k, n and $\Gamma(z)$ denote a natural number and a gamma function respectively,

$$n! = n!_k\cdot(n-1)!_k\cdot(n-2)!_k\cdot\ \cdots\ \cdot\{n-(k-1)\}!_k \qquad (2)$$

$$(kn-s)!_k = k^n\Gamma\left(n+\frac{k-s}{k}\right) / \Gamma\left(\frac{k-s}{k}\right) \quad , \quad s=1,2,\cdots k-1 \qquad (3^{s})$$

$$(kn)!_k = k^n\Gamma\left(n+\frac{k}{k}\right) / \Gamma\left(\frac{k}{k}\right) = k^n\Gamma(n+1) = k^n n! \qquad (3^{0})$$

$$(-kn)!_k = \infty, \qquad 0!_k = 1 \qquad (4)$$

$$(-1)!_k\cdot(-2)!_k\cdot\ \cdots\ \cdot\{-(k-1)\}!_k = 1 \qquad (5)$$

$$\{-(kn+1)\}!_k\cdot\{-(kn+2)\}!_k\cdot\cdots\ \cdot\{-(kn+k-1)\}!_k$$

$$= \frac{k}{(kn-1)!_k\cdot(kn-2)!_k\cdot\cdots\ \cdot\{kn-(k-1)\}!_k} \qquad (6)$$

Proof

First, calculate $(1^{k-1})\times\cdots\times(1^{1})\times(1^{0})$, as follows.

$$\{kn-(k-1)\}!_k\cdot\ \cdots\ \cdot(kn-1)!_k\cdot(kn)!_k$$

$$= \{kn-(k-1)\}\{kn-(2k-1)\}\{kn-(3k-1)\}\ \cdots\ (k+1) \quad \cdot 1$$

$$\times\{kn-(k-2)\}\{kn-(2k-2)\}\{kn-(3k-2)\}\ \cdots\ (k+2) \quad \cdot 2$$

$$\vdots$$

$$\times \quad (kn-1) \quad \{kn-(k+1)\}\{kn-(2k+1)\} \cdots \{k+(k-1)\}\,(k-1)$$

$$\times \quad kn \quad (kn-k) \quad (kn-2k) \quad \cdots (k+k) \quad \cdot\; k$$

$$=(kn)!$$

Replacing $2n$ with n, we obtain (2).

Next, since $\Gamma(1+z)=z\Gamma(z)$,

$$\Gamma\left(1+\frac{k-s}{k}\right)=\frac{k-s}{k}\Gamma\left(\frac{k-s}{k}\right)$$

$$\Gamma\left(2+\frac{k-s}{k}\right)=\Gamma\left(1+\frac{2k-s}{k}\right)=\frac{2k-s}{k}\Gamma\left(\frac{2k-s}{k}\right)=\frac{2k-s}{k}\frac{k-s}{k}\Gamma\left(\frac{k-s}{k}\right)$$

$$\Gamma\left(3+\frac{k-s}{k}\right)=\Gamma\left(1+\frac{3k-s}{k}\right)=\frac{3k-s}{k}\frac{2k-s}{k}\frac{k-s}{k}\Gamma\left(\frac{k-s}{k}\right)$$

$$\vdots$$

$$\Gamma\left(n+\frac{k-s}{k}\right)=\frac{(nk-s)\cdot\cdots\cdot(2k-s)(k-s)}{k^n}\Gamma\left(\frac{k-s}{k}\right)=\frac{(kn-s)!_k}{k^n}\Gamma\left(\frac{k-s}{k}\right)$$

$$\therefore \quad (kn-s)!_k=k^n\Gamma\left(n+\frac{k-s}{k}\right)/\,\Gamma\left(\frac{k-s}{k}\right) \qquad (3^s)$$

Especially substituting $s=0$ for (3^s) , we obtain (3^0)

Substitute $n=0$ for (3^0) ; then

$$0!_k=k^0\Gamma(1)=1$$

Replace n with $-n$ in (3^0) ; then

$$(-kn)!_k=k^{-n}\Gamma(1-n)=\infty \qquad \text{(approach from +)} \qquad (4)$$

Next, from (2) and (3^0)

$$(kn-1)!_k\cdot(kn-2)!_k\cdot\ \cdots\ \cdot\{kn-(k-1)\}!_k=\frac{(kn)!}{(kn)!_k}=\frac{\Gamma(1+kn)}{k^n\Gamma(1+n)} \qquad \text{(w)}$$

Substitute $n=0$ for (w) ; then

$$(-1)!_k\cdot(-2)!_k\cdot\ \cdots\ \cdot\{-(k-1)\}!_k=\frac{\Gamma(1)}{\Gamma(1)k^0}=\frac{0!}{0!\cdot 1}=1$$

This secures the justification of a definition (1^-) conjointly with $0!_k=1$.

Finally, replace n with $-n$ n (w) , then

$$\{-(kn+1)\}!_k\cdot\{-(kn+2)\}!_k\cdot\cdots\ \cdot\{-(kn+k-1)\}!_k$$

$$=\frac{\Gamma(1-kn)}{k^{-n}\Gamma(1-n)}=\frac{k^n\Gamma\{-(kn-1)\}}{\Gamma\{-(n-1)\}}$$

According to Frmulas1.1 in **1.3** **(**Singular Point Formulas) ,

$$\frac{\Gamma\{-(kn-1)\}}{\Gamma\{-(n-1)\}}=(-1)^{(n-1)-(kn-1)}\frac{(n-1)!}{(kn-1)!}=(-1)^{-(k-1)n}\frac{(n-1)!}{(kn-1)!}$$

Tthen

$$\{-(kn+1)\}!_k\cdot\{-(kn+2)\}!_k\cdot\cdots\ \cdot\{-(kn+k-1)\}!_k$$

$$= (-1)^{-(k-1)n}\frac{(n-1)!\,k^n}{(kn-1)!} = (-1)^{-(k-1)n}\frac{kn(n-1)!\,k^n}{(kn)!}$$

$$= (-1)^{-(k-1)n}\frac{kn!\,k^n}{(kn)!_k\cdot(kn-1)!_k\cdot(kn-2)!_k\cdot\ \cdots\ \cdot\{kn-(k-1)\}!_k} \qquad \{\text{ from (2) }\}$$

$$= (-1)^{-(k-1)n}\frac{kn!\,k^n}{k^n n!\cdot(kn-1)!_k\cdot(kn-2)!_k\cdot\ \cdots\ \cdot\{kn-(k-1)\}!_k} \qquad \{\text{ from } (3^0)\ \}$$

$$= (-1)^{-(k-1)n}\frac{k}{(kn-1)!_k\cdot(kn-2)!_k\cdot\ \cdots\ \cdot\{kn-(k-1)\}!_k} \qquad (6)$$

Example 1

$$27!_5 = (5\times 6-3)!_5 = 5^6\Gamma\left(6+\frac{5-3}{5}\right) / \Gamma\left(\frac{5-3}{5}\right) = 5^6\Gamma\left(\frac{32}{5}\right)/\Gamma\left(\frac{2}{5}\right)$$

$$= 15625\times 240.83377994/2.21815954 = 1696464.0025$$

Example 2

$$(-6)!_5\cdot(-7)!_5\cdot(-8)!_5\cdot(-9)!_5 = \frac{(-1)^{-(5-1)}\cdot 5}{1!_5\cdot 2!_5\cdot 3!_5\cdot 4!_5} = \frac{5}{24}$$

2.3.3 Expression by the k-fold factorial of Maclaurin expansion

Formula 2.3.3

$$f\left(\frac{x}{k}\right) = \frac{f(0)}{0!_k}x^0+\frac{f^{(1)}(0)}{k!_k}x^1+\frac{f^{(2)}(0)}{(2k)!_k}x^2+\frac{f^{(3)}(0)}{(3k)!_k}x^3+\ \cdots \qquad (7)$$

Calculation

$$f(x) = f(0)+\frac{f^{(1)}(0)}{1!}x^1+\frac{f^{(2)}(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\ \cdots$$

$$= \frac{f(0)}{k^0\cdot 0!}kx^0+\frac{f^{(1)}(0)}{k^1\cdot 1!}k^1x^1+\frac{f^{(2)}(0)}{k^2\cdot 2!}k^2x^2+\frac{f^{(3)}(0)}{k^3\cdot 3!}k^3x^3+\ \cdots$$

$$= \frac{f(0)}{0!_k}(kx)^0+\frac{f^{(1)}(0)}{k!_k}(kx)^1+\frac{f^{(2)}(0)}{(2k)!_k}(kx)^2+\frac{f^{(3)}(0)}{(3k)!_k}(kx)^3+\ \cdots$$

Replacing x with x/k, we obtain (7).

2.3.4 Expansion by the multifactorial of an elementary function

Applying Formula 2.3.3 to an elementary function, we obtain the following expressions.

(1) Expansion by the multifactorial of the exponental function

$$e^{\pm\frac{x}{n}} = \frac{x^0}{0!_n} \pm \frac{x^1}{n!_n} + \frac{x^2}{(2n)!_n} \pm \frac{x^3}{(3n)!_n} + \frac{x^4}{(4n)!_n} \pm \cdots$$

(2) Expansions by the multifactorial of the trigonometric function

$$\cos\frac{x}{n} = \frac{x^0}{0!_n} - \frac{x^2}{(2n)!_n} + \frac{x^4}{(4n)!_n} - \frac{x^6}{(6n)!_n} + \frac{x^8}{(8n)!_n} - \cdots$$

$$\sin\frac{x}{n} = \frac{x^1}{n!_n} - \frac{x^3}{(3n)!_n} + \frac{x^5}{(5n)!_n} - \frac{x^7}{(7n)!_n} + \frac{x^9}{(9n)!_n} - \cdots$$

(3) Expansions by the multifactorial of the hyperbolic function

$$\cosh\frac{x}{n} = \frac{x^0}{0!_n} + \frac{x^2}{(2n)!_n} + \frac{x^4}{(4n)!_n} + \frac{x^6}{(6n)!_n} + \frac{x^8}{(8n)!_n} + \cdots$$

$$\sinh\frac{x}{n} = \frac{x^1}{n!_n} + \frac{x^3}{(3n)!_n} + \frac{x^5}{(5n)!_n} + \frac{x^7}{(7n)!_n} + \frac{x^9}{(9n)!_n} + \cdots$$

(4) Expansions by the multifactorial of the logarithmic function

$$\log\left(1\pm\frac{x}{n}\right) = \pm\frac{0!}{n!_n}x - \frac{1!}{(2n)!_n}x^2 \pm \frac{2!}{(3n)!_n}x^3 - \frac{3!}{(4n)!_n}x^4 \pm \cdots$$

$$\log\sqrt{\frac{n+x}{n-x}} = \frac{0!}{n!_n}x + \frac{2!}{(3n)!_n}x^3 + \frac{4!}{(5n)!_n}x^5 + \frac{6!}{(7n)!_n}x^7 + \cdots$$

$$\frac{1}{2i}\log\frac{n+xi}{n-xi} = \frac{0!}{n!_n}x - \frac{2!}{(3n)!_n}x^3 + \frac{4!}{(5n)!_n}x^5 - \frac{6!}{(7n)!_n}x^7 + \cdots$$

(5) Expansions by the multifactorial of the inverse trigonometric function

$$\tan^{-1}\frac{x}{n} = \frac{0!}{n!_n}x - \frac{2!}{(3n)!_n}x^3 + \frac{4!}{(5n)!_n}x^5 - \frac{6!}{(7n)!_n}x^7 + \cdots$$

$$\therefore\ \tan^{-1}\frac{x}{n} = \frac{1}{2i}\log\frac{n+xi}{n-xi}$$

(6) Expansions by the multifactorial of the inverse hyperbolic function

$$\tanh^{-1}\frac{x}{n} = \frac{0!}{n!_n}x + \frac{2!}{(3n)!_n}x^3 + \frac{4!}{(5n)!_n}x^5 + \frac{6!}{(7n)!_n}x^7 + \cdots$$

$$\therefore\ \tanh^{-1}\frac{x}{n} = \log\sqrt{\frac{n+x}{n-x}} \qquad |x| < n$$

(7) Expansions by the multifactorial of the irrational function

$$(1\pm x)^{\frac{1}{n}} = 1\pm\frac{(-1)!_n}{n!_n}x - \frac{(n-1)!_n}{(2n)!_n}x^2 \pm \frac{(2n-1)!_n}{(3n)!_n}x^3 - \frac{(3n-1)!_n}{(4n)!_n}x^4 \pm \cdots$$

$$(1\pm x)^{-\frac{1}{n}} = 1\mp\frac{1!_n}{n!_n}x + \frac{(n+1)!_n}{(2n)!_n}x^2 \mp \frac{(2n+1)!_n}{(3n)!_n}x^3 + \frac{(3n+1)!_n}{(4n)!_n}x^4 \mp \cdots$$

$$\int_0^x \sqrt[n]{1\pm x^n}\,dx = x \pm \frac{(-1)!_n}{n!_n}\frac{x^{n+1}}{n+1} - \frac{(n-1)!_n}{(2n)!_n}\frac{x^{2n+1}}{2n+1} \pm \frac{(2n-1)!_n}{(3n)!_n}\frac{x^{3n+1}}{3n+1} - \cdots$$

$$\int_0^x \frac{dx}{\sqrt[n]{1\pm x^n}} = x \mp \frac{1!_n}{n!_n}\frac{x^{n+1}}{n+1} + \frac{(n+1)!_n}{(2n)!_n}\frac{x^{2n+1}}{2n+1} \mp \frac{(2n+1)!_n}{(3n)!_n}\frac{x^{3n+1}}{3n+1} + \cdots$$

2.3.5 The multifactorial series of the elementary function

The following series are obtained as the special values of **2.3.4** .

$$e^{\pm\frac{1}{n}} = \frac{1}{0!_n} \pm \frac{1}{n!_n} + \frac{1}{(2n)!_n} \pm \frac{1}{(3n)!_n} + \frac{1}{(4n)!_n} \pm \cdots$$

$$\cos\frac{1}{n} = \frac{1}{0!_n} - \frac{1}{(2n)!_n} + \frac{1}{(4n)!_n} - \frac{1}{(6n)!_n} + \frac{1}{(8n)!_n} - \cdots$$

$$\sin\frac{1}{n} = \frac{1}{n!_n} - \frac{1}{(3n)!_n} + \frac{1}{(5n)!_n} - \frac{1}{(7n)!_n} + \frac{1}{(9n)!_n} - \cdots$$

$$\cosh\frac{1}{n} = \frac{1}{0!_n} + \frac{1}{(2n)!_n} + \frac{1}{(4n)!_n} + \frac{1}{(6n)!_n} + \frac{1}{(8n)!_n} + \cdots$$

$$\sinh\frac{1}{n} = \frac{1}{n!_n} + \frac{1}{(3n)!_n} + \frac{1}{(5n)!_n} + \frac{1}{(7n)!_n} + \frac{1}{(9n)!_n} + \cdots$$

$$\log\left(1\pm\frac{1}{n}\right) = \pm\frac{0!}{n!_n} - \frac{1!}{(2n)!_n} \pm \frac{2!}{(3n)!_n} - \frac{3!}{(4n)!_n} \pm \cdots$$

$$\log\sqrt{\frac{n+1}{n-1}} = \frac{0!}{n!_n} + \frac{2!}{(3n)!_n} + \frac{4!}{(5n)!_n} + \frac{6!}{(7n)!_n} + \cdots \qquad (n>1)$$

$$\tan^{-1}\frac{1}{n} = \frac{0!}{n!_n} - \frac{2!}{(3n)!_n} + \frac{4!}{(5n)!_n} - \frac{6!}{(7n)!_n} + \cdots$$

$$\tanh^{-1}\frac{1}{n} = \frac{0!}{n!_n} + \frac{2!}{(3n)!_n} + \frac{4!}{(5n)!_n} + \frac{6!}{(7n)!_n} + \cdots$$

$$\sqrt[n]{2} = 1 + \frac{(-1)!_n}{n!_n} - \frac{(n-1)!_n}{(2n)!_n} + \frac{(2n-1)!_n}{(3n)!_n} - \frac{(3n-1)!_n}{(4n)!_n} + \cdots \qquad (n>1)$$

$$\frac{1}{\sqrt[n]{2}} = 1 - \frac{1!_n}{n!_n} + \frac{(n+1)!_n}{(2n)!_n} - \frac{(2n+1)!_n}{(3n)!_n} + \frac{(3n+1)!_n}{(4n)!_n} - \cdots$$

$$1 = \frac{(-1)!_n}{n!_n} + \frac{(n-1)!_n}{(2n)!_n} + \frac{(2n-1)!_n}{(3n)!_n} + \frac{(3n-1)!_n}{(4n)!_n} + \cdots \qquad (n>1)$$

Especially when $n=1$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

$$\cos 1 = \frac{1}{0!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \cdots$$

$$\sin 1 = \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \cdots$$

$$\cosh 1 = \frac{1}{0!} + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} + \cdots$$

$$\sinh 1 = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \frac{1}{9!} + \cdots$$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\tanh^{-1} 1 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \qquad = \infty$$

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - \cdots$$

2.3.6 Expansions by the multifactorial of the Pochhammer Symbol

Pochhammer Symbol of a rational number smaller than 1 can be expressed by the multi-factorial.

Formula 2.3.4

When n , k are two or more natural numbers and a is an integer such that $|a| < n$,

$$\left(\frac{a}{n}\right)_k = \frac{\{a+(k-1)n\}!_n}{n^k} \qquad (when\ a>0) \tag{8}$$

$$= \frac{a\{a+(k-1)n\}!_n}{n^k} \qquad (when\ a<0) \tag{8'}$$

Calculation

When $a>0$,

$$\left(\frac{a}{n}\right)_k = \frac{a}{n}\left(\frac{a}{n}+1\right)\cdots\left(\frac{a}{n}+k-1\right) = \frac{a(a+1n)\cdots\{a+(k-1)n\}}{n^k}$$

$$= \frac{\{a+(k-1)n\}!_n}{n^k}$$

When $a<0$, since $a\,!_n=1,\ a+1n>0,\ \cdots\ ,\ a+(k-1)n > 0$, then

$$a = a \cdot a\,!_n \qquad (k=1)$$

$$a(a+1n) = a \cdot (a+1n)!_n \qquad (k=2)$$

$$a(a+1n)(a+2n) = a \cdot (a+2n)!_n \qquad (k=3)$$

hence by inducton

$$\frac{a(a+1n)\cdots\{a+(k-1)n\}}{n^k} = \frac{a \cdot \{a+(k-1)n\}!_n}{n^k}$$

Example

$$\left(\frac{1}{3}\right)_k = \frac{1\cdot 4\cdot 7\cdots\{1+3(k-1)\}}{3^k} = \frac{\{1+3(k-1)\}!!!}{3^k}$$

$$\left(\frac{-1}{3}\right)_k = \frac{-1\cdot 2\cdot 5\cdots\{-1+3(k-1)\}}{3^k} = \frac{-1\cdot\{-1+3(k-1)\}!!!}{3^k} \quad (-1)!!! = 1$$

$$\left(\frac{-2}{3}\right)_k = \frac{-2\cdot 1\cdot 4\cdots\{-2+3(k-1)\}}{3^k} = \frac{-2\cdot\{-2+3(k-1)\}!!!}{3^k} \quad (-2)!!! = 1$$

2.3.7 Expansions by the multifactorial of the Hypergeometric Function

The hypergeometric function of which parameters are rational number smaller than 1 can be exprssed by the multifactorial.

Formula 2.3.5

When n is two or more natural number and a is an integer such that $|a| < n$,

$$ {}_2F_1\left(\frac{a}{n}, b, c; x\right) = 1 + \sum_{k=1}^{\infty} \frac{\{a+n(k-1)\}!_n\cdot(b)_k}{(c)_k}\frac{x^k}{(nk)!_n} \qquad (a>0) \tag{9}$$

$$ = 1 + a\sum_{k=1}^{\infty} \frac{\{a+n(k-1)\}!_n\cdot(b)_k}{(c)_k}\frac{x^k}{(nk)!_n} \qquad (a<0) \tag{9'}$$

When n is two or more natural number and a,b,c are integers such that $|a|,|b|,|c| < n$,

$$ {}_2F_1\left(\frac{a}{n}, \frac{b}{n}, \frac{c}{n}; x\right)$$

$$ = 1 + \sum_{k=1}^{\infty} \frac{\{a+n(k-1)\}!_n\cdot\{b+n(k-1)\}!_n}{\{c+n(k-1)\}!_n}\frac{x^k}{(nk)!_n} \qquad (a,b,c>0) \tag{10}$$

$$ = 1 + a\sum_{k=1}^{\infty} \frac{\{a+n(k-1)\}!_n\cdot\{b+n(k-1)\}!_n}{\{c+n(k-1)\}!_n}\frac{x^k}{(nk)!_n} \qquad (\text{only } a<0) \tag{10'}$$

$$ = 1 + \frac{1}{c}\sum_{k=1}^{\infty} \frac{\{a+n(k-1)\}!_n\cdot\{b+n(k-1)\}!_n}{\{c+n(k-1)\}!_n}\frac{x^k}{(nk)!_n} \qquad (\text{only } c<0) \tag{10''}$$

Calculation

From (8),(8') and $n^k\cdot k! = (nk)!_n$

$$ {}_2F_1\left(\frac{a}{n}, b, c; x\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{a}{n}\right)_k (b)_k}{(c)_k}\frac{x^k}{k!}$$

$$ = 1 + \sum_{k=1}^{\infty} \frac{\{a+(k-1)n\}!_n(b)_k}{(c)_k}\frac{x^k}{(nk)!_n} \qquad (a>0) \tag{9}$$

$$ = 1 + a\sum_{k=1}^{\infty} \frac{\{a+(k-1)n\}!_n(b)_k}{(c)_k}\frac{x^k}{(nk)!_n} \qquad (a<0) \tag{9'}$$

$$ {}_2F_1\left(\frac{a}{n},\frac{b}{n},\frac{c}{n};x\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{a}{n}\right)_k \left(\frac{b}{n}\right)_k}{\left(\frac{c}{n}\right)_k} \frac{x^k}{k!} $$

$$ = 1 + \sum_{k=1}^{\infty} \frac{\frac{\{a+(k-1)n\}!_n}{n^k} \frac{\{b+(k-1)n\}!_n}{n^k}}{\frac{\{c+(k-1)n\}!_n}{n^k}} \frac{x^k}{k!} \qquad (a,b,c>0) $$

$$ = 1 + \sum_{k=1}^{\infty} \frac{\{a+n(k-1)\}!_n \cdot \{b+n(k-1)\}!_n}{\{c+n(k-1)\}!_n} \frac{x^k}{(nk)!_n} \qquad (a,b,c>0) \tag{10} $$

Next, because a symbol whose sign is negative among a,b,c serves as a coefficient of $\sum$, (10') and (10'') are obvious.

Example 1

$$ {}_2F_1\left(\frac{1}{3},c,c;x\right) $$

$$ = 1 + \sum_{k=1}^{\infty} \frac{\{1+3(k-1)\}!_3 \cdot (c)_k}{(c)_k} \frac{x^k}{(3k)!_3} = 1 + \sum_{k=1}^{\infty} \frac{(3k-2)!!!}{(3k)!!!} x^k $$

$$ = 1 + \frac{1!!!}{3!!!}x + \frac{4!!!}{6!!!}x^2 + \frac{7!!!}{9!!!}x^3 + \frac{10!!!}{12!!!}x^4 + \cdots = (1-x)^{-\frac{1}{3}} $$

Example 2

$$ {}_2F_1\left(\frac{-1}{3},c,c;x\right) $$

$$ = 1 + \sum_{k=1}^{\infty} \frac{\{-1+3(k-1)\}!_3 \cdot (c)_k}{(c)_k} \frac{x^k}{(3k)!_3} = 1 + (-1) \sum_{k=1}^{\infty} \frac{(3k-4)!!!}{(3k)!!!} x^k $$

$$ = 1 - \frac{(-1)!!!}{3!!!}x - \frac{2!!!}{6!!!}x^2 - \frac{5!!!}{9!!!}x^3 - \frac{8!!!}{12!!!}x^4 - \cdots = (1-x)^{\frac{1}{3}} $$

Example 3

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2\right) $$

$$ = 1 + \sum_{k=1}^{\infty} \frac{\{1+2(k-1)\}!_2 \cdot \{1+2(k-1)\}!_2}{\{3+2(k-1)\}!_2} \frac{x^{2k}}{(2k)!_2} $$

$$ = 1 + \sum_{k=1}^{\infty} \frac{(2k-1)!! \cdot (2k-1)!!}{(2k+1)!!} \frac{x^{2k}}{(2k)!!} $$

$$ = 1 + \frac{1!!}{2!!}\frac{x^2}{3} + \frac{3!!}{4!!}\frac{x^4}{5} + \frac{5!!}{6!!}\frac{x^6}{7} + \cdots = \frac{sin^{-1}x}{x} $$

Example 4

$$
{}_2F_1\left(\frac{1}{n},\frac{n-1}{n},\frac{n+1}{n};x^n\right)
$$

$$
=1+\sum_{k=1}^{\infty}\frac{\{1+n(k-1)\}!_n\cdot\{n-1+n(k-1)\}!_n}{\{n+1+n(k-1)\}!_n}\frac{x^{nk}}{(nk)!_n}
$$

$$
=1+\sum_{k=1}^{\infty}\frac{\{n(k-1)+1\}!_n\cdot(nk-1)!_n}{(nk+1)!_n}\frac{x^{nk}}{(nk)!_n}
$$

$$
=1+\frac{(n-1)!_n}{n!_n}\frac{x^n}{n+1}+\frac{(2n-1)!_n}{(2n)!_n}\frac{x^{2n}}{2n+1}+\frac{(3n-1)!_n}{(3n)!_n}\frac{x^{3n}}{3n+1}+\cdots
$$

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Alien's Mathematics