Summary of Super Calculus

01 Gamma Function & Digamma Function

Although the factorial $n!$ and the harmonic number H_n ($=1+1/2+\cdots+1/n$) are usuallydefined for a natural number, if a gamma function and a digamma function are used, these can be defined for the real number p. That is,

$$p! = \Gamma(1+p) \quad , \quad H_p = \psi(1+p)+\gamma \quad (\gamma=0.57721\cdots)$$

The former is convenient to express the coefficient of the higher order primitive or derivative of a power function, and the latter is indispensable to non-integer order calculus of the logarithmic function.

Although some formulas about these functions are described here, the following two formulas proved in Section 3 are especially important. That is, when $m, n=0,1,2,3,\cdots$,

$$\frac{\Gamma(-n)}{\Gamma(-m)} = (-1)^{m-n}\frac{m!}{n!} \qquad , \qquad \frac{\psi(-n)}{\Gamma(-n)} = (-1)^{n+1}n!$$

These show that the ratios of the singular points of $\Gamma(z)$ or $\psi(z)$ reduce to integers or its reciprocals. The former is necessary to express the higher order derivative of fractional functions, and the latter is required for the super calculus of a logarithmic function.

02 Multifactorial

The relational expression of multifactorial and the gamma function is shown here.
For instance, in case of double factorial,

$$(2n-1)!! = 2^n\Gamma\left(n+\frac{1}{2}\right) / \Gamma\left(\frac{1}{2}\right) = 2^n\Gamma\left(n+\frac{1}{2}\right) / \sqrt{\pi}$$

$$(2n)!! = 2^n\Gamma\left(n+\frac{2}{2}\right) / \Gamma\left(\frac{2}{2}\right) = 2^n\Gamma(n+1) = 2^n n!$$

These are used to express the half-integration of a integer-power function later.
Moreover, we obtain the following Maclaurin expansions as by-products.

$$f\left(\frac{x}{2}\right) = \frac{f(0)}{0!!}x^0+\frac{f'(0)}{2!!}x^1+\frac{f''(0)}{4!!}x^2+\frac{f^{(3)}(0)}{6!!}x^3+\cdots$$

$$f\left(\frac{x}{3}\right) = \frac{f(0)}{0!!!}x^0+\frac{f'(0)}{3!!!}x^1+\frac{f''(0)}{6!!!}x^2+\frac{f^{(3)}(0)}{9!!!}x^3+\cdots$$

03 Generalized Multinomial Theorem

First, the binomial theorem and a generalized binomial theorem are mentioned. The Leibniz rule and the Leibniz rule about super-differentiation are expressed just like these later.

Next, multinomial thorem and generalized multinomial thorem are shown as follows.

Theorem 3.3.1

For real numbers $x_1, x_2, \cdots, x_m$ and a natural number n, the following expressions hold.

$$\left(x_1+x_2+\cdots+x_m\right)^n = \sum_{r_1=0}^{n}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{m-1}=0}^{r_{m-2}}\binom{n}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{m-2}}{r_{m-1}}x_1^{n-r_1}x_2^{r_1-r_2}\cdots x_{m-1}^{r_{m-2}-r_{m-1}}x_m^{r_{m-1}} \tag{1.1}$$

$$= \sum_{r_1=0}^{n}\sum_{r_2=0}^{n}\cdots\sum_{r_{m-1}=0}^{n}\binom{n}{r_1+r_2+\cdots+r_{m-1}}\binom{r_1+r_2+\cdots+r_{m-1}}{r_2+\cdots+r_{m-1}}\cdots\binom{r_{m-2}+r_{m-1}}{r_{m-1}} \times x_1^{n-r_1-\cdots-r_{m-1}}x_2^{r_1}x_3^{r_2}\cdots x_m^{r_{m-1}}$$

Theorem 3.4.1

The following expressions hold for real numbers α and $x_1, x_2, \cdots, x_m$ s.t. $|x_1| \ge |x_2+x_3+\cdots+x_m|$.

$$(x_1+x_2+\cdots+x_m)^{\alpha} = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{m-1}=0}^{r_{m-2}}\binom{\alpha}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{m-2}}{r_{m-1}} x_1^{\alpha-r_1}x_2^{r_1-r_2}\cdots x_{m-1}^{r_{m-2}-r_{m-1}}x_m^{r_{m-1}} \tag{1.1}$$

$$= \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_{m-1}=0}^{\infty}\binom{\alpha}{r_1+r_2+\cdots+r_{m-1}}\binom{r_1+r_2+\cdots+r_{m-1}}{r_2+\cdots+r_{m-1}}\cdots\binom{r_{m-2}+r_{m-1}}{r_{m-1}} \times x_1^{\alpha-r_1-\cdots-r_{m-1}}x_2^{r_1}x_3^{r_2}\cdots x_m^{r_{m-1}} \tag{1.2}$$

Where, $|x_1| = |x_2+x_3+\cdots+x_m|$ is allowed at $\alpha>0$.

Higher order calculus of the product of many functions and super order calculus of the product of many functions are expressed just like these later.

What should be paid attention here are the following property of generalized binomial coefficients.

$$\sum_{r=0}^{n}{}_nC_r = \sum_{r=0}^{\infty}\binom{n}{r}$$

That is, once generalized binomial coefficients was used, the upper limit of Σ should be ∞. Therefore, when n is a natural number and p is a real number, the following holds in most cases.

$$n \to p \quad \Longrightarrow \quad \sum_{r=0}^{n}{}_nC_r\, f(n,x) \to \sum_{r=0}^{\infty}\binom{p}{r}f(p,x)$$

When the original coefficient is not binomial coefficient e.g. 1 ,

$$n \to p \quad \Longrightarrow \quad \sum_{r=0}^{n}{}_nC_r\, f(n,x) \to \sum_{r=0}^{\infty}\binom{p}{r}f(p,x)$$

Although $\sum_{r=0}^{p}1\cdot f(p,x)$ is difficult, $\sum_{r=0}^{\infty}\binom{p}{r}f(p,x)$ is satisfactory. What enables super calculus in this text is just this property of the generalized binomial coefficient. Newton is great !

04 Higher Integral

(1) Definitions and Notations

The 1st order primitive function of $f(x)$ is usually denoted $F(x)$. However, such a notation is unsuitable for the description of the 2nd or more order primitive functions. Then, $f^{\langle 1\rangle}(x), f^{\langle 2\rangle}(x), \cdots, f^{\langle n\rangle}(x)$ denote the each order primitive functions of $f(x)$ in this text. Here, for example, when $f(x)=\sin x$, $f^{\langle 1\rangle}(x)$ might mean $-\cos x$ or might mean $-\cos x+c$. Which it means follows the definition at that time.

Furthermore, each order integrals of $f(x)$ are denoted as follows.

$$\int_{a_1}^{x} f(x)dx^1 , \int_{a_2}^{x}\int_{a_1}^{x} f(x)dx^2 , \cdots , \int_{a_n}^{x}\cdots\int_{a_1}^{x} f(x)dx^n$$

And these are called ***higher integral with variable lower limits*** . On the other hand,

$$\int_{a}^{x} f(x)dx^1 , \int_{a}^{x}\int_{a}^{x} f(x)dx^2 , \cdots , \int_{a}^{x}\cdots\int_{a}^{x} f(x)dx^n$$

are called ***higher integral with a fixed lower limit*** .

(2) Fundamental Theorem of Higher Integral

There is Fundamental Theorem of Calculus for the 1st order integral. The same theorem holds for the higher order integral.

Theorem 4.1.3

Let $f^{<r>}\ r=0,1,\cdots,n$ be continuous functions defined on a closed interval I and $f^{<r+1>}$ be the arbitrary primitive function of $f^{<r>}$. Then the following expression holds for $a_r,\ x\in I$.

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x}f(x)dx^n = f^{<n>}(x) - \sum_{r=0}^{n-1}f^{<n-r>}(a_{n-r})\int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x}dx^r$$

Especially, when $a_r=a\ \ for\ \ r=1,2,\cdots,n$,

$$\int_{a}^{x}\cdots\int_{a}^{x}f(x)dx^n = f^{<n>}(x) - \sum_{r=0}^{n-1}f^{<n-r>}(a)\frac{(x-a)^r}{r!}$$

(3) Lineal and Collateral

We call ***Constant-of-integration Polynomial*** the 2nd term of the right sides of these. And when Constant-of-integration Polynomial is 0, we call the left side ***Lineal Higher Integral*** and we call $f^{<n>}(x)$ ***Lineal Higher Primitive Function.***

Oppositely, when Constant-of-integration Polynomial is not 0, we call the left side ***Collateral Higher Integral*** and we call the right side ***Collateral Higher Primitive Function***.

For example,

$$\int_{\frac{3\pi}{2}}^{x}\int_{\frac{2\pi}{2}}^{x}\int_{\frac{1\pi}{2}}^{x}\sin x\,dx^3=\cos x$$

Left: Lineal 3rd order integral
Right: Lineal 3rd order primitive

$$\int_0^x\int_0^x\int_0^x\sin x\,dx^3=\cos x+\frac{x^2}{2}-1$$

Left: Collateral 3rd order integral
Right: Collateral 3rd order primitive

Furthermore, from Theorem 4.1.3, we see that a_r must be all zeros of $f^{<r>}\ \ for\ r=1,2,\cdots,n$ in order for the higher integral of $f(x)$ to be lineal.

(4) Higher Integral and Reimann-Liouville Integral

The higher integral with a fixed lower limit reduce to the 1st order integral which called **Reimann-Liouville Integral.**

Theorem 4.2.1 (Cauchy formula for repeated integration)

When $f(x)$ is continuously integrable function and $\Gamma(z)$ denotes a gamma function ,

$$\int_{a}^{x}\cdots\int_{a}^{x}f(x)dx^n = \frac{1}{\Gamma(n)}\int_a^x(x-t)^{n-1}f(t)dt$$

Reimann-Liouville Integral of the right side is important. However, in the higher integral, since the left side itself has an operation functions, the right side is not indispensable.

By the way, replacing the left side in Theorem 4.1.3 for Reimann-Liouville Integral and shifting the index by $-n$, we obtain the following.

$$f^{<0>}(x)=\sum_{r=0}^{n-1}f^{(r)}(a)\frac{(x-a)^r}{r!} + \frac{1}{\Gamma(n)}\int_a^x(x-t)^{n-1}f^{(n)}(t)dt$$

This is the Taylor expansion of $f(x)$ around a. And Reimann-Liouville Integral of $f^{(n)}$ is the remainder term called *Bernoulli form*

(5) Higher Integrals of Elementary Functions.

When m is a natural number, the 2nd order integral of x^m becomes as follows.

$$\int_0^x\int_0^x x^m dx^2 = \frac{1}{(m+1)(m+2)}x^{m+2} = \frac{m!}{(m+2)!}x^{m+2}$$

Then, when α is a positive number, it is as follows.

$$\int_0^x\int_0^x x^\alpha dx^2 = \frac{\alpha!}{(\alpha+2)!}x^{\alpha+2}$$

Here, $\alpha!$ can be expressed by gamma function $\Gamma(1+\alpha)$. Thus

$$\int_0^x\int_0^x x^\alpha dx^2 = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+2)}x^{\alpha+2}$$

By such an easy calculation, we obtain the following expressions for elementary functions. Where, $\uparrow, \downarrow$ denote the ceiling function and the floor function respectively.

Higher Integrals of Power Function etc.

$$\int_0^x\cdots\int_0^x x^\alpha dx^n = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n)}x^{\alpha+n} \qquad (\alpha \geqq 0)$$

$$\int_\infty^x\cdots\int_\infty^x x^\alpha dx^n = (-1)^n\frac{\Gamma(-\alpha-n)}{\Gamma(-\alpha)}x^{\alpha+n} \qquad (\alpha < -n)$$

$$\int_{\mp\infty}^x\cdots\int_{\mp\infty}^x e^{\pm x} dx^n = (\pm 1)^n e^{\pm x}$$

$$\int_0^x\cdots\int_0^x \log x\, dx^n = \frac{x^n}{n!}\left(\log x - \sum_{k=1}^{n}\frac{1}{k}\right) \qquad x > 0$$

$$\int_{\frac{n\pi}{2}}^x\cdots\int_{\frac{2\pi}{2}}^x\int_{\frac{1\pi}{2}}^x \sin x\, dx^n = \sin\left(x - \frac{n\pi}{2}\right)$$

$$\int_{\frac{(n-1)\pi}{2}}^x\cdots\int_{\frac{1\pi}{2}}^x\int_{\frac{0\pi}{2}}^x \cos x\, dx^n = \cos\left(x - \frac{n\pi}{2}\right)$$

$$\int_{\frac{n\pi i}{2}}^x\cdots\int_{\frac{2\pi i}{2}}^x\int_{\frac{1\pi i}{2}}^x \sinh x\, dx^n = \frac{e^x - (-1)^n e^{-x}}{2}$$

$$\int_{\frac{(n-1)\pi i}{2}}^x\cdots\int_{\frac{1\pi i}{2}}^x\int_{\frac{0\pi i}{2}}^x \cosh x\, dx^n = \frac{e^x + (-1)^n e^{-x}}{2}$$

Higher Integrals of Inverse Trigonometric Functions

$$\begin{aligned}\int_0^x\cdots\int_0^x \tan^{-1}x\, dx^n = {} & \frac{\tan^{-1}x}{n!}\sum_{k=0}^{n/2\downarrow}(-1)^k {}_nC_{n-2k}\, x^{n-2k} \\ & + \frac{\log(1+x^2)}{2\cdot n!}\sum_{k=1}^{n/2\uparrow}(-1)^k {}_nC_{n+1-2k}\, x^{n+1-2k} \\ & - \frac{1}{n!}\sum_{r=1}^{n/2\downarrow}(-1)^r {}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}x^{n+1-2r}\end{aligned}$$

$$\begin{aligned}\int_0^x\cdots\int_0^x \cot^{-1}x\, dx^n = {} & \frac{x^n}{n!}\cot^{-1}x - \frac{\tan^{-1}x}{n!}\sum_{k=1}^{n/2\downarrow}(-1)^k {}_nC_{n-2k}\, x^{n-2k} \\ & - \frac{\log(x^2+1)}{2\cdot n!}\sum_{k=1}^{n/2\uparrow}(-1)^k {}_nC_{n+1-2k}\, x^{n+1-2k} \\ & + \frac{1}{n!}\sum_{r=1}^{n/2\downarrow}(-1)^r {}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}x^{n+1-2r}\end{aligned}$$

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x}\sin^{-1}x\,dx^n = \sum_{r=0}^{n/2\downarrow}\frac{x^{n-2r}}{(2r)!!^2(n-2r)!}\sin^{-1}x$$

$$+\sum_{r=1}^{n/2\uparrow}\sum_{s=0}^{n-2r+1}(-1)^s{}_{n-2r+1}C_s\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r+1}}{(n-2r+1)!}\sqrt{1-x^2}$$

Where, $a_1 = i\cdot 1.508879\cdots$, $a_2 = 0$, $a_3 = -i\cdot 0.475883\cdots$, $a_4 = 0$, $\cdots$

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x}\cos^{-1}x\,dx^n = \frac{x^n}{n!}\cos^{-1}x - \sum_{r=1}^{n/2\downarrow}\frac{x^{n-2r}}{(2r)!!^2(n-2r)!}\sin^{-1}x$$

$$-\sum_{r=1}^{n/2\uparrow}\sum_{s=0}^{n-2r+1}(-1)^s{}_{n-2r+1}C_s\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r+1}}{(n-2r+1)!}\sqrt{1-x^2}$$

Where, $a_1 = 1$, $a_2 = 0$, $a_3 = ?$, $a_4 = 0$, $\cdots$

$$\int_{1}^{x}\cdots\int_{1}^{x}\sec^{-1}x\,dx^n = \frac{x^n}{n!}\sec^{-1}x - \sum_{r=0}^{(n-1)/2\downarrow}\frac{(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{x^{n-2r-1}}{(n-2r-1)!}\log\left(x+\sqrt{x^2-1}\right)$$

$$+\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^s\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{x^2-1}$$

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x}\csc^{-1}x\,dx^n = \frac{x^n}{n!}\csc^{-1}x + \sum_{r=0}^{(n-1)/2\downarrow}\frac{(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{x^{n-2r-1}}{(n-2r-1)!}\log\left(x+\sqrt{x^2-1}\right)$$

$$-\sum_{r=1}^{n/2\downarrow}\sum_{s=0}^{n-2r}(-1)^s\frac{{}_{n-2r}C_s}{2r+s}\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r}}{(n-2r)!}\sqrt{x^2-1}$$

Where, $a_1 , a_2 , \cdots , a_n$ are all complex numbers.

Higher Integrals of Inverse Hyperbolic Functions

$$\int_{0}^{x}\cdots\int_{0}^{x}\tanh^{-1}x\,dx^n = \frac{\tanh^{-1}x}{n!}\sum_{k=0}^{n/2\downarrow}{}_nC_{n-2k}x^{n-2k} + \frac{\log\left(1-x^2\right)}{2\cdot n!}\sum_{k=1}^{n/2\uparrow}{}_nC_{n+1-2k}x^{n+1-2k}$$

$$-\frac{1}{n!}\sum_{r=1}^{n/2\downarrow}{}_nC_{n+1-2r}\left\{\psi(1+n)-\psi(2r)\right\}x^{n+1-2r}$$

$$\int_{0}^{x}\cdots\int_{0}^{x}\coth^{-1}x\,dx^n = \frac{x^n}{n!}\coth^{-1}x + \frac{\tanh^{-1}x}{n!}\sum_{k=1}^{n/2\downarrow}{}_nC_{n-2k}x^{n-2k}$$

$$+\frac{\log\left(1-x^2\right)}{2\cdot n!}\sum_{k=1}^{n/2\uparrow}{}_nC_{n+1-2k}x^{n+1-2k}$$

$$-\frac{1}{n!}\sum_{r=1}^{n/2\downarrow}{}_nC_{n+1-2r}\left\{\psi(1+n)-\psi(2r)\right\}x^{n+1-2r} \qquad |x|<1$$

$$\int_{a_n}^{x}\cdots\int_{a_1}^{x}\sinh^{-1}x\,dx^n = \sum_{r=0}^{n/2\downarrow}\frac{(-1)^r x^{n-2r}}{(2r)!!^2(n-2r)!}\sinh^{-1}x$$

$$+\sum_{r=1}^{n/2\uparrow}\sum_{s=0}^{n-2r+1}(-1)^{r+s}{}_{n-2r+1}C_s\frac{(s-1)!!^2}{(s+2r-1)!!^2}\frac{x^{n-2r+1}}{(n-2r+1)!}\sqrt{1+x^2}$$

Where, $a_1 = 1.508879\cdots$, $a_2 = 0$, $a_3 = -0.475883\cdots$, $a_4 = 0$, $\cdots$

$$\int_{1}^{x}\cdots\int_{1}^{x}\cosh^{-1}x\,dx^n = \sum_{r=0}^{n/2\downarrow}\frac{x^{n-2r}}{(2r)!!^2(n-2r)!}\cosh^{-1}x$$

$$- \sum_{r=1}^{n/2\uparrow} \sum_{s=0}^{n-2r+1} (-1)^s {}_{n-2r+1}C_s \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r+1}}{(n-2r+1)!} \sqrt{x^2-1}$$

$$\int_{a_n}^{x} \cdots \int_{a_1}^{x} sech^{-1} x \, dx^n = \frac{x^n}{n!} sech^{-1} x + \sum_{r=0}^{(n-1)/2\downarrow} \frac{(2r-1)!!}{(2r)!!(2r+1)!} \frac{x^{n-2r-1}}{(n-2r-1)!} sin^{-1} x$$

$$+ \sum_{r=1}^{n/2\downarrow} \sum_{s=0}^{n-2r} (-1)^s \frac{{}_{n-2r}C_s}{2r+s} \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r}}{(n-2r)!} \sqrt{1-x^2}$$

Where, $a_1 = a_3 = a_5 = \cdots = 0$, $a_2, a_4, a_6, \cdots$ are complex numbers.

$$\int_{a_n}^{x} \cdots \int_{a_1}^{x} csch^{-1} x \, dx^n = \frac{x^n}{n!} csch^{-1} x + \sum_{r=0}^{(n-1)/2\downarrow} \frac{(-1)^r (2r-1)!!}{(2r)!!(2r+1)!} \frac{x^{n-2r-1}}{(n-2r-1)!} sinh^{-1} x$$

$$+ \sum_{r=1}^{n/2\downarrow} \sum_{s=0}^{n-2r} (-1)^{r+s} \frac{{}_{n-2r}C_s}{2r+s} \frac{(s-1)!!^2}{(s+2r-1)!!^2} \frac{x^{n-2r}}{(n-2r)!} \sqrt{x^2+1}$$

Where, $a_1 = 0$, $a_2 = 0.6079\cdots$, $a_3 = 0$, $a_4 = 1.5539$, $\cdots$

(6) Termwise Higher Integral and Taylor series of higher primitive function

Theorem 4.6.1

Let $f^{<r>}$ $r=0,1,\cdots,n$ be continuous functions defined on $[a,b]$ and $f^{<r+1>}$ be the arbitrary primitive function of $f^{<r>}$. At this time, if $f(x)$ can be expanded to the Taylor series around a, the following expressions hold for $x \in [a,b]$.

$$f^{<n>}(x) = \sum_{r=0}^{n-1} f^{<n-r>}(a) \frac{(x-a)^r}{r!} + \sum_{r=0}^{\infty} f^{(r)}(a) \frac{(x-a)^{n+r}}{(n+r)!} = \sum_{r=0}^{\infty} f^{<n-r>}(a) \frac{(x-a)^r}{r!}$$

This expression shows that the Taylor series of $f^{<n>}$ consists of the constant-of-integration polynomial and the termwise higher integral of $f(x)$. The following can be said from this.

(1) A termwise higher integral with a fixed lower limit is collateral generally.
(2) It is the following case that a termwise higher integral with a fixed lower limit is lineal.

$$f^{<r>}(a) = 0 \quad for \ r=1,2,\cdots,n \quad \& \quad f^{(s)}(a) \neq 0, \pm\infty \quad for\ at\ least\ one \ s \geqq 0$$

For example,

$$\int_{a}^{x} \cdots \int_{a}^{x} e^x dx^n = \sum_{r=0}^{\infty} e^a \frac{(x-a)^{n+r}}{(n+r)!} \qquad a \neq -\infty$$ is collateral higher integral.

$$\int_{0}^{x} \cdots \int_{0}^{x} tan^{-1} x \, dx^n = \sum_{r=0}^{\infty} (-1)^r \frac{(2r)!}{(n+2r+1)!} x^{n+2r+1}$$ is lineal higher integral.

Next, the following is obtained from the Taylor series of the higher integral of $log\ x$.

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)\cdots(r+n)} = \frac{(-1)^{n-1}}{n!} \sum_{r=0}^{n-1} (-1)^r {}_nC_r H_{n-r} \qquad \left\{ H_n = \sum_{k=1}^{n} \frac{1}{k} \right\}$$

$$\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r(r+1)\cdots(r+n)} = \frac{2^n}{n!} (log 2 - H_n) + \frac{1}{n!} \sum_{r=0}^{n-1} {}_nC_r H_{n-r}$$

$$\sum_{r=0}^{n-1} (-1)^r {}_nC_r H_{n-r} = \frac{(-1)^{n-1}}{n}$$

$$\int_0^1 \frac{(1-x)^p}{1+x}dx = 2^p\{\log 2 - \psi(1+p) - \gamma\} + \sum_{r=1}^{\infty}\binom{p}{r}\{\psi(1+r)+\gamma\}$$

Example

$$\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\frac{1}{3\cdot 4\cdot 5\cdot 6}+\frac{1}{4\cdot 5\cdot 6\cdot 7}+\cdots = \frac{1}{18}$$

$$\frac{1}{1\cdot 2\cdot 3\cdot 4}-\frac{1}{2\cdot 3\cdot 4\cdot 5}+\frac{1}{3\cdot 4\cdot 5\cdot 6}-\frac{1}{4\cdot 5\cdot 6\cdot 7}+-\cdots = \frac{4}{3}\log 2 - \frac{8}{9}$$

05 Termwise Higher Integral (Trigonometric, Hyperbolic)

In this chapter, for the function which second or more order integral cannot be expressed with the elementary functions among trigonometric functions and hyperbolic functions, we integrate the series expansion of these function termwise and obtain the following expressions. Where, $\uparrow, \downarrow$ denote the ceiling function and the floor function respectively. And Bernoulli Numbers and Euler Numbers are as follows.

$$B_0=1,\ B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ B_6=\frac{1}{42},\ B_8=-\frac{1}{30},\ B_{10}=\frac{5}{66},\ \cdots$$

$$E_0=1,\ E_2=-1,\ E_4=5,\quad E_6=-61,\ E_8=1385,\ E_{10}=-50521,\ \cdots$$

(1) Taylor Series

$$\int_0^x\cdots\int_0^x \tan x\, dx^n = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,(2k+n-1)!}x^{2k+n-1} \qquad |x|<\frac{\pi}{2}$$

$$\int_0^x\cdots\int_0^x \tanh x\, dx^n = \sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,(2k+n-1)!}x^{2k+n-1} \qquad |x|<\frac{\pi}{2}$$

$$\int_0^x\cdots\int_0^x \sec x\, dx^n = \sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+n)!}x^{2k+n} \qquad |x|<\frac{\pi}{2}$$

(2) Fourier Series

When $\eta(x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^x}$, $\beta(n)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^n}$,

$$\int_0^x\cdots\int_0^x \tan x\, dx^n = \frac{1}{2^{n-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^n}\sin\left(2kx-\frac{n\pi}{2}\right) + \sum_{k=1}^{n/2\uparrow}(-1)^{k-1}\frac{\eta(2k-1)}{2^{2k-2}}\frac{x^{n+1-2k}}{(n+1-2k)!} \qquad |x|<\frac{\pi}{2}$$

$$\int_0^x\cdots\int_0^x \tanh x\, dx^n = \frac{(-1)^{n-1}}{2^{n-1}}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-2kx}}{k^n} + \frac{x^n}{n!} - \sum_{k=0}^{n-1}(-1)^k\frac{\eta(k+1)}{2^k}\frac{x^{n-1-k}}{(n-1-k)!} \qquad x>0$$

$$\int_0^x\cdots\int_0^x \sec x\, dx^n = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^n}\cos\left\{(2k+1)x-\frac{n\pi}{2}\right\} + \sum_{k=1}^{n/2\downarrow}(-1)^{k-1}\frac{2\beta(2k)}{(n-2k)!}x^{n-2k} \qquad |x|<\frac{\pi}{2}$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} \operatorname{sech} x\, dx^n = (-1)^n 2\sum_{k=0}^{\infty}(-1)^k \frac{e^{-(2k+1)x}}{(2k+1)^n} \qquad x>0$$

(3) Riemann Odd Zeta & Dirichlet Odd Eta

Comparing Taylor series and Fourier series, we obtain Riemann Odd Zeta and Dirichlet Odd Eta. For example,

$$\zeta(2n+1) = \frac{(-1)^n}{\pi}\frac{2^{2n}}{2^{2n}-1}\left\{\sum_{k=1}^{\infty}\frac{(2^{2k}-1)|B_{2k}|}{2k\,(2k+2n+1)\,!}\pi^{2k+2n+1} - \frac{\pi^{2n+1}}{(2n+1)\,!}\log 2\right\}$$

$$-\frac{(-1)^n}{\pi}\frac{2^{2n}}{2^{2n}-1}\sum_{k=1}^{n-1}(-1)^k\frac{\pi^{2n+1-2k}}{(2n+1-2k)\,!}\frac{2^{2k}-1}{2^{2k}}\zeta(2k+1)$$

$$\zeta(2n+1) = (-1)^n\frac{1}{\pi}\left\{\frac{\pi^{2n+1}}{(2n)\,!}\left(\log\pi - \sum_{k=1}^{2n+1}\frac{1}{k}\right) - \sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k\,(2k+2n+1)\,!}\pi^{2k+2n+1}\right\}$$

$$+(-1)^n\frac{1}{\pi}\sum_{k=1}^{n-1}(-1)^{k-1}\frac{\pi^{2n-2k+1}}{(2n-2k+1)\,!}\zeta(2k+1)$$

$$\zeta(n) = \frac{2^{n-1}}{2^{n-1}-1}\left\{\sum_{k=1}^{\infty}(-1)^{k-1}\frac{e^{-k}}{k^n} + \sum_{j=1}^{n-2}(-1)^{j-1}\frac{2^{n-1-j}-1}{2^{n-1-j}}\frac{1}{j\,!}\zeta(n-j)\right\}$$

$$-\frac{(-2)^{n-1}}{2^{n-1}-1}\left\{\sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{2k\,(2k+n-1)\,!} - \frac{1}{2}\frac{1}{n\,!} + \frac{\log 2}{(n-1)\,!}\right\}$$

$$\beta(2n) = \frac{(-1)^{n-1}}{2}\sum_{k=0}^{\infty}\frac{|E_{2k}|}{(2k+2n)\,!}\left(\frac{\pi}{2}\right)^{2k+2n} + \sum_{k=1}^{n}(-1)^{k-1}\frac{\beta(2n-2k)}{(2k)\,!}\left(\frac{\pi}{2}\right)^{2k}$$

06 Termwise Higher Integral (Inv-Trigonometric, Inv-Hyperbolic)

Here, we integrate the series of an inverse trigonometric function or an inverse hyperbolic function term by term. Then, we obtain formulas simpler than what were obtained in "04 HIgher Integral". Moreover, both are compared and we obtain various by-products.

(1) Taylor Series

$$\int_0^x\cdots\int_0^x \tan^{-1}x\, dx^n = \sum_{k=0}^{\infty}(-1)^k\frac{(2k)\,!}{(2k+n+1)!}x^{2k+n+1} \qquad |x|<1$$

$$\int_0^x\cdots\int_0^x \cot^{-1}x\, dx^n = \frac{\pi}{2}\frac{x^n}{n\,!} - \sum_{k=0}^{\infty}(-1)^k\frac{(2k)\,!}{(2k+n+1)!}x^{2k+n+1} \qquad 0<x\leqq 1$$

$$\int_0^x\cdots\int_0^x \sin^{-1}x\, dx^n = \sum_{k=0}^{\infty}\frac{\{(2k-1)\,!!\}^2}{(2k+n+1)!}x^{2k+n+1} \qquad |x|<1 \quad \text{collateral}$$

$$\int_0^x\cdots\int_0^x \cos^{-1}x\, dx^n = \frac{\pi}{2}\frac{x^n}{n\,!} - \sum_{k=0}^{\infty}\frac{\{(2k-1)\,!!\}^2}{(2k+n+1)!}x^{2k+n+1} \qquad |x|<1 \quad \text{collateral}$$

$$\int_0^x\cdots\int_0^x \tanh^{-1}x\, dx^n = \sum_{k=0}^{\infty}\frac{(2k)\,!}{(2k+n+1)!}x^{2k+n+1} \qquad |x|<1$$

$$\int_0^x\cdots\int_0^x \sinh^{-1}x\, dx^n = \sum_{k=0}^{\infty}(-1)^k\frac{\{(2k-1)\,!!\}^2}{(2k+n+1)!}x^{2k+n+1} \qquad |x|<1 \quad \text{collateral}$$

$$\int_0^x\cdots\int_0^x \operatorname{sech}^{-1}x\, dx^n = \frac{x^n}{n\,!}\left(\log\frac{2}{x} + \sum_{j=1}^{n}\frac{1}{j}\right) - \sum_{k=1}^{\infty}\frac{\{(2k-1)\,!!\}^2}{2k\,(2k+n)\,!}x^{2k+n}$$

$$0<x<1 \quad \text{collateral}$$

$$\int_0^x \cdots \int_0^x csch^{-1}x\, dx^n = \frac{x^n}{n!}\left(log\frac{2}{x} + \sum_{j=1}^{n}\frac{1}{j}\right) - \sum_{k=1}^{\infty}(-1)^k \frac{\{(2k-1)!!\}^2}{2k\,(2k+n)!}x^{2k+n}$$

$0< x<1$ **collateral**

(2) By-products

$$\int_0^1 \frac{(1-x)^n}{1-x^2}dx = 2^{n-1}log\,2 - \sum_{r=1}^{n/2\downarrow} {}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}$$

$$\int_0^1 \frac{(1-x)^n}{1+x^2}dx = \frac{\pi}{4}\sum_{k=0}^{n/2\downarrow}(-1)^k {}_nC_{n-2k} + \frac{log\,2}{2}\sum_{k=1}^{n/2\uparrow}(-1)^k {}_nC_{n+1-2k}$$

$$- \sum_{r=1}^{n/2\downarrow}(-1)^r {}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}$$

$$\sum_{k=0}^{\infty}\frac{(2k)!}{(2k+n+1)!} = \frac{2^{n-1}}{n!}log\,2 - \frac{1}{n!}\sum_{r=1}^{n/2\downarrow} {}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}$$

$$\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2k+n+1)!} = \frac{\pi}{4\,n!}\sum_{k=0}^{n/2\downarrow}(-1)^k {}_nC_{n-2k} + \frac{log\,2}{2\cdot n!}\sum_{k=1}^{n/2\uparrow}(-1)^k {}_nC_{n+1-2k}$$

$$- \frac{1}{n!}\sum_{r=1}^{n/2\downarrow}(-1)^r {}_nC_{n+1-2r}\{\psi(1+n)-\psi(2r)\}$$

$$\sum_{k=0}^{\infty}\frac{\{(2k-1)!!\}^2}{(2k+n+1)!} = \frac{{}_{2n-1}C_{n-1}}{(2n)!!}\pi - \sum_{k=1}^{n/2\uparrow}\frac{1}{(2k-1)!!^2(n-2k+1)!}$$

$$\sum_{k=1}^{\infty}\frac{\{(2k-1)!!\}^2}{2k\,(2k+n)!} = \frac{1}{n!}\left(log\,2 + \sum_{j=1}^{n}\frac{1}{j}\right) + \sum_{r=1}^{n/2\downarrow}\frac{1}{2r(2r-1)!!^2}\frac{1}{(n-2r)!}$$

$$- \frac{\pi}{2}\sum_{r=0}^{(n-1)/2\downarrow}\frac{(2r-1)!!}{(2r)!!\,(2r+1)!}\frac{1}{(n-2r-1)!}$$

Example

$$\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5\cdot6}+\frac{1}{5\cdot6\cdot7\cdot8}+\frac{1}{7\cdot8\cdot9\cdot10}+\cdots = \frac{2}{3}log\,2 - \frac{5}{12}$$

$$\frac{1}{1\cdot2\cdot3\cdot4}-\frac{1}{3\cdot4\cdot5\cdot6}+\frac{1}{5\cdot6\cdot7\cdot8}-\frac{1}{7\cdot8\cdot9\cdot10}+-\cdots = \frac{5}{12}-\frac{\pi}{12}-\frac{log\,2}{6}$$

$$\frac{(-1)!!^2}{2!}+\frac{1!!^2}{4!}+\frac{3!!^2}{6!}+\frac{5!!^2}{8!}+\cdots = \frac{\pi}{2}-1$$

$$\frac{1!!^2}{2\cdot3!}+\frac{3!!^2}{4\cdot5!}+\frac{5!!^2}{6\cdot7!}+\frac{7!!^2}{8\cdot9!}+\cdots = \frac{1}{1!}(log\,2+1)-\frac{\pi}{2}$$

07 Super Integral (Non-integer order Integral)

It is $\sum_{j=0}^{p} a = ap$ which extended the domain of index j of $\sum_{j=1}^{m} a$ to the real number interval $[0,p]$ from the natural number interval $[1,m]$. And it is $\prod_{k=0}^{q} b = b^q$ which extended the domain of index k of $\prod_{k=1}^{n} b$ to the real number interval $[0,q]$ from the natural number interval $[1,n]$. It is called *analytic continuation* to extend the domain generaly. Although usually analytic continuation is used for extending the domain of a function, it can be used also for extending the domain of the index of a operator. Here, extending the domain of the index of the integration operator, we obtain Super Integral (Non-integer order Integral).

(1) Definitions and Notations

$f^{<p>}(x)$ denotes the non-integer order primitive function of $f(x)$. And we call this ***Super Primitive Function of $f(x)$***. Since there is a super primitive function innumerably, which $f^{<p>}(x)$ means follows the definition at that time.

We call it ***Super Integral*** to integrate a function f with respect to an independent variable x from $a(0)$ to $a(p)$ continuously. And it is described as follows.

$$\int_{a(p)}^{x} \sim \int_{a(0)}^{x} f(x)\,dx^{p} \quad \left\{ = \int_{a(p)}^{x} \sim \int_{a(0)}^{x} f(x)\,dx \sim dx \right\}$$

And

when $a(k) = a \ \ for\ all \ \ k \in [0, p]$, we call it ***super integral with a fixed lower limit***,

when $a(k) \neq a \ \ for\ some\ k \in [0, p]$, we call it ***super integral with variable lower limits***.

(2) Fundamental Theorem of Super Integral

Let $f^{<r>}\ r \in [0, p]$ be an continuous function on the closed interval I and be arbitrary the r-th order primitive function of f. And let $a(r)$ be a continuous function on the closed interval $[0, p]$.

Then the following expression holds for $a(r),\ x \in I$.

$$\int_{a(p)}^{x} \sim \int_{a(0)}^{x} f(x)\,dx^{p} = f^{<p>}(x) - \sum_{r=0}^{p-1} f^{<p-r>}\{a(p-r)\} \int_{a(p)}^{x} \sim \int_{a(p-r+1)}^{x} dx^{r}$$

Especially, when $a(r) = a \ \ for\ all \ \ k \in [0, p]$,

$$\int_{a}^{x} \sim \int_{a}^{x} f(x)\,dx^{n} = f^{<p>}(x) - \sum_{r=0}^{p-1} f^{<p-r>}(a) \frac{(x-a)^{r}}{\Gamma(1+r)}$$

(3) Lineal and Collateral

We call ***Constant-of-integration Function*** the 2nd term of the right sides of these.

And when Constant-of-integration Function is 0, we call the left side ***Lineal Super Integral .*** and we call $f^{<p>}(x)$ ***Lineal Super Primitive Function.***

Oppositely, when Constant-of-integration Function is not 0, we call the left side ***Collateral Super Integral*** and we call the right side ***Collateral Super Primitive Function***.

For example,

$$\int_{\frac{p\pi}{2}}^{x} \sim \int_{\frac{0\pi}{2}}^{x} \sin x\,dx^{p} = \sin\left(x - \frac{p\pi}{2} \right)$$

Left: Lineal p th order integral

Right: Lineal p th order primitive

$$\int_{0}^{x} \sim \int_{0}^{x} \sin x\,dx^{p} = \sum_{r=0}^{\infty} \frac{(-1)^{r}}{\Gamma(2r+2+p)} x^{2r+1+p}$$

Left: Collateral p th order integral

Right: Collateral p th order primitive

(4) Super Integral and Reimann-Liouville Integral

The super integral with a fixed lower limit reduce to the 1st order integral which called **Reimann-Liouville Integral.**

$$\int_{a}^{x} \sim \int_{a}^{x} f(x)\,dx^{p} = \frac{1}{\Gamma(p)} \int_{a}^{x} (x-t)^{p-1} f(t)\,dt$$

This is what extended the parameter n to the real number in Cauchy formula for repeated integration. Since the left side has lost the operating function, Reimann-Liouville Integral of the right side is very important. All the super integral with a fixed lower limit can be verified numerically by this. On the other hand, since the super integral with variable lower limits cannot apply Reimann-Liouville Integral, the verification is vary difficult.

(5) Fractional Integral & Super Integral

In traditional *Fractional Integral*, the super primitive function is drawn from Riemann-Liouville Integral.

For example, in the case of $f(x) = x^{\alpha}$, it is as follws.

Let

$$f(t) = t^{\alpha} \ , \ g(x-t) = \frac{(x-t)^{p-1}}{\Gamma(p)}$$

Then

$$(x^{\alpha})^{<p>} = \frac{1}{\Gamma(p)} \int_0^x t^{\alpha}(x-t)^{p-1} dt = \int_0^x f(t) g(x-t) dt$$

We find out that this is a convolution $(f*g)(x)$. Then we take the Laplace transform of $(f*g)(x)$

$$(f*g)(x) \longrightarrow F(s) \cdot G(s)$$

$$f(x) = x^{\alpha} \longrightarrow \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} = F(s)$$

$$g(x) = \frac{x^{p-1}}{\Gamma(p)} \longrightarrow \frac{1}{\Gamma(p)} \frac{\Gamma(p)}{s^p} = \frac{1}{s^p} = G(s)$$

$$\therefore \ F(s) \cdot G(s) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \frac{1}{s^p} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+p+1)} \frac{\Gamma(\alpha+p+1)}{s^{\alpha+p+1}}$$

Finally, taking the inverse Laplace transform, we obtain $\dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+p+1)} x^{a+p}$.

Because the technique of Fractional Integral is difficult like this, it is more difficult to obtain the super primitive function of $log\ x$ by this technique.

Above all, the problem is that Fractional Integral cannot treat the lineal non-integer order integral such as sin x. Because Riemann-Liouville Integral cannot be applied to the integral with variable lower limets.

On the other hand, in Super Integral that I advocates, first of all, we obtain the following higher integral.

$$\int_0^x \cdots \int_0^x x^{\alpha} dx^n = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n)} x^{\alpha+n} \qquad (\alpha \geqq 0)$$

And replacing the index of the integration operator n with a real number p, we obtain the following very easily.

$$\int_0^x \sim \int_0^x x^{\alpha} dx^p = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p)} x^{\alpha+p} \qquad (\alpha \geqq 0)$$

Furthermore, performing the higher integral with variable lower limits and replacing the index of the integration operator with a real number, we can obtain the super integral such as sin x easily.

(6) Super Integrals of Elementary Functions.

In this way, the following super integrals obtined from " 04 Higher Integral " ,

$$\int_0^x \sim \int_0^x x^{\alpha} dx^p = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p)} x^{\alpha+p} \qquad (\alpha \geqq 0)$$

$$\int_{\infty}^x \sim \int_{\infty}^x x^{\alpha} dx^p = (-1)^p \frac{\Gamma(-\alpha-p)}{\Gamma(-\alpha)} x^{\alpha+p} \qquad (\alpha < -p)$$

$$\int_{\mp\infty}^x \sim \int_{\mp\infty}^x e^{\pm x} dx^p = (\pm 1)^p e^{\pm x}$$

$$\int_0^x \sim \int_0^x \log x\, dx^p = \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} x^p$$

$$\int_{\frac{p\pi}{2}}^x \sim \int_{\frac{0\pi}{2}}^x \sin x\, dx^p = \sin\left(x - \frac{p\pi}{2}\right)$$

$$\int_{\frac{(p-1)\pi}{2}}^x \sim \int_{\frac{-1\pi}{2}}^x \cos x\, dx^p = \cos\left(x - \frac{p\pi}{2}\right)$$

$$\int_{\frac{p\pi i}{2}}^{x} \sim \int_{\frac{0\pi i}{2}}^{x} \sinh x\, dx^p = \frac{e^x - (-1)^p e^{-x}}{2}$$

$$\int_{\frac{(p-1)\pi i}{2}}^{x} \sim \int_{\frac{-1\pi i}{2}}^{x} \cosh x\, dx^p = \frac{e^x + (-1)^p e^{-x}}{2}$$

$$\int_0^x \sim \int_0^x \tan^{-1} x\, dx^p = \frac{\tan^{-1} x}{\Gamma(1+p)} \sum_{k=0}^{\infty} (-1)^k \binom{p}{p-2k} x^{p-2k}$$
$$+ \frac{\log(1+x^2)}{2\Gamma(1+p)} \sum_{k=1}^{\infty} (-1)^k \binom{p}{p+1-2k} x^{p+1-2k}$$
$$- \frac{1}{\Gamma(1+p)} \sum_{r=1}^{\infty} (-1)^r \binom{p}{p+1-2r} \{\psi(1+p) - \psi(2r)\} x^{p+1-2r}$$

$$\int_0^x \sim \int_0^x \cot^{-1} x\, dx^p = \frac{x^p}{\Gamma(1+p)} \cot^{-1} x - \frac{\tan^{-1} x}{\Gamma(1+p)} \sum_{k=1}^{\infty} (-1)^k \binom{p}{p-2k} x^{p-2k}$$
$$- \frac{\log(1+x^2)}{2\Gamma(1+p)} \sum_{k=1}^{\infty} (-1)^k \binom{p}{p+1-2k} x^{p+1-2k}$$
$$+ \frac{1}{\Gamma(1+p)} \sum_{r=1}^{\infty} (-1)^r \binom{p}{p+1-2r} \{\psi(1+p) - \psi(2r)\} x^{p+1-2r}$$

$$\int_0^x \sim \int_0^x \tanh^{-1} x\, dx^p = \frac{\tanh^{-1} x}{\Gamma(1+p)} \sum_{k=0}^{\infty} \binom{p}{p-2k} x^{p-2k}$$
$$+ \frac{\log(1-x^2)}{2\Gamma(1+p)} \sum_{k=1}^{\infty} \binom{p}{p+1-2k} x^{p+1-2k}$$
$$- \frac{1}{\Gamma(1+p)} \sum_{r=1}^{\infty} \binom{p}{p+1-2r} \{\psi(1+p) - \psi(2r)\} x^{p+1-2r}$$

(7) By-products

$$\sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k} = \frac{(2n)!!}{(2n+1)!!} \quad , \quad \sum_{k=0}^{n} \frac{(-1)^k}{mk+1} \binom{n}{k} = \frac{B(1+n,\, 1/m)}{m}$$

$$(q,p) = \sum_{r=0}^{\infty} \frac{(-1)^r}{p+r} \binom{q-1}{r} \quad , \quad (x^q)^{\langle p \rangle} = \frac{x^{q+p}}{\Gamma(p)} \sum_{r=0}^{\infty} \frac{(-1)^r}{p+r} \binom{q}{r}$$

Where, $B(x,y)$ is the beta function.

08 Termwise Super Integral

The following termwise super integrals are obtained from " 05 Termwise Higher Integral (Trigonometric,Hyperbolic) " and " 06 Termwise Higher Integral (Inv-Trigonometric, Inv-Hyperbolic) ".

Where, ↑, ↓ denote the ceiling function and the floor function respectively. And Bernoulli Numbers and Euler Numbers are as follows.

$$B_0 = 1,\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ \cdots$$

$$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\quad E_6 = -61,\ E_8 = 1385,\ E_{10} = -50521,\ \cdots$$

$$\int_0^x \sim \int_0^x \tan x\,dx^p = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,\Gamma(2k+p)} x^{2k+p-1} \qquad 0<x<\frac{\pi}{2}$$

$$\int_0^x \sim \int_0^x \tanh x\,dx^p = \sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,\Gamma(2k+p)} x^{2k+p-1} \qquad 0<x<\frac{\pi}{2}$$

$$\int_0^x \sim \int_0^x \sec x\,dx^p = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{\Gamma(2k+p+1)} x^{2k+p} \qquad 0<x<\frac{\pi}{2} \quad \text{collateral}$$

$$\int_0^x \sim \int_0^x \operatorname{sech} x\,dx^p = \sum_{k=0}^{\infty} \frac{E_{2k}}{\Gamma(2k+p+1)} x^{2k+p} \qquad 0<x<\frac{\pi}{2} \quad \text{collateral}$$

$$\int_{\frac{\pi}{2}}^x \sim \int_{\frac{\pi}{2}}^x \cot x\,dx^p = -\sum_{k=1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,\Gamma(2k+p)} \left(x-\frac{\pi}{2}\right)^{2k+p-1} \qquad \frac{\pi}{2}<x<\pi$$

$$\int_{\frac{\pi}{2}}^x \sim \int_{\frac{\pi}{2}}^x \csc x\,dx^p = \sum_{k=0}^{\infty} \frac{|E_{2k}|}{\Gamma(2k+p+1)} \left(x-\frac{\pi}{2}\right)^{2k+p} \qquad \frac{\pi}{2}<x<\pi \quad \text{collateral}$$

$$\int_{\infty}^x \sim \int_{\infty}^x \operatorname{csch} x\,dx^p = (-1)^p 2 \sum_{k=0}^{\infty} \frac{e^{-(2k+1)x}}{(2k+1)^p} \qquad x>0$$

$$\int_{\infty}^x \sim \int_{\infty}^x \operatorname{sech} x\,dx^p = (-1)^p 2 \sum_{k=0}^{\infty} (-1)^k \frac{e^{-(2k+1)x}}{(2k+1)^p} \qquad x>0$$

$$\int_0^x \sim \int_0^x \tan^{-1} x\,dx^p = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{\Gamma(2k+p+2)} x^{2k+p+1} \qquad 0<x<1$$

$$\int_0^x \sim \int_0^x \cot^{-1} x\,dx^p = \frac{\pi}{2}\frac{x^p}{\Gamma(1+p)} - \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{\Gamma(2k+p+2)} x^{2k+p+1} \qquad 0<x<1$$

$$\int_0^x \sim \int_0^x \sin^{-1} x\,dx^p = \sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{\Gamma(2k+p+2)} x^{2k+p+1} \qquad 0<x<1 \quad \text{collateral}$$

$$\int_0^x \sim \int_0^x \cos^{-1} x\,dx^p = \frac{\pi}{2}\frac{x^p}{\Gamma(1+p)} - \sum_{k=0}^{\infty} \frac{\{(2k-1)!!\}^2}{\Gamma(2k+p+2)} x^{2k+p+1} \qquad 0<x<1 \quad \text{collateral}$$

$$\int_0^x \sim \int_0^x \tanh^{-1} x\,dx^p = \sum_{k=0}^{\infty} \frac{(2k)!}{\Gamma(2k+p+2)} x^{2k+p+1} \qquad 0<x<1$$

$$\int_0^x \sim \int_0^x \sinh^{-1} x\,dx^p = \sum_{k=0}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{\Gamma(2k+p+2)} x^{2k+p+1} \qquad 0<x<1 \quad \text{collateral}$$

$$\int_0^x \sim \int_0^x \operatorname{sech}^{-1} x\,dx^p = \frac{x^p}{\Gamma(1+p)}\left\{\log\frac{2}{x}+\psi(1+p)+\gamma\right\} - \sum_{k=1}^{\infty} \frac{\{(2k-1)!!\}^2}{2k\,\Gamma(2k+p+1)} x^{2k+p}$$
$$0<x<1 \quad \text{collateral}$$

$$\int_0^x \sim \int_0^x \operatorname{csch}^{-1} x\,dx^p = \frac{x^p}{\Gamma(1+p)}\left\{\log\frac{2}{x}+\psi(1+p)+\gamma\right\} - \sum_{k=1}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{2k\,\Gamma(2k+p+1)} x^{2k+p}$$
$$0<x<1 \quad \text{collateral}$$

09 Higher Derivative

(1) Definitions and Notations

When $f^{(n)}(x)$ denotes the derivative function of $f^{(n-1)}(x)$ for $n=1,2,3,\cdots$, we call $f^{(n)}(x)$ ***Higher Derivative***

of $\boldsymbol{f(x)}$..

Moreover, we call it ***Higher Differentiation*** to differentiate a function f with respect to an independent variable x repeatedly. And it is described as follows.

$$\frac{d^n}{dx^n}f(x) \qquad \left\{ = \frac{d}{dx}\left(\cdots \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}f(x) \right) \right) \cdots \right) \frac{d}{dx} : n \text{ pieces} \right\}$$

(2) Fundamental Theorem of Higher Differentiation

The following theorem holds from Theorem 4.1.3 .

When $f^{(r)}$ $r=0,1,\cdots,n$ are continuous functions on a closed interval I and are the rth derivative functions of f, the following expression holds for $x \in I$.

$$\frac{d^n}{dx^n}f(x) = f^{(n)}(x)$$

This theorem guarantees that **only the lineal exists in the higher differentiation**.

(3) Higher Derivative of Elementary Functions

When m is a natural number, the 2nd order derivative of x^m becomes as follows.

$$\left(x^m\right)^{(2)} = m(m-1)x^{m-2} = \frac{m!}{(m-2)!}x^{m-2}$$

Then, when α is a positive number, it is as follows.

$$\left(x^\alpha\right)^{(2)} = \frac{\alpha!}{(\alpha-2)!}x^{\alpha-2}$$

Here, $\alpha!$ can be expressed by gamma function $\Gamma(1+\alpha)$. Thus

$$\left(x^\alpha\right)^{(2)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-2)}x^{\alpha+2}$$

By such an easy calculation, we obtain the following expressions for elementary functions.

$$\left(x^\alpha\right)^{(n)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)}x^{\alpha-n} \qquad (\alpha \geqq 0)$$

$$= (-1)^{-n}\frac{\Gamma(-\alpha+n)}{\Gamma(-\alpha)}x^{\alpha-n} \qquad (\alpha < 0)$$

$$\left(e^{\pm x}\right)^{(n)} = (\pm 1)^{-n}e^{\pm x}$$

$$(\log x)^{(n)} = (-1)^{n-1}(n-1)!\ x^{-n}$$

$$(\sin x)^{(n)} = \sin\left(x+\frac{n\pi}{2}\right)$$

$$(\cos x)^{(n)} = \cos\left(x+\frac{n\pi}{2}\right)$$

$$(\sinh x)^{(n)} = \frac{e^x-(-1)^{-n}e^{-x}}{2}$$

$$(\cosh x)^{(n)} = \frac{e^x+(-1)^{-n}e^{-x}}{2}$$

etc.

(4) Higher Derivative of Inverse Trigonometric Functions

When $\uparrow, \downarrow$ denote the ceiling function and the floor function and n is a natural number,

$$\left(\tan^{-1}x\right)^{(n)} = (-1)^n \frac{(n-1)!}{\left(x^2+1\right)^n} \sum_{r=1}^{n/2\uparrow} (-1)^r {}_nC_{n+1-2r}\, x^{n+1-2r}$$

$$\left(\cot^{-1}x\right)^{(n)} = (-1)^{n-1} \frac{(n-1)!}{\left(x^2+1\right)^n} \sum_{r=1}^{n/2\uparrow} (-1)^r {}_nC_{n+1-2r}\, x^{n+1-2r}$$

$$\left(\sin^{-1}x\right)^{(n)} = \sum_{r=0}^{n/2\downarrow} \binom{n-1}{n-1-2r} \frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(1-x^2\right)^{n-r-\frac{1}{2}}}$$

$$\left(\cos^{-1}x\right)^{(n)} = -\sum_{r=0}^{n/2\downarrow} \binom{n-1}{n-1-2r} \frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(1-x^2\right)^{n-r-\frac{1}{2}}}$$

etc.

(5) Higher Derivative of Inverse Hyperbolic Functions

When $\uparrow, \downarrow$ denote the ceiling function and the floor function and n is a natural number,

$$\left(\tanh^{-1}x\right)^{(n)} = \left(\coth^{-1}x\right)^{(n)} = (-1)^n \frac{(n-1)!}{\left(x^2-1\right)^n} \sum_{r=1}^{n/2\uparrow} {}_nC_{n+1-2r}\, x^{n+1-2r}$$

$$\left(\sinh^{-1}x\right)^{(n)} = (-1)^{n-1} \sum_{r=0}^{n/2\downarrow} (-1)^r \binom{n-1}{n-1-2r} \frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(x^2+1\right)^{n-r-\frac{1}{2}}}$$

$$\left(\cosh^{-1}x\right)^{(n)} = (-1)^{n-1} \sum_{r=0}^{n/2\downarrow} (-1)^r \binom{n-1}{n-1-2r} \frac{(2r-1)!!\,(2n-3-2r)!!\,x^{n-1-2r}}{\left(x^2-1\right)^{n-r-\frac{1}{2}}}$$

etc.

(6) By-Products

$$\sum_{k=1}^{n/2\uparrow} (-1)^k {}_nC_{n+1-2k} = -2^{n/2} \sin\frac{n\pi}{4}$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{\{k+(n-1)\}!}{k!} = \frac{(n-1)!}{2^n}$$

Example

$$-{}_5C_4 + {}_5C_2 - {}_5C_0 = 4$$

$$1\cdot 2 - 2\cdot 3 + 3\cdot 4 - 4\cdot 5 +- \cdots = \frac{1}{4}$$

10 Termwise Higher Derivative (Trigonometric, Hyperbolic)

In this chapter, for the function which second or more order integral cannot be expressed with the elementary functions among trigonometric functions and hyperbolic functions, we differentiate the series expansion of these function termwise and obtain the following expressions. Where, $\uparrow, \downarrow$ denote the ceiling function and the floor function respectively. And Bernoulli Numbers and Euler Numbers are as follows.

$$B_0=1,\ B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ B_6=\frac{1}{42},\ B_8=-\frac{1}{30},\ B_{10}=\frac{5}{66},\ \cdots$$

$$E_0=1,\ E_2=-1,\ E_4=5,\quad E_6=-61,\ E_8=1385,\ E_{10}=-50521,\ \cdots$$

(1) Taylor Series

$$(\tan x)^{(n)} = \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-n-1)!}x^{2k-n-1} \qquad |x|<\frac{\pi}{2}$$

$$(\tanh x)^{(n)} = \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k(2k-n-1)!}x^{2k-n-1} \qquad |x|<\frac{\pi}{2}$$

$$(\cot x)^{(n)} = (-1)^n\frac{n!}{x^{n+1}} - \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k-n-1)!}x^{2k-n-1} \qquad 0<x<\pi$$

$$(\coth x)^{(n)} = (-1)^n\frac{n!}{x^{n+1}} - \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{2^{2k}B_{2k}}{2k(2k-n-1)!}x^{2k-n-1} \qquad 0<x<\pi$$

$$(\csc x)^{(n)} = (-1)^n\frac{n!}{x^{n+1}} + \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k\,(2k-n-1)!}\,x^{2k-n-1} \qquad 0<x<\pi$$

$$(\mathrm{csch}\, x)^{(n)} = (-1)^n\frac{n!}{x^{n+1}} + \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{\left(2^{2k}-2\right)B_{2k}}{2k(2k-n-1)!}\,x^{2k-n-1} \qquad 0<x<\pi$$

$$(\sec x)^{(n)} = \sum_{k=\frac{n+1}{2}\downarrow}^{\infty} \frac{|E_{2k}|}{(2k-n)!}x^{2k-n} \qquad |x|<\frac{\pi}{2}$$

$$(\mathrm{sech}\, x)^{(n)} = \sum_{k=\frac{n+1}{2}\downarrow}^{\infty} \frac{E_{2k}}{(2k-n)!}x^{2k-n} \qquad |x|<\frac{\pi}{2}$$

(2) Fourier Series

$$(\tanh x)^{(n)} = (-1)^{n-1}2^{n+1}\sum_{k=1}^{\infty}(-1)^{k-1}k^n e^{-2kx} \qquad x>0$$

$$(\coth x)^{(n)} = (-1)^n 2^{n+1}\sum_{k=1}^{\infty}k^n e^{-2kx} \qquad x>0$$

$$(\mathrm{csch}\, x)^{(n)} = (-1)^n 2\sum_{k=0}^{\infty}(2k+1)^n e^{-(2k+1)x} \qquad x>0$$

$$(\mathrm{sech}\, x)^{(n)} = (-1)^n 2\sum_{k=0}^{\infty}(-1)^k(2k+1)^n e^{-(2k+1)x} \qquad x>0$$

(3) Dirichlet Odd Eta (minus) & Even Beta (minus)

Comparing Taylor series and Fourier series, we obtain Dirichlet Odd Eta (minus) and Even Beta (minus) .

$$\eta(1-2n) = \frac{(-1)^{n-1}}{2^{4n-1}}\sum_{k=n}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-2n)!}\left(\frac{\pi}{4}\right)^{2k-2n}$$

$$\eta(1-2n) = \frac{(-1)^{n-1}}{2^{4n-1}}\left(\frac{4}{\pi}\right)^{2n}\left\{(2n-1)! + \sum_{k=n}^{\infty}\frac{2^{2k}|B_{2k}|}{2k(2k-2n)!}\left(\frac{\pi}{4}\right)^{2k}\right\}$$

$$\eta(1-2n) = \frac{\left(2^{2n}-1\right)B_{2n}}{2n}$$

$$\beta(-2n) = \frac{(-1)^n}{2^{2n+1}} \sum_{k=n+1}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k(2k-2n-1)!}\left(\frac{\pi}{4}\right)^{2k-2n-1}$$

$$\beta(-2n) = \frac{(-1)^n}{2^{2n+1}}\left(\frac{4}{\pi}\right)^{2n}\left\{(2n)! - \sum_{k=n+1}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k-2n-1)!}\left(\frac{\pi}{4}\right)^{2k}\right\}$$

$$\beta(-2n) = \frac{E_{2n}}{2}$$

(4) Other by-products

$$\frac{1^n}{e^1} + \frac{2^n}{e^2} + \frac{3^n}{e^3} + \frac{4^n}{e^4} + \cdots = n! + (-1)^n \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{B_{2k}}{2k(2k-n-1)!}$$

$$\frac{1^n}{e^1} - \frac{2^n}{e^2} + \frac{3^n}{e^3} - \frac{4^n}{e^4} + - \cdots = (-1)^{n-1} \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{\left(2^{2k}-1\right)B_{2k}}{2k\,(2k-n-1)!}$$

$$\frac{1^n}{e^1} + \frac{3^n}{e^3} + \frac{5^n}{e^5} + \frac{7^n}{e^7} + \cdots = \frac{n!}{2} - \frac{(-1)^n}{2} \sum_{k=\frac{n+1}{2}\uparrow}^{\infty} \frac{\left(2^{2k}-2\right)B_{2k}}{2k(2k-n-1)!}$$

$$\frac{1^n}{e^1} - \frac{3^n}{e^3} + \frac{5^n}{e^5} - \frac{7^n}{e^7} + - \cdots = \frac{(-1)^n}{2} \sum_{k=\frac{n+1}{2}\downarrow}^{\infty} \frac{E_{2k}}{(2k-n)!}$$

$$\frac{1^p}{e^1} + \frac{2^p}{e^2} + \frac{3^p}{e^3} + \frac{4^p}{e^4} + \cdots \fallingdotseq \Gamma(1+p) \qquad p>0$$

$$\frac{1^p}{e^1} + \frac{3^p}{e^3} + \frac{5^p}{e^5} + \frac{7^p}{e^7} + \cdots \fallingdotseq \frac{\Gamma(1+p)}{2} \qquad p>0$$

$$(2n-1)! = \sum_{k=n}^{\infty} \frac{\left(2^{2k}-2\right)|B_{2k}|}{2k(2k-2n)!}\left(\frac{\pi}{2}\right)^{2k}$$

$$(2n)! = \sum_{k=n+1}^{\infty} \frac{2^{2k}|B_{2k}|}{2k(2k-2n-1)!}\left(\frac{\pi}{2}\right)^{2k}$$

11 Termwise Higher Derivative (Inv-Trigonometric, Inv-Hyperbolic)

Among trigonometric functions and hyperbolic functions, there are functions that it is difficult to obtain the general form of the second or more order derivative. In this chapter, we differentiate the series expansion of these functions termwise and obtain the following expressions. Where, ↑, ↓ denote the ceiling function and the floor function respectively.

(1) Taylor Series

$$\left(\tan^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{(2k+1-n)!} x^{2k+1-n} \qquad |x|<1$$

$$\left(\cot^{-1}x\right)^{(n)} = -\sum_{k=\frac{n-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{(2k+1-n)!} x^{2k+1-n} \qquad |x|<1$$

$$\left(\sin^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+1-n)!} x^{2k+1-n} \qquad |x|<1$$

$$\left(\cos^{-1}x\right)^{(n)} = -\sum_{k=\frac{n-1}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{(2k+1-n)!} x^{2k+1-n} \qquad |x|<1$$

$$\left(\csc^{-1}x\right)^{(n)} = (-1)^n \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{(2k+n)!}{(2k+1)!} x^{-2k-n-1} \qquad |x|>1$$

$$\left(\sec^{-1}x\right)^{(n)} = (-1)^{n-1} \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!} \frac{(2k+n)!}{(2k+1)!} x^{-2k-n-1} \qquad |x|>1$$

$$\left(\tanh^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty} \frac{(2k)!}{(2k+1-n)!} x^{2k+1-n} \qquad |x|<1$$

$$\left(\coth^{-1}x\right)^{(n)} = (-1)^{(n)} \sum_{k=0}^{\infty} \frac{(2k+n)!}{(2k+1)!} x^{-2k-1-n} \qquad |x|>1$$

$$\left(\sinh^{-1}x\right)^{(n)} = \sum_{k=\frac{n-1}{2}\uparrow}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{(2k+1-n)!} x^{2k+1-n} \qquad |x|<1$$

$$\left(\cosh^{-1}x\right)^{(n)} = (-1)^{n-1} \frac{(n-1)!}{x^n} + (-1)^{n-1} \sum_{k=1}^{\infty} \frac{(2k+n-1)!}{\{(2k)!!\}^2} x^{-2k-n} \qquad x>1$$

$$\left(\operatorname{csch}^{-1}x\right)^{(n)} = (-1)^n \frac{(n-1)!}{x^n} - \sum_{k=\frac{n}{2}\uparrow}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{2k\,(2k-n)!} x^{2k-n} \qquad 0<x<1$$

$$\left(\operatorname{sech}^{-1}x\right)^{(n)} = (-1)^n \frac{(n-1)!}{x^n} - \sum_{k=\frac{n}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{2k\,(2k-n)!} x^{2k-n} \qquad 0<x<1$$

(4) By-Products

$$\sum_{k=\frac{n-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{(2k+1-n)!} = (-1)^n \frac{(n-1)!}{2^n} \sum_{k=1}^{n/2\uparrow} (-1)^k {}_nC_{n+1-2k}$$

Example

$$1\cdot 2 - 3\cdot 4 + 5\cdot 6 - 7\cdot 8 +- \cdots \quad = -\frac{1}{2}$$

12 Super Derivative (Non-integer times Derivative)

Here, extending the domain of the index function of a differenciation oprator, we obtain Super Derivative (non-integer order derivative).

(1) Definitions and Notations

$f^{<p>}(x)$ denotes the non-integer order derivative function of $f(x)$. And we call this ***Super Derivaive of*** $\boldsymbol{f(x)}$.

Since there is a super derivative function innumerably, which $f^{<p>}(x)$ means follows the definition at that time.

We call it ***Super Differentiation*** to differentiate a function f with respect to an independent variable continuously. And it is described as follows.

$$\frac{d^p}{dx^p} f(x) \qquad \left\{ = \frac{d}{dx} \sim \frac{d}{dx} f(x) \qquad \frac{d}{dx} : \ p \text{ pieces} \right\}$$

(2) Fundamental Theorem of Super Differentiation

Let $f^{(r)}$ $r \in [0, p]$ be an continuous function on the closed interval I and be arbitrary the r-th order derivative function of f. And let $a(r)$ be a continuous function on the closed interval $[0, p]$.

Then the following expression holds for $a(r), x \in I$.

$$\frac{d^p}{dx^p} f(x) = f^{(p)}(x) + \frac{d^p}{dx^p} \sum_{r=0}^{p-1} f^{(r)}\{a(p-r)\} \int_{a(p)}^{x} \sim \int_{a(p-r)}^{x} dx^r$$

Especially, when $a(r) = a \ \ for\ all\ \ k \in [0, p]$,

$$\frac{d^p}{dx^p} f(x) = f^{(p)}(x) + \frac{d^p}{dx^p} \sum_{r=0}^{p-1} f^{(r)}(a) \frac{(x-a)^r}{\Gamma(1+r)}$$

(3) Lineal and Collateral

We call the 2nd term of the right sides of these ***Constant-of-differentiation Function***.

And when Constant-of-differentiation Function is 0, we call the left side ***Lineal Super Differentiation*** and we call $f^{(p)}(x)$ **Lineal Super Derivaive Function** .

Oppositely, when Constant-of-differentiation Function is not 0, we call the left side ***Collateral Super Differentiation*** and we call the right side ***Collateral Super Derivaive Function.***

For example,

when $a \neq -\infty$,

$$\frac{d^p}{dx^p} e^x = e^x + \frac{d^p}{dx^p} \sum_{r=0}^{p-1} e^a \frac{(x-a)^r}{\Gamma(1+r)}$$

Left: Collateral p-th order differenciation

Right: Collateral p-th order derivative

when $a = -\infty$

$$\frac{d^p}{dx^p} e^x = e^x$$

Left: Lineal p-th order differenciation

Right: Lineal p-th order derivative

As seen from this example, the lineal and the collateral exist in the super differentiation unlike the higher differenciation

(4) Riemann-Liouville Differintegral

When the super integral of $f(x)$ is the one with a fixed lower limit, the super derivative of $f(x)$ is obtained by the following formula. In this formula, the n th order differenciation is subtracted from the $n-p$ th order integration and the p th order derivative is obtained. Then, this formula is called ***Riemann-Liouville Differintegral.***

$$\frac{d^p}{dx^p} f(x) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-p-1} f(t)\,dt \qquad n = ceil(p)$$

Since the left side has lost the operating function, Reimann-Liouville Differintegral of the right side is very important.

(5) Fractional Derivative & Super Derivative

In traditional *Fractional Derivative*, the super derivative is drawn from Riemann-Liouville Differintegral.

For example, in the case of $f(x) = log\, x$, it is as follws.

$$(log\, x)^{\left(\frac{1}{2}\right)} = \frac{1}{\Gamma(1-1/2)} \frac{d^1}{dx^1} \int_0^x (x-t)^{1-\frac{1}{2}-1} log\, t\, dt$$

$$= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_0^x (x-t)^{-\frac{1}{2}} log\, t\, dt$$

$$= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \left[-4\sqrt{x}\, tanh^{-1}\left(\frac{\sqrt{x-t}}{\sqrt{x}} \right) - 2\sqrt{x-t}\, (log\, t - 2) \right]_0^x$$

Long calculation continues.

⋮

$$= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \left\{ 4\sqrt{x}\,(\log 2-1) + 2\sqrt{x}\log x \right\} = \frac{\log x + 2\log 2}{\sqrt{\pi}} x^{-\frac{1}{2}}$$

Because the technique of Fractional Derivative is difficult like this, it is unknown in whether the case of $p = 1/3$ is calculable in this way.

Also, it is the same as the case of super integral that Riemann-Liouville Differintegral cannot be applied to the non-integer order derivative such as $\sin x$.

On the other hand, in Super Derivative that I advocates, first of all, we obtain the following higher derivative.

$$\int_0^x \sim \int_0^x \log x \, dx^p = \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} x^p$$

Since differentiation is the reverse operation of integration, reversing the sign of the indexof the integration operator, we obtain the following immediately.

$$(\log x)^{(p)} = \frac{\log x - \psi(1-p) - \gamma}{\Gamma(1-p)} x^{-p}$$

Substituting $p = 1/2$ for this,

$$(\log x)^{\left(\frac{1}{2}\right)} = \frac{\log x - \psi(1/2) - \gamma}{\Gamma(1/2)} x^{-\frac{1}{2}}$$

$$= \frac{\log x - (-\gamma - 2\log 2) - \gamma}{\sqrt{\pi}} x^{-\frac{1}{2}} = \frac{\log x + 2\log 2}{\sqrt{\pi}} x^{-\frac{1}{2}}$$

Thus, we obtain the desired super derivative very easily.

By the way, when $p = 2$, according to the formulas

$$\frac{\psi(-n)}{\Gamma(-n)} = (-1)^{n+1} n! \quad , \quad \Gamma(-n) = \pm\infty$$

it is as follows.

$$(\log x)^{(2)} = \frac{\log x - \psi(-1) - \gamma}{\Gamma(-1)} x^{-2} = -(-1)^{1+1} 1!\, x^{-2} = -\frac{1}{x^2}$$

Furthermore, in a similar way, we can obtain the super derivative such as sin x easily.

(6) Super Derivatives of Elementary Functions

In this way, the following super derivatives obtined from " 09 Higher Derivative " .

$$\left(x^{\alpha}\right)^{(p)} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-p)} x^{\alpha-p} \qquad (\alpha \geqq 0)$$

$$= (-1)^{-p} \frac{\Gamma(-\alpha+p)}{\Gamma(-\alpha)} x^{\alpha-p} \qquad (\alpha < 0)$$

$$\left(e^{\pm x}\right)^{(p)} = (\pm 1)^{-p} e^{\pm x}$$

$$(\log x)^{(p)} = \frac{\log x - \psi(1-p) - \gamma}{\Gamma(1-p)} x^{-p}$$

$$(\sin x)^{(p)} = \sin\left(x + \frac{p\pi}{2}\right)$$

$$(\cos x)^{(p)} = \cos\left(x + \frac{p\pi}{2}\right)$$

$$(\sinh x)^{(p)} = \frac{e^x - (-1)^{-p} e^{-x}}{2}$$

$$(\cosh x)^{(p)} = \frac{e^x + (-1)^{-p} e^{-x}}{2}$$

$$\left(\tan^{-1}x\right)^{(p)} = \frac{\tan^{-1}x}{\Gamma(1-p)} \sum_{k=0}^{\infty} (-1)^k \binom{-p}{-p-2k} x^{-p-2k}$$
$$+ \frac{\log\left(1+x^2\right)}{2\Gamma(1-p)} \sum_{k=1}^{\infty} (-1)^k \binom{-p}{-p+1-2k} x^{-p+1-2k}$$
$$- \frac{1}{\Gamma(1-p)} \sum_{r=1}^{\infty} (-1)^r \binom{-p}{-p+1-2r} \{\psi(1-p)-\psi(2r)\} x^{-p+1-2r}$$

$$\left(\cot^{-1}x\right)^{(p)} = -\frac{\tan^{-1}x}{\Gamma(1-p)} \sum_{k=0}^{\infty} (-1)^k \binom{-p}{-p-2k} x^{-p-2k}$$
$$- \frac{\log\left(1+x^2\right)}{2\Gamma(1-p)} \sum_{k=1}^{\infty} (-1)^k \binom{-p}{-p+1-2k} x^{-p+1-2k}$$
$$+ \frac{1}{\Gamma(1-p)} \sum_{r=1}^{\infty} (-1)^r \binom{-p}{-p+1-2r} \{\psi(1-p)-\psi(2r)\} x^{-p+1-2r}$$

(7) By-Products

$$\sum_{k=0}^{n} \frac{(-1)^k}{mk+(m-1)} \binom{n}{k} = -\frac{B(1+n,\,-1/m)}{m(mn+m-1)}$$ $B(\,)$ is Beta function

$$\left(x^q\right)^{(p)} = \frac{q+1-p}{\Gamma(1-p)} \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1-p} \binom{q}{r} \cdot x^{q-p}$$

$$B(q,-p) = -\frac{q-p}{p} \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1-p} \binom{q-1}{r}$$ $B(\,)$ is Beta function

13 Termwise Super Derivative

The following termwise super derivatives are obtained from "10 Termwise Higher Derivative (Trigonometric, Hyperbolic)" and "11 Termwise Higher Derivative (Inv-Trigonometric, Inv-Hyperbolic)" .

Where, $\uparrow, \downarrow$ denote the ceiling function and the floor function respectively. And Bernoulli Numbers and Euler Numbers are as follows.

$$B_0=1,\ B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ B_6=\frac{1}{42},\ B_8=-\frac{1}{30},\ B_{10}=\frac{5}{66},\ \cdots$$
$$E_0=1,\ E_2=-1,\ E_4=5,\quad E_6=-61,\ E_8=1385,\ E_{10}=-50521,\ \cdots$$

$$(\tan x)^{(p)} = \sum_{k=\frac{p+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)\left|B_{2k}\right|}{2k\,\Gamma(2k-p)} x^{2k-p-1} \qquad 0<x<\frac{\pi}{2}$$

$$(\tanh x)^{(p)} = \sum_{k=\frac{p+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{2k\,\Gamma(2k-p)} x^{2k-p-1} \qquad 0<x<\frac{\pi}{2}$$

$$(\sec x)^{(p)} = \sum_{k=\frac{p+1}{2}\downarrow}^{\infty} \frac{\left|E_{2k}\right|}{\Gamma(2k-p+1)} x^{2k-p} \qquad 0<x<\frac{\pi}{2}$$ **collateral**

$$(\mathrm{sech}\, x)^{(p)} = \sum_{k=\frac{p+1}{2}\downarrow}^{\infty} \frac{E_{2k}}{\Gamma(2k-p+1)} x^{2k-p} \qquad 0<x<\frac{\pi}{2}$$ **collateral**

$$(\cot x)^{(p)} = -\sum_{k=\frac{p+1}{2}\uparrow}^{\infty} \frac{2^{2k}\left(2^{2k}-1\right)|B_{2k}|}{2k\,\Gamma(2k-p)}\left(x-\frac{\pi}{2}\right)^{2k-p-1} \qquad \frac{\pi}{2}<x<\pi$$

$$(\csc x)^{(p)} = \sum_{k=\frac{p+1}{2}\downarrow}^{\infty} \frac{|E_{2k}|}{\Gamma(2k-p+1)}\left(x-\frac{\pi}{2}\right)^{2k-p} \qquad \frac{\pi}{2}<x<\pi$$

$$(\operatorname{csch} x)^{(p)} = (-1)^{-p}2\sum_{k=0}^{\infty} \frac{(2k+1)^p}{e^{(2k+1)x}} \qquad x>0$$

$$(\operatorname{sech} x)^{(p)} = (-1)^{-p}2\sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^p}{e^{(2k+1)x}} \qquad x>0$$

$$\left(\tan^{-1}x\right)^{(p)} = \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{\Gamma(2k+2-p)} x^{2k+1-p} \qquad 0<x<1$$

$$\left(\cot^{-1}x\right)^{(p)} = \frac{\pi}{2}\frac{x^{-p}}{\Gamma(1-p)} - \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} (-1)^k \frac{(2k)!}{\Gamma(2k+2-p)} x^{2k+1-p} \qquad 0<x<1$$

$$\left(\sin^{-1}x\right)^{(p)} = \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{\Gamma(2k+2-p)} x^{2k+1-p} \qquad 0<x<1 \quad \textbf{collateral}$$

$$\left(\cos^{-1}x\right)^{(p)} = \frac{\pi}{2}\frac{x^{-p}}{\Gamma(1-p)} - \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{\Gamma(2k+2-p)} x^{2k+1-p} \qquad 0<x<1 \quad \textbf{collateral}$$

$$\left(\tanh^{-1}x\right)^{(p)} = \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} \frac{(2k)!}{\Gamma(2k+2-p)} x^{2k+1-p} \qquad 0<x<1$$

$$\left(\sinh^{-1}x\right)^{(p)} = \sum_{k=\frac{p-1}{2}\uparrow}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{\Gamma(2k+2-p)} x^{2k+1-p} \qquad 0<x<1 \quad \textbf{collateral}$$

$$\left(\operatorname{sech}^{-1}x\right)^{(p)} = \frac{x^{-p}}{\Gamma(1-p)}\left\{\log\frac{2}{x}+\psi(1-p)+\gamma\right\} - \sum_{k=\frac{p}{2}\uparrow}^{\infty} \frac{\{(2k-1)!!\}^2}{2k\Gamma(2k-p+1)} x^{2k-p}$$

$$0<x<1 \quad \textbf{collateral}$$

$$\left(\operatorname{csch}^{-1}x\right)^{(p)} = \frac{x^{-p}}{\Gamma(1-p)}\left\{\log\frac{2}{x}+\psi(1-p)+\gamma\right\} - \sum_{k=\frac{p}{2}\uparrow}^{\infty} (-1)^k \frac{\{(2k-1)!!\}^2}{2k\Gamma(2k-p+1)} x^{2k-p}$$

$$0<x<1 \quad \textbf{collateral}$$

14 Higher and Super Calculus of Logarithmic Integral etc.

Here, the higher integrals and the super calculus of the double logarithm function and the following four functions are shown.

$$Ei(x) = \int_{-\infty}^{x}\frac{e^t}{t}dt \ , \ Ci(x) = \int_{\infty}^{x}\frac{\cos t}{t}dt \ , \ Si(x) = \int_{0}^{x}\frac{\sin t}{t}dt \ , \ li(x) = \int_{0}^{x}\frac{1}{\log t}dt$$

(1) Higher Integrals of Logarithmic Integral etc.

$$\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} Ei(x)\,dx^n = \frac{1}{n!}\left\{x^n Ei(x) - e^x\sum_{r=0}^{n-1} r!\,x^{n-1-r}\right\}$$

$$\int_{\infty}^{x}\cdots\int_{\infty}^{x} Ci(x)dx^n = \frac{1}{n!}\left\{ Ci(x)x^n - \sin x \sum_{r=0}^{(n-1)/2\downarrow}(-1)^r(2r)!\, x^{n-1-2r} + \cos x \sum_{r=0}^{(n-2)/2\downarrow}(-1)^r(2r+1)!\, x^{n-2-2r}\right\} \qquad n \geqq 2$$

$$\int_{0}^{x}\cdots\int_{0}^{x} Si(x)dx^n = \frac{1}{n!}\left\{ Si(x)x^n + \cos x \sum_{r=0}^{(n-1)/2\downarrow}(-1)^r(2r)!\, x^{n-1-2r} + \sin x \sum_{r=0}^{(n-2)/2\downarrow}(-1)^r(2r+1)!\, x^{n-2-2r}\right\} - \sum_{r=0}^{(n-1)/2\downarrow}(-1)^r \frac{x^{n-1-2r}}{(2r+1)(n-1-2r)!}$$

$$\int_{0}^{x}\cdots\int_{0}^{x} li(x)dx^n = \frac{1}{n!}\sum_{r=0}^{n}(-1)^r {}_nC_r x^{n-r} Ei\{(r+1)\log x\}$$

$$\int_{0}^{x}\cdots\int_{0}^{x} \log|\log x|dx^n = \frac{1}{n!}\left\{ x^n \log|\log x| + \sum_{r=1}^{n}(-1)^r {}_nC_r x^{n-r} Ei(r\log x)\right\}$$

(2) Super Calculus of Logarithmic Integral etc.

$$\int_{0}^{x}\sim\int_{0}^{x} li(x)dx^p = \frac{1}{\Gamma(1+p)}\sum_{r=0}^{\infty}(-1)^r\binom{p}{r} x^{p-r} Ei\{(r+1)\log x\} \qquad x \geqq 0$$

$$\{li(x)\}^{(p)} = \frac{1}{\Gamma(1-p)}\sum_{r=0}^{\infty}(-1)^r\binom{-p}{r} x^{-p-r} Ei\{(r+1)\log x\} \qquad x \geqq 0$$

$$\int_{0}^{x}\sim\int_{0}^{x} \log|\log x|dx^p = \frac{1}{\Gamma(1+p)}\left\{ x^p \log|\log x| + \sum_{r=1}^{\infty}(-1)^r\binom{p}{r} x^{p-r} Ei(r\log x)\right\}$$

$$(\log|\log x|)^{(p)} = \frac{1}{x^p\Gamma(1-p)}\left\{ \log|\log x| + \sum_{r=1}^{\infty}(-1)^r\binom{-p}{r}\frac{Ei(r\log x)}{x^r}\right\} \qquad x \geqq 0$$

15 Higher and Super Calculus of Elliptic Integral

When $|x| \leqq 1$, $|k| \leqq 1$, $|c| \leqq 1$, Elliptic Integrals of the 1st~3rd kind are expressed as follows.

$$F(x,k) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad , \quad E(x,k) = \int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

$$\Pi(x,c,k) = \int_0^x \frac{dx}{(1+cx^2)\sqrt{(1-x^2)(1-k^2x^2)}}$$

In this chapter, we expand these to a double series or a triple series and calculate the arc length of an ellipse and a lemniscate using thiese. Next, we calculate these term by term.

(1) Double (triple) Series Expansion

$$F(x,k) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r}{2r+1}\binom{-1/2}{r-s}\binom{-1/2}{s} k^{2s}x^{2r+1}$$

$$E(x,k) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^r}{2r+1}\binom{-1/2}{r-s}\binom{1/2}{s} k^{2s}x^{2r+1}$$

$$\Pi(x,c,k) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{(-1)^r}{2r+1}c^{r-s}\binom{-1/2}{s-t}\binom{-1/2}{t} k^{2t}x^{2r+1}$$

(2) Termwise Higher Calculus

$$\int_0^x \cdots \int_0^x F(x,k)\,dx^n = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r(2r)!}{(2r+n+1)!}\binom{-1/2}{r-s}\binom{-1/2}{s} k^{2s} x^{2r+n+1}$$

$$\int_0^x \cdots \int_0^x E(x,k)\,dx^n = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r(2r)!}{(2r+n+1)!}\binom{-1/2}{r-s}\binom{1/2}{s} k^{2s} x^{2r+n+1}$$

$$\int_0^x \cdots \int_0^x \Pi(x,c,k)\,dx^n = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^r(2r)!}{(2r+n+1)!} c^{r-s}\binom{-1/2}{s-t}\binom{-1/2}{t} k^{2t} x^{2r+n+1}$$

$$\frac{d^n}{dx^n}F(x,k) = \sum_{r=\frac{n-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r(2r)!}{(2r-n+1)!}\binom{-1/2}{r-s}\binom{-1/2}{s} k^{2s} x^{2r-n+1}$$

$$\frac{d^n}{dx^n}E(x,k) = \sum_{r=\frac{n-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r(2r)!}{(2r-n+1)!}\binom{-1/2}{r-s}\binom{1/2}{s} k^{2s} x^{2r-n+1}$$

$$\frac{d^n}{dx^n}\Pi(x,c,k) = \sum_{r=\frac{n-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^r(2r)!}{(2r-n+1)!} c^{r-s}\binom{-1/2}{s-t}\binom{-1/2}{t} k^{2t} x^{2r-n+1}$$

(3) Termwise Super Calculus

$$\int_0^x \sim \int_0^x F(x,k)\,dx^p = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r\Gamma(2r+1)}{\Gamma(2r+p+2)}\binom{-1/2}{r-s}\binom{-1/2}{s} k^{2s} x^{2r+p+1}$$

$$\int_0^x \sim \int_0^x E(x,k)\,dx^p = \sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r\Gamma(2r+1)}{\Gamma(2r+p+2)}\binom{-1/2}{r-s}\binom{1/2}{s} k^{2s} x^{2r+p+1}$$

$$\int_0^x \sim \int_0^x \Pi(x,c,k)\,dx^p = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^r\Gamma(2r+1)}{\Gamma(2r+p+2)} c^{r-s}\binom{-1/2}{s-t}\binom{-1/2}{t} k^{2t} x^{2r+p+1}$$

$$\frac{d^p}{dx^p}F(x,k) = \sum_{r=\frac{p-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r\Gamma(2r+1)}{\Gamma(2r-p+2)}\binom{-1/2}{r-s}\binom{-1/2}{s} k^{2s} x^{2r-p+1}$$

$$\frac{d^p}{dx^p}E(x,k) = \sum_{r=\frac{p-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r} \frac{(-1)^r\Gamma(2r+1)}{\Gamma(2r-p+2)}\binom{-1/2}{r-s}\binom{1/2}{s} k^{2s} x^{2r-p+1}$$

$$\frac{d^p}{dx^p}\Pi(x,c,k) = \sum_{r=\frac{p-1}{2}\uparrow}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s} \frac{(-1)^r\Gamma(2r+1)}{\Gamma(2r-n+2)} c^{r-s}\binom{-1/2}{s-t}\binom{-1/2}{t} k^{2t} x^{2r-p+1}$$

16 Higher Integral of the Product of Two Functions

We obtain the following thorem for the product of two functions.

(1) Theorem 16.1.2

Let $f^{<r>}$ be the arbitrary rth order primitive function of $f(x)$ and $g^{<r>}$ be the rth order derivative function of $g(x)$ for $r=1,2,\cdots,m+n-1$. Let $f_{a_k}^{<r>}, g_{a_k}^{(r)}$ be the function values of $f^{<r>}, g^{(r)}$ on a_k for $k=1,2,\cdots,n$.

And let $B(n,m)$ be the beta function. Then, the following formulas hold.

$$\int_{a_n}^x \cdots \int_{a_1}^x f^{<0>}g^{(0)}dx^n = \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>}g^{(r)}$$
$$-\sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\binom{-n+r}{s} f_{a_{n-r}}^{<n-r+s>} g_{a_{n-r}}^{(s)} \int_{a_n}^x \cdots \int_{a_{n-r+1}}^x dx^r$$

$$+\ (-1)^m \sum_{r=1}^{n-1}\sum_{s=0}^{r-1}\sum_{t=s}^{r-1} {}_tC_s \cdot {}_{m+n-1-r+t}C_{m-1}\, f_{a_{n-r}}^{\langle m+n-r+s\rangle} g_{a_{n-r}}^{(m+s)} \int_{a_n}^{x}\cdots\int_{a_{n-r+1}}^{x} dx^r$$

$$+\ \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1} \frac{{}_{n-1}C_k}{m+k} \int_{a_n}^{x}\cdots\int_{a_1}^{x} f^{<m+k>} g^{(m+k)} dx^n$$

Especially, when $a_r = a \ \ for\ \ r=1,2,\cdots,n$ and

when $f^{<r>}(a)=0\ \ (r=1,2,\cdots,m+n-1)\ \ or\ \ g^{(s)}(a)=0\ \ (s=0,1,\cdots,m+n-2)$,

$$\int_a^x\cdots\int_a^x f^{<0>} g^{(0)} dx^n = \sum_{r=0}^{m-1}\binom{-n}{r} f^{<n+r>} g^{(r)} + \frac{(-1)^m}{B(n,m)} \sum_{k=0}^{n-1}\frac{{}_{n-1}C_k}{m+k}\int_a^x\cdots\int_a^x f^{<m+k>} g^{(m+k)} dx^n$$

(2) Higher Integral of the Product of Two Functions

We obtain the following expressions using this theorem.

$$\int_{-\frac{b}{a}}^{x}\cdots\int_{-\frac{b}{a}}^{x} (ax+b)^p (cx+d)^q dx^n$$

$$= \sum_{r=0}^{\infty}\binom{-n}{r} \frac{(1/a)^{n+r}}{(1/c)^r} \frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p+n+r)\Gamma(1+q-r)} \frac{(ax+b)^{p+n+r}}{(cx+d)^{r-q}}$$

$$\int_{-\frac{b}{a}}^{x}\cdots\int_{-\frac{b}{a}}^{x} (ax+b)^p\, (cx+d)^m dx^n$$

$$= \sum_{r=0}^{m}\binom{-n}{r} \frac{(1/a)^{n+r}}{(1/c)^r} \frac{\Gamma(1+p)\Gamma(1+m)}{\Gamma(1+p+n+r)\Gamma(1+m-r)} \frac{(ax+b)^{p+n+r}}{(cx+d)^{r-m}}$$

$$\int_0^x\cdots\int_0^x x^\alpha \log x\, dx^n = \sum_{r=0}^{\infty}\binom{-n}{r} \frac{\log x - \psi(1+n+r) - \gamma}{\Gamma(1+n+r)} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} x^{\alpha+n}$$

$$\int_0^x\cdots\int_0^x x^m \log x\, dx^n = \sum_{r=0}^{m}\binom{-n}{r} \frac{\log x - \psi(1+n+r) - \gamma}{\Gamma(1+n+r)} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m+n}$$

$$\int_{a_n}^x\cdots\int_{a_1}^x x^m \sin x\, dx^n = \sum_{r=0}^{m}\binom{-n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \sin\left\{x - \frac{(n+r)\pi}{2}\right\}$$

$$\int_{a_n}^x\cdots\int_{a_1}^x x^m \cos x\, dx^n = \sum_{r=0}^{m}\binom{-n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \cos\left\{x - \frac{(n+r)\pi}{2}\right\}$$

$$\int_{a_n}^x\cdots\int_{a_1}^x x^m \sinh x\, dx^n = \sum_{r=0}^{m}\binom{-n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \frac{e^x - (-1)^{n+r} e^{-x}}{2}$$

$$\int_{a_n}^x\cdots\int_{a_1}^x x^m \cosh x\, dx^n = \sum_{r=0}^{m}\binom{-n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r} \frac{e^x + (-1)^{n+r} e^{-x}}{2}$$

$$\int_0^x\cdots\int_0^x \log^2 x\, dx^n = \log x\left(\log x - \sum_{k=1}^{n}\frac{1}{k}\right)\frac{x^n}{n!}$$

$$+ \sum_{r=1}^{\infty}(-1)^{r-1}\binom{-n}{r} \frac{\Gamma(r)}{\Gamma(1+n+r)} x^n \left(\log x - \sum_{k=1}^{n+r}\frac{1}{k}\right)$$

$$\int_{-\infty}^x\cdots\int_{-\infty}^x e^x x^\alpha dx^n = \frac{x^\alpha}{(-x)^\alpha} \sum_{r=0}^{n-1} \frac{{}_{n-1}C_r\, \Gamma(n-r+\alpha,\ -x)\, x^r}{(n-1)!} + R$$

$$R = \begin{cases} 0 & x \leqq 0 \\ 2i \sin\alpha\pi \sum_{r=0}^{n-1} \frac{{}_{n-1}C_r \Gamma(n-r+\alpha) x^r}{(n-1)!} & x>0 \end{cases}$$

$$\int_{-\infty}^{x} \cdots \int_{-\infty}^{x} e^x x^m dx^n = e^x \sum_{r=0}^{m} \binom{-n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r}$$

$$\int_{-\infty}^{x} \cdots \int_{-\infty}^{x} e^x \log x \, dx^n = e^x \left\{ \log|x| + \sum_{r=0}^{n-2} \sum_{s=0}^{n-2-r} \frac{s!\, x^r}{(r+s+1)!} \right\} - Ei(x) \sum_{r=0}^{n-1} \frac{x^r}{r!}$$

$$\int_{-\infty}^{x} \cdots \int_{-\infty}^{x} e^x \sin x dx^n = \left(\sin\frac{\pi}{4} \right)^n e^x \sin\left(x - \frac{n\pi}{4} \right)$$

$$\int_{-\infty}^{x} \cdots \int_{-\infty}^{x} e^x \cos x dx^n = \left(\sin\frac{\pi}{4} \right)^n e^x \cos\left(x - \frac{n\pi}{4} \right)$$

$$\int_{\infty}^{x} \cdots \int_{\infty}^{x} e^{-x} x^\alpha dx^n = \frac{(-1)^n}{(n-1)!} \sum_{r=0}^{n-1} (-1)^r {}_{n-1}C_r \Gamma(n-r+\alpha, x) x^r$$

$$\int_{\infty}^{x} \cdots \int_{\infty}^{x} e^{-x} x^m dx^n = \frac{(-1)^n}{e^x} \sum_{r=0}^{m} (-1)^r \binom{-n}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r}$$

$$\int_{\infty}^{x} \cdots \int_{\infty}^{x} e^{-x} \log x \, dx^n = (-1)^n e^{-x} \left\{ \log|x| + \sum_{r=0}^{n-2} \sum_{s=0}^{n-2-r} \frac{s!\,(-x)^r}{(r+s+1)!} \right\}$$
$$- (-1)^n Ei(-x) \sum_{r=0}^{n-1} \frac{(-x)^r}{r!}$$

$$\int_{\infty}^{x} \cdots \int_{\infty}^{x} e^{-x} \sin x dx^n = (-1)^n \left(\sin\frac{\pi}{4} \right)^n e^{-x} \sin\left(x + \frac{n\pi}{4} \right)$$

$$\int_{\infty}^{x} \cdots \int_{\infty}^{x} e^{-x} \cos x dx^n = (-1)^n \left(\sin\frac{\pi}{4} \right)^n e^{-x} \cos\left(x + \frac{n\pi}{4} \right)$$

(3) By-Products

$$\sum_{k=0}^{n/2\downarrow} \frac{(-1)^k}{2k+1} {}_nC_{2k} = \frac{1}{n+1} \left(\sin\frac{\pi}{4} \right)^{-n-1} \sin\frac{(n+1)\pi}{4}$$

$$\sum_{k=1}^{n/2\uparrow} \frac{(-1)^k}{2k} {}_nC_{2k-1} = \frac{1}{n+1} \left\{ \left(\sin\frac{\pi}{4} \right)^{-n-1} \cos\frac{(n+1)\pi}{4} - 1 \right\}$$

Example

$$\frac{{}_8C_0}{1} - \frac{{}_8C_2}{3} + \frac{{}_8C_4}{5} - \frac{{}_8C_6}{7} + \frac{{}_8C_8}{9} = \frac{16}{9}$$

$$-\frac{{}_8C_1}{2} + \frac{{}_8C_3}{4} - \frac{{}_8C_5}{6} + \frac{{}_8C_7}{8} = \frac{5}{3}$$

17 Super Integral of the Product of Two Functions

We obtain the following thorems for the product of two functions.

(1) Theorem 17.1.2

Let r, p are positive numbers, $f^{<r>}$ be an arbitrary r th order primitive function of $f(x)$, $g^{(r)}$ be the r th order derivative function of $g(x)$, $B(n,m)$, $\Gamma(p)$ are the beta function and the gamma function respectively. At this time, if there is a number a such that

$$f^{<r>}(a)=0 \quad r\in[0,m+p] \qquad or \qquad g^{(s)}(a)=0 \quad s\in[0,m+p-1],$$

then the following expression holds.

$$\int_a^x\sim\int_a^x fg\,dx^p=\sum_{r=0}^{m-1}\binom{-p}{r}f^{<p+r>}g^{(r)}+R_m^p$$

$$R_m^p=\frac{(-1)^m}{B(p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{p-1}{k}\int_a^x\sim\int_a^x f^{<m+k>}g^{(m+k)}dx^p$$

(2) Super Integral of the Product of Two Functions

We obtain the following expressions using this theorem.

$$\int_{-\frac{b}{a}}^x\sim\int_{-\frac{b}{a}}^x (ax+b)^p(cx+d)^q dx^s$$

$$=\sum_{r=0}^{\infty}\binom{-s}{r}\frac{(1/a)^{s+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p+s+r)\Gamma(1+q-r)}\frac{(ax+b)^{p+s+r}}{(cx+d)^{r-q}}$$

$$\int_{-\frac{b}{a}}^x\sim\int_{-\frac{b}{a}}^x (ax+b)^p\,(cx+d)^m dx^s$$

$$=\sum_{r=0}^{m}\binom{-s}{r}\frac{(1/a)^{s+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+m)}{\Gamma(1+p+s+r)\Gamma(1+m-r)}\frac{(ax+b)^{p+s+r}}{(cx+d)^{r-m}}$$

$$\int_0^x\sim\int_0^x x^{\alpha}\log x\,dx^p=\sum_{r=0}^{\infty}\binom{-p}{r}\frac{\log x-\psi(1+p+r)-\gamma}{\Gamma(1+p+r)}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha+p}$$

$$\int_0^x\sim\int_0^x x^{m}\log x\,dx^p=\sum_{r=0}^{m}\binom{-p}{r}\frac{\log x-\psi(1+p+r)-\gamma}{\Gamma(1+p+r)}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m+p}$$

$$\int_{a_p}^x\sim\int_{a_0}^x x^m\sin x\,dx^p=\sum_{r=0}^{m}\binom{-p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\sin\left\{x-\frac{(p+r)\pi}{2}\right\}$$

$$\int_{a_p}^x\sim\int_{a_0}^x x^m\cos x\,dx^p=\sum_{r=0}^{m}\binom{-p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\cos\left\{x-\frac{(p+r)\pi}{2}\right\}$$

$$\int_{a_p}^x\sim\int_{a_0}^x x^m\sinh x\,dx^p=\sum_{r=0}^{m}\binom{-p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\frac{e^x-(-1)^{p+r}e^{-x}}{2}$$

$$\int_{a_p}^x\sim\int_{a_0}^x x^m\cosh x\,dx^p=\sum_{r=0}^{m}\binom{-p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\frac{e^x+(-1)^{p+r}e^{-x}}{2}$$

$$\int_0^x\sim\int_0^x \log^2 x\,dx^p=\frac{x^p\{\log x-\psi(1+p)-\gamma\}\log x}{\Gamma(1+p)}$$

$$+x^p\sum_{r=1}^{\infty}(-1)^{r-1}\binom{-p}{r}\frac{\{\log x-\psi(1+p+r)-\gamma\}\Gamma(r)}{\Gamma(1+p+r)}$$

$$\int_{-\infty}^x\sim\int_{-\infty}^x e^x x^{\alpha}dx^p=\frac{(-1)^{\alpha}}{\Gamma(p)}\sum_{r=0}^{\infty}\binom{p-1}{r}\Gamma(p-r+\alpha,\,-x)x^r$$

$$\int_{-\infty}^x\sim\int_{-\infty}^x e^x x^m dx^p=e^x\sum_{r=0}^{m}\binom{-p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}$$

$$\int_{-\infty}^{x}\sim\int_{-\infty}^{x} e^{x}\sin x\, dx^{p} = \left(\sin\frac{\pi}{4}\right)^{p} e^{x}\sin\left(x-\frac{p\pi}{4}\right)$$

$$\int_{-\infty}^{x}\sim\int_{-\infty}^{x} e^{x}\cos x\, dx^{p} = \left(\sin\frac{\pi}{4}\right)^{p} e^{x}\cos\left(x-\frac{p\pi}{4}\right)$$

$$\int_{\infty}^{x}\sim\int_{\infty}^{x} e^{-x} x^{\alpha} dx^{p} = \frac{(-1)^{p}}{\Gamma(p)}\sum_{r=0}^{\infty}(-1)^{r}\binom{p-1}{r}\Gamma(p-r+\alpha, x)x^{r}$$

$$\int_{\infty}^{x}\sim\int_{\infty}^{x} e^{-x} x^{m} dx^{p} = \frac{(-1)^{p}}{e^{x}}\sum_{r=0}^{m}(-1)^{r}\binom{-p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}$$

$$\int_{\infty}^{x}\sim\int_{\infty}^{x} e^{-x}\sin x\, dx^{p} = (-1)^{p}\left(\sin\frac{\pi}{4}\right)^{p} e^{-x}\sin\left(x+\frac{p\pi}{4}\right)$$

$$\int_{\infty}^{x}\sim\int_{\infty}^{x} e^{-x}\cos x\, dx^{p} = (-1)^{p}\left(\sin\frac{\pi}{4}\right)^{p} e^{-x}\cos\left(x+\frac{p\pi}{4}\right)$$

(3) By-Products

$$\sum_{k=0}^{\infty}\frac{(-1)^{k}}{2k+1}\binom{p}{2k} = \frac{1}{p+1}\left(\sin\frac{\pi}{4}\right)^{-p-1}\sin\frac{(p+1)\pi}{4}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k}}{2k}\binom{p}{2k-1} = \frac{1}{p+1}\left\{\left(\sin\frac{\pi}{4}\right)^{-p-1}\cos\frac{(p+1)\pi}{4} - 1\right\}$$

18 Higher Derivative of the Product of Two Functions

The following Leibniz rule is drawn from the Theorem 16.1.2.

Theorem 18.1.1 (Leibniz)

When functions $f(x)$ and $g(x)$ are n times differentiable, the following expression holds.

$$\{f(x)g(x)\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r} f^{(n-r)}(x)g^{(r)}(x)$$

(2) Higher Derivative of the Product of Two Functions

We obtain the following expressions using this theorem. Where, $p>0$, α are real numbers and m is a non-negative integer. Furthermore, if $\alpha = -1,-2,-3,\cdots$, it shall read as follows.

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} \longrightarrow (-1)^{-r}\frac{\Gamma(-\alpha+r)}{\Gamma(-\alpha)}$$

$$\{(ax+b)^{p}(cx+d)^{q}\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r}\frac{(1/a)^{-n+r}}{(1/c)^{r}}\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p-n+r)\Gamma(1+q-r)}\frac{(ax+b)^{p-n+r}}{(cx+d)^{r-q}}$$

$$\{(ax+b)^{p}(cx+d)^{m}\}^{(n)} = \sum_{r=0}^{m}\binom{n}{r}\frac{(1/a)^{-n+r}}{(1/c)^{r}}\frac{\Gamma(1+p)\Gamma(1+m)}{\Gamma(1+p-n+r)\Gamma(1+m-r)}\frac{(ax+b)^{p-n+r}}{(cx+d)^{r-m}}$$

$$(x^{\alpha}\log x)^{(n)} = -\sum_{r=0}^{n-1}(-1)^{n-r}\binom{n}{r}\frac{\Gamma(n-r)\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-n} + \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)}x^{\alpha-n}\log x$$

$$(x^{m}\log x)^{(n)} = -\sum_{r=0}^{n-1}(-1)^{n-r}\binom{n}{r}\frac{\Gamma(n-r)\Gamma(1+m)}{\Gamma(1+m-r)}x^{\alpha-n} \qquad n>m=0,1,2,3,\cdots$$

$$\left(x^{\alpha}\sin x\right)^{\{n\}}=\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}\sin\left\{x+\frac{(n-r)\pi}{2}\right\}$$

$$\left(x^{\alpha}\cos x\right)^{\{n\}}=\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}\cos\left\{x+\frac{(n-r)\pi}{2}\right\}$$

$$\left(x^{m}\sin x\right)^{\{n\}}=\sum_{r=0}^{m}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\sin\left\{x+\frac{(n-r)\pi}{2}\right\}$$

$$\left(x^{m}\cos x\right)^{\{n\}}=\sum_{r=0}^{m}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\cos\left\{x+\frac{(n-r)\pi}{2}\right\}$$

$$\left(x^{\alpha}\sinh x\right)^{(n)}=\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}\frac{e^{x}-(-1)^{-(n-r)}e^{-x}}{2}$$

$$\left(x^{\alpha}\cosh x\right)^{(n)}=\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}\frac{e^{x}+(-1)^{-(n-r)}e^{-x}}{2}$$

$$\left(x^{m}\sinh x\right)^{(n)}=\sum_{r=0}^{m}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\frac{e^{x}-(-1)^{-(n-r)}e^{-x}}{2}$$

$$\left(x^{m}\cosh x\right)^{(n)}=\sum_{r=0}^{m}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\frac{e^{x}+(-1)^{-(n-r)}e^{-x}}{2}$$

$$\left(\log^{2}x\right)^{(n)}=\frac{(-1)^{n-1}}{x^{n}}\left\{2\Gamma(n)\log x-\sum_{r=1}^{n-1}\binom{n}{r}\Gamma(n-r)\Gamma(r)\right\}$$

$$\left(e^{x}x^{\alpha}\right)^{(n)}=e^{x}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}$$

$$\left(e^{x}x^{m}\right)^{(n)}=e^{x}\sum_{r=0}^{m}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}$$

$$\left(e^{x}\log x\right)^{(n)}=e^{x}\log x+e^{x}\sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}\frac{\Gamma(r)}{x^{r}}$$

$$\left(e^{x}\sin x\right)^{(n)}=\left(\sin\frac{\pi}{4}\right)^{-n}e^{x}\sin\left(x+\frac{n\pi}{4}\right)$$

$$\left(e^{x}\cos x\right)^{(n)}=\left(\sin\frac{\pi}{4}\right)^{-n}e^{x}\cos\left(x+\frac{n\pi}{4}\right)$$

$$\left(e^{x}\sinh x\right)^{(n)}=e^{x}\sum_{r=0}^{n}\binom{n}{r}\frac{e^{x}-(-1)^{-r}e^{-x}}{2}$$

$$\left(e^{x}\cosh x\right)^{(n)}=e^{x}\sum_{r=0}^{n}\binom{n}{r}\frac{e^{x}+(-1)^{-r}e^{-x}}{2}$$

$$\left(e^{-x}x^{\alpha}\right)^{(n)}=e^{-x}\sum_{r=0}^{n}(-1)^{-(n-r)}\binom{n}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r}$$

$$\left(e^{-x}x^{m}\right)^{(n)}=e^{-x}\sum_{r=0}^{m}(-1)^{-(n-r)}\binom{n}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}$$

$$\left(e^{-x}\log x\right)^{(n)}=\frac{(-1)^{-n}}{e^{x}}\left\{\log x-\sum_{r=1}^{n}\binom{n}{r}\frac{\Gamma(r)}{x^{r}}\right\}$$

$$\left(e^{-x}\sin x\right)^{(n)} = \left(-\sin\frac{\pi}{4}\right)^{-n} e^{-x}\sin\left(x-\frac{n\pi}{4}\right)$$

$$\left(e^{-x}\cos x\right)^{(n)} = \left(-\sin\frac{\pi}{4}\right)^{-n} e^{-x}\cos\left(x-\frac{n\pi}{4}\right)$$

$$\left(e^{-x}\sinh x\right)^{(n)} = e^{-x}\sum_{r=0}^{n}(-1)^{-n+r}\binom{n}{r}\frac{e^{x}-(-1)^{-r}e^{-x}}{2}$$

$$\left(e^{-x}\cosh x\right)^{(n)} = e^{-x}\sum_{r=0}^{n}(-1)^{-n+r}\binom{n}{r}\frac{e^{x}+(-1)^{-r}e^{-x}}{2}$$

$$\left(\sin^2 x\right)^{(n)} = -2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right)$$

$$\left(\cos^2 x\right)^{(n)} = 2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right)$$

$$\left(\sin^3 x\right)^{(n)} = \frac{3}{4}\sin\left(x+\frac{n\pi}{2}\right) - \frac{3^n}{4}\sin\left(3x+\frac{n\pi}{2}\right)$$

$$\left(\cos^3 x\right)^{(n)} = \frac{3}{4}\cos\left(x+\frac{n\pi}{2}\right) + \frac{3^n}{4}\cos\left(3x+\frac{n\pi}{2}\right)$$

etc.

(3) By-Products

$$\sum_{r=0}^{n/2\downarrow}(-1)^r{}_nC_{2r} = 2^{\frac{n}{2}}\cos\frac{n\pi}{4} \quad , \quad \sum_{r=0}^{(n-1)/2\downarrow}(-1)^r{}_nC_{2r+1} = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$$

$$\sum_{r=0}^{n/2\downarrow}{}_nC_{2r} = 2^{n-1} \quad , \quad \sum_{r=0}^{(n-1)/2\downarrow}{}_nC_{2r+1} = 2^{n-1}$$

$$\sum_{r=0}^{n/2\downarrow}2^{2r-1}{}_nC_{2r} = \frac{3^n+(-1)^n}{4} \quad , \quad \sum_{r=0}^{(n-1)/2\downarrow}2^{2r}{}_nC_{2r+1} = \frac{3^n-(-1)^n}{4}$$

19 Super Derivative of the Product of Two Functions

The following Leibniz rule for Super Derivative is drawn from the Theorem 16.1.2.

Theorem 19.1.1

Let $B(x,y)$ be the beta function and p be a positive number. And for $r=0,1,2,\cdots$, let $f^{<-p+r>}$ be arbitrary primitive function of $f(x)$ and $g^{(r)}$ be the r th order derivative function of $g(x)$. Then the following expressions hold

$$\{f(x)g(x)\}^{(p)} = \sum_{r=0}^{m-1}\binom{p}{r}f^{(p-r)}(x)g^{(r)}(x) + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-p-1}{k}\left\{f^{<m+k>}(x)g^{(m+k)}(x)\right\}^{(p)}$$

Especially, when $n=0,1,2,\cdots$

$$\{f(x)g(x)\}^{(n)} = \sum_{r=0}^{n}\binom{n}{r}f^{(n-r)}(x)g^{(r)}(x) \qquad \text{(Leibniz)}$$

(2) Super Derivative of the Product of Two Functions

We obtain the following expressions using this theorem. Where, $p>0$, α are real numbers and m is a non-negative integer. Furthermore, if $\alpha = -1,-2,-3,\cdots$, it shall read as follows.

$$\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)} \longrightarrow (-1)^{-r}\frac{\Gamma(-\alpha+r)}{\Gamma(-\alpha)}$$

$$\{(ax+b)^p(cx+d)^q\}^{(s)} = \sum_{r=0}^{\infty}\binom{s}{r}\frac{(1/a)^{-s+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+q)}{\Gamma(1+p-s+r)\Gamma(1+q-r)}\frac{(ax+b)^{p-s+r}}{(cx+d)^{r-q}}$$

$$\{(ax+b)^p(cx+d)^m\}^{(s)} = \sum_{r=0}^{m}\binom{s}{r}\frac{(1/a)^{-s+r}}{(1/c)^r}\frac{\Gamma(1+p)\Gamma(1+m)}{\Gamma(1+p-s+r)\Gamma(1+m-r)}\frac{(ax+b)^{p-s+r}}{(cx+d)^{r-m}}$$

$$(x^\alpha \log x)^{(p)} = \sum_{r=0}^{\infty}\binom{p}{r}\frac{\log x-\psi(1-p+r)-\gamma}{\Gamma(1-p+r)}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-p}$$

$$(x^m \log x)^{(p)} = \sum_{r=0}^{m}\binom{p}{r}\frac{\log x-\psi(1-p+r)-\gamma}{\Gamma(1-p+r)}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-p}$$

$$(x^m \sin x)^{(p)} = \sum_{r=0}^{m}\binom{p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\sin\left\{x+\frac{(p-r)\pi}{2}\right\}$$

$$(x^m \cos x)^{(p)} = \sum_{r=0}^{m}\binom{p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\cos\left\{x+\frac{(p-r)\pi}{2}\right\}$$

$$(x^m \sinh x)^{\{p\}} = \sum_{r=0}^{m}\binom{p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\frac{e^x-(-1)^{-p+r}e^{-x}}{2}$$

$$(x^m \cosh x)^{\{p\}} = \sum_{r=0}^{m}\binom{p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}\frac{e^x+(-1)^{-p+r}e^{-x}}{2}$$

$$(\log^2 x)^{(p)} = \frac{x^{-p}\log x\{\log x-\psi(1-p)-\gamma\}}{\Gamma(1-p)}$$

$$-x^{-p}\sum_{r=1}^{\infty}(-1)^r\binom{p}{r}\frac{\{\log x-\psi(1-p+r)-\gamma\}\Gamma(r)}{\Gamma(1-p+r)}$$

$$(e^x x^\alpha)^{(p)} = e^x\sum_{r=0}^{m-1}\binom{p}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)}\sum_{k=0}^{\infty}\frac{1}{m+k}\binom{-p-1}{k}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-k)}\left(e^x x^{\alpha-m-k}\right)^{(p)}$$

$$(e^x x^m)^{(p)} = e^x\sum_{r=0}^{m}\binom{p}{r}\frac{\Gamma(1+m)}{\Gamma(1+m-r)}x^{m-r}$$

$$(e^x \sin x)^{(p)} = \left(\sin\frac{\pi}{4}\right)^{-p}e^x\sin\left(x+\frac{p\pi}{4}\right)$$

$$(e^x \cos x)^{(p)} = \left(\sin\frac{\pi}{4}\right)^{-p}e^x\cos\left(x+\frac{p\pi}{4}\right)$$

$$(e^x \sinh x)^{(p)} = e^x\sum_{r=0}^{\infty}\binom{p}{r}\frac{e^x-(-1)^{-r}e^{-x}}{2}$$

$$(e^x \cosh x)^{(p)} = e^x\sum_{r=0}^{\infty}\binom{p}{r}\frac{e^x+(-1)^{-r}e^{-x}}{2}$$

$$(e^{-x} x^\alpha)^{(p)} = \frac{(-1)^{-p}}{e^x}\sum_{r=0}^{m-1}(-1)^r\binom{p}{r}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-r)}x^{\alpha-r} + R_m^p$$

$$R_m^p = \frac{1}{B(-p,m)} \sum_{k=0}^{\infty} \frac{(-1)^k}{m+k} \binom{-p-1}{k} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-m-k)} \left(\frac{x^{\alpha-m-k}}{e^x} \right)^{(p)}$$

$$\left(e^{-x}x^m\right)^{(p)} = \frac{(-1)^{-p}}{e^x} \sum_{r=0}^{m} (-1)^r \binom{p}{r} \frac{\Gamma(1+m)}{\Gamma(1+m-r)} x^{m-r}$$

$$\left(e^{-x}\sin x\right)^{(p)} = (-1)^{-p} \left(\sin\frac{\pi}{4}\right)^{-p} e^{-x} \sin\left(x - \frac{p\pi}{4}\right)$$

$$\left(e^{-x}\cos x\right)^{(p)} = (-1)^{-p} \left(\sin\frac{\pi}{4}\right)^{-p} e^{-x} \cos\left(x - \frac{p\pi}{4}\right)$$

$$\left(e^{-x}\sinh x\right)^{(p)} = e^{-x} \sum_{r=0}^{\infty} (-1)^{-p+r} \binom{p}{r} \frac{e^x - (-1)^{-r} e^{-x}}{2}$$

$$\left(e^{-x}\cosh x\right)^{(p)} = e^{-x} \sum_{r=0}^{\infty} (-1)^{-p+r} \binom{p}{r} \frac{e^x + (-1)^{-r} e^{-x}}{2}$$

$$\left(\sin^2 x\right)^{(p)} = -2^{p-1} \cos\left(2x + \frac{p\pi}{2}\right)$$

$$\left(\cos^2 x\right)^{(p)} = 2^{p-1} \cos\left(2x + \frac{p\pi}{2}\right)$$

$$\left(\sin^3 x\right)^{(p)} = \frac{3}{4} \sin\left(x + \frac{p\pi}{2}\right) - \frac{3^p}{4} \sin\left(3x + \frac{p\pi}{2}\right)$$

$$\left(\cos^3 x\right)^{(p)} = \frac{3}{4} \cos\left(x + \frac{p\pi}{2}\right) + \frac{3^p}{4} \cos\left(3x + \frac{p\pi}{2}\right)$$

etc.

(3) By-Products

$$\sum_{k=0}^{\infty} (-1)^k \binom{p}{2k} = 2^{\frac{p}{2}} \cos\frac{p\pi}{4} \ , \qquad \sum_{k=0}^{\infty} (-1)^k \binom{p}{2k+1} = 2^{\frac{p}{2}} \sin\frac{p\pi}{4}$$

$$\sum_{r=0}^{\infty} \binom{p}{2r} = 2^{p-1} \ , \qquad \sum_{r=0}^{\infty} \binom{p}{2r+1} = 2^{p-1} \ , \qquad \sum_{r=0}^{\infty} \binom{p}{r} = 2^p \qquad p > -1$$

20 Higher Calculus of the product of many functions

20.1 Higher Derivative of the product of many functions

The following theorems are drawn from the Theorem 18.1.1.

Theorem 20.1.1

When $f_k^{(r)}$ denotes the r th order derivative function of $f_k(x)$ $(k=1,2,\cdots,\lambda)$,

$$\left(f_1 f_2 \cdots f_\lambda\right)^{(n)} = \sum_{r_1=0}^{n} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{n}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{(n-r_1)} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})}$$

Theorem 20.1.2

When $f^{(r)}$ denotes the r th order derivativefunction of $f(x)$ and λ is a natural number,

$$\left\{f^\lambda(x)\right\}^{(n)} = \sum_{r_1=0}^{n} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{n}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f^{(n-r_1)} f^{(r_1-r_2)} \cdots f^{(r_{\lambda-1})}$$

Example

$$\left(x^{\alpha}e^{x}\sin x\right)^{(n)}=\sum_{r=0}^{n}\sum_{s=0}^{r}\binom{n}{r}\binom{r}{s}\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n+r)}x^{\alpha-n+r}e^{x}\sin\left(x+\frac{s\pi}{2}\right)$$

$$\left\{\log^{3}x\right\}^{(n)}=(-1)^{n-1}\frac{3(n-1)!}{x^{n}}\log^{2}x+(-1)^{n}\frac{3\log x}{x^{n}}\sum_{r=1}^{n-1}\frac{n!}{(n-r)r}$$

$$+\frac{(-1)^{n-1}}{x^{n}}\sum_{r=2}^{n-1}\sum_{s=1}^{r-1}\frac{n!}{(n-r)(r-s)s}$$

Higher Derivatives of $\cos^{m}x$, $\sin^{m}x$

$$\left(\cos^{m}x\right)^{(n)}=\frac{1}{2^{m-1}}\sum_{r=0}^{m/2\downarrow}{}_{m}C_{r}\,(m-2r)^{n}\cos\left\{(m-2r)x+\frac{n\pi}{2}\right\}$$

$$\left(\sin^{m}x\right)^{(n)}=\frac{1}{2^{m-1}}\sum_{r=0}^{m/2\downarrow}{}_{m}C_{r}\,(m-2r)^{n}\cos\left\{(m-2r)\left(x-\frac{\pi}{2}\right)+\frac{n\pi}{2}\right\}$$

$$\left(\cos^{\alpha}x\right)^{(n)}=\frac{1}{2^{\alpha}}\sum_{r=0}^{\infty}\binom{\alpha}{r}(\alpha-2r)^{n}\cos\left\{(\alpha-2r)x+\frac{n\pi}{2}\right\}\qquad \alpha>0$$

$$\left(\sin^{\alpha}x\right)^{(n)}=\frac{1}{2^{\alpha}}\sum_{r=0}^{\infty}\binom{\alpha}{r}(\alpha-2r)^{n}\cos\left\{(\alpha-2r)\left(x-\frac{\pi}{2}\right)+\frac{n\pi}{2}\right\}\qquad \alpha>0$$

20.2 Higher Integral of the product of many functions

The following theorems are drawn from the Theorem 16.1.2 and Theorem 20.1.1.

Theorem 20.2.1

Let $f_k^{(r)}$ be the r th order derivative function of $f_k(x)$ $(k=1,2,\cdots,\lambda)$, $f_k^{<r>}$ be the arbitrary r th order primitive function of $f_k(x)$, m, n are natural numbers and $B(n,m)$ be the beta function. If there is a number a such that $f_1^{<r>}(a)=0$ $(r=1,2,\cdots,m+n-1)$ or $f_k^{(s)}(a)=0$ $(s=0,1,\cdots,m+n-2)$ for at least one $k>1$, then the following expression holds.

$$\int_a^x\!\!\cdots\!\int_a^x f_1f_2\cdots f_\lambda dx^n=\sum_{r_1=0}^{m-1}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{\lambda-1}=0}^{r_{\lambda-2}}\binom{-n}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{\lambda-2}}{r_{\lambda-1}}f_1^{\langle n+r_1\rangle}f_2^{(r_1-r_2)}\cdots f_\lambda^{(r_{\lambda-1})}$$
$$+R_m^n$$

$$R_m^n=\frac{(-1)^m}{B(n,m)}\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{m+k_1}\sum_{k_3=0}^{k_2}\cdots\sum_{k_{\lambda-1}=0}^{k_{\lambda-2}}\frac{{}_{n-1}C_{k_1}}{m+k_1}\binom{m+k_1}{k_2}\binom{k_2}{k_3}\cdots\binom{k_{\lambda-2}}{k_{\lambda-1}}$$
$$\times\int_a^x\!\!\cdots\!\int_a^x f_1^{\langle m+k_1\rangle}f_2^{(m+k_1-k_2)}f_3^{(k_2-k_3)}\cdots f_\lambda^{(k_{\lambda-1})}dx^n$$

Theorem 20.2.2

Let $f^{(r)}$ be the r th order derivative function of $f(x)$, $f_k^{<r>}$ be the arbitrary r th order primitive function of $f(x)$, m, n are natural numbers and $B(n,m)$ be the beta function. At this time , if there is a number a such that

$$f^{<r>}(a)=0\ \ (r=1,2,\cdots,m+n-1)\qquad or\qquad f^{(s)}(a)=0\ \ (s=0,1,\cdots,m+n-2)$$

then the following expression holds for $\lambda=2,3,4,\cdots$.

$$\int_a^x\!\!\cdots\!\int_a^x f^{\lambda}dx^n=\sum_{r_1=0}^{m-1}\sum_{r_2=0}^{r_1}\cdots\sum_{r_{\lambda-1}=0}^{r_{\lambda-2}}\binom{-n}{r_1}\binom{r_1}{r_2}\cdots\binom{r_{\lambda-2}}{r_{\lambda-1}}f^{\langle n+r_1\rangle}f^{(r_1-r_2)}\cdots f^{(r_{\lambda-1})}+R_m^n$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{{}_{n-1}C_{k_1}}{m+k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$

$$\times \int_a^x \cdots \int_a^x f^{\langle m+k_1 \rangle} f^{(m+k_1-k_2)} f^{(k_2-k_3)} \cdots f^{(k_{\lambda-1})} dx^n$$

Example

$$\int_{-\infty}^x \cdots \int_{-\infty}^x x^\alpha e^x \sin x \, dx^n = \sum_{r=0}^{m-1} \sum_{s=0}^{r} \binom{-n}{r} \binom{r}{s} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+r)} x^{\alpha+n+r} e^x \sin\left(x + \frac{s\pi}{2}\right) + R_m^n$$

$$R_m^n = \frac{(-1)^m}{B(n,m)} \sum_{r=0}^{n-1} \sum_{s=0}^{m+r} \frac{{}_{n-1}C_r}{m+r} \binom{m+r}{s} \int_{-\infty}^x \cdots \int_{-\infty}^x \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} e^x \sin\left(x + \frac{s\pi}{2}\right) dx^n$$

$$\int_0^x \cdots \int_0^x \log^3 x \, dx^n = \frac{\log x - \psi(1+n) - \gamma}{\Gamma(1+n)} x^n (\log x)^2$$

$$- 2x^n \log x \sum_{r=1}^{\infty} (-1)^r \binom{-n}{r} \frac{\log x - \psi(1+n+r) - \gamma}{\Gamma(1+n+r)} \Gamma(r)$$

$$+ x^n \sum_{r=2}^{\infty} \sum_{s=1}^{r-1} (-1)^r \binom{-n}{r} \binom{r}{s} \frac{\log x - \psi(1+n+r) - \gamma}{\Gamma(1+n+r)} \Gamma(r-s) \Gamma(s)$$

Higher Integrls of $\cos^m x$, $\sin^m x$

$$\int_{\frac{(n-1)\pi}{2}}^x \cdots \int_{\frac{1\pi}{2}}^x \int_{\frac{0\pi}{2}}^x \cos^{2m+1} x \, dx^n = \frac{1}{2^{2m}} \sum_{r=0}^{m} \frac{{}_{2m+1}C_r}{(2m-2r+1)^n} \cos\left\{ (2m-2r+1)x - \frac{n\pi}{2} \right\}$$

$$\int_{\frac{n\pi}{2}}^x \cdots \int_{\frac{2\pi}{2}}^x \int_{\frac{1\pi}{2}}^x \sin^{2m+1} x \, dx^n = \frac{1}{2^{2m}} \sum_{r=0}^{m} \frac{(-1)^{m-r} {}_{2m+1}C_r}{(2m-2r+1)^n} \sin\left\{ (2m-2r+1)x - \frac{n\pi}{2} \right\}$$

$$\int_{a_n}^x \cdots \int_{a_1}^x \cos^{2m} x \, dx^n = \frac{1}{2^{2m-1}} \sum_{r=0}^{m-1} \frac{{}_{2m}C_m}{(2m-2r)^n} \cos\left\{ (2m-2r)x - \frac{n\pi}{2} \right\} + \frac{{}_{2m}C_m}{2^{2m}} \int_{a_n}^x \cdots \int_{a_1}^x dx^n$$

$$\int_{a_n}^x \cdots \int_{a_1}^x \sin^{2m} x \, dx^n = \frac{1}{2^{2m-1}} \sum_{r=0}^{m-1} \frac{(-1)^{m-r} {}_{2m}C_m}{(2m-2r)^n} \cos\left\{ (2m-2r)x - \frac{n\pi}{2} \right\} + \frac{{}_{2m}C_m}{2^{2m}} \int_{a_n}^x \cdots \int_{a_1}^x dx^n$$

Where, $a_1, a_2, \cdots, a_n$ are the solutions of the following transcendental equations.

$$\sum_{r=0}^{m-1} \frac{{}_{2m}C_m}{(2m-2r)^n} \cos\left\{ (2m-2r)x - \frac{n\pi}{2} \right\} = 0 \qquad k=1,2,\cdots,n$$

$$\sum_{r=0}^{m-1} \frac{(-1)^{m-r} {}_{2m}C_m}{(2m-2r)^n} \cos\left\{ (2m-2r)x - \frac{n\pi}{2} \right\} = 0 \qquad k=1,2,\cdots,n$$

$$\int_0^x \cos^\alpha x \, dx = \frac{1}{2^\alpha} \sum_{r=0}^{\infty} \binom{\alpha}{r} \frac{1}{\alpha-2r} \sin\{(\alpha-2r)x\}$$

$$\int_{\frac{\pi}{2}}^x \sin^\alpha x \, dx = \frac{1}{2^\alpha} \sum_{r=0}^{\infty} \binom{\alpha}{r} \frac{1}{\alpha-2r} \sin\left\{ (\alpha-2r)\left(x - \frac{\pi}{2}\right) \right\}$$

$$\int_0^x \cos^{2m} x \, dx = \frac{1}{2^{2m-1}} \sum_{r=0}^{m-1} \frac{{}_{2m}C_r}{2m-2r} \sin\{(2m-2r)x\} + \frac{{}_{2m}C_m}{2^{2m}} x$$

$$\int_{\frac{\pi}{2}}^{x} \sin^{2m} x\, dx = \frac{1}{2^{2m-1}} \sum_{r=0}^{m-1} (-1)^{m-r} \frac{{}_{2m}C_r}{2m-2r} \sin\{(2m-2r)x\} + \frac{{}_{2m}C_m}{2^{2m}} \left(x - \frac{\pi}{2} \right)$$

21 Super Calculus of the product of many functions

21.1 Super Integrals of the product of many functions

The following theorems are drawn from the Theorem 20.2.1.

Theorem 21.1.1

Let p, r are positive numbers, m be a natural number, $f_k^{(r)}$ be the rth order derivative function of $f_k(x)$ $(k=1,2,\cdots,\lambda)$, $f_k^{<r>}$ be arbitrary r th order primitive function of $f_k(x)$ and $B(p,m)$ be the beta function. At this time, if there is a number a such that

$$f_1^{<r>}(a)=0 \quad r\in[0,m+p] \quad or \quad f_k^{(s)}(a)=0 \quad s\in[0,m+p-1] \text{ for at least one } k>1,$$

the following expression holds

$$\int_a^x \sim \int_a^x f_1 f_2 \cdots f_\lambda dx^p = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{\langle p+r_1 \rangle} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{1}{m+k_1} \binom{p-1}{k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$

$$\times \int_a^x \sim \int_a^x f_1^{\langle m+k_1 \rangle} f_2^{(m+k_1-k_2)} f_3^{(k_2-k_3)} \cdots f_\lambda^{(k_{\lambda-1})} dx^p$$

Theorem 21.1.2

Let p, r are positive numbers, m be a natural number, $f^{(r)}$ be the rth order derivative function of $f(x)$, $f^{<r>}$ be arbitrary rth order primitive function of $f(x)$ and $B(p,m)$ be the beta function. At this time, if there is a number a such that

$$f^{<r>}(a)=0 \quad r\in[0,m+p] \quad or \quad f^{(s)}(a)=0 \quad s\in[0,m+p-1]$$

the following expression holds for $\lambda = 2,3,4,\cdots$.

$$\int_a^x \sim \int_a^x f^\lambda dx^p = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{-p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f^{\langle p+r_1 \rangle} f^{(r_1-r_2)} \cdots f^{(r_{\lambda-1})} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{1}{m+k_1} \binom{p-1}{k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$

$$\times \int_a^x \sim \int_a^x f^{\langle m+k_1 \rangle} f^{(m+k_1-k_2)} f^{(k_1-k_2)} \cdots f^{(k_{\lambda-1})} dx^p$$

Example

$$\int_{-\infty}^x \sim \int_{-\infty}^x x^\alpha e^x \sin x\, dx^p = \sum_{r=0}^{m-1} \sum_{s=0}^{r} \binom{-p}{r} \binom{r}{s} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+p+r)} x^{\alpha+p+r} e^x \sin\left(x + \frac{s\pi}{2} \right) + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(p,m)} \sum_{r=0}^{\infty} \sum_{s=0}^{m+r} \binom{p-1}{r} \frac{{}_{m+r}C_s}{m+r} \int_{-\infty}^x \sim \int_{-\infty}^x \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m+r)} x^{\alpha+m+r} e^x \sin\left(x + \frac{s\pi}{2} \right) dx^p$$

$$\int_0^x \sim \int_0^x \log^3 x\, dx^p = \frac{\log x - \psi(1+p) - \gamma}{\Gamma(1+p)} x^p (\log x)^2$$

$$- 2x^p \log x \sum_{r=1}^{\infty} (-1)^r \binom{-p}{r} \frac{\log x - \psi(1+p+r) - \gamma}{\Gamma(1+p+r)} \Gamma(r)$$

$$+ x^p \sum_{r=2}^{\infty} \sum_{s=1}^{r-1} (-1)^r \binom{-p}{r} \binom{r}{s} \frac{\log x - \psi(1+p+r) - \gamma}{\Gamma(1+p+r)} \Gamma(r-s) \Gamma(s)$$

Super Integrls of $cos^m x$, $sin^m x$

$$\int_{a(p)}^{x} \sim \int_{a(0)}^{x} cos^{2m+1} x\, dx^p = \frac{1}{2^{2m}} \sum_{r=0}^{m} \frac{{}_{2m+1}C_r}{(2m-2r+1)^p} cos\left\{ (2m-2r+1)x - \frac{p\pi}{2} \right\}$$

$$\int_{a(p)}^{x} \sim \int_{a(0)}^{x} sin^{2m+1} x\, dx^p = \frac{1}{2^{2m}} \sum_{r=0}^{m} \frac{(-1)^{m-r} {}_{2m+1}C_r}{(2m-2r+1)^p} sin\left\{ (2m-2r+1)x - \frac{p\pi}{2} \right\}$$

Where, $a(s)$ $s \in [0,p]$ are zeros of lineal super primitives of $cos^{2m+1}x$ or $sin^{2m+1}x$.

21.2 Super Derivatives of the product of many functions

The following theorems are drawn from the Theorem 21.1.1.

Theorem 21.2.1

Let p, r are positive numbers, m be a natural number, $f_k^{(r)}$ be the rth order derivative function of $f_k(x)$ $(k=1,2,\cdots,\lambda)$, $f_k^{<r>}$ be arbitrary r th order primitive function of $f_k(x)$ and $B(p,m)$ be the beta function. At this time, if there is a number a such that

$$f_1^{<r>}(a) = 0 \quad r \in [0, m-p] \quad or \quad f_k^{(s)}(a) = 0 \quad s \in [0, m-p-1]$$ for at least one $k > 1$,

the following expression holds

$$(f_1 f_2 \cdots f_\lambda)^{(p)} = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f_1^{(p-r_1)} f_2^{(r_1-r_2)} \cdots f_\lambda^{(r_{\lambda-1})} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{1}{m+k_1} \binom{-p-1}{k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$

$$\times \left\{ f_1^{\langle m+k_1 \rangle} f_2^{(m+k_1-k_2)} f_3^{(k_2-k_3)} \cdots f_\lambda^{(k_{\lambda-1})} \right\}^{(p)}$$

Theorem 21.2.2

Let p, r are positive numbers, m be a natural number, $f^{(r)}$ be the rth order derivative function of $f(x)$, $f^{<r>}$ be arbitrary rth order primitive function of $f(x)$ and $B(p,m)$ be the beta function. At this time, if there is a number a such that

$$f^{<r>}(a) = 0 \quad r \in [0, m-p] \quad or \quad f^{(s)}(a) = 0 \quad s \in [0, m-p-1]$$

the following expression holds for $\lambda = 2,3,4,\cdots$.

$$(f^\lambda)^{(p)} = \sum_{r_1=0}^{m-1} \sum_{r_2=0}^{r_1} \cdots \sum_{r_{\lambda-1}=0}^{r_{\lambda-2}} \binom{p}{r_1} \binom{r_1}{r_2} \cdots \binom{r_{\lambda-2}}{r_{\lambda-1}} f^{\langle p+r_1 \rangle} f^{(r_1-r_2)} \cdots f^{(r_{\lambda-1})} + R_m^p$$

$$R_m^p = \frac{(-1)^m}{B(-p,m)} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{m+k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{\lambda-1}=0}^{k_{\lambda-2}} \frac{1}{m+k_1} \binom{-p-1}{k_1} \binom{m+k_1}{k_2} \binom{k_2}{k_3} \cdots \binom{k_{\lambda-2}}{k_{\lambda-1}}$$

$$\times \left\{ f^{\langle m+k_1 \rangle} f^{(m+k_1-k_2)} f^{(k_2-k_3)} \cdots f^{(k_{\lambda-1})} \right\}^{(p)}$$

Super Derivatives of $cos^m x$, $sin^m x$

$$\left(cos^m x\right)^{(p)} = \frac{1}{2^{m-1}} \sum_{r=0}^{m/2\downarrow} {}_mC_r\,(m-2r)^p\, cos\left\{(m-2r)x + \frac{p\pi}{2}\right\}$$

$$\left(sin^m x\right)^{(p)} = \frac{1}{2^{m-1}} \sum_{r=0}^{m/2\downarrow} {}_mC_r\,(m-2r)^p\, cos\left\{(m-2r)\left(x - \frac{\pi}{2}\right) + \frac{p\pi}{2}\right\}$$

22 Higher Derivative of Composition

22.1 Formulas of Higher Derivative of Composition

About the formula of the higher derivative of composition, the following formula was shown by ***Faà di Bruno*** about 150 years ago.

Formula 22.1.1 (Faà di Bruno)

Let $j_1, j_2, \cdots, j_n$ are non-negative integers. Let $g^{(n)}$, f_n are derivative functions and $B_{n,r}(f_1, f_2, \ldots)$ are ***the 2nd kind of Bell polynomials*** such that

$$g^{(n)} = g^{(n)}(f)\ ,\quad f_n = f^{(n)}(x) \qquad (n=1,2,3,\cdots)$$

$$B_{n,r}(f_1, f_2, \ldots, f_n) = \sum \frac{n!}{j_1!\ j_2! \ldots j_n!}\left(\frac{f_1}{1!}\right)^{j_1}\left(\frac{f_2}{2!}\right)^{j_2} \cdots \left(\frac{f_n}{n!}\right)^{j_n}$$

$$\left(j_1 + j_2 + \cdots + j_n = r \quad \& \quad j_1 + 2j_2 + \cdots + nj_n = n\right)$$

Then, the higher derivative function with respect to x of the composition $g\{f(x)\}$ is expressed as follows.

$$\{g\{f(x)\}\}^{(n)} = \sum_{r=1}^{n} g^{(r)} B_{n,r}(f_1, f_2, \ldots, f_n)$$

Next, the algorithm that generates $B_{n,r}(f_1, f_2, \ldots, f_n)$ without an omission is shown. And the derivatives up to the 8 th order of $z = g\{f(x)\}$ are calculated using this.

Even if the above algorithm is used, it is not so easy to obtain $B_{n,r}(f_1, f_2, \ldots, f_n)$.
However, when the core function $f(x)$ is the 1st degree, it becomes remarkably easy.

Formula 22.1.4

When $g^{(n)} = g^{(n)}(f)$, $f_n = f^{(n)}(x)$ $(n=1,2,3,\cdots)$, if $f(x)$ is the 1st degree, the higher derivative function with respect to x of the composition $g\{f(x)\}$ is expressed as follows.

$$\{g\{f(x)\}\}^{(n)} = g^{(n)} f_1^n$$

And, this can be easily enhanced to the super-differentiation in this case. That is, the following expression holds for $p > 0$. This is the grounds for which we have used "Linear form" since "**12 Super Deribative**" as a fait accompli .

$$\{g\{f(x)\}\}^{(p)} = g^{(p)} f_1^p$$

Next, trying the higher differentiation of some compositions, we obtain the following formulas.

$$\left\{e^{f(x)}\right\}^{(n)} = e^{f(x)} \sum_{r=1}^{n} B_{n,r}(f_1, f_2, \ldots, f_n)$$

$$\left\{e^{-f(x)}\right\}^{(n)} = e^{-f(x)} \sum_{r=1}^{n} (-1)^r B_{n,r}(f_1, f_2, \ldots, f_n)$$

$$\left\{g\left(e^x\right)\right\}^{(n)} = \sum_{r=1}^{n} S(n,r) g^{(r)} e^{rx} \quad , \quad S(n,r) = \frac{1}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} (r-s)^n$$

$$\{g(e^{-x})\}^{(n)} = (-1)^n \sum_{r=1}^{n} S(n,r) g^{(r)} e^{-rx}$$

$$\{\log f(x)\}^{(n)} = \sum_{r=1}^{n} (-1)^{r-1} (r-1)!\, B_{n,r}(f_1, f_2, \cdots, f_n) f^{-r}$$

$$\{g(\log x)\}^{(n)} = \sum_{r=1}^{n} g^{(r)} B_{n,r}\left(\frac{0!}{x}, -\frac{1!}{x^2}, \cdots, (-1)^{n-1}\frac{(n-1)!}{x^n} \right)$$

In $\{e^{f(x)}\}^{(n)}, \{e^{-f(x)}\}^{(n)}$, especially when $f(z) = \log\Gamma(z)$, we obtain the following formula. In addition, this formula was discovered by ***Masayuki Ui*** living in Yokohama City in early December 2016.

Formula 22.3.1 (Masayuki Ui)

When $\Gamma(z)$ is the gamma function, $\psi_n(z)$ is the polygamma function and $B_{n,r}(f_1, f_2, \cdots)$ are Bell polynomials , the following expressions hold.

$$\frac{d^n}{dz^n}\Gamma(z) = \Gamma(z) \sum_{k=1}^{n} B_{n,k}(\psi_0(z), \psi_1(z), \cdots, \psi_{n-1}(z))$$

$$\frac{d^n}{dz^n}\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z)} \sum_{k=1}^{n} (-1)^k B_{n,k}(\psi_0(z), \psi_1(z), \cdots, \psi_{n-1}(z))$$

23 Higher Integral of Composition

Formula 23.1.1

Let $f = f(x)$, $g^{<n>}$ be the lineal higher primitive function of $g(f)$ and $h, h^{(k)}$ are the functions of f such that

$$h = \frac{dx}{df} = \frac{1}{f^{(1)}} \quad , \quad h^{(k)} = \frac{d^k h}{df^k} \qquad k = 1, 2, 3, \cdots$$

Let S_k, M_k, r_k are the polynomials such taht

$$S_k = \sum_{r_{k1}=0}^{m_k-1} \binom{-1}{r_{k1}} \sum_{r_{k2}=0}^{r_{k1}} \binom{r_{k1}}{r_{k2}} \sum_{r_{k3}=0}^{r_{k2}} \binom{r_{k2}}{r_{k3}} \cdots \sum_{r_{kk}=0}^{r_{kk-1}} \binom{r_{kk-1}}{r_{kk}} \qquad k=1,2,\cdots,n$$

$$M_k = (-1)^{m_k} \sum_{r_{k2}=0}^{m_k} \binom{m_k}{r_{k2}} \sum_{r_{k3}=0}^{r_{k2}} \binom{r_{k2}}{r_{k3}} \sum_{r_{k4}=0}^{r_{k3}} \binom{r_{k3}}{r_{k4}} \cdots \sum_{r_{kk}=0}^{r_{kk-1}} \binom{r_{kk-1}}{r_{kk}} \qquad k=2,3,\cdots,n$$

$$R_{jk} = \sum_{i=k}^{j} r_{ik} \qquad j,k = 1,2,\cdots,n$$

And let a , f_a are the zeros of the lineal higher primitive functions of $g\{f(x)\}, gh$ respectively.
Then, the lineal higher integral with respect to x of the composition $g\{f(x)\}$ is expressed as follows.

$$\int_a^x \cdots \int_a^x \{g(f(x))\} dx^n = S_1 S_2 \cdots S_n\, g^{\langle n+R_{n1}\rangle} h^{(R_{n1}-R_{n2})} \cdots h^{(R_{nn-1}-R_{nn})} h^{(R_{nn})} + R_{m_1}^n$$

$$R_{m_1}^n = (-1)^{m_1} \int_{f_a}^{f} \left(\int_{f_a}^{f} \left(\int_{f_a}^{f} \cdots \left(\int_{f_a}^{f} g^{\langle m_1\rangle} h^{(m_1)} df \right) \cdots hdf \right) hdf \right) hdf \qquad \text{(n-fold nest)}$$

$$+ S_1 M_2 \int_{f_a}^{f} \left(\int_{f_a}^{f} \cdots \left(\int_{f_a}^{f} g^{\langle 1+R_{11}+m_2\rangle} h^{(R_{11}-R_{22}+m_2)} h^{(R_{22})} df \right) \cdots hdf \right) hdf$$

$$+ S_1 S_2 M_3 \int_{f_a}^{f} \cdots \left(\int_{f_a}^{f} g^{\langle 2+R_{21}+m_3\rangle} h^{(R_{21}-R_{32}+m_3)} h^{(R_{32}-R_{33})} h^{(R_{33})} df \right) \cdots hdf$$

$$\vdots$$

$$+ S_1 S_2 \cdots S_{n-1} M_n \int_{f_a}^{f} g^{\langle n-1+R_{n-11}+m_n \rangle} h^{(R_{n-11}-R_{n2}+m_n)} h^{(R_{n2}-R_{n3})} \cdots h^{(R_{nn})} df$$

If it writes down up to the 3rd order without using S_k, M_k, r_k, it is s follows.

$$\int_a^x \{g(f(x))\} dx = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} g^{\langle 1+r_{11} \rangle} h^{(r_{11})} + R_{m_1}^1$$

$$R_{m_1}^1 = (-1)^{m_1} \int_{f_a}^{f} g^{\langle m_1 \rangle} h^{(m_1)} df$$

$$\int_a^x \int_a^x \{g(f(x))\} dx^2 = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} \sum_{r_{21}=0}^{m_2-1} \sum_{r_{22}=0}^{r_{21}} \binom{-1}{r_{21}} \binom{r_{21}}{r_{22}} g^{\langle 2+r_{11}+r_{21} \rangle} h^{(r_{11}+r_{21}-r_{22})} h^{(r_{22})} + R_{m_1}^2$$

$$R_{m_1}^2 = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} (-1)^{m_2} \sum_{r_{22}=0}^{m_2} \binom{m_2}{r_{22}} \int_{f_a}^{f} g^{\langle 1+r_{11}+m_2 \rangle} h^{(r_{11}-r_{22}+m_2)} h^{(r_{22})} df + (-1)^{m_1} \int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle m_1 \rangle} h^{(m_1)} df \right) h df$$

$$\int_a^x \int_a^x \int_a^x \{g(f(x))\} dx^3 = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} \sum_{r_{21}=0}^{m_2-1} \sum_{r_{22}=0}^{r_{21}} \binom{-1}{r_{21}} \binom{r_{21}}{r_{22}} \sum_{r_{31}=0}^{m_3-1} \sum_{r_{32}=0}^{r_{31}} \sum_{r_{33}=0}^{r_{32}} \binom{-1}{r_{31}} \binom{r_{31}}{r_{32}} \binom{r_{32}}{r_{33}}$$
$$\times g^{\langle 3+r_{11}+r_{21}+r_{31} \rangle} h^{(r_{11}+r_{21}+r_{31}-r_{22}-r_{32})} h^{(r_{22}+r_{32}-r_{33})} h^{(r_{33})} + R_{m_1}^3$$

$$R_{m_1}^3 = \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} \sum_{r_{21}=0}^{m_2-1} \sum_{r_{22}=0}^{r_{21}} \binom{-1}{r_{21}} \binom{r_{21}}{r_{22}} (-1)^{m_3} \sum_{r_{32}=0}^{m_3} \sum_{r_{33}=0}^{r_{32}} \binom{m_3}{r_{32}} \binom{r_{32}}{r_{33}}$$
$$\times \int_{f_a}^{f} g^{\langle 2+r_{11}+r_{21}+m_3 \rangle} h^{(r_{11}+r_{21}-r_{22}-r_{32}+m_3)} h^{(r_{22}+r_{32}-r_{33})} h^{(r_{33})} df$$
$$+ \sum_{r_{11}=0}^{m_1-1} \binom{-1}{r_{11}} (-1)^{m_2} \sum_{r_{22}=0}^{m_2} \binom{m_2}{r_{22}} \int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle 1+r_{11}+m_2 \rangle} h^{(r_{11}-r_{22}+m_2)} h^{(r_{22})} df \right) h df$$
$$+ (-1)^{m_1} \int_{f_a}^{f} \left(\int_{f_a}^{f} \left(\int_{f_a}^{f} g^{\langle m_1 \rangle} h^{(m_1)} df \right) h df \right) h df$$

Next, trying the higher integration of some compositions, we obtain the following formulas

$$\int_{\sqrt[\beta]{c}}^{x} \cdots \int_{\sqrt[\beta]{c}}^{x} (x^\beta - c)^\alpha dx^n = (x^\beta - c)^\alpha \frac{x^n}{\beta^n} S_1 S_2 \cdots S_n \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+n+R_{n1})} \left(1 - \frac{1}{x^\beta}\right)^{n+R_{n1}}$$
$$\times \frac{\Gamma(1/\beta)}{\Gamma(1/\beta - R_{n1}+R_{n2})} \cdots \frac{\Gamma(1/\beta)}{\Gamma(1/\beta - R_{nn-1}+R_{nn})} \frac{\Gamma(1/\beta)}{\Gamma(1/\beta - R_{nn})}$$

$$\int_0^x \cdots \int_0^x (\log x)^\alpha dx^n = x^n \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} (-1)^{\sum_{k=1}^{n} r_k} \prod_{k=1}^{n} k^{r_k} \frac{\Gamma(1+\alpha)}{\Gamma\left(1+\alpha+n+\sum_{k=1}^{n} r_k\right)} (\log x)^{\alpha+n+\sum_{k=1}^{n} r_k}$$
$$+ (-1)^\alpha \frac{\Gamma(1+\alpha)}{n!} \sum_{k=1}^{n} (-1)^{k-1} {}_nC_k \frac{x^{n-k}}{k^\alpha}$$

$$\int_0^x \cdots \int_0^x \frac{1}{\log x} dx^n = -x^n \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} (-1)^{\sum_{k=1}^{n} r_k} \prod_{k=1}^{n} k^{r_k} \frac{\psi\left(n+\sum_{k=1}^{n} r_k\right)}{\Gamma\left(n+\sum_{k=1}^{n} r_k\right)} (\log x)^{n-1+\sum_{k=1}^{n} r_k}$$

$$+\frac{1}{(n-1)!}\left\{(x-1)^{n-1}\log|\log x| - \sum_{r=0}^{n-1}(-1)^{n-r}{}_{n-1}C_r x^r \log(n-r)\right\}$$

$$\int_0^x\cdots\int_0^x \log|\log x|\,dx^n = -x^n\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}(-1)^{\sum_{k=1}^{n}r_k}\prod_{k=1}^{n}k^{r_k}\frac{\psi\left(1+n+\sum_{k=1}^{n}r_k\right)}{\Gamma\left(1+n+\sum_{k=1}^{n}r_k\right)}(\log x)^{n+\sum_{k=1}^{n}r_k}$$

$$+\frac{1}{n!}\left\{(\log|\log x|-\gamma)(x-1)^n+\sum_{r=1}^{n}(-1)^r{}_nC_r(\gamma+\log r)x^{n-r}\right\}$$

Formula 23.1.1 is so complicated and is not applicable to any composition. The 1st, the inverse function of the core function must be known. The 2nd, the higher primitive function of the enclosing function $g(f)$ must have the property such as $\lim_{n\to\infty} g^{<n>}(f)=c$. Considering these, there are not many compositions that Formula 23.1.1 is applicable. However, when the core function $f(x)$ is the 1st degree, it becomes remarkably easy.

Formula 23.1.2

When $f(x)=cx+d$,

$$\int_a^x\cdots\int_a^x\{g(f(x))\}dx^n=\left(\frac{1}{c}\right)^n\int_{f_a}^f\cdots\int_{f_a}^f g(f)\,df^n$$

And, this can be easily enhanced to the super integral in this case. That is, the following expression holds for $p>0$. This is the grounds for which we have used "Linear form" since " **07 Super Integral** " as a fait accompli.

$$\int_a^x\sim\int_a^x\{g(f(x))\}dx^p=\left(\frac{1}{c}\right)^p\int_{f_a}^f\sim\int_{f_a}^f g(f)\,df^p \qquad f(x)=cx+d$$

24 Sugioka's Theorem on the Series of Higher (Repeated) Integrals

Mikio Sugioka showed the following in his work " 数学の研究 ".

1. The series of the higher integral of a function $f(x)$ results in one integral of $e^{\pm x}$, $\sin x, \cos x, \sinh x, \cosh x$,
2. The series of the higher integral of a function $f(x)$ is expanded into the series of the higher integrals of $e^{\pm x}$, etc.

These are introduced and proved in this chapter. In this summary, we assume $\lim_{m\to\infty}\int_a^x\cdots\int_a^x f(x)\,dx^m=0$. Also, we describe the n th order differential coefficient at a with $f_a^{(n)}\ \left(=f^{(n)}(x)\,|_{x=a}\right)$

(1) Sum of the Series of Higher Integrals

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^r = e^x\int_a^x f(x)e^{-x}dx$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^r = e^{-x}\int_a^x f(x)e^{x}dx$$

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \frac{1}{2}\left\{e^x\int_a^x f(x)e^{-x}dx + e^{-x}\int_a^x f(x)e^{x}dx\right\}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \cos x\int_a^x f(x)\cos x\,dx + \sin x\int_a^x f(x)\sin x\,dx$$

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \frac{1}{2}\left\{e^x\int_a^x f(x)e^{-x}dx - e^{-x}\int_a^x f(x)e^{x}dx\right\}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sin x\int_a^x f(x)\cos x\,dx - \cos x\int_a^x f(x)\sin x\,dx$$

Example

$$\int_a^x e^x dx + \int_a^x\int_a^x e^x dx^2 + \int_a^x\int_a^x\int_a^x e^x dx^3 + \int_a^x\cdots\int_a^x e^x dx^4 + \cdots = e^x\int_a^x e^x e^{-x} dx = e^x(x-a)$$

$$\int_a^x e^x dx - \int_a^x\int_a^x\int_a^x e^x dx^3 + \int_a^x\cdots\int_a^x e^x dx^5 - \int_a^x\cdots\int_a^x e^x dx^7 + - \cdots$$

$$= \cos x\int_a^x e^x\cos x\,dx + \sin x\int_a^x e^x\sin x\,dx = \frac{e^x}{2} - \frac{e^a}{2}\{\cos(x-a) - \sin(x-a)\}$$

$$\int_0^x\int_0^x x^2 dx^2 - \int_0^x\cdots\int_0^x x^2 dx^4 + \int_0^x\cdots\int_0^x x^2 dx^6 - \int_0^x\cdots\int_0^x x^2 dx^8 + - \cdots$$

$$= \sin x\int_0^x x^2\cos x\,dx - \cos x\int_0^x x^2\sin x\,dx = x^2 - 2 + 2\cos x$$

(2) Integrals Series Expansion

$$\sum_{r=0}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^r = \sum_{r=0}^{\infty} f_a^{(r)}\sum_{s=0}^{\infty}\frac{(x-a)^{s*r}}{(s+r)!} = \sum_{r=0}^{\infty} f_a^{(r)}\int_a^x\cdots\int_a^x e^{x-a}dx^r$$

$$\sum_{r=0}^{\infty}(-1)^r\int_a^x\cdots\int_a^x f(x)\,dx^r = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}(-1)^s\frac{(x-a)^{s+r}}{(s+r)!} = \sum_{r=0}^{\infty} f^{(r)}(a)\int_a^x\cdots\int_a^x e^{-(x-a)}dx^r$$

$$\sum_{r=1}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \sum_{r=0}^{\infty} f_a^{(r)}\sum_{s=0}^{\infty}\frac{(x-a)^{2s+1+r}}{(2s+1+r)!} = \sum_{r=0}^{\infty} f_a^{(r)}\int_a^x\cdots\int_a^x \sinh(x-a)\,dx^r$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \sum_{r=0}^{\infty} f_a^{(r)}\sum_{s=0}^{\infty}(-1)^s\frac{(x-a)^{2s+1+r}}{(2s+1+r)!} = \sum_{r=0}^{\infty} f_a^{(r)}\int_a^x\cdots\int_a^x \sin(x-a)\,dx^r$$

$$\sum_{r=0}^{\infty}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f_a^{(r)}\sum_{s=0}^{\infty}\frac{(x-a)^{2s+r}}{(2s+r)!} = \sum_{r=0}^{\infty} f_a^{(r)}\int_a^x\cdots\int_a^x \cosh(x-a)\,dx^{2r}$$

$$\sum_{r=0}^{\infty}(-1)^r\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f_a^{(r)}\sum_{s=0}^{\infty}(-1)^s\frac{(x-a)^{2s+r}}{(2s+r)!} = \sum_{r=0}^{\infty} f_a^{(r)}\int_a^x\cdots\int_a^x \cos(x-a)\,dx^r$$

Example

$$\log x + \int_1^x \log x\,dx + \int_1^x\int_1^x \log x\,dx^2 + \int_1^x\int_1^x\int_1^x \log x\,dx^3 + \cdots = \sum_{r=1}^{\infty}(-1)^{r-1}(r-1)!\sum_{s=0}^{\infty}\frac{(x-1)^{s+r}}{(s+r)!}$$

$$= 0!\int_1^x e^{x-1}dx - 1!\int_1^x\int_1^x e^{x-1}dx^2 + 2!\int_1^x\int_1^x\int_1^x e^{x-1}dx^3 - 3!\int_1^x\cdots\int_1^x e^{x-1}dx^4 + - \cdots$$

$$\int_0^x \sec x\,dx - \int_0^x\int_0^x\int_0^x \sec x\,dx^3 + \int_0^x\cdots\int_0^x \sec x\,dx^5 - \int_0^x\cdots\int_0^x \sec x\,dx^7 + - \cdots$$

$$= \sum_{r=0}^{\infty}|E_{2r}|\sum_{s=0}^{\infty}(-1)^s\frac{x^{2s+1+2r}}{(2s+1+2r)!} \qquad E_{2r}:\ \textit{Euler Number}$$

$$= \sin x + \int_0^x\int_0^x \sin x\,dx^2 + 5\int_0^x\cdots\int_0^x \sin x\,dx^4 + 61\int_0^x\cdots\int_0^x \sin x\,dx^6 + \cdots$$

$$\tan^{-1}x + \int_1^x\int_1^x \tan^{-1}x\,dx^2 + \int_1^x\cdots\int_1^x \tan^{-1}x\,dx^4 + \int_1^x\cdots\int_1^x \tan^{-1}x\,dx^6 + \cdots$$

$$= \frac{\pi}{4}\sum_{s=0}^{\infty}\frac{(x-1)^{2s}}{(2s)!} + \sum_{r=1}^{\infty}\frac{(r-1)!}{2^{r/2}}\sin\left(\frac{3r\pi}{4}\right)\sum_{s=0}^{\infty}\frac{(x-1)^{2s+r}}{(2s+r)!}$$

$$= \frac{\pi}{4}\cosh(x-1) + \frac{1}{2}\int_1^x \cosh(x-1)\,dx - \frac{1}{2}\int_1^x\!\!\int_1^x \cosh(x-1)\,dx^2 + \cdots$$

(3) Series of Higher Integrals with coefficients

$$\sum_{r=1}^{\infty} r\int_a^x\!\cdots\!\int_a^x f(x)\,dx^r = e^x\int_a^x f(x)e^{-x}dx + e^x\int_a^x\!\!\int_a^x f(x)e^{-x}dx^2$$

$$\sum_{r=1}^{\infty}(-1)^{r-1} r\int_a^x\!\cdots\!\int_a^x f(x)\,dx^r = e^{-x}\int_a^x f(x)e^{x}dx - e^{-x}\int_a^x\!\!\int_a^x f(x)e^{x}dx^2$$

Example

$$1\int_a^x dx + 2\int_a^x\!\!\int_a^x dx^2 + 3\int_a^x\!\!\int_a^x\!\!\int_a^x dx^3 + 4\int_a^x\!\cdots\!\int_a^x dx^4 + \cdots = e^x\int_a^x e^{-x}dx + e^x\int_a^x\!\!\int_a^x e^{-x}dx^2$$

$$= e^{x-a}(x-a)$$

$$1\int_0^x \log x\,dx - 2\int_0^x\!\!\int_0^x \log x\,dx^2 + 3\int_0^x\!\!\int_0^x\!\!\int_0^x \log x\,dx^3 - 4\int_0^x\!\cdots\!\int_0^x \log x\,dx^4 + - \cdots$$

$$= e^{-x}\int_0^x \log x\,e^x dx - e^{-x}\int_a^x\!\!\int_a^x \log x\,e^x dx^2$$

$$= e^{-x}x\{Ei(x) - \gamma\} + e^{-x} - 1$$

25 Series of Higher Integral with Geometric Coefficients

This chapter is a generalization of " **24 Sugioka's Theorem on the Series of Higher Integral** "

In this summary, we introduce the formulas in the case of $\lim_{m\to\infty} c^m\int_a^x\!\cdots\!\int_a^x f(x)\,dx^m = 0 \quad (c>0)$.

(1) Sum of the Series of Higher Integrals with Geometric Coefficient

$$\sum_{r=1}^{\infty} c^r\int_a^x\!\cdots\!\int_a^x f(x)\,dx^r = ce^{cx}\int_a^x f(x)e^{-cx}dx$$

$$\sum_{r=1}^{m}(-1)^{r-1}c^r\int_a^x\!\cdots\!\int_a^x f(x)\,dx^r = ce^{-cx}\int_a^x f(x)e^{cx}dx$$

$$\sum_{r=1}^{\infty} c^{2r-1}\int_a^x\!\cdots\!\int_a^x f(x)\,dx^{2r-1} = \frac{c}{2}\left\{e^{cx}\int_a^x f(x)e^{-cx}dx + e^{-cx}\int_a^x f(x)e^{cx}dx\right\}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}c^{2r-1}\int_a^x\!\cdots\!\int_a^x f(x)\,dx^{2r-1} = c\cos cx\int_a^x f(x)\cos cx\,dx + c\sin cx\int_a^x f(x)\sin cx\,dx$$

$$\sum_{r=1}^{\infty} c^{2r}\int_a^x\!\cdots\!\int_a^x f(x)\,dx^{2r} = \frac{c}{2}\left\{e^{cx}\int_a^x f(x)e^{-cx}dx - e^{-cx}\int_a^x f(x)e^{cx}dx\right\}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}c^{2r}\int_a^x\!\cdots\!\int_a^x f(x)\,dx^{2r} = c\sin cx\int_a^x f(x)\cos cx\,dx - c\cos cx\int_a^x f(x)\sin cx\,dx$$

Example

$$c^1\int_a^x e^x dx + c^2\int_a^x\!\!\int_a^x e^x dx^2 + c^3\int_a^x\!\!\int_a^x\!\!\int_a^x e^x dx^3 + c^4\int_a^x\!\cdots\!\int_a^x e^x dx^4 + \cdots$$

$$= ce^{cx}\int_a^x e^x e^{-cx}dx = \frac{ce^{cx}\{e^{(1-c)x} - e^{(1-c)a}\}}{1-c}$$

$$c^1\int_a^x e^x dx - c^3\int_a^x\int_a^x\int_a^x e^x dx^3 + c^5\int_a^x\cdots\int_a^x e^x dx^5 - c^7\int_a^x\cdots\int_a^x e^x dx^7 +- \cdots$$

$$= c\cos cx\int_a^x e^x\cos cx\,dx + c\sin cx\int_a^x e^x\sin cx\,dx$$

$$= \frac{ce^x}{1+c^2} - \frac{ce^a}{1+c^2}\left[\cos\{c(x-a)\} - c\sin\{c(x-a)\}\right]$$

$$c^2\int_0^x\int_0^x x^2 dx^2 - c^4\int_0^x\cdots\int_0^x x^2 dx^4 + c^6\int_0^x\cdots\int_0^x x^2 dx^6 - c^8\int_0^x\cdots\int_0^x x^2 dx^8 +- \cdots$$

$$= c\sin cx\int_0^x x^2\cos cx\,dx - c\cos cx\int_0^x x^2\sin cx\,dx = \frac{c^2x^2 - 2 + 2\cos cx}{c^2}$$

(2) Series of Higher Integrals with coefficients

$$\sum_{r=1}^{\infty} rc^r\int_a^x\cdots\int_a^x f(x)\,dx^r = ce^{cx}\int_a^x f(x)e^{-cx}dx + c^2e^{cx}\int_a^x\int_a^x f(x)e^{-cx}dx^2$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}rc^r\int_a^x\cdots\int_a^x f(x)\,dx^r = ce^{-cx}\int_a^x f(x)e^{cx}dx - c^2e^{-cx}\int_a^x\int_a^x f(x)e^{cx}dx^2$$

Example

$$1c^1\int_a^x dx + 2c^2\int_a^x\int_a^x dx^2 + 3c^3\int_a^x\int_a^x\int_a^x dx^3 + 4c^4\int_a^x\cdots\int_a^x dx^4 + \cdots$$

$$= ce^{cx}\int_a^x e^{-cx}dx + ce^{cx}\int_a^x\int_a^x e^{-cx}dx^2 = ce^{c(x-a)}(x-a)$$

(3) Calculation by Double Series

$$\sum_{r=0}^{\infty} c^r\int_a^x\cdots\int_a^x f(x)\,dx^r = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}\frac{c^s(x-a)^{s+r}}{(s+r)!}$$

$$\sum_{r=0}^{\infty}(-1)^r c^r\int_a^x\cdots\int_a^x f(x)\,dx^r = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}(-1)^s\frac{c^s(x-a)^{s+r}}{(s+r)!}$$

$$\sum_{r=1}^{\infty} c^{2r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}\frac{c^{2s+1}(x-a)^{2s+1+r}}{(2s+1+r)!}$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}c^{2r-1}\int_a^x\cdots\int_a^x f(x)\,dx^{2r-1} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}(-1)^s\frac{c^{2s+1}(x-a)^{2s+1+r}}{(2s+1+r)!}$$

$$\sum_{r=0}^{\infty} c^{2r}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}\frac{c^{2s}(x-a)^{2s+r}}{(2s+r)!}$$

$$\sum_{r=0}^{\infty}(-1)^r c^{2r}\int_a^x\cdots\int_a^x f(x)\,dx^{2r} = \sum_{r=0}^{\infty} f^{(r)}(a)\sum_{s=0}^{\infty}(-1)^s\frac{c^{2s}(x-a)^{2s+r}}{(2s+r)!}$$

Example

$$\log x + c^1\int_1^x \log x\,dx + c^2\int_1^x\int_1^x \log x\,dx^2 + c^3\int_1^x\int_1^x\int_1^x \log x\,dx^3 + \cdots$$

$$= \sum_{r=1}^{\infty}(-1)^{r-1}(r-1)!\sum_{s=0}^{\infty}\frac{c^s(x-1)^{s+r}}{(s+r)!}$$

$$c^1\int_0^x \sec x\,dx - c^3\int_0^x\int_0^x\int_0^x \sec x\,dx^3 + c^5\int_0^x\cdots\int_0^x \sec x\,dx^5 - c^7\int_0^x\cdots\int_0^x \sec x\,dx^7 +- \cdots$$

$$= \sum_{r=0}^{\infty} |E_{2r}| \sum_{s=0}^{\infty} (-1)^s \frac{c^{2s+1} x^{2s+1+2r}}{(2s+1+2r)!} \qquad E_{2r}: \text{ Euler Number}$$

$$\tan^{-1}x + c^2\int_1^x\int_1^x \tan^{-1}x\,dx^2 + c^4\int_1^x\cdots\int_1^x \tan^{-1}x\,dx^4 + c^6\int_1^x\cdots\int_1^x \tan^{-1}x\,dx^6 + \cdots$$

$$= \frac{\pi}{4}\sum_{s=0}^{\infty}\frac{c^{2s}(x-1)^{2s}}{(2s)!} + \sum_{r=1}^{\infty}\frac{(r-1)!}{2^{r/2}}\sin\left(\frac{3r\pi}{4}\right)\sum_{s=0}^{\infty}\frac{c^{2s}(x-1)^{2s+r}}{(2s+r)!}$$

26 Higher and Super Calculus of Zeta Function etc

In this chapter, n is a natural number, p is a complex number, $\Gamma(p)$ is gamma function, $\psi(p)$ is digamma function, H_n is a harmonic number and γ_r is a Stieltjes constant defined by the following equation.

$$\gamma_r = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{(\log k)^r}{k} - \frac{(\log n)^{r+1}}{r+1}\right\}$$

Then, the following higher calculus and super calculus are obtain. These are all lineal.

26.1 Higher and Super Calculus of Riemann Zeta Function

$$\zeta^{<n>}(z) = \frac{(z-1)^{n-1}}{(n-1)!}\{\log(z-1) - H_{n-1}\} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r+n}}{(r+n)!}$$

$$\zeta^{<p>}(z) = \frac{\log(z-1) - \psi(p) - \gamma_0}{\Gamma(p)}(z-1)^{p-1} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r+p}}{\Gamma(1+r+p)}$$

$$\zeta^{(n)}(z) = \frac{(-1)^{-n}n!}{(z-1)^{n+1}} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r-n}}{\Gamma(1+r-n)}$$

$$\zeta^{(p)}(z) = \frac{\log(z-1) - \psi(-p) - \gamma_0}{\Gamma(-p)}(z-1)^{-p-1} + \sum_{r=0}^{\infty}(-1)^r\gamma_r\frac{(z-1)^{r-p}}{\Gamma(1+r-p)}$$

26.2 Higher and Super Calculus of Dirichlet Lambda Function

$$\lambda^{<n>}(z) = \frac{(z-1)^{n-1}}{2(n-1)!}\{\log(z-1) - H_{n-1}\}$$
$$+ \frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r + \frac{\log^{r+1}2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(\log 2)^{r-s}\right\}\frac{(z-1)^{r+n}}{(r+n)!}$$

$$\lambda^{<p>}(z) = \frac{\log(z-1) - \psi(p) - \gamma_0}{2\Gamma(p)}(z-1)^{p-1}$$
$$+ \frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r + \frac{\log^{r+1}2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(\log 2)^{r-s}\right\}\frac{(z-1)^{r+p}}{\Gamma(1+r+p)}$$

$$\lambda^{(n)}(z) = \frac{(-1)^{-n}n!}{2(z-1)^{n+1}}$$
$$+ \frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r + \frac{\log^{r+1}2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(\log 2)^{r-s}\right\}\frac{(z-1)^{r-n}}{\Gamma(1+r-n)}$$

$$\lambda^{(p)}(z) = \frac{\log(z-1) - \psi(-p) - \gamma_0}{2\Gamma(-p)}(z-1)^{-p-1}$$
$$+ \frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\left\{\gamma_r + \frac{\log^{r+1}2}{r+1} - \sum_{s=0}^{r-1}\binom{r}{s}\gamma_s(\log 2)^{r-s}\right\}\frac{(z-1)^{r-p}}{\Gamma(1+r-p)}$$

26.3 Higher and Super Calculus of Dirichlet Eta Function

Formula 26.3.1h (Higher Integral)

$$\eta^{<n>}(z) = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^{-n} r}{r^z} \qquad Re(z) > 0$$

$$\eta^{<n>}(z) = \frac{z^n}{n!} + (-1)^n \sum_{k=2}^{\infty} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^{-n} r}{r^z}$$

Formula 26.3.1s (Super Integral)

$$\eta^{<p>}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i} \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^{-p} r}{r^z} \qquad Re(z) > 0$$

$$\eta^{<p>}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i} \sum_{k=2}^{\infty} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^{-p} r}{r^z}$$

Formula 26.3.2h (Higher Derivative)

$$\eta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^{-n} \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\log^n r}{r^z} \qquad Re(z) > 0$$

$$\eta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^{-n} \sum_{k=2}^{\infty} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^n r}{r^z}$$

Formula 26.3.2s (Super Derivative)

$$\eta^{(p)}(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i} \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\log^p r}{r^z} \qquad Re(z) > 0$$

$$\eta^{(p)}(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i} \sum_{k=2}^{\infty} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^p r}{r^z}$$

26.4 Higher and Super Calculus of Dirichlet Beta Function

Formula 26.4.1h (Higher Integral)

$$\beta^{<n>}(z) = \frac{z^n}{n!} + (-1)^n \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^{-n}(2r-1)}{(2r-1)^z} \qquad Re(z) > 0$$

$$\beta^{<n>}(z) = \frac{z^n}{n!} + (-1)^{-n} \sum_{k=2}^{\infty} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^{-n}(2r-1)}{(2r-1)^z}$$

Formula 26.4.1s (Super Integral)

$$\beta^{<p>}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i} \sum_{r=2}^{\infty} (-1)^{r-1} \frac{\log^{-p}(2r-1)}{(2r-1)^z} \qquad Re(z) > 0$$

$$\beta^{<p>}(z) = \frac{z^p}{\Gamma(1+p)} + e^{p\pi i} \sum_{k=2}^{\infty} \sum_{r=2}^{k} \frac{(-1)^{r-1}}{2^{k+1}} \binom{k}{r} \frac{\log^{-p}(2r-1)}{(2r-1)^z}$$

Formula 26.4.2h (Higher Derivative)

$$\beta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^{-n}\sum_{r=2}^{\infty}(-1)^{r-1}\frac{\log^{n}(2r-1)}{(2r-1)^{z}} \qquad Re(z)>0$$

$$\beta^{(n)}(z) = \frac{z^{-n}}{\Gamma(1-n)} + (-1)^{-n}\sum_{k=2}^{\infty}\sum_{r=2}^{k}\frac{(-1)^{r-1}}{2^{k+1}}\binom{k}{r}\frac{\log^{n}(2r-1)}{(2r-1)^{z}}$$

Formula 26.4.2s (Super Derivative)

$$\beta^{(p)}(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i}\sum_{r=2}^{\infty}(-1)^{r-1}\frac{\log^{p}(2r-1)}{(2r-1)^{z}} \qquad Re(z)>0$$

$$\beta^{(p)}(z) = \frac{z^{-p}}{\Gamma(1-p)} + e^{-p\pi i}\sum_{k=2}^{\infty}\sum_{r=2}^{k}\frac{(-1)^{r-1}}{2^{k+1}}\binom{k}{r}\frac{\log^{p}(2r-1)}{(2r-1)^{z}}$$

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Kano Kono

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