1 Gamma Function & Digamma Function

1.1 Gamma Function

The gamma function is defined to be an extension of the factorial to real number arguments. By this, for example, a definition of (1/2) ! and the calculation is enabled.

1.1.1 Gauss expression

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)} \tag{1.1}$$

Calculation.

Deform $(z-1)!$ as follwс

$$(z-1)! = \frac{z!}{z} = \frac{1}{z}\cdot\frac{1\cdot 2\cdot 3\cdots z\cdot(z+1)(z+2)\cdots(z+n)}{(z+1)(z+2)\cdots(z+n)}$$

$$= \frac{n!(n+1)(n+2)\cdots(n+z)}{z(z+1)(z+2)\cdots(z+n)} = \frac{n!\,n^z \dfrac{(n+1)}{n}\dfrac{(n+2)}{n}\cdots\dfrac{(n+z)}{n}}{z(z+1)(z+2)\cdots(z+n)}$$

Because natural number n may be arbitrary, when $n\to\infty$, as follows:

$$\frac{n+1}{n} \to 1\,,\ \frac{n+2}{n} \to 1\ ,\cdots,\ \frac{n+z}{n} \to 1$$

Hence

$$\lim_{n\to\infty}\frac{n!\,n^z \dfrac{(n+1)}{n}\dfrac{(n+2)}{n}\cdots\dfrac{(n+z)}{n}}{z(z+1)(z+2)\cdots(z+n)} = \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)(z+2)\cdots(z+n)}$$

Since z is already good in anything besides $0,-1,-2,\cdots$, replacing function $(z-1)!$ with $\Gamma(z)$, we obtain the desired expression

1.1.2 Euler expression

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}\right\} \tag{2.1}$$

Calculation.

Further deform (1.1) as follows

$$\lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)(z+2)\cdots(z+n)}$$

$$= \lim_{n\to\infty}\frac{1}{z}\cdot\frac{1\cdot 2\cdot 3\cdots n}{(z+1)(z+2)\cdots(z+n)}\cdot\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n}{n-1}\right)^z$$

$$= \lim_{n\to\infty}\frac{1}{z}\cdot\frac{1\cdot 2\cdot 3\cdots n}{(z+1)(z+2)\cdots(z+n)}\cdot\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n}\right)^z \qquad \because \frac{n+1}{n}\to 1$$

$$= \lim_{n\to\infty} \frac{1}{z} \cdot \frac{\left\{ \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right) \right\}^z}{\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right)}$$

$$= \frac{1}{z} \prod_{n=1}^{\infty} \left\{ \left(1+\frac{1}{n}\right)^z \left(1+\frac{z}{n}\right)^{-1} \right\}$$

1.1.3 Integral expression

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad Re(z)>0 \tag{3.1}$$

Proof.

$$\Gamma_n(z) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt \tag{3.2}$$

Here

$$\lim_{n\to\infty} \left(1-\frac{t}{n}\right)^n = e^{-t}$$

so

$$\lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt = \int_0^\infty e^{-t} t^{z-1}\, dt \tag{3.3}$$

Further, in (3.2) let $t = ns$. Then because $t:0\sim n \rightarrow s:0\sim 1$, $dt = n ds$

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt = \int_0^1 (1-s)^n n^{z-1} s^{z-1} n\, ds = n^z \int_0^1 (1-s)^n\, s^{z-1}\, ds$$

$$= n^z \left[\frac{s^z}{z} (1-s)^n \right]_0^1 + \frac{n^z \cdot n}{z} \int_0^1 (1-s)^{n-1}\, s^z\, ds$$

$$= \frac{n^z \cdot n}{z} \left[\frac{s^{z+1}}{z+1} (1-s)^{n-1} \right]_0^1 + \frac{n^z \cdot n\,(n-1)}{z(z+1)} \int_0^1 (1-s)^{n-2}\, s^{z+1}\, ds$$

$$\vdots$$

$$= \frac{n^z \cdot n(n-1)\cdots 3\cdot 2\cdot 1}{z(z+1)(z+2)\cdots(z+n-1)} \int_0^1 s^{z+n-1}\, ds = \frac{n^z \cdot\, n!}{z(z+1)(z+2)\cdots(z+n)}$$

Hence

$$\lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)}$$

Thus, from (1.1),(3.3),(3.4) we obtain

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt \qquad Re(z)>0 \tag{3.1}$$

1.1.4 Weierstrass expression

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} \tag{4.1}$$

Calculation.

We employ a reciprocal of Gauss expression (1.1). Then

$$\frac{z(z+1)(z+2)\cdots(z+n)}{n^z \cdot n!} = \frac{1}{n^z} \cdot z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right)$$

$$= \frac{e^{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)z}}{e^{(\log n)z}} \cdot z\left(1+\frac{z}{1}\right)e^{-z}\left(1+\frac{z}{2}\right)e^{-\frac{z}{2}}\cdots\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

$$= e^{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)z} \cdot z\left(1+\frac{z}{1}\right)e^{-z}\left(1+\frac{z}{2}\right)e^{-\frac{z}{2}}\cdots\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

$$\therefore \quad \lim_{n\to\infty} e^{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)z} \cdot z\left(1+\frac{z}{1}\right)e^{-z}\left(1+\frac{z}{2}\right)e^{-\frac{z}{2}}\cdots\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

$$= e^{\gamma z} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} \qquad where\ \gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$

$$\therefore \quad \frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) e^{-\frac{z}{n}}$$

1.1.5 Properties of the Gamma Function (The 1)

$$\Gamma(z+1) = z\Gamma(z) \tag{5.1}$$

$$\Gamma(1-z) = -z\Gamma(-z) \tag{5.1'}$$

$$\Gamma(1) = \Gamma(2) = 1 \tag{5.2}$$

$$\Gamma(n+1) = n! \qquad n \text{ is a nonnegative integer} \tag{5.3}$$

$$\frac{\Gamma(z+n)}{\Gamma(z)} = z(z+1)(z+2)\cdots(z+n-1) \qquad n \text{ is a natural number} \tag{5.4}$$

$$\frac{\Gamma(z)}{\Gamma(z-n)} = (z-1)(z-2)\cdots(z-n) \qquad n \text{ is a natural number} \tag{5.4'}$$

$$\frac{\Gamma(-z)}{\Gamma(-z-n)} = (-1)^{-n}\frac{\Gamma(1+z+n)}{\Gamma(1+z)} \qquad n \text{ is a nonnegative integer} \tag{5.5}$$

$$\lim_{n\to\infty}\frac{\Gamma(z+n)}{\Gamma(n)n^z} = 1 \tag{5.6}$$

Proof.

From Gauss expression (1.1)

$$\frac{\Gamma(z+1)}{\Gamma(z)} = \lim_{n\to\infty}\left\{\frac{n!\,n^{z+1}z(z+1)(z+2)\cdots(z+n)}{(z+1)(z+2)\cdots(z+n+1)\cdot n!\,n^{z}}\right\} = \lim_{n\to\infty} z\cdot\frac{n}{z+n+1} = z$$

Then (5.1) follows. And reversing the sign of (5.1), we obtain (5.1').

Next,

$$\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)e^{-\frac{1}{k}} = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)\cdot e^{-1}\cdot e^{-\frac{1}{2}}\cdots e^{-\frac{n+1}{n}}$$

$$= \frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n}\cdot e^{-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)} = \frac{(n+1)}{n}\cdot e^{Log\,n}\cdot e^{-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}$$

$$= \frac{(n+1)}{n}\cdot e^{-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-Log\,n\right)}$$

$$\therefore\quad \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}} = \lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)e^{-\frac{1}{k}} = e^{-\gamma}$$

Substitute this for Weierstrass expression (4.1) as follows.

$$\frac{1}{\Gamma(1)} = e^{\gamma\cdot 1}\,1\cdot\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}} = e^{\gamma}\cdot e^{-\gamma} = 1$$

Hence $\Gamma(1)=1$, and from (5.1) $\Gamma(2)=1\times\Gamma(1)=1$ i.e. (5.2) follows.

Substitute positive integer n for (5.1) one by one as follows.

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots \qquad = n!\,\Gamma(1) = n!$$

i.e. (5.2) follows. And because $\Gamma(1)=0!$ (5.3) holds also for $n=0$.

From (5.1)

$$\frac{\Gamma(z+1)}{\Gamma(z)} = z\,,\ \frac{\Gamma(z+2)}{\Gamma(z+1)} = z+1\,,\ \cdots\ \frac{\Gamma(z+n)}{\Gamma(z+n-1)} = z+n-1$$

Multiply these on each other as follows.

$$\frac{\Gamma(z+1)}{\Gamma(z)}\cdot\frac{\Gamma(z+2)}{\Gamma(z+1)}\cdot\ \cdots\ \frac{\Gamma(z+n)}{\Gamma(z+n-1)} = z(z+1)(z+2)\cdots(z+n-1)$$

Hence (5.4) follows. and in (5.4) by replacing z with $z-n$, (5.4') follows.

Let replace z with $-z$ in (5.4') , then

$$\frac{\Gamma(-z)}{\Gamma(-z-n)} = (-z-1)(-z-2)\cdots(-z-n)$$

$$= (-1)^{-n}(z+1)(z+2)\cdots(z+n) = (-1)^{-n}\frac{\Gamma(1+z+n)}{\Gamma(1+z)}$$

This equation holds also for $n=0$. Moreover, this equation holds obviously for the positive integer z.

Then,this quation includs a part of Singular Point Formulas (Later **1.3**).

From (5.4)

$$\Gamma(z+n) = z(z+1)(z+2)\cdots(z+n-1)\Gamma(z)$$

$$=z(z+1)(z+2)\cdots(z+n-1)(z+n)\cdot\Gamma(z)\cdot\frac{1}{(z+n)}$$

$$\therefore\quad \lim_{n\to\infty}\frac{\Gamma(z+n)}{(n-1)!\,n^z}=\lim_{n\to\infty}\frac{z(z+1)(z+2)\cdots(z+n-1)(z+n)\cdot\Gamma(z)}{(n-1)!\,n^z(z+n)}$$

$$=\lim_{n\to\infty}\frac{z(z+1)(z+2)\cdots(z+n-1)(z+n)}{n!\,n^z}\cdot\frac{n}{z+n}\cdot\Gamma(z)$$

$$=\frac{1}{\Gamma(z)}\cdot 1\cdot\Gamma(z)=1$$

By substituting $(n-1)!=\Gamma(n)$ for this, (5.6) follows.

1.1.6 Properties of the Gamma Function (The 2)

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^n}(2n-1)!!$$

$$\Gamma\left(\frac{1}{2}-n\right)=(-1)^n\frac{2^n\sqrt{\pi}}{(2n-1)!!}$$

$$\Gamma\left(n+\frac{1}{3}\right)=\frac{1\cdot 4\cdot 7\cdots(3n-2)}{3^n}\Gamma\left(\frac{1}{3}\right)$$

$$\Gamma\left(n+\frac{2}{3}\right)=\frac{2\cdot 5\cdot 8\cdots(3n-1)}{3^n}\Gamma\left(\frac{2}{3}\right)$$

$$\Gamma\left(n+\frac{1}{4}\right)=\frac{1\cdot 5\cdot 9\cdots(4n-3)}{4^n}\Gamma\left(\frac{1}{4}\right)$$

$$\Gamma\left(n+\frac{3}{4}\right)=\frac{3\cdot 7\cdot 11\cdots(4n-1)}{4^n}\Gamma\left(\frac{3}{4}\right)$$

$$\Gamma(z)\Gamma(-z)=-\frac{\pi}{z\sin\pi z},\qquad \Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$$

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)=\frac{\pi}{\cos\pi z},\qquad \Gamma(1+z)\Gamma(1-z)=\frac{\pi z}{\sin\pi z}$$

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

$$\Gamma(3z)=\frac{3^{3z-1/2}}{2\pi}\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right)$$

Proof.: Omitted.

1.1.7 Logarithmic Function & Gamma Function

$$\int_x^{x+1}\frac{\Gamma'(z)}{\Gamma(z)}dz=\log x \tag{7.1}$$

$$\int_1^{n+1} \frac{\Gamma'(z)}{\Gamma(z)} dz = log(n!) \tag{7.2}$$

Calculation.

Just do logarithm integral calculus obediently, as follows:

$$\int_x^{x+1} \frac{\Gamma'(z)}{\Gamma(z)} dz = [log\Gamma(z)]_x^{x+1} = log\Gamma(x+1) - log\Gamma(x)$$
$$= log\{x\Gamma(x)\} - log\Gamma(x) = log\, x + log\Gamma(x) - log\Gamma(x)$$
$$= log\, x$$

Also

$$\int_1^{n+1} \frac{\Gamma'(z)}{\Gamma(z)} dz = [log\Gamma(z)]_1^{n+1} = log\Gamma(n+1) - log\Gamma(1)$$
$$= log(n!) - log 1$$
$$= log(n!)$$

1.1.8 2nd order differential calculus of the Gamma Function

$$\frac{\Gamma''(z)}{\Gamma(z)} = \left\{\frac{\Gamma'(z)}{\Gamma(z)}\right\}' + \left\{\frac{\Gamma'(z)}{\Gamma(z)}\right\}^2 = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2} + \left\{\frac{\Gamma'(z)}{\Gamma(z)}\right\}^2 \tag{8.1}$$

Calculation.

$$\left\{\frac{\Gamma'(z)}{\Gamma(z)}\right\}' = \frac{\Gamma''(z)\Gamma(z) - \Gamma'(z)^2}{\Gamma(z)^2} = \frac{\Gamma''(z)}{\Gamma(z)} - \left\{\frac{\Gamma'(z)}{\Gamma(z)}\right\}^2$$

From this

$$\frac{\Gamma''(z)}{\Gamma(z)} = \left\{\frac{\Gamma'(z)}{\Gamma(z)}\right\}' + \left\{\frac{\Gamma'(z)}{\Gamma(z)}\right\}^2$$

On the other hand, from next section (**2.8** Trigamma Function)

$$\left\{\frac{\Gamma'(z)}{\Gamma(z)}\right\}' = \frac{d}{dz}\psi(z) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$$

Hence (8.1) follows.

1.1.9 Spcial Values of the Gamma Function

From properties of the gamma function (**1.1.5 , 1.1.6**), the following special values are obtained. Because these are used frequently, we write here.

(1) The 1

$$\Gamma(0) = \infty$$

$$\Gamma(1) = 1 \quad , \quad \Gamma(2) = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \frac{-1!!}{2^0}\sqrt{\pi} = \sqrt{\pi} \quad , \quad \Gamma\left(\frac{3}{2}\right) = \frac{1!!}{2^1}\sqrt{\pi} = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3!!}{2^2}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi} \quad , \quad \Gamma\left(\frac{7}{2}\right) = \frac{5!!}{2^3}\sqrt{\pi} = \frac{15}{8}\sqrt{\pi}$$

$$\Gamma\left(\frac{9}{2}\right) = \frac{7!!}{2^4}\sqrt{\pi} = \frac{105}{16}\sqrt{\pi} \quad , \quad \Gamma\left(\frac{11}{2}\right) = \frac{9!!}{2^5}\sqrt{\pi} = \frac{945}{32}\sqrt{\pi}$$

$$\Gamma\left(\frac{1}{3}\right) = 2.678938\cdots \quad , \quad \Gamma\left(\frac{2}{3}\right) = 1.354118\cdots$$

$$\Gamma\left(\frac{1}{4}\right) = 3.625600\cdots \quad , \quad \Gamma\left(\frac{3}{4}\right) = 1.225417\cdots$$

(2) The 2

$$\Gamma\left(-\frac{1}{2}\right) = -\frac{2^1}{1!!}\sqrt{\pi} = -2\sqrt{\pi} \quad , \quad \Gamma\left(-\frac{3}{2}\right) = \frac{2^2}{3!!}\sqrt{\pi} = \frac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(-\frac{5}{2}\right) = -\frac{2^3}{5!!}\sqrt{\pi} = -\frac{8}{15}\sqrt{\pi} \quad , \quad \Gamma\left(-\frac{7}{2}\right) = \frac{2^4}{7!!}\sqrt{\pi} = \frac{16}{105}\sqrt{\pi}$$

$$\Gamma\left(-\frac{9}{2}\right) = -\frac{2^5}{9!!}\sqrt{\pi} = -\frac{32}{945}\sqrt{\pi} \quad , \quad \Gamma\left(-\frac{11}{2}\right) = \frac{2^6}{11!!}\sqrt{\pi}$$

1.2 Digamma Function

1.2.1 The definition of the Digamma Function

1st order derivative of the logarithm of the gamma function $\Gamma(z)$ is called Digamma Function, and is defined in the following expression.

$$\psi(z)=\frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{1.0}$$

$$\psi(z)=\int_0^{\infty}\left(\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\right)dt \qquad \text{(integral expression)} \tag{1.0'}$$

In addition, the derivative more than 2nd order are callecd Trigamma ,Tetragamma, Pentagamma, etc., and generaly, n th order derivative is called Polygamma Function.

These are dnoted like $\psi^{(0)}(z),\psi^{(1)}(z),\psi^{(2)}(z),\cdots,\psi^{(n)}(z)$ including the digamma, and named generally with Psi Function.

1.2.2 Properties of the Digamma Function

$$\psi(z) = -\gamma-\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right) = \lim_{n\to\infty}\left(\log n - \sum_{k=0}^{n}\frac{1}{k+z}\right) \tag{2.1}$$

$$\psi(1)= -\gamma , \qquad \psi(2) = 1-\gamma \tag{2.2}$$

$$\psi(z+1) = \psi(z) + \frac{1}{z} = -\gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right) \tag{2.3}$$

$$\psi(z+n) = \psi(z) + \sum_{k=0}^{n-1}\frac{1}{z+k} , \quad \psi(z-n) = \psi(z) - \sum_{k=1}^{n}\frac{1}{z-k} \tag{2.4}$$

$$\psi(1+n) = -\gamma + \sum_{k=1}^{n}\frac{1}{k} \tag{2.5}$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2 \tag{2.6}$$

$$\psi\left(\frac{1}{2}\pm n\right) = -\gamma - 2\log 2 + 2\sum_{k=0}^{n-1}\frac{1}{2k+1} \tag{2.7}$$

$$\psi^{(1)}(z) = \frac{d}{dz}\psi(z) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2} \qquad \text{(Trigamma Function)} \tag{2.8}$$

where $\gamma = 0.57721566490153286060651209008240243104215\cdots$

Proof

Let invers Weierstrass expression in the previous section (**1.1.4**) as follows.

$$\Gamma(z) = e^{-\gamma z}\cdot z^{-1}\cdot\prod_{n=1}^{\infty}\left(\frac{n}{n+z}\cdot e^{\frac{z}{n}}\right)$$

And differentiate the logarithm of this with respect to z, then

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{\gamma e^{-rz}}{e^{-\gamma z}} - \frac{z^{-2}}{z^{-1}} + \sum_{n=1}^{\infty}\frac{e^{z/n}/n}{e^{z/n}} - \sum_{n=1}^{\infty}\frac{n/(n+z)^2}{n/(n+z)}$$

$$= -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right)$$

Hence the first half of (2.1) follows.

Next,

$$-\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right) = -\gamma + \sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=0}^{\infty}\frac{1}{n+z}$$

$$= -\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) + \sum_{n=1}^{\infty}\frac{1}{n} - \sum_{n=0}^{\infty}\frac{1}{n+z}$$

$$= \lim_{n\to\infty}\left(\log n - \sum_{k=0}^{n}\frac{1}{k+z}\right)$$

Hence the latter half of (2.1) follows.

Substitute $z=1,\ z=2$ for the first half of (2.1), then

$$\psi(1) = -\gamma - \frac{1}{1} + \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = -\gamma$$

$$\psi(2) = -\gamma - \frac{1}{2} + \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots$$

$$= 1 - \gamma - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) = 1 - \gamma$$

Thus (2.2) follows.

Differentiate both side of $\Gamma(z+1) = z\Gamma(z)$ (the previous (5.1)) with respect to z , then

$$\Gamma'(z+1) = z\Gamma'(z) + \Gamma(z)$$

Divide both side of this by $\Gamma(z+1) = z\Gamma(z)$, then

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{\Gamma'(z)}{\Gamma(z)} + \frac{1}{z}, \qquad \text{i.e.} \qquad \psi(z+1) = \psi(z) + \frac{1}{z}$$

Hence the first half of (2.3) follows, and by substituting (2.1) for this, the latter half of (2.3) follows.

Next, substitute $z+1, z+2, \cdots, z+n-1$ and $z, z-1, \cdots, z-n$ for (2.3) sequentially; then

$$\psi(z+1) - \psi(z) = \frac{1}{z}, \qquad \psi(z) - \psi(z-1) = \frac{1}{z-1}$$

$$\psi(z+2) - \psi(z+1) = \frac{1}{z+1}, \qquad \psi(z-1) - \psi(z-2) = \frac{1}{z-2}$$

$$\vdots$$

$$\psi(z+n) - \psi(z+n-1) = \frac{1}{z+n-1}, \qquad \psi(z-n+1) - \psi(z-n) = \frac{1}{z-n}$$

Add these on each other, then

$$\psi(z+n) - \psi(z) = \sum_{k=0}^{n-1} \frac{1}{z+k} \qquad , \qquad \psi(z) - \psi(z-n) = \sum_{k=1}^{n} \frac{1}{z-k}$$

Hence (2.4) follows.

Substitute $z=1$ for the first half of (2.4); then

$$\psi(1+n) = \psi(1) + \sum_{k=0}^{n-1} \frac{1}{1+k} = -\gamma + \sum_{k=1}^{n} \frac{1}{k}$$

Hence (2.5) follows.

Substitute $z=1/2$ for (2.1) ; then

$$\psi\left(\frac{1}{2}\right) = -\gamma + \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k=0}^{\infty} \frac{2}{2k+1} = -\gamma + 2\sum_{k=1}^{\infty} \frac{1}{2k} - 2\sum_{k=0}^{\infty} \frac{1}{2k+1}$$

$$= -\gamma - 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = -\gamma - 2\log 2$$

Hence (2.6) follows.

Substitute $z=1/2 \pm n$ for (2.1) ; then

$$\psi\left(\frac{1}{2}+n\right) = -\gamma + \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k=0}^{\infty} \frac{1}{k+\frac{1}{2}+n}$$

$$= -\gamma + 2\sum_{k=1}^{\infty} \frac{1}{2k} - 2\sum_{k=0}^{\infty} \frac{1}{2k+1+2n} - 2\sum_{k=0}^{n-1} \frac{1}{2k+1} + 2\sum_{k=0}^{n-1} \frac{1}{2k+1}$$

$$= -\gamma + 2\sum_{k=1}^{\infty} \frac{1}{2k} - 2\sum_{k=0}^{\infty} \frac{1}{2k+1} + 2\sum_{k=0}^{n-1} \frac{1}{2k+1}$$

$$= -\gamma - 2\log 2 + 2\sum_{k=1}^{n} \frac{1}{2k-1}$$

$$\psi\left(\frac{1}{2}-n\right) = -\gamma + \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k=0}^{\infty} \frac{1}{k+\frac{1}{2}-n} = -\gamma + 2\sum_{k=1}^{\infty} \frac{1}{2k} - 2\sum_{k=0}^{\infty} \frac{1}{2k+1-2n}$$

$$= -\gamma + 2\sum_{k=1}^{\infty} \frac{1}{2k} - 2\sum_{k=0}^{n-1} \frac{1}{2k+1-2n} - 2\sum_{k=n}^{\infty} \frac{1}{2k+1-2n}$$

$$= -\gamma + 2\sum_{k=1}^{\infty} \frac{1}{2k} + 2\sum_{k=1}^{n} \frac{1}{2k-1} - 2\sum_{k=1}^{\infty} \frac{1}{2k+1}$$

$$= -\gamma - 2\log 2 + 2\sum_{k=1}^{n} \frac{1}{2k-1}$$

Hence (2.7) follows.

Finally,

$$\frac{d}{dz}\psi(z) = \frac{d}{dz}\left\{-\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right)\right\}$$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{(n+z)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}$$

Hence (2.8) follows.

1.2.3 Spcial values of the Digamma Function

From properties of the digamma function (**1.2.2**), the following special values are obtained.

(1) The 1

$$\psi(0) = -\infty$$

$$\psi(1) = -\gamma \quad , \quad \psi(2) = -\gamma + 1$$

$$\psi(3) = -\gamma + \frac{3}{2} \quad , \quad \psi(4) = -\gamma + \frac{11}{6}$$

$$\psi(5) = -\gamma + \frac{5}{4} \quad , \quad \psi(6) = -\gamma + \frac{29}{20}$$

$$\psi(7) = -\gamma + \frac{97}{60} \quad , \quad \psi(8) = -\gamma + \frac{739}{420}$$

(2) The 2

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2$$

$$\psi\left(\frac{3}{2}\right) = \psi\left(-\frac{1}{2}\right) = -\gamma - 2\log 2 + 2$$

$$\psi\left(\frac{5}{2}\right) = \psi\left(-\frac{3}{2}\right) = -\gamma - 2\log 2 + 2\times\frac{4}{3}$$

$$\psi\left(\frac{7}{2}\right) = \psi\left(-\frac{5}{2}\right) = -\gamma - 2\log 2 + 2\times\frac{23}{15}$$

$$\psi\left(\frac{9}{2}\right) = \psi\left(-\frac{7}{2}\right) = -\gamma - 2\log 2 + 2\times\frac{176}{105}$$

$$\psi\left(\frac{11}{2}\right) = \psi\left(-\frac{9}{2}\right) = -\gamma - 2\log 2 + 2\times\frac{563}{315}$$

$$\psi\left(\frac{13}{2}\right) = \psi\left(-\frac{11}{2}\right) = -\gamma - 2\log 2 + 2\times\frac{6508}{3465}$$

1.3 Singular Point Formulas

Formula 1.3.1

When $\Gamma(z)$, $\psi(z)$, $\psi_m(z)$ denote gamma function, digamma function and polygamma function respectively, following expressions hold for $n=0,1,2,3,\cdots$.

$$\frac{\Gamma(0)}{\Gamma(-n)} = (-1)^n\Gamma(1+n) = (-1)^n n! \tag{1.1}$$

$$\frac{\Gamma(-n)}{\Gamma(-m)} = (-1)^{m-n}\frac{m!}{n!} \qquad (m=0,1,2,3,\cdots) \tag{1.2}$$

$$\frac{\psi(-n)}{\Gamma(-n)} = (-1)^{n+1}n! \tag{1.3}$$

$$\frac{\psi(-n)}{\psi\{-(n+1)\}} = 1 \tag{1.4}$$

$$\frac{\psi_m(-n)}{\psi_m\{-(n+1)\}} = 1 \qquad \left\{\psi_m(z)=\frac{d^m}{dz^m}\psi(z)\right\} \tag{1.5}$$

Proof

From Gauss expression (**1.1.1**), the following expression holds.

$$\frac{\Gamma(z+1)}{\Gamma(z)} = \lim_{n\to\infty}\frac{n!\,n^{z+1}\ z(z+1)(z+2)\cdots(z+n)}{n!\,n^z\ (z+1)(z+2)\cdots(z+n+1)}$$

When $z = -1,-2\cdots$, this expression becomes the indeterminate form, and the value is not decided.

Now assume $z \ne -1$, $\ne -2$, $\cdots$, then $z+1$, $z+2$, $\cdots$, $z+n$ become nonzero all.

Hence, we can divide the numerator and the denominator by $z+1$, $z+2$, $\cdots$, $z+n$ as follows:

$$\frac{\Gamma(z+1)}{\Gamma(z)} = \lim_{n\to\infty}\frac{nz}{z+n+1} = \lim_{n\to\infty}\frac{z}{\frac{z}{n}+1+\frac{1}{n}} = z \tag{1.0}$$

Here substitute $z=0$ for this; then

$$\frac{\Gamma(z+1)}{\Gamma(z)} = \frac{\Gamma(1)}{\Gamma(0)} = 0$$

Because we have supposed it to be $z \ne -1$, we cannot substitute $z=-1$ for (1.0).

Therefore let $z\to -1$. Then

$$\lim_{z\to -1}\frac{\Gamma(z+1)}{\Gamma(z)} = \frac{\Gamma(0)}{\Gamma(-1)} = -1$$

Similarly let $z\to -2,-3,\cdots, -k, \cdots$, then

$$\frac{\Gamma(-1)}{\Gamma(-2)} = -2,\ \frac{\Gamma(-2)}{\Gamma(-3)} = -3,\ \cdots,\ \frac{\Gamma(-k+1)}{\Gamma(-k)} = -k,\ \cdots$$

Multiply these from 1 to k ; then

$$\frac{\Gamma(0)}{\Gamma(-k)} = \frac{\Gamma(0)}{\Gamma(-1)}\frac{\Gamma(-1)}{\Gamma(-2)}\cdots\frac{\Gamma(-k+1)}{\Gamma(-k)} = (-1)^k k! = (-1)^k\Gamma(1+k)$$

Replacing k with n gives (1.1). And using this

$$\frac{\Gamma(-n)}{\Gamma(-m)} = \frac{\Gamma(-n)}{\Gamma(0)}\frac{\Gamma(0)}{\Gamma(-m)} = \frac{(-1)^m m!}{(-1)^n n!} = (-1)^{m-n}\frac{m!}{n!}$$

i.e. we obtain (1.2). This expression also holds for $m=0, n=0$.

Next

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!\ n^z}{z(z+1)(z+2)\cdots(z+n)}$$

$$\psi(z) = \frac{d\log\Gamma(z)}{dz} = \lim_{n\to\infty}\left\{\log n - \left(\frac{1}{z}+\frac{1}{z+1}+\frac{1}{z+2}+\cdots+\frac{1}{z+n}\right)\right\}$$

From these two expressions

$$\frac{\psi(z)}{\Gamma(z)} = \lim_{n\to\infty}\frac{\log n - \left(\frac{1}{z}+\frac{1}{z+1}+\frac{1}{z+2}+\cdots+\frac{1}{z+n}\right)}{\dfrac{n!\ n^z}{z(z+1)(z+2)\cdots(z+n)}}$$

Assum $z\neq 0$ and multiply $z+k$ $(k=0,1,2,\cdots)$ by numerator and denominator; then

$$\frac{\psi(z)}{\Gamma(z)} = \lim_{n\to\infty}\frac{(z+k)\log n - \left(\frac{z+k}{z}+\frac{z+k}{z+1}+\cdots+\frac{z+k}{z+k-1}+1+\frac{z+k}{z+k+1}+\cdots+\frac{z+k}{z+n}\right)}{\dfrac{n!\ n^z}{z(z+1)\cdots(z+k-1)\cdot 1\cdot(z+k+1)\cdots(z+n)}}$$

Here let $z\to -k$ $(k=0,1,2,\cdots)$, then as follows.

$$\frac{\psi(-k)}{\Gamma(-k)} = \lim_{n\to\infty}\frac{0\cdot\log n - \left(\frac{0}{-k}+\frac{0}{-k+1}+\cdots+\frac{0}{-1}+1+\frac{0}{1}+\cdots+\frac{0}{n}\right)}{\dfrac{n!\ n^{-k}}{(-k)\cdots(-2)(-1)\cdot 1\cdot\{1\cdot 2\cdots(n-k)\}}}$$

$$= \lim_{n\to\infty} -\frac{(-k)\cdots(-2)(-1)\cdot 1\cdot\{1\cdot 2\cdots(n-k)\}}{n!\ n^{-k}}$$

Calculate this in detail ;

When $z\to -k$ $(k=0)$

$$\frac{\psi(-k)}{\Gamma(-k)} = \lim_{n\to\infty} -\frac{n^0 n!}{n!} = 1$$

When $z\to -k$ $(k=1,2,3,\cdots)$

$$\frac{\psi(-k)}{\Gamma(-k)} = \lim_{n\to\infty}(-1)^{k+1}k!\cdot\frac{n^k}{(n-k+1)(n-k+2)\cdots(n-k+k)}$$

$$= \lim_{n\to\infty}(-1)^{k+1}k!\cdot\frac{1}{\left(1-\frac{k}{n}+\frac{1}{n}\right)\left(1-\frac{k}{n}+\frac{2}{n}\right)\cdots\left(1-\frac{k}{n}+\frac{k}{n}\right)}$$

$$= (-1)^{k+1} k!$$

Hence Replacing k with n gives (1.3).

Next using (1.2),(1.3)

$$\frac{\psi(-n)}{\psi\{-(n+1)\}} = \frac{\psi(-n)}{\Gamma(-n)} \frac{\Gamma(-n)}{\Gamma\{-(n+1)\}} \frac{\Gamma\{-(n+1)\}}{\psi\{-(n+1)\}}$$

$$= (-1)^{n+1} n! \{-(n+1)\} \frac{1}{(-1)^{n+2}(n+1)!} = 1$$

Thus we obtain (1.4).

Finally,

$$\psi_m(z) = (-1)^{m+1} m! \left\{ \frac{1}{z^{m+1}} + \frac{1}{(z+1)^{m+1}} + \frac{1}{(z+2)^{m+1}} + \cdots \right\}$$

From this

$$\frac{\psi_m\{-(z+1)\}}{\psi_m(-z)} = \frac{\dfrac{1}{(-z-1)^{m+1}} + \dfrac{1}{(-z)^{m+1}} + \dfrac{1}{(-z+1)^{m+1}} + \dfrac{1}{(-z+2)^{m+1}} + \cdots}{\dfrac{1}{(-z)^{m+1}} + \dfrac{1}{(-z+1)^{m+1}} + \dfrac{1}{(-z+2)^{m+1}} + \cdots}$$

Assuming $z \neq 0$, denominator is also non-zero and positive. Then

$$\frac{\psi_m\{-(z+1)\}}{\psi_m(-z)} = 1 + \frac{\dfrac{1}{(-z-1)^{m+1}}}{\dfrac{1}{(-z)^{m+1}} + \dfrac{1}{(-z+1)^{m+1}} + \cdots + \dfrac{1}{(-z+n)^{m+1}} + \cdots}$$

Let $z \to n$ then $\dfrac{1}{(-z+n)^{m+1}} \to \infty$. Therefor the summation of terms after $\dfrac{1}{(-z+n+1)^{m+1}}$ converges to a finite value. $(\because m>0)$. Consequently, second term in the right hand side converges to 0.

Thus

$$\frac{\psi_m\{-(n+1)\}}{\psi_m(-n)} = \lim_{z \to n} \frac{\psi_m\{-(z+1)\}}{\psi_m(-z)} = 1$$

i.e. (1.5) was poroved.

Examples

The results actually calculated with computational software are as follows.

```
f[z_] := Gamma[z + 1] / Gamma[z]                     Limit [f[z], z → -2]   -2

g[z_] := PolyGamma[z] / Gamma[z]                     Limit [g[z], z → -3]    6

h[z_] := PolyGamma[3, z] / PolyGamma[3, z - 1]       Limit [h[z], z → -4]    1
```

Significance of these formulas

$z = 0, -1, -2 \cdots$ are singular points (pole of order 1). The value of these functions is $\pm\infty$. Nevertheless, the arbitrary ratio of $\Gamma(z), \psi(z)$ on these points is removable singular point. esides, all those ratios reduce to the integer or the reciprocal number of the integer. formulas (1.1) ~ (1.4) mentioned above insists on this.

For exsample

$$\frac{\Gamma(-3)}{\Gamma(-7)} = \frac{\Gamma(-3)}{\Gamma(0)}\frac{\Gamma(0)}{\Gamma(-7)} = \frac{-7!}{-3!} = 840$$

$$\frac{\Gamma(-3)}{\psi(-7)} = \frac{\Gamma(-3)}{\Gamma(-7)}\frac{\Gamma(-7)}{\psi(-7)} = \frac{-7!}{-3!}\frac{1}{(-1)^8 7!} = \frac{1}{6}$$

$$\frac{\psi(-8)}{\Gamma(-5)} = \frac{\psi(-8)}{\Gamma(-8)}\frac{\Gamma(-8)}{\Gamma(-5)} = (-1)^9 8!\frac{-5!}{8!} = 120$$

etc.

These are phenomena which is peculiar to the gamma function and the digamma function and do not occur in the polygamma functon more than the trigamma function.

For example, if we adopt the ratio $\psi_1(z)/\psi_0(z)$ of the trigamma functkion and the digamma function, $z=0,-1,-3,\cdots$ are all poles of order 1 of this function (ratio) , and generaly, if we adopt the ratio $\psi_m(z)/\psi_n(z)$ $(m>n)$, $z=0,-1,-2,-3,\cdots$ become poles of order $(m-n)$. Functions more than the trigamma are right different from Functions less than the digamma in a dimension.

What I can say in polygamma functions more than the trigamma is only that the ratio of polygamma functions of the same dimension becomes 1 in $z=0,-1,-2,-3,\cdots$ entirely.

It is indispensable for Super Calculus (non-integer order calculus) of the power function and the logarithmic function that the ratio between singular points of the gamma function or the digamma function is a rational number.

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Alien's Mathematics