同济大学

高等代数与解析几何

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§9.2 标准正交基与QR分解

定义. 设V是内积空间, $\alpha_1, \cdots, \alpha_s$是$V$中非零向量, 如果它们两两正交, 则称$\alpha_1, \cdots, \alpha_s$为正交向量组.

命题. 设$\alpha_1, \cdots, \alpha_s$是正交向量组, 则$\alpha_1, \cdots, \alpha_s$线性无关.

证明. 设

$$\sum_{i=1}^{s} k_i \alpha_i = \mathbf{0}$$

两边与向量α_j作内积$(j = 1, 2, \cdots, s)$, 得

$$(\sum_{i=1}^{s} k_i \alpha_i, \alpha_j) = (\mathbf{0}, \alpha_j) = 0$$

另一方面, $(\sum_{i=1}^{s} k_i \alpha_i, \alpha_j) = k_j(\alpha_j, \alpha_j)$, 因为$\alpha_j \neq \mathbf{0}$, 所以由$k_j(\alpha_j, \alpha_j) = 0$推出$k_j = 0 (j = 1, 2, \cdots, s)$. 这说明$\alpha_1, \cdots, \alpha_s$线性无关.

说明. (1) 在n维线性空间V中, 两两正交的非零向量不能超过n 个.

(2) 如果n个向量构成一个正交向量组, 则它是n维线性空间V的一个基.

定义. 设V是内积空间, 如果$\alpha_1, \cdots, \alpha_n$两两正交, 而且每一个向量$\alpha_i$都是单位向量, 则称$\alpha_1, \cdots, \alpha_n$为$V$的标准正交基.

命题. n个向量$\alpha_1, \cdots, \alpha_n$构成标准正交基当且仅当矩阵$G = ((\alpha_i, \alpha_j))_{n\times n}$是单位阵.

证明. 由标准正交基的定义直接推出. 请同学们给出详细证明.

命题. 设V是一个内积空间, $\alpha_1, \cdots, \alpha_n$是$V$的一个基, $G=((\alpha_i,\alpha_j))$, 对于$\alpha=\sum\limits_{i=1}^{n} x_i\alpha_i, \beta=\sum\limits_{i=1}^{n} y_i\alpha_i, \in V$, 令

$$X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, Y=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

则

$$(\alpha,\beta)=X^T G\overline{Y}$$

特别当$\alpha_1, \cdots, \alpha_n$是$V$的标准正交基时,

$$(\alpha,\beta)=\sum_{i=1}^{n} x_i\overline{y}_i$$

说明. 矩阵$G=((\alpha_i,\alpha_j))$称为度量矩阵

证明.

$$(\alpha,\beta)=(\sum_{i=1}^{n}x_i\alpha_i,\sum_{j=1}^{n}y_j\alpha_j)=\sum_{i=1}^{n}x_i(\alpha_i,\sum_{j=1}^{n}y_j\alpha_j)=\sum_{i=1}^{n}\sum_{j=1}^{n}x_i\bar{y}_j(\alpha_i,\alpha_j)=X^TG\overline{Y}$$

定理.(格拉姆-施密特Gram-Schmidt正交化过程) 设V为内积空间, $\beta_1,\beta_2,\cdots,\beta_m$是线性无关的向量组. 令

$$\begin{cases}\alpha_1=\beta_1\\ \alpha_r=\beta_r-\sum\limits_{k=1}^{r-1}\frac{(\beta_r,\alpha_k)}{(\alpha_k,\alpha_k)}\alpha_k\end{cases}$$

则$\alpha_1,\alpha_2,\cdots,\alpha_m$是正交向量组, 并且

$$\mathrm{Span}\{\beta_1,\beta_2,\cdots,\beta_m\}=\mathrm{Span}\{\alpha_1,\alpha_2,\cdots,\alpha_m\}.$$

证明. 对m进行归纳，当$m=1$时，定理显然成立.
归纳假设,$\alpha_1,\alpha_2,\cdots,\alpha_{m-1}$是正交向量组，并且

$$\mathrm{Span}\{\beta_1,\beta_2,\cdots,\beta_{m-1}\}=\mathrm{Span}\{\alpha_1,\alpha_2,\cdots,\alpha_{m-1}\}.$$

由于$\beta_1,\beta_2,\cdots,\beta_{m-1}$是线性无关的向量组，所以$\alpha_1,\alpha_2,\cdots,\alpha_{m-1}$也线性无关，所以$(\alpha_k,\alpha_k)\neq 0,(k=1,2,\cdots,m-1)$，所以$\alpha_m$的定义有意义. 对于$j<m$,

$$\begin{aligned}(\alpha_m,\alpha_j)&=(\beta_m-\sum_{k=1}^{m-1}\frac{(\beta_m,\alpha_k)}{(\alpha_k,\alpha_k)}\alpha_k,\alpha_j)=(\beta_m,\alpha_j)-\sum_{k=1}^{m-1}\frac{(\beta_m,\alpha_k)}{(\alpha_k,\alpha_k)}(\alpha_k,\alpha_j)\\&=(\beta_m,\alpha_j)-\sum_{k=1}^{m-1}\frac{(\beta_m,\alpha_k)}{(\alpha_k,\alpha_k)}\delta_{jk}(\alpha_k,\alpha_j)=(\beta_m,\alpha_j)-(\beta_m,\alpha_j)=0\end{aligned}$$

所以$\alpha_1,\alpha_2,\cdots,\alpha_m$是正交向量组,

$$\mathrm{Span}\{\beta_1,\beta_2,\cdots,\beta_m\}=\mathrm{Span}\{\alpha_1,\alpha_2,\cdots,\alpha_{m-1},\beta_m\}==\mathrm{Span}\{\alpha_1,\alpha_2,\cdots,\alpha_{m-1},\alpha_m\}.$$

推论. 每一n维内积空间V中, 标准正交基必存在.

证明. 设$\{\beta_1 \cdots, \beta_n\}$是$V$的一个基, 将$\{\beta_1 \cdots, \beta_n\}$正交化得$V$的一个正交基$\alpha_1, \cdots, \alpha_n$, 再将$\alpha_1, \cdots, \alpha_n$ 单位化, 即$\xi_i = \frac{\alpha_i}{\|\alpha_i\|}$, 则$\xi_1, \cdots, \xi_n$就是$V$的一组标准正交基.

推论. n维内积空间V中, 任一正交向量组$\{\alpha_1, \cdots, \alpha_s\}$都可扩充成一个正交基.

证明. 由于$\{\alpha_1, \cdots, \alpha_s\}$是正交向量组, 所以$\{\alpha_1, \cdots, \alpha_s\}$线性无关, 于是由$\{\alpha_1, \cdots, \alpha_s\}$可以扩充成$V$的一个基$\{\alpha_1, \cdots, \alpha_s, \beta_{s+1}, \cdots, \beta_n\}$, 将这个基正交化后得到$V$的正交基$\{\alpha_1, \cdots, \alpha_s, \alpha_{s+1}, \cdots, \alpha_n\}$.

推论. 设A是数域$\mathbb{F}$上n阶可逆方阵, 则存在可逆的上三角矩阵R和矩阵Q使得$\overline{Q}^tQ=E$并且$A=QR$. (称为A的QR分解)

证明. 考虑内积空间$V=\mathbb{F}^n$, 它的内积为标准内积, 因为A是可逆阵, 所以A的列向量组$\beta_1,\cdots,\beta_n$构成了V的一组基, 由正交化公式得

$$\begin{cases} \alpha_1=\beta_1 \\ \alpha_r=\beta_r-\sum\limits_{k=1}^{r-1}\frac{(\beta_r,\alpha_k)}{(\alpha_k,\alpha_k)}\alpha_k \end{cases} \Rightarrow \begin{cases} \beta_1=\alpha_1 \\ \beta_r=\alpha_r+\sum\limits_{k=1}^{r-1}\frac{(\beta_r,\alpha_k)}{(\alpha_k,\alpha_k)}\alpha_k \end{cases}$$

写成矩阵形式

$$(\beta_1,\cdots,\beta_n)=(\alpha_1,\cdots,\alpha_n)R_1$$

这里$R_1=(b_{ij})$

$$b_{ij}=\begin{cases} 0 & i>j \\ \frac{(\beta_j,\alpha_i)}{(\alpha_i,\alpha_i)} & i\le j \end{cases}$$

令$\xi_i=\frac{\alpha_i}{\|\alpha_i\|}$, 则

$$(\alpha_1,\cdots,\alpha_n)=(\xi_1,\cdots,\xi_n)\mathrm{diag}(\|\alpha_1\|,\cdots,\|\alpha_n\|)$$

令$Q=(\xi_1,\cdots,\xi_n)$, $R=\mathrm{diag}(\|\alpha_1\|,\cdots,\|\alpha_n\|)R_1$则

$$A=(\beta_1,\cdots,\beta_n)=(\alpha_1,\cdots,\alpha_n)R_1=(\xi_1,\cdots,\xi_n)\mathrm{diag}(\|\alpha_1\|,\cdots,\|\alpha_n\|)R_1=QR$$

并且$\overline{Q}^tQ=E$

例. 设

$$A=\begin{pmatrix}1&1&-1&1\\1&0&0&-1\\0&1&0&-1\\0&0&1&1\end{pmatrix}$$

求A的QR分解

解. 设$\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{F}^4$依次是$A$的第$1,2,3,4$列向量, 它们线性无关. 将它们正交化

$$\beta_1=\alpha_1=\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\ \beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=\alpha_2-\frac{1}{2}\beta_1=\begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\\0\end{pmatrix}$$

$$\beta_3=\alpha_3-\frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1-\frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2=\alpha_3+\frac{1}{2}\beta_1+\frac{1}{3}\beta_2=\begin{pmatrix}-\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\\1\end{pmatrix}$$

$$\beta_4=\alpha_4-\frac{(\alpha_4,\beta_1)}{(\beta_1,\beta_1)}\beta_1-\frac{(\alpha_4,\beta_2)}{(\beta_2,\beta_2)}\beta_2-\frac{(\alpha_4,\beta_3)}{(\beta_3,\beta_3)}\beta_3=\alpha_4=\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$$

所以

$$
\begin{aligned}
\alpha_1 &= \beta_1 \\
\alpha_2 &= \frac{1}{2}\beta_1 + \alpha_2 \\
\alpha_3 &= -\frac{1}{2}\beta_1 - \frac{1}{3}\beta_2 + \beta_3 \\
\alpha_4 &= \beta_4
\end{aligned}
$$

即

$$
(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\beta_1, \beta_2, \beta_3, \beta_4)\begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

将$\beta_1,\beta_2,\beta_3,\beta_4$单位化:

$$\eta_1=\frac{1}{\sqrt{2}}\beta_1=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\\ 0\end{pmatrix},\eta_2=\sqrt{\frac{2}{3}}\beta_2=\begin{pmatrix}\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\\ 0\end{pmatrix},$$

$$\eta_3=\frac{\sqrt{3}}{2}\beta_3=\begin{pmatrix}-\frac{1}{\sqrt{12}}\\ \frac{1}{\sqrt{12}}\\ \frac{1}{\sqrt{12}}\\ \frac{3}{\sqrt{12}}\end{pmatrix},\eta_4=\frac{1}{2}\beta_4=\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ -\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}$$

所以

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(\eta_1,\eta_2,\eta_3,\eta_4)\begin{pmatrix}\sqrt{2} & & & \\ & \sqrt{\frac{3}{2}} & & \\ & & \frac{2}{\sqrt{3}} & \\ & & & 2\end{pmatrix}\begin{pmatrix}1 & \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 1 & -\frac{1}{3} & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

由此得

$$R=\begin{pmatrix}\sqrt{2} & & & \\ & \sqrt{\frac{3}{2}} & & \\ & & \frac{2}{\sqrt{3}} & \\ & & & 2\end{pmatrix}\begin{pmatrix}1 & \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 1 & -\frac{1}{3} & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}\sqrt{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ 0 & \frac{\sqrt{6}}{2} & -\frac{1}{\sqrt{6}} & 0\\ 0 & 0 & \frac{\sqrt{12}}{3} & 0\\ 0 & 0 & 0 & 2\end{pmatrix}$$

$$Q=(\eta_1,\eta_2,\eta_3,\eta_4)=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & -\frac{1}{2}\\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} & -\frac{1}{2}\\ 0 & 0 & \frac{3}{\sqrt{12}} & \frac{1}{2}\end{pmatrix}$$

$A=QR.$

说明. 矩阵的QR分解提供了解线性方程组$AX=\beta$的一种新的数值计算方法. 事实上由$AX=\beta$和$A=QR$, 得$RX=Q^{-1}\beta$. 而R是上三角阵, 这样就能很方便地求出解X.

推论. 设$A_{m\times n}$是列满秩矩阵, 则存在列向量组为正交单位化的m行n列矩阵Q和可逆上三角阵R, 使得$A=QR$.

证明. 请同学们完成证明.

练习1. 习题9.2:3,4,5,

练习2. 设

$$A=\begin{pmatrix}1 & -1 & 2\\ 1 & 2 & 0\\ 1 & 1 & 2\end{pmatrix}$$

求A QR分解.