Summary

1 Zeta Generating Functions

Both of hyperbolic functions and trigonometric functions can be expanded to Fourier series and Taylor series. And if the termwise higher order integration of these is carried out, Riemann Zeta Functions are obtained.

Where, these are automorphisms which are expressed by lower zetas. However, in this chapter, we stop those so far. The work that obtain the non-automorphism formulas by removing lower zetas from these are performed subsequent to Chapter2 .

In this chapter, we obtain the following polynomials from the zeta generating functions

Where, Riemann Zeta, Dirichlet Eta and Dirichlet Lambda are as follows.

$$\zeta(x)=\sum_{r=1}^{\infty}\frac{1}{r^x}\ ,\ \eta(x)=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^x}\ ,\ \lambda(x)=\sum_{r=1}^{\infty}\frac{1}{(2r-1)^x}$$

Bernoulli numbers and Euler numbers are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$$

$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$$

Harmonic number is $H_s=\sum_{t=1}^{s}1/t=\psi(1+s)+\gamma$

$$\zeta(n)=\frac{(-x)^{n-1}}{(n-1)!}\left(\log x-H_{n-1}\right)+\frac{(-1)^n}{2}\frac{x^n}{n!}+\sum_{r=1}^{\infty}\frac{e^{-rx}}{r^n}$$
$$-(-1)^n\sum_{r=1}^{\infty}\frac{B_{2r}x^{2r+n-1}}{2r(2r+n-1)!}-\sum_{s=1}^{n-2}\frac{(-1)^s x^s}{s!}\zeta(n-s)$$

$$\eta(n)=-\frac{(-1)^n}{2}\frac{x^n}{n!}-\sum_{r=1}^{\infty}(-1)^r\frac{e^{-rx}}{r^n}$$
$$+(-1)^n\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)B_{2r}x^{2r+n-1}}{2r(2r+n-1)!}-\sum_{s=1}^{n-1}\frac{(-1)^s x^s}{s!}\eta(n-s)$$

$$\lambda(n)=\frac{(-1)^{n-1}}{2}\frac{x^{n-1}}{(n-1)!}\left(\log\frac{x}{2}-H_{n-1}\right)+\sum_{r=1}^{\infty}\frac{e^{-(2r-1)x}}{(2r-1)^n}$$
$$+\frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)B_{2r}x^{2r+n-1}}{2r(2r+n-1)!}-\sum_{s=1}^{n-2}\frac{(-1)^s x^s}{s!}\lambda(n-s)$$

$$\sum_{r=1}^{\infty}\frac{\cos rx}{r^{2n-1}}-\frac{(-1)^n x^{2n-2}}{(2n-2)!}\left(\log x-H_{2n-2}\right)+(-1)^n\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+2n-2}}{2r(2r+2n-2)!}$$
$$=\sum_{s=0}^{n-2}\frac{(-1)^s x^{2s}}{(2s)!}\zeta(2n-1-2s)$$

$$\sum_{r=1}^{\infty}\frac{\sin rx}{r^{2n}}-\frac{(-1)^n x^{2n-1}}{(2n-1)!}\left(\log x-H_{2n-1}\right)+(-1)^n\sum_{r=1}^{\infty}\frac{|B_{2r}|x^{2r+2n-1}}{2r(2r+2n-1)!}$$
$$=-\sum_{s=1}^{n-1}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\zeta(2n+1-2s)$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\cos rx}{r^{2n-1}}-(-1)^n\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r+2n-2}}{2r(2r+2n-2)!}=\sum_{s=0}^{n-1}\frac{(-1)^s x^{2s}}{(2s)!}\eta(2n-1-2s)$$

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{\sin rx}{r^{2n}}-(-1)^n\sum_{r=1}^{\infty}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r+2n-1}}{2r(2r+2n-1)!}=-\sum_{s=1}^{n}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\eta(2n+1-2s)$$

$$\sum_{r=1}^{\infty}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n-1}}-\frac{(-1)^n x^{2n-2}}{2(2n-2)!}\left(\log\frac{x}{2}-H_{2n-2}\right)-\frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|\,x^{2r+2n-2}}{2r(2r+2n-2)!}$$
$$=\sum_{s=0}^{n-2}\frac{(-1)^s x^{2s}}{(2s)!}\lambda(2n-1-2s)$$

$$\sum_{r=1}^{\infty}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n}}-\frac{(-1)^n x^{2n-1}}{2(2n-1)!}\left(\log\frac{x}{2}-H_{2n-1}\right)-\frac{(-1)^n}{2}\sum_{r=1}^{\infty}\frac{\left(2^{2r}-2\right)|B_{2r}|\,x^{2r+2n-1}}{2r(2r+2n-1)!}$$
$$=-\sum_{s=1}^{n-1}\frac{(-1)^s x^{2s-1}}{(2s-1)!}\lambda(2n+1-2s)$$

Furthermore, if the termwise higher order differentiation of the Fourier series of each family of tanh, cot and tan are carried out, the following expressions are obtained.

$$\zeta(-n)=\frac{1}{2^{n+1}\left(1-2^{n+1}\right)}\sum_{r=1}^{n}(-1)^{r-1}{}_nD_r \qquad n=1,2,3,\cdots$$

$$\zeta(1-2n)=\frac{(-1)^{n-1}}{2^{2n}\left(1-2^{2n}\right)}T_{2n-1} \qquad n=1,2,3,\cdots$$

$$=(-1)^{2n-1}\frac{B_{2n}}{2n} \qquad n=1,2,3,\cdots$$

Where, ${}_nD_r$ are the Eulderian Numbers and T_{n-1} are the tangent numbers. These are defined as follows respectively.

$${}_nD_r=\sum_{k=0}^{r-1}(-1)^k\binom{n+1}{k}(r-k)^n \quad , \quad T_{n-1}=2^n\left(2^n-1\right)\frac{|B_n|}{n}$$

By-products

$$-\log 0=\zeta(1)=\zeta(1)-\frac{\pi i}{2}$$

2 Formulas for Riemann Zeta at natural number

In this chapter, removing the lower zetas from automorphism formulas in the previous chapter, we obtain non-automorphism formulas for Riemann Zeta at natural number.
Where, Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$
$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$

And Harmonic number is $H_s=\sum_{t=1}^{s}1/t=\psi(1+s)+\gamma$

For $0<x\le 2\pi$,

$$\zeta(n)=\frac{x^{n-1}}{(n-1)!}\left(\frac{1}{n-1}-\frac{x}{2n}\right)+\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}}-\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

$$\zeta(n)=\frac{x^{n-1}}{(n-1)!}\left\{\frac{1}{n-1}+\frac{(n-1)x}{2n}-\log x\right\}$$
$$+\sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(xr)^s}{s!}\frac{1}{r^n e^{xr}}-\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}$$

For $0<x\le\pi$,

$$\zeta(n)=\frac{2^{n-1}}{2^{n-1}-1}\left\{\frac{x^n}{2n!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(xr)^s}{s!}\frac{(-1)^r}{r^n e^{xr}}+\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{x^{n-1}}{2(n-1)!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{\{x(2r-1)\}^s}{s!}\frac{e^{-x(2r-1)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{x^{n-1}}{2(n-1)!}\left(\frac{1}{n-1}-\log\frac{x}{2}\right) + \sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{\{x(2r-1)\}^s}{s!}\frac{e^{-x(2r-1)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{\left(2^{2r}-2\right)B_{2r}x^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Especially,

$$\zeta(n) = \frac{n+1}{2n!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{r^s}{s!}\frac{1}{r^n e^r} - \sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}}{2r(n-1+2r)!}$$

$$\zeta(n) = \frac{2^{n-1}}{n!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2r)^s}{s!}\frac{1}{r^n e^{2r}} - \sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{B_{2r}2^{n-1+2r}}{2r(n-1+2r)!}$$

$$\zeta(n) = \frac{n^2+1}{2n!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{r^s}{s!}\frac{1}{r^n e^r} - \sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{B_{2r}}{2r(n-1+2r)!}$$

$$\zeta(n) = \frac{2^{n-1}}{2^{n-1}-1}\left\{\frac{1}{2n!} - \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{r^s}{s!}\frac{(-1)^r}{r^n e^r} + \sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{1}{2(n-1)!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(2r-1)^s}{s!}\frac{e^{-(2r-1)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{-n}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{1}{2(n-1)!}\left(\frac{1}{n-1}+\log 2\right) + \sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(2r-1)^s}{s!}\frac{e^{-(2r-1)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(n-1+2r)!}\right\}$$

$$\zeta(n) = \frac{2^n}{2^n-1}\left\{\frac{2^{n-2}}{(n-1)!\,(n-1)} + \sum_{r=1}^{\infty}\sum_{s=0}^{n-2}\frac{(4r-2)^s}{s!}\frac{e^{-(4r-2)}}{(2r-1)^n} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{1-n}{2r}\frac{\left(2^{2r}-2\right)B_{2r}2^{n-1+2r}}{2r(n-1+2r)!}\right\}$$

Example

$$\zeta(5) = \frac{6}{2\cdot 5!\,4} + \sum_{r=1}^{\infty}\left(1+\frac{r^1}{1!}+\frac{r^2}{2!}+\frac{r^3}{3!}+\frac{r^4}{4!}\right)\frac{1}{r^5 e^r} - \sum_{r=1}^{\infty}\binom{-5}{2r-1}\frac{B_{2r}}{2r(4+2r)!}$$

$$\zeta(5) = \frac{2^4}{5!\,4} + \sum_{r=1}^{\infty}\left\{1+\frac{2r}{1!}+\frac{(2r)^2}{2!}+\frac{(2r)^3}{3!}+\frac{(2r)^4}{4!}\right\}\frac{1}{r^5 e^{2r}} - \sum_{r=1}^{\infty}\binom{-5}{2r-1}\frac{B_{2r}2^{4+2r}}{2r(4+2r)!}$$

$$\zeta(5) = \frac{5^2+1}{2\cdot 5!\,4} + \sum_{r=1}^{\infty}\left(1+\frac{r^1}{1!}+\frac{r^2}{2!}+\frac{r^3}{3!}\right)\frac{1}{r^5 e^r} - \sum_{r=1}^{\infty}\binom{-4}{2r}\frac{B_{2r}}{2r(4+2r)!}$$

$$\zeta(5) = \frac{2^4}{2^4-1}\left\{\frac{1}{2\cdot 5!} - \sum_{r=1}^{\infty}\left(1+\frac{r^1}{1!}+\frac{r^2}{2!}+\frac{r^3}{3!}+\frac{r^4}{4!}\right)\frac{(-1)^r}{r^5 e^r} + \sum_{r=1}^{\infty}\binom{-5}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}}{2r(4+2r)!}\right\}$$

$$\zeta(3) = \frac{8}{7}\left\{\frac{1}{8}+\sum_{r=1}^{\infty}\left(1+\frac{2r-1}{1!}+\frac{(2r-1)^2}{2!}\right)\frac{e^{-(2r-1)}}{(2r-1)^3}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-3}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(2+2r)!}\right\}$$

$$\zeta(3) = \frac{8}{7}\left\{\frac{1}{4}\left(\frac{1}{2}+\log 2\right)+\sum_{r=1}^{\infty}\left(1+\frac{2r-1}{1!}\right)\frac{e^{-(2r-1)}}{(2r-1)^3}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-2}{2r}\frac{\left(2^{2r}-2\right)B_{2r}}{2r(2+2r)!}\right\}$$

$$\zeta(3) = \frac{8}{7}\left\{\frac{1}{2}+\sum_{r=1}^{\infty}\left(1+\frac{4r-2}{1!}\right)\frac{e^{-(4r-2)}}{(2r-1)^3}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-2}{2r}\frac{\left(2^{2r}-2\right)B_{2r}\,2^{2+2r}}{2r(2+2r)!}\right\}$$

3 Formulas for Riemann Zeta at odd number

In this chapter, we obtain non-automorphism formulas for Riemann Zeta at odd number. Where, Bernoulli numbers, Euler numbers and tangent numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\ \cdots$

$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$

$T_1=1,\ T_3=2,\ T_5=16,\ T_7=272,\ T_9=7936,\ \cdots$

And Harmonic number is $H_s=\sum_{t=1}^{s}1/t = \psi(1+s)+\gamma$

For $0<x<2\pi$,

$$\zeta(2n+1) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}(-1)^s\frac{\left(2^{2s}-2\right)B_{2s}(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+2}}$$
$$+(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

$$\zeta(2n+1) = \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n+1}}$$
$$-(-1)^n x^{2n}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{|B_{2r}|x^{2r}}{2r}\right\}$$

For $0<x\le\pi$,

$$\zeta(2n+1) = \frac{2^{2n}}{2^{2n}-1}\left\{\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)(rx)^{2s}}{(2s)!}\frac{(-1)^r\sin rx}{r^{2n+2}}\right.$$
$$\left.-(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r}\right\}$$

$$\zeta(2n+1) = -\frac{2^{2n}}{2^{2n}-1}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{(-1)^r\cos rx}{r^{2n+1}}\right.$$
$$\left.-(-1)^n x^{2n}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}}{(2s)!(2n+2r-2s)!}\frac{\left(2^{2r}-1\right)|B_{2r}|x^{2r}}{2r}\right\}$$

$$\zeta(2n+1) = \frac{2^{2n+1}}{2^{2n+1}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{|E_{2s}|\{(2r-1)x\}^{2s}}{(2s)!}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n+1}}$$
$$-\frac{(-1)^n(2x)^{2n}}{2^{2n+1}-1}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!}-\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}\left(2^{2r}-2\right)|B_{2r}|}{(2s)!(2n+2r-2s)!}\frac{x^{2r}}{2r}\right\}$$

Especially,

$$\zeta(2n+1) = (-1)^n\pi^{2n}\left\{\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}H_{2n+1-2s}}{(2s)!(2n+1-2s)!}+\sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n+1+2r-2s)!}\frac{|B_{2r}|\pi^{2r}}{2r}\right\}$$

$$\zeta(2n+1) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2^{2n}-1}\sum_{r=1}^{\infty}\left\{\sum_{s=0}^{n}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!(2n+1+2r-2s)!}\right\}T_{2r-1}\left(\frac{\pi}{2}\right)^{2r}$$

$$\zeta(2n+1) = \frac{(-1)^{n-1}\pi^{2n}}{2^{2n+1}-1}\left\{\sum_{s=0}^{n}\frac{E_{2s}H_{2n-2s}}{(2s)!(2n-2s)!} - \sum_{r=1}^{\infty}\sum_{s=0}^{n}\frac{E_{2s}(2^{2r}-2)|B_{2r}|}{(2s)!(2n+2r-2s)!}\frac{1}{2r}\left(\frac{\pi}{2}\right)^{2r}\right\}$$

Example

$$\zeta(5) = \pi^4\left\{\frac{269}{21600}+\sum_{r=1}^{\infty}\left(-\frac{1}{(2r+5)!}+\frac{1}{6(2r+3)!}-\frac{7}{360(2r+1)!}\right)\frac{|B_{2r}|\pi^{2r}}{2r}\right\}$$

$$\zeta(5) = -\frac{16\pi^4}{15}\sum_{r=1}^{\infty}\left\{-\frac{1}{(2r+5)!}+\frac{1}{6(2r+3)!}-\frac{7}{360(2r+1)!}\right\}\frac{(2^{2r}-1)|B_{2r}|\pi^{2r}}{2r}$$

$$\zeta(5) = \frac{\pi^4}{31}\left\{\frac{83}{288}+\sum_{r=1}^{\infty}\left(\frac{1}{(4+2r)!}-\frac{1}{2(2+2r)!}+\frac{5}{24(2r)!}\right)\frac{(2^{2r}-2)|B_{2r}|}{2r}\left(\frac{\pi}{2}\right)^{2r}\right\}$$

4 Formulas for Riemann Zeta at even number

In this chapter, we obtain non-automorphism formulas for Riemann Zeta at even number.
Where, Bernoulli numbers and Euler numbers are as follows.

$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$

$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$

For $0<x<2\pi$,

$$\zeta(2n) = -\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}(2^{2s}-2)(rx)^{2s}}{(2s)!}\frac{\sin rx}{r^{2n+1}} - \frac{|B_{2n}|x^{2n}}{2(2n)!}\left\{(2^{2n}-2)-\frac{\pi(2^{2n+1}-2)}{x}\right\}$$

$$\zeta(2n)=\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{\cos rx}{r^{2n}} - \frac{|E_{2n}|x^{2n}}{2(2n)!} + \frac{\pi}{2}\frac{2^{2n}(2^{2n}-1)|B_{2n}|x^{2n-1}}{(2n)!}$$

For $0<x\le\pi$,

$$\zeta(2n)=\frac{2^{2n}}{2^{2n}-2}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}(2^{2s}-2)(rx)^{2s-1}}{(2s)!}\frac{(-1)^r\sin rx}{r^{2n}} + \frac{|B_{2n}|(2x)^{2n}}{2(2n)!}$$

$$\zeta(2n) = -\frac{2^{2n}}{2^{2n}-2}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(rx)^{2s}}{(2s)!}\frac{(-1)^r\cos rx}{r^{2n}} - \frac{|E_{2n}|x^{2n}}{2(2n)!}\right\}$$

$$\zeta(2n) = -\frac{2^{2n}}{2^{2n}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}(2^{2s}-2)\{(2r-1)x\}^{2s-1}}{(2s)!}\frac{\sin\{(2r-1)x\}}{(2r-1)^{2n}} + \frac{\pi}{2}\frac{2^{2n}|B_{2n}|x^{2n-1}}{(2n)!}$$

$$\zeta(2n)=\frac{2^{2n}}{2^{2n}-1}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\{(2r-1)x\}^{2s}}{(2s)!}\frac{\cos\{(2r-1)x\}}{(2r-1)^{2n}} + \frac{\pi}{4}\frac{2^{4n}|B_{2n}|x^{2n-1}}{(2n)!}$$

Especially,

$$\zeta(2n) = \frac{|B_{2n}|(2\pi)^{2n}}{2(2n)!}$$

$$\zeta(2n) = -\frac{1}{2^{2n}-2}\left\{\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|(\pi r)^{2s}}{(2s)!}\frac{(-1)^r}{r^{2n}} - \frac{|E_{2n}|\pi^{2n}}{2(2n)!}\right\}$$

By-products

$$\sum_{s=0}^{n-1}\frac{(2^{2s}-2)B_{2s}}{(2s)!(2n-2s)!} = -\frac{(2^{2n+1}-2)B_{2n}}{(2n)!0!}$$

$$\sum_{s=0}^{n-1}\frac{E_{2s}}{(2s)!\,(2n-1-2s)!}=\frac{2^{2n}\left(2^{2n}-1\right)B_{2n}}{(2n)!}$$

5 Formulas for Riemann Zeta at complex number

In this chapter, we obtain the formulas for Riemann Zeta at a complex number by processing " 2 Formulas for Riemann Zeta at natural number "

Where, Bernoulli numbers are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$$

And gamma function and incomplete gamma function are

$$\Gamma(p)=\int_0^\infty t^{p-1}e^{-t}dt \quad , \quad \Gamma(p,x)=\int_x^\infty t^{p-1}e^{-t}dt$$

And p is a complex number such that $p\neq 1,0,-1,-2,\cdots$.

For $x=u+vi$ s.t. $0<|x|\leqq 2\pi$, $u\geqq 0$,

$$\zeta(p)=\frac{x^{p-1}}{\Gamma(p)}\left(\frac{1}{p-1}-\frac{x}{2p}\right)+\sum_{r=1}^{\infty}\frac{\Gamma(p,xr)}{\Gamma(p)r^p}-\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}$$

$$\zeta(p)=\frac{x^{p-1}}{\Gamma(p)}\left\{\frac{1}{p-1}+\frac{(p-1)x}{2p}-\log x\right\}+\sum_{r=1}^{\infty}\frac{\Gamma(p-1,xr)}{\Gamma(p-1)r^p}-\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}$$

For $x=u+vi$ s.t. $0<|x|\leqq \pi$, $u\geqq 0$,

$$\zeta(p)=\frac{2^{p-1}}{2^{p-1}-1}\left\{\frac{x^p}{2\Gamma(p+1)}-\sum_{r=1}^{\infty}\frac{\Gamma(p,xr)}{\Gamma(p)}\frac{(-1)^r}{r^p}+\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}\right\}$$

$$\zeta(p)=\frac{2^p}{2^p-1}\left\{\frac{x^{p-1}}{2(p-1)\Gamma(p)}+\sum_{r=1}^{\infty}\frac{\Gamma\{p,x(2r-1)\}}{\Gamma(p)(2r-1)^p}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}\right\}$$

$$\zeta(p)=\frac{2^p}{2^p-1}\left\{\frac{x^{p-1}}{2\Gamma(p)}\left(\frac{1}{p-1}-\log\frac{x}{2}\right)+\sum_{r=1}^{\infty}\frac{\Gamma\{p-1,x(2r-1)\}}{\Gamma(p-1)(2r-1)^p}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{\left(2^{2r}-2\right)B_{2r}x^{p-1+2r}}{2r\Gamma(p+2r)}\right\}$$

Especially,

$$\zeta(p)=\frac{p+1}{2(p-1)\Gamma(p+1)}+\sum_{r=1}^{\infty}\frac{\Gamma(p,r)}{\Gamma(p)r^p}-\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{B_{2r}}{2r\Gamma(p+2r)}$$

$$\zeta(p)=\frac{p^2+1}{2(p-1)\Gamma(p+1)}+\sum_{r=1}^{\infty}\frac{\Gamma(p-1,r)}{\Gamma(p-1)r^p}-\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{B_{2r}}{2r\Gamma(p+2r)}$$

$$\zeta(p)=\frac{2^{p-1}}{2^{p-1}-1}\left\{\frac{1}{2\Gamma(p+1)}-\sum_{r=1}^{\infty}\frac{\Gamma(p,r)}{\Gamma(p)}\frac{(-1)^r}{r^p}+\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{\left(2^{2r}-1\right)B_{2r}}{2r\Gamma(p+2r)}\right\}$$

$$\zeta(p)=\frac{2^p}{2^p-1}\left\{\frac{1}{2(p-1)\Gamma(p)}+\sum_{r=1}^{\infty}\frac{\Gamma(p,2r-1)}{\Gamma(p)(2r-1)^p}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{-p}{2r-1}\frac{\left(2^{2r}-2\right)B_{2r}}{2r\Gamma(p+2r)}\right\}$$

$$\zeta(p)=\frac{2^p}{2^p-1}\left\{\frac{1}{2\Gamma(p)}\left(\frac{1}{p-1}+\log 2\right)+\sum_{r=1}^{\infty}\frac{\Gamma(p-1,2r-1)}{\Gamma(p-1)(2r-1)^p}+\frac{1}{2}\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{\left(2^{2r}-2\right)B_{2r}}{2r\Gamma(p+2r)}\right\}$$

$$\zeta(p) = \frac{2^p}{2^p-1}\left\{\frac{2^{p-2}}{(p-1)\Gamma(p)} + \sum_{r=1}^{\infty}\frac{\Gamma(p-1,4r-2)}{\Gamma(p-1)(2r-1)^p} + \frac{1}{2}\sum_{r=1}^{\infty}\binom{1-p}{2r}\frac{(2^{2r}-2)B_{2r}2^{p-1+2r}}{2r\Gamma(p+2r)}\right\}$$

6 Global definition of Riemann Zeta, and generalization of related coefficients

From Euler to Riemann, the zeta function was defined with patches as the domain was expanded.

$$\zeta(p) = \begin{cases} \displaystyle\sum_{r=1}^{\infty}\frac{1}{r^p} & Re(p)>1 \\ \displaystyle\frac{1}{1-2^{1-p}}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^p} & 0 \leqq Re(p) \leqq 1 \\ & p \ne 1 \\ \displaystyle\frac{2\Gamma(1-p)}{(2\pi)^{1-p}}\sin\left(\frac{p\pi}{2}\right)\frac{1}{1-2^p}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{r^{1-p}} & Re(p) \leqq 0 \\ & p \ne 0 \end{cases}$$

This is inconvenient. so, we focus on the following sequence.

$${}_nB_r = \sum_{s=1}^{r}(-1)^{r-s}\binom{r}{s}s^n \qquad r=0,1,2,\cdots,n$$

Using this sequence, we can define the Zeta function on the whole complex plane as follows.

Definition 6.2.1

We difine the Riemann Zeta Function on the complex plane as follows.

$$\zeta(p) = \frac{1}{1-2^{1-p}}\sum_{r=1}^{\infty}\frac{1}{2^{r+1}}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}s^{-p} \qquad p \ne 1$$

Furthermore, by using this sequence, the following various coefficients can be generalized.

Generalized Stirling Number of the 2nd kind

$$S_2(p,r) = \frac{1}{r!}\sum_{s=1}^{\infty}(-1)^{s-1}\binom{r}{s}s^p \qquad r=1,2,3,\cdots$$

Generalized Tangent Number

$$T_p = \begin{cases} 0 & p=0 \\ \displaystyle\sum_{r=1}^{\infty}2^{p-r}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}s^p & p \ne 0 \end{cases}$$

Generalized Bernoulli Number

$$B_p = \begin{cases} -\dfrac{1}{2} & p=1 \\ \displaystyle\frac{p}{2^p-1}\sum_{r=1}^{\infty}\frac{1}{2^{r+1}}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}s^{p-1} & p \ne 1 \end{cases}$$

7 Completed Riemann Zeta

In **7.1**, we consider even function and odd function for complex function. Generally, we obtain the same results as for real-valued functions, but the results unique to complex-valued functions are as follows.

Theorem 7.1.3

Let $f(z)$ be a complex function in the domain D.

(1) If $f(z)$ is an even function , both the real part and the imaginary part are even functions.

(2) If $f(z)$ is an odd function , both the real part and the imaginary part are odd functions.

Theorem 7.1.4

Let $f(z)$ be a complex function in the domain D. Then,

if $f(z)$ is an even function or an odd function , $|f(z)|^2$ is an even function.

In **7.2**, we study complex conjugate properties. Especially when the function $f(z)$ is an even function or an odd function with complex conjugate properties, two important theorems are obtained.

Theorem 7.2.3

When $f(x,y)=u(x,y)+iv(x,y)$ is a function with the complex conjugate property in the domain D ,

(1) if $f(x,y)$ is an even function,

$$u(x,y) = u(x,-y) = u(-x,y) = u(-x,-y)$$
$$v(x,y) = -v(x,-y) = -v(-x,y) = v(-x,-y)$$

(2) if $f(x,y)$ is an odd function,

$$u(x,y) = u(x,-y) = -u(-x,y) = -u(-x,-y)$$
$$v(x,y) = -v(x,-y) = v(-x,y) = -v(-x,-y)$$

Corollary 7.2.3

Let $f(x,y)=u(x,y)+iv(x,y)$ be a function with the complex conjugate property in the domain D .

Then, the followings hold for any real number $x,y \in D$.

(1) When $f(x,y)$ is an even function , $v(x,0)=0$, $v(0,y)=0$.

(2) When $f(x,y)$ is an odd function , $u(0,y)=0$, $v(x,0)=0$.

Theorem 7.2.4

When $f(z)$ is a function with the complex conjugate property in the domain D and has a zero $z_1=x_1+iy_1$ $(x_1 \neq 0)$,

(1) if $f(z)$ is an even function, $-x_1-iy_1$, x_1-iy_1 , $-x_1+iy_1$ are also zeros of $f(z)$.

(2) if $f(z)$ is an odd function, $-x_1-iy_1$, x_1-iy_1 , $-x_1+iy_1$ are also zeros of $f(z)$.

In **7.3**, symmetric functional equations are derived from functional equations.

Formula 7.3.1 (Riemann)

$$\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \qquad z \neq 0,1$$

$$\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right) = \pi^{-\frac{1}{2}\left(\frac{1}{2}-z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-z\right)\right\}\zeta\left(\frac{1}{2}-z\right)$$

Where, $z \neq \pm 1/2$

In **7.4**, we define the completed Riemann zeta functions $\xi(z), \Xi(z)$ as follows, respectively. These are a little different from Landau's definition.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$

Then, the following equations hold from Formula 7.3.1 .

$$\xi(z) = \xi(1-z)$$
$$\Xi(z) = \Xi(-z)$$

From the latter, we can see that $\Xi(z)$ is an even function. Therefore, Thoerem 7.2.3 (1) and Corollary 7.2.3 hold for the real part $u(x,y)$ and the imaginary part $v(x,y)$ of $\Xi(z)$ as they are. And from Theorem 7.2.4 , the following very important theorem is obtained.

Theorem 7.4.1

If Riemann zeta function $\zeta(z)$ has a non-trivial zero whose real part is not $1/2$, the one set consists of the following four.

$$1/2+\alpha_1 \pm i\beta_1 \ , \quad 1/2-\alpha_1 \pm i\beta_1 \qquad (\ 0<\alpha_1<1/2\ )$$

08 Factorization of Completed Riemann Zeta

In **8.1**, the following Hadamard product is shown.

Formula 8.1.1 (Hadamard product of $\xi(z)$)

Let completed zeta function be as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

When non-trivial zeros of $\zeta(z)$ are $z_k = x_k \pm iy_k \quad k=1,2,3,\cdots$ and γ is Euler-Mascheroni constant, $\xi(z)$ is expressed by the Hadamard product as follows.

$$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

$$\xi(z) = e^{\left(\log 2+\frac{\log\pi}{2}-1-\frac{\gamma}{2}\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}}$$

And, the following special values are obtained.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2}} = 1.02336448\cdots$$

$$\prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\} = \frac{\pi}{3}$$

In **8.2**, we consider how the formulas in the previous section are expressed when non-trivial zeros whose real part is 1/2 and non-trivial zeros whose real part is not 1/2 are mixed. Then, we obtain the following theorems.

Theorem 8.2.2

Let γ be Euler-Mascheroni constant, non-trivial zeros of Riemann zeta function are $x_n+iy_n \quad n=1,2,3,\cdots$.

Among them, zeros whose real part is 1/2 are $1/2\pm iy_r \quad r=1,2,3,\cdots$ and zeros whose real parts is not 1/2 are $1/2\pm\alpha_s\pm i\beta_s \ (0<\alpha_s<1/2) \quad s=1,2,3,\cdots$. Then the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right) = 1$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = \sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\beta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\beta_s^2}\right\}$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = 1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2} = 0.0230957\cdots$$

Formula 8.2.3 (Special values)

When non-trivial zeros of Riemann zeta function are $x_k \pm iy_k \quad k=1,2,3,\cdots$, the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right) = 1$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right) = \frac{\pi}{3}$$

Theorem 8.2.4

Let non-trivial zeros of Riemann zeta function are $x_n+iy_n \quad n=1,2,3,\cdots$ and γ be Euler-Mascheroni constant. If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = 1+\frac{\gamma}{2}-\log 2-\frac{\log\pi}{2} = 0.0230957\cdots$$

Incidentally, when this was calculated using 200000 y_r, both sides coincided with the decimal point 4 digits.

In **8.3**, we show that $\xi(z)$ is factored completely.

Theorem 8.3.1 (Factorization of $\xi(z)$)

Let Riemann zeta function be $\zeta(z)$, the non-trivial zeros are $z_n=x_n\pm iy_n \quad n=1,2,3,\cdots$ and completed zeta function be as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Then, $\xi(z)$ is factorized as follows.

$$\xi(z) = \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)$$

In **8.4**, we first derive the factorization of $\Xi(z)$.

Theorem 8.4.1 (Factorization of $\Xi(z)$)

Let Riemann zeta function be $\zeta(z)$, the non-trivial zeros are $z_n=x_n\pm iy_n \quad n=1,2,3,\cdots$ and completed zeta function be as follows.

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$

Then, $\Xi(z)$ is factorized as follows.

$$\Xi(z) = \Xi(0)\prod_{n=1}^{\infty}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\}$$

Where, $\Xi(0)=\prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2} = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.99424155\cdots$

And, using this theorem and Theorem 7.4.1 in the previous section, we obtaine the following theorem.

Theorem 8.4.4

When Riemann zeta function is $\zeta(z)$ and the non-trivial zeros sre $z_n=x_n\pm iy_n \quad n=1,2,3,\cdots$,
If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\prod_{r=1}^{\infty}\frac{y_r^2}{1/4+y_r^2} = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.99424155\cdots$$

Incidentally, when this was calculated using 100000 y_r, both sides coincided with the decimal point 5 digits.

09 Maclaurin Series of Completed Riemann Zeta

In **9.1**, completed Riemann zeta $\xi(z)$ is expanded in Maclaurin series.

Theorem 9.1.3 (Maclaurin series of $\xi(z)$)

Let completed Riemann zeta be

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$$

Then, the following expression holds on the whole complex plane.

$$\xi(z) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{(-1)^{s-t}g_{s-t}(3/2)}{2^{s-t}(s-t)!}c_t z^r$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \dots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \dots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ -\frac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

The first few are as folows.

$$\begin{aligned}\xi(z) = 1 &+ \left(\frac{\log^1\pi}{2^1 1!} - \frac{g_1(3/2)}{2^1 1!} - \frac{\gamma_0}{0!}\right)z^1 \\ &+ \left(\frac{\log^2\pi}{2^2 2!} + \frac{g_2(3/2)}{2^2 2!} - \frac{\gamma_1}{1!} - \frac{\log^1\pi}{2^1 1!}\frac{g_1(3/2)}{2^1 1!} + \frac{g_1(3/2)}{2^1 1!}\frac{\gamma_0}{0!} - \frac{\log^1\pi}{2^1 1!}\frac{\gamma_0}{0!}\right)z^2 \\ &+ \left(\frac{\log^3\pi}{2^3 3!} - \frac{g_3(3/2)}{2^3 3!} - \frac{\gamma_2}{2!} - \frac{\log^2\pi}{2^2 2!}\frac{g_1(3/2)}{2^1 1!} - \frac{\log^2\pi}{2^2 2!}\frac{\gamma_0}{0!} - \frac{g_2(3/2)}{2^2 2!}\frac{\gamma_0}{0!} \right. \\ &\quad + \frac{\log^1\pi}{2^1 1!}\frac{g_2(3/2)}{2^2 2!} - \frac{\log^1\pi}{2^1 1!}\frac{\gamma_1}{1!} + \frac{g_1(3/2)}{2^1 1!}\frac{\gamma_1}{1!} \\ &\quad \left. + \frac{\log^1\pi}{2^1 1!}\frac{g_1(3/2)}{2^1 1!}\frac{\gamma_0}{0!}\right)z^3 \\ &+ \\ &\vdots\end{aligned}$$

$$= 1. - 0.0230957\,z + 0.0233439\,z^2 - 0.000497984\,z^3 + 0.000253182\,z^4$$
$$-5.05025\times10^{-6}\,z^5 + 1.72099\times10^{-6}\,z^6 - 3.23784\times10^{-8}\,z^7 + 8.31597\times10^{-9}\,z^8$$
$$\vdots$$

In **9.2**, completed Riemann zeta $\Xi(z)$ is expanded in Maclaurin series.

Theorem 9.2.3 (Maclaurin Series of $\Xi(z)$)

Let completed Riemann zeta be

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$

Then, the following expression holds on the whole complex plane.

$$\Xi(z) = \Xi(0)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{g_{s-t}(5/4)}{2^{s-t}(s-t)!}c_t z^r$$

$$\Xi(0) = -\frac{1}{4\pi^{1/4}}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.9942415563\cdots$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \ldots, \psi_{r-1}\left(\frac{5}{4}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ \frac{2}{\zeta(1/2)} \sum_{s=r}^{\infty} (-1)^r \frac{\gamma_{s-1}}{(s-1)!} \binom{s}{r} \left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

The first few are as folows.

$$\begin{aligned} \Xi(z) = \Xi(0)\Bigg\{1 &+ \left(-\frac{\log^1\pi}{2^1 1!} + \frac{g_1(5/4)}{2^1 1!} + c_1\right) z^1 \\ &+ \left(\frac{\log^2\pi}{2^2 2!} + \frac{g_2(5/4)}{2^2 2!} + c_2 - \frac{\log^1\pi}{2^1 1!}\frac{g_1(5/4)}{2^1 1!} + \frac{g_1(5/4)}{2^1 1!}c_1 - \frac{\log^1\pi}{2^1 1!}c_1\right) z^2 \\ &+ \left(-\frac{\log^3\pi}{2^3 3!} + \frac{g_3(5/4)}{2^3 3!} + c_3 + \frac{\log^2\pi}{2^2 2!}\frac{g_1(5/4)}{2^1 1!} + \frac{\log^2\pi}{2^2 2!}c_1 + \frac{g_2(5/4)}{2^2 2!}c_1 \right. \\ &\quad \left. - \frac{\log^1\pi}{2^1 1!}\frac{g_2(5/4)}{2^2 2!} - \frac{\log^1\pi}{2^1 1!}c_2 + \frac{g_1(5/4)}{2^1 1!}c_2 - \frac{\log^1\pi}{2^1 1!}\frac{g_1(5/4)}{2^1 1!}c_1\right) z^3 \\ &+ \cdots \Bigg\} \end{aligned}$$

$$\begin{aligned} = 0.994242\,(1. &+ 4.44089\times 10^{-16} z + 0.023105 z^2 + 1.38778\times 10^{-16} z^3 \\ &+ 0.000248334 z^4 + 2.08167\times 10^{-17} z^5 + 1.67435\times 10^{-6} z^6 \\ &+ 7.37257\times 10^{-18} z^7 + 8.0307\times 10^{-9} z^8 + 1.0842\times 10^{-18} z^9 \\ &+ 2.94014\times 10^{-11} z^{10}) \end{aligned}$$

We can see that the coefficients of the odd degree are almost zero.

10 Vieta's Formulas on Completed Riemann Zeta

In **10.1**, the relations between the zeros of completed Riemann zeta $\xi(z)$ and the coefficients of the Maclaurin series are shown by the two theorems.

Theorem 10.1.1

Let completed Riemann zeta $\xi(z)$ and the Maclaurin series are as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \sum_{r=0}^{\infty} A_r z^r$$

Then, these coefficients A_r $r=0,1,2,3,\cdots$ are given by

$$A_r = \sum_{s=0}^{r}\sum_{t=0}^{s} \frac{\log^{r-s}\pi}{2^{r-s}(r-s)!} \frac{(-1)^{s-t} g_{s-t}(3/2)}{2^{s-t}(s-t)!} c_t$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{3}{2}\right) = \begin{cases} 1 & r=0 \\ \sum_{k=1}^{r} B_{r,k}\left(\psi_0\left(\frac{3}{2}\right), \psi_1\left(\frac{3}{2}\right), \ldots, \psi_{r-1}\left(\frac{3}{2}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ -\frac{\gamma_{r-1}}{(r-1)!} & r=1,2,3,\cdots \end{cases}$$

Theorem 10.1.2

Let completed Riemann zeta $\xi(z)$ and the Maclaurin series are as follows.

$$\xi(z) = -z(1-z)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \sum_{r=0}^{\infty} B_r z^r$$

Then,

(1) The following expressions hold for non-trivial zeros $z_k = x_k \pm i y_k$, $y_k \neq 0$ $k=1,2,3,\cdots$ of $\zeta(z)$.

$$B_1 = -\sum_{r_1=1}^{\infty} \frac{2x_{r_1}}{x_{r_1}^2+y_{r_1}^2}$$

$$B_2 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^2 x_{r_1}x_{r_2}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)} + \sum_{r_1=1}^{\infty} \frac{2^0}{x_{r_1}^2+y_{r_1}^2}$$

$$B_3 = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{2^3 x_{r_1}x_{r_2}x_{r_3}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} - \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^1\left(x_{r_1}+x_{r_2}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$$B_4 = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty}\sum_{r_4=r_3+1}^{\infty} \frac{2^4 x_{r_1}x_{r_2}x_{r_3}x_{r_4}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)\left(x_{r_4}^2+y_{r_4}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\sum_{r_3=r_2+1}^{\infty} \frac{2^2\left(x_{r_1}x_{r_2}+x_{r_1}x_{r_3}+x_{r_2}x_{r_3}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\left(x_{r_3}^2+y_{r_3}^2\right)} + \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty} \frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)}$$

$$\vdots$$

$$B_{2n-1} = -\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty} \frac{2^{2n-1} x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-2}=r_{2n-3}+1}^{\infty} \frac{2^{2n-3}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-3}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-2}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-2}}^2+y_{r_{2n-2}}^2\right)}$$

$$\vdots$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} \frac{2^1\left(x_{r_1}+x_{r_2}+\cdots+x_{r_n}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

$$B_{2n} = \sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n}=r_{2n-1}+1}^{\infty} \frac{2^{2n} x_{r_1}x_{r_2}\cdots x_{r_{2n}}}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n}}^2+y_{r_{2n}}^2\right)}$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_{2n-1}=r_{2n-2}+1}^{\infty} \frac{2^{2n-2}\left(x_{r_1}x_{r_2}\cdots x_{r_{2n-2}}+x_{r_1}x_{r_2}\cdots x_{r_{2n-1}}+\cdots+x_{r_2}x_{r_3}\cdots x_{r_{2n-1}}\right)}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_{2n-1}}^2+y_{r_{2n-1}}^2\right)}$$

$$\vdots$$

$$+\sum_{r_1=1}^{\infty}\sum_{r_2=r_1+1}^{\infty}\cdots\sum_{r_n=r_{n-1}+1}^{\infty} \frac{2^0}{\left(x_{r_1}^2+y_{r_1}^2\right)\left(x_{r_2}^2+y_{r_2}^2\right)\cdots\left(x_{r_n}^2+y_{r_n}^2\right)}$$

(2) When A_n is a coefficient in **Theorem 10.1.1** , $B_n = A_n$ $n=1,2,3,\cdots$.

And, if Riemann Hypothesis is true, the following proposition equivalent to this must hold.

Proposition 10.1.3

When $z_k = 1/2 \pm i y_k$, $y_k \neq 0$ ($k=1,2,3,\cdots$) are the non-trivial zeros of Riemann zeta $\zeta(z)$ and

$A_r \;\; r=1,2,3,\cdots$ are constants given by Theorem 10.1.1 , the following expressions hold.

$$\sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2} = -A_1 = 0.0230957089\cdots$$

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)} = A_2+A_1 = 0.0002481555\cdots$$

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)} = -A_3 - 2\left(A_2+A_1\right) = 0.0000016727\cdots$$

$$\sum_{r=1}^{\infty}\sum_{s=r+1}^{\infty}\sum_{t=s+1}^{\infty}\sum_{u=t+1}^{\infty}\frac{1}{\left(1/4+y_r^2\right)\left(1/4+y_s^2\right)\left(1/4+y_t^2\right)\left(1/4+y_u^2\right)}$$

$$= A_4+3A_3+5\left(A_2+A_1\right) = 8.021073428\times 10^{-9}$$

Proposition 10.1.3'

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^2 = A_1^2 - 2\left(A_1+A_2\right) = 0.00003710063\cdots$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^3 = -A_1^3+3\left(A_1-2\right)\left(A_1+A_2\right) - 3A_3 = 0.00000014367786\cdots$$

$$\sum_{r=1}^{\infty}\left(\frac{1}{1/4+y_r^2}\right)^4 = A_1^4 - 4A_1^3+4A_1^2\left(\frac{5}{2}-A_2\right)+4A_1\left(3A_2+A_3-5\right)$$

$$+ 2A_1^2 - 20A_2 - 12A_3 - 4A_4 = 6.59827915\times 10^{-10}$$

In **10.2**, the relations between the zeros of completed Riemann zeta $\Xi(z)$ and the coefficients of the Maclaurin series are shown by the two theorems.

Theorem 10.2.1

Let completed Riemann zeta $\Xi(z)$ and the Maclaurin series are as follows.

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$

$$= \Xi(0)\left(1+A_1z^1+A_2z^2+A_3z^3+A_4z^4+\cdots\right)$$

Then, these coefficients $A_r \;\; r=0,1,2,3,\cdots$ are given by

$$A_r = \sum_{s=0}^{r}\sum_{t=0}^{s}(-1)^{r-s}\frac{\log^{r-s}\pi}{2^{r-s}(r-s)!}\frac{g_{s-t}(5/4)}{2^{s-t}(s-t)!}c_t$$

Where, $\psi_n(z)$ is the polygamma function, $B_{n,k}(f_1, f_2, \ldots)$ is Bell polynomials, γ_r is Stieltjes constant,

$$g_r\left(\frac{5}{4}\right) = \begin{cases} 1 & r=0 \\ \displaystyle\sum_{k=1}^{r}B_{r,k}\left(\psi_0\left(\frac{5}{4}\right), \psi_1\left(\frac{5}{4}\right), \ldots, \psi_{r-1}\left(\frac{5}{4}\right)\right) & r=1,2,3,\cdots \end{cases}$$

$$c_r = \begin{cases} 1 & r=0 \\ \dfrac{2}{\zeta(1/2)}\displaystyle\sum_{s=r}^{\infty}(-1)^r\frac{\gamma_{s-1}}{(s-1)!}\binom{s}{r}\left(\frac{1}{2}\right)^{s-r} & r=1,2,3,\cdots \end{cases}$$

Theorem 10.2.2

Let completed Riemann zeta $\Xi(z)$ and the Maclaurin series are as follows.

$$\Xi(z) = -\left(\frac{1}{2}+z\right)\left(\frac{1}{2}-z\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+z\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+z\right)\right\}\zeta\left(\frac{1}{2}+z\right)$$

$$= \Xi(0)\left(1+B_1z^1+B_2z^2+B_3z^3+B_4z^4+\cdots\right)$$

Then,

(1) The following expressions hold for non-trivial zeros $z_k = x_k \pm i y_k$, $y_k \neq 0$ $k=1,2,3,\cdots$ of $\zeta(z)$.

$$\Xi(0) = \prod_{n=1}^{\infty} \frac{(x_n - 1/2)^2 + y_n^2}{x_n^2 + y_n^2} = -\frac{1}{4\pi^{1/4}} \Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) = 0.9942415563\cdots$$

$$B_1 = -\sum_{r_1=1}^{\infty} \frac{2(x_{r_1}-1/2)}{(x_{r_1}-1/2)^2 + y_{r_1}^2}$$

$$B_2 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^2 (x_{r_1}-1/2)(x_{r_2}-1/2)}{\{(x_{r_1}-1/2)^2 + y_{r_1}^2\}\{(x_{r_2}-1/2)^2 + y_{r_2}^2\}} + \sum_{r_1=1}^{\infty} \frac{2^0}{\{(x_{r_1}-1/2)^2 + y_{r_1}^2\}}$$

$$B_3 = -\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{2^3 (x_{r_1}-1/2)(x_{r_2}-1/2)(x_{r_3}-1/2)}{\{(x_{r_1}-1/2)^2 + y_{r_1}^2\}\{(x_{r_2}-1/2)^2 + y_{r_2}^2\}\{(x_{r_3}-1/2)^2 + y_{r_3}^2\}}$$
$$-\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^1 \{(x_{r_1}-1/2)+(x_{r_2}-1/2)\}}{\{(x_{r_1}-1/2)^2 + y_{r_1}^2\}\{(x_{r_2}-1/2)^2 + y_{r_2}^2\}}$$

$$B_4 = \sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \cdots \sum_{r_4=r_3+1}^{\infty} \frac{2^4 (x_{r_1}-1/2)(x_{r_2}-1/2)\cdots(x_{r_4}-1/2)}{\{(x_{r_1}-1/2)^2 + y_{r_1}^2\}\{(x_{r_2}-1/2)^2 + y_{r_2}^2\}\cdots\{(x_{r_4}-1/2)^2 + y_{r_4}^2\}}$$
$$+\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{2^2 \{(x_{r_1}-1/2)(x_{r_2}-1/2) + \cdots + (x_{r_2}-1/2)(x_{r_3}-1/2)\}}{\{(x_{r_1}-1/2)^2 + y_{r_1}^2\}\{(x_{r_2}-1/2)^2 + y_{r_2}^2\}\{(x_{r_3}-1/2)^2 + y_{r_3}^2\}}$$
$$+\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{2^0}{\{(x_{r_1}-1/2)^2 + y_{r_1}^2\}\{(x_{r_2}-1/2)^2 + y_{r_2}^2\}}$$

$\vdots$

(2) When A_n is a coefficient in **Theorem 10.2.1** , $B_n = A_n$ $n=1,2,3,\cdots$.

And, if Riemann Hypothesis is true, the following proposition equivalent to this must hold.

Proposition 10.2.3

When $z_k = 1/2 \pm i y_k$, $y_k \neq 0$ ($k=1,2,3,\cdots$) are the non-trivial zeros of Riemann zeta $\zeta(z)$ and A_r $r=1,2,3,\cdots$ are constants given by Theorem 10.1.1 , the following expressions hold.

$$\sum_{r_1=1}^{\infty} \frac{1}{y_{r_1}^2} = A_2 = 0.0231049931\cdots$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2} = A_4 = 0.0002483340\cdots$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2 y_{r_3}^2} = A_6 = 0.00000167435\cdots$$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \sum_{r_3=r_2+1}^{\infty} \sum_{r_4=r_3+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2 y_{r_3}^2 y_{r_4}^2} = A_8 = 8.030697\times 10^{-9}$$

$\vdots$

$$\sum_{r_1=1}^{\infty} \sum_{r_2=r_1+1}^{\infty} \cdots \sum_{r_{2n}=r_{2n-1}+1}^{\infty} \frac{1}{y_{r_1}^2 y_{r_2}^2 \cdots y_{r_{2n}}^2} = A_{2n}$$

Proposition 10.2.3'

$$\sum_{r=1}^{\infty}\frac{1}{y_r^{\,4}} = A_2^2 - 2A_4 = 0.00003717259\cdots$$

$$\sum_{r=1}^{\infty}\frac{1}{y_r^{\,6}} = A_2^3 - 3A_2A_4 + 3A_6 = 0.00000014417393\cdots$$

$$\sum_{r=1}^{\infty}\frac{1}{y_r^{\,8}} = A_2^4 + 2A_4^2 - 4A_2^2A_4 + 4A_2A_6 - 4A_8 = 6.6303\times 10^{-10}$$

11 Zeros on the Critical Line of Riemann Zeta

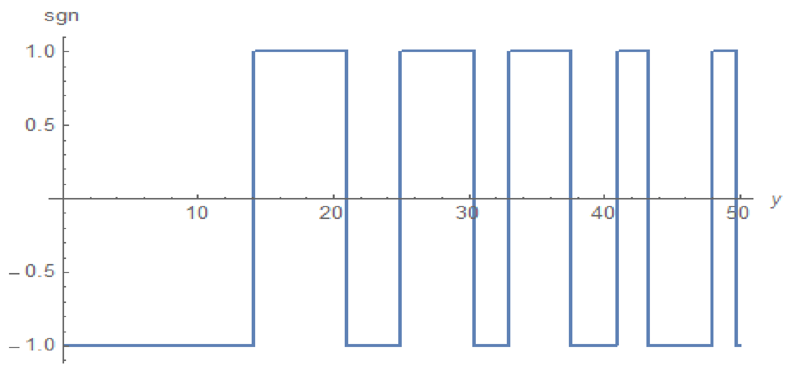
In **11.1**, substituting $z = 0 + iy$ for the completed Riemann zeta $\Xi(z)$,

$$\Xi_h(y) = -\left(\frac{1}{2}+iy\right)\left(\frac{1}{2}-iy\right)\pi^{-\frac{1}{2}\left(\frac{1}{2}+iy\right)}\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right)$$

We use this to calculate the zeros on the critical line. However, this function is too small in absolute value and can only find the zeros up to $y=917$.

So we normalize $\Xi_h(y)$ and define the following sign function.

$$sgn(y) = -\frac{\Xi_h(y)}{|\Xi_h(y)|} = \pi^{-\frac{iy}{2}}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right)}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\zeta\left(\frac{1}{2}+iy\right)\right|}$$

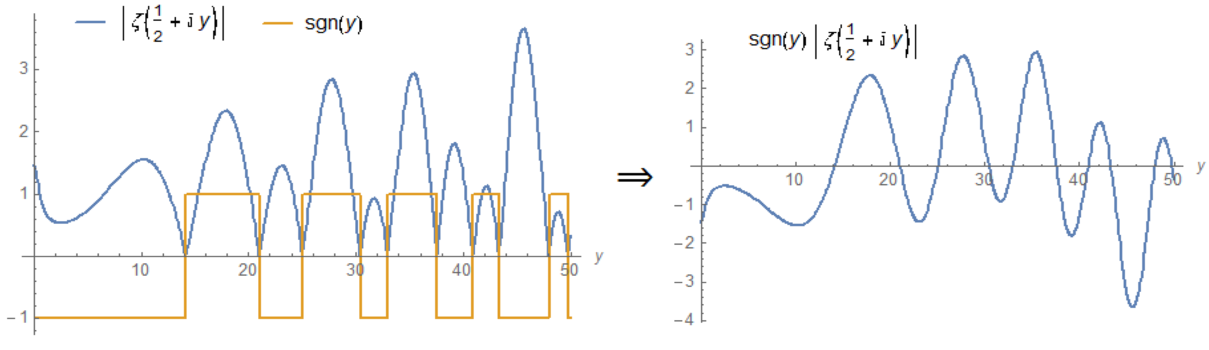


Using this sign function $sgn(y)$, we can find the zeros at large y.

However, this sign function $sgn(y)$ has the disadvantage that it is easy to miss Lehmer's phenomenon.

In **11.2**, multiplying this sign function $sgn(y)$ by the absolute value of the Riemann zeta $\zeta(1/2+iy)$, we obtain a smooth function $Z(y)$.

$$Z(y) = sgn(y)\left|\zeta\left(\frac{1}{2}+iy\right)\right| = \pi^{-\frac{iy}{2}}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\right|}\zeta\left(\frac{1}{2}+iy\right)$$



Using this $Z(y)$ function, we can find the zeros on the critical line of $\zeta(z)$ by the intersection of the curve and the y -axis Therefore, the risk of missing the Lehmer's phenomenon is reduced.

In **11.3**, first, a lemma is prepared.

Lemma

When $f(z)$ is a complex function defined on the domain D , the following expression holds.

$$e^{i\,Im\,log\,f(z)} = \frac{f(z)}{|f(z)|}$$

Applying this lemma to the gamma function in the **11.2** ,

$$Z(y) = \pi^{-\frac{iy}{2}} \frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}\right|}\zeta\left(\frac{1}{2}+iy\right) = \pi^{-\frac{iy}{2}} e^{i\,Im\,log\,\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\}}\zeta\left(\frac{1}{2}+iy\right)$$

From this, we obtain

$$Z(y) = e^{i\theta(y)}\zeta\left(\frac{1}{2}+iy\right) \qquad \text{where,} \quad \theta(y) = Im\,log\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+iy\right)\right\} - \frac{y}{2}log\pi$$

This is a definitional equation of Riemann-Siegel Z function .

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⋮

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